# Administrivia

Mid-Term Exam: Thursday, March 9

In class, 8-1/2x11 cheat sheet, no electronic device

Practice problems will be posted on Canvas soon

Review: definitions, reasoning from basic concepts

practice problems in Rosen's textbook

Students with accommodations: I'll send an email today, please respond

# Honor Code

Solutions you submit must reflect your own work

- Acknowledge any help from a source other than me or the TAs
- Do NOT post problems on websites
- NEVER trust solutions that appear on external websites

We're here to help you succeed

- Of course, we'll give you enough hints to solve any problem
- But you MUST write the solution in your own words

Curved grading $\longrightarrow$ cheating is unfair to those who do not cheat!

# Lem's Bogus "Proof"

$$\begin{array}{c} x \vee \neg y \\ \neg x \vee y \\ \hline y \vee \neg y \end{array}$$

| | | |
|---|---|---|
| | $P(A) = P(B) \leftrightarrow A = B$ | Given |
| $\equiv$ | $((P(A) = P(B)) \rightarrow (A = B)) \wedge ((A = B) \rightarrow (P(A) = P(B)))$ | Conditional identity |
| $\equiv$ | $(\neg(P(A) = P(B)) \vee (A = B)) \wedge ((A = B) \rightarrow (P(A) = P(B)))$ | Conditional identity |
| $\equiv$ | $(\neg(P(A) = P(B)) \vee (A = B)) \wedge (\neg(A = B) \vee (P(A) = P(B)))$ | Conditional identity |
| $\equiv$ | $((A = B) \vee \neg(P(A) = P(B))) \wedge (\neg(A = B) \vee (P(A) = P(B)))$ | Commutative law |
| $\Rightarrow$ | $(\neg(P(A) = P(B)) \vee (P(A) = P(B))$ | Resolution rule |
| $\equiv$ | $(P(A) = P(B)) \vee \neg(P(A) = P(B))$ | Commutative law |
| $\equiv$ | T | Complement law |

$$(P(A) = P(B) \leftrightarrow A = B) \longrightarrow T$$

# The Principle of Induction (PI)

$$\frac{\begin{array}{c} P(0) \\ \forall k \geq 0:\ \ P(k) \Rightarrow P(k+1) \end{array}}{\therefore\ \forall n \in \mathbb{N}:\ \ P(n)}$$

To prove that $P(x)$ is TRUE for all natural numbers:

Step 1. **(BASIS)** Show that $P(0)$ is TRUE

Step 2. **(INDUCTIVE HYPOTHESIS)** State $P(k)$

Step 3. **(INDUCTIVE STEP)** Show that $P(k) \Rightarrow P(k+1)$ is TRUE for all $k$

# Exercise 1

Prove that $\forall n \geq 0 \;\; n^3 - n \;\, is\ divisible\ by\ 3$.

Proof by Induction:

**Basis:** $\boldsymbol{n = 0}$. $n^3 - n = 0^3 - 0 = 0$

0 is divisible by 3.

**Inductive Hypothesis:** $\boldsymbol{k^3 - k}$ **is divisible by 3,** $\boldsymbol{k \geq 0}$.

**Inductive Step:** $\boldsymbol{(k+1)^3 - (k+1)}$

$$= (k^3 + 3k^2 + 3k + 1) - (k + 1)$$
$$= (k^3 - k) + 3(k^2 + k) + (1 - 1)$$
$$= (k^3 - k) + 3(k^2 + k)$$

The first term is divisible by 3 (inductive hypothesis) and the second term is a multiple of 3.

The sum of two numbers, each divisible by 3, is divisible by 3.

This establishes the inductive step, and the claim follows from PI.

# All horses are the same color!

Theorem. For every set of *n* horses, all horses are the same color.

Proof. (By induction.)

Base case: $k = 1$. $P(1)$ is true - there's only one horse!

Inductive hypothesis: $P(k)$: For every set of *k* horses, all horses are the same color.

Inductive step: $P(k) \rightarrow P(k+1)$ (and the theorem will follow by PI).

We have horses $\{H_1, H_2, \ldots H_k, H_{k+1}\}$

Divide these into two subsets of size $k$: $A = \{H_1, H_2, \ldots H_k\}$ and $B = \{H_2, \ldots H_k, H_{k+1}\}$

From the inductive hypothesis $P(k)$ it follows that all horses in $A$ are the same color and that all horses in $B$ are the same color.

But $H_1, H_2$ are the same color and also $H_2, H_{k+1}$ are the same color, so $H_1, H_{k+1}$ are the same color. Therefore all horses are the same color.

# All horses are the same color!

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Base case: $k = 1$. $P(1)$ is true - there's only one horse!

Inductive hypothesis: $P(k)$: For every set of *k* horses, all horses are the same color.

Inductive step: $P(k) \rightarrow P(k+1)$ (and the theorem will follow by PI).

We have horses $\{H_1, H_2, \ldots H_k, H_{k+1}\}$

Divide these into two subsets, each of size $k$: $A = \{H_1, H_2, \ldots H_k\}$ and $B = \{H_2, \ldots H_k, H_{k+1}\}$

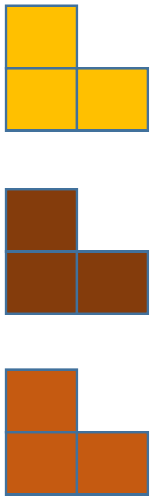
Does P(1) imply P(2)? We start with $\{H_1, H_2\}$, $A = \{H_1\}, B = \{H_2\}$

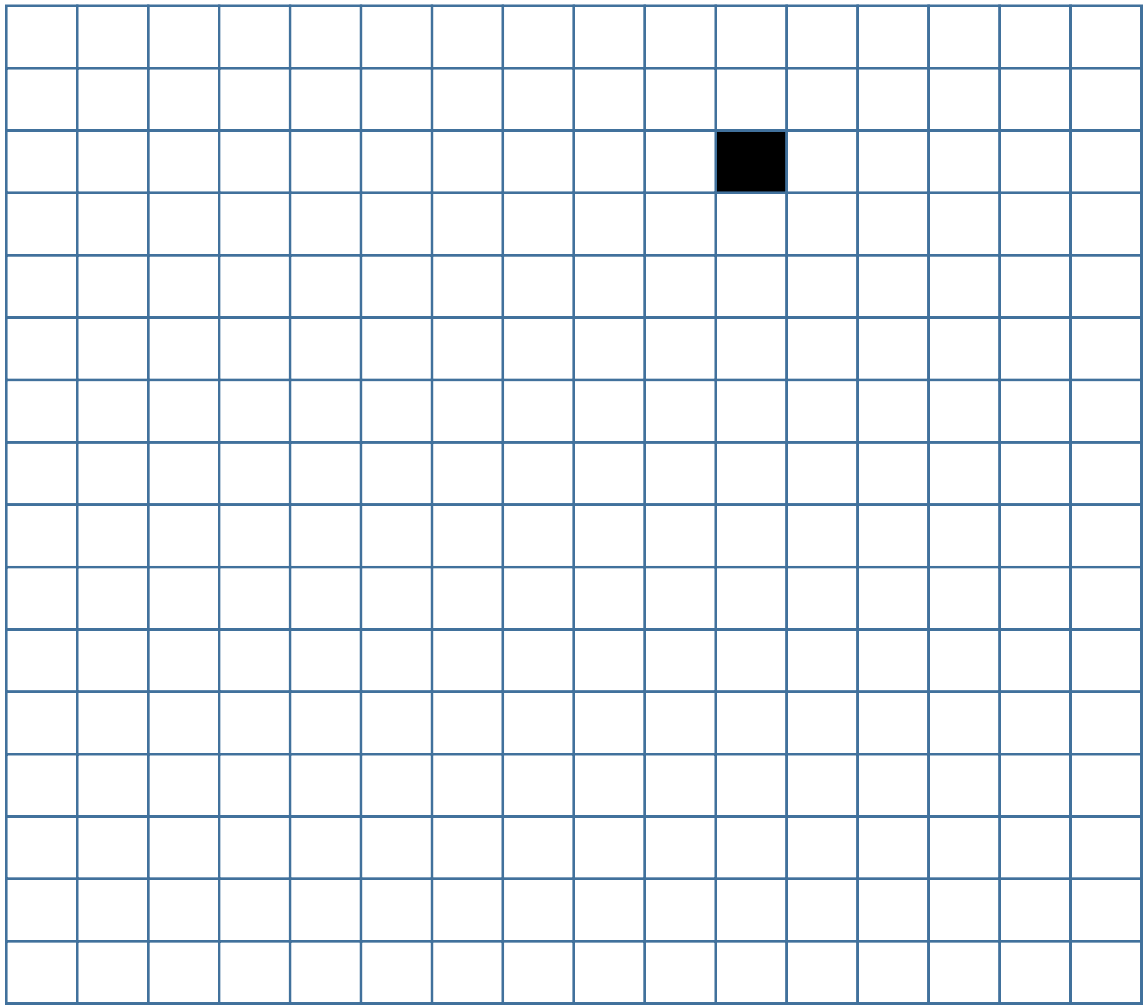
From the inductive hypothesis $P(k)$ it follows that all horses in $A$ are the same color and that all horses in $B$ are the same color. Sure, I'll buy that.

But $H_1, H_2$ are the same color and also $H_2, H_{k+1}$ are the same color, so $H_1, H_{k+1}$ are the same color. Therefore, all horses are the same color.

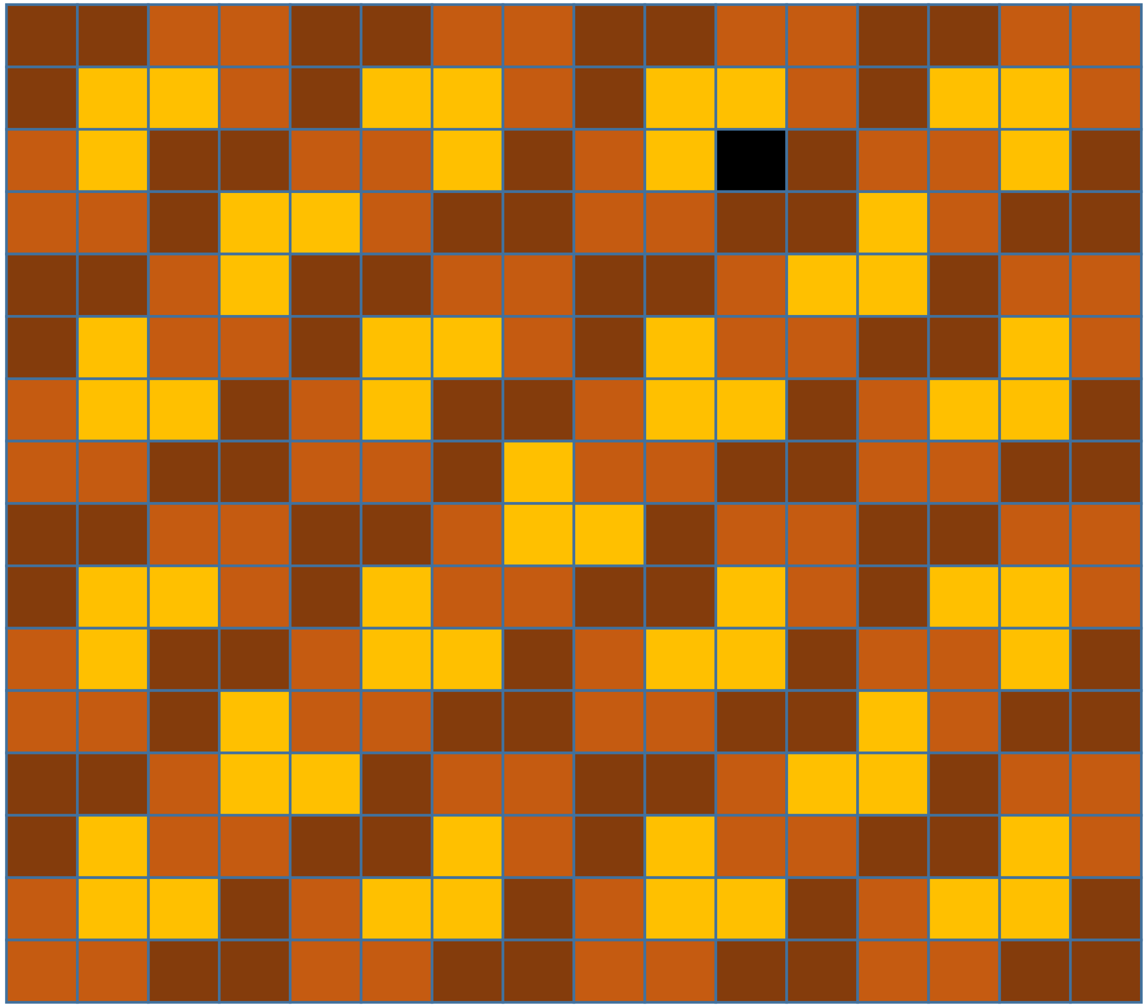
WHOA! Hold your horses, not so fast! No common horse in $A = \{H_1\}, B = \{H_2\}$!!!

# Induction: Tiling a Courtyard





Tile the 16 x 16 courtyard using multiple tiles of the given shape.



# Tiling the Courtyard

Define the predicate $P(n)$: Every $2^n \times 2^n$ courtyard with one hole can be tiled using triominoes.

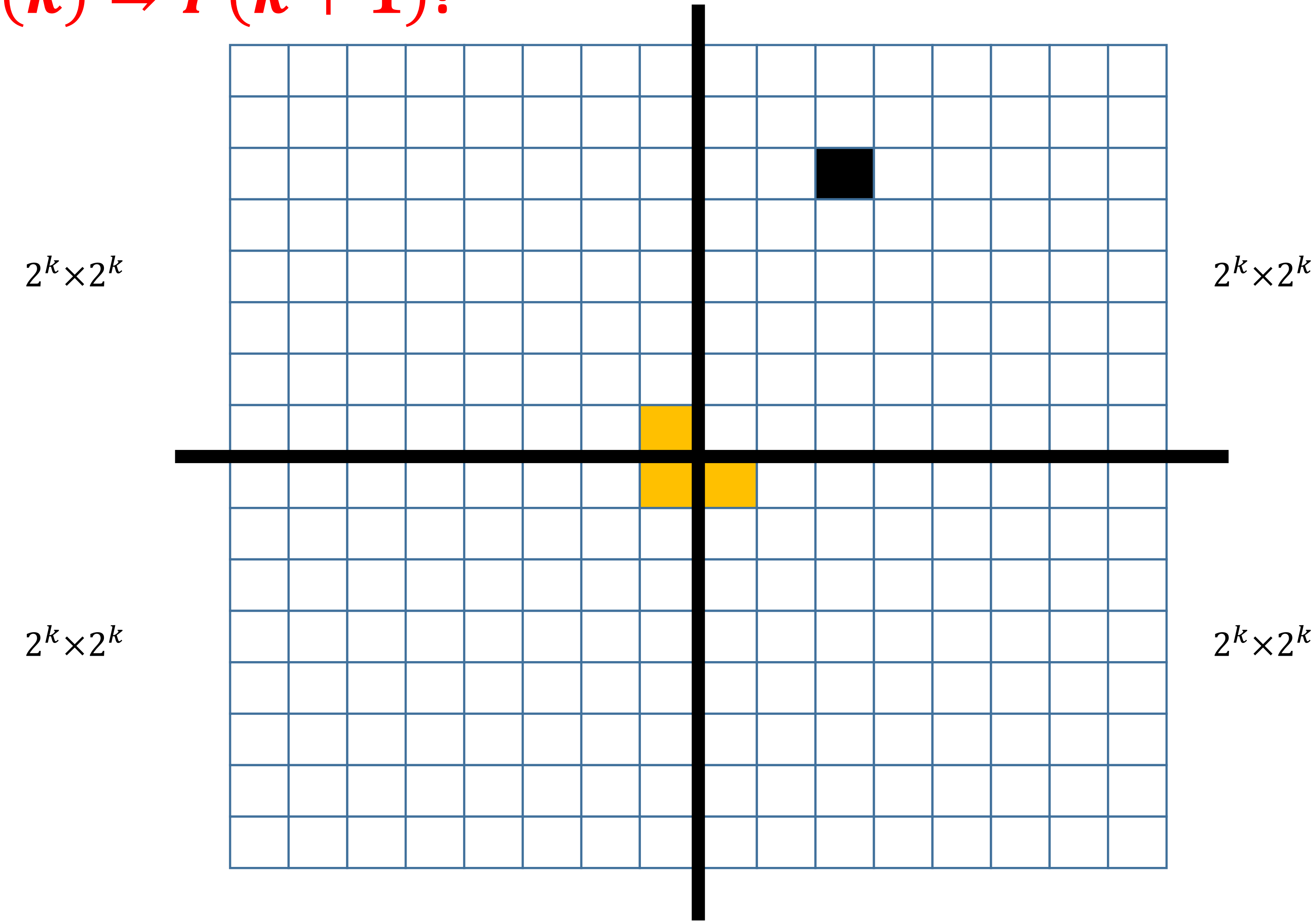
Theorem: $\forall n \in \mathbb{N}\ \ P(n)$.

**Basis:** $P(0)$: $1 \times 1$ courtyard with a hole (!). Vacuously TRUE.

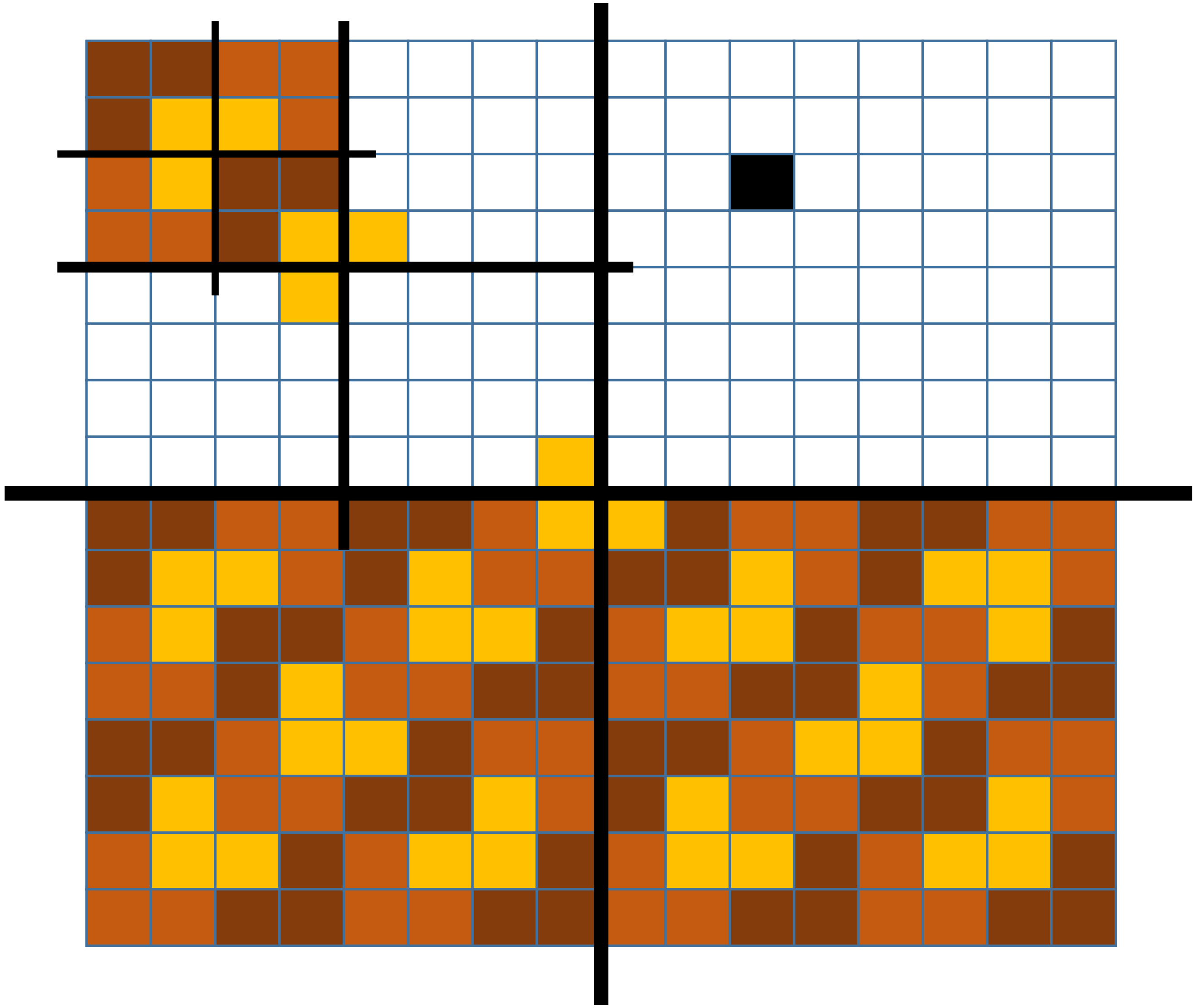
**Inductive hypothesis:** $P(k)$: Claim is true , $k \geq 0$.

**Inductive step:** Show that $P(k) \Rightarrow P(k+1)$.

$$P(k) \Rightarrow P(k+1)!$$



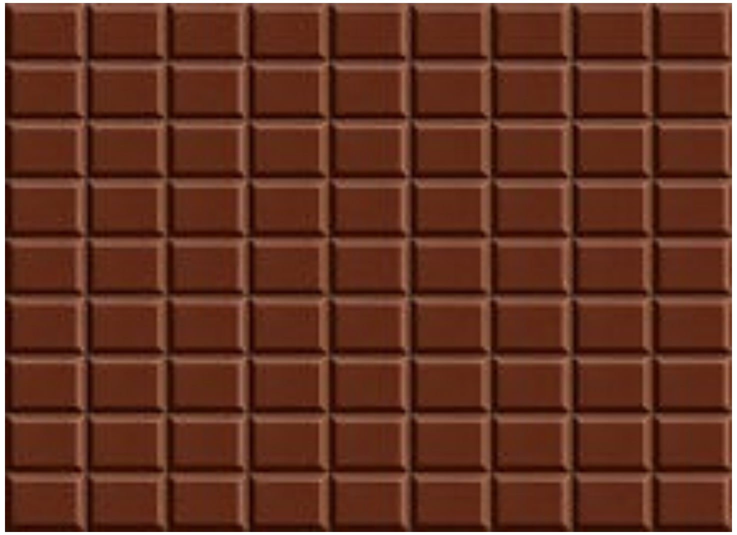
# Recursively tiling the courtyard



# The Chocolate Bar Problem

How many cuts are needed to split a rectangular chocolate bar into individual pieces?

Can only cut one piece at a time.



Make 8 vertical cuts, then 8 cuts in each of 9 smaller rectangles. #cuts = 8 + (8 x 9) = 80

Can we do it with fewer cuts?

# The Chocolate Bar Problem

$\forall n \geq 1$: $n - 1$ cuts are necessary to split a rectangular bar into $n$ individual pieces.

Proof: (by induction on the number $n$ of pieces)

Basis: $n = 1$. #cuts needed = $0 = 1 - 1$.

Inductive Hypothesis: $P(k)$: $k - 1$ cuts needed to split every $k$ piece bar.

Inductive Step: Start with a $k + 1$ piece bar.

The first cut may not result in a $k$-piece bar! 😟

So how do we make an inductive argument? 🤔

# Another Look at Dominoes

The first domino falls.
For any $k$, if the $k$th domino falls, then so does the $k+1$st.
Therefore, every domino falls.

$$\frac{P(0) \ \wedge \ \forall k \geq 0{:}\,(P(k) \rightarrow P(k+1))}{\forall n \in \boldsymbol{N}{:}\, P(n)}$$

The first domino falls.
For any $k$, if the first $k$ dominoes all fall, then so does the $k+1$st.
Therefore, every domino falls.

$$\frac{P(0) \ \wedge \ \forall k \geq 0{:}\,\big(\forall i \leq k{:}\ P(i)\big) \rightarrow P(k+1))}{\forall n \in \boldsymbol{N}{:}\, P(n)}$$

This is called proof by strong induction. The inductive hypothesis is stronger.

Every proof by strong induction can be recast as a proof by induction – logically the two proof techniques are equivalent.

# The Chocolate Bar Problem

$\forall n \geq 1$: $n - 1$ cuts are necessary to split a rectangular bar into $n$ individual pieces.

Proof: (by induction on the number $n$ of pieces)

Basis: $n = 1$. #cuts needed = 0 $= 1 - 1$.

Inductive Hypothesis: $\forall i \leq k$: $P(i)$ $(i - 1$ cuts needed to split every $i$ piece bar).

Inductive Step: Start with a $k + 1$ piece bar.

The first cut splits the bar into two rectangles of sizes $n_1, n_2$. $(n_1 + n_2 = k + 1)$

By the (strong) inductive hypothesis, these require $n_1 - 1, n_2 - 1$ cuts.

$$\begin{aligned}\text{So total \#cuts } &= 1 + (n_1 - 1) + (n_2 - 1)\\ &= (n_1 + n_2) - 1\\ &= (k + 1) - 1\end{aligned}$$