Quantified Propositions

$\forall x \in A\colon\ P(x)$

"for every element x in the domain A, the proposition $P(x)$ is true"

When A is implicit, we write $\forall x\colon P(x)$.

Similarly, $\exists x \in A\colon P(x)$

"there is at least one element in A for which $P(\cdot)$ is true"

(there may be multiple such elements)

When A is implicit, we write $\exists x\colon P(x)$.

Examples

Everybody who lives in the dorm has a roommate who is not their friend.

$$D(x) \;:\; x \textit{ lives in the dorm}$$
$$F(x,y):\; x \textit{ and } y \textit{ are friends}$$
$$R(x,y):\; x \textit{ and } y \textit{ are roommates}$$

Step 1: $x \textit{ has a roommate who is not a friend}$

$$\exists y: R(x,y) \land \neg F(x,y)$$

Step 2: $\textit{The above is true for all } x \textit{ who lives in the dorm}$

$$\forall x: D(x) \rightarrow (\exists y: R(x,y) \land \neg F(x,y))$$

Examples

If anyone who lives in the dorm has a friend who has Covid then everyone in the dorm is quarantined.

$D(x): x\ lives\ in\ the\ dorm$ $\quad$ $F(x,y):\ \ x\ and\ y\ are\ friends$

$Q(x): x\ is\ quarantined$ $\quad$ $C(x)\ \ :\ \ x\ is\ infected\ with\ Covid$

Someone who lives in the dorm has a friend who has Covid

$$\exists x: [D(x) \wedge \exists y\ (F(x,y) \wedge C(y))]$$

Everyone in the dorm is quarantined

$$\forall x: (D(x) \to Q(x))$$

$$\exists x: \left[D(x) \wedge \exists y\ \left(F(x,y) \wedge C(y)\right)\right] \to\ \ \forall x: \left(D(x) \to Q(x)\right)$$

The x in the antecedent is bound to the ∃ but in the consequent it is bound to ∀.

$$\exists x: \left[D(x) \wedge \exists y\ \left(F(x,y) \wedge C(y)\right)\right] \to\ \ \forall z: \left(D(z) \to Q(z)\right)$$

Each variable is bound to its nearest enclosing quantifier.

More Examples

$$L(x, y): \;\; x \; loves \; y$$

Domain: Set of all people

At least one person loves Layla.

$$\exists x: L(x, Layla)$$

At most one person loves Layla.

$$\exists x: Loves(x, Layla) \rightarrow \forall y: \left(y \neq x \rightarrow \; \neg Loves(y, Layla)\right)$$

Exactly one person loves Layla.

$$\exists x: Loves(x, Layla) \wedge \; \forall y: \left(y \neq x \rightarrow \; \neg Loves(y, Layla)\right)$$

Exactly two people love Layla.

$$\exists \; x, y: (x \neq y) \wedge Loves(x, Layla) \wedge Loves(y, Layla)$$

$$\wedge \, \forall z: \left((z \neq x \wedge z \neq y) \rightarrow \neg Loves(z, Layla)\right)$$

De Morgan's Laws for Quantifiers

$$\neg\forall x:\ P(x) \quad \equiv \quad \exists x:\ \neg P(x)$$

$$\neg\exists x:\ P(x) \quad \equiv \quad \forall x:\ \neg P(x)$$

What is the truth value of $\exists x \in A: P(x)$ when $A = \phi$?

Is there any element in ϕ which, when plugged in for x will make $P(x)$ true?

NO! There is no element in ϕ to plug in for x.

So, the quantified proposition $\exists x \in A: P(x)$ is FALSE when $A = \phi$.

De Morgan's Laws for Quantifiers

$$\neg\forall x:\ P(x) \quad \equiv \quad \exists x:\ \neg P(x)$$

$$\neg\exists x:\ P(x) \quad \equiv \quad \forall x:\ \neg P(x)$$

What is the truth value of $\exists x \in A: \neg P(x)$ when $A = \phi$?

Is there any element in ϕ which, when plugged in for x will make $\neg P(x)$ true?

NO! There is no element in ϕ to plug in for x.

So, the quantified proposition $\exists x \in A: \neg P(x)$ is FALSE when $A = \phi$.

Both $\exists x \in A: P(x)$ and $\exists x \in A: \neg P(x)$ are FALSE when $A = \phi$.

Somewhat less intuitively …

What is the truth value of $\forall x \in A: P(x)$ when $A = \phi$?

If any element of ϕ is plugged in for x will $P(x)$ be true?

But there isn't any element in ϕ, so what does this even mean?

$$\begin{aligned} \forall x \in A: P(x) &\equiv \neg\neg\forall x: P(x) \\ &\equiv \neg(\exists x: \neg P(x)) \\ &\equiv \neg F \\ &\equiv T \end{aligned}$$

When $A = \phi, \forall x: P(x)\ is\ TRUE!$

We say that $\forall x: P(x)$ is VACUOUSLY TRUE when $A = \phi$.

Rules of Inference with Quantifiers

Suppose Domain $\mathbb{N} = \{0,1,2,\dots\}$

Universal Instantiation:

$$\frac{\forall x: P(x)}{\therefore P(77)}$$

Universal Generalization:

$$\frac{x \in \mathbb{N} \Rightarrow P(x)}{\therefore \forall x: P(x)}$$

Existential Instantiation:

$$\frac{\exists x: P(x)}{\therefore c \in \mathbb{N} \ \wedge P(c)}$$

Existential Generalization:

$$\frac{P(a)}{\therefore \exists x: P(x)}$$

Proving Equivalence

A: $\exists x\,\big(P(x) \vee Q(x)\big)$ B: $\exists x P(x) \vee \exists y Q(y)$

Prove that $A \iff B$

Prove $A \Rightarrow B$

$\exists x\,\big(P(x) \vee Q(x)\big)$

$\Rightarrow P(a) \vee Q(a)$

existential instantiation

$\Rightarrow \exists x\, P(x) \vee \exists y\, Q(y)$

existential generalization

Prove $B \Rightarrow A$

$\exists x P(x) \vee \exists y Q(y)$

$\Rightarrow P(a) \vee Q(b)$

existential instantiation

$\Rightarrow \big(P(a) \vee Q(a)\big) \vee \big(P(b) \vee Q(b)\big)$

addition

$\Rightarrow \exists x\text{:}\ \big(P(x) \vee Q(x)\big)$

existential generalization

Inferences with quantified propositions

All the world loves a lover
Majnu loves Layla

∴ Archie loves Betty

1. $\forall x\ \forall y\ \left(\exists z\colon\ L(y, z) \rightarrow\ L(x, y)\right)$
2. $L(Majnu, Layla)$
3. $\forall x\ L(x, Majnu)$ 1,2 Instantiate Majnu for y and Layla for z and simplify
4. $L(Betty, Majnu)$ 1,3 Universal Instantiation
5. $\forall x\ L(x, Betty)$ 1,4 Instantiate Betty for y and Majnu for z and simplify
6. $L(Archie, Betty)$ 1,5 Universal Instantiation

Logical Deductions

Everybody loves my baby
<u>My baby loves nobody but me</u>

∴ I am my baby

1. $\forall x\, L(x, baby)$ Hypothesis
2. $\forall x\, (L(baby, x) \Rightarrow E(x, me))$ $E(x, y)\colon\ x = y$
3. $\neg E(baby, me)$ negate the conclusion
4. $L(baby, baby)$ Universal instantiation, 1
5. $L(baby, baby) \Rightarrow E(baby, me)$ Universal Instantiation, 2

$\neg L(baby, baby)$ $E(baby, me)$

Another Example

Everyone has a parent

∴ Everyone has a grandparent

$\forall x\, \exists y\colon P(x,y)$ (parent of x is y)

$\forall x\, \exists y\, \exists z\colon P(x,y) \land P(y,z)$ (every x has some grandparent z)

$\neg \forall x\, \exists y\, \exists z\colon P(x,y) \land P(y,z)$ (negate the conclusion)

$\exists x\, \neg \exists y\, \exists z\colon P(x,y) \land P(y,z)$ (1)

$\neg \exists y\, \exists z\colon P(a,y) \land P(y,z)$ (2, Existential Instantiation)

$\exists y : P(a,y)$ (hypothesis)

$P(a,b)$ (Existential Instantiation)

$\exists y : P(b,y)$ (hypothesis)

$P(b,c)$ (Existential Instantiation)

$\forall y\, \neg \exists z\colon P(a,y) \land P(y,z)$ (3, from 2)

$\neg \exists z\colon P(a,b) \land P(b,z)$ (Universal Instantiation)

$\forall z\colon \neg\big(P(a,b) \land P(b,z)\big)$

$\neg(P(a,b) \land P(b,c))$

$\neg P(a,b) \lor \neg P(b,c)$

Gödel's Incompleteness Theorem (1931)

There are quantified propositions (over $\mathbb{N}$) that are tautologies but for which there is no proof

(Mathematics is incomplete)

OR

There are proofs for statements (over $\mathbb{N}$) that are false!

(Mathematics is inconsistent)