Getting beyond propositions

John is taller than Dan

<u>Dan is taller than Bill</u>

∴ John is taller than Bill

Is this argument valid or invalid?

We need to specify the logical meaning of “is taller than”

Propositions only deal with nouns

We need to capture the relationship between pairs of nouns in this example.

Predicates

Dictionary definition: *" ...the part of a sentence or clause containing a verb and stating something about the subject ..."*

$isTaller(x, y):\ \ "x\ is\ taller\ than\ y"$

$isTaller\ is\ a\ predicate, not\ a\ proposition$

$x, y:\ \ arguments, variables\ of\ \ the\ predicate$ (over set of people, in this case)

The arguments/variables are elements of a domain (set) that is specified.

The value of the predicate is a boolean value (True/False).

Predicates

John is taller than Dan $isTaller(John, Dan)$

Dan is taller than Bill $isTaller(Dan, Bill)$

Each of these statements is a proposition.

But we still cannot conclude that John is taller than Bill.

So far, we've only defined the syntax of the predicate.

Next, we must define its property.

Predicates

$isTaller(John, Dan) \land isTaller(Dan, Bill)) \Rightarrow isTaller(John, Bill)$
$isTaller(John, Dan)$
$isTaller(Dan, Bill)$

With this we can conclude $isTaller(John, Bill)$

What about: John is taller than David

<u>David is taller than Bill</u>

∴ John is taller than Bill

We need another rule like the first one but involving David.

Will we have to create a separate rule for every set of three persons?

Quantification

For any three persons x, y, z $(isTaller(x,y) \wedge isTaller(y,z)) \Rightarrow isTaller(x,z)$

Formally we must define the domain of discourse (set of all persons, buildings, …)

Often the domain is implicit but understood from the context.

$$\forall x \forall y \forall z: \ (isTaller(x,y) \wedge isTaller(y,z)) \Rightarrow isTaller(x,z)$$

$\forall x$: "for all x" the universal quantifier

The quantifier *binds* each variable to the domain of discourse. A quantified statement in which every variable is bound is a *quantified proposition*.

A variable that is not bound to a quantifier is a *free variable*.

A statement with one or more free variables is NOT a quantified proposition.

Predicates over numbers

$1 + 2 = 3$

$1 + 2 + 3 = 6$

$1 + 2 + 3 + 4 \; = \; 10$

$P(n)$: $\quad 1 + \cdots + n = \frac{n(n+1)}{2}$ is a predicate (not a proposition)

$P(1), P(2), P(3)$ are all propositions that are true.

The quantified proposition $\forall n \in \mathbb{N}: P(n)$ is true!

Finite Domains

Let $People = \{John, Bill, Dan\}$

$$\forall x \in People: isTaller(John, x)$$

is equivalent to

$$isTaller(John, Bill) \land isTaller(John, Dan) \land isTaller(John, John)$$

Finite Domains

Let $People = \{John, Bill, Dan\}$

$$\exists x \in People\colon isTaller(John, x)$$

is equivalent to

$$isTaller(John, Bill) \vee isTaller(John, Dan) \vee \ isTaller(John, John)$$

$$\forall\ and\ \exists\ are\ necessary\ when\ the\ domain\ is\ infinite.$$

Quantification

John is taller than everyone else.

$$\forall x: (x \neq John) \Rightarrow isTaller(John, x)$$

Everyone is taller than Bill.

$$\forall x: (x \neq Bill) \Rightarrow isTaller(x, Bill)$$

No one is taller than John.

$$\forall x: \neg isTaller(x, John)$$

John is taller than someone else. $???$

There exists a person x $: isTaller(John, x) \wedge (x \neq John)$

$$\exists x: isTaller(John, x) \wedge (x \neq John)$$

$\exists$ "$there\ exists$" is the *existential quantifier*

Examples

H(x) = “x is a horse”

A(y) = “y is an animal”

Every horse is an animal.

$$\forall x\colon\ H(x)\ \Rightarrow\ A(x)$$

Some animals are horses.

$$\exists x\colon\ H(x)\ \land\ A(x)$$

Nested Quantifiers

$L(x, y)$ = "x loves y"

$$L(Majnu, Layla) = True$$

Everyone loves someone.

$$\forall x \, \exists y \; : \; L(x, y)$$

Someone loves everyone.

$$\exists x \, \forall y : \; L(x, y)$$

Someone is loved by everyone.

$$\exists x \, \forall y : \; L(y, x)$$

No one loves the Grinch

$$\forall x \, \neg L(x, Grinch)$$

Nested quantifiers

No one loves everyone.

not(someone loves everyone)

$\neg(\exists x\ \forall y\colon\ L(x, y))$

Everyone does not love everyone.

$\forall x\ \neg(\forall y\colon\ L(x, y))$

For every person there is someone that person does not love.

$\forall x\ \exists y\colon\ \neg L(x, y)$

These statements are all equivalent!

Pushing negation through a layer changes the quantifier!

De Morgan's Law for nested quantifiers

Examples

Everybody who lives in the dorm has a roommate who is not their friend.

$$D(x) \quad : \quad x\ lives\ in\ the\ dorm$$
$$F(x,y): \quad x\ and\ y\ are\ friends$$
$$R(x,y): \quad x\ and\ y\ are\ roommates$$

Step 1: $x\ has\ a\ roommate\ who\ is\ not\ a\ friend$

$$\exists y: R(x,y) \wedge \neg F(x,y)$$

Step 2: $The\ above\ is\ true\ for\ all\ x\ who\ lives\ in\ the\ dorm$

$$\forall x:\ D(x) \rightarrow (\exists y: R(x,y) \wedge \neg F(x,y))$$

Examples

If anyone who lives in the dorm has a friend who has Covid then everyone in the dorm is quarantined.

$D(x): x \text{ lives in the dorm}$ $\quad$ $F(x,y): \quad x \text{ and } y \text{ are friends}$

$Q(x): x \text{ is quarantined}$ $\quad$ $C(x) \quad : \quad x \text{ is infected with Covid}$

Someone who lives in the dorm has a friend who has Covid

$$\exists x: [D(x) \wedge \exists y \left(F(x,y) \wedge C(y)\right)]$$

Everyone in the dorm is quarantined

$$\forall x: (D(x) \rightarrow Q(x))$$

$$\exists x: \left[D(x) \wedge \exists y \left(F(x,y) \wedge C(y)\right)\right] \rightarrow \ \forall x: \left(D(x) \rightarrow Q(x)\right)$$

The x in the antecedent is bound to the $\exists$ but in the consequent it is bound to $\forall$.

$$\exists x: \left[D(x) \wedge \exists y \left(F(x,y) \wedge C(y)\right)\right] \rightarrow \ \forall z: \left(D(z) \rightarrow Q(z)\right)$$

Each variable is bound to its nearest enclosing quantifier.

More Examples

$$L(x, y):\ \ x\ loves\ y$$

Domain: Set of all people

At least one person loves Layla.

$$\exists x: L(x, Layla)$$

At most one person loves Layla.

$$\exists x: Loves(x, Layla) \rightarrow \forall y: \left(y \neq x \rightarrow\ \neg Loves(y, Layla)\right)$$

Exactly one person loves Layla.

$$\exists x: Loves(x, Layla) \wedge\ \forall y: \left(y \neq x \rightarrow\ \neg Loves(y, Layla)\right)$$

Exactly two people love Layla.

$$\exists\ x, y: (x\ \neq\ y) \wedge Loves(x, Layla) \wedge Loves(y, Layla)$$

$$\wedge \forall z: \left((z \neq x \wedge z \neq y) \rightarrow \neg Loves(z, Layla)\right)$$