Administrivia

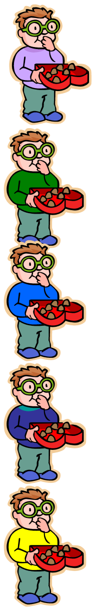
- Final Exam
  - Monday, May 8
  - Section A: Room GN-204
  - Section B: Room GS-216
  - Accommodations: TBD
  - Start at 8am, End at 10:00am. (accommodations until 11am)
  - Topics include everything we studied this semester, but more weight on number theory and graph theory.
- Review Sessions
  - Bring questions to class next Tuesday.
  - Review Session in Labs next Tuesday
- Canvas grades
  - Problem Sets, Labs, Zybooks grades will all be updated next week.
  - Please check that all your grades have been entered correctly.

Stable Matching

The Stable Matching Problem:

- There are n boys and n girls.
- Each boy has a preference list of the girls.
- Each girl has a preference list of the boys.

Boys



1: CBEAD

2 : ABECD

3 : DCBAE

4 : ACDBE

5 : ABDEC

Girls

A : 35214

B : 52143

C : 43512

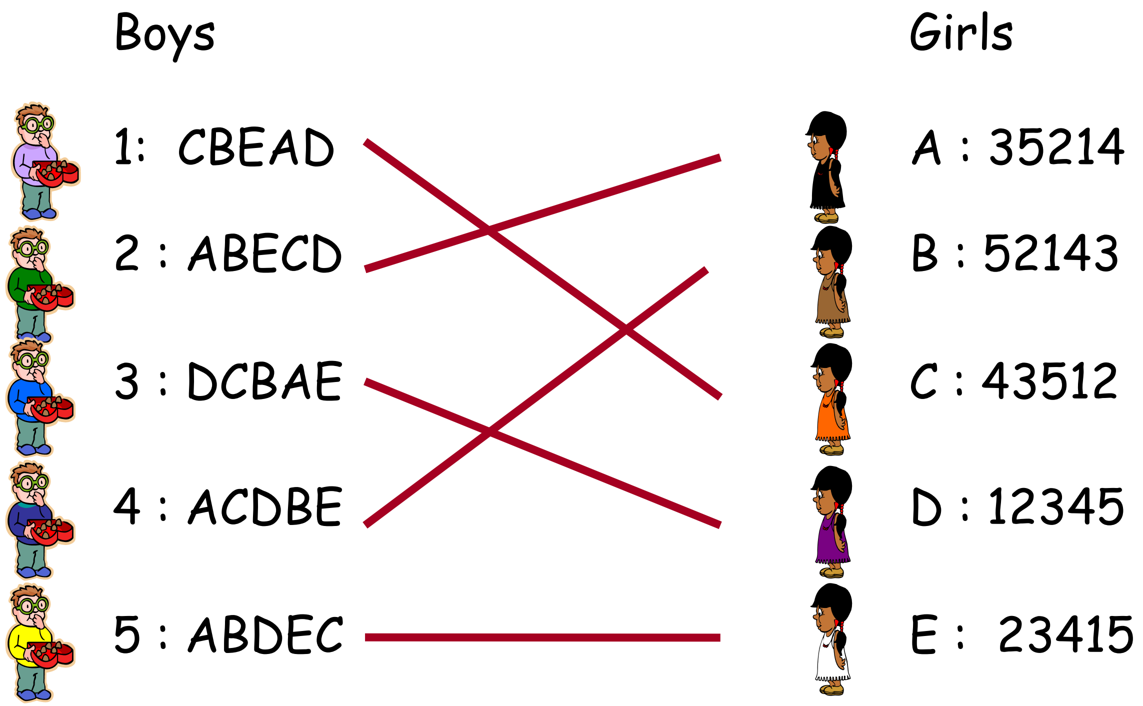
D : 12345

E : 23415

Stable Matching

What is a ***stable*** matching?

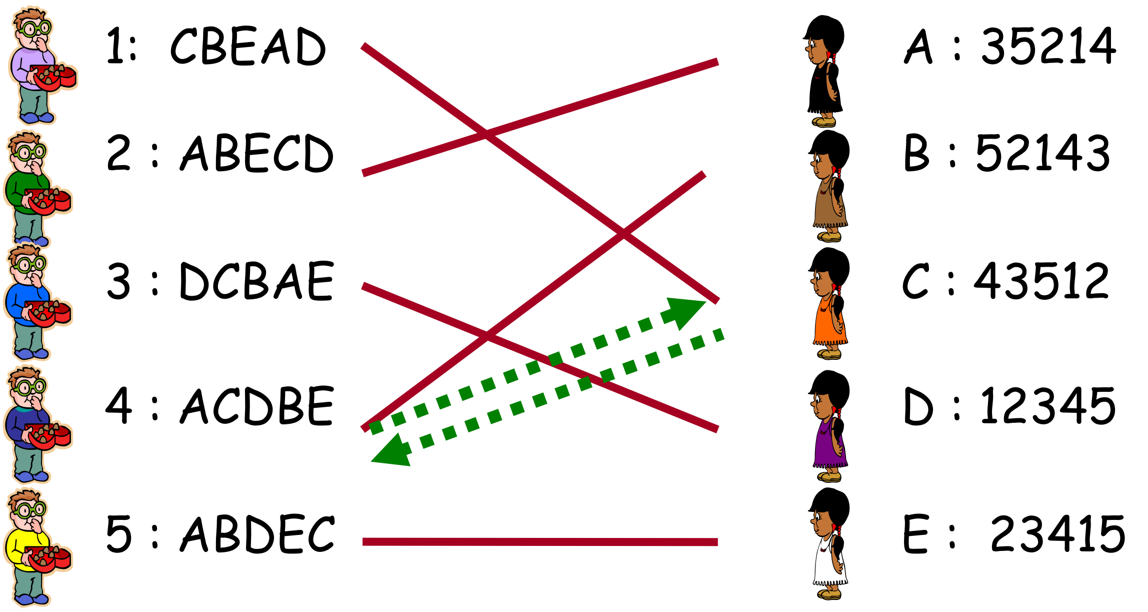
Here's a matching:



Stable Matching

Suppose 4 and B meet up with 1 and C.

- Boy 4 prefers girl C to girl B (his current partner).
- Girl C prefers boy 4 to boy 1 (her current partner).



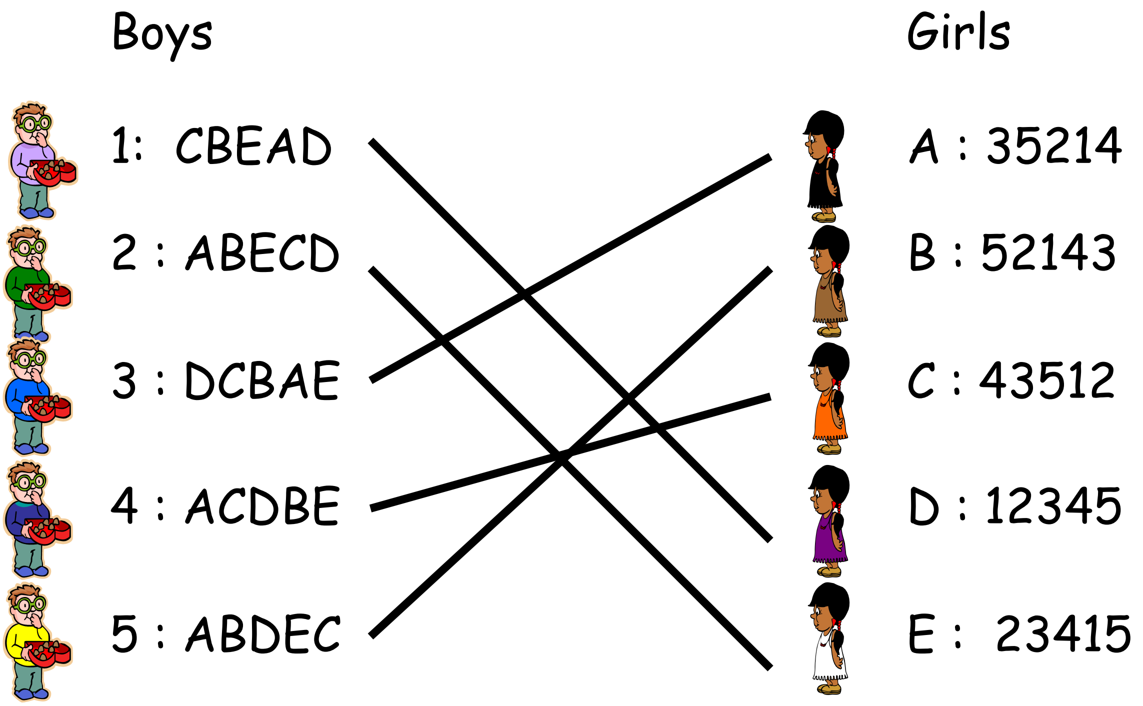
4 and C leave together, saying farewell to 1 and B.

Stable Matching

What is a ***stable*** matching?

A stable matching is a matching with no unstable pair, and everyone is matched.

Here's a stable matching.



Stable Matching

What is a ***stable*** matching?

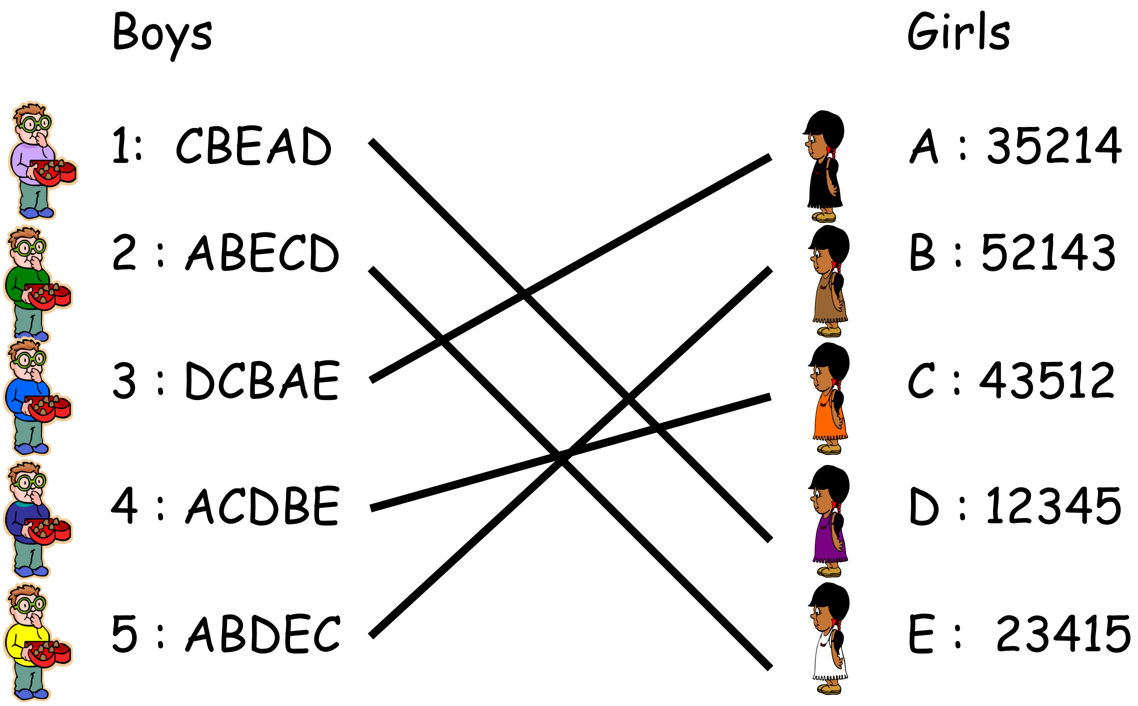
A stable matching is a matching with no unstable pair, and everyone is matched.

Stability is a measure of the collective satisfaction of the group, rather than of any one participant’s happiness.

Stable Matching

Can we always find a stable matching?

Not obvious …



Stable Matching

Gale,Shapley [1962]:

A stable matching always exists and can be found.

This is more than a solution to a puzzle:

- College Admissions (original Gale & Shapley paper, 1962)
- Matching Hospitals & Residents.
- Matching Dancing Partners.
- Matching kidney donors to patients.
- Matching end-users to nearest caching servers on the Internet.

The proof is based on a matching procedure…

2012 Nobel Prize in Economics

https://www.nobelprize.org/prizes/economic-sciences/2012/press-release/

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012 More

This year's Prize concerns a central economic problem: how to match different agents as well as possible. For example, students have to be matched with schools, and donors of human organs with patients in need of a transplant. How can such matching be accomplished as efficiently as possible? What methods are beneficial to what groups? The prize rewards two scholars who have answered these questions on a journey from abstract theory on stable allocations to practical design of market institutions.

Lloyd Shapley used so-called cooperative game theory to study and compare different matching methods. A key issue is to ensure that a matching is stable in the sense that two agents cannot be found who would prefer each other over their current counterparts. Shapley and his colleagues derived specific methods – in particular, the so-called Gale-Shapley algorithm – that always ensure a stable matching. These methods also limit agents' motives for manipulating the matching process. Shapley was able to show how the specific design of a method may systematically benefit one or the other side of the market.

Alvin Roth recognized that Shapley's theoretical results could clarify the functioning of important markets in practice. In a series of empirical studies, Roth and his colleagues demonstrated that stability is the key to understanding the success of particular market institutions. Roth was later able to substantiate this conclusion in systematic laboratory experiments. He also helped redesign existing institutions for matching new doctors with hospitals, students with schools, and organ donors with patients. These reforms are all based on the Gale-Shapley algorithm, along with modifications that take into account specific circumstances and ethical restrictions, such as the preclusion of side payments.

Even though these two researchers worked independently of one another, the combination of Shapley's basic theory and Roth's empirical investigations, experiments and practical design has generated a flourishing field of research and improved the performance of many markets. This year's prize is awarded for an outstanding example of economic engineering.

The Matching Procedure

Morning: each boy offers a proposal to the first girl on his list.

Billy Bob

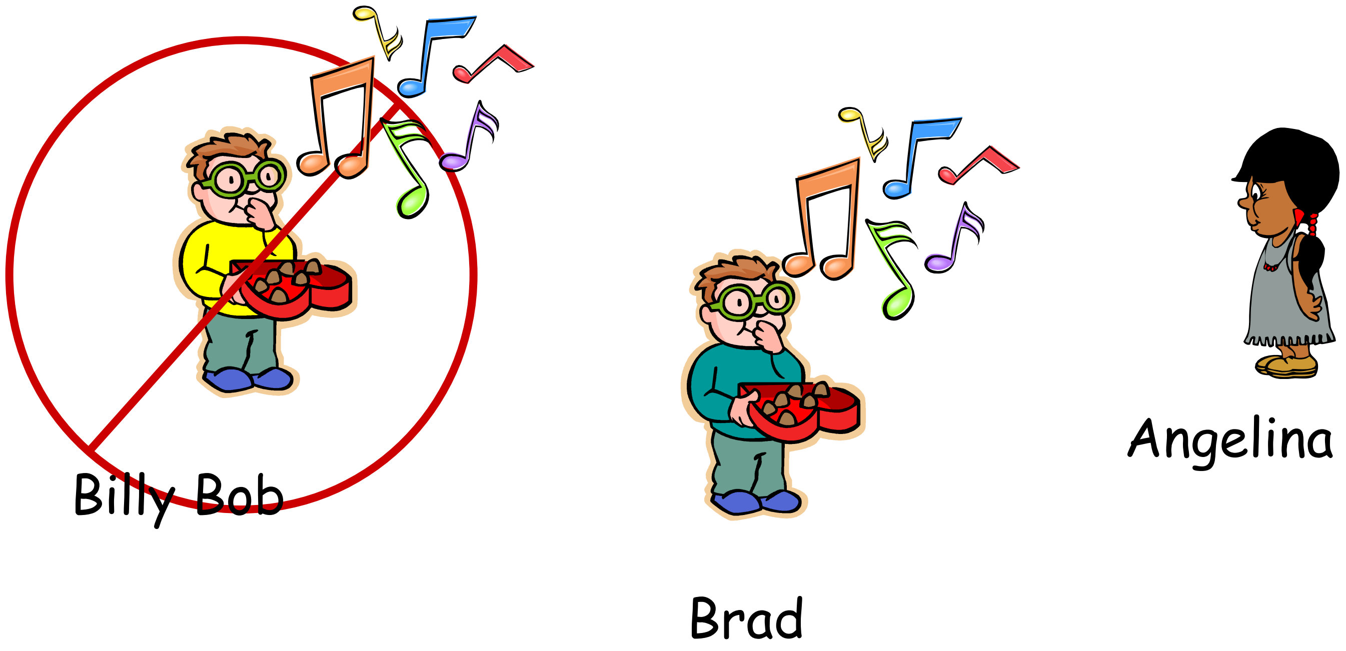
Brad

Angelina

The Matching Procedure

Morning: each boy offers a proposal to the first girl on his list.

Afternoon: girl rejects all but favorite



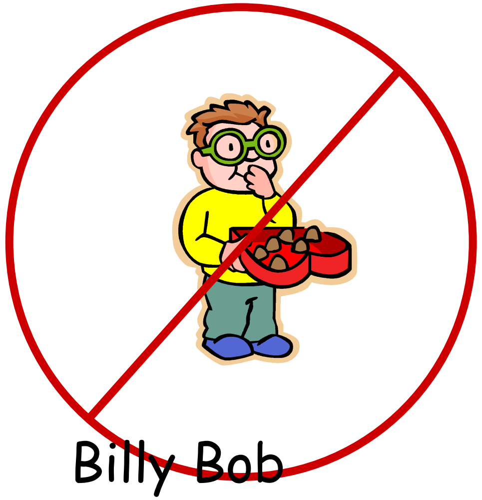
The Matching Procedure

Morning: each boy offers a proposal to the first girl on his list.

Afternoon: girl rejects all but favorite

Evening: rejected boy writes off girl

Repeat this procedure, each unmatched boy goes down his list.

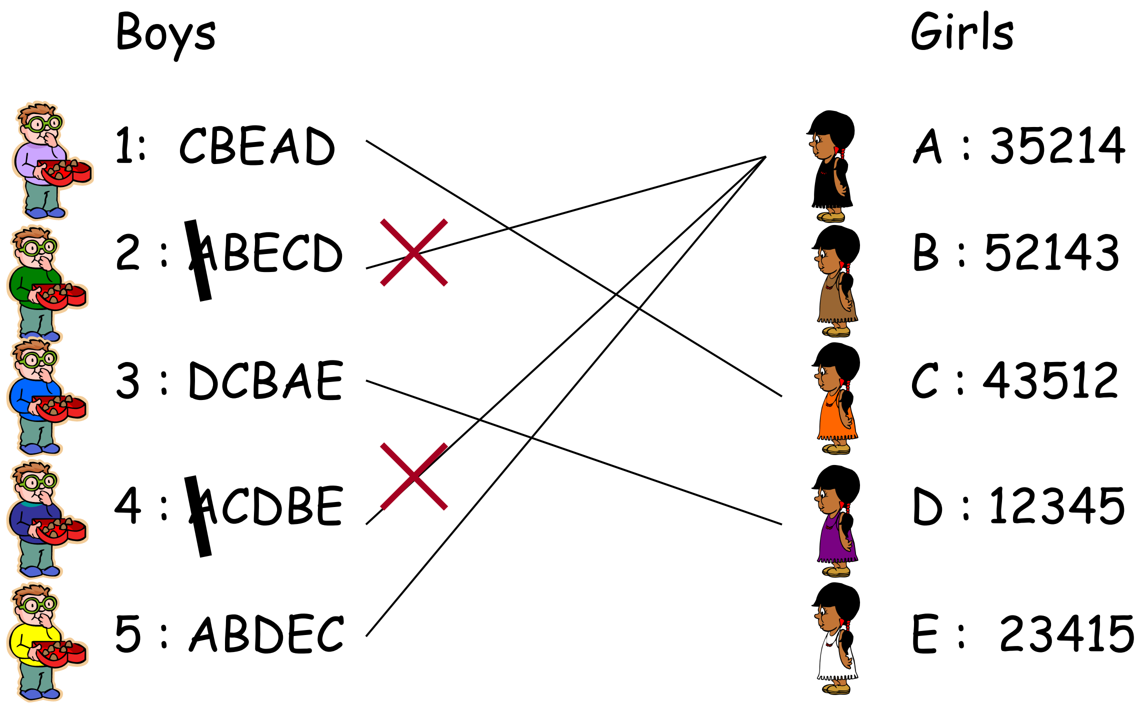


Day One

Morning: each boy offers a proposal to the first girl on his list.

Afternoon: girl rejects all but favorite

Evening: rejected boy writes off girl

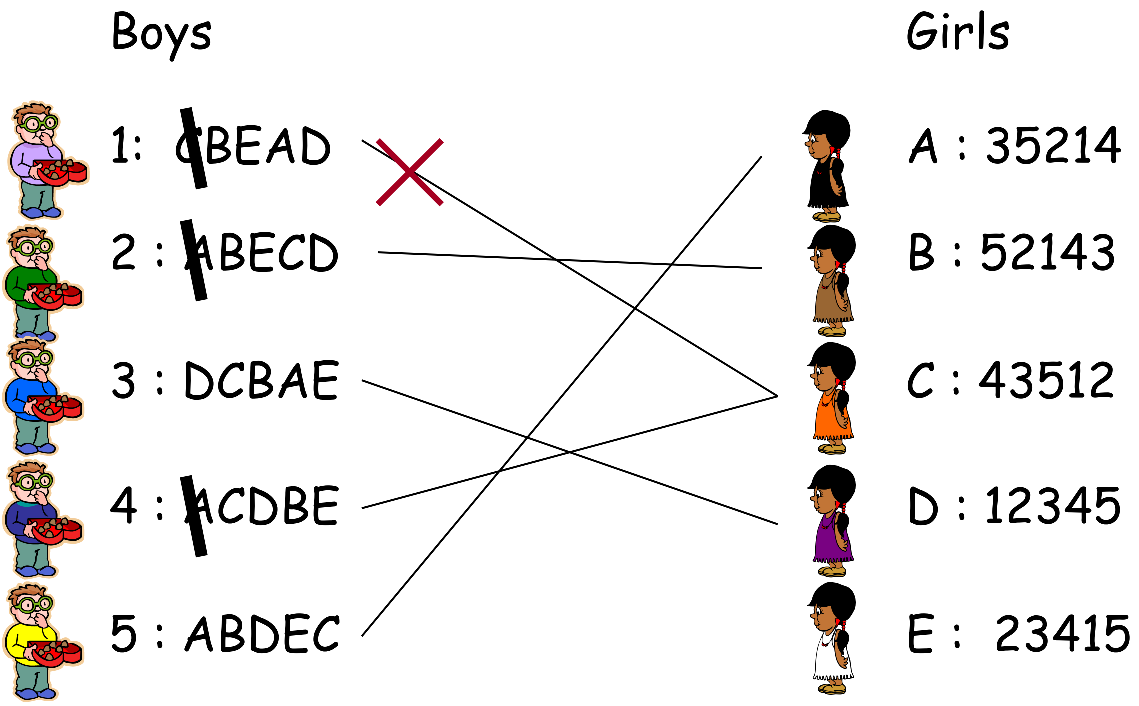


Day Two

Morning: each boy offers a proposal to the first girl on his list.

Afternoon: girls reject all but favorite

Evening: rejected boy writes off girl

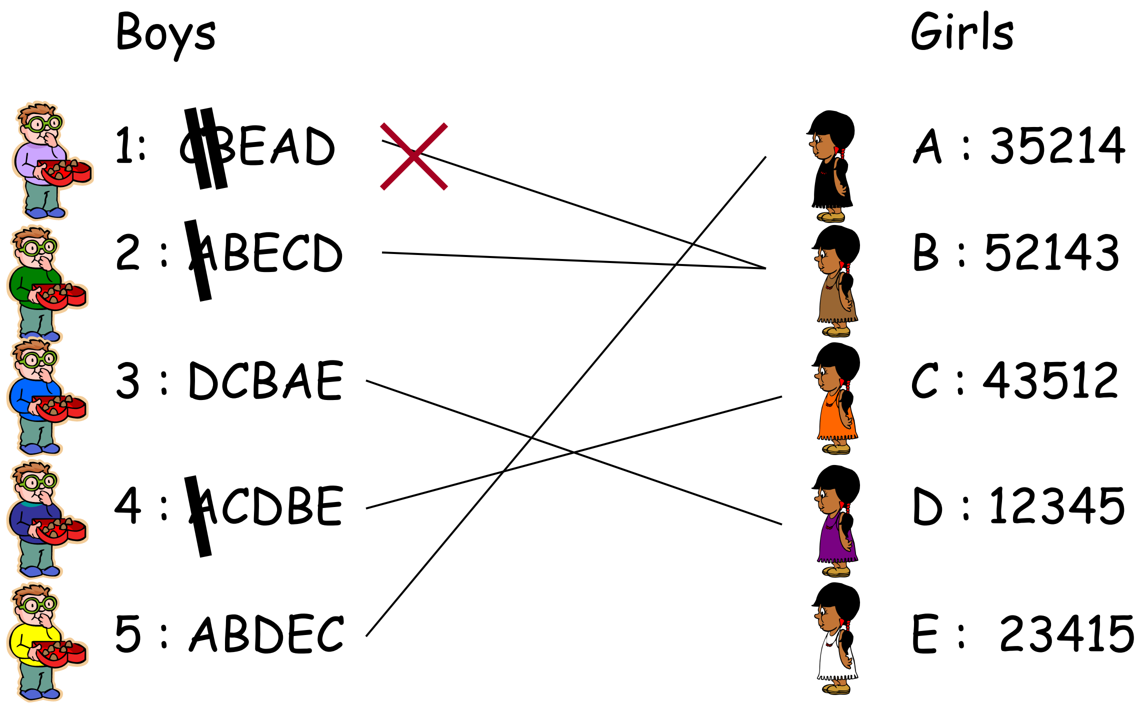


Day Three

Morning: each boy offers a proposal to the first girl on his list.

Afternoon: girls reject all but favorite

Evening: rejected boy writes off girl



Day Four

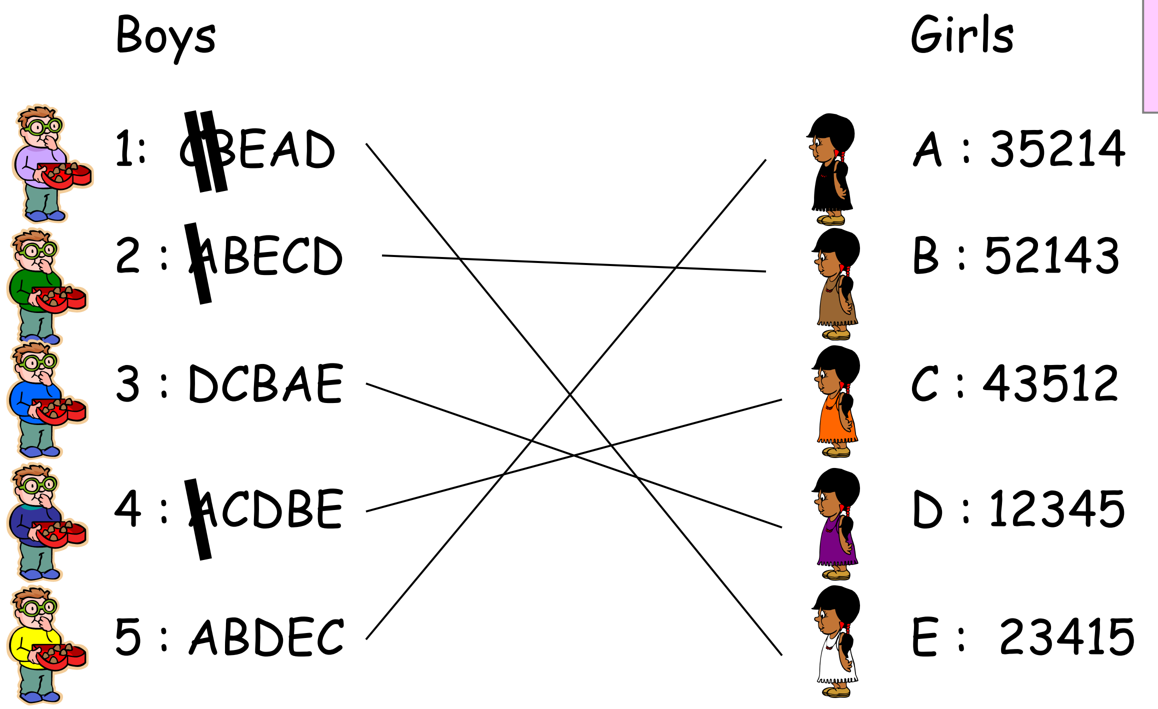
Morning: each boy offers a proposal to the first girl on his list.

Afternoon: girl rejects all but favorite

Evening: rejected boy writes off girl

OKAY, matching day!

Every girl has accepted a proposal.



Proof of Gale-Shapley Theorem

Gale,Shapley [1962]: This procedure always finds a stable matching.

What do we need to check?

1. The procedure will terminate.
2. Everyone is matched.
3. No unstable pairs.

Step 1 of the Proof

Claim 1. The procedure will terminate in at most n^2 days.

- The procedure terminates when every girl has accepted a proposal.
- If the procedure does not terminate on a given day, then at least one boy was rejected that day.
- No boy gets more than n-1 rejections.
- The maximum number of rejections overall is no greater than n(n-1).
- The number of days is no greater than n(n-1), which is less than n^2.

- A more careful analysis shows the number of days is at most $n^2 -2n+2$.

Step 2 of the Proof

Claim 2. Everyone is matched when the procedure stops.

Proof: by contradiction.

1. Suppose boy *B* is not matched at the end.
2. Some girl G is also not matched at the end.
3. *B* was rejected by every girl, including G.
4. G must have accepted a previous proposal.
5. Once a girl has accepted a proposal, she will be matched at the end.
6. G is matched at the end. ><
7. Every boy and every girl is matched at the end.

Step 3 of the Proof

Claim 3. There is no unstable pair when the algorithm terminates.

Fact. If a girl G rejects a boy B,

then G will be matched to a boy she prefers over B.

Consider B and G, who are NOT matched to each other.

Case 1. If G is still on B's list, then B is matched to a girl her prefers over G.

So, B has no incentive to leave.

Case 2. If G is not on B's list, then G previously rejected B in favor of a boy she prefers.

Since every girl can only improve her match with every rejection, G has no incentive to leave.

Proof of Gale-Shapley Theorem

Gale,Shapley [1962]:

A stable matching always exists and can be found.

Claim 1. The procedure will terminate in at most n^2 days.

Claim 2. Everyone is matched when the procedure stops.

Claim 3. There is no unstable pair.

So, the theorem follows.

Further Questions

Is this matching procedure better for boys or for girls??

- All boys get the best partner simultaneously!
- All girls get the worst partner simultaneously!

Why?

Out of all possible stable matchings, boys get the best possible partners simultaneously.

Can boys do better by lying? NO!

Can girls do better by lying? YES!