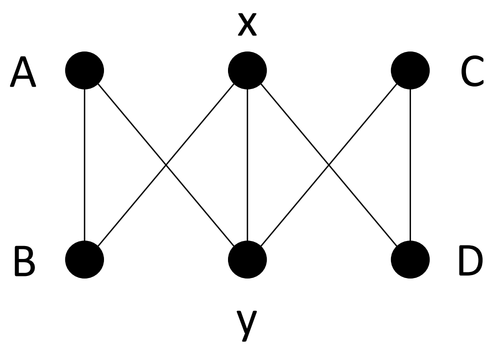
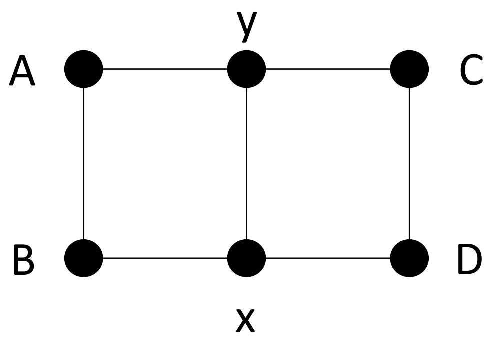
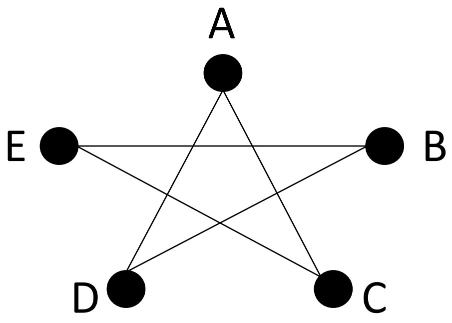
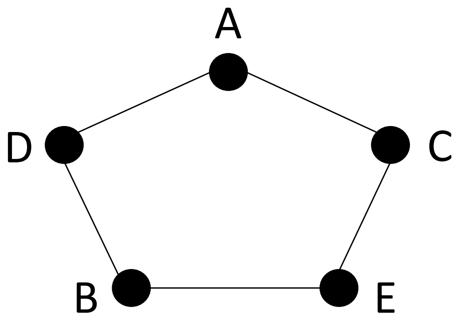
Planar graphs

A graph is planar if it can be drawn on the plane without crossing edges.
We will focus exclusively on connected planar graphs.



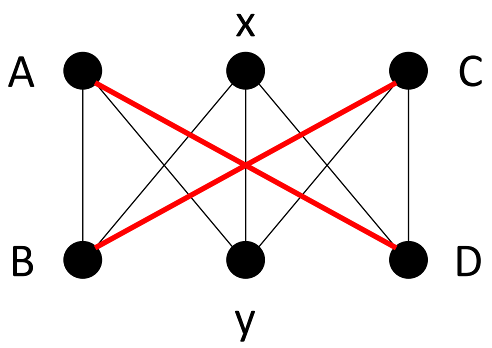


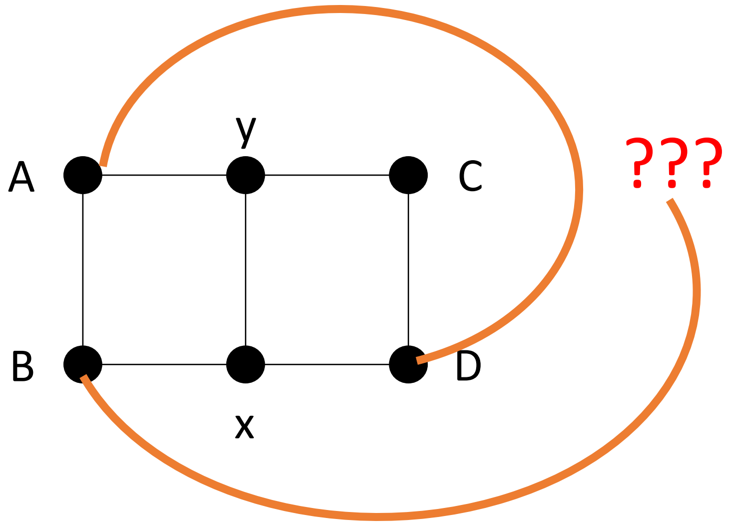




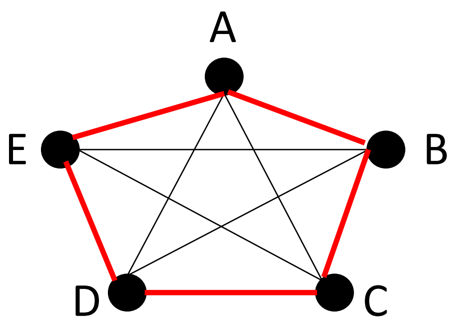
Planar graphs?

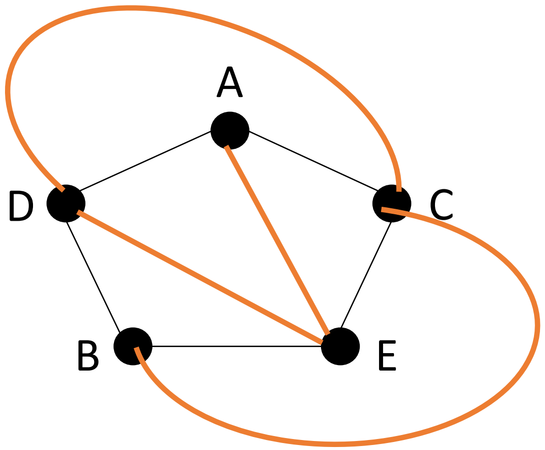
$K_{3,3}$





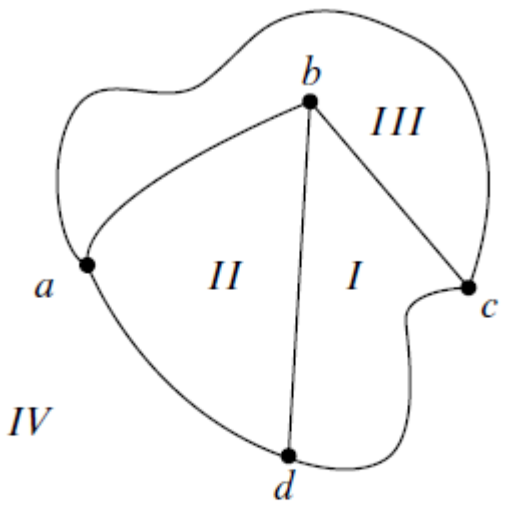
K_5

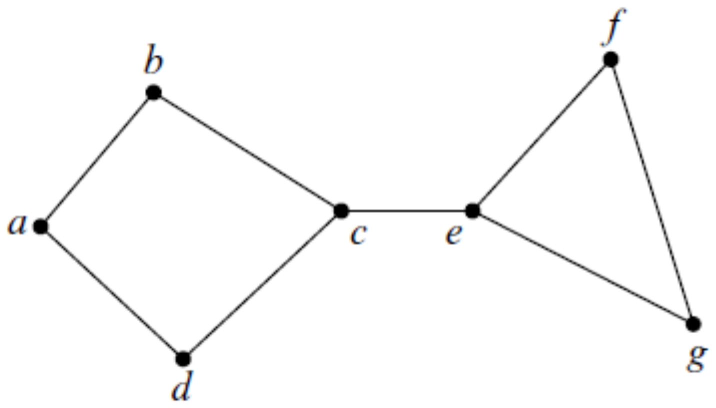




Connect A to B ???

The boundary of a region





We define the boundary of a region as a closed walk in clockwise order of all edges that lie within the region.

This is well-defined when the graph is connected.

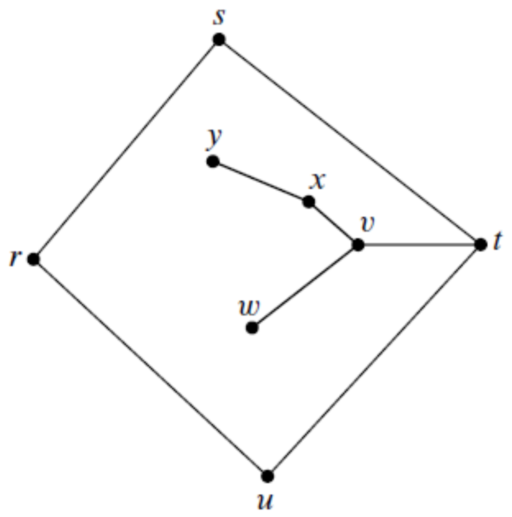
Region I : $\{c, d\}, \{d, b\}, \{b, c\}$
Region II : $\{d, a\}, \{a, b\}, \{b, d\}$
Region III : $\{a, c\}, \{c, b\}, \{b, a\}$
Region IV : $\{c, d\}, \{d, a\}, \{a, c\}$

Boundary of outer region :
$\{f, g\}, \{g, e\}, \{e, c\}, \{c, d\}, \{d, a\}, \{a, b\}, \{b, c\}, \{c, e\}, \{e, f\}$

Note that the edge $\{e, c\}$ occurs twice on the boundary of the outer region.

Each edge lies once on the boundary of 2 regions, or twice on the boundary of one region.
Each region has 3 or more bounding edges

What about dongles?



Boundary of outer region: $\{t, u\}, \{u, r\}, \{r, s\}, \{s, t\}$

Boundary of inner region: $\{t, u\}\{u, r\}, \{r, s\}, \{s, t\}, \{t, v\}, \{v, x\}, \{x, y\}, \{y, x\}, \{x, v\}, \{v, w\}, \{w, v\}, \{v, t\}$

Each edge lies once on the boundary of 2 regions, or twice on the boundary of one region.
Therefore, X = Sum of the number of edges of every region boundary = $2m$

Also, if the number of vertices is at least 3, and since each region has 3 or more bounding edges, $X \geq 3r$.

Therefore, $2m \geq 3r$ for every connected planar graph with at least 3 vertices.

In general, if every cycle has length c or greater, then $2m \geq cr$.

Euler’s Formula

Theorem: For every connected planar graph with n vertices, m edges, and r regions: $n - m + r = 2$

Corollary: The number of regions in all drawings of a planar graph is invariant.

Proof: Induction on the structure of the graph G.

Idea: Start with a single node, and form a sequence of connected subgraphs $G_0 G_1 \dots G_m$

such that G_0 is a single node,

G_i is formed by adding one edge to G_{i-1},

$G_m = G$

and at each step G_i satisfies the formula.

Base Case: $n = 1, e = 0, r = 1.\quad 1 - 0 + 1 = 2$

Inductive Hypothesis: The formula is true for connected subgraph G_k

Inductive Step: Insert an edge incident to at least one vertex in G_k.

Case 1: Only one end point is in G_k, so the edge is a dongle.

This adds one new vertex, one new edge, but the number of regions stays the same.

So the value of the LHS remains 2.

Case 2: Both end-points are in G_k. This creates a new region but the number of nodes stays the same.

So the value of the LHS remains 2.

Planar graphs have few edges

Theorem. For every connected planar graph G with $n \geq 3$ vertices: $m \leq 3n - 6$.

Proof: We showed that $2m \geq 3r$

From Euler's Theorem, $n - m + r = 2 \Rightarrow r = 2 + m - n$

Therefore, $2m \geq 3(2 + m - n)$

which implies $2m \geq 6 + 3m - 3n$

which yields $m \leq 3n - 6$

Corollary 1: K_5 is not planar.
Proof: $n = 5, m = 10$.
But $10 > 3 \cdot 5 - 6 = 9$, violating Euler's formula so the graph is not planar.

Corollary 2: $K_{3,3}$ is not planar.
Proof: $K_{3,3}$ has only even length cycles, so if it were planar $2m \geq 4r$, implying $2m \geq 4(2 + m - n)$,
But $2m = 18$, $4(2 + m - n) = 4(2 + 9 - 6) = 20$!

A Corollary

Claim: Every connected planar graph has a vertex with degree 5 or less.

Proof: (By contradiction)

Suppose each of the n vertices has degree 6 or more.

Number of edges $m = \frac{1}{2}\sum_v degree(v)$

$$\geq \frac{1}{2}6n$$

$$= 3n > 3n - 6$$

This contradicts:

Theorem. For every connected planar graph G with $n \geq 3$ vertices: $m \leq 3n - 6$.

The Five-Color Theorem

Theorem: Every planar graph can be colored with 5 or fewer colors.

Proof: By induction on the number n of vertices.

Base Case: $n \leq 5$ Use a different color for each vertex.

Inductive Hypothesis: Every planar graph with k or fewer vertices has a 5-coloring.

Inductive Step: Let G be a graph with $k + 1$ vertices.

a. If there is a graph with degree 4 or less, remove it
    1. By the inductive hypothesis, the remaining planar graph is 5-colorable
    2. Reinsert the vertex removed in Step 1 and use a color different from its (at most) 4 neighbors.

b. Every vertex has degree at least 5.

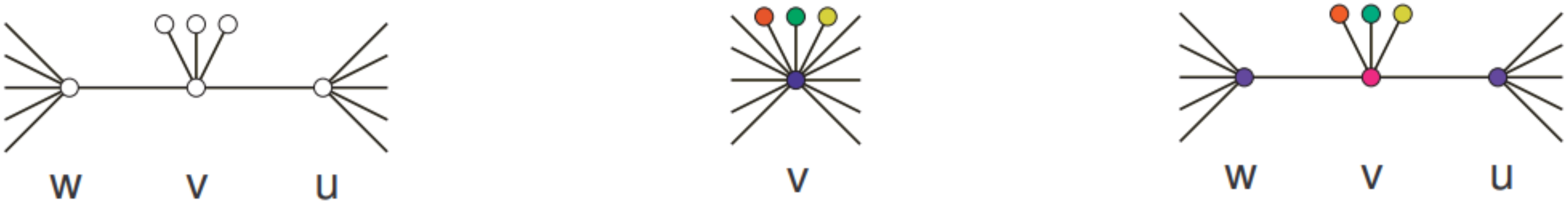
The Five-Color Theorem

b. Every vertex has degree at least 5.

Pick a vertex v of degree 5.

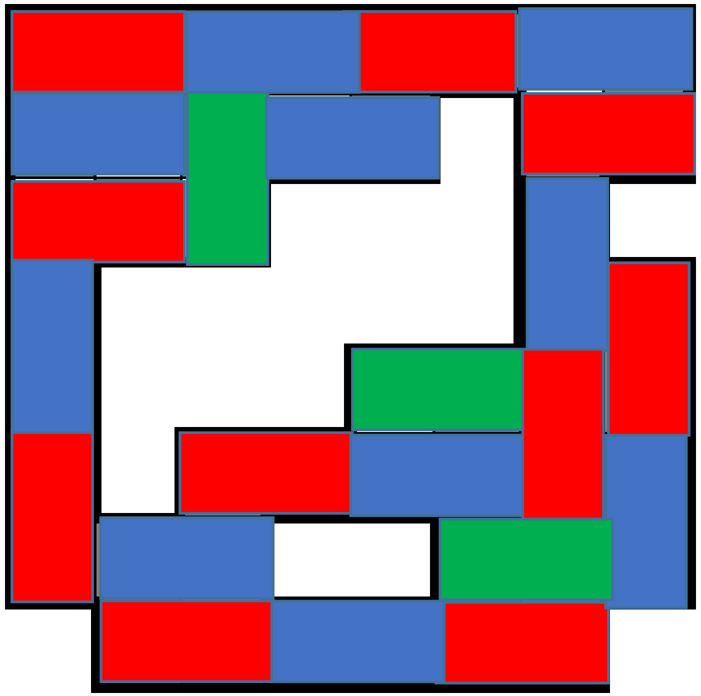
At least one pair of its neighbors u, w don't have an edge between them. Why?

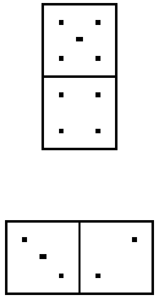
Merge u, v, w into one vertex. The graph remains planar; color it recursively.



Separate u, v, w and color u, w with one color!

Covering a chess board with dominoes





dominos

Can we place dominoes so that:

no two dominoes overlap, and
every board square is covered, and
every domino covers two board squares?

Latin Squares

Latin Square: Fill the nxn with numbers from 1 to n so that:

- Each row contains every number from 1 to n.
- Each column contains every number from 1 to n.

1	2	3	4
3	4	2	1
2	1	4	3
4	3	1	2

The 5 x 5 Latin Square

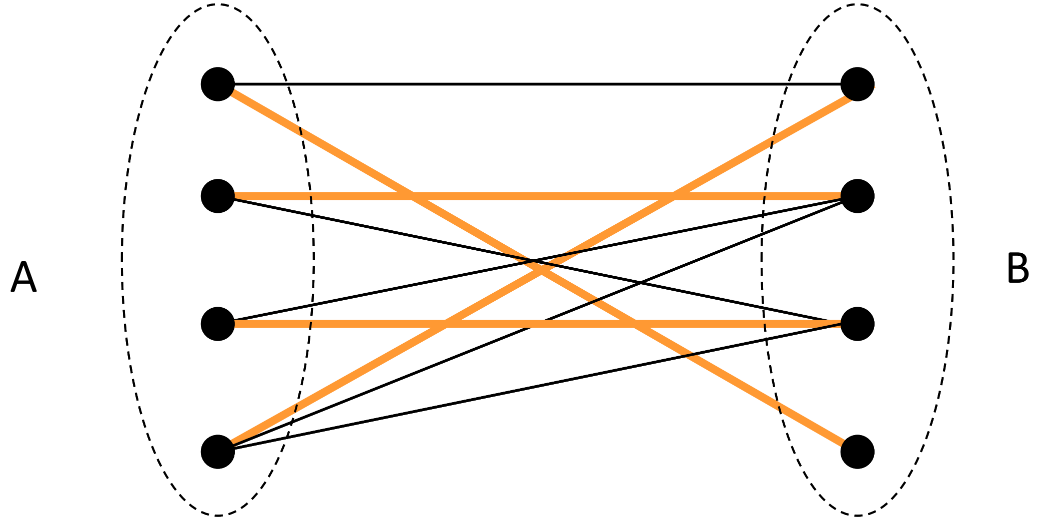
Given a partially filled Latin Square with some rows are already filled in.

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4

Can you always extend it to a Latin Square?

Bipartite Graph Matching

A graph $G(A, B, E)$ is **bipartite** if its vertex set can be partitioned into two subsets A and B so that each edge has one endpoint in A and the other endpoint in B.



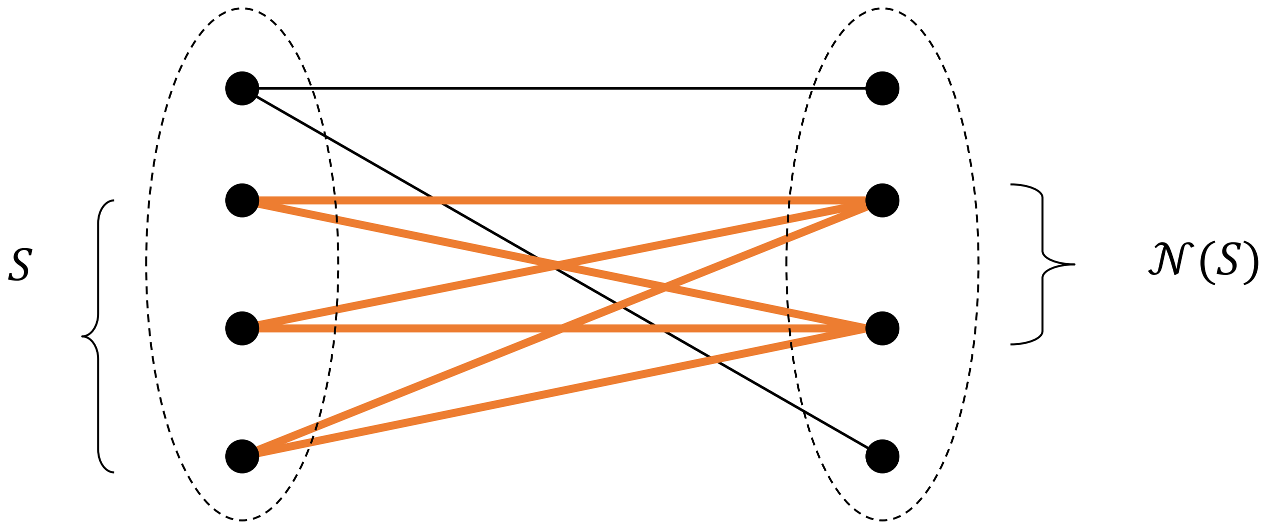
A **perfect matching** is a subset of edges which matches every vertex in A with a unique vertex in B.

Question: Given a bipartite graph, does it have a perfect matching?

Perfect Matching

Does a perfect matching always exist? NO!

If there are more vertices on one side, then definitely not.



$\mathcal{N}(S)$ is the set of neighbors of S.

If $|\mathcal{N}(S)| < |S|$ for any subset $S \subseteq A$, then no perfect matching exists.

Hall's Theorem

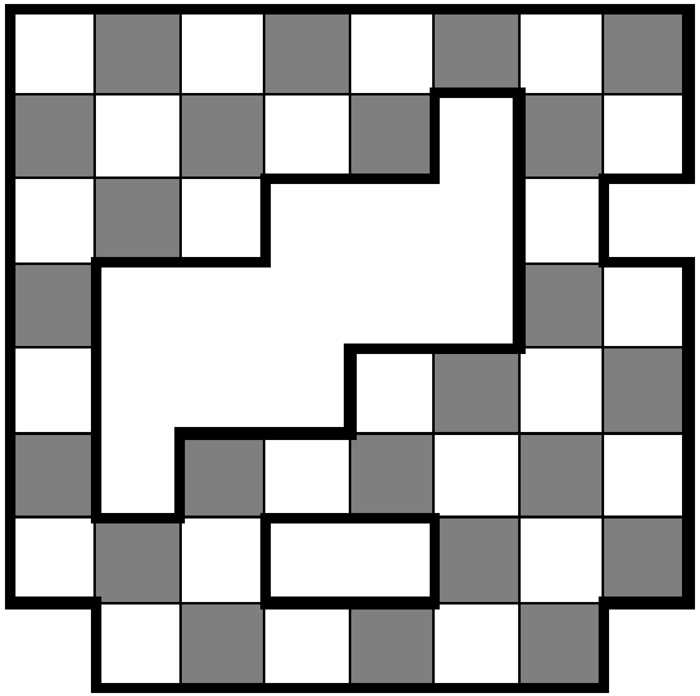
A bipartite graph $G(A, B, E)$ has a perfect matching

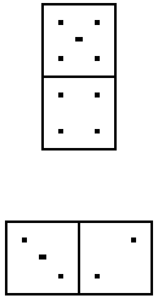
if and only if

$|S| \leq |\mathcal{N}(S)|$ for every subset $S \subseteq A$.

the matching condition

Application of Bipartite Matching





A: a vertex for each white square, B: a vertex for each black square

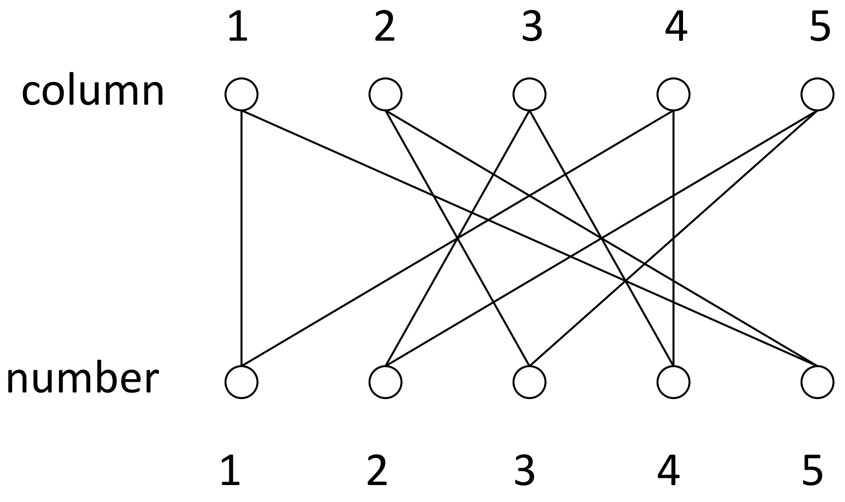
Add an edge for every two squares that are adjacent. The resulting graph is bipartite.

A perfect matching in this graph corresponds to a perfect covering by dominoes.

Application of Bipartite Matching

Given a partial Latin square, we construct a bipartite graph to fill in the next row.

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4



We want to "match" the numbers to the columns.

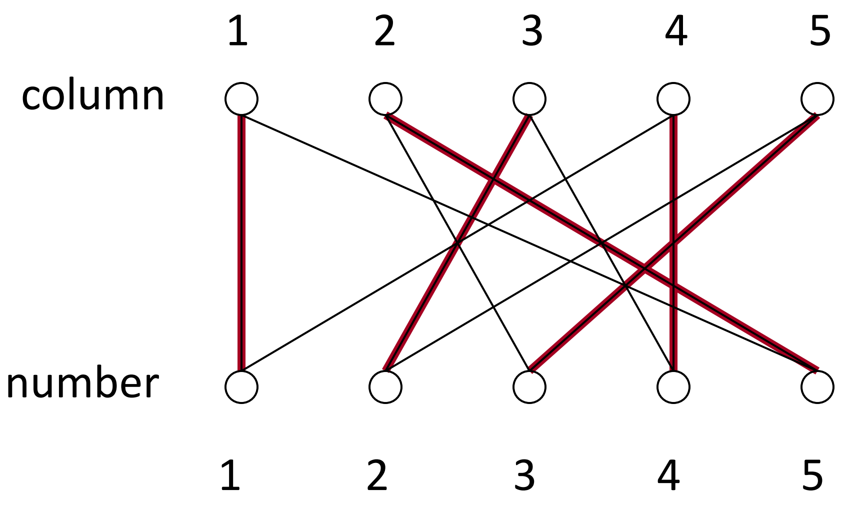
Add one vertex for each column, and one vertex for each number.

Add an edge between column i and color j if color j can be put in column i.

Application of Bipartite Matching

Given a partial Latin square, we construct a bipartite graph to fill in the next row.

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4
1	5	2	4	3



A perfect matching corresponds to a valid assignment of the next row.

If we can always complete the next row, then by induction we are done.

Hall's Theorem can be used to prove that the bipartite graph at each step has a perfect matching.

Proof of Hall's Theorem

A bipartite graph $G(A, B, E)$ has a perfect matching

if and only if $|\mathcal{N}(S)| \geq |S|$ for every subset $S \subseteq A$.

The matching condition does not hold.

There is a subset $S \subseteq A$, such that $|S| > \mathcal{N}(S)|$

Clearly, no perfect matching exists.

The matching condition does hold.

Proof by induction on the size of A.

The Matching Condition Holds

Proof by induction on n, the number of elements in A.

Basis: $n = 1$.

Trivial, the single edge in the graph is the matching.

I.H.: $P(k)$: if the matching condition holds for any bipartite graph with $|A| = |B| \leq k$
then the graph has a perfect matching.

I.S.: Case 1: $|\mathcal{N}(S)| > |S|$ for every subset $S \subseteq A$.

Pick any edge $(a, b) \in E$ and match vertex $a \in A, b \in B$.
Remove a, b and associated edges from the graph.
The remaining subgraph on k vertices satisfies the matching condition.
The I.H. applies, and the graph has a perfect matching
(the removed edge together with the inductive matching for k)

The Matching Condition Holds

I.S.: Case 2: $|\mathcal{N}(S)| \geq |S|$ for every subset $S \subseteq A$, and $|\mathcal{N}(T)| = |T|$ for some subset $T \subseteq A$.

Separate the graph into two parts: $T, \mathcal{N}(T)$ and $A - T, B - \mathcal{N}(T)$

Graph G_2 on at most k vertices, satisfies the matching condition.

Graph G_1 also on at most k vertices, satisfies the matching condition (pf by contradiction).

Voila!