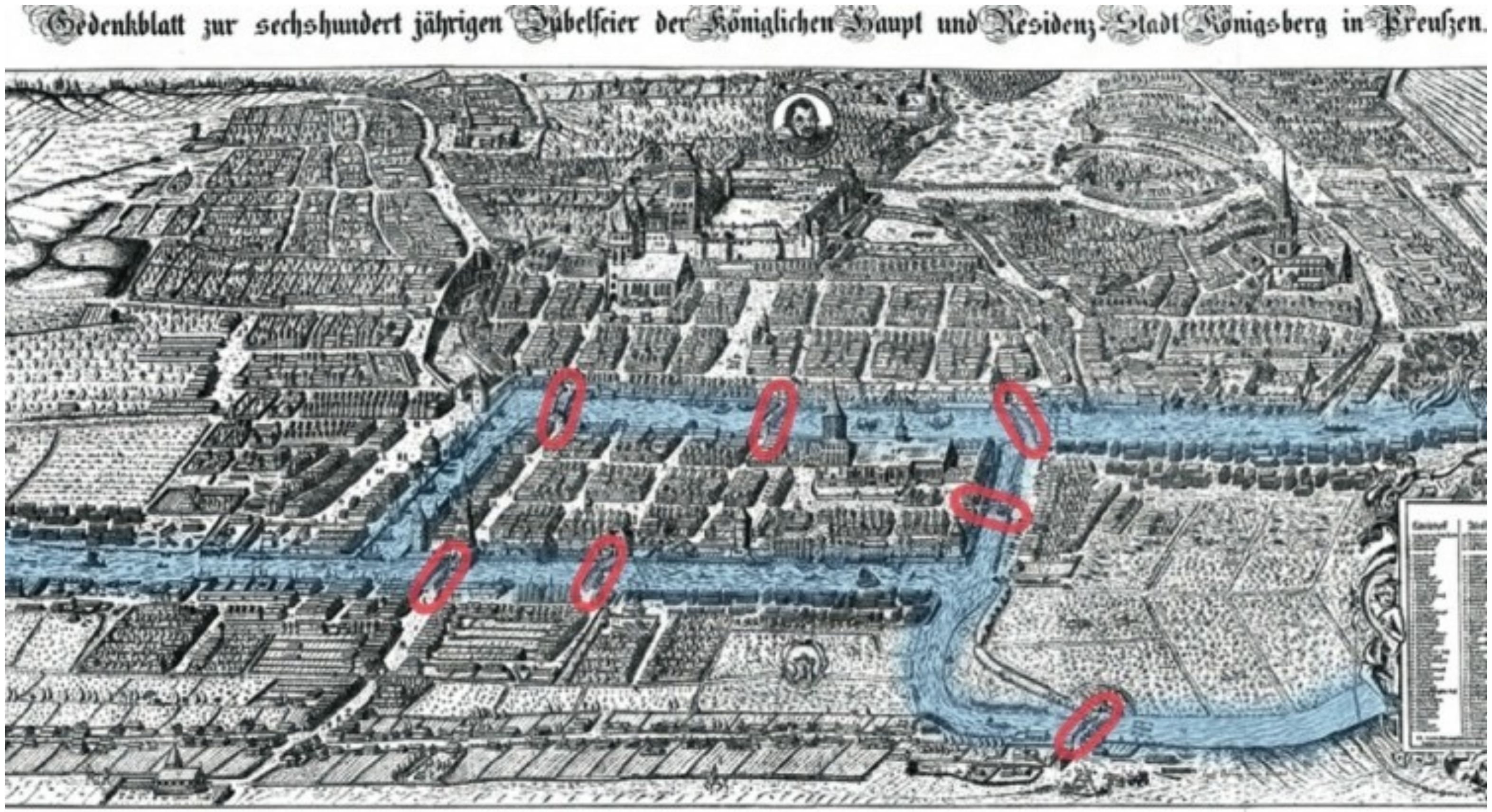
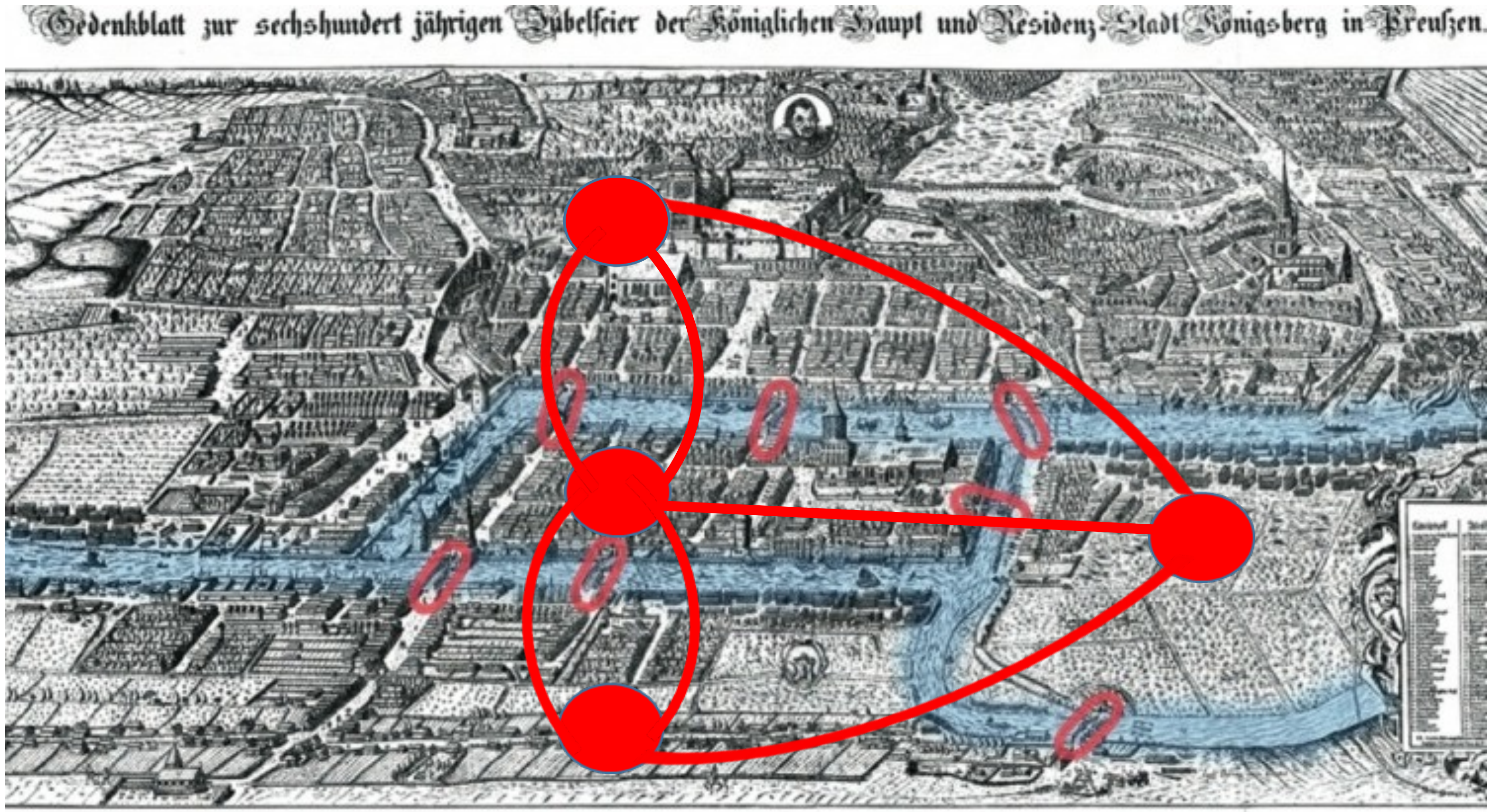
The Seven Bridges of Konigsberg



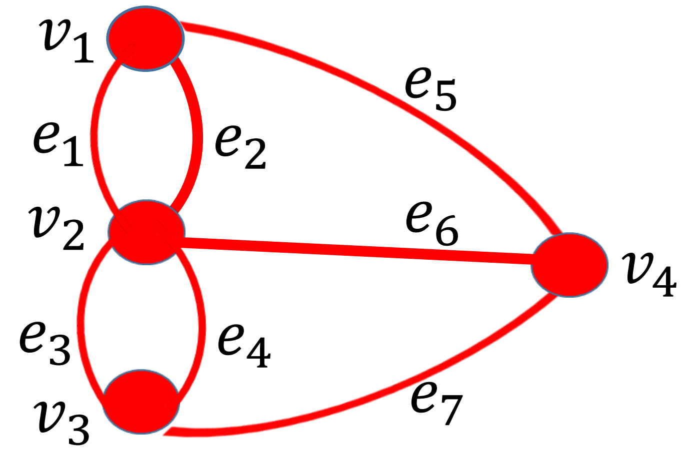
On a walking tour can a person cross each of the seven bridges exactly once?

Euler’s Graph Formulation



This *graph* models the land masses and the bridges connecting them.

Terminology



Vertex (node)

Edge (a set of two nodes)

An undirected graph $G(V, E)$ consists of a finite set V of vertices and a set of edges $E = \{e: e = \{a, b\}, a, b \in V\}$.

Wait a minute! This graph has two edges

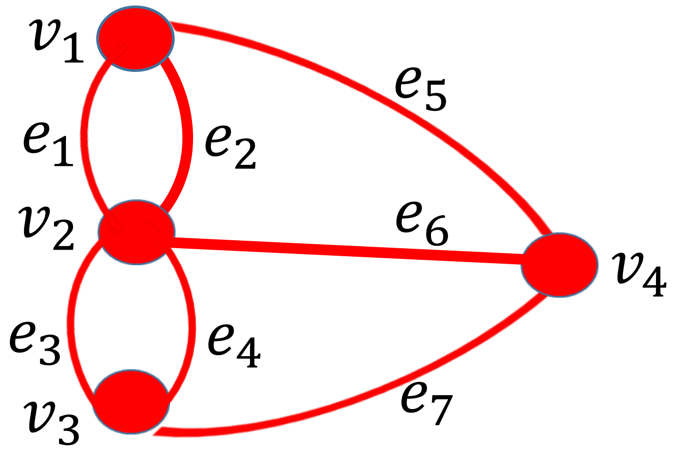
$$e_1 = \{v_1, v_2\},\ \ e_2 = \{v_1, v_2\}$$

These are identical … and yet different?

My bad ….. between any two vertices there's either 0 or 1 edge.
Also, no self-loops allowed.

But let it slide … only for this lecture!

Terminology



Vertex (node)

Edge (a set of two nodes)

Degree of a vertex
= # edges incident to it

Walk: A sequence of alternating vertices and edges that starts and ends in vertices in which the vertices before and after each edge are the two endpoints of that edge.
E.g. v_1, e_5, v_4, e_7, v_3 (Open walk)
v_1, e_2, v_2, e_2, v_1 (closed walk)

Trail: A walk in which no edge is repeated. v_1, e_1, v_2, e_6, v_4

Circuit: A closed walk in which no edge is repeated. $v_1, e_1, v_2, e_3, v_3, e_4, v_2, e_2, v_1$

Path: A trail in which no vertex is repeated. v_1, e_1, v_2, e_3, v_3

Cycle: A circuit of length ≥ 1 with the same first and last vertices and no repeated vertex.

$v_1, e_1, v_2, e_3, v_3, e_7, v_4, e_5, v_1$

Eulerian Trail/Circuit: A trail/circuit that traverses every edge exactly once.

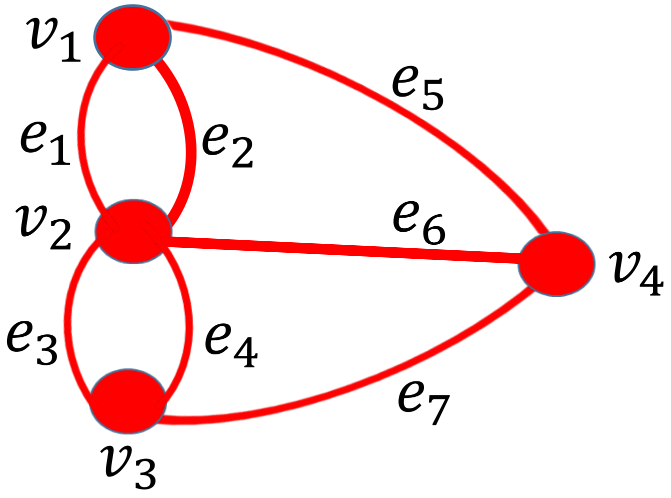
Does the graph above have an eulerian trail?

Property of an eulerian trail

Lemma: In an eulerian trail every vertex, other than possibly the start and end vertices, has even degree.

Proof: Consider any vertex other than the start or end vertices of the eulerian trail.

Each time an edge is used to enter the vertex, a different edge is used to exit the vertex
Therefore, the degree of the vertex is even.



Corollary: The Konigsberg graph has no vertex of even degree.
By the lemma, it does not have an eulerian trail.

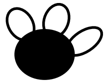
Eulerian Circuits

Lemma: If a graph has an eulerian circuit then every vertex has even degree.

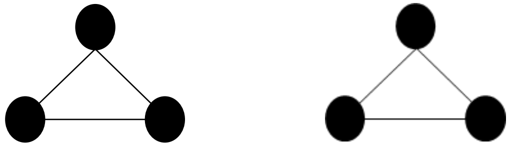
What about the converse? If every vertex has even degree does there exist an eulerian circuit?

How about this graph?

Degree = 6 (each edge is incident twice to the same vertex!)



What about



It’s not connected!

What if the graph is connected and every vertex has even degree – must it have an eulerian circuit?

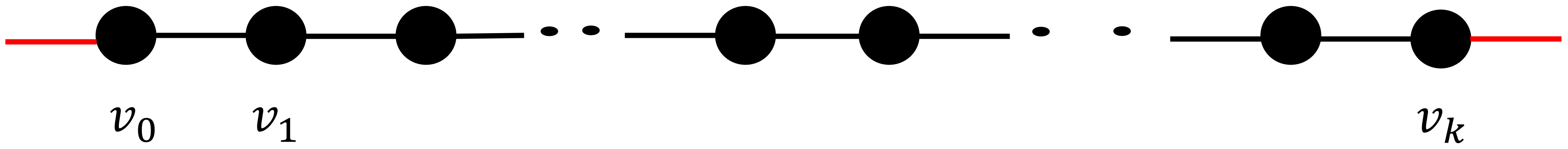
Euler's Theorem

Theorem: A connected graph has an eulerian circuit if and only if every vertex has even degree.

Proof: $\Rightarrow$ already done.

We'll prove $\Leftarrow$ next. We start with a graph G in which every vertex has even degree.

Let $p = v_0 v_1 \cdots \cdots v_k$ be a longest trail in the graph.



Question 1: Can an edge not on the longest trail connect to either v_0 or to v_k ?

NO!

FACT 1: Every edge incident to v_0 and to v_k is contained in the longest trail.

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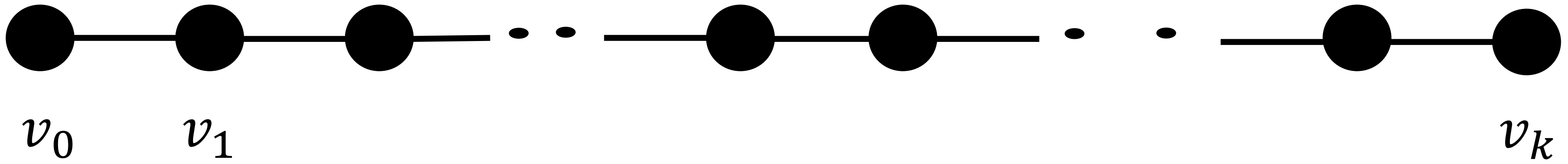

FACT 1: Every edge incident to v_0 and to v_k is contained in the longest trail.

Question 2: How many edges in the trail are incident to v_0 and to v_k ?

$1 + 2 * \big($# Occurrences of v_0 (v_k) along the trail, not counting the first (last)$\big)$.

This is an ODD number.

But the degrees of both vertices is even, and all incident edges are accounted for (FACT 1).

How do we explain this?

FACT 2: v_0 and v_k are the same vertex! The longest trail is a circuit!

Euler's Theorem

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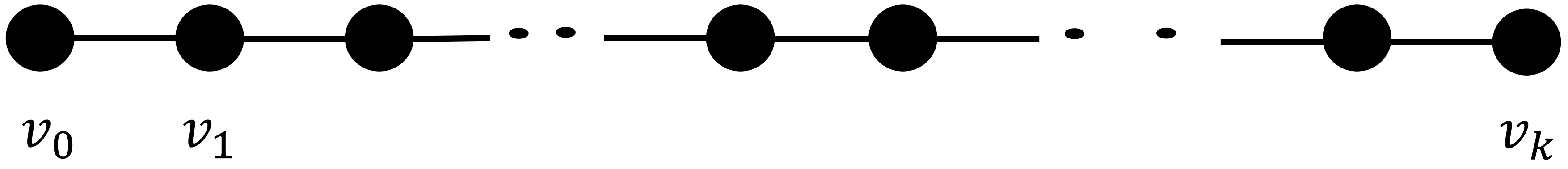


FACT 1: Every edge incident to v_0 and to v_k is contained in the longest trail.

FACT 2: v_0 and v_k are the same vertex! The longest trail is a circuit!

But is it an eulerian circuit?

Euler's Theorem … II

Let $p = v_0 v_1 \cdots \cdots v_k$ be a longest trail in the graph.
Fact 1: Every edge incident to v_k is contained in p
Fact 2: p is a circuit: $v_0 = v_k$

Claim: p is an eulerian circuit
Proof by contradiction!

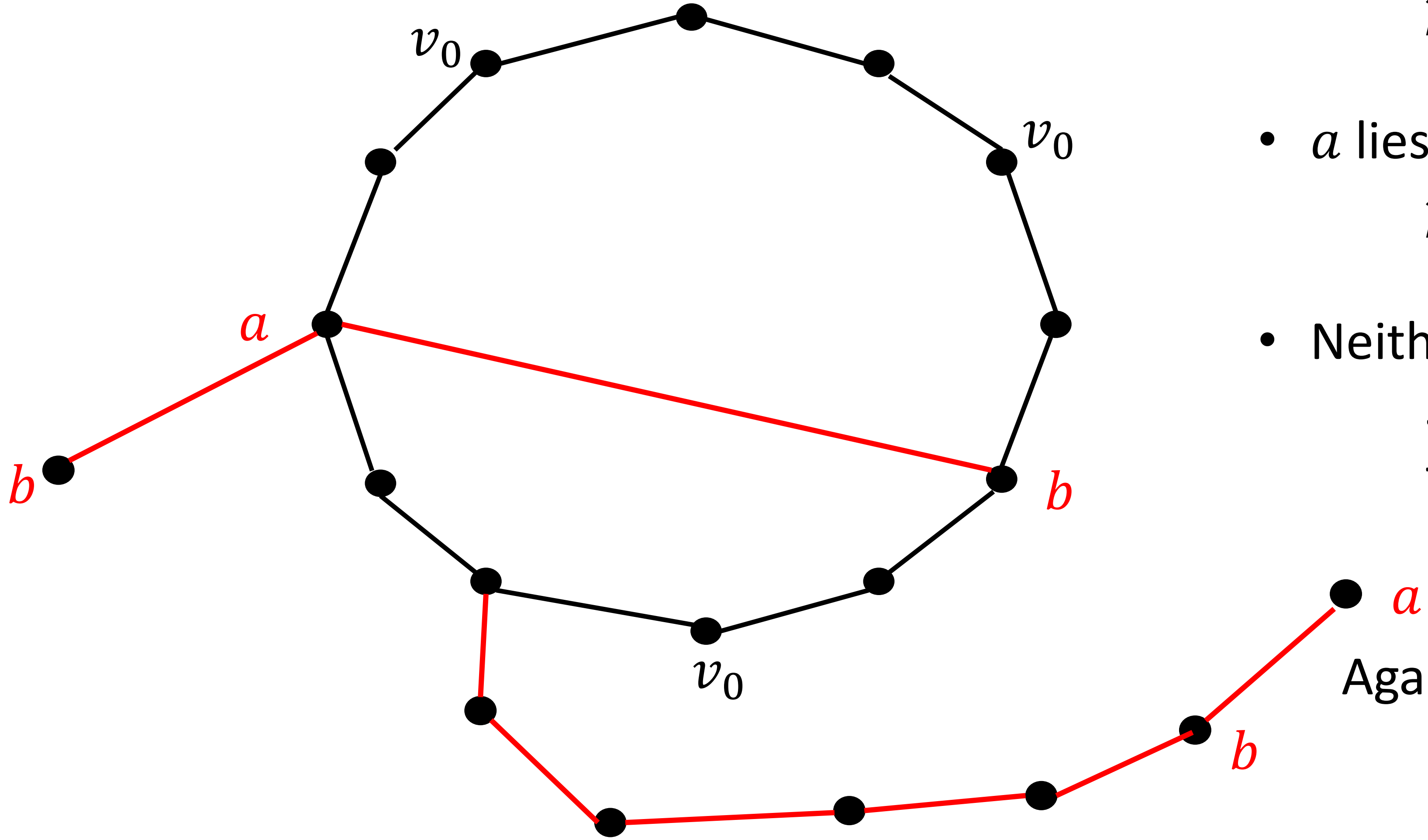
Assume that p is not an eulerian circuit, and let $\{a, b\}$ be an edge that is not contained in p.

Let's check every possible way to accommodate $\{a, b\}$.

Every choice will result in a contradiction!

Where can we place $\{a, b\}$?

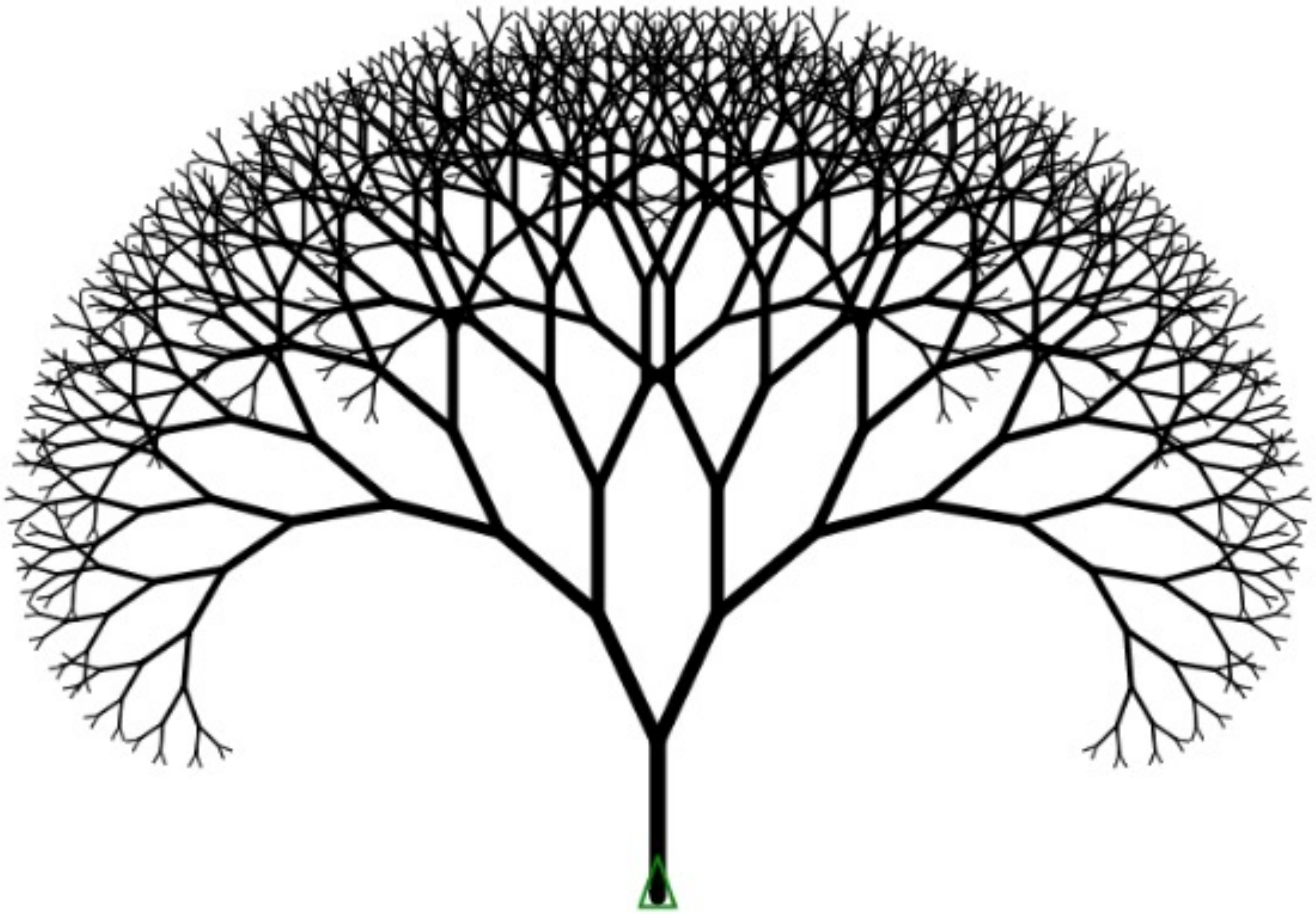
The longest trail p.



- a, b both lie on p.
  p is not the longest trail $><$
- a lies on p, but b does not
  p is not the longest trail $><$
- Neither a nor b lies on p
  Since the graph is connected,
  there is a path from a to p

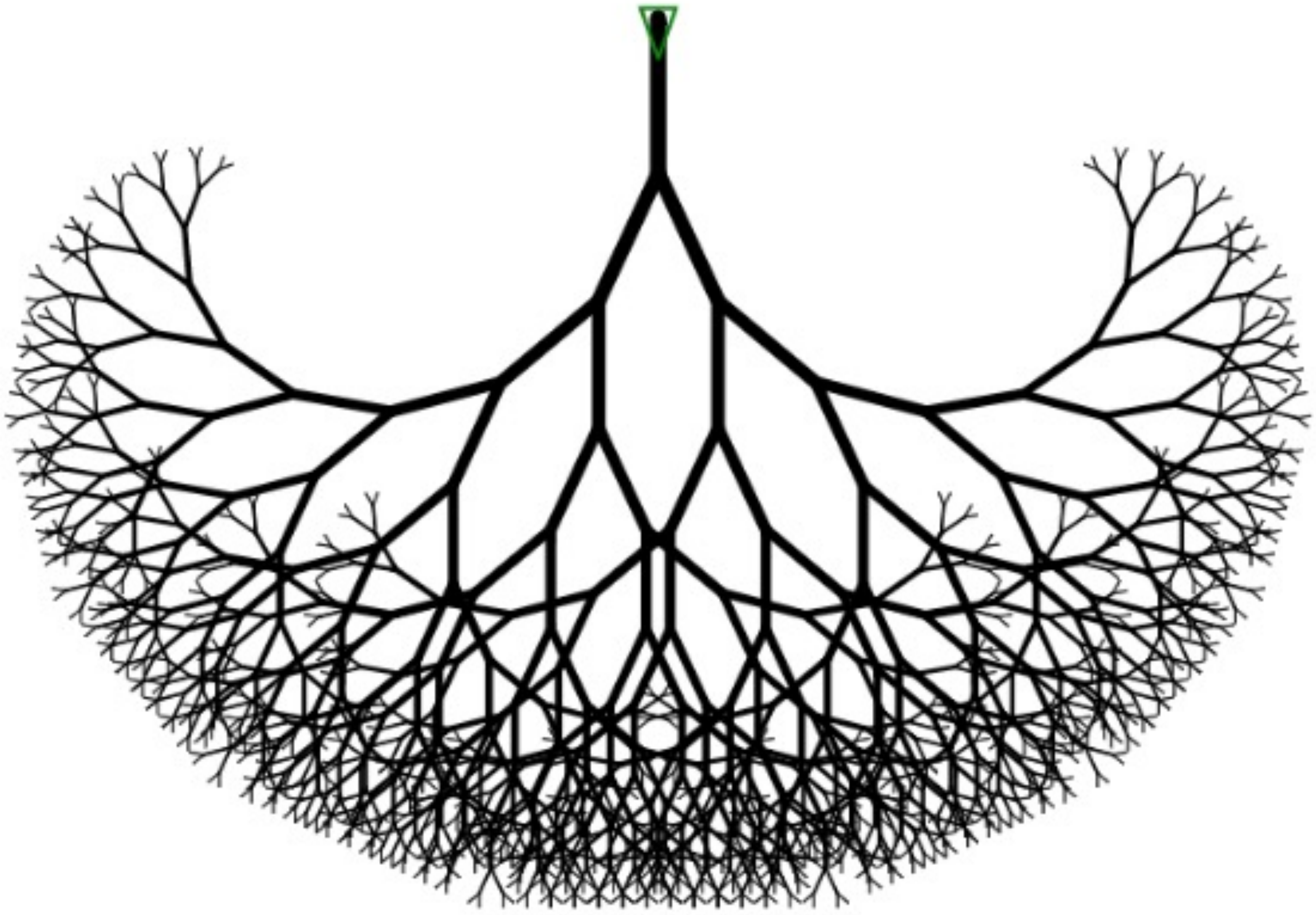
Again, p is not the longest trail $><$

Trees in Nature

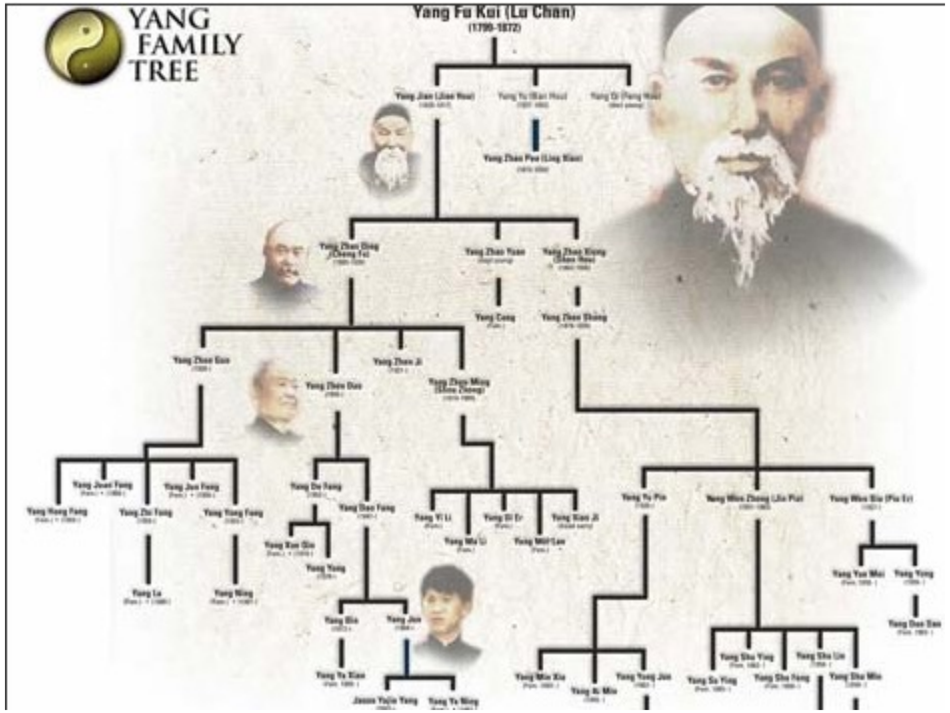


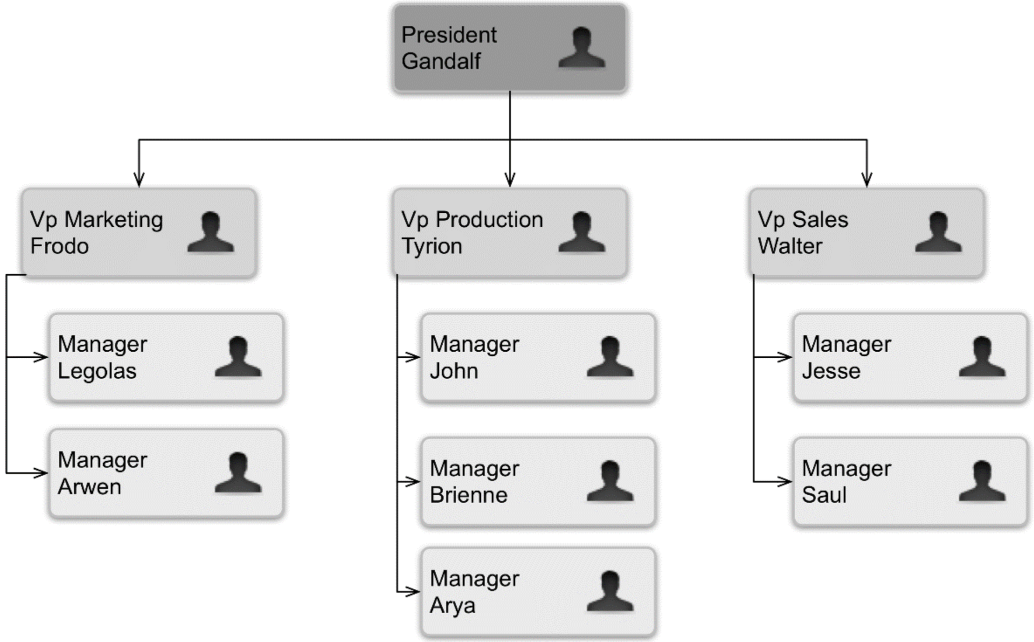
A complete binary tree in nature

Trees in Computer Science

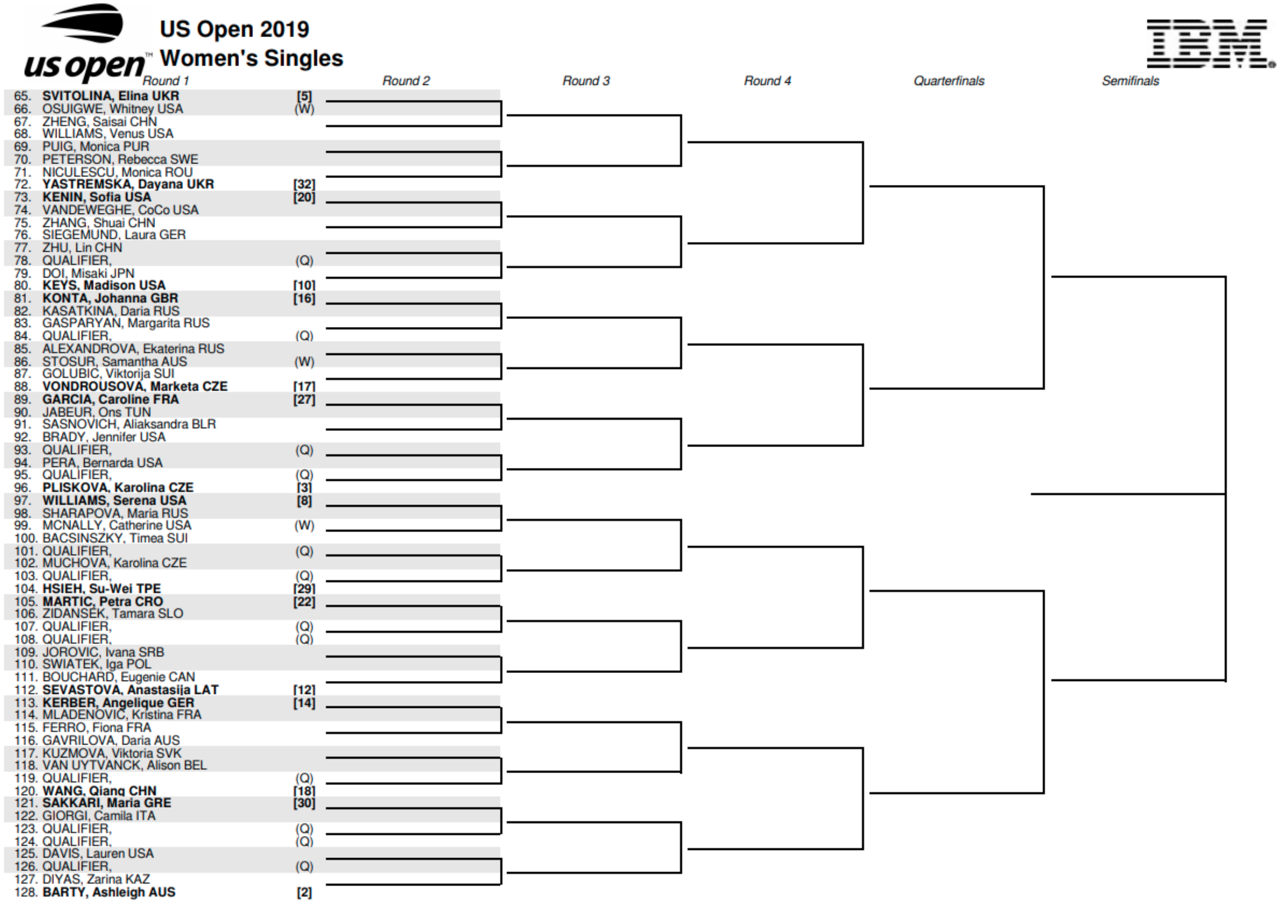


Family trees and Organization Charts



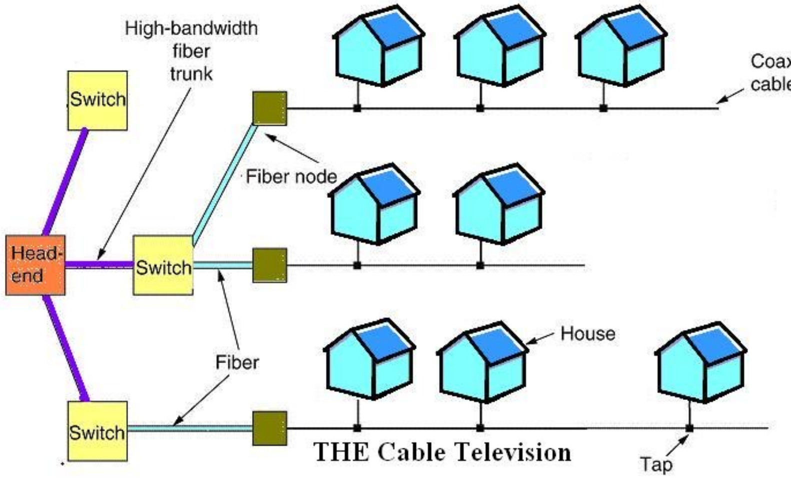


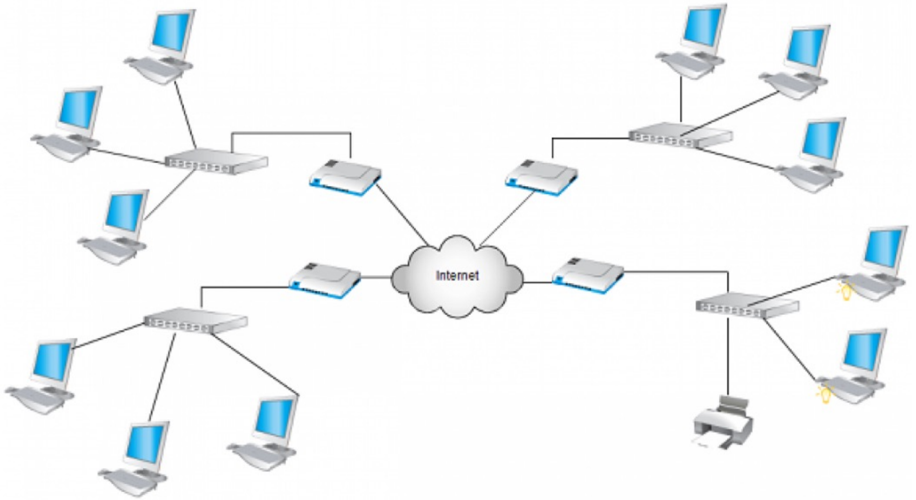
Tournament trees

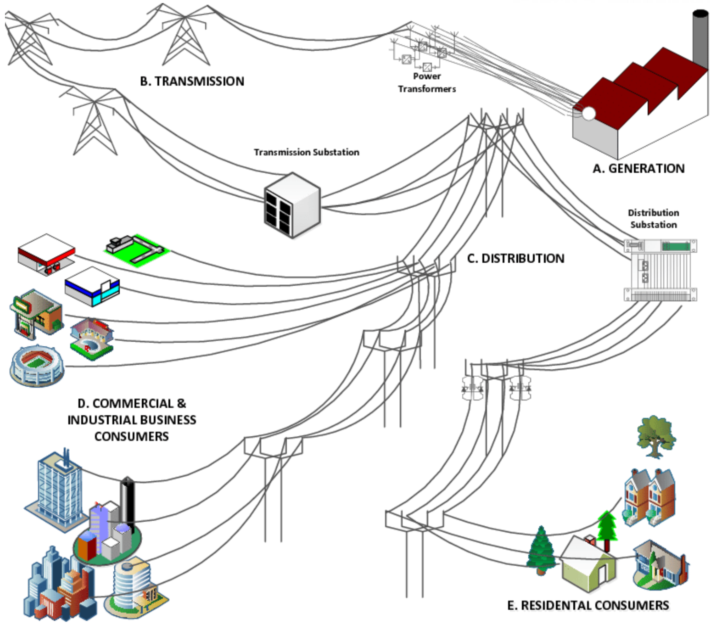


If there are 73 players, how many games must be played to determine the champion?

More man-made trees





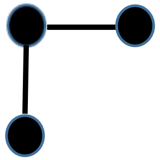


Trees

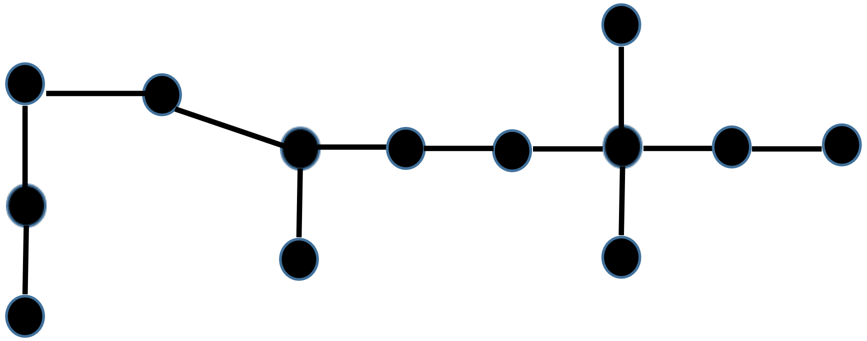
A connected acyclic graph is called a tree.

Tree with one vertex
and zero edges

n = 2
m = 1

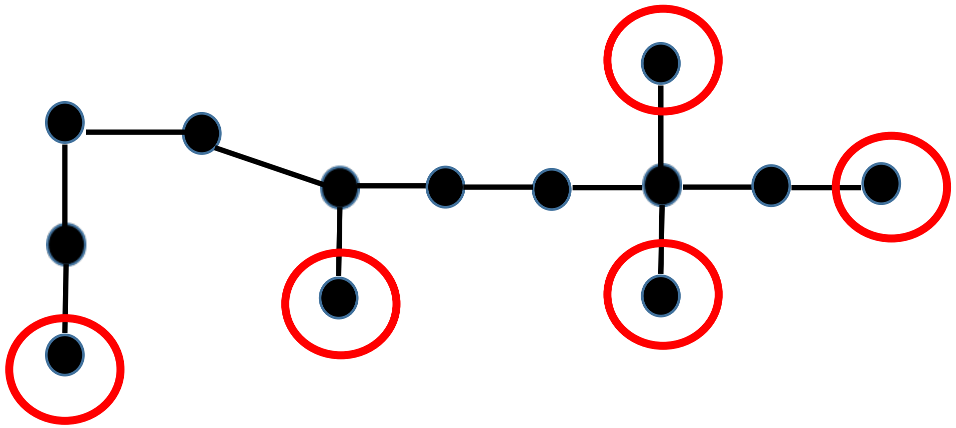


n = 3
m = 2



n = 13
m = 12

Leaves of a tree



A vertex with degree 1 is called a leaf of the tree.

Some simple observations

1. Every connected subgraph of a tree T is also a tree.

    If the subgraph has a cycle, then T must have a cycle. But T is acyclic!

2. There is a unique path between every pair of vertices.

    There must be one because the tree is a connected graph.

    Why can't there be two different paths between a pair of vertices?

More simple observations

3. Adding an edge between any two nonadjacent vertices in a tree creates a cycle.

    The new edge and the unique path connecting the vertices in the tree creates a cycle.

4. Removing any tree edge disconnects some pair of vertices.

    The edge that is removed was the unique path between the two end points.
    Removing it disconnects the end points.

Still more simple observations

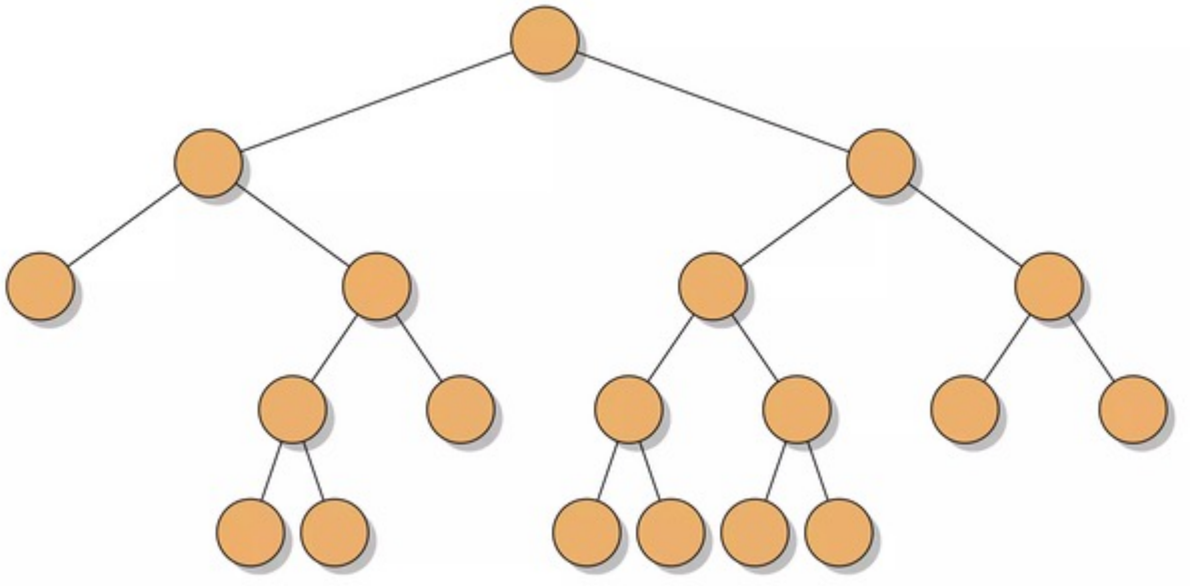
5. Every tree with at least two vertices contains at least two leaves.

   The end points of a longest path in the tree are both leaves!

6. Every tree with n vertices has n-1 edges.

   Proof by induction on number of vertices.

Full binary trees



Every vertex is either a leaf or has exactly 2 children.

Theorem:
If n = #leaves then # non-leaves = n-1

Lemma: Some two siblings are both leaves.

Prove theorem using lemma and induct on n.

Every tournament tree is a full binary tree.
So, a tournament with 73 players will take …….. games!