

$$\alpha = E\left[\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2\right] = E\left[\frac{1}{n} \sum_{i=1}^n x_i^2 + \frac{1}{n} \sum_{i=1}^n \bar{x}^2 - \frac{2\bar{x}}{n} \sum_{i=1}^n x_i\right]$$

$E[x_i^2] = E[x^2]$ bc exp. value of single element has the same value if it is from pop. or sample

$$\frac{\sum_{i=1}^n x_i}{n} = \bar{x}$$

$$\Rightarrow \alpha = \frac{1}{n} \sum_{i=1}^n E[x^2] + \frac{1}{n} \sum_{i=1}^n E[\bar{x}^2] - 2E[\bar{x}^2]$$

$$= \frac{1}{n} n E[x^2] + \frac{1}{n} n E[\bar{x}^2] - 2E[\bar{x}^2]$$

$$\alpha = E[x^2] - E[\bar{x}^2]$$

$$E[\bar{x}^2] = \frac{1}{n} \sum_{i=1}^n x_i \cdot \frac{1}{n} \sum_{j=1}^n x_j = \frac{1}{n^2} (x_1 + x_2 + \dots + x_n)(x_1 + x_2 + \dots + x_n)$$

$$= E \left[\frac{1}{n^2} \left\{ \sum_{i=j}^n x_i^2 + \sum_{i=1}^n x_i \sum_{\substack{j=1 \\ i \neq j}}^{n-1} x_j \right\} \right]$$

<-- with $i=j$ there's no need for the coefficient 2 because the sum will Account for the ij, ji degeneracy. The sum over j is still $n-1$ to ensure that the $i=j$ element is skipped

$$= \frac{1}{n^2} \left[n E[x^2] + n E[x] (n-1) E[x] \right]$$

$$E[\bar{x}^2] = \frac{1}{n} E[x^2] + \frac{n-1}{n} E[x]^2$$

$$\alpha = E[x^2] - E[\bar{x}^2]$$

$$\Rightarrow \alpha = E[x^2] - \frac{1}{n} E[x^2] - \frac{n-1}{n} E[x]^2$$

$$= \left(1 - \frac{1}{n}\right) E[x^2] - \frac{n-1}{n} E[x]^2$$

$$= \frac{n-1}{n} E[x^2] - \frac{n-1}{n} E[x]^2$$

$$\alpha = \frac{n-1}{n} (E[x^2] - E[x]^2) = \frac{n-1}{n} \sigma^2$$

$$\Rightarrow \alpha = E\left[\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2\right] = \frac{n-1}{n} \sigma^2$$

$$\frac{E\left[\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2\right]}{n-1} = \frac{\sigma^2}{n}$$