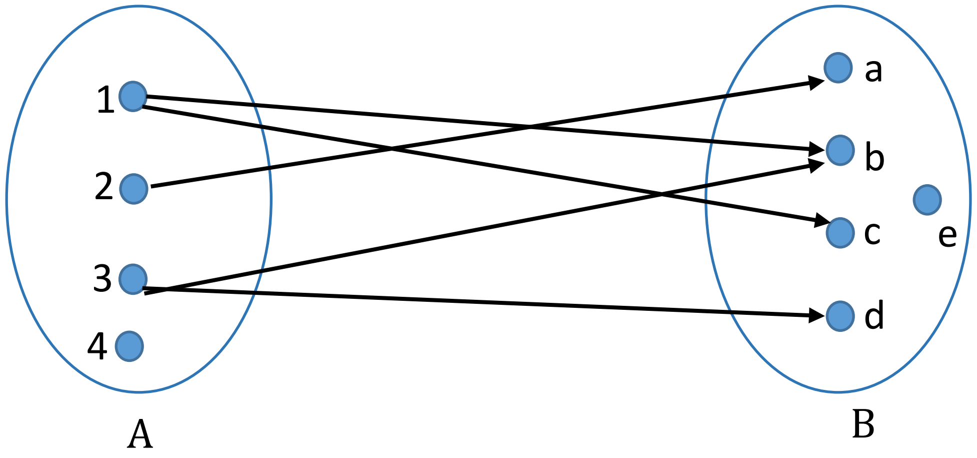
Relations

A relation R from a domain A to a target B is a subset of A x B.

Example:

R: {1,2,3,4 →{a,b,c,d,e}

R = {(1,b), (1,c), (2,a), (3,b), (3,d)}



Properties of Relations

A relation R over a set A is:

- ***Reflexive*** if $\forall x \in A: (x, x) \in R$

$$DIVIDES = \{(a, b): a, b \in \mathbb{N}^+ \ \wedge \ a|b\}$$

- ***Symmetric*** if $\forall x, y \in A: \ (x, y) \in R \iff (y, x) \in R$

$$CLOSEBY = \{(a, b): a, b \in \mathbb{N} \wedge |a - b| \leq 2\}$$

- ***Transitive*** if $\forall x, y, z \in A: \big((x, y) \in R \ \wedge (y, z) \in R\big) \Rightarrow (x, z) \in R$

$$DIVIDES = \{(a, b): a, b \in \mathbb{N} \ \wedge a|b\} \quad IMPLIES = \{(P, Q): P \Rightarrow Q\}$$

Equivalence Relations

A relation R over a set A that is reflexive, symmetric and transitive is called an ***equivalence*** relation.

Examples:

$$\{(P, Q): P \iff Q\}$$

$$\{(a, b): rem(a, 3) = rem(b, 3)\}$$

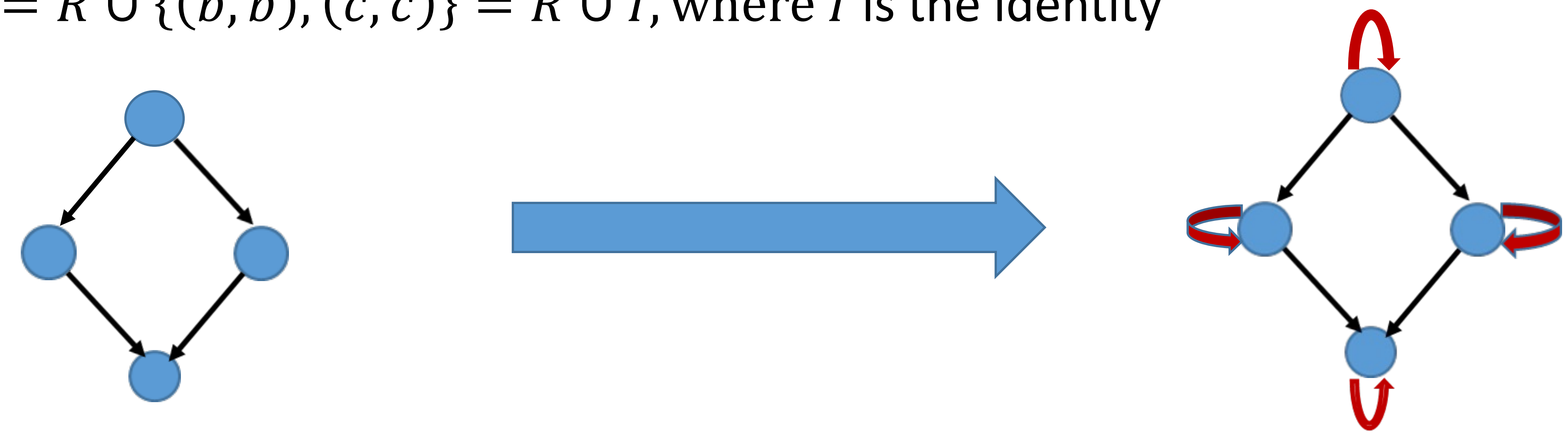
Reflexive Closure

The ***reflexive closure*** of relation R is the smallest reflexive relation $r(R)$: $r(R) \supseteq R$.

Example:

$$R = \{(a,a),(a,b),(b,c)\}$$

$r(R) = R \cup \{(b,b),(c,c)\} = R \cup I$, where I is the identity



Symmetric Closure

The ***symmetric closure*** of relation R is the smallest symmetric relation $s(R)$: $s(R) \supseteq R$.

Example:

$$R = \{(a,a),(a,b),(b,c)\}$$

$$s(R) = R \cup \{(b,a),(c,b)\} \quad = \quad R \cup R^{-} \quad \text{where } R^{-} \text{ is the inverse of } R$$

Transitive Closure

The ***transitive closure*** of relation R is the smallest transitive relation $R^+ \supseteq R$.

Example:

$$R = \{(a,a),(a,b),(b,c)\}$$

$$R^+ = R \cup \{(a,c)\}$$

Composing Relations

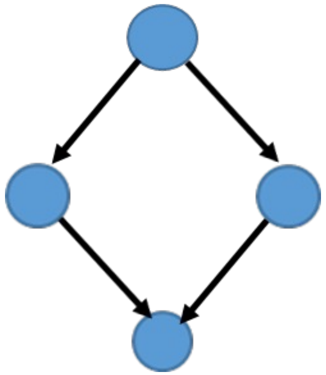
Given two relations $R: A \to B$, $\quad S: B \to C$

we define the composition

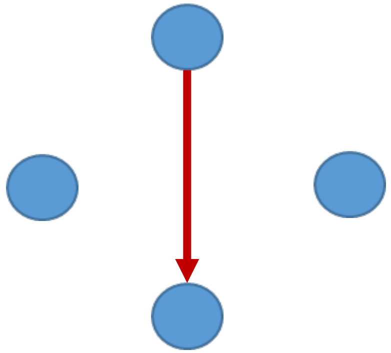
$S \circ R: A \to C$ as

$$\{(a, c): a \in A \ \wedge c \in C \wedge \exists b \in B: (a, b) \in R \wedge (b, c) \in S\}$$

If R is a relation over a set A then $R \circ R = \{(a, b): \exists x \in A \ (a, x) \in R \ \wedge (x, b) \in R\}$



R: direct flights



$R \circ R$: one-stop flights

Composing Relations

If R is a relation over a set A then $R \circ R = \{(a, b) \colon \exists x \in A \; (a, x) \in R \; \wedge (x, b) \in R\}$

$$R \circ (R \circ R) = \{(a, b) \colon \exists x, y \in A \; (a, x) \in R \; \wedge (x, y) \in R \wedge (y, b) \in R\}$$

R: direct flights

$R \circ R = R^2$: one-stop flights

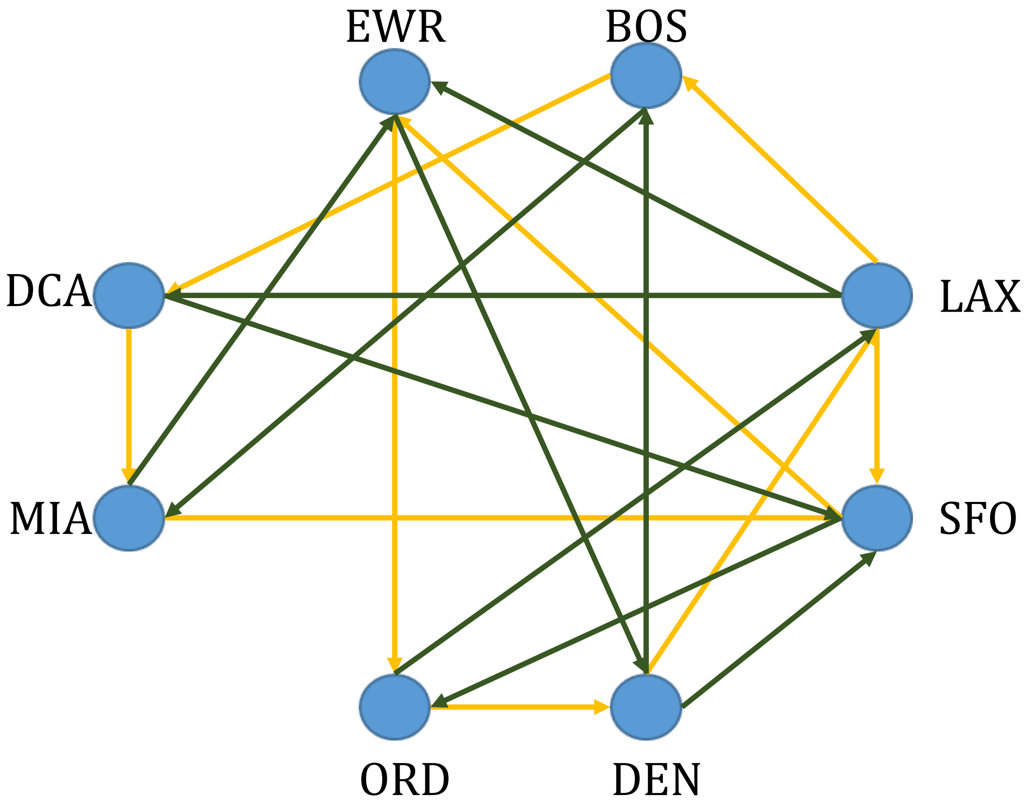
$R \circ R \circ R = R^3$: two-stop flights

In general: $(a, b) \in R^k$ iff there is a sequence of k flights from a to b.

Our little airline

$$A = \{\text{EWR}, \text{BOS}, \text{DCA}, \text{LAX}, \text{SFO}, \text{ORD}, \text{DEN}, \text{MIA}\}$$

$$FLIGHTS = \{(\text{EWR}, \text{ORD}), (\text{BOS}, \text{DCA}), (\text{LAX}, \text{SFO}), (\text{DEN}, \text{LAX}), (\text{DCA}, \text{MIA}), (\text{SFO}, \text{EWR}), (\text{ORD}, \text{DEN}), (\text{LAX}, \text{BOS}), (\text{MIA}, \text{SFO})\}$$



$\boldsymbol{R}$

$\boldsymbol{R \circ R}$

Composing Relations

Suppose A consists of n cities and that one can fly (directly or indirectly) from a to b

Then there is a sequence of k flights where $1 \leq k \leq n$. (Why not $n-1$?)

In other words, $(a, b) \in R \cup R^2 \cup R^3 \cup \cdots \cup R^n$

Theorem: For any relation R over a set A, $|A| = n$,

$$R^+ = R \cup R^2 \cup R^3 \cup \cdots \cup R^n$$

Corollary: If R is reflexive then $R^+ = R^n$

since $R \subseteq R^2 \subseteq R^3 \cdots \subseteq R^n$

Functions

A function f from a domain A to a target B is a relation such that:

$$\forall x \in A\ \exists b \in B: f(x) = b$$
$$\land \quad \forall x \in A, \forall b_1 \in B, \forall b_2 \in B: (f(x) = b_1\ \land f(x) = b_2) \Rightarrow b_1 = b_2$$

"Every domain element is mapped to exactly one element in the target."

Example: $Domain = \mathbb{N}, Target = \mathbb{N}$

$$f(x) = x^2$$

Example: $Domain = \mathbb{N}, Target = \mathbb{R}$

$$f(x) = x^2$$

Example: $f(x) = \sqrt{x}$

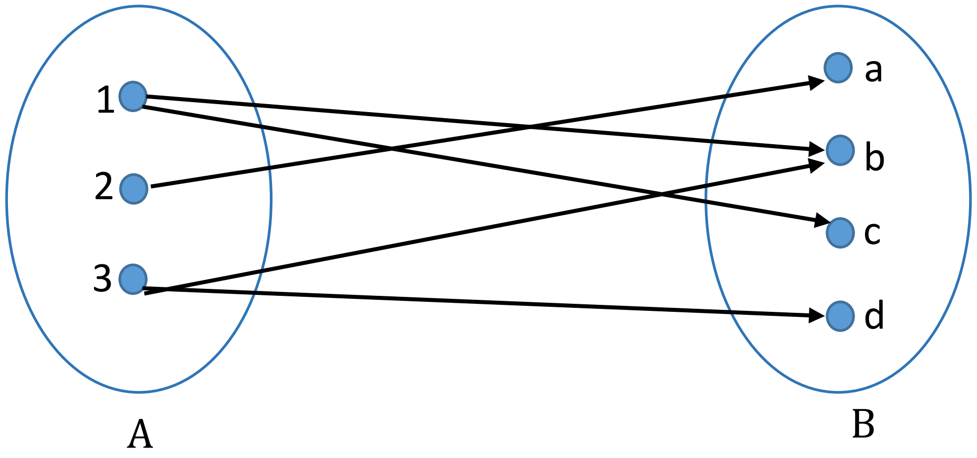
$$Domain = \mathbb{N}, Target = \boldsymbol{non-negative\ reals}$$

$$Domain = \mathbb{N}, Target = \mathbb{R}$$

Functions

$$f\colon \{1,2,3\} \to \{a,b,c,d\}$$
$$f = \{(1,b),(1,c),(2,a),(3,b),(3,d)\}$$

Is f a function?



Types of Functions

Definiton 1: A function *f:A→B is* *one-to-one* (also called *injective*) if

$$\forall x_1, x_2 \in A\colon\ (x_1 \neq x_2)\ \Rightarrow f(x_1) \neq f(x_2)$$

"every domain element is mapped to a unique element in the target."

Definition 2: A function *f:A→B is* *onto* (also called *surjective*) if

$$\forall y \in B\ \exists x \in A\colon f(x) = y$$

"every element in the target is the target of at least one domain element."

Definition 3: A function *f:A→B* is a one-to-one correspondence (also called bijective) if *f* is both injective and surjective.

"every domain element is matched with exactly one element in the target, and vice versa."

Examples

Let $f: \mathbb{N} \to \mathbb{N}$

$$f(x) = x^2$$

one-to-one but not onto

$$f(x) = x^2 - 1$$

not a function!

$$f(x) = (x-1)^2$$

not one-to-one, not onto!

$$f(x) = x$$

one-to-one and onto

$$f(0) = 0, \qquad \forall x > 0\colon\ f(x) = x - 1$$

not one-to-one, but onto

The Pigeonhole Principle

If $k + 1$ pigeons occupy k pigeonholes, then at least two pigeons share a pigeonhole.

No function from a domain of size $k + 1$ to a target of size k is injective.

The well-ordering principle

Every non-empty subset of $\mathbb{N}$ *has a least element.*

Theorem:

The pigeonhole principle is logically equivalent to the well-ordering principle.

(We'll see a proof of this later!)

Proof Techniques

To prove that proposition P is a tautology:

A. Direct method: Show that P follows logically from known true statements.

Establish that the implication $(True \rightarrow P)$ is true.

B. Proof by contradiction: Show that if P is false, then so it True!

Establish that the implication $(\neg P \rightarrow False)$ is true.

Example: Proof by Contradiction

Theorem. $\forall n \in \mathbb{N}$: $(n > 1\ and\ n\ is\ not\ prime) \rightarrow n\ can\ be\ factored\ as\ a\ product\ of\ primes$.

Proof. (By contradiction.) Suppose the statement is false, and that counterexamples exist.

Let C be the non-empty set of counterexamples

(numbers that are not prime and cannot be factored into primes).

Then, by the WOP, C has a least element. Let's call it m.

m is not prime and $m > 1$ and m cannot be factored as a product of primes.

Since m is not prime, $m = a \cdot b$ where $1 < a, b < m$.

a, b are not in C: *(because m is the smallest element in C)*

$$a = p_1 . p_2 \dots p_k\ and\ b = q_1 . q_2 \dots q_l,\ \text{where } \forall i, j\ p_i\ and\ q_j\ are\ primes.$$

So, $m = p_1 . p_2 \dots p_k . q_1 . q_2 \dots q_l$

But then, $m \notin C$, a contradiction!

This contradicts the assumption that C is non-empty.

Therefore, C is empty.

Which is bigger?

1. {1, 2, 3} or {Alice, Bob, Charlie}
2. {10, 20, 25} or {234, 567}
3. {10, 20, 25} or {10, 20, 250}
4. {0, 1, 2, …} or {1, 2, 3, …}
5. {0, 2, 3, …} or {1, 2, 3, …}

What does it even mean for two sets to be equal in size?