MA 323

Lecture # 12

Van der Corput Sequences: We introduce a specific class of one-dimensional low-discrepancy sequences called Van der Corput sequences.

By a "base" we mean an integer $b \geq 2$. Every positive integer k has a unique representation (called its base-b or b-ary expansion) as a linear combination of non-negative powers of b with coefficients in $\{0, 1, .., b-1\}$.

We can write this as, $$k = \sum_{j=0}^{\infty} a_j(k)\, b^j,$$

with all but finitely many coefficients $a_j(k)$ equal to zero.

The "radical inverse function" ψ_b maps each k to a point in $[0, 1)$ by flipping the coefficients of k about the base-b "decimal" point to get the base-b fraction

$. a_0 a_1 a_2 \ldots$ More precisely,

$$\psi_b(k) = \sum_{j=0}^{\infty} \frac{a_j(k)}{b^{j+1}}$$

The base-b Van der Corput sequence is the sequence

$0 = \psi_b(0), \psi_b(1), \psi_b(2), \ldots$

k	0	1	2	3	4	5	6	7
k Binary	0	1	10	11	100	101	110	111
$\psi_2(k)$ Binary	0	0.1	0.01	0.11	0.001	0.101	0.011	0.111
$\psi_2(k)$	0	$1/2$	$1/4$	$3/4$	$1/8$	$5/8$	$3/8$	$7/8$

Halton Sequence : Halton provided the simplest construction and first analysis of the low-discrepancy sequences in arbitrary dimension d. The coordinates of a Halton sequence follow Van der Corput sequences in distinct bases. Thus, let $b_1, b_2, \ldots, b_d$ be relatively prime integers greater than 1, and set,

$$x_k = \left(\psi_{b_1}(k), \psi_{b_2}(k), \ldots, \psi_{b_d}(k) \right), \quad k = 0, 1, 2, \ldots$$

where ψ_b is the radical inverse function

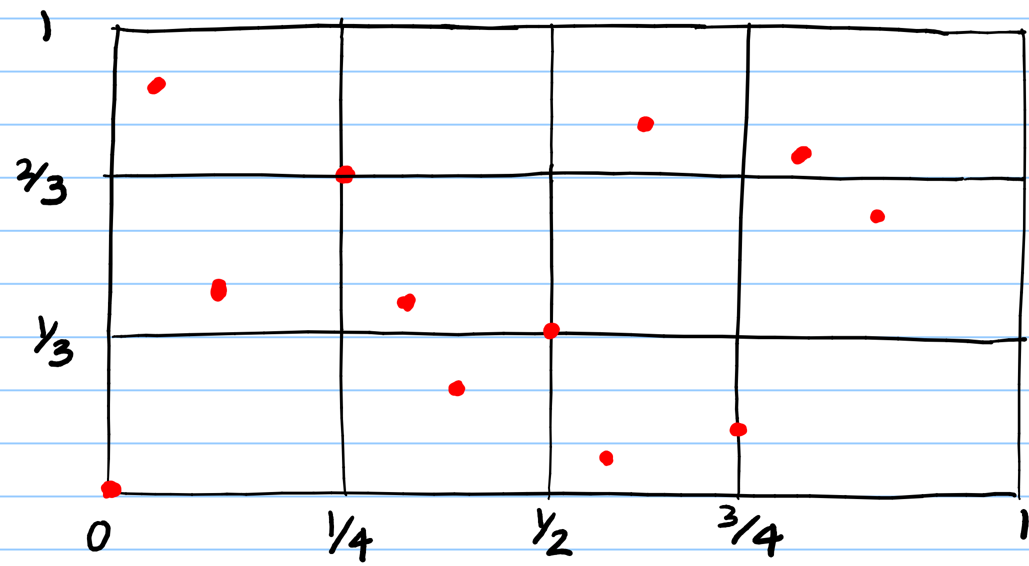
$$\left(\psi_b(k) = \sum_{j=0}^{\infty} \frac{a_j(k)}{b^{j+1}} \right)$$

The requirement that the b_i's be relatively prime is necessary for the sequence to fill the hypercube.

<u>Example:</u> The two-dimensional sequence defined by $b_1 = 2$ and $b_2 = 6$ has no points in $\left[0, \frac{1}{2}\right) \times \left[\frac{5}{6}, 1\right)$.

Since smaller bases are preferred to larger bases, we choose $b_1, b_2, \ldots, b_d$ to be the first d prime numbers.

k	0	1	2	3	4	5	6	7	8	9	10	11
$\psi_2(k)$	0	$1/2$	$1/4$	$3/4$	$1/8$	$5/8$	$3/8$	$7/8$	$1/16$	$9/16$	$5/16$	$13/16$
$\psi_3(k)$	0	$1/3$	$2/3$	$1/9$	$4/9$	$7/9$	$2/9$	$5/9$	$8/9$	$1/27$	$10/27$	$19/27$



First twelve points of a two-dimensional Halton sequence.

Halton points form an infinite sequence. We can achieve slightly better uniformity if we are willing to fix the number of points "n" in advance. The "n" points

$$\left\{ \left(\frac{k}{n}, \psi_{b_1}(k), \ldots, \psi_{b_{d-1}}(k) \right), k = 0, 1, 2, \ldots, n-1 \right\}$$

with relatively prime $b_1, b_2, \ldots, b_{d-1}$ form a "Hammersley point set" in dimension d.

<u>Faure Sequence:</u> Faure developed a different extension of Van der Corput sequences to multiple dimensions in which all coordinates use a common base.

This base must be at least as large as the dimension itself, but can be much smaller than the largest base used for a Halton sequence of equal dimension.

In a d-dimensional Faure sequence, the coordinates are constructed by permuting segments of a single Van der Corput sequence.

For the base b, we choose the smallest prime number greater than or equal to d. Let $a_l(k)$ denote the coefficients in the base-b expansion of k, so that

$$k = \sum_{l=0}^{\infty} a_l(k)\, b^l .$$

The i-th coordinate, $i = 1, 2, \dots, d$, of the kth point in the Faure sequence is given by,

$$\sum_{j=1}^{\infty} \frac{y_j^{(i)}(k)}{b^j},$$

where $$y_j^{(i)}(k) = \sum_{l=0}^{\infty} \binom{l}{j-1} (i-1)^{l-j+1} a_l(k) \bmod b,$$

with $$\binom{m}{n} = \begin{cases} m!/(m-n)!\,n!\,, & m \geq n, \\ 0, & \text{otherwise}, \end{cases}$$

and $0! = 1$.