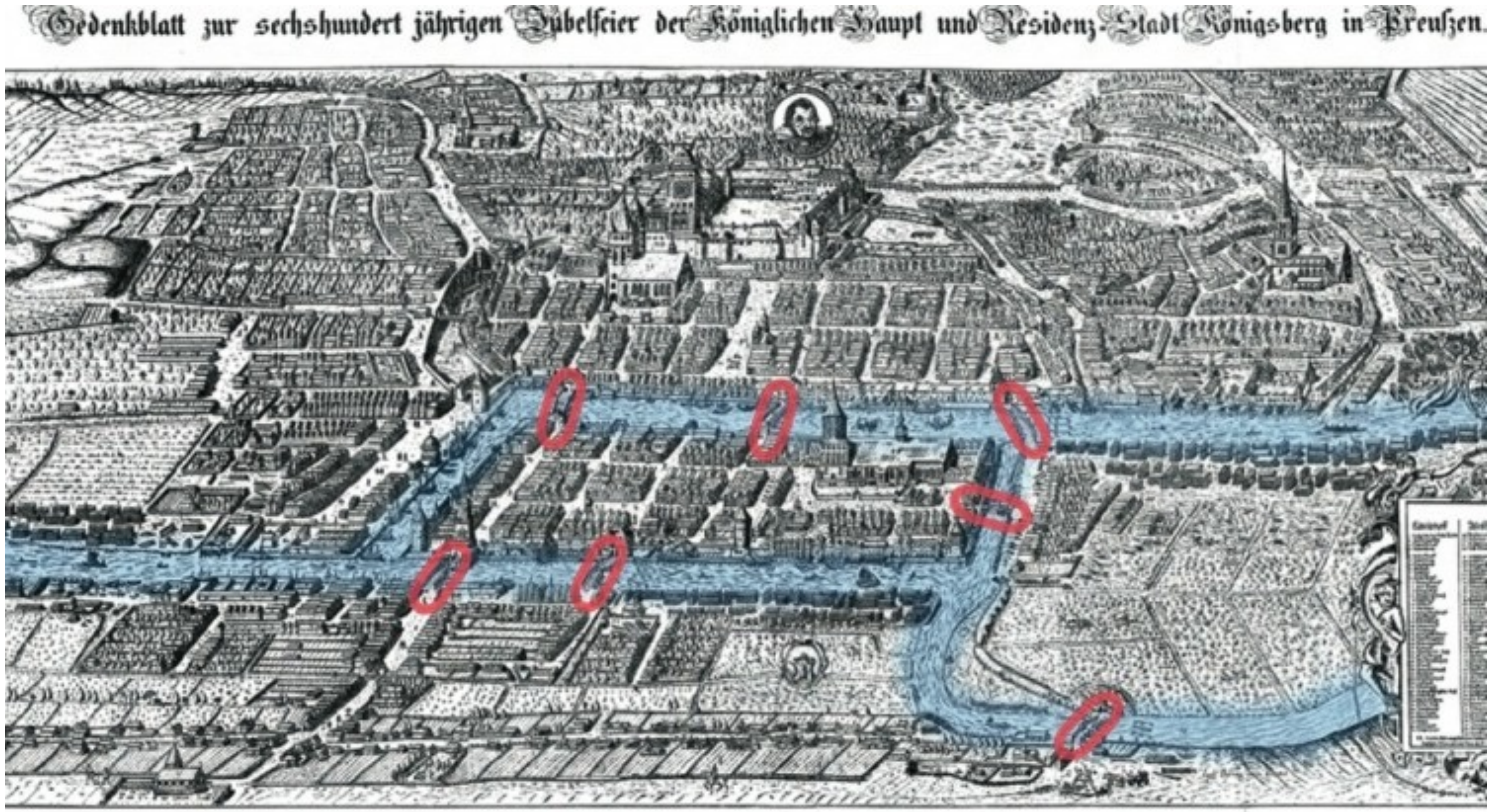
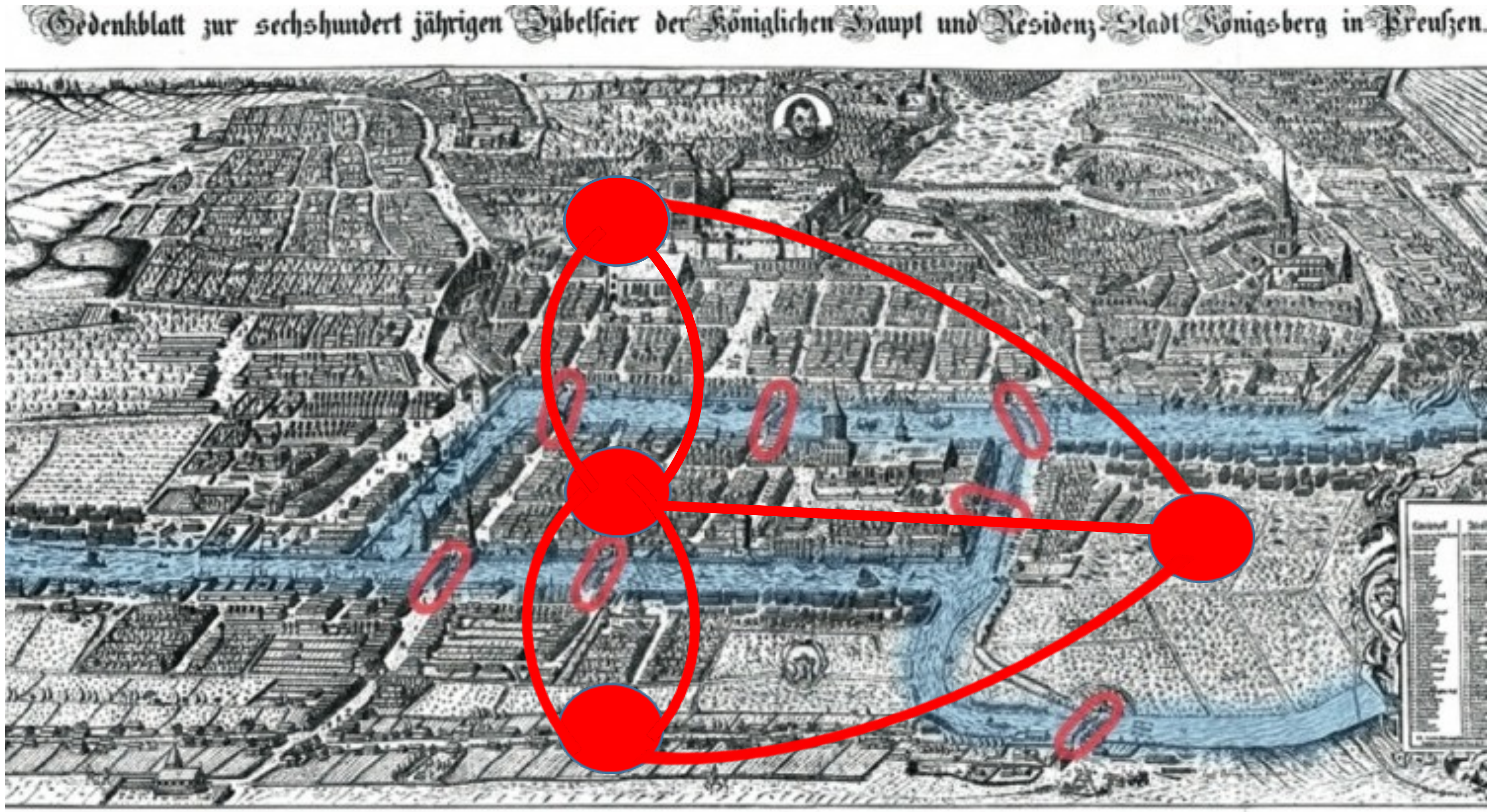
The Seven Bridges of Konigsberg



On a walking tour can a person cross each of the seven bridges exactly once?

Euler's Graph Formulation



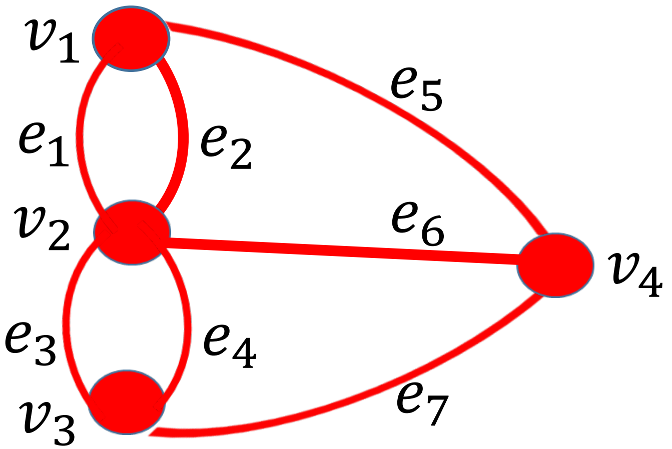
This *graph* models the land masses and the bridges connecting them.

Terminology

Vertex (node)

Edge (a set of two nodes)

Degree of a vertex
 = # edges incident to it



Walk: A sequence of alternating vertices and edges that starts and ends in vertices in which the vertices before and after each edge are the two endpoints of that edge.
E.g. v_1, e_5, v_4, e_7, v_3 (Open walk)
v_1, e_2, v_2, e_2, v_1 (closed walk)

Trail: A walk in which no edge is repeated. v_1, e_1, v_2, e_6, v_4

Circuit: A closed walk in which no edge is repeated. $v_1, e_1, v_2, e_3, v_3, e_4, v_2, e_2, v_1$

Path: A trail in which no vertex is repeated. v_1, e_1, v_2, e_3, v_3

Cycle: A circuit of length ≥ 1 with the same first and last vertices and no repeated vertex.

$v_1, e_1, v_2, e_3, v_3, e_7, v_4, e_5, v_1$

Eulerian Trail/Circuit: A trail/circuit that traverses every edge exactly once.

Does the graph above have an eulerian trail?

Directed Graphs

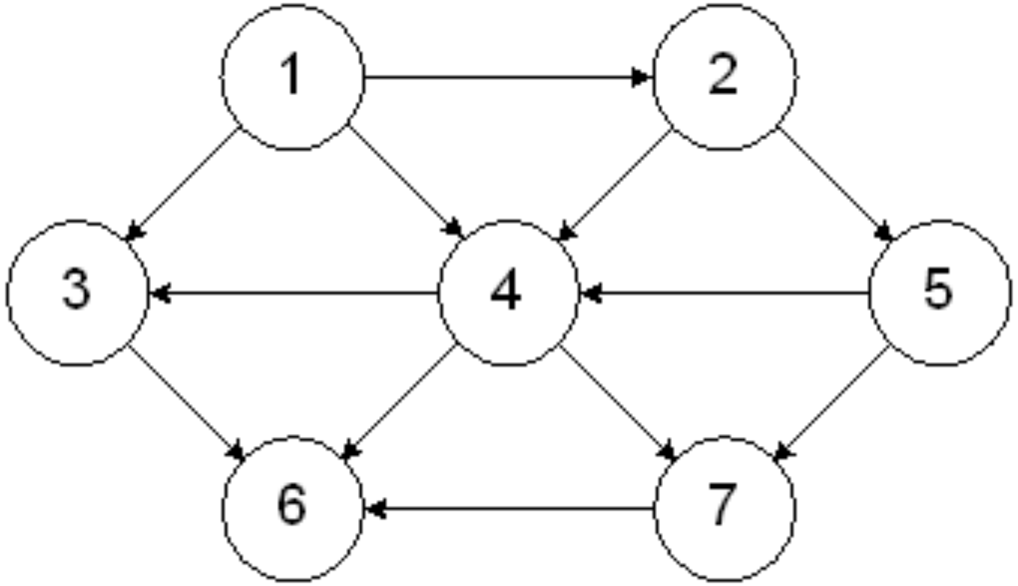
A directed graph $G = (V, E)$ consists of:

V – a set of vertices, and

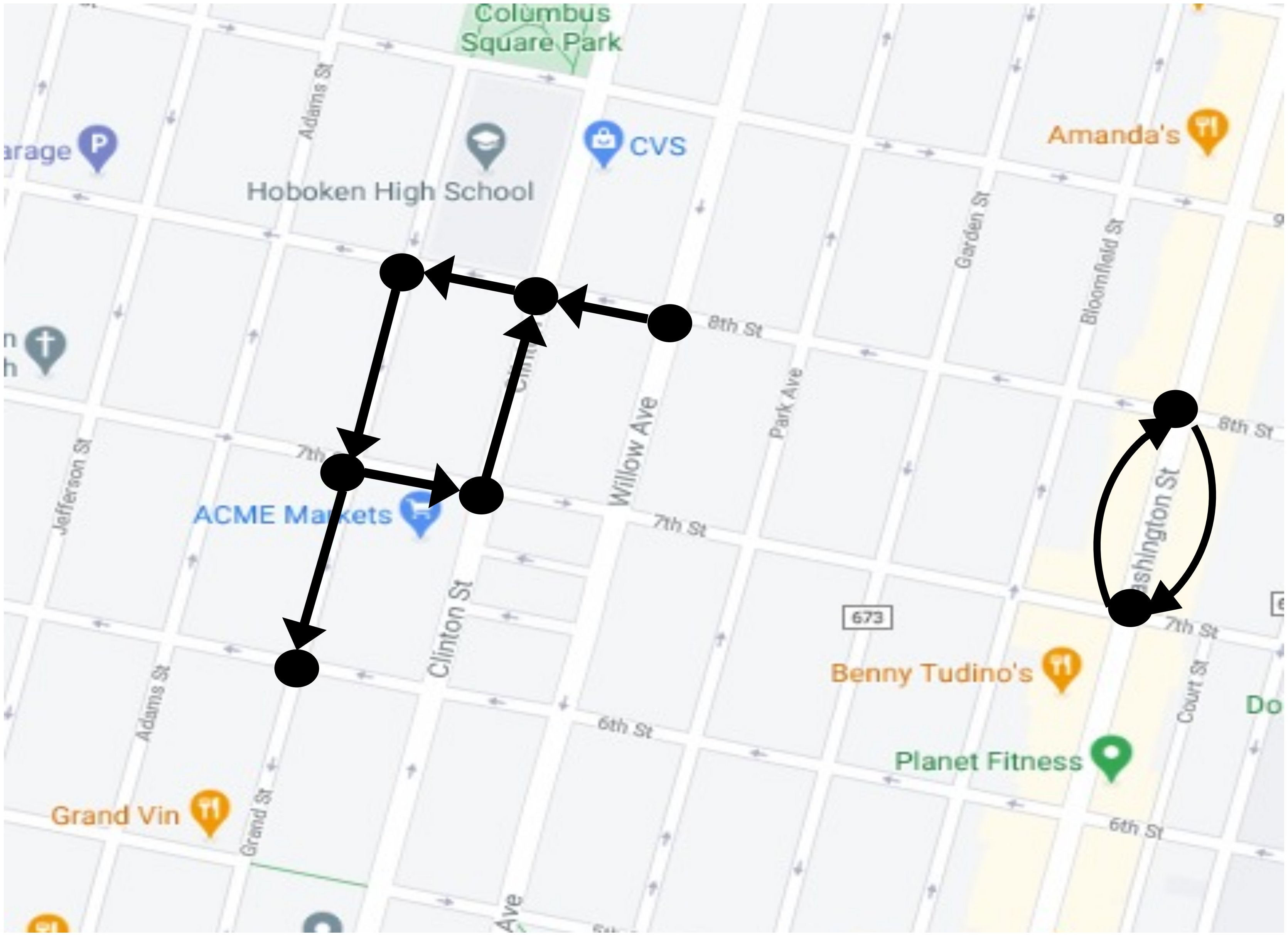
$E \subseteq V \times V$ – a set of directed edges

each edge is an ordered pair $(u, v), u, v \in V$.

Directed Graphs



Directed Graphs: Examples



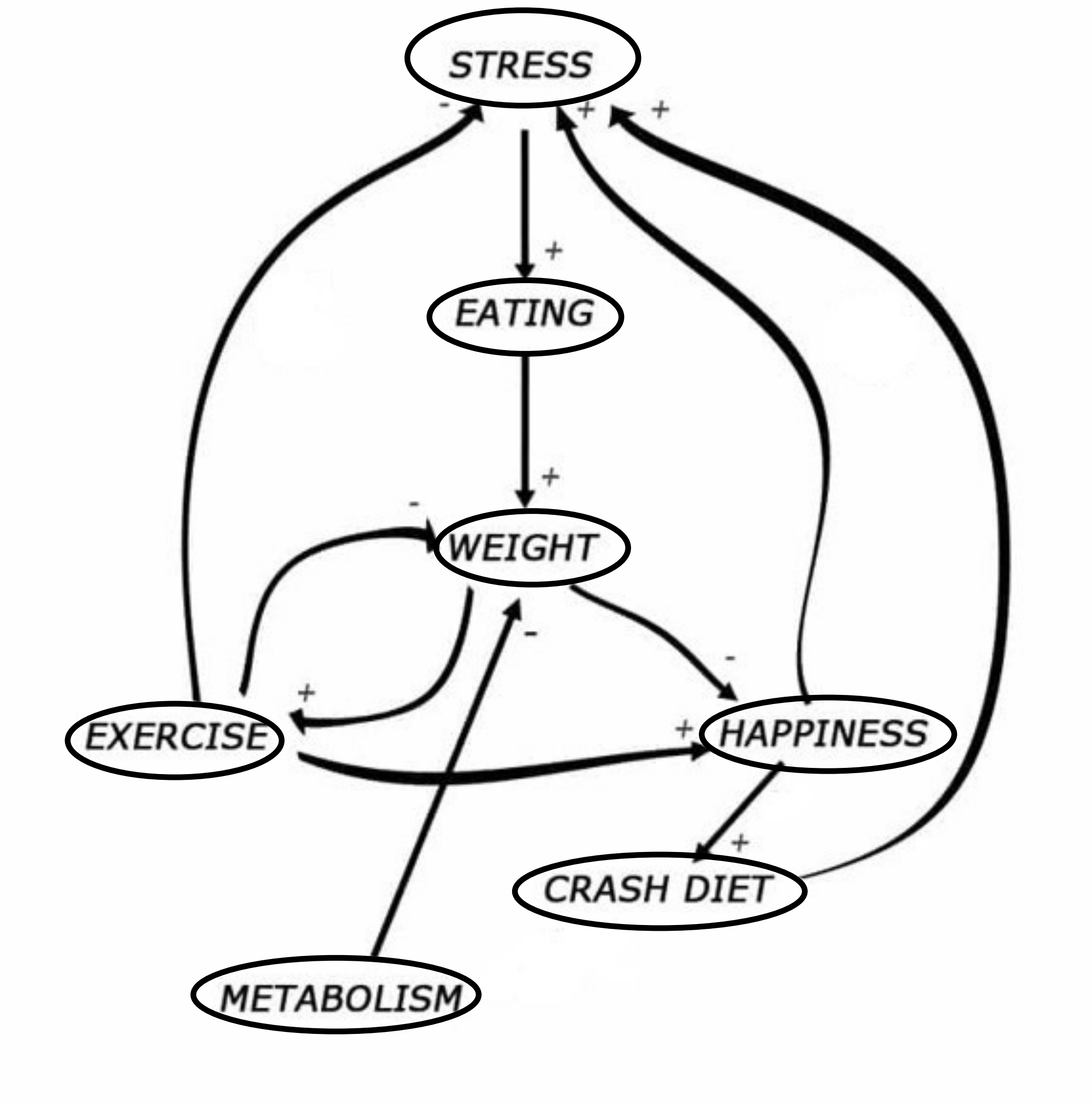
Node

Each intersection

Edges

Street segments from one intersection to another

Directed Graphs: Examples



Node

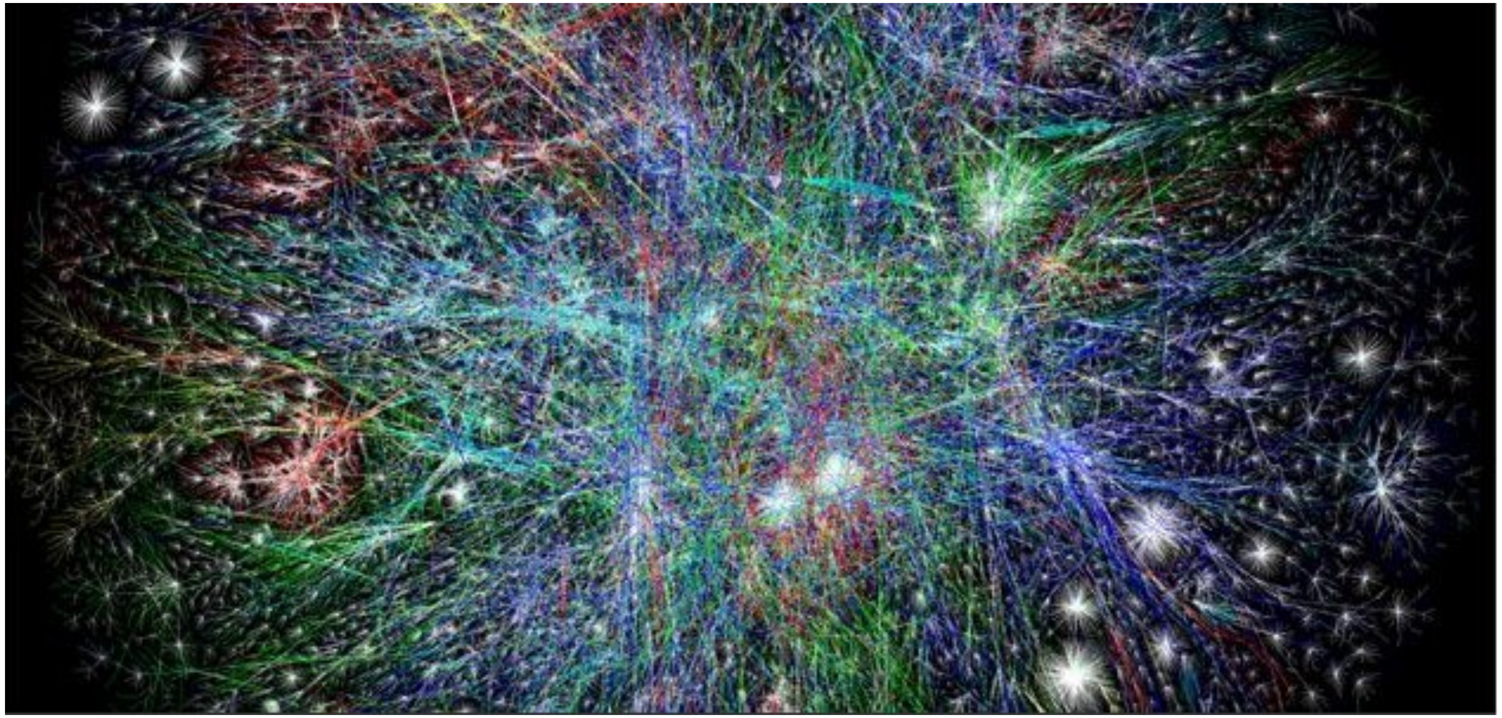
Each behavior/action

Edges

Causal relationships

Picture from https://agilepainrelief.com/glossary/causal-loop-diagram

The web as a directed graph



Vertex for each web page.
A directed edge from page p_1 to page p_2 if p_1 contains a hyperlink to page p_2

- Social networks
- Airline Flights
- Email/Phone communications
- Program Analysis: calling patterns among functions
- Prerequisite structure among courses in the catalog
- Epidemiology
  - Contact tracing
  - Causality relationships

Directed Graphs: Terminology

- **Outdegree of v :** $\deg^{+}(\boldsymbol{v})$

  # of outgoing edges

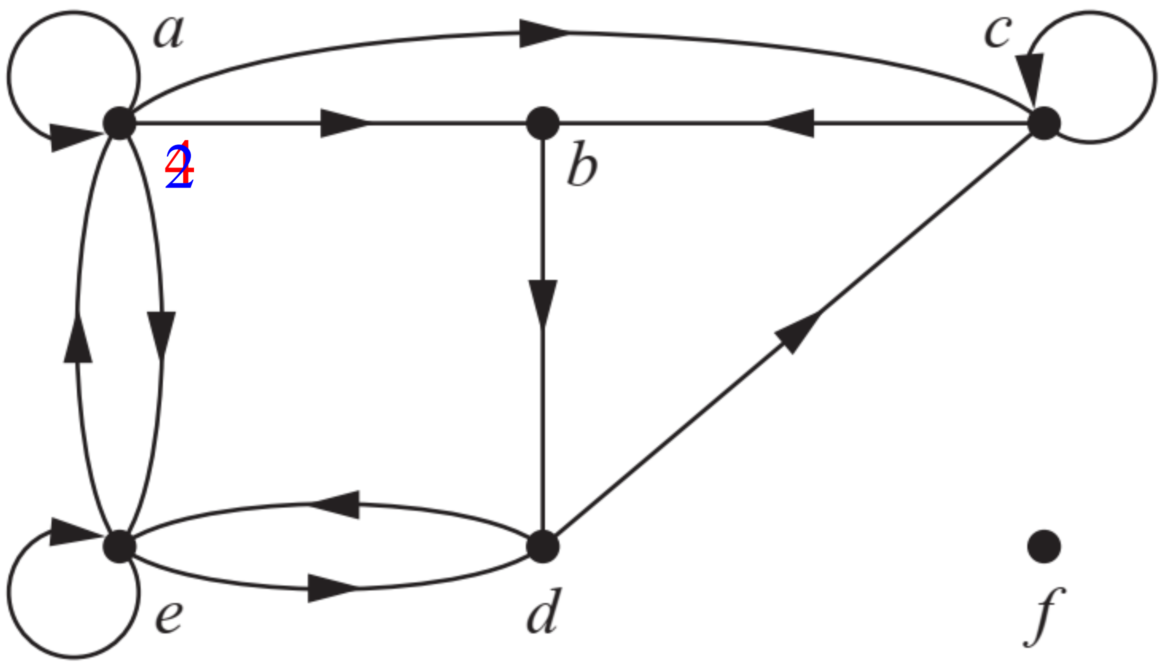
  # of edges with $\boldsymbol{v}$ as initial node

- **Indegree of v :** $\deg^{-}(\boldsymbol{v})$

  # of incoming edges

  # of edges with $\boldsymbol{v}$ as end node

- For a directed graph $G = (V, E)$



G

$$\sum_{v \in V} \deg^{+}(\boldsymbol{v}) = |\boldsymbol{E}|$$

Size of E: # of elements in E

- Each edge has ONE initial node
- # of edges = # of initial nodes
- $\deg^{+}(\boldsymbol{v})$: # of times $\boldsymbol{v}$'s role is an initial node

Directed Graphs: Terminology

- **Outdegree of $\boldsymbol{v}$:** $\deg^{+}(\boldsymbol{v})$

  # of outgoing edges

  # of edges with $\boldsymbol{v}$ as initial node

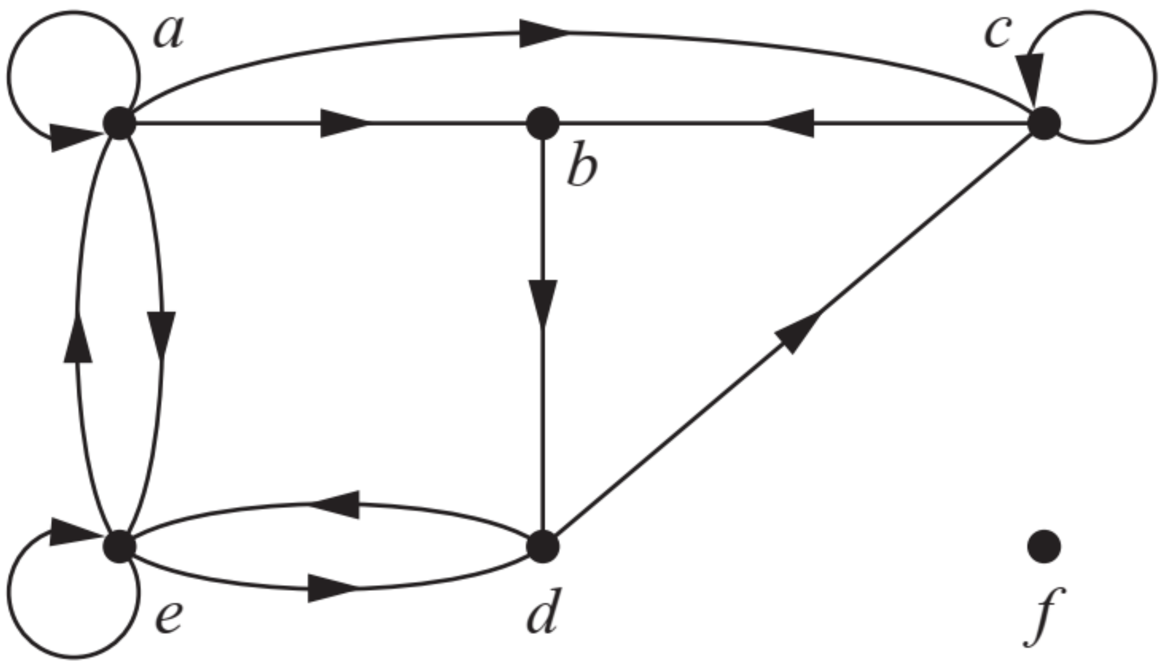
- **Indegree of $\boldsymbol{v}$:** $\deg^{-}(\boldsymbol{v})$

  # of incoming edges

  # of edges with $\boldsymbol{v}$ as end node

- For a directed graph $G = (V, E)$

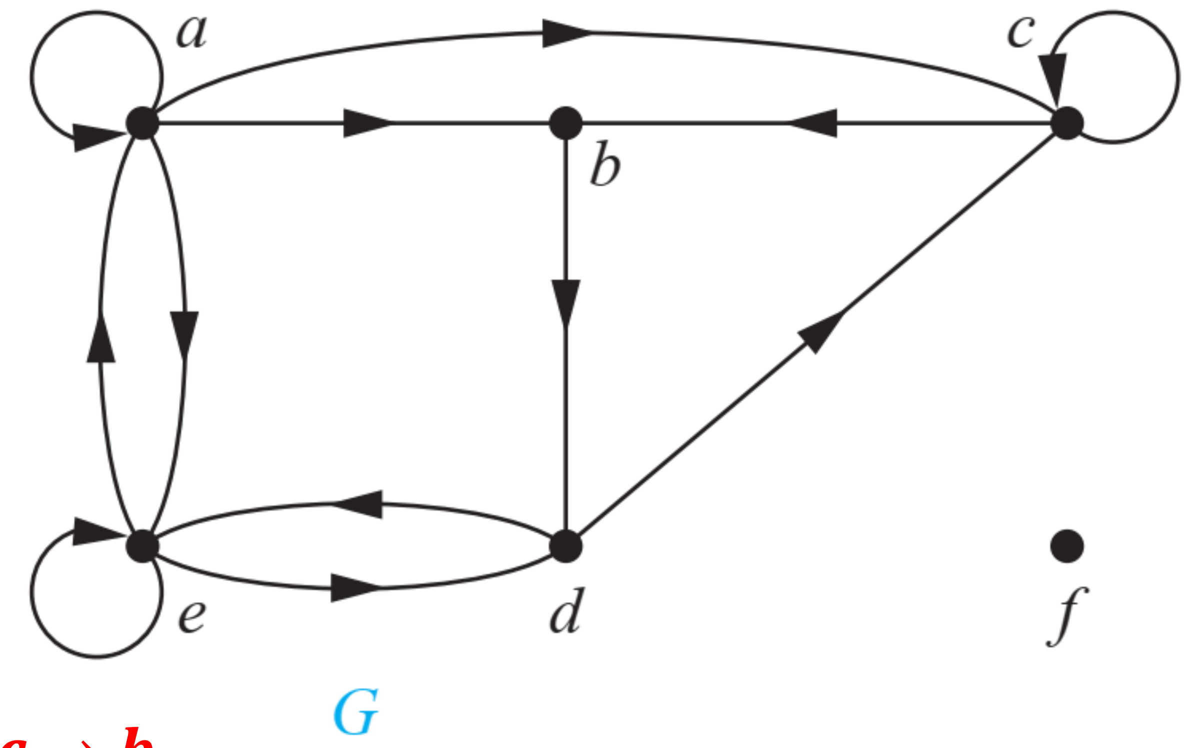
$$\sum_{v \in V} \deg^{-}(\boldsymbol{v}) = |\boldsymbol{E}|$$



G

Directed Graphs: Terminology

- **Walk:** A *walk* from $\boldsymbol{x_0}$ to $\boldsymbol{x_n}$ is a sequence $x_0, e_1, x_1, e_2, x_2, \ldots, e_n, x_n$
- ***Length*** of a walk: # of edges in the walk
- **Path:** A *walk* with no repeated nodes
- **Cycle:** A *walk* that begins and ends at a node and has no repeated nodes.



$\mathbf{1}.\, \boldsymbol{a \to a}$ $\quad \mathbf{2}.\, \boldsymbol{e \to d \to c}$ $\quad \mathbf{3}.\, \boldsymbol{b \to d \to c \to b}$

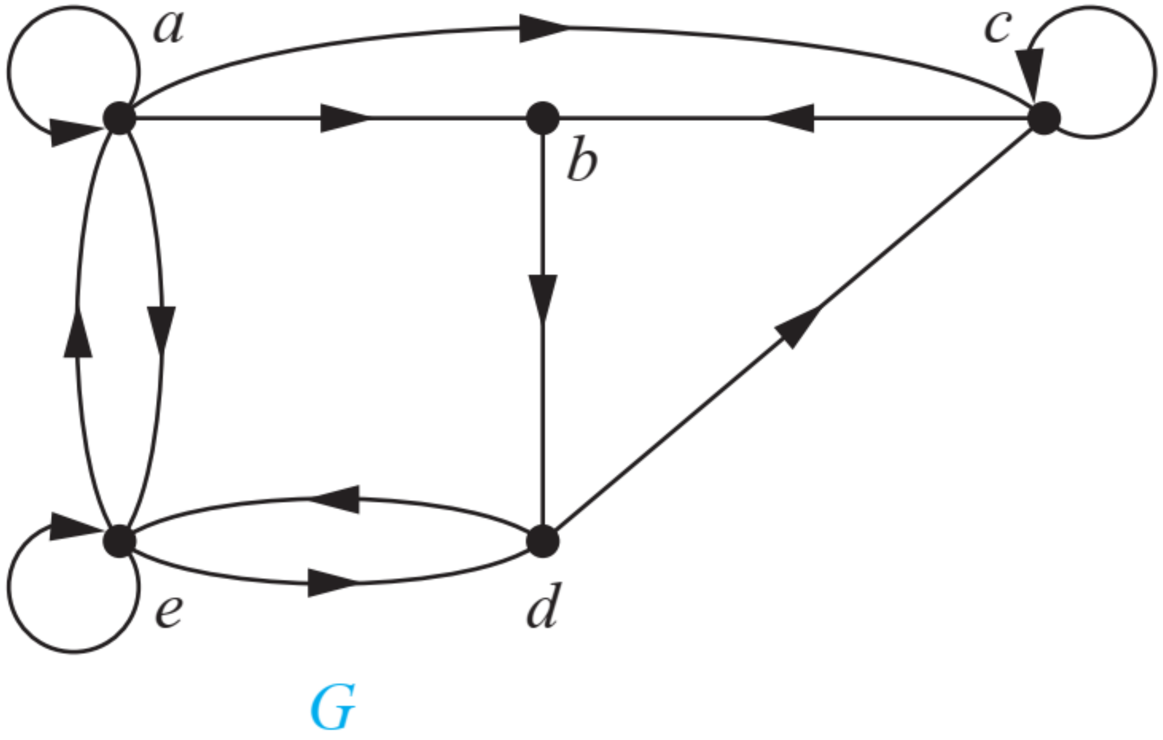
How many walks from a to b? Infinitely many!

How many paths from a to b? 3: $a \to b,\ a \to c \to b,\ a \to e \to d \to c \to b$

Directed Graphs: Terminology

- **Strongly Connected Graph:**

  A directed graph where there is a directed path from **every node** to **every other node**.



Is this graph strongly connected?

Yes! Strongly connected component: $\boldsymbol{a} \rightarrow \boldsymbol{c} \rightarrow \boldsymbol{b} \rightarrow \boldsymbol{d} \rightarrow \boldsymbol{e} \rightarrow \boldsymbol{a}$

Directed Graphs: Terminology

- **Strongly Connected Graph:**

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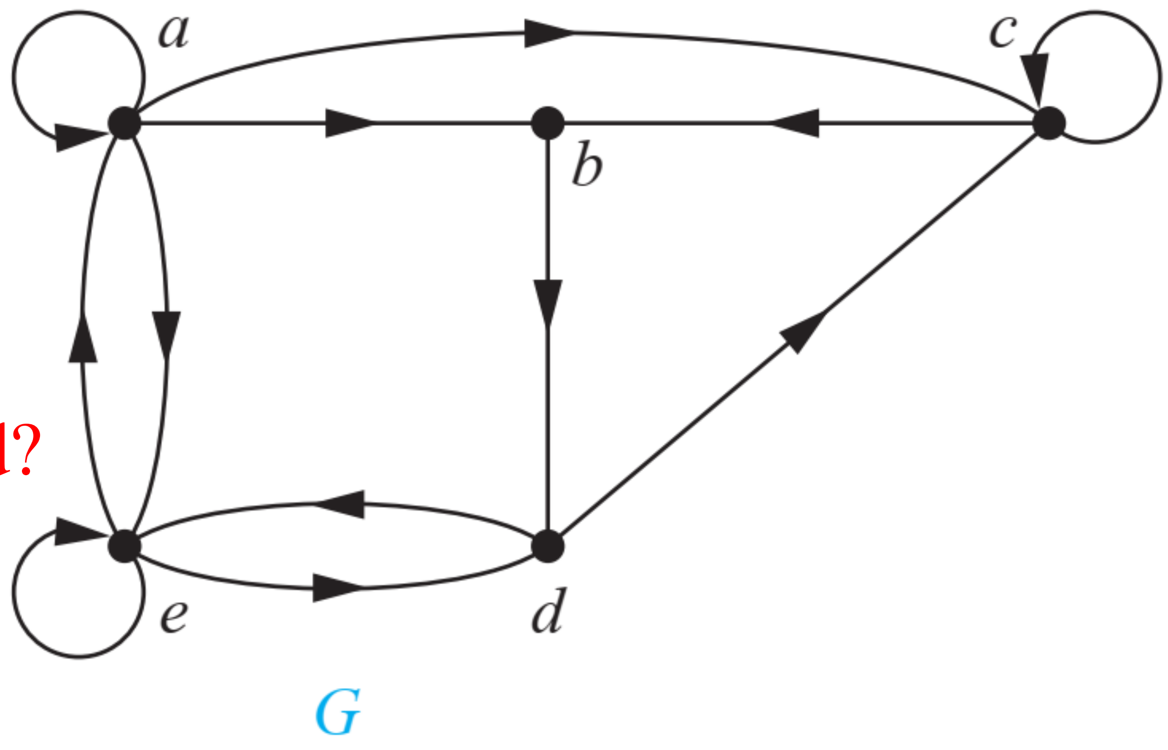
Q4. Is this graph strongly connected?



Directed Graphs: Terminology

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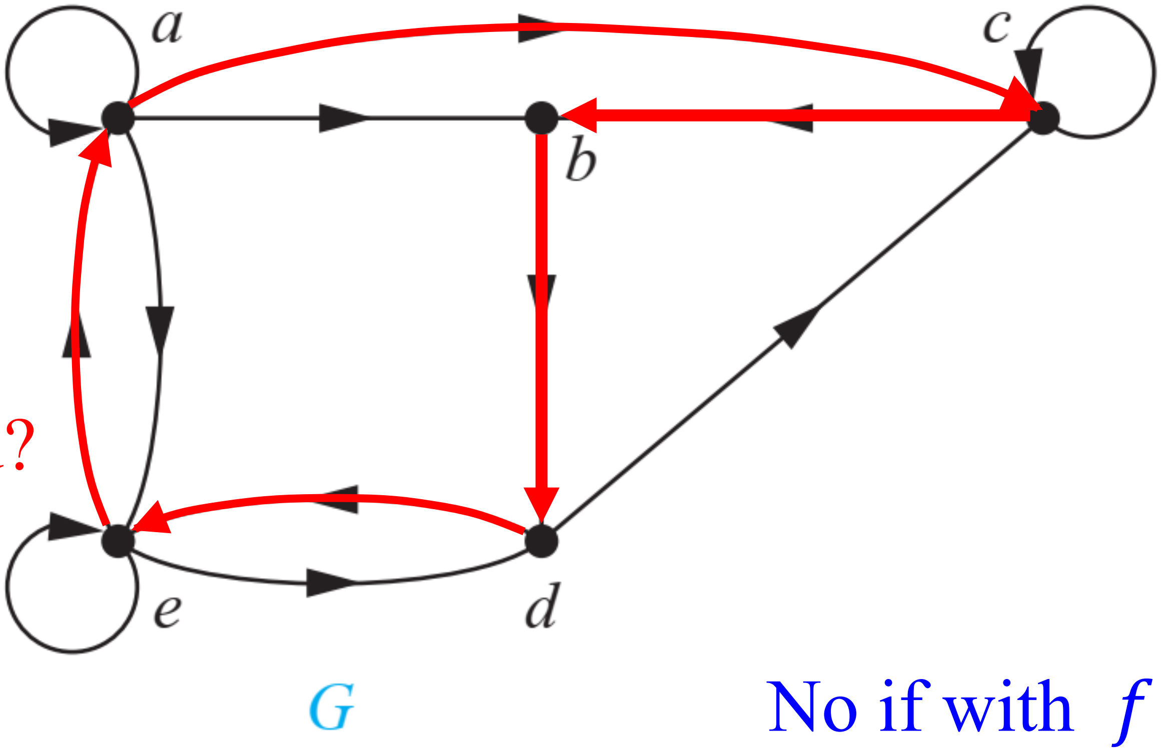


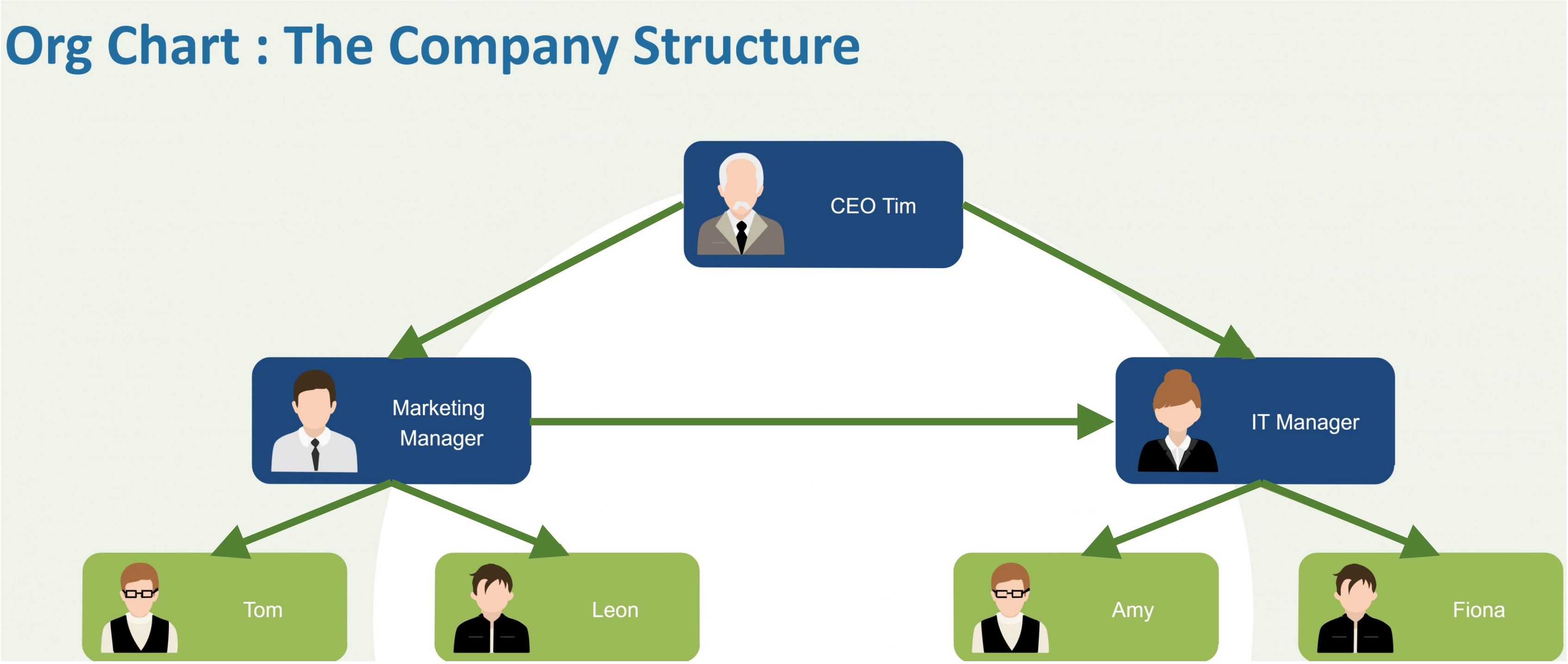
Yes.

No if with f

Strongly connected component: $\boldsymbol{a} \rightarrow \boldsymbol{c} \rightarrow \boldsymbol{b} \rightarrow \boldsymbol{d} \rightarrow \boldsymbol{e} \rightarrow \boldsymbol{a}$

Directed Acyclic Graphs

A directed graph with NO cycles.



Directed Acyclic Graphs

Review Article | Open Access | Published: 04 June 2018

Directed acyclic graphs: a tool for causal studies in paediatrics

Thomas C Williams, Cathrine C Bach, Niels B Matthiesen, Tine B Henriksen & Luigi Gagliardi

Pediatric Research **84**, 487–493(2018) | Cite this article

JMLR: Workshop and Conference Proceedings 6: 59–86 NIPS 2008 Workshop on Causality

Beware of the DAG!

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Article PDF Available

Using Directed Acyclic Graphs in Epidemiological Research in Psychosis: An Analysis of the Role of Bullying in Psychosis

May 2017 · Schizophrenia Bulletin 43(6)
DOI: 10.1093/schbul/sbx013
Project: Directed acyclic graphs in epidemiological research in psychology

Authors:

Graphical presentation of confounding in directed acyclic graphs FREE

Marit M. Suttorp, Bob Siegerink, Kitty J. Jager, Carmine Zoccali, Friedo W. Dekker

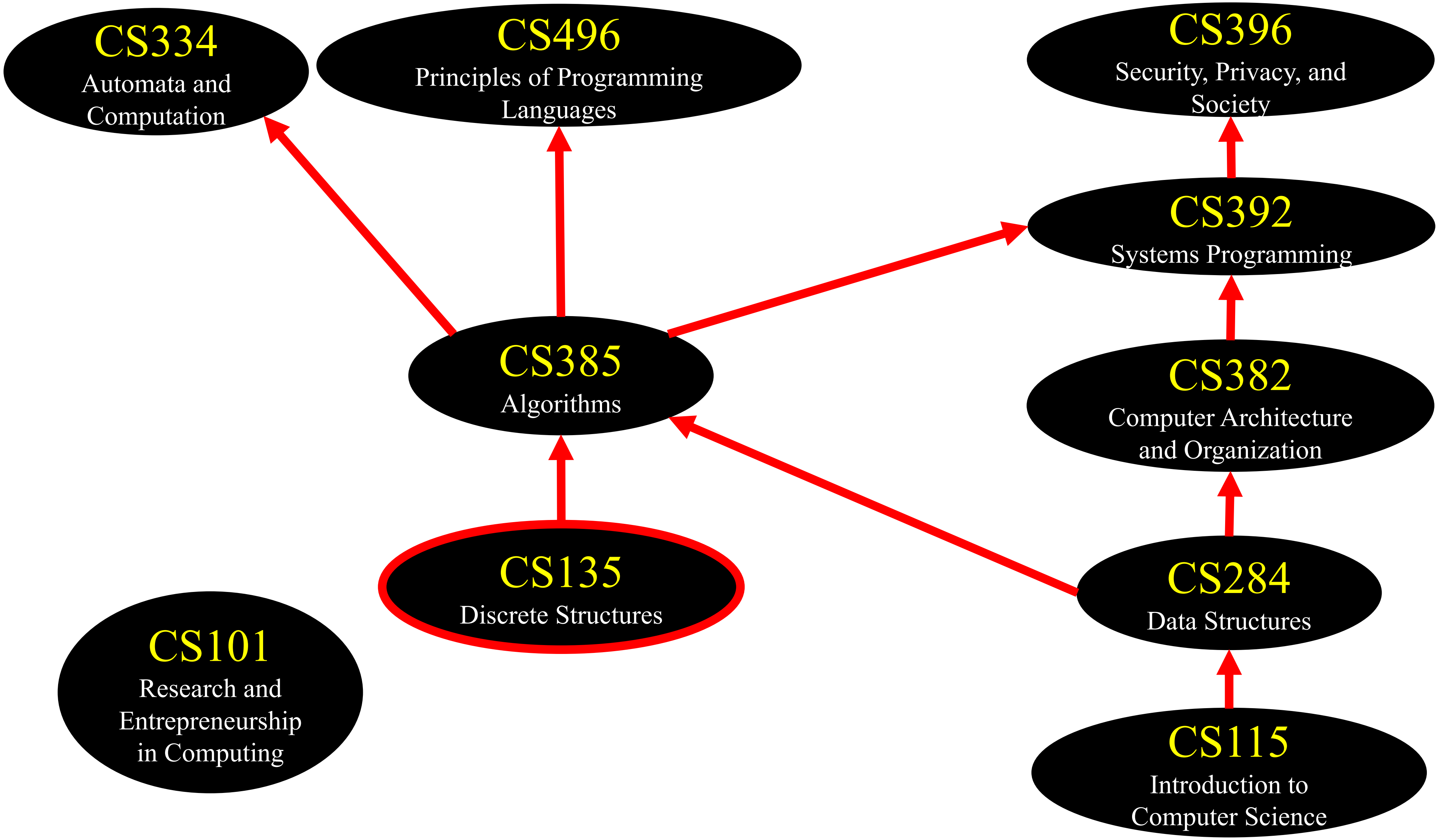
Nephrology Dialysis Transplantation, Volume 30, Issue 9, September 2015, Pages 1418–1423, https://doi.org/10.1093/ndt/gfu325
Published: 16 October 2014 **Article history ▾**

Beware of Dag???

What happens if you refuse to give Dag his axe?

From a lore standpoint, the Danes in the game believe that dying with your axe in hand grants passage to Valhalla, where they will become one of Odin's warriors. Denying Dag this is a pretty big deal, but then again he did try to kill you. Even so, Sigurd will look down on this decision to refuse him entrance into Valhalla, so choose carefully.

Directed Graphs: DAG

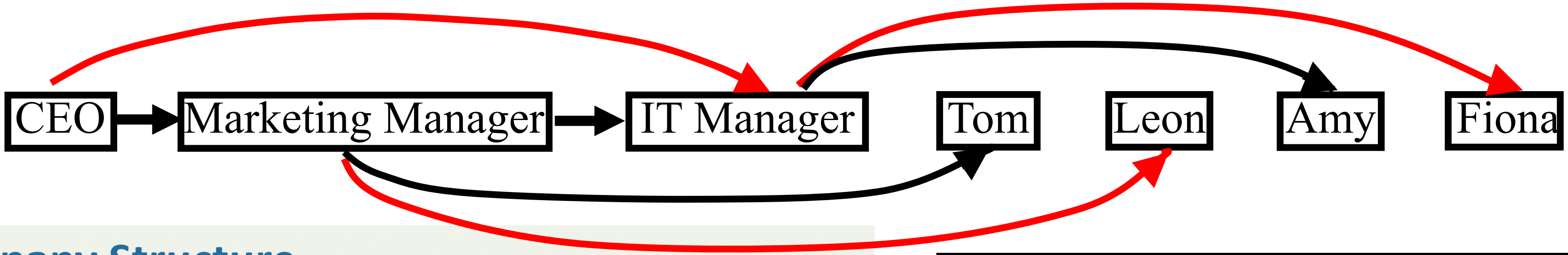


Directed Graphs: Problems

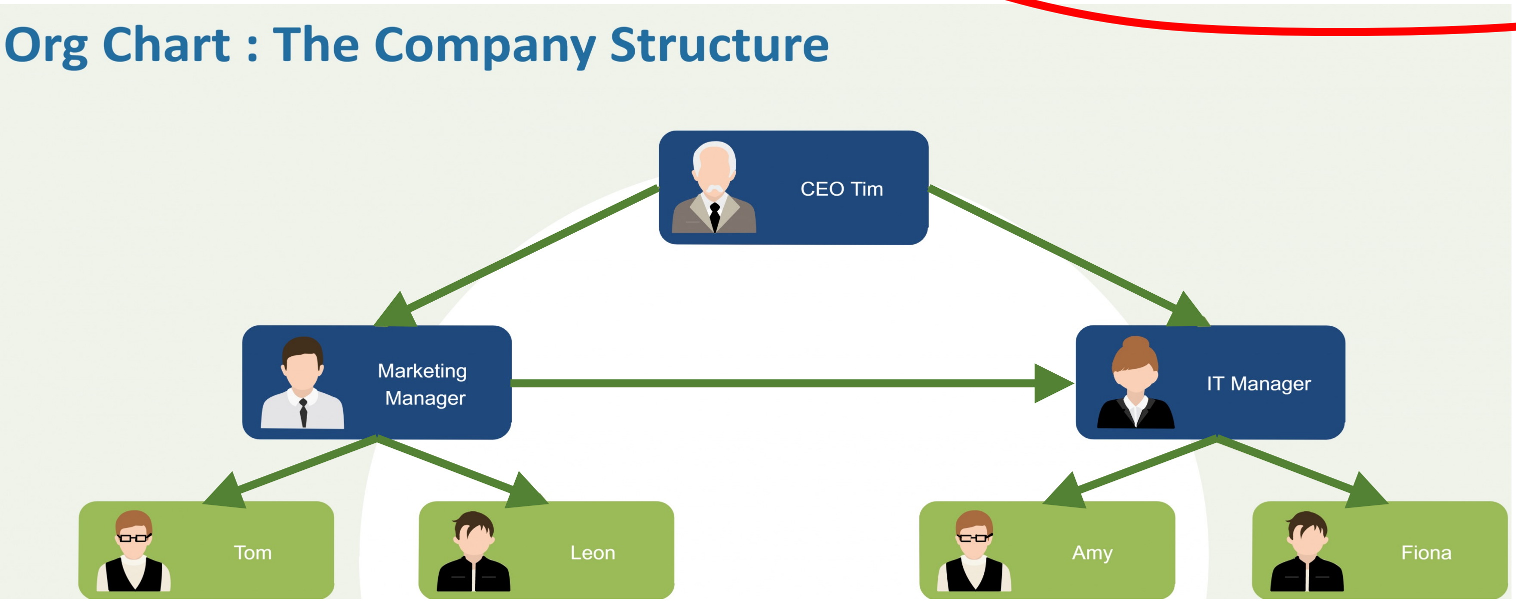
Prove: If G = (V, E) is a DAG, then G has a topological ordering.

- **A topological ordering of a directed graph G:**

A linear ordering of nodes s.t. for every directed edge $(u, v) \in G$, node u comes before v in the ordering.

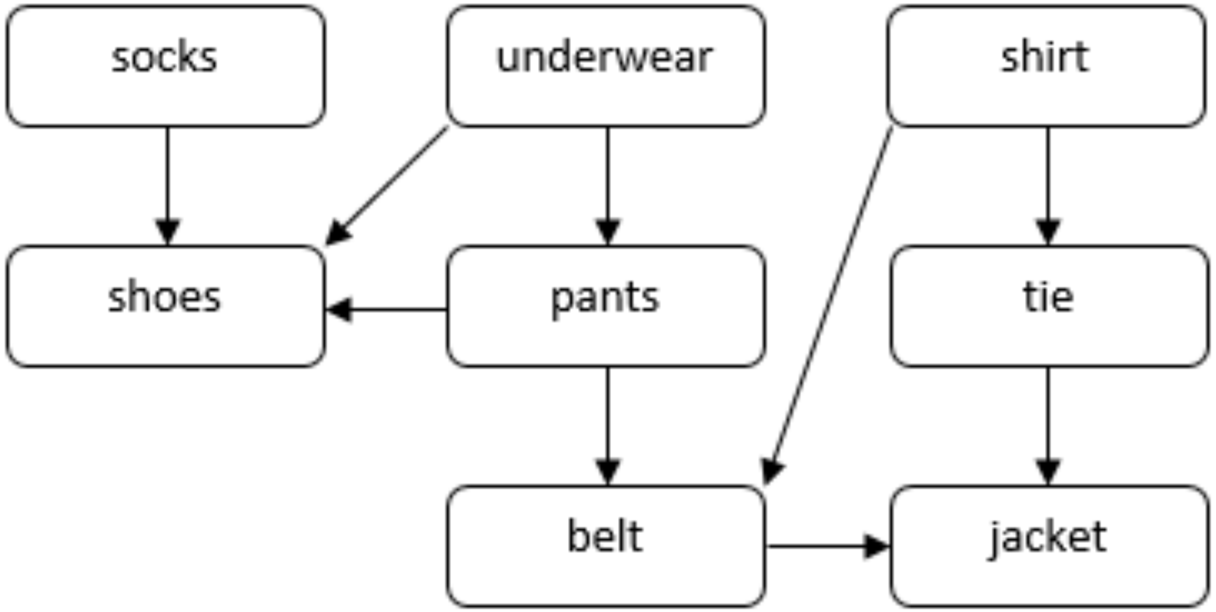


All edges are pointing to the right



Task Scheduling: Getting Dressed

Precedence Constraints



A Schedule

Directed Graphs: Problems

Prove: If G = (V, E) is a DAG, then G has a node with indegree 0.

Direct proof

Proof by Contradiction

Proof by Induction

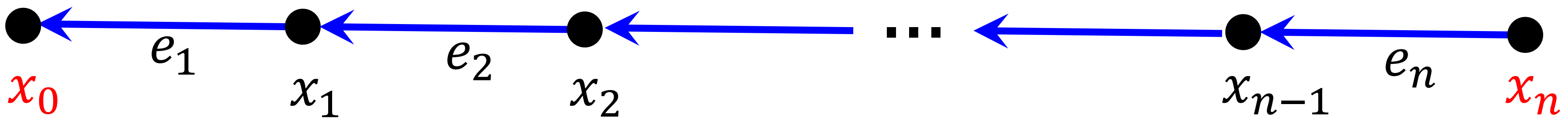
Prove: If G = (V, E) is a DAG, then G has a node with indegree 0.

Proof:

- Given G is a DAG, we assume G has **no** node with indegree 0, i.e., every node in G has indegree ≥ 1. Let $|V| = \boldsymbol{n}$.

Proof by Contradiction

- Then for **each node**, we can **move backwards** through an **incoming edge**.
- Pick any node x_0, if we walk backwards **n** times, it forms a backward walk $P = e_1, e_2, \dots, e_n$ s.t. $e_1 = (x_1, x_0), e_2 = (x_2, x_1), \dots, e_n = (x_n, x_{n-1},)$



- P passes through $\boldsymbol{n+1}$ nodes and there are only $\boldsymbol{n}$ distinct nodes,
- By the pigeonhole principle: P must have passed through some node **twice**.
- Thus, there is a cycle, which contradict the fact that G is a DAG.

Directed Graphs: Problems

Prove: If G = (V, E) is a DAG, then G has a topological ordering.

Proof by Induction on # of nodes $|V|$

- Base Case: If G is a DAG with 1 node, $G = (\{v\}, \emptyset)$, the topological ordering is v
- Inductive Hypothesis: Assume if G is a DAG with n nodes, G has a topological ordering.
- Inductive Step: When G' is a DAG with $n + 1$ nodes,

Using definitions/known theorems, IH

G' has a topological ordering.

Prove: If G = (V, E) is a DAG, then G has a topological ordering.

$\boxed{\text{Proof by Induction on \# of nodes } |V|}$

- Base Case: If G is a DAG with 1 node, $G = (\{v\}, \emptyset)$, the topological ordering is v
- Inductive Hypothesis: Assume if G is a DAG with n nodes, G has a topological ordering.
- Inductive Step: When G' is a DAG with $n + 1$ nodes,
  - Pick a node $v \in G'$ with no indegree 0.
  - $G'' = \text{G}' - \{v\}$ is a DAG since deleting nodes/edges does not create cycles.
  - By IH, the DAG G'' with n nodes has a topological ordering.
  - v, followed by the topological ordering of G'' is a topological ordering of G'

G' has a topological ordering.

Scheduling my required CS courses

101 115 284 135 382 385 392 396 496 334