Administrivia

Email me (and/or inform your lab CA) if you are interested in being a CA for CS 135 in Fall 2023 and/or Spring 2024.

- Fall class is small, so require at most 2-3 CAs.
- Spring ‘24 classes will require at least 12 CAs.

Course Evaluations begin today – take a few minutes to respond!

Trees

A connected, acyclic graph is called a tree.

1. Every connected subgraph of a tree T is also a tree.

   If the subgraph has a cycle, then T must have a cycle. But T is acyclic!

2. There is a unique path between every pair of vertices.

   There must be one because the tree is a connected graph.

   Why can't there be two different paths between a pair of vertices?

More simple observations

3. Adding an edge between any two nonadjacent vertices in a tree creates a cycle.

The new edge and the unique path connecting the vertices in the tree creates a cycle.

4. Removing any tree edge disconnects some pair of vertices.

The edge that is removed was the unique path between the two end points. Removing it disconnects the end points.

Still more simple observations

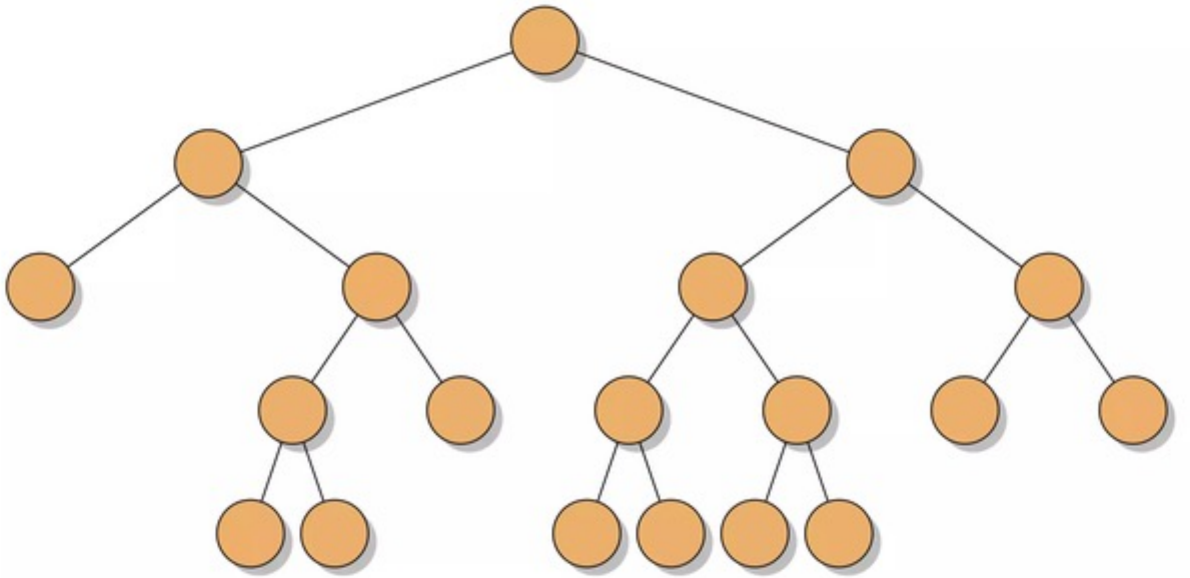
5. Every tree with at least two vertices contains at least two leaves.

   The end points of a longest path in the tree are both leaves!

6. Every tree with n vertices has n-1 edges.

   Proof by induction on number of vertices.

Full binary trees



Every vertex is either a leaf or has exactly 2 children.

Theorem:
If n = #leaves then # non-leaves = n-1

Lemma: Some two siblings are both leaves.

Prove theorem using lemma and induct on n.

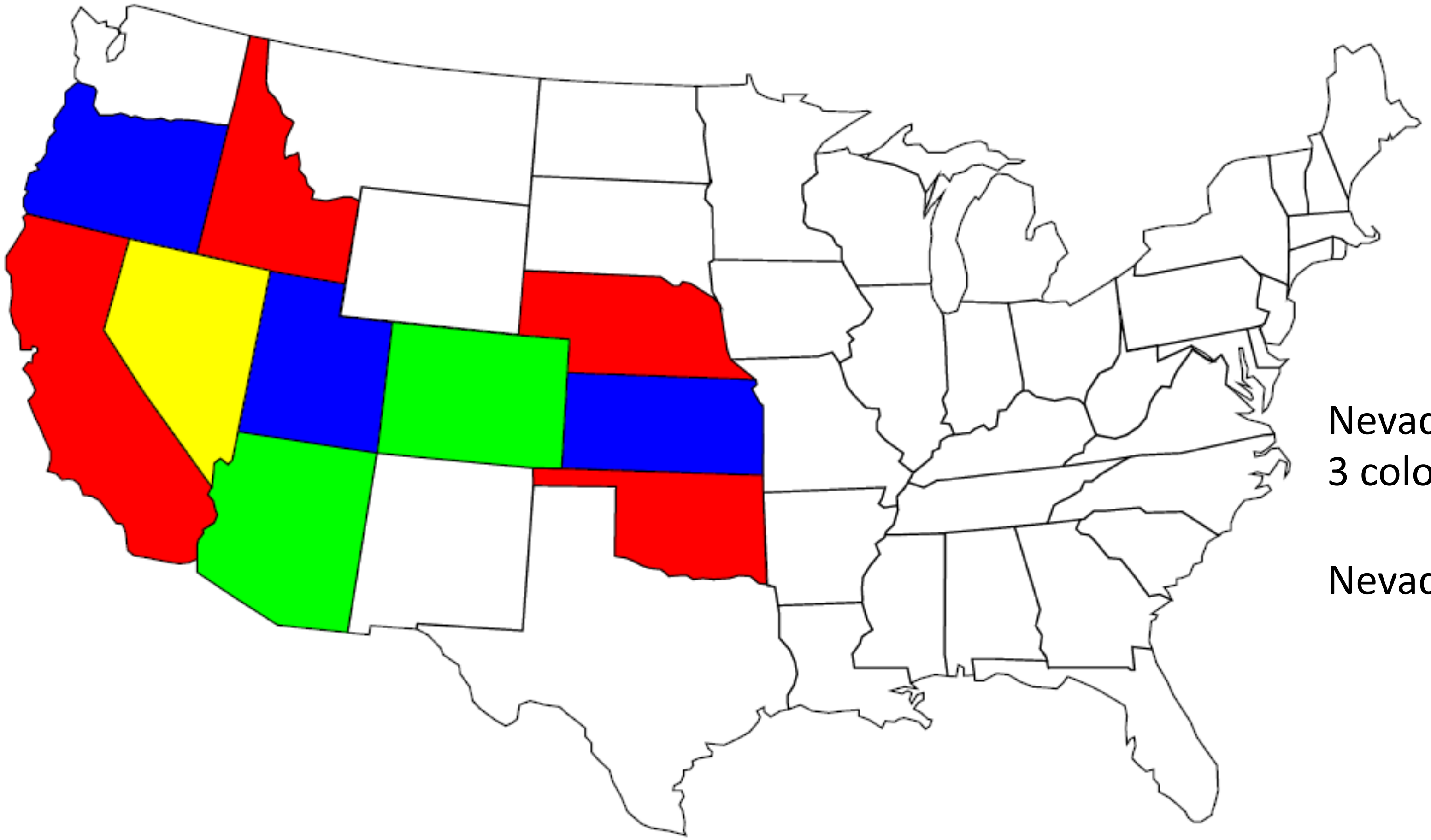
Every tournament tree is a full binary tree.
So, a tournament with 73 players will take …….. games!

Map Coloring

Any two states that share a border must be colored differently.

How many colors suffice?

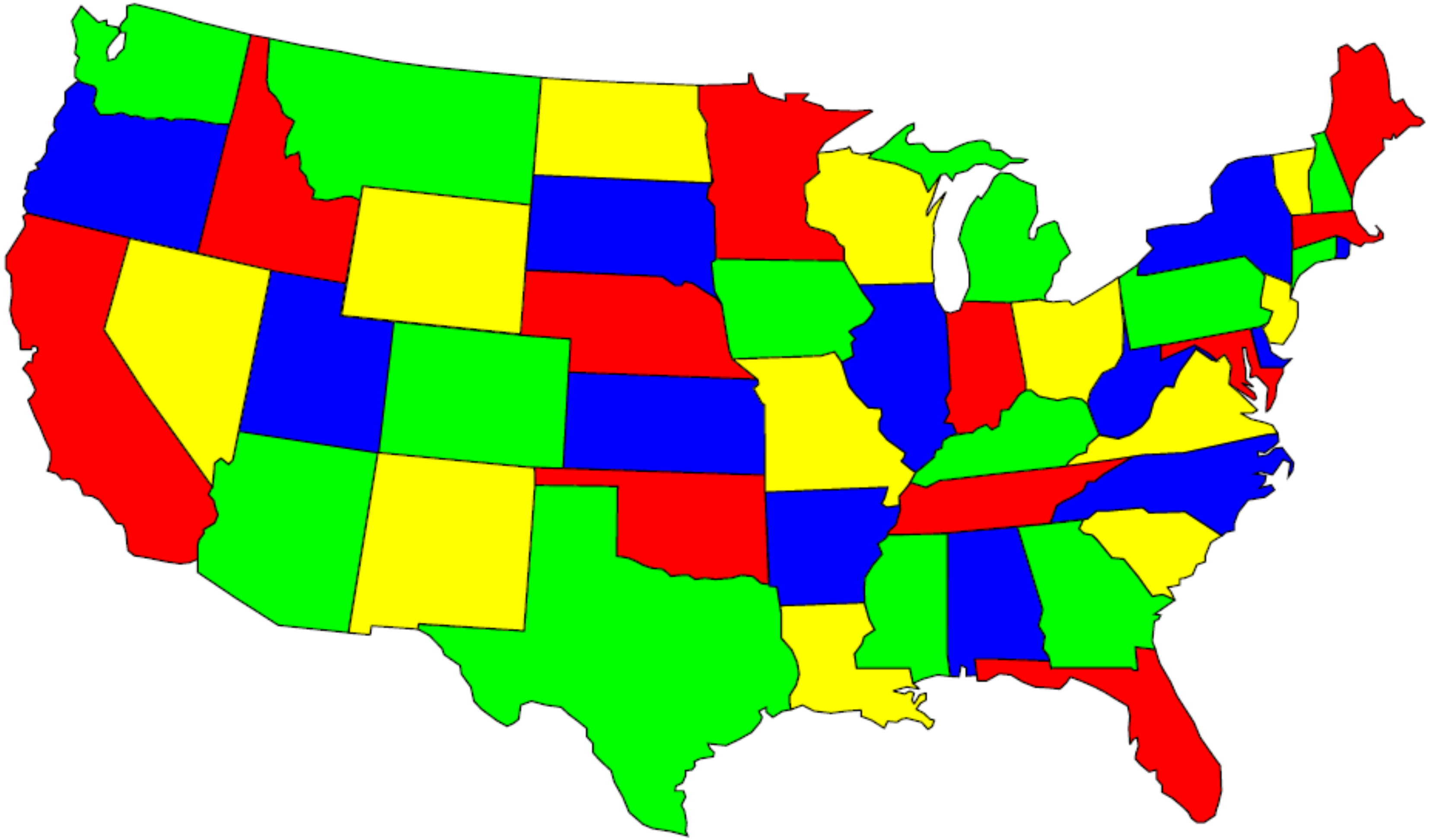
4 colors are necessary



Nevada has 5 neighbors that require 3 colors.

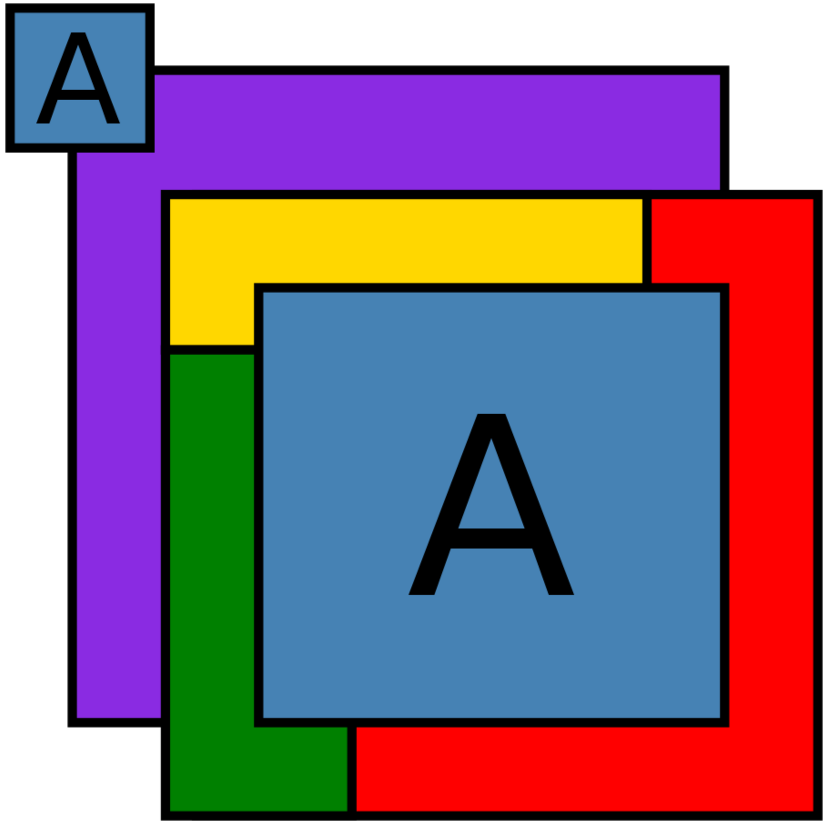
Nevada needs a fourth color.

4 colors are sufficient

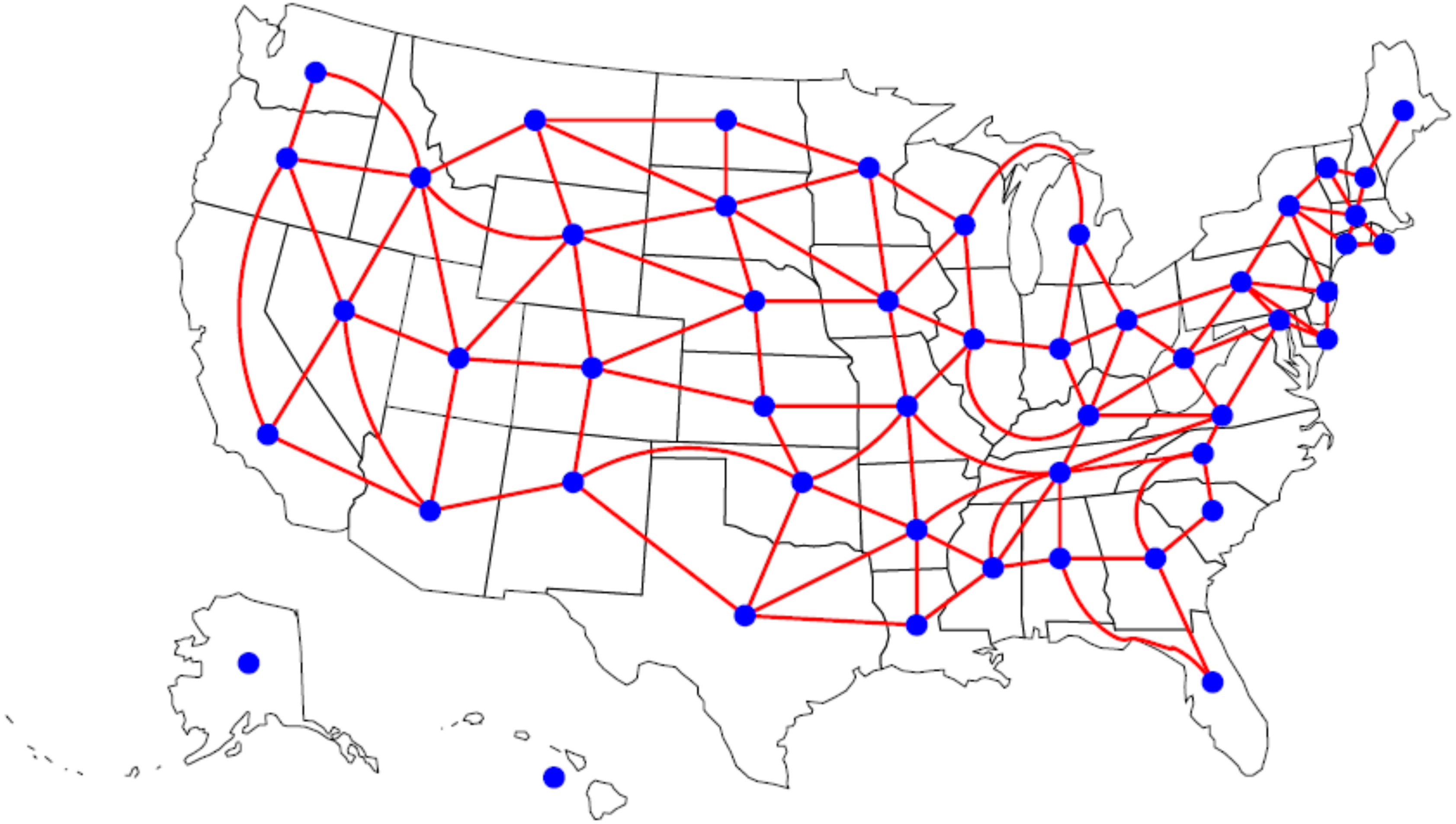


Non-Contiguous States

This map requires 5 colors!



Planar Graph Representation



Vertex for each state

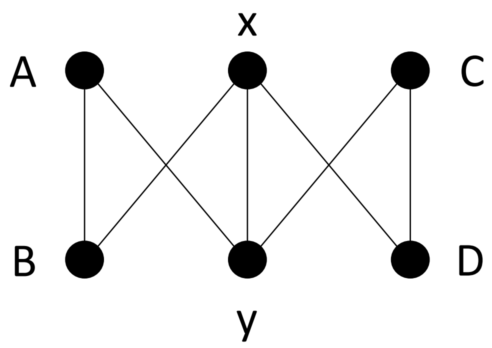
Edge between states that share a boundary

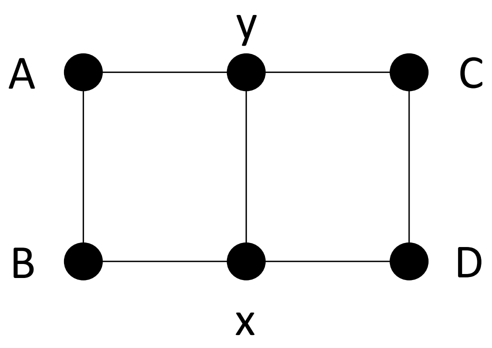
The Four-Color Theorem

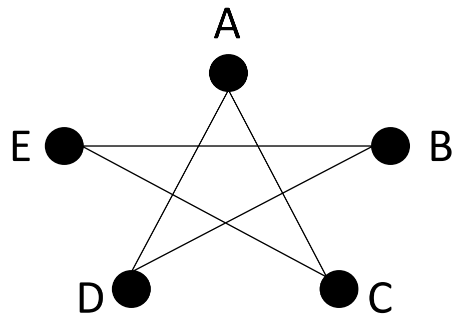
- Conjectured in 1852
- Many mathematicians thought they had a proof, only to find a fatal flaw
- Finally proved in 1976. The proof required examining numerous cases by a computer, sparking debate on what a proof really is.
- Recognizing planar graphs that can be colored using 3 colors:
  - no efficient algorithm known
  - Harder than factoring!

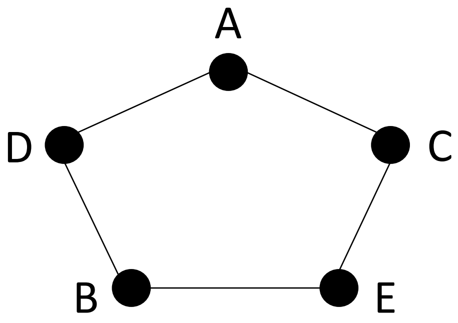
Planar graphs

A graph is planar if it can be drawn on the plane without crossing edges.
We will focus exclusively on connected planar graphs.



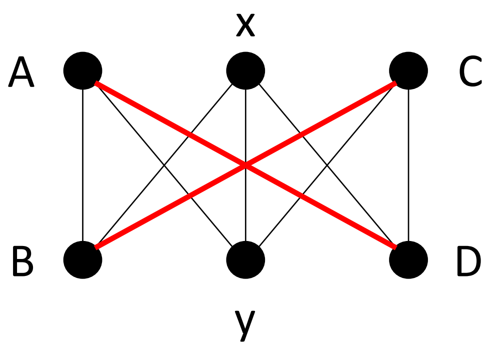


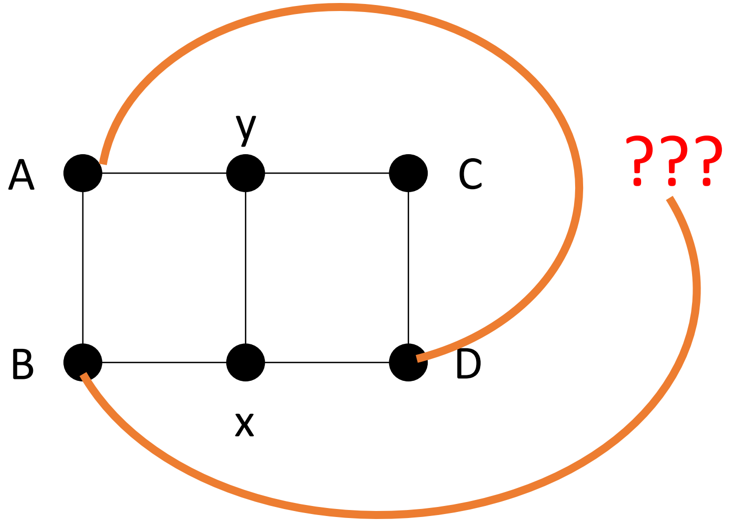




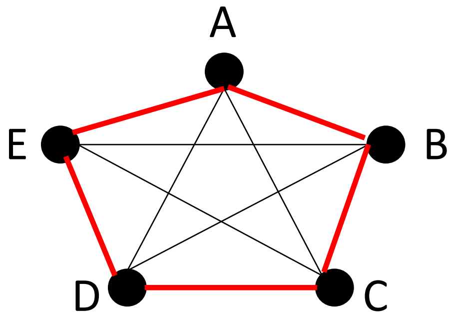
Planar graphs?

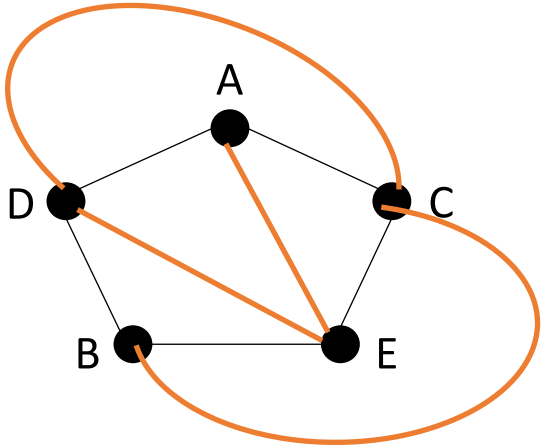
$K_{3,3}$





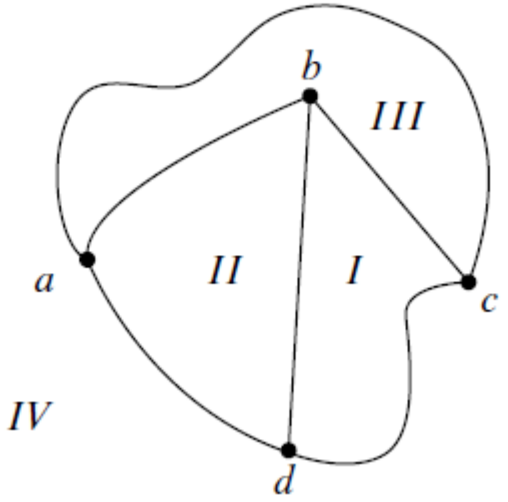
K_5

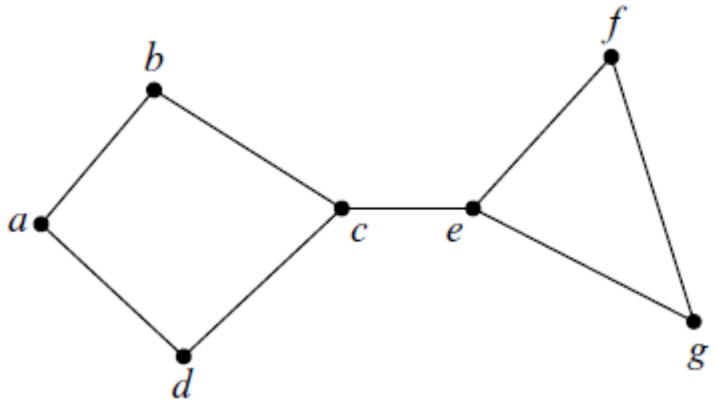




Connect A to B ???

Planar drawings





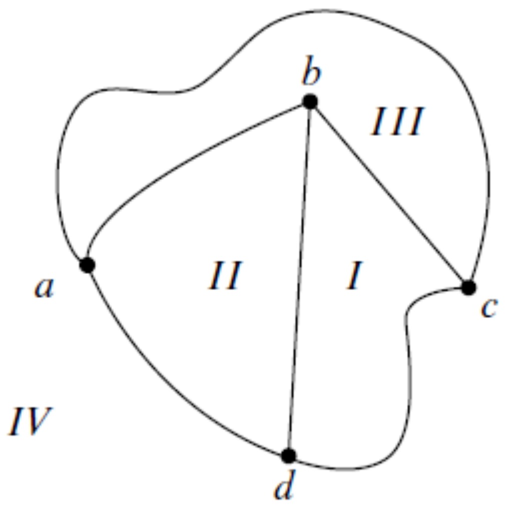
The areas labeled I, II, III, IV are called *regions* or *faces.*

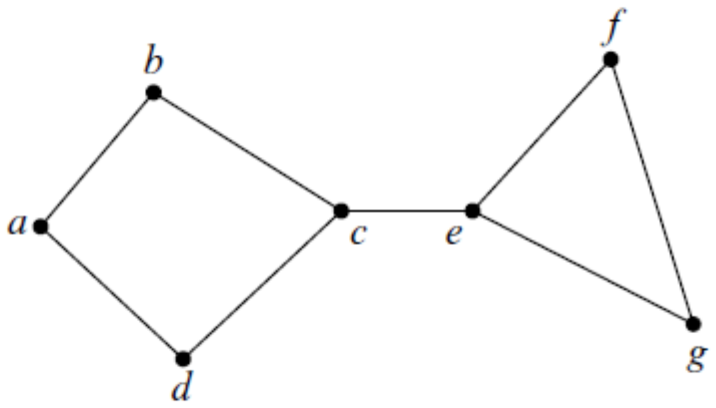
Not every edge divides a region.

Each region is enclosed within edges of the graph.

What is the relationship between the numbers of vertices, edges and regions?

The boundary of a region





We define the boundary of a region as a closed walk in clockwise order of all edges that lie within the region.

This is well-defined when the graph is connected.

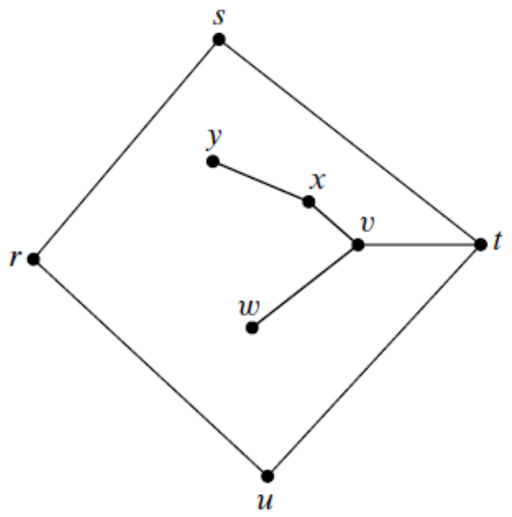
Region I : $\{c,d\},\{d,b\},\{b,c\}$
Region II : $\{d,a\},\{a,b\},\{b,d\}$
Region III : $\{a,c\},\{c,b\},\{b,a\}$
Region IV : $\{c,d\},\{d,a\},\{a,c\}$

Boundary of outer region :
$\{f,g\},\{g,e\},\{e,c\},\{c,d\},\{d,a\},\{a,b\},\{b,c\},\{c,e\},\{e,f\}$

Note that the edge $\{e,c\}$ occurs twice on the boundary of the outer region.

Each edge lies once on the boundary of 2 regions, or twice on the boundary of one region.
Each region has 3 or more bounding edges

What about dongles?



Boundary of outer region: $\{t, u\}, \{u, r\}, \{r, s\}, \{s, t\}$

Boundary of inner region: $\{t, u\}\{u, r\}, \{r, s\}, \{s, t\}, \{t, v\}, \{v, x\},$
$\{x, y\}, \{y, x\}, \{x, v\}, \{v, w\}, \{w, v\}, \{v, t\}$

Each edge lies once on the boundary of 2 regions, or twice on the boundary of one region.
Therefore, X = Sum of the number of edges of every region boundary = $2m$

Also, if the number of vertices is at least 3, and since each region has 3 or more bounding edges, $X \geq 3r$.

Therefore, $2m \geq 3r$ for every connected planar graph with at least 3 vertices.

In general, if every cycle has length c or greater, then $2m \geq cr$.

Euler's Formula

Theorem: For every connected planar graph with n vertices, m edges, and r regions: $n - m + r = 2$

Corollary: The number of regions in all drawings of a planar graph is invariant.

Proof: Induction on the structure of the graph G.

Idea: Start with a single node, and form a sequence of connected subgraphs $G_0 G_1 \ldots G_m$

such that G_0 is a single node,

G_i is formed by adding one edge to G_{i-1},

$G_m = G$

and at each step G_i satisfies the formula.

Base Case: $n = 1, e = 0, r = 1.\quad 1 - 0 + 1 = 2$

Inductive Hypothesis: The formula is true for connected subgraph G_k

Inductive Step: Insert an edge incident to at least one vertex in G_k.

Case 1: Only one end point is in G_k, so the edge is a dongle.

This adds one new vertex, one new edge, but the number of regions stays the same.

So the value of the LHS remains 2.

Case 2: Both end-points are in G_k. This creates a new region but the number of nodes stays the same.

So the value of the LHS remains 2.

Planar graphs have few edges

Theorem. For every connected planar graph G with $n \geq 3$ vertices: $m \leq 3n - 6$.

Proof: We showed that $2m \geq 3r$

From Euler's Theorem, $n - m + r = 2 \;\; \Rightarrow \;\; r = 2 + m - n$

Therefore, $2m \geq 3(2 + m - n)$

which implies $2m \geq 6 + 3m - 3n$

which yields $m \leq 3n - 6$

Corollary 1: K_5 is not planar.
Proof: $n = 5, m = 10.$
But $10 > 3 \cdot 5 - 6 = 9$, violating Euler's formula so the graph is not planar.

Corollary 2: $K_{3,3}$ is not planar.
Proof: $K_{3,3}$ has only even length cycles, so if it were planar $2m \geq 4r$, implying $2m \geq 4(2 + m - n)$,
But $2m = 18$, $4(2 + m - n) = 4(2 + 9 - 6) = 20!$

The Five-Color Theorem

Theorem: Every planar graph can be colored with 5 or fewer colors.

Proof: By induction on the number n of vertices.

Base Case: $n \leq 5$ Use a different color for each vertex.

Inductive Hypothesis: Every planar graph with k or fewer vertices has a 5-coloring.

Inductive Step: Let G be a graph with $k + 1$ vertices.

a. If there is a graph with degree 4 or less, remove it
   1. By the inductive hypothesis, the remaining planar graph is 5-colorable
   2. Reinsert the vertex removed in Step 1 and use a color different from its (at most) 4 neighbors.

b. Every vertex has degree at least 5.

But First …

Claim: Every connected planar graph has a vertex with degree 5 or less.

Proof: (By contradiction)

Suppose each of the n vertices has degree 6 or more.

Number of edges $m = \frac{1}{2}\sum_v degree(v)$

$$\geq \frac{1}{2}6n$$

$$= 3n > 3n - 6$$

This contradicts:

Theorem. For every connected planar graph G with $n \geq 3$ vertices: $m \leq 3n - 6$.

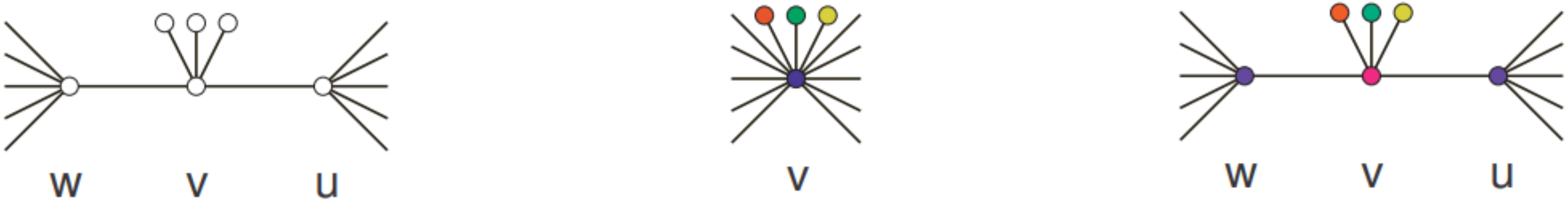
The Five-Color Theorem

b. Every vertex has degree at least 5.

Pick a vertex v of degree 5.

At least one pair of its neighbors u, w don’t have an edge between them. Why?

Merge u, v, w into one vertex. The graph remains planar; color it recursively.



Separate u, v, w and color u, w with one color!