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A novel approach to concept-cognitive learning in interval-valued formal contexts: a granular computing viewpoint

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Abstract
Concept-cognitive learning (CCL) is to make machines like human beings have the ability of summarizing and reasoning. Automatically learn and find concepts from given information clues is a research focus of CCL. The existing researches mainly focuses on the concept learning methods in classical and fuzzy formal contexts, but there are few researches on the CCL of interval-valued contexts. In view of the universality of interval values in practical applications, we study the mechanism of CCL in interval-valued formal contexts. Firstly, we propose interval-valued formal contexts and a pair of dual cognitive operators as the fundamental foundation of concept learning. Then we mine the relationship between interval-valued information granules and concepts from cognitive learning and granular computing perspective. Then we systematically study the mechanism of interval-valued CCL from the establishment of interval-valued information granules (IvIGs) and its mathematical properties, and the transformation between different information granules (IGs) and clue oriented concept learning. Moreover, three algorithms are established to automatically learn concepts from different clue information. Finally, we download eight public data sets to verify the effectiveness and feasibility of the proposed algorithms from the perspective of the size of extension of concepts, running time of concept learning algorithms and the number of concepts learned by the concept learning algorithms. The experimental comparison indicates that the proposed algorithms are effective and feasible for interval-valued CCL.

Keywords Interval-valued formal contexts · Granular computing · Formal concept analysis · Formal concepts · Interval-valued concept learning

1 Introduction

Concept-cognitive learning (CCL) is to learn concepts from a given clue by simulating human thought processes including perception, attention and thinking [53]. As the basic unit of human thinking, the concept of things is the basis of human cognition. For human beings, it is easy to recognize zebra although some people have never seen zebra, because the concept of zebra has been formed in the process of human growth. For machines, learning concepts from given information unconsciously and automatically like human beings is difficult. CCL can help machines simulate the behavior of human brain to learn concepts. Since 1982, Wille [43] proposed formal concept analysis (FCA) for CCL from formal contexts. After decades of development, CCL has been applied in information sciences [6, 21, 45], machine leaning [9, 19, 28, 29], feature selection [14–16] and decision analysis [8, 22, 49]. There are many methods to study CCL from different perspectives, such as fuzzy FCA [2, 18], granular computing (GrC) [30, 51] and cognitive process [23, 25, 37].

A formal context as a basis of FCA, it consists of a set of objects, a set of attributes and a binary relation between the two sets [7]. In classical formal contexts, we usually use "0" represents the object not having the attribute and "1" represents the object having the attribute in the relation. There are only two kinds of crisp relation between objects and attributes in classical formal contexts. To express the degrees to which objects have attributes, Yahia and Jaoua

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[47] introduced fuzzy sets into classical formal contexts to establish fuzzy formal contexts, where the relation in fuzzy formal contexts is expressed by a fuzzy relation table with [0, 1]. In classical formal contexts, formal concepts were obtained by the so-called Galois connection. In fuzzy formal contexts, the Galois connection is generalized to fuzzy Galois connection. Wu et al. [44] calculated granular reducts and determined granular consistent sets in formal contexts by utilizing Boolean functions and discernibility matrices. Mi et al. [27] established a kind of new Galois connection between power sets to perform attribute reduction of concept lattices. To approximate crisp and fuzzy sets, Shao and Yang [33] put forward two pairs of operators to show two types of multi-level concepts in a fuzzy formal context, and proposed two pairs of approximation operators by using the multi-level concepts to measure roughness of target concepts. To learn concepts from incomplete information, Zhao et al. [53] put forward a pair of novel approximation operators to define approximation concepts in incomplete data, then established an approximation computing system to learn approximation concepts by updating dynamically objects and attributes. Wei et al. [41] summed up the current research status and future development trend of triadic concept analysis from basic notions of triadic concept analysis, triadic fuzzy concept analysis, triadic factor analysis, and triadic implications and triadic association rules. Wei et al. [42] employed three-way concept lattices, which include the negative decision rule information and positive decision rule information simultaneously, to perform rules acquisition in formal decision contexts, and analyzed the relations among the three kinds of consistences obtained. To improve the robustness of the learned concepts and reduce computational complexity of concept learning, Zhang et al. [52] put forward an incremental concept-cognitive learning algorithm by using the topological structure of attributes for incremental concept-cognitive learning and showed the processes and results of concept-cognitive learning by using a concept tree. These researches are mainly concerned with the formation of concepts and the description of the relationship between concepts. Considering that in the big data environment, the number of data samples and features are increasing rapidly, it is very time-consuming to calculate all concepts and establish concept lattice. This paper focuses on how to learn concepts automatically from given object or attribute information.

Granular computing (GrC) is an important approach to solve complex problems by simulating human thinking in the field of computational intelligence research. It can help human to deal with complex problems by abstracting and dividing these problems into some simple subproblems. Then the solutions of the complex problems are obtained by combined with the solutions of the subproblems. The concept of GrC was explored by Zadeh [50] in 1979. GrC is a method to construct, represent and process IGs [31]. Bargiela and Pedrycz [1] believed that GrC is to solve complex applications by using granules (subsets, groups, classes, and so on) to reduce the computing scale of the applications. Yao [48] interpreted cognitive concept learning by using a layered model of knowledge discovery from the perspective of GrC and cognitive informatics. In order to make the consistency of relations as high as possible, Cabrerizo et al. [3] used the information granularity concept to estimate the missing values for incomplete preference relations. To improve learning efficiency and deal with more complex data, Niu et al. [30] investigated parallel computing techniques to perform concept learning of multi-source data and big data from the viewpoints of information fusion and GrC. From the perspective of cognitive informatics, Shivhare et al. [35] systematically established methods to distinguish the cognitive relations of objects and the cognitive relations of attributes. Moreover, they proposed objects and attributes induced forward and backward cognitive memory methods by introducing bidirectional associative memory to 3WFCA [36]. Recently, some scholars study three-way decision (3WD) [12, 13, 17, 32], rough set theory (RST) [4, 11, 20] and concept learning [24, 34, 39] from perspective of GrC and cognitive learning.

In particular, CCL based granular computing (GrC) provides an effective method for automatic concept learning. The power set of the object set and the power set of the attribute set constitute complete lattices, respectively. In the cognitive system established by two complete lattices and the two mappings between them, predecessors have explored the relationship between any information granule and a concept, and what kind of conditions of information granule is called concept. In literatures [45, 46], researchers have systematically studied the relationship between information granules and concepts under the background of classical formal context and fuzzy formal context, respectively. In many practical applications, the attribute values of objects are intervals rather than fuzzy or Boolean values, such as the patient's blood oxygen saturation and body temperature over a period of time. Therefore, it is very necessary to study the concept learning of interval-valued data. At present, the concept learning based on interval-valued data mainly focuses on the definition of concept and the establishment of lattice [40]. Singh et al. [38] combined the interval-valued fuzzy graph and the fuzzy concept lattice to reduce the number of fuzzy formal concepts and simplify the corresponding fuzzy concept lattice structure. Ma et al. [10, 26] introduced interval-set theory into concept lattice to study interval-set concept lattices, and studied mathematical properties and attribute reduction of interval-set concept lattices. Currently, there is little research on automatic concept learning in the context of interval-valued attributes. Cognitive concept learning provides an effective method for automatic concept learning in formal contexts [35, 36, 45, 46]. In this paper, we will learn concepts from interval-valued formal contexts by transformation between IGs. Firstly, the interval-valued formal context is introduced by using the interval-valued information system to express the interval-valued relations between objects and attributes. Then we define a pair of dual cognitive operators $(*, *)$ on interval-valued formal contexts as the basis of the CCL and discuss the corresponding properties. Further, we introduce the interval-valued cognitive system to define four kinds of IvIGs. Moreover, we describe the cognitive process in the interval-valued cognitive system by transforming IvIGs. Finally, we design three concept learning algorithms to learn concepts from different clue information and download eight public UCI data sets to evaluate the effectiveness and feasibility of the established concept learning algorithms.

The main innovations of this paper are as follows: (1) We propose a dual cognitive operator for concept learning in interval-valued formal contexts. (2) We establish a basic theoretical framework of granular computing based CCL in interval-valued formal contexts. (3) We develop algorithms to automatically learn concepts according to the given cue information.

The remainder of this paper is organized as follows. In Sect. 2, we briefly review the related knowledge of the formal contexts and FCA. In Sect. 3, we define interval-valued formal contexts and a pair of dual cognitive operators $(*, *)$ as basis of interval-valued cognitive concept learning. In Sect. 4, four kinds of IvIGs are defined to explore cognitive process of concept in the interval-valued cognitive system. In Sect. 5, we present the formation processes of concepts by transforming the general IvIGs into the sufficient and necessary IvIGs and design three concept learning algorithms (O-IvCL, IvA-IvCL and OIvA-IvCL) to learn concepts from different clue information. In Sect. 6, eight data sets are downloaded from UCI Machine Learning Repository to evaluate the effectiveness and feasibility of the proposed algorithms. Finally, we summarize this article and look forward to the future work.

2 Preliminaries

Before studying the concept learning in the context of interval-valued data, we first briefly review basic knowledge of formal contexts and formal concept analysis (FCA). More detailed information can be found in [7]. Throughout this paper, AT, $\widetilde{AT}$ and $\overline{AT}$ represent a classical attribute set, a fuzzy attribute set and an interval-valued attribute set, respectively.

Let (U, AT, I) be a formal context, where U and AT are the set of objects and the set of attributes, respectively. I is a relation between U and AT. If an object $x \in U$ is in the relation I with an attribute $a \in AT$, we denote $(x, a) \in I$ or xIa and read it as “the object x has the attribute a”. For example, there is a formal context with watching movies in Table 1, where U and AT are a set of movie viewers and a set of movies, respectively. a_1, a_2, a_3, a_4 and a_5 represent *Forrest Gump*, *Schindler’s List*, *The Pursuit of Happyness*, *Flipped* and *The Shawshank Redemption*, respectively. The relation I between U and AT is denoted by ✓ in Table 1, where ✓ indicates that the viewer watched the movie.

In a formal context (U, AT, I), for any subsets $X \subseteq U$ and $B \subseteq AT$, a pair of dual cognitive operators with respect to X and B is defined as $X' = \{a \in AT \mid (x, a) \in I, \forall x \in X\}$ and $B' = \{x \in U \mid (x, a) \in I, \forall a \in B\}$. The elements of X' represent the common attributes owned by all objects in X and the elements of B' represent all objects which have all attributes in B. For any pair (X, B), iff $X' = B$ and $B' = X$, we call (X, B) is a formal concept of formal context (U, AT, I), where X and B are called the extension and intension of concept (X, B), respectively. Let $\mathcal{B}(U, AT, I)$ denote the set of all concepts of formal context (U, AT, I).

Further considering Table 1, we take $X = \{x_1, x_2, x_4\}$ and $B = \{a_1, a_3, a_5\}$, and compute $X' = \{a_1, a_3, a_5\}$ and $B' = \{x_1, x_2, x_4\}$. Obviously, $X' = B$ and $B' = X$. So (X, B) is a formal concept. What this concept says is in this formal context, viewers x_1, x_2, x_4 have all watched movies *Forrest Gump*(a_1), *The Pursuit of Happyness*(a_3) and *The Shawshank Redemption* (a_5), and only viewers x_1, x_2, x_4 have watched movies *Forrest Gump*(a_1), *The Pursuit of Happyness*(a_3) and *The Shawshank Redemption*(a_5). This shows that the extension and intension of concepts are mutually and uniquely determined.

Further, we want to know which movies have been watched by each of the viewers x_1, x_2 and x_6. We take $X_1 = \{x_1, x_2, x_6\}$ and compute X_1', and obtain $X_1' = \{a_1, a_3\}$. Therefore, viewers x_1, x_2 and x_6 have watched *Forrest Gump* and *The Pursuit of Happyness*. Correspondingly, we want to know who has watched *Forrest Gump* and *The Pursuit of Happyness*. We take $B_1 = \{a_1, a_3\}$ and calculate B_1', and get $B_1' = \{x_1, x_2, x_4, x_6\}$. Obviously, in addition to viewers x_1, x_2, x_6, there is viewer x_4 who has also watched *Forrest Gump* and *The Pursuit of Happyness*. That is to say, $X_1' = B_1$

Table 1 Record of watching movies

AT	a_1	a_2	a_3	a_4	a_5
U					
x_1	✓		✓		✓
x_2	✓	✓	✓		✓
x_3		✓	✓	✓	
x_4	✓		✓	✓	✓
x_5		✓		✓	✓
x_6	✓	✓	✓		✓

but $B_1^{'} \neq X_1$. Therefore, the pair (X_1, B_1), also called information granule (IG), is not a formal concept in Table 1. This shows that a pair (an information granule) derived from any subset of objects or any subset of objects attributes is not necessarily a formal concept. So how to find the formal concepts from the given object information, attribute information, object and attribute information is the key problem of concept learning. Next, we will study concept learning in the context of interval-valued data.

3 Interval-valued formal contexts and dual cognitive operators

Formal concept analysis (FCA) is a concept learning tool based on formal contexts. We will study concept learning methods for interval-valued data based on formal concept analysis (FCA). Firstly, we define interval-valued formal contexts corresponding to interval-valued data. Then, the dual cognitive operators of FCA is studied in the interval-valued formal context. Meanwhile, we discuss mathematical properties of two cognitive operators.

Definition 1 Let $(U, AT, \overline{I})$ be an interval-valued formal context, where

(1) $U = \{x_1, x_2, \dots, x_n\}$, which is called the object set, and x_i is an object.
(2) $AT = \{a_1, a_2, \dots, a_m\}$ is the attribute set, and a_j is an attribute.
(3) $\overline{I} = \{< (x,a), [a^L(x), a^U(x)] > |(x,a) \in U \times AT\}$ and $a(x) = [a^L(x), a^U(x)]: U \times AT \to R \times R$, $a^U(x) \geq a^L(x)$.

For a given interval-valued formal context, $Ma^U = \underset{x\in U}{\vee} a^U(x)$ and $ma^L = \underset{x\in U}{\wedge} a^L(x)$ are the upper and lower limits of the attribute a on U. We denote $a(x) = [a^L(x), a^U(x)] = \overline{I}(x,a)$. The set of relation $\overline{I}(x,a)$ is represented by $\mathcal{U} = \{\overline{I}(x,a) | a \in AT, x \in U\}$. For any $x, y \in U$, $a \in AT$ and $\overline{I}(x,a), \overline{I}(y,a) \in \mathcal{U}$, there is $\overline{I}(x,a) \geq \overline{I}(y,a) \Leftrightarrow \overline{I}(x,a) \subseteq \overline{I}(y,a)$ $(i.e.\ a^L(y) \leq a^L(x) \leq a^U(x) \leq a^U(y))$.

Note: For any $a \in AT$ and $x \in U$, when $[a^L(x), a^U(x)] \subseteq [0,1]$, $(U, AT, \overline{I})$ is called fuzzy interval-valued formal context. For any interval-valued formal contexts, they can be normalized into fuzzy interval-valued formal contexts by using max-min normalization $(\frac{value-min}{max-min})$. That is to say, the fuzzy interval-valued formal context is a special case of interval-valued formal contexts. The interval-valued formal context is more general than fuzzy interval-valued formal context. This paper studies concept learning in interval-valued formal contexts, which are more widely used than fuzzy interval-valued formal contexts.

In an interval-valued formal context $(U, AT, \overline{I})$, for $X \subseteq U$ and $B \subseteq AT$, we call the pair $(X, \overline{B})$ the interval-valued information granule (IvIG), where $\overline{B} = \{< a, [\overline{B}^L(a), \overline{B}^U(a)] > | a \in AT\}$. $\overline{B}^L(a)$ and $\overline{B}^U(a)$ are the lower limit and upper limit of attribute a on $\overline{B}$, respectively. If $a \in AT$ and $a \notin B$, then $[\overline{B}^L(a), \overline{B}^U(a)] = \phi$.

Definition 2 Let $(U, AT, \overline{I})$ be an interval-valued formal context, for $X \subseteq U$ and $B \subseteq AT$, a pair of dual cognitive operators $(*, *)$ with respect to X and $\overline{B}$ is defined as

(1) $X^* = \{< a, [ma^L(X), Ma^U(X)] > | a \in AT\}$,
(2) $\overline{B}^* = \{x \in U | a^L(x) \geq \overline{B}^L(a), a^U(x) \leq \overline{B}^U(a), \forall a \in B\}$.

Where $ma^L(X) = \underset{x\in X}{\wedge} a^L(x)$ and $Ma^U(X) = \underset{x\in X}{\vee} a^U(x)$. We rule $\phi^* = \overline{A} = \{< a, \phi > | a \in AT\}$. For $X \subseteq U, B \subseteq AT$, if $X = \overline{B}^*$ and $X^* = \overline{B}$, we call pair $(X, \overline{B})$ an interval formal concept (Abbreviated as interval concept). X and $\overline{B}$ are the extension and intension of the interval concept $(X, \overline{B})$, respectively. The extension of interval concepts is a crisp object set and the intension of interval concepts is an interval-valued attribute set.

For $B_1, B_2 \subseteq AT$, if $[\overline{B}_1^L(a), \overline{B}_1^U(a)] \subseteq [\overline{B}_2^L(a), \overline{B}_2^U(a)]$, $\forall a \in AT$, we denote $\overline{B}_2 \Subset \overline{B}_1$. We define $(\overline{B}_1 \wedge \overline{B}_2)(a) = [\overline{B}_1^L(a) \wedge \overline{B}_2^L(a), \overline{B}_1^U(a) \vee \overline{B}_2^U(a)]$ and rule $\overline{B}_1 \wedge \phi^* = \overline{B}_1$, and define $(\overline{B}_1 \vee \overline{B}_2)(a) = [\overline{B}_1^L(a), \overline{B}_1^U(a)] \cap [\overline{B}_2^L(a), \overline{B}_2^U(a)]$.

Theorem 1 *Let $(U, AT, \overline{I})$ be an interval-valued formal context, X_1, X_2 and $X \subseteq U$, B_1, B_2 and $B \subseteq AT$, the following conclusions are true.*

(1) $X_1 \subseteq X_2 \Rightarrow X_2^* \Subset X_1^*, \overline{B}_1 \Subset \overline{B}_2 \Rightarrow \overline{B}_2^* \subseteq \overline{B}_1^*$;
(2) $X \subseteq X^{**}, \overline{B} \Subset \overline{B}^{**}$;
(3) $X^* = X^{***}, \overline{B}^* = \overline{B}^{***}$;
(4) $X \subseteq \overline{B}^* \Leftrightarrow \overline{B} \Subset X^*$;
(5) $(X_1 \cup X_2)^* = X_1^* \wedge X_2^*, (\overline{B}_1 \vee \overline{B}_2)^* = \overline{B}_1^* \cap \overline{B}_2^*$;
(6) $X_1^* \vee X_2^* \Subset (X_1 \cap X_2)^*, \overline{B}_1^* \cup \overline{B}_2^* \subseteq (\overline{B}_1 \wedge \overline{B}_2)^*$.

From the above properties, we know that (X^{**}, X^*) and (B^*, B^{**}) are two trivial interval concepts.

The interval concept lattice $\overline{E}(U, AT, \overline{I})$ is constructed by all interval concepts in the interval-valued formal context $(U, AT, \overline{I})$ according to an ordered relation. The ordered relation is given as $(X_2, \overline{B}_2) \geq (X_1, \overline{B}_1) \Leftrightarrow \overline{B}_2 \Subset \overline{B}_1 \Leftrightarrow X_1 \subseteq X_2$, where $(X_1, \overline{B}_1)$ and $(X_2, \overline{B}_2)$ are interval concepts. $(X_2, \overline{B}_2)$ is called a supper-concept of $(X_1, \overline{B}_1)$, and $(X_1, \overline{B}_1)$ is called a sub-concept of $(X_2, \overline{B}_2)$.

Table 2 Interval-valued formal context $(U, AT, \bar{I})$

U	a	b	c	d
x_1	[1.2,1.5]	[0.3,0.7]	[1.3,1.5]	[3.1,3.7]
x_2	[1.3,1.8]	[0.4,0.5]	[1.7,2.4]	[3.3,4.1]
x_3	[0.8,1.5]	[0.1,0.9]	[1.3,1.6]	[2.8,3.7]
x_4	[0.4,1.9]	[0.4,0.8]	[1.4,2.9]	[2.6,4.2]

$$\begin{tikzcd}[column sep=tiny]
 & & C(U) \arrow[dl, dash] \arrow[dr, dash] & & \\
 & C(\{x_1,x_2,x_4\}) \arrow[dl, dash] \arrow[dr, dash] & & C(\{x_1,x_2,x_3\}) \arrow[dl, dash] \arrow[dr, dash] & \\
C(\{x_2,x_4\}) \arrow[dr, dash] & & C(\{x_1,x_2\}) \arrow[dl, dash] \arrow[dr, dash] & & C(\{x_1,x_3\}) \arrow[dl, dash] \\
 & C(\{x_1\}) \arrow[dr, dash] & & C(\{x_2\}) \arrow[dl, dash] & \\
 & & C(\phi) & &
\end{tikzcd}$$

Fig. 1 The interval concept lattice of $(U, AT, \bar{I})$

Next, we use a small example to visually illustrate interval concepts and the relationships between different interval concepts.

Example 1 An interval-valued formal context $(U, AT, \bar{I})$ is shown in Table 2. There are four objects $U = \{x_1, x_2, x_3, x_4\}$ and four attributes $AT = \{a, b, c, d\}$.

By the definition of dual cognitive operator $(*, *)$, we can judge whether an information granule is a formal concept or not. Next, we find all the concepts in the formal context shown in Table 2 by exhausting all subsets of the universe. For example, we take $X = \{x_1, x_2\}$, then calculate $X^* = \{< a, [1.2, 1.8] >, < b, [0.3, 0.7] >, < c, [1.3, 2.4] >, < d, [3.1, 4.1] >\} \triangleq \overline{B}$. Further, we calculate $\overline{B}^*$, then get $\overline{B}^* = \{x_1, x_2\} = X$. This shows that the information granule $(X, \overline{B})$ is a formal concept. After $2^{|U|}$ rounds of calculation, we get all the formal concepts, which are shown in Table 3. From Table 3, we find that only 9 of the 16 information granules are formal concepts.

Then according to the above ordered relation $\succeq$ of interval concepts, we establish an interval concept lattice, which is shown in Fig. 1. The lattice consists of nine concepts, where $C(\phi)$ and $C(U)$ are two trivial concepts. $C(\{x_2, x_4\})$ is called a supper-concept of $C(\{x_1\})$, and $C(\{x_1\})$ is called a sub-concept of $C(\{x_2, x_4\})$. $C(\{x_2, x_4\})$ is also called a sub-concept of $C(\{x_1, x_2, x_4\})$. Through lattice structure, the relationship between concepts is clearly visualized.

Through Example 1, we can see that learning concept by exhaustive information granules will increase exponentially with the scale of objects. Obviously, it is only suitable for concept learning on small-scale data. In the big data environment, the establishment of scientific search methods of concept is the inevitable requirement of the development of formal concept analysis. Next, we will explore what kind of information granule is a concept? How to learn the concept efficiently? At the same time, we discuss how to learn interval concepts from given information of objects and attributes in interval-valued formal contexts, which provides a systematic approach to learning concepts.

4 Mining the relationship between information granules and concepts in interval-valued cognitive systems

In this section, we will establish a new cognitive system in interval-valued formal contexts. In the cognitive system, we will systematically study the relationship between information granules and concepts.

Table 3 The extension and intension of all concepts

No.	Extension (X)	Intension ($\overline{B}$)	Abbreviation of concepts
1	ϕ	$\{< a, \phi >, < b, \phi >, < c, \phi >, < d, \phi >\}$	$C(\phi)$
2	$\{x_1\}$	$\{< a, [1.2, 1.5] >, < b, [0.3, 0.7] >, < c, [1.3, 1.5] >, < d, [3.1, 3.7] >\}$	$C(\{x_1\})$
3	$\{x_2\}$	$\{< a, [1.3, 1.8] >, < b, [0.4, 0.5] >, < c, [1.7, 2.4] >, < d, [3.3, 4.1] >\}$	$C(\{x_2\})$
4	$\{x_1, x_2\}$	$\{< a, [1.2, 1.8] >, < b, [0.3, 0.7] >, < c, [1.3, 2.4] >, < d, [3.1, 4.1] >\}$	$C(\{x_1, x_2\})$
5	$\{x_1, x_3\}$	$\{< a, [0.8, 1.5] >, < b, [0.1, 0.9] >, < c, [1.3, 1.6] >, < d, [2.8, 3.7] >\}$	$C(\{x_1, x_3\})$
6	$\{x_2, x_4\}$	$\{< a, [0.4, 1.9] >, < b, [0.4, 0.8] >, < c, [1.4, 2.9] >, < d, [2.6, 4.2] >\}$	$C(\{x_2, x_4\})$
7	$\{x_1, x_2, x_3\}$	$\{< a, [0.8, 1.8] >, < b, [0.1, 0.9] >, < c, [1.3, 2.4] >, < d, [2.8, 4.1] >\}$	$C(\{x_1, x_2, x_3\})$
8	$\{x_1, x_2, x_4\}$	$\{< a, [0.4, 1.9] >, < b, [0.3, 0.8] >, < c, [1.3, 2.9] >, < d, [2.6, 4.2] >\}$	$C(\{x_1, x_2, x_4\})$
9	U	$\{< a, [0.4, 1.9] >, < b, [0.1, 0.9] >, < c, [1.3, 2.9] >, < d, [2.6, 4.2] >\}$	$C(U)$

First, we define two mappings between two concept lattices and establish an interval-valued cognitive system. We use E to represent a lattice, 1_E and 0_E are the unit element and the zero element, respectively.

Definition 3 Let E_1 and $\overline{E}_2$ be a complete lattice and an interval-valued complete lattice, respectively, for any $o_1, o_2 \in E_1$, $\overline{\mathcal{T}}: E_1 \to \overline{E}_2$ is an interval-valued operator, for any $\overline{a}_1, \overline{a}_2 \in \overline{E}_2$, $\mathcal{R}: \overline{E}_2 \to E_1$ is an operator, $\hbar$ and $\mathfrak{R}$ are called a pair of dual cognitive operators, iff $\hbar$ and $\mathfrak{R}$ satisfy the following conditions:

(1) $\hbar(o_1 \vee o_2) = \hbar(o_1) \wedge \hbar(o_2)$;
(2) $\mathfrak{R}(\overline{a}_1 \vee \overline{a}_2) = \mathfrak{R}(\overline{a}_1) \wedge \mathfrak{R}(\overline{a}_2)$;
(3) $\hbar(0_{E_1}) = 1_{\overline{E}_2}, \hbar(1_{E_1}) = 0_{\overline{E}_2}$;
(4) $\mathfrak{R}(1_{\overline{E}_2}) = 0_{E_1}, \mathfrak{R}(0_{\overline{E}_2}) = 1_{E_1}$.

Definition 4 A four-tuple $(E_1, \overline{E}_2, \hbar, \mathfrak{R})$ is an interval-valued cognitive system, for $o \in E_1$ and $\overline{a} \in \overline{E}_2$, if operators $\hbar$ and $\mathfrak{R}$ further satisfy:

(1) $\hbar \circ \mathfrak{R}(\overline{a}) \geq \overline{a}$;
(2) $\mathfrak{R} \circ \hbar(o) \geq o$;

where $\hbar \circ \mathfrak{R}(\overline{a})$ and $\mathfrak{R} \circ \hbar(o)$ can be abbreviated as $\hbar\mathfrak{R}(\overline{a})$ and $\mathfrak{R}\hbar(o)$, respectively.

From the above two definitions, we know that the two operators $\hbar$ and $\mathfrak{R}$ are used to characterize an object set and its interval-valued attribute set for the interval-valued cognitive system, respectively.

Theorem 2 *For any $o_1, o_2, o \in E_1$, and $\overline{a}_1, \overline{a}_2, \overline{a} \in \overline{E}_2$, an interval-valued cognitive system $(E_1, \overline{E}_2, \hbar, \mathfrak{R})$ has the following properties*:

(1) *If $o_2 \geq o_1$, then $\hbar(o_1) \geq \hbar(o_2)$*;
(2) *If $\overline{a}_2 \geq \overline{a}_1$, then $\mathfrak{R}(\overline{a}_1) \geq \mathfrak{R}(\overline{a}_2)$*;
(3) $\hbar(o_2) \vee \hbar(o_1) \leq \hbar(o_2 \wedge o_1)$;
(4) $\mathfrak{R}(\overline{a}_2) \vee \mathfrak{R}(\overline{a}_1) \leq \mathfrak{R}(\overline{a}_2 \wedge \overline{a}_1)$;
(5) $\overline{a} \leq \mathcal{T}(o) \Leftrightarrow o \leq \mathfrak{R}(\overline{a}), \mathcal{T}(o) \leq \overline{a} \Leftrightarrow \mathfrak{R}(\overline{a}) \leq o$;
(6) *For any $o \in E_1$, $\hbar \circ \mathfrak{R} \circ \hbar(o) = \hbar(o)$*;
(7) *For any $\overline{a} \in \overline{E}_2$, $\mathfrak{R} \circ \hbar \circ \mathfrak{R}(\overline{a}) = \mathfrak{R}(\overline{a})$*.

Proof From Definition 3, (1)–(5) can be easily proven.

For property (6): According to $\mathfrak{R} \circ \hbar(o) \geq o$ of Definition 4 and property (1), we obtain $\hbar \circ \mathfrak{R} \circ \hbar(o) \leq \hbar(o)$. In contrast, from $\hbar \circ \mathfrak{R}(\overline{a}) \geq \overline{a}$ of Definition 4, if we take $\overline{a} = \hbar(o)$, then obtain $\hbar \circ \mathfrak{R} \circ \hbar(o) \geq \hbar(o)$. Thus $\hbar \circ \mathfrak{R} \circ \hbar(o) = \hbar(o)$. Property (7) can be easily proven by using the same way as property (6). □

Theorem 3 *Let $(U, AT, \overline{I})$ be an interval-valued formal context, where $U = \{x_1, x_2, \cdots x_n\}$ is an objects set and $\underline{AT} = \{a_1, a_2, \cdots a_m\}$ is an attribute set. If $E_1 = P(U)$ and $\overline{E}_2 = P(\overline{AT})$, then the operators $(*, *)$ defined in Sect. 3 are a pair of dual cognitive operators on the interval-valued formal context, where $P(U)$ is the power set of U, and $P(\overline{AT})$ is a set composed by all sets of constructed by AT and interval numbers*.

Proof Theorem 3 can be easily proven by six properties of operators $(*, *)$ in Sect. 3 and Definition 3. □

From the above definitions and theorems, we know that the interval-valued formal context can be regarded as an interval relation between objects and their interval-valued attributes in cognitive learning process. To learn interval concepts from the interval-valued formal context, we need to find a pair of dual cognitive operators satisfying Definition 3. In fact, $(*, *)$ with respect to the object set and interval-valued attribute set is a pair of dual cognitive operators.

To understand the description of interval-valued information granules (IvIGs) of the interval-valued cognitive system, the pair $(o, \overline{a})$ is denoted by an IvIG, where o and $\overline{a}$ are the object set and the interval-valued attribute set, respectively.

Definition 5 Let $E_1 = P(U)$ and $\overline{E}_2 = P(\overline{AT})$ be the complete lattice and complete interval-valued lattice, respectively, and $\hbar$, $\mathfrak{R}$ be a pair of cognitive operators, i.e.$(E_1, \overline{E}_2, \hbar, \mathfrak{R})$ is an interval-valued cognitive system. In interval-valued cognitive system $(E_1, \overline{E}_2, \hbar, \mathfrak{R})$, $\forall o \in E_1$ and $\overline{a} \in \overline{E}_2$, there are two sets:

$$\mathcal{G}_1 = \{(o, \overline{a}) \mid o \leq \mathfrak{R}(\overline{a}),\ \overline{a} \leq \hbar(o)\},$$
$$\mathcal{G}_2 = \{(o, \overline{a}) \mid \mathfrak{R}(\overline{a}) \leq o,\ \hbar(o) \leq \overline{a}\}.$$

• If $(o, \overline{a}) \in \mathcal{G}_1$, then $(o, \overline{a})$ is a necessary IvIG of the interval-valued cognitive system and $\overline{a}$ is a necessary interval-valued attribute set of o. Therefore, we call $\mathcal{G}_1$ a necessary IvIG set of the interval-valued cognitive system.

• If $(o, \overline{a}) \in \mathcal{G}_2$, then $(o, \overline{a})$ is a sufficient IvIG of the interval-valued cognitive system and $\overline{a}$ is a sufficient interval-valued attribute set of o. Therefore, $\mathcal{G}_2$ is called the sufficient IvIG set of the interval-valued cognitive system.

• If $(o, \overline{a}) \in \mathcal{G}_1 \cap \mathcal{G}_2$, in other words, $\overline{a} = \hbar(o),\ o = \mathfrak{R}(\overline{a})$, then $(o, \overline{a})$ is a sufficient and necessary IvIG of the interval-valued cognitive system and $\overline{a}$ is a sufficient and necessary interval-valued attribute set of o. $\mathcal{G}_1 \cap \mathcal{G}_2$ is a sufficient and necessary IvIG set of the interval-valued cognitive system.

• If $(o,\overline{a}) \in \mathcal{G}_1 \cup \mathcal{G}_2$, then $(o,\overline{a})$ is an IvIG of the interval-valued cognitive system and $\mathcal{G}_1 \cup \mathcal{G}_2$ is an IvIG set of the interval-valued cognitive system.

• If $(o,\overline{a}) \notin \mathcal{G}_1 \cup \mathcal{G}_2$, then $(o,\overline{a})$ is an inconsistent IvIG of the interval-valued cognitive system.

From the above definitions, sufficient and necessary IvIGs are interval concepts of the interval-valued cognitive system. According to the first three points of Definition 5, from the global perspective, we can obtain all concepts by calculating $\mathcal{G}_1 \cap \mathcal{G}_2$ directly.

In fact, the process of finding concepts is to learn sufficient and necessary information granules from unknown or partial information. For any information interval-valued information granules, how do we get the concept? In the process of human cognition of things, people tends to first grab sufficient or necessary information of things, and then gradually mines the sufficient and necessary information. The cognitive process is also the concept learning process.

Theorem 4 *Let $(U, AT, \overline{I})$ be an interval-valued formal context and $(*, *)$ be a pair of dual cognitive operators, where $U = \{x_1, x_2, \cdots x_n\}$ and $AT = \{a_1, a_2, \cdots, a_m\}$. $E_1 = P(U)$ and $\overline{E}_2 = P(\overline{AT})$, for any $X \in E_1$ and $\overline{B} \in \overline{E}_2$, there are two conclusions.*

(1) *If $\overline{B}^* \subseteq X$ and $X^* \subseteq \overline{B}$, then $\overline{B}$ is the set of the sufficient interval-valued attributes of X;*
(2) *If $\overline{B}^* \supseteq X$ and $X^* \supseteq \overline{B}$, then $\overline{B}$ is the set of the necessary interval-valued attributes of X;*

Proof From Theorem 3 and Definition 5, (1) and (2) can be easily proven. □

From the above definitions and theorems, when we learn the concept of a thing from the given object or interval-valued attribute information, we seek necessary or sufficient interval-valued attribute information at the beginning in cognitive process.

Theorem 5 *Let $(E_1, \overline{E}_2, \hbar, \mathfrak{R})$ be an interval-valued cognitive system, for $o_1, o_2 \in E_1$ and $\overline{a}_1, \overline{a}_2 \in \overline{E}_2$, and $\mathcal{G}_1$ be a necessary IvIG set of the interval-valued cognitive system. There are two operators $\vee$ and $\wedge$ defined as follows:*

$$(o_2,\overline{a}_2) \vee (o_1,\overline{a}_1) = (\mathfrak{R}\circ\hbar(o_2 \vee o_1), \overline{a}_2 \wedge \overline{a}_1),$$
$$(o_2,\overline{a}_2) \wedge (o_1,\overline{a}_1) = (o_2 \wedge o_1, \hbar\circ\mathfrak{R}(\overline{a}_2 \vee \overline{a}_1)).$$

$(\mathcal{G}_1, \leq)$ is computationally closed for operators $\vee$ and $\wedge$.

Proof Assume $(o_2,\overline{a}_2), (o_1,\overline{a}_1) \in \mathcal{G}_1$, then

$$o_1 \leq \mathfrak{R}(\overline{a}_1), \quad o_2 \leq \mathfrak{R}(\overline{a}_2),$$
$$\overline{a}_1 \leq \hbar(o_1), \quad \overline{a}_2 \leq \hbar(o_2),$$

and

$$\mathfrak{R}\circ\hbar\circ\mathfrak{R}(\overline{a}_1 \vee \overline{a}_2) = \mathfrak{R}(\overline{a}_1 \vee \overline{a}_2) = \mathfrak{R}(\overline{a}_1) \wedge \mathfrak{R}(\overline{a}_2) \geq o_1 \wedge o_2$$

Moreover, by Theorem 2, we find that

$$\hbar(o_1 \wedge o_2) \geq \hbar(\mathfrak{R}(\overline{a}_1) \wedge \mathfrak{R}(\overline{a}_1)) = \hbar\circ\mathfrak{R}(\overline{a}_1 \vee \overline{a}_2).$$

Therefore, $(o_2,\overline{a}_2) \wedge (o_1,\overline{a}_1)$ is a necessary IvIG, that is to say $(o_2,\overline{a}_2) \wedge (o_1,\overline{a}_1) \in \mathcal{G}_1$.

Similarly, we can use the similar manner to prove $(o_2,\overline{a}_2) \vee (o_1,\overline{a}_1) \in \mathcal{G}_1$.

Therefore, $(\mathcal{G}_1, \leq)$ is closed for operators $\vee$ and $\wedge$. □

Theorem 5 shows that we can find all the necessary information granules by calculating set $\mathcal{G}_1$.

Theorem 6 *Let $(E_1, \overline{E}_2, \hbar, \mathfrak{R})$ be an interval-valued cognitive system, for $o_1, o_2 \in G_1$ and $\overline{a}_1, \overline{a}_2 \in \overline{E}_2$, and $\mathcal{G}_2$ be a sufficient IvIG set of the interval-valued cognitive system. There are two operators $\vee$ and $\wedge$ defined as follows:*

$$(o_2,\overline{a}_2) \vee (o_1,\overline{a}_1) = (\mathfrak{R}\circ\hbar(o_2 \vee o_1), \overline{a}_2 \wedge \overline{a}_1),$$
$$(o_2,\overline{a}_2) \wedge (o_1,\overline{a}_1) = (o_2 \wedge o_1, \hbar\circ\mathfrak{R}(\overline{a}_2 \vee \overline{a}_1)),$$

$(\mathcal{G}_2, \leq)$ is computationally closed for operators $\vee$ and $\wedge$.

Proof Let $(o_2,\overline{a}_2), (o_1,\overline{a}_1) \in \mathcal{G}_2$, then

$$\mathfrak{R}(\overline{a}_1) \leq o_1, \quad \mathfrak{R}(\overline{a}_2) \leq o_2,$$
$$\hbar(o_1) \leq \overline{a}_1, \quad \hbar(o_2) \leq \overline{a}_2,$$

and

$$\mathfrak{R}\circ\hbar\circ\mathfrak{R}(\overline{a}_2 \vee \overline{a}_1) = \mathfrak{R}(\overline{a}_2 \vee \overline{a}_1) = \mathfrak{R}(\overline{a}_2) \wedge \mathfrak{R}(\overline{a}_1) \leq o_2 \wedge o_1.$$

Moreover, by Theorem 2, we find that

$$\hbar(o_2 \wedge o_1) \leq \hbar(\mathfrak{R}(\overline{a}_2) \wedge \mathfrak{R}(\overline{a}_1)) = \hbar\circ\mathfrak{R}(\overline{a}_2 \vee \overline{a}_1).$$

Therefore, $(o_2,\overline{a}_2) \wedge (o_1,\overline{a}_1)$ is a sufficient IvIG, that is to say $(o_2,\overline{a}_2) \wedge (o_1,\overline{a}_1) \in \mathcal{G}_2$.

Similarly, we can use the similar manner to prove $(o_2,\overline{a}_2) \vee (o_1,\overline{a}_1) \in \mathcal{G}_2$.

Therefore, $(\mathcal{G}_2, \leq)$ is closed for operators $\vee$ and $\wedge$. □

Theorem 6 shows that we can find all the sufficient information granules by calculating set $\mathcal{G}_2$.

From the first three points of Definition 5 and Theorems 5–6, we can know that there are two ways to find a concept (i.e. a sufficient and necessary information granule): a general information granule is transformed into a sufficient information granule and then transformed into a sufficient and necessary information granule; or a general information granule is transformed into a necessary information granule and then transformed into a sufficient and necessary information granule. In the next section, we will study these two ways of concept learning systematically.

5 The cognitive mechanism of concept learning in interval-valued cognitive systems

Essentially, an interval concept is a sufficient and necessary IvIG. Cognitive process of concepts is actually a transformation process of information granules. In the initial stage of cognition, there is generally no sufficient and necessary interval-valued information granules. We first transform general IvIGs into necessary IvIGs or sufficient IvIGs, and then transform the obtained IvIGs into sufficient and necessary IvIGs. Next, we will study the cognitive mechanism of concept learning based on granular computing (GrC).

5.1 Concept-cognitive learning based GrC

Case 1. The transformation process from general IvIGs to necessary IvIGs is given as follows.

Theorem 7 *Let $(E_1, \overline{E}_2, \hbar, \mathfrak{R})$ be an interval-valued cognitive system, $o \in E_1$ and $\overline{a} \in \overline{E}_2$, and $\mathcal{G}_1$ be a necessary IvIG set of the interval-valued cognitive system. There are six ways to transform $(o, \overline{a})$ into a necessary IvIG.*

(1) $(o \wedge \mathfrak{R}(\overline{a}), \overline{a} \vee \hbar(o)) \in \mathcal{G}_1$;
(2) $(o \vee \mathfrak{R}(\overline{a}), \overline{a} \wedge \hbar(o)) \in \mathcal{G}_1$;
(3) $(\mathfrak{R}(\overline{a}), \overline{a} \wedge \hbar(o)) \in \mathcal{G}_1$;
(4) $(o \wedge \mathfrak{R}(\overline{a}), \hbar(o)) \in \mathcal{G}_1$;
(5) $(\mathfrak{R}\circ\hbar(o), \overline{a} \wedge \hbar(o)) \in \mathcal{G}_1$;
(6) $(o \wedge \mathfrak{R}(\overline{a}), \hbar\circ\mathfrak{R}(\overline{a}) \in \mathcal{G}_1$;

Proof (1) Since $(E_1, \overline{E}_2, \hbar, \mathfrak{R})$ is an interval-valued cognitive system, From Definition 5 and Theorem 2, we obtain

$$\hbar(\mathfrak{R}(\overline{a}) \wedge o) \geq \hbar(\mathfrak{R}(\overline{a})) \vee \hbar(o) \geq \overline{a} \vee \hbar(o) \tag{1}$$

and

$$\mathfrak{R}(\hbar(o) \vee \overline{a}) = \mathfrak{R}(\hbar(o)) \wedge \mathfrak{R}(\overline{a}) \geq o \wedge \mathfrak{R}(\overline{a}). \tag{2}$$

Therefore, $(o \wedge \mathfrak{R}(\overline{a}), \overline{a} \vee \hbar(o)) \in \mathcal{G}_1$.

(2) We can prove (2) by using the same way as (1).

(3) Since $(E_1, \overline{E}_2, \hbar, \mathfrak{R})$ is an interval-valued cognitive system, according to Definition 5 and Theorem 2, we obtain $\hbar\circ\mathfrak{R}(\overline{a}) \geq \overline{a} \geq \overline{a} \wedge \hbar(o)$ and $\mathfrak{R}(\overline{a} \wedge \hbar(o)) \geq \mathfrak{R}(\overline{a}) \vee \mathfrak{R}\circ\hbar(o) \geq \mathfrak{R}(\overline{a})$. Thus, we have $(\mathfrak{R}(\overline{a}), \overline{a} \wedge \hbar(o)) \in \mathcal{G}_1$.

(4) It can be proven by using the same way as (3).

(5) From Definition 5 and Theorem 2, we know that $\overline{a} \wedge \hbar(o) \leq \hbar(o) = \hbar\circ\mathfrak{R}\circ\hbar(o)$ and $\mathfrak{R}(\overline{a} \wedge \hbar(o)) \geq \mathfrak{R}(\overline{a}) \vee \mathfrak{R}\circ\hbar(o) \geq \hbar(o)$. Therefore, we obtain $(\mathfrak{R}\circ\hbar(o), \overline{a} \wedge \hbar(o)) \in \mathcal{G}_1$.

(6) It can be proven by using the same way as (6). □

Example 2 (Further considering Example 1) To understand the transformation process of information granules , we take $o = \{x_2, x_4\}$ and $\overline{a} = \{< a, [1.1, 2.3] >, < b, [0.3, 0.8] >, < c, [1.3, 3.2] >, < d, [3.1, 5.2] >\}$ to show calculation process of Cases 1. From the definition of dual cognitive operators $(*, *)$, we can obtain $\hbar(o) = \{< a, [0.4, 1.9] >, < b, [0.4, 0.8] >, < c, [1.4, 2.9] >, < d, [2.6, 4.2] >\}$ and $\mathfrak{R}(\overline{a}) = \{x_1, x_2\}$. So $o \wedge \mathfrak{R}(\overline{a}) = \{x_2\}$, $o \vee \mathfrak{R}(\overline{a}) = \{x_1, x_2, x_4\}$, $\mathfrak{R}\circ\hbar(o) = \{x_2, x_4\}$, $\overline{a} \vee \hbar(o) = \{< a, [1.1, 1.9] >, < b, [0.4, 0.8] >, < c, [1.4, 2.9] >, < d, [3.1, 4.2] >\}$, $\overline{a} \wedge \hbar(o) = \{< a, [0.4, 2.3] >, < b, [0.3, 0.8] >, < c, [1.3, 3.2] >, < d, [2.6, 5.2] >\}$ and $\hbar\circ\mathfrak{R}(\overline{a}) = \{< a, [1.2, 1.8] >, < b, [0.3, 0.7] >, < c, [1.3, 2.4] >, < d, [3.1, 4.1] >\}$. According to (1)–(6) in Case 1, we get six necessary IvIGs by transforming general IvIG $(o, \overline{a})$, which are given as follows:

(1) $(\{x_2\}, \{< a, [1.1, 1.9] >, < b, [0.4, 0.8] >, < c, [1.4, 2.9] >, < d, [3.1, 4.2] >\})$;
(2) $(\{x_1, x_2, x_4\}, \{< a, [0.4, 2.3] >, < b, [0.3, 0.8] >, < c, [1.3, 3.2] >, < d, [2.6, 5.2] >\})$;
(3) $(\{x_1, x_2\}, \{< a, [0.4, 2.3] >, < b, [0.3, 0.8] >, < c, [1.3, 3.2] >, < d, [2.6, 5.2] >\})$;
(4) $(\{x_2\}, \{< a, [0.4, 1.9] >, < b, [0.4, 0.8] >, < c, [1.4, 2.9] >, < d, [2.6, 4.2] >\})$;
(5) $(\{x_2, x_4\}, \{< a, [0.4, 2.3] >, < b, [0.3, 0.8] >, < c, [1.3, 3.2] >, < d, [2.6, 5.2] >\})$;
(6) $(\{x_2\}, \{< a, [1.2, 1.8] >, < b, [0.3, 0.7] >, < c, [1.3, 2.4] >, < d, [3.1, 4.1] >\})$.

The six IvIGs are necessary but not sufficient IvIGs.

From the Theorem 7, we know that necessary IvIGs can be obtained by using the above six transformation methods. Next, we will discuss the transformation process from general IvIGs to sufficient IvIGs.

Case 2. The transformation process from general IvIGs to sufficient IvIGs is given as follows.

Theorem 8 *Let $(E_1, \overline{E}_2, \hbar, \mathfrak{R})$ be an interval-valued cognitive system, $o \in E_1$ and $\overline{a} \in \overline{E}_2$, and $\mathcal{G}_2$ be a sufficient IvIG set of the interval-valued cognitive system. There are two ways to transform $(o, \overline{a})$ into a sufficient IvIG.*

(1) $(\mathfrak{R}\circ\hbar(o), \overline{a} \vee \hbar(o)) \in \mathcal{G}_2$;
(2) $(o \vee \mathfrak{R}(\overline{a}), \hbar\circ\mathfrak{R}(\overline{a})) \in \mathcal{G}_2$;

Proof (1) Since $(E_1, \overline{E}_2, \hbar, \mathfrak{R})$ is an interval-valued cognitive system, from Definition 5 and Theorem 2, we obtain $\overline{a} \vee \hbar(o) \geq \hbar(o) = \hbar\circ\mathfrak{R}\circ\hbar(o)$ and

$\mathfrak{R}(\overline{a} \vee \hbar(o)) = \mathfrak{R}(\overline{a}) \wedge \mathfrak{R}\circ\hbar(o) \leq \mathfrak{R}\circ\hbar(o)$. Therefore, we have $(\mathfrak{R}\circ\hbar(o), \overline{a} \vee \hbar(o)) \in \mathcal{G}_2$.

(2) From Definition 5 and Theorem 2, we can see that $\hbar(o \vee \mathfrak{R}(\overline{a})) = \hbar(o) \wedge \hbar\circ(\overline{a}) \leq \hbar\circ(\overline{a})$ and $\mathfrak{R}\circ\hbar\circ\mathfrak{R}(\overline{a}) = \mathfrak{R}(\overline{a}) \leq o \vee \mathfrak{R}(\overline{a})$. Therefore, we have $(o \vee \mathfrak{R}(\overline{a}), \hbar\circ\mathfrak{R}(\overline{a})) \in \mathcal{G}_2$. □

Example 3 (Further considering Example 2) There are $\mathfrak{R}\circ\hbar(o) = \{x_2, x_4\}$, $o \vee \mathfrak{R}(\overline{a}) = \{x_1, x_2, x_4\}$, $\overline{a} \vee \hbar(o) = \{< a, [1.1, 1.9] >, < b, [0.4, 0.8] >, < c, [1.4, 2.9] >, < d, [3.1, 4.2] >\}$ and $\hbar\circ\mathfrak{R}(\overline{a}) = \{< a, [1.2, 1.8] >, < b, [0.3, 0.7] >, < c, [1.3, 2.4] >, < d, [3.1, 4.1] >\}$. According to (1)-(2) in **Case 2**, we obtain 2 sufficient IvIGs by transforming general IvIG $(o, \overline{a})$, which are given as follows:

(1) $(\{x_2, x_4\}, \{< a, [1.1, 1.9] >, < b, [0.4, 0.8] >, < c, [1.4, 2.9] >, < d, [3.1, 4.2] >\})$;
(2) $(\{x_1, x_2, x_4\}, \{< a, [1.2, 1.8] >, < b, [0.3, 0.7] >, < c, [1.3, 2.4] >, < d, [3.1, 4.1] >\})$.

The above two IvIGs are sufficient but not necessary IvIGs.

From Theorems 7 and 8, we can transform the general IvIGs into sufficient or necessary IvIGs, but we can not get sufficient and necessary IvIGs (namely interval concepts) from the given general IvIGs. The goal of concept learning is from general information to learn sufficient and necessary information. Next, we will show transformation process from information granules obtained by Theorems 7 and 8 to sufficient and necessary IvIGs. This reveals the cognitive learning process of interval concepts based on granular computing.

Case 3. The transformation process from the necessary IvIGs, which are obtained in **Case 1**, to sufficient and necessary IvIGs is given as follows.

Theorem 9 *Let $(E_1, \overline{E}_2, \hbar, \mathfrak{R})$ be an interval-valued cognitive system, $o_1 \in E_1$ and $\overline{a}_1 \in \overline{E}_2$, and $\mathcal{G}_1$ be a necessary IvIG set and $\mathcal{G}_2$ be a sufficient IvIG set. If $(o_1, \overline{a}_1) \in \mathcal{G}_1$, then*

(1) $(o_1 \vee \mathfrak{R}(\overline{a_1}), \hbar(o_1 \vee \mathfrak{R}(\overline{a_1}))) \in \mathcal{G}_1 \cap \mathcal{G}_2$;
(2) $(\mathfrak{R}(\overline{a}_1 \vee \hbar(o_1)), \overline{a}_1 \vee \hbar(o_1)) \in \mathcal{G}_1 \cap \mathcal{G}_2$;

Proof (1) From Definition 5, we have $\hbar(o_1)) \geq \overline{a}_1$ and $\mathfrak{R}(\overline{a}_1) \geq o_1$. Thus, $\overline{a}_1 \vee \hbar(o_1)) = \hbar(o_1))$ and $o_1 \vee \mathfrak{R}(\overline{a}_1) = \mathfrak{R}(\overline{a}_1)$. Therefore, there exists $\hbar(o_1 \vee \mathfrak{R}(\overline{a}_1)) = \hbar\circ\mathfrak{R}(\overline{a}_1) = \hbar(o_1 \vee \mathfrak{R}(\overline{a}_1))$ and $\mathfrak{R}(\hbar(o_1 \vee \mathfrak{R}(\overline{a}_1))) = \mathfrak{R}\circ\hbar\circ\mathfrak{R}(\overline{a}_1) = \mathfrak{R}(\overline{a}_1) = o_1 \vee \mathfrak{R}(\overline{a_1})$. So, $(o_1 \vee \mathfrak{R}(\overline{a_1}), \hbar(o_1 \vee \mathfrak{R}(\overline{a_1})))$ is a sufficient and necessary IvIG. That is to say $(o_1 \vee \mathfrak{R}(\overline{a_1}), \hbar(o_1 \vee \mathfrak{R}(\overline{a_1}))) \in \mathcal{G}_1 \cap \mathcal{G}_2$.

(2) It can be proven by using the same way as (1). □

Example 4 (Further considering Case 1 and Example 2) We apply the six necessary IvIGs obtained in **Case 1** to learn sufficient and necessary IvIGs. Each of the six necessary IvIGs is applied in (1)and (2) in Case 3 respectively to learn sufficient and necessary IvIGs, and the repeated IvIGs are merged. These sufficient and necessary IvIGs are given as follows:

(1) $(\{x_2\}, \{< a, [1.3, 1.8] >, < b, [0.4, 0.5] >, < c, [1.7, 2.4] >, < d, [3.3, 4.1] >\})$;
(2) $(\{x_1, x_2, x_4\}, \{< a, [0.4, 1.9] >, < b, [0.3, 0.8] >, < c, [1.3, 2.9] >, < d, [2.6, 4.2] >\})$;
(3) $(\{x_1, x_2\}, \{< a, [1.2, 1.8] >, < b, [0.3, 0.7] >, < c, [1.3, 2.4] >, < d, [3.1, 4.1] >\})$;
(4) $(\{x_2, x_4\}, \{< a, [0.4, 1.9] >, < b, [0.4, 0.8] >, < c, [1.4, 2.9] >, < d, [2.6, 4.2] >\})$.

From Table 3, we know that the four information granules are interval concepts. That is to say, transformation process from general IvIGs to sufficient and necessary IvIGs is also the process of interval-valued concept learning.

Case 4. The transformation process from the sufficient IvIGs, which are obtained in Case 2, to sufficient and necessary IvIGs is given as follows.

Theorem 10 *Let $(E_1, \overline{E}_2, \hbar, \mathfrak{R})$ be an interval-valued cognitive system, $o_2 \in E_1$ and $\overline{a}_2 \in \overline{E}_2$, and $\mathcal{G}_1$ be a necessary IvIG set and $\mathcal{G}_2$ be a sufficient IvIG set. If $(o_2, \overline{a}_2) \in \mathcal{G}_1$, then*

(1) $(o_2 \wedge \mathfrak{R}(\overline{a_2}), \hbar(o_2 \wedge \mathfrak{R}(\overline{a_2}))) \in \mathcal{G}_1 \cap \mathcal{G}_2$;
(2) $(\mathfrak{R}(\overline{a}_2 \wedge \mathcal{T}(o_2)), \overline{a}_2 \wedge \hbar(o_2)) \in \mathcal{G}_1 \cap \mathcal{G}_2$;

Proof (1) From Definition 5, we have $o_2 \geq \mathfrak{R}(\overline{a}_2)$ and $\overline{a}_2 \geq \hbar(o_2))$. Thus, $o_2 \wedge \mathfrak{R}(\overline{a}_2) = \mathfrak{R}(\overline{a}_2)$ and $\overline{a}_2 \wedge \hbar(o_2)) = \hbar(o_2))$. Therefore, we have $\hbar(o_2 \wedge \mathfrak{R}(\overline{a}_2)) = \hbar\circ\mathfrak{R}(\overline{a}_2) = \hbar(o_2 \wedge \mathfrak{R}(\overline{a}_2))$ and $\mathfrak{R}(\hbar(o_2 \wedge \mathfrak{R}(\overline{a}_2))) = \mathfrak{R}\circ\hbar\circ\mathfrak{R}(\overline{a}_2) = \mathfrak{R}(\overline{a}_2) = o_2 \wedge \mathfrak{R}(\overline{a_2})$. So, $(o_2 \wedge \mathfrak{R}(\overline{a_2}), \hbar(o_2 \wedge \mathfrak{R}(\overline{a_2})))$ is a sufficient and necessary IvIG. That is to say $(o_2 \wedge \mathfrak{R}(\overline{a_2}), \hbar(o_2 \wedge \mathfrak{R}(\overline{a_2}))) \in \mathcal{G}_1 \cap \mathcal{G}_2$.

(2) It can be proven by using the same way as (1).

Example 5 (Further considering Case 2 and Example 3) We apply the two sufficient IvIGs obtained in Case 2 to learn sufficient and necessary IvIGs. Each of the two sufficient IvIGs is applied in (1)and (2) in Case 4 respectively to learn sufficient and necessary IvIGs, and the repeated IvIGs are merged. These sufficient and necessary IvIGs are given as follows:

(1) $(\{x_2\}, \{< a, [1.3, 1.8] >, < b, [0.4, 0.5] >, < c, [1.7, 2.4] >, < d, [3.3, 4.1] >\})$;
(2) $(\{x_2, x_4\}, \{< a, [0.4, 1.9] >, < b, [0.4, 0.8] >, < c, [1.4, 2.9] >, < d, [2.6, 4.2] >\})$;
(3) $(\{x_1, x_2\}, \{< a, [1.2, 1.8] >, < b, [0.3, 0.7] >, < c, [1.3, 2.4] >, < d, [3.1, 4.1] >\})$;
(4) $(\{x_1, x_2, x_4\}, \{< a, [0.4, 1.9] >, < b, [0.3, 0.8] >, < c, [1.3, 2.9] >, < d, [2.6, 4.2] >\})$.

Obviously, the four information granules are interval concepts.

Based on above discussion, we know that there are several methods to obtain sufficient IvIGs, necessary IvIGs, and sufficient and necessary IvIGs from the given IvIGs.

From Cases 1 and 3, through necessary IvIGs transfer, we have 6×2 methods to obtain sufficient and necessary IvIGs. From Cases 2 and 4, through sufficient IvIGs transfer, we have 2×2 methods to obtain sufficient and necessary IvIGs. So there are 14 methods to obtain sufficient and necessary IvIGs from general IvIGs. The framework of transformation process of information granules is shown in Fig. 2. From Fig. 2, we know that there are 2 ways (16 branches) to transform general IvIGs into sufficient and necessary IvIGs. The one is to first get necessary IvIGs, and then learn sufficient and necessary IvIGs. The other is to first get sufficient IvIGs, and then learn sufficient and necessary IvIGs.

In Sect. 5.1, we show in depth how to derive a interval concept from any interval-valued information granule. However, in practical applications, we may need to learn concepts according to actual needs (such as given object information, given attribute information or given both object and attribute information). In the next Sect. 5.2, we will give the concept learning algorithms under given clue information.

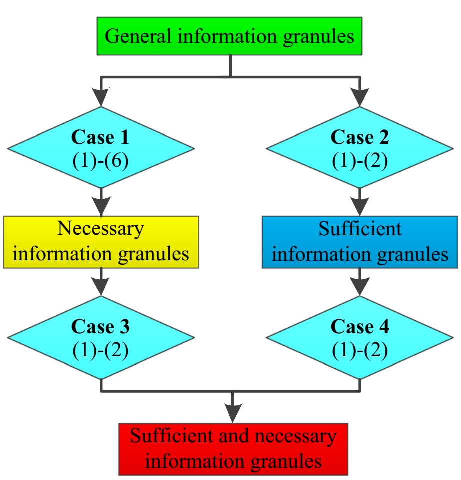

Fig. 2 Cognitive learning process of concepts based on granular computing

5.2 Concept learning algorithms from given clue information with objects or interval-valued attributes

From the above discussion, we will design 3 algorithms to learn concepts from given object and interval-valued attribute clue information.

First, concept learning algorithm based on object information in an interval-valued formal context is shown in Algorithm 1. In Algorithm 1, we compute the extension of concept C in steps 2–8 by using X^* and compute the intension of concept C in steps 9–20 by using X^{**}. Obviously, $C = (X^{**}, X^*)$.

Algorithm 1: Objects oriented interval-valued concept learning (O-IvCL)

Input: An interval-valued formal context $(U, AT, \overline{I})$ and a set of objects X.
Output: An interval concept C.
1: $\overline{A} \leftarrow \phi$, $O \leftarrow \phi$ and $C = (O, \overline{A})$; // $\overline{A}$ and O are the intension and extension of concept C, respectively.
2: **for** each $a \in AT$ **do**
3: $\quad \overline{A}^L(a) \leftarrow +\infty$ and $\overline{A}^U(a) \leftarrow -\infty$;
4: $\quad$ **for** each $x \in X$ **do**
5: $\quad\quad \overline{A}^L(a) \leftarrow min(a^L(x), \overline{A}^L(a))$;
6: $\quad\quad \overline{A}^U(a) \leftarrow max(a^U(x), \overline{A}^U(a))$;
7: $\quad$ **end for** // update $\overline{A}(a)$ by computing X^*.
8: **end for** //update $\overline{A}$ by computing X^*.
9: **for** each $x \in U$ **do**
10: $\quad flag \leftarrow true$; // $true$ represents that x belong to the extension of concept C.
11: $\quad$ **for** each $a \in AT$ **do**
12: $\quad\quad$ **if** $a^L(x) < \overline{A}^L(a)$ **or** $a^U(x) > \overline{A}^U(a)$ **then**
13: $\quad\quad\quad flag \leftarrow false$; //$false$ represents that x does not belong to the extension of concept C.
14: $\quad\quad\quad$ **break**; // stop searching.
15: $\quad\quad$ **end if**
16: $\quad$ **end for** // computing X^{**}.
17: $\quad$ **if** $flag$ **then**
18: $\quad\quad O \leftarrow O \cup \{x\}$; // x is added to the extension of concept C.
19: $\quad$ **end if**
20: **end for** // update O by computing X^{**}.
21: **return** $C = (O, \overline{A})$.

Next, we will learn concepts from given interval-valued attribute information. Concept learning algorithm based on interval-valued attribute information in an interval-valued formal context is shown in Algorithm 2. In Algorithm 2, we calculate the intension of concept C in steps 2–13 by using $\overline{B}^*$ and calculate the extension of concept C in steps 14–20 by using $\overline{B}^*$. Obviously, $C = (\overline{B}^*, \overline{B}^{**})$.

Algorithm 2: Interval-valued attributes oriented interval-valued concept learning (IvA-IvCL)

Input: An interval-valued formal context $(U, AT, \overline{I})$ and a set of interval-valued attributes $\overline{B}$.
Output: An interval concept C.

$\overline{A} \leftarrow \phi$, $O \leftarrow \phi$ and $C = (O, \overline{A})$; // $\overline{A}$ and O are the intension and extension of concept C, respectively.
for each $x \in U$ **do**
  $flag \leftarrow true$; // $true$ represents that x belong to the extension of concept C.
  for each $a \in AT$ **do**
    if $a^L(x) \geq \overline{B}^L(a)$ **and** $a^U(x) \leq \overline{A}^U(a)$ **then**
      $flag \leftarrow false$; // $false$ **represents that** x **does not belong to the extension of concept** C**.**
      break; // **stop searching.**
    end if
  end for // computing $\overline{B}^*$.
  if $flag$ **then**
    $O \leftarrow O \cup \{x\}$; // x is added to the extension of concept C.
  end if
end for
for each $a \in AT$ **do**
  $\overline{A}^L(a) \leftarrow +\infty$ and $\overline{A}^U(a) \leftarrow -\infty$;
  for each $x \in O$ **do**
    $\overline{A}^L(a) \leftarrow min(a^L(x), \overline{A}^L(a))$;
    $\overline{A}^U(a) \leftarrow max(a^U(x), \overline{A}^U(a))$;
  end for // update $\overline{A}(a)$ by computing B^{**}.
end for // update $\overline{A}$ by computing B^{**}.
return $C = (O, \overline{A})$.

From Algorithms 1 and 2, we know that concepts can be learned from given object or interval-valued attribute clue information. Sometimes, object and attribute information is given at the same time in concept learning. For example, we query the zebra on the internet, and the search results will show the characteristics of zebra, and also give some pictures of zebra. So we will learn concepts from both given object and interval-valued attribute information.

Concept learning algorithm based on both object and interval-valued attribute information is given as Algorithm 3. There are 2 ways (total 16 branches) to learn concepts from given object and interval-valued attribute information in Algorithm 3. The first way is to learn necessary information granules, then to transform them into sufficient and necessary information granules, and the process is shown in steps 1–5. The second way is to learn sufficient information granules, then transform them into sufficient and necessary information granules, and the calculation process is shown in steps 6–10.

Algorithm 3: Objects and interval-valued attributes oriented interval-valued concept learning (OIvA-IvCL)

Input: An interval-valued formal context $(U, AT, \overline{I})$ and a general information granule $(X, \overline{B})$.
Output: Interval concepts $C_1, C_2, \cdots, C_k$.

1: Transform $(X, \overline{B})$ into necessary information granules by (1)-(6) in **Case 1**, the results are denoted as $N_1, N_2, \cdots, N_6$;
2: **for** $(o, \overline{a}) \in \{N_1, N_2, \cdots, N_6\}$ **do**
3:   Learn concepts from $(o, \overline{a})$ by using (1)-(2) in **Case 3**;
4: **end for**
5: Merge the same concepts and get concepts $C_1, C_2, \cdots, C_i$;
6: Transform $(X, \overline{B})$ into sufficient information granules by (1)-(2) in **Case 2**, the results are denoted as S_1 and S_2;
7: **for** $(o, \overline{a}) \in \{S_1, S_2\}$ **do**
8:   Learn concepts from $(o, \overline{a})$ by using (1)-(2) in **Case 4**;
9: **end for**
10: Merge the repeated concepts and get concepts $C_1, C_2, \cdots, C_k$;
11: **return** $C_1, C_2, \cdots, C_k$.

6 Experimental analysis

Based on the cognitive mechanism of concept learning in interval-valued cognitive systems, we can not only find all concepts in interval-valued formal context form the global perspective, but also find the corresponding concepts from the local perspective by using the given clue information. The proposed interval-valued concept learning algorithms have been verified in simple examples, and they can accurately calculate concepts by using given clues. In particular, concept learning by using given clue has many challenges in the practical application of big data, such as object scales and attribute scales.

In this section, to evaluate the effectiveness and feasibility of the proposed concept learning algorithms (O-IvCL, IvA-IvCL and OIvA-IvCL), we will design a series of experiments to illustrate that (1) O-IvCL and IvA-IvCL algorithms can quickly find the corresponding concept under the given object or attribute information clues, then verify their effectiveness; (2) OIvA-IvCL algorithm is similar to two-way concept learning algorithm [46] under fuzzy formal context, in that it can also find the concept under the given clue information of both objects and attributes. Moreover, because interval-valued data is more noise resistant than fuzzy data with precise values, the resulting concept of OIvA-IvCL may be more accurate. The main evaluation indexes are the size of extension of concepts, running time of concept learning algorithms, and number of concepts learned by the concept learning algorithms.

We download eight data sets from UCI Machine Learning Repository [5]. Their detailed information is shown in Table 4. The attribute values of all data sets are normalized by using $\frac{a(x)-a_{min}}{a_{max}-a_{min}}$, where a_{max} and a_{min} represent the max value and min value of all objects under attribute a, respectively.

To learn concepts from interval-valued cognitive systems, we construct interval-valued attributes by using the following methods.

(1) $a^L(x) = max(0, (1-\epsilon) \times a(x))$;
(2) $a^U(x) = min(1, (1+\epsilon) \times a(x))$;

where $a(x)$ represents the value of object x with respect to attribute a in the original information systems. We take $\epsilon = 0.2$ to construct the interval-valued attributes.

We will compare the proposed algorithms with granular computing approach to two-way learning (GrC-TWL) algorithm [46]. These algorithms need some object, interval-valued attribute and fuzzy attribute information. First, we randomly divide the set of objects into 10 subsets, take one of them and denote it as S. Then we use the following methods to obtain the object, fuzzy attribute and interval-valued attribute information.

(1) $X = \complement_S Y$;
(2) $\tilde{B}(a) = \bigwedge_{x \in Y} \mu_a(x), \forall a \in AT$;
(3) $\overline{B}^L(a) = \bigwedge_{x \in Y} a^L(x), \overline{B}^U(a) = \bigvee_{x \in Y} a^U(x), \forall a \in AT$;

where Y is the first 50% objects in S, $\mu_a(x) = a(x)$ and $\complement_S Y$ is the complement of Y with respect to S. Then we use the $\tilde{B}$, $\overline{B}$ and X as the fuzzy attribute information, interval-valued attribute information and object information of the above algorithms to learn concept. The object, fuzzy attribute and interval-valued attribute information of the remaining 9 subsets are obtained by the same way. After 10 times, we take average performance as the final performance.

The parameter of the objects oriented interval concept learning (O-IvCL) algorithm is X, the parameter of the interval-valued attributes oriented interval concept learning (IvA-IvCL) algorithm is $\overline{B}$, the parameter of the objects and interval-valued attributes oriented concept learning (OIvA-IvCL) algorithm is $(X, \overline{B})$, and the parameter of granular computing approach to two-way learning (GrC-TWL) algorithm is $(X, \tilde{B})$. For the 4 algorithms, we use the above information to learn concepts, and then compute size of the extension, running time of the algorithms and number of concepts learned by the algorithms. We use the same hardware and software

Table 4 Description of data sets

No.	Data sets	Abbreviation	Objects	Attributes	Decisions
1	Wine	Wine	178	13	3
2	Wpbc	Wpbc	198	33	2
3	Seeds	Seeds	210	7	3
4	Wdbc	Wdbc	569	30	2
5	Winequality-red	Wine-R	1599	11	6
6	Segment	Segment	2310	19	7
7	Spambase	Spambase	4601	58	2
8	Winequality-white	Wine-W	4898	11	7

to run all the experiments. The information of hardware: Inter(R) Core(TM) i7-7700K CPU @ 4.20 GHz 4.20 GHz and 16 GB RAM. The information of software: Windows 10 and MATLAB R2015b. After 10 times, the average performances are shown in Tables 5 and 6. The source code of the proposed algorithms is presented in https://github.com/humeng24/Interval-valued-concept-learning.

From the experimental results of Table 5, we can see that from the given object information or given attribute information, algorithm O-IvCL and algorithm IvA-IvCL can find a corresponding concept in a short time (no more than 13 ms). This shows that algorithms O-IvCL and IvA-IvCL are feasible in finding concept and effective in computation.

Next, we focus on comparing the performance of the two algorithms under both given object information and given attribute information, namely OIvA-IvCL and GrC-TWL. The number of concepts reflects the ability of the algorithm to find concepts. From the experimental results of Table 6, we find that both algorithm OIvA-IvCL and algorithm GrC-TWL can find multiple concepts related to the given cue information. Moreover, because the given data value is an interval and the other is the exact real number in the interval, the extension of the concepts calculated by the two algorithms is similar. This also shows the feasibility of our algorithm OIvA-IvCL in finding the concept. In terms of running time, we analyze that the most essential difference between the two algorithms is that the definition of dual cognitive operator adds more compact constraints on the interval than on the real number, so the computing time of algorithm OIvA-IvCL is more than that of algorithm GrC-TWL. However, the performance of the proposed algorithm is acceptable as the computing time on the largest data set (Winequality-white) is less than 1 s.

The extension of a concept is composed of objects. For same clue information, the smaller the extension of the concept learned by the concept learning algorithm is, the more accurate the algorithm is. For example, *white horse* is a sub concept of *horse*, if two algorithms use the same

Table 5 Experimental results of algorithms O-IvCL and IvA-IvCL under given object or attribute information

Data sets	Number of concepts		Size of extension		Running time(Unit: ms)	
	O-IvCL	IvA-IvCL	O-IvCL	IvA-IvCL	O-IvCL	IvA-IvCL
Wine	1 ± 0	1 ± 0	31 ± 9.4	23.6 ± 8.8	0.380 ± 0.026	0.367 ± 0.019
Wpbc	1 ± 0	1 ± 0	25.5 ± 6.9	20.8 ± 6.6	0.441 ± 0.050	0.435 ± 0.010
Seeds	1 ± 0	1 ± 0	108.6 ± 36.5	103.3 ± 19.8	0.429 ± 0.010	0.428 ± 0.010
Wdbc	1 ± 0	1 ± 0	286.1 ± 49.7	255.1 ± 36.3	1.224 ± 0.027	1.221 ± 0.018
Wine-R	1 ± 0	1 ± 0	1350.1 ± 51.1	1381.4 ± 57.6	3.078 ± 0.047	3.101 ± 0.039
Segment	1 ± 0	1 ± 0	2163.7 ± 75.3	2136.9 ± 71.8	4.574 ± 0.077	4.634 ± 0.093
Spambase	1 ± 0	1 ± 0	3821.5 ± 126.3	3810 ± 130.8	10.701 ± 0.224	12.256 ± 0.196
Wine-W	1 ± 0	1 ± 0	4596.4 ± 112.4	4619 ± 75.5	9.158 ± 0.090	9.471 ± 0.137
Average	1 ± 0	1 ± 0	1547.9 ± 58.5	1543.8 ± 50.9	3.748 ± 0.060	3.989 ± 0.065

Table 6 Experimental results of algorithms OIvA-IvCL and GrC-TWL under given object and attribute information

Data sets	Number of concepts		Size of extension		Running time(Unit: ms)	
	OIvA-IvCL	GrC-TWL	OIvA-IvCL	GrC-TWL	OIvA-IvCL	GrC-TWL
Wine	4.5 ± 0.5	$\mathbf{5 \pm 0}$	29.5 ± 6.3	58.9 ± 13.2	6.437 ± 1.257	0.883 ± 0.070
Wpbc	4.1 ± 0.7	$\mathbf{5 \pm 0}$	30.6 ± 6.9	59.3 ± 13.2	7.250 ± 1.241	1.105 ± 0.065
Seeds	$\mathbf{5 \pm 0}$	$\mathbf{5 \pm 0}$	93.4 ± 21.7	134.7 ± 18.4	8.663 ± 0.098	0.946 ± 0.024
Wdbc	$\mathbf{5 \pm 0}$	$\mathbf{5 \pm 0}$	231.2 ± 15.6	359 ± 21.5	25.017 ± 0.277	2.540 ± 0.050
Wine-R	$\mathbf{5 \pm 0}$	4.9 ± 0.3	1318.5 ± 49.8	1462.9 ± 40.5	62.687 ± 0.221	5.003 ± 0.083
Segment	$\mathbf{5 \pm 0}$	4.9 ± 0.3	2110.8 ± 66.8	2186.3 ± 49.2	96.702 ± 0.187	8.293 ± 0.357
Spambase	$\mathbf{5 \pm 0}$	2 ± 0.8	3653.5 ± 105.9	4584.4 ± 18.3	234.500 ± 1.358	32.883 ± 0.317
Wine-W	$\mathbf{5 \pm 0}$	$\mathbf{5 \pm 0}$	4550.2 ± 63.7	4711.5 ± 48.7	190.660 ± 0.468	14.503 ± 0.121
Average	4.83 ± 0.15	4.60 ± 0.18	1502.2 ± 42.1	1694.6 ± 27.9	78.990 ± 0.638	8.270 ± 0.136

Bold indicates the maximum number of concepts learned by the concept learning algorithms under the given object and attribute information

clue information to learn concepts, the concept learned by the first algorithm is *white horse* and the concept learned by the second algorithm is *horse*, then the first algorithm is more accurate than the second algorithm, and the second algorithm is rougher than the first algorithm in concept learning. Further, when we search for goods in online store, we want to find as few results as possible to meet our needs from the ocean of goods. Therefore, we hope that concepts learned by concept learning algorithms are as accurate as possible. That is to say, the extension of the concept should be as small as possible. The average size of extension of concepts learned by the four algorithms is shown in Fig. 3. From Fig. 3, the size of extension of concepts learned by OIvA-IvCL is the smallest in most cases.

Moreover, we compare the ability of algorithms OIvA-IvCL and GrC-TWL to find concepts under almost the same given information. From Fig. 4, we can see that the average size of extension of concepts learned by OIvA-IvCL (4.83)and GrC-TWL (4.60) is similar.

To sum up, in interval-valued formal context, our proposed concept learning theory and algorithms can better help to learn and find concepts.

7 Conclusion and future work

How to learn concepts from given information is a hot issue in knowledge discovery. In this work, we introduce interval-valued formal contexts as the basis of interval-valued cognitive concept learning. Interval-valued FCA is a generalized model of classical FCA and fuzzy FCA. We define a pair of dual cognitive operators on interval-valued formal contexts to construct an interval-valued Galois connection. The interval-valued cognitive system and four kinds of IvIGs are defined to describe the cognitive mechanism of interval-valued concept learning. We propose 6 methods to learn necessary IvIGs from general IvIGs and 2 methods to learn sufficient IvIGs from general IvIGs, respectively. On the basis of necessary or sufficient

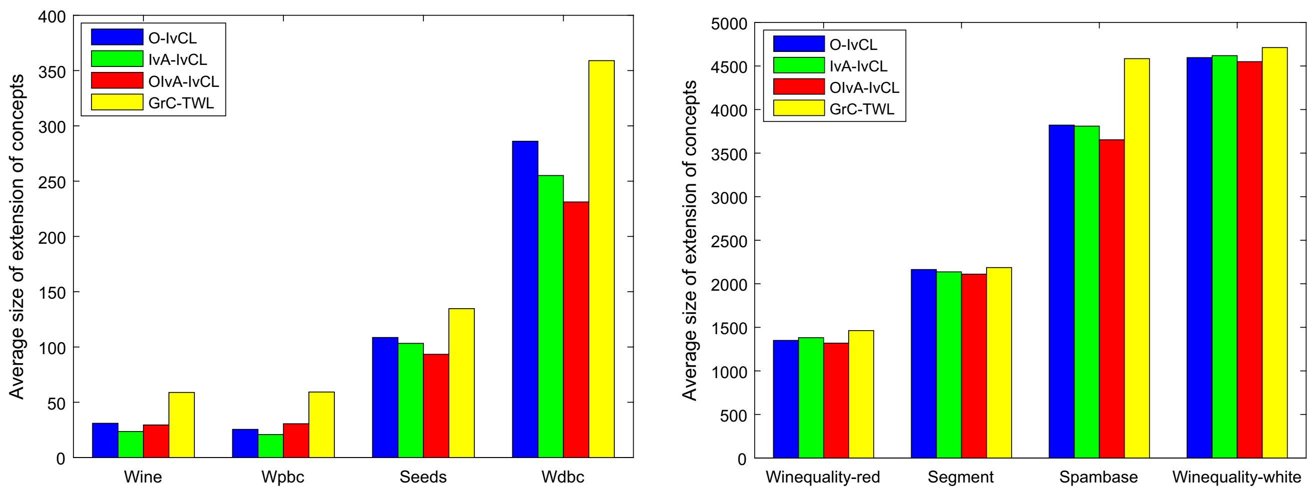

Fig. 3 Average size of extension of concepts learned on eight data sets

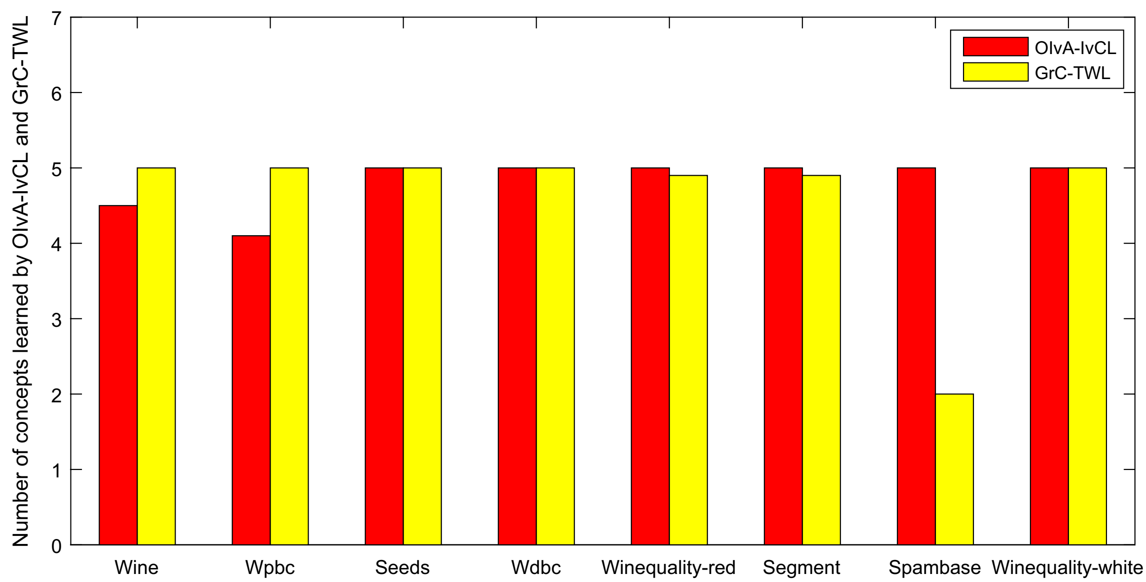

Fig. 4 Average number of concepts learned by OIvA-IvCL and GrC-TWL

IvIGs, we obtain 2 methods to transform necessary IvIGs into sufficient and necessary IvIGs, and 2 methods to transform sufficient IvIGs into sufficient and necessary IvIGs, respectively. From general IvIGs to sufficient and necessary IvIGs, there are 2 ways (16 branches) to learn concepts. We design three algorithms (O-IvCL, IvA-IvCL and OIvA-IvCL) to achieve the concepts of automatically learning from different clue information. We download eight public UCI data sets to verify the feasibility and effectiveness of O-IvCL, IvA-IvCL and OIvA-IvCL algorithms from the perspectives of the size of extension of concepts, running time of concept learning algorithms and number of concepts. The experimental results show that O-IvCL, IvA-IvCL and OIvA-IvCL algorithms are effective and feasible in concept learning of interval-valued formal contexts.

In this paper, we propose a basic theoretical framework for interval-valued concept learning from the perspective of granular computing and cognitive learning. In the future, we will study dynamic interval-valued concept learning on dynamic interval-valued data sets.

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References

1. Bargiela A, Pedrycz W (2008) Toward a theory of granular computing for human-centred information processing. IEEE Trans Fuzzy Syst 16(2):320–330
2. Benítez-Caballero MJ, Medina J, Ramírez-Poussa E, Ślęzak D (2020) Rough-set-driven approach for attribute reduction in fuzzy formal concept analysis. Fuzzy Sets Syst 391:117–138
3. Cabrerizo FJ, AI-Hmouz R, Morfeq A, Martínez MÁ, Pedrycz W, Herrera-Viedma E, (2020) Estimating incomplete information in group decision making: a framework of granular computing. Appl Soft Comput 86:105930
4. Che XY, Chen DG, Mi JS (2020) A novel approach for learning label correlation with application to feature selection of multi-label data. Inf Sci 512:795–812
5. Dua D, Graff C (2019) UCI machine learning repository. University of California, School of Information and Computer Science, Irvine. http://archive.ics.uci.edu/ml
6. Formica A (2019) Similarity reasoning in formal concept analysis: from one-to many-valued contexts. Knowl Inf Syst 60(2):715–739
7. Ganter B, Wille R (1999) Formal concept analysis: mathematical foundations. Springer, Berlin
8. Gjicali K, Finn BM, Hebert D (2020) Effects of belief generation on social exploration, culturally-appropriate actions, and cross-cultural concept learning in a game-based social simulation. Comput Educ 156:103959
9. Guan ZY, Zhang LJ, Peng JY, Fan J (2015) Multi-view concept learning for data representation. IEEE Trans Knowl Data Eng 27(11):3016–3028
10. Guo QC, Ma JM (2020) Judgment methods of interval-set consistent sets of dual interval-set concept lattices (in Chinese). Comput Sci 3:98–102
11. Guo Y, Tsang ECC, Xu W et al (2019) Local logical disjunction double-quantitative rough sets. Inf Sci 500:87–112
12. Guo Y, Tsang ECC, Xu W, Chen D (2020) Adaptive weighted generalized multi-granulation interval-valued decision-theoretic rough sets. Knowl Based Syst 187:104804
13. Guo Y, Tsang ECC, Hu M et al (2020) Incremental updating approximations for double-quantitative decision-theoretic rough sets with the variation of objects. Knowl Based Syst 189:105082
14. Hu M, Tsang ECC, Guo YT, Xu WH (2021) Fast and robust attribute reduction based on the separability in fuzzy decision systems. IEEE Trans Cybern. https://doi.org/10.1109/TCYB.2020.3040803
15. Hu M, Tsang ECC, Guo YT, Chen DG, Xu WH (2021) A novel approach to attribute reduction based on weighted neighborhood rough sets. Knowl Based Syst 220:Art. 106908. https://doi.org/10.1016/j.knosys.2021.106908
16. Hu M, Tsang ECC, Guo YT, Xu WH, Chen DG (2020) A fast reduction algorithm with attribute pre-sort based on neighborhood rough set. In: 2020 international conference on machine learning and cybernetics (ICMLC), pp 279–284
17. Ishwarya MS, Kumar AC (2020) Decision-making in cognitive paradoxes with contextuality and quantum formalism. Appl Soft Comput 95(5):106521
18. Kumar CA, Mouliswaran SC, Amriteya P, Arun SR (2015) Fuzzy formal concept analysis approach for information retrieval. In: Proc. 5th int. conf. fuzzy neuro comput (FANCCO), pp 255–271
19. Kuznetsov SO (2004) Machine learning and formal concept analysis. In: Proc. int. conf. formal concept anal. (ICFCA), pp 23–26
20. Leung Y, Fischer MM, Wu WZ, Mi JS (2008) A rough set approach for the discovery of classification rules in interval valued information systems. Int J Approx Reason 47(2):233–246
21. Li JH, Mei CL, Xu WH, Qian Y (2015) Concept learning via granular computing: a cognitive viewpoint. Inf Sci 298:447–467
22. Li JH, Huang CC, Qi JJ, Qian Y, Liu W (2017) Three-way cognitive concept learning via multi-granularity. Inf Sci 378:244–263
23. Li JH, Mi YL, Liu WQ (2019) Incremental cognition of concepts: theories and methods (in Chinese). Chin J Comput 42:2233–2250
24. Loia V, Orciuoli F, Pedrycz W (2018) Towards a granular computing approach based on formal concept analysis for discovering periodicities in data. Knowl Based Syst 146:1–11
25. Ma JM, Cai MJ, Zou CJ (2017) Concept acquisition approach of object-oriented concept lattices. Int J Mach Learn Cybern 8(1):123–134
26. Ma JM, Hu LL, Qian YH (2019) Object-oriented interval-set concept lattices. Int J Approx Reason 110:64–81
27. Mi JS, Leung Y, Wu WZ (2010) Approaches to attribute reduction in concept lattices induced by axialities. Knowl Based Syst 23(6):504–511
28. Mi YL, Shi Y, Li JH, Liu WQ, Yan MY (2020) Fuzzy-based concept learning method: Exploiting data with fuzzy conceptual clustering. IEEE Trans Cybern. https://doi.org/10.1109/TCYB.2020.2980794
29. Mi YL, Liu WQ, Shi Y, Li JH (2020) Semi-supervised concept learning by concept-cognitive learning and concept space. IEEE Trans Knowl Data Eng. https://doi.org/10.1109/TKDE.2020.3010918
30. Niu JJ, Huang CC, Li JH, Fan M (2018) Parallel computing techniques for concept-cognitive learning based on granular computing. Int J Mach Learn Cybern 9:1785–1805
31. Pedrycz W (2013) Granular computing analysis and design of intelligent systems. CRC Press, Boca Raton
32. Savchenko AV (2016) Fast multi-class recognition of piecewise regular objects based on sequential three-way decisions and granular computing. Knowl Based Syst 19:252–262
33. Shao M, Yang H (2013) Two kinds of multi-level formal concepts and its application for sets approximations. Int J Mach Learn Cybern 4(6):621–630
34. Shi Y, Mi Y, Li J, Liu W (2019) Concurrent concept-cognitive learning model for classification. Inf Sci 496:65–81
35. Shivhare R, Kumar CA, Li JH (2017) Establishment of cognitive relations based on cognitive informatics. Cogn Comput 9(5):721–729
36. Shivhare R, Kumar CA (2017) Three-way conceptual approach for cognitive memory functionalities. Int J Mach Learn Cybern 8(1):21–34
37. Singh PK, Kumar CA (2014) Bipolar fuzzy graph representation of concept lattice. Inf Sci 288:437–448
38. Singh PK, Kumar CA, Li JH (2016) Knowledge representation using interval-valued fuzzy formal concept lattice. Soft Comput 20(4):1485–1502
39. Singh PK (2019) Bipolar fuzzy concept learning using next neighbor and Euclidean distance. Soft Comput 23(12):4503–4520
40. Wang H, Cui PZ (2015) Analysis of interval-valued decision formal contexts. J Intell Fuzzy Syst 29(4):1565–1574
41. Wei L, Qian T, Wan Q, Qi JJ (2018) A research summary about triadic concept analysis. Int J Mach Learn Cybern 9(4):699–712
42. Wei L, Liu L, Qi JJ, Qian T (2020) Rules acquisition of formal decision contexts based on three-way concept lattices. Inf Sci 516:529–544
43. Wille R (1982) Restructuring lattice theory: an approach based on hierarchies of concepts. In: Rival I (ed) Ordered sets. Reidel, Dordrecht, pp 445–470
44. Wu WZ, Leung Y, Mi JS (2009) Granular computing and knowledge reduction in formal contexts. IEEE Trans Knowl Data Eng 21(10):1461–1474

45. Xu WH, Pang JZ, Luo SQ (2014) A novel cognitive system model and approach to transformation of information granules. Int J Approx Reason 55:853–866
46. Xu WH, Li WT (2016) Granular computing approach to two-way learning based on formal concept analysis in fuzzy datasets. IEEE Trans Cybern 46(2):366–379
47. Yahia BS, Jaoua A (2001) Discovering knowledge from fuzzy concept lattice. In: Data mining and computational intelligence. Physica, Berlin, pp 167–190
48. Yao YY (2009) Interpreting concept learning in cognitive informatics and granular computing. IEEE Trans Syst Man Cybern B Cybern 39(4):855–866
49. Yao YY (2020) Three-way granular computing, rough sets, and formal concept analysis. Int J Approx Reason 116:106–125
50. Zadeh LA (1979) Fuzzy sets and information granularity, advances in fuzzy set theory and applications. North-Holland Publishing, Amsterdam
51. Zhang WX, Xu WH (2007) Cognitive model based on granular computing. Chin J Eng Math 24:957–971
52. Zhang T, Li HH, Liu MQ, Rong M (2020) Incremental concept-cognitive learning based on attribute topology. Int J Approx Reason 118:173–189
53. Zhao YX, Li JH, Liu WQ, Xu WH (2017) Cognitive concept learning from incomplete information. Int J Mach Learn Cybern 8(1):159–170