Naïve Set Theory

- A set is an unordered collection of objects, called members or elements of the set.
  $x \in S$ represents the proposition "x is a member of S."
  $x \notin S \equiv \neg(x \in S)$ (x is not a member of S).
- A set can be an element of another set.
- The empty set ϕ contains no element.
- No set can contain itself as a member, either directly or indirectly.

Subsets

$A \subseteq B$ means that every member of A is also a member of B

or $\forall x\colon (x \in A \Rightarrow x \in B)$

$A \subset B$ means that every member of A is a member of B, and B has members that are not members of A

or $\forall x\colon (x \in A \Rightarrow x \in B) \land (\exists x\colon x \in B \land x \notin A)$

Set Notation

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$**:** sets of natural numbers, integers, rationals, real numbers

Sets can be represented by:

- Listing elements in the set {1, 2, 3}
- By a predicate that describes properties of elements (Set builder notation)

$$\{x \colon P(x)\}$$

$$\{x \in \mathbb{N} : \exists y \in \mathbb{N}, x = 2y\}$$

This is the set of even numbers.

Operations on Sets

Set Union: $A \cup B = \{x\colon (x \in A) \vee (x \in B)\}$

Intersection: $A \cap B = \{x\colon (x \in A) \wedge (x \in B)\}$

Difference: $A - B = \{x\colon (x \in A) \wedge (x \notin B)\}$

Complement (with respect to a universe U of elements):

$$\bar{A} = U - A = \{x\colon (x \in U) \wedge (x \notin A)\}$$

Cartesian Product: $A \times B = \{(a, b) : (a \in A) \wedge (b \in B)\}$

Example: $\{1,2\} \times \{a, b, c\} = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

Note: $(1, a) \neq (a, 1)$!

Power Sets

The power set $P(S)$ of a set S is defined as:

$$P(S) = \{X : X \subseteq S\}$$

"The set of all subsets of S"

$$P(\{1,2\}) = \left\{\emptyset, \{1\}, \{2\}, \{1,2\}\right\}$$

$$P(\{a,b,c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

If a finite set S has m elements, then $P(S)$ has $2^m > m$ elements.

Proving set identities

Prove that $A \cup (A \cap B) = A$

$$
\begin{aligned}
A \cup (A \cap B) &= \{x : (x \in A) \vee (x \in A \cap B)\} \\
&= \{x : (x \in A) \vee (x \in A \wedge x \in B)\} \\
&= \{x : (x \in A)\}, \qquad \text{because } \big(p \vee (p \wedge q)\big) \equiv p \\
&= A
\end{aligned}
$$

Anything look familiar?

Table 3.5.1: Set identities.

Name	Identities	
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap$
Identity laws	$A \cup \emptyset = A$	$A \cap U = A$
Domination laws	$A \cap \emptyset = \emptyset$	$A \cup U = U$
Double Complement law	$\overline{\overline{A}} = A$	
Complement laws	$A \cap \overline{A} = \emptyset$ $\overline{U} = \emptyset$	$A \cup \overline{A} = U$ $\overline{\emptyset} = U$
De Morgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$

Table 1.5.1: Laws of propositional logic.

Idempotent laws:	$p \vee p \equiv p$	$p \wedge p \equiv p$
Associative laws:	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Distributive laws:	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	$p \vee F \equiv p$	$p \wedge T \equiv p$
Domination laws:	$p \wedge F \equiv F$	$p \vee T \equiv T$
Double negation law:	$\neg\neg p \equiv p$	
Complement laws:	$p \wedge \neg p \equiv F$ $\neg T \equiv F$	$p \vee \neg p \equiv T$ $\neg F \equiv T$
De Morgan's laws:	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Conditional identities:	$p \rightarrow q \equiv \neg p \vee q$	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

An Example

Prove that $\big((A - B) \cup (A - C) = A\big) \Rightarrow (A \cap B \cap C = \phi)$

$$A \cap B \cap C = \boxed{\big((A - B) \cup (A - C)\big)} \cap (B \cap C) \quad \textit{substitute for } A, \textit{associative}$$

$$= \big((A - B) \cap (B \cap C)\big) \cup \big((A - C) \cap \boxed{(C \cap B)}\big) \quad \textit{distributive, commutative}$$

$$= \Big(\big((\boxed{A \cap \bar{B}}) \cap B\big) \cap C\Big) \cup \big(((\boxed{A \cap \bar{C}}) \cap C\big) \cap B \quad \textit{associative}$$

$$= \Big(\big(A \cap (\boxed{\bar{B} \cap B})\big) \cap C\Big) \cup \Big(\big(A \cap (\boxed{\bar{C} \cap C})\big) \cap B\Big) \quad \textit{associative}$$

$$= (A \cap \phi \cap C) \cup (A \cap \phi \cap B) \quad \textit{complement}$$

$$= \phi \cup \phi \quad \textit{domination}$$

$$= \phi \quad \textit{identity}$$

Another Method

Prove that $\big((A - B) \cup (A - C) = A\big) \;\Rightarrow (A \cap B \cap C = \phi)$

$\forall x: x \in A \quad \Leftrightarrow \quad x \in (A - B) \cup (A - C)$

$\Leftrightarrow (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)$

$\Leftrightarrow x \in A \;\wedge\; (x \notin B \;\vee\; x \notin C)$ distributive law

So, $\forall x: \;x \in A \;\Rightarrow\; x \notin B \;\vee\; x \notin C$ absorption

$\equiv \forall x: \;x \notin A \;\vee\; x \notin B \;\vee\; x \notin C$ conditional

$\equiv \forall x: \;\neg(x \in A \;\wedge\; x \in B \;\wedge\; x \in C)$ deMorgan's

$\equiv \neg\exists x: \;x \in A \cap B \cap C$

$\Rightarrow \;A \cap B \cap C \;=\; \phi$

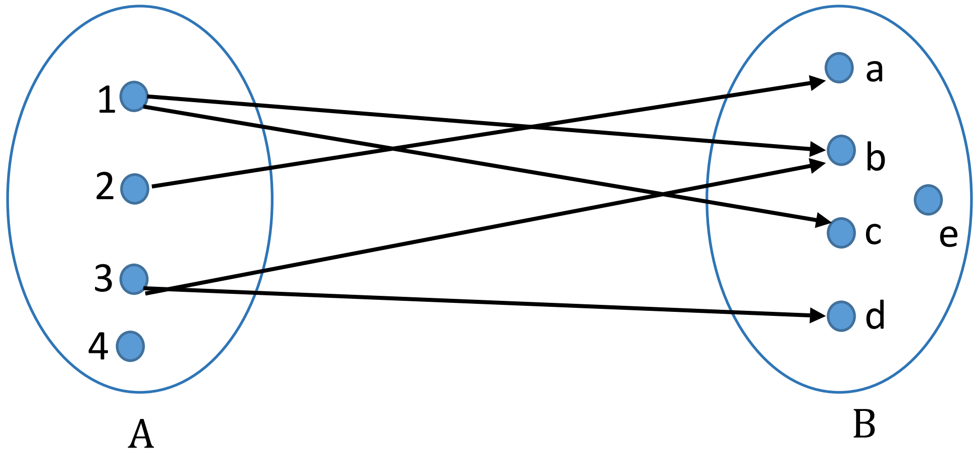
Relations

A relation R from a domain A to a range B is a subset of $A \times B$.

Example:

$R: \{1,2,3,4\} \rightarrow \{a,b,c,d,e\}$

$R = \{(1,b), (1,c), (2,a), (3,b), (3,d)\}$



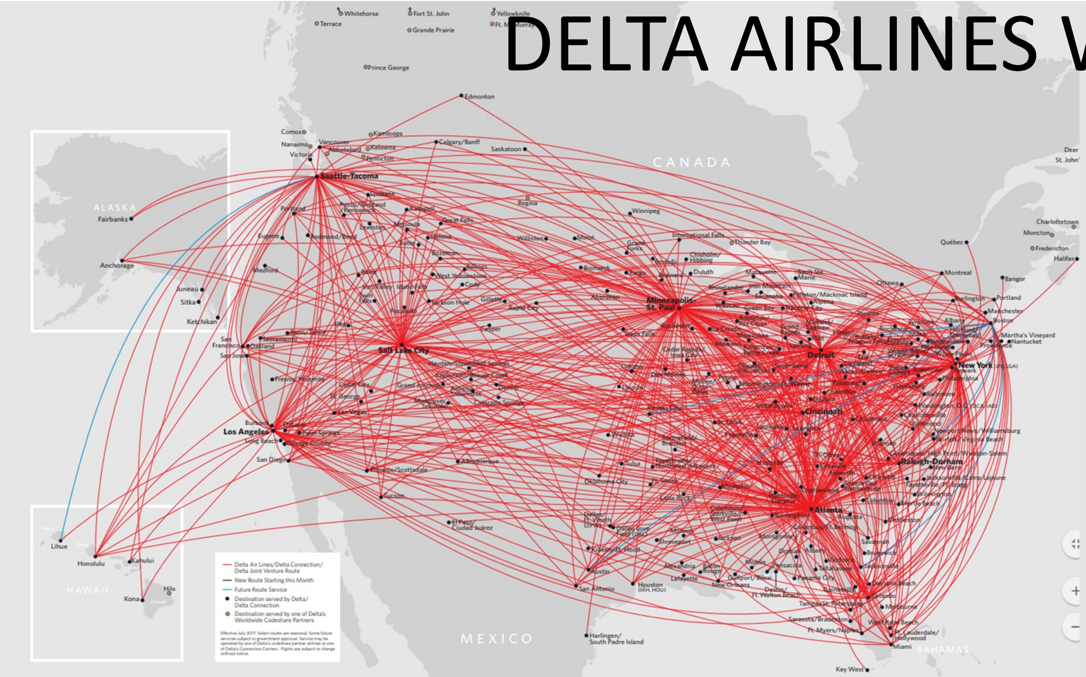
Relations

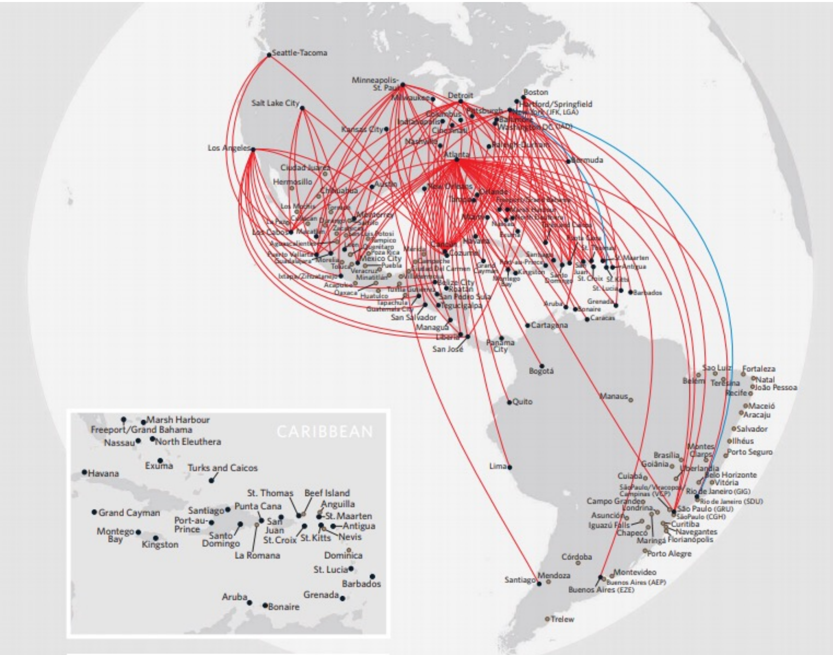
A relation R with domain A and range B is a subset of $A \times B$

A relation R over a set A is a subset of $A \times A$.

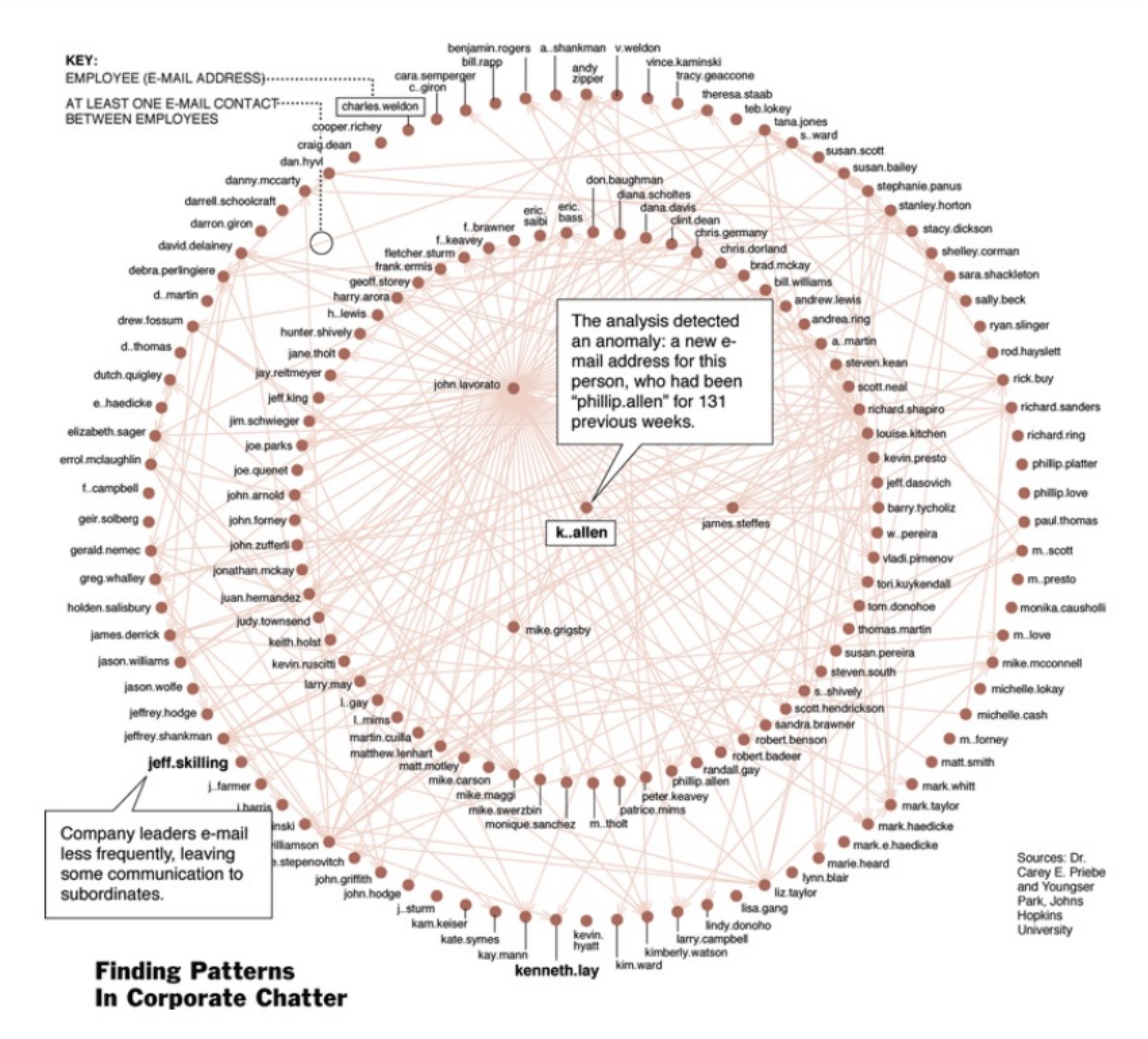
$$\begin{aligned} A &= \{\mathrm{EWR}, \mathrm{BOS}, \mathrm{DCA}, \mathrm{LAX}, \mathrm{SFO}, \mathrm{ORD}, \mathrm{DEN}, \mathrm{MIA}\} \\ FLIGHTS &= \{(\mathrm{EWR}, \mathrm{ORD}), (\mathrm{BOS}, \mathrm{DCA}), (\mathrm{LAX}, \mathrm{SFO}), \\ &\quad (\mathrm{DEN}, \mathrm{LAX}), (\mathrm{DCA}, \mathrm{MIA}), (\mathrm{SFO}, \mathrm{EWR}), \\ &\quad (\mathrm{ORD}, \mathrm{DEN}), (\mathrm{LAX}, \mathrm{BOS}), (\mathrm{MIA}, \mathrm{SFO})\} \end{aligned}$$

DELTA AIRLINES WORLDWIDE FLIGHTS

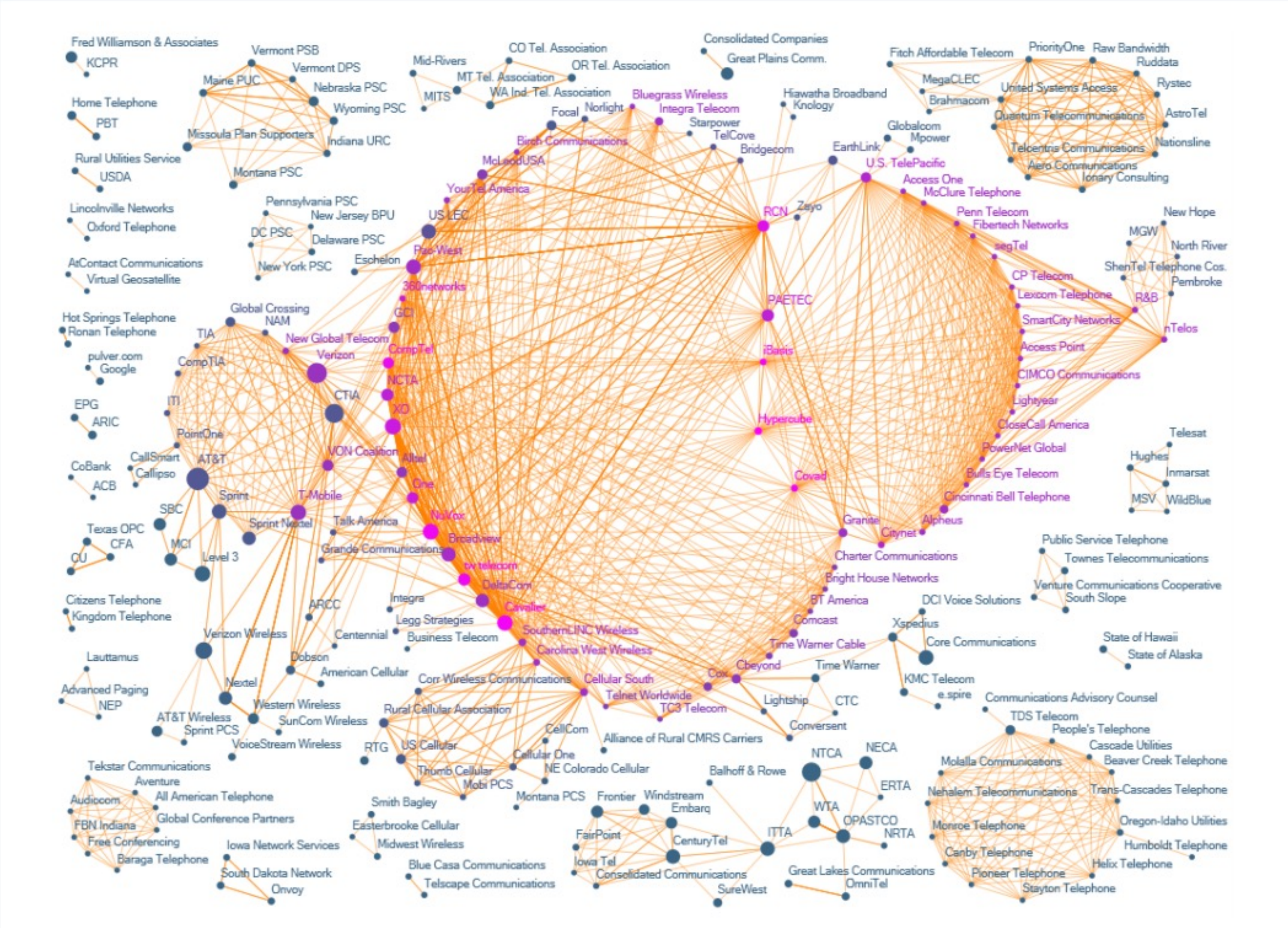




One week of Enron emails



The evolution of FCC lobbying coalitions



Properties of Relations

A relation R over a set A is:

- ***Reflexive*** if $\forall x \in A: (x, x) \in R$

$$DIVIDES = \{(a, b): a, b \in \mathbb{N}^+ \ \wedge \ a|b\}$$

- ***Anti-Reflexive*** if $\forall x \in A: (x, x) \notin R$

$$GREATER = \{(a, b): a, b \in \mathbb{N} \ \wedge \ a > b\}$$

Properties of Relations

A relation R over a set A is:

- ***Symmetric*** if $\forall x, y \in A$: $(x, y) \in R \Leftrightarrow (y, x) \in R$

$$CLOSEBY = \{(a, b): a, b \in \mathbb{N} \wedge |a - b| \leq 2\}$$

- ***Anti-Symmetric*** if $\forall x, y \in A$: $\big((x, y) \in R \wedge (y, x) \in R\big) \Rightarrow (x = y)$

$$DIVIDES = \{(a, b): a, b \in \mathbb{N}^{+} \wedge a|b\}$$

Properties of Relations

A relation R over a set A is:

- ***Transitive*** if $\forall x, y, z \in A: \big((x, y) \in R \ \wedge (y, z) \in R\big) \Rightarrow (x, z) \in R$

$$DIVIDES = \{(a, b): a, b \in \mathbb{N} \ \ \wedge a|b\}$$

$$IMPLIES = \{(P, Q): P \Rightarrow Q\}$$

Equivalence Relations

A relation R over a set A that is reflexive, symmetric and transitive is called an ***equivalence*** relation.

Examples:

$$\{(P, Q): P \iff Q\}$$

$$\{(a, b): rem(a, 3) = rem(b, 3)\}$$

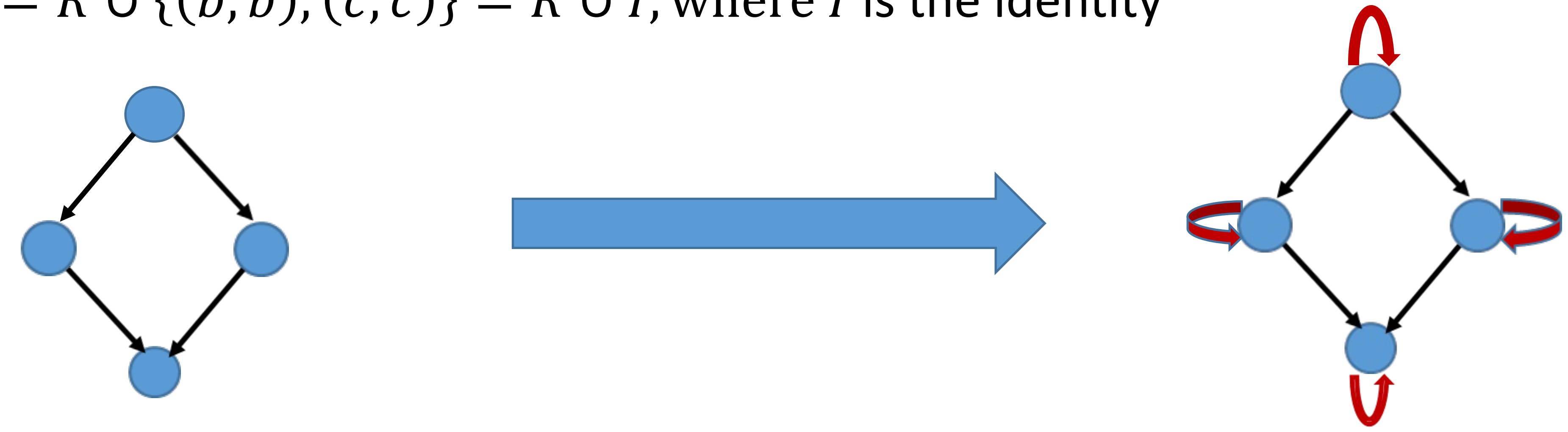
Reflexive Closure

The ***reflexive closure*** of relation R is the smallest reflexive relation $r(R)$: $r(R) \supseteq R$.

Example:

$$R = \{(a,a),(a,b),(b,c)\}$$

$r(R) = R \cup \{(b,b),(c,c)\} = R \cup I$, where I is the identity



Symmetric Closure

The ***symmetric closure*** of relation R is the smallest symmetric relation $s(R)$: $s(R) \supseteq R$.

Example:

$$R = \{(a,a),(a,b),(b,c)\}$$

$$s(R) = R \cup \{(b,a),(c,b)\} \quad = \quad R \cup R^{-} \quad \text{where } R^{-} \text{ is the inverse of } R$$

Transitive Closure

The ***transitive closure*** of relation R is the smallest transitive relation $R^+ \supseteq R$.

Example:

$$R = \{(a,a),(a,b),(b,c)\}$$
$$R^+ = R \cup \{(a,c)\}$$

Composing Relations

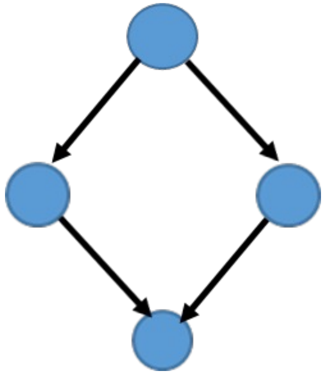
Given two relations $R: A \to B, \quad S: B \to C$

we define the composition

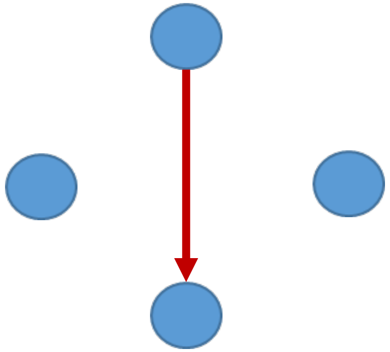
$S \circ R: A \to C$ as

$$\{(a, c): a \in A \ \wedge c \in C \wedge \exists b \in B: (a, b) \in R \wedge (b, c) \in S\}$$

If R is a relation over a set A then $R \circ R = \{(a, b): \exists x \in A \ (a, x) \in R \ \wedge (x, b) \in R\}$



R: direct flights



$R \circ R$: one-stop flights

Composing Relations

If R is a relation over a set A then $R \circ R = \{(a, b): \exists x \in A\ (a, x) \in R\ \wedge (x, b) \in R\}$

$$R \circ (R \circ R) = \{(a, b): \exists x, y \in A\ (a, x) \in R\ \wedge (x, y) \in R \wedge (y, b) \in R\}$$

R: direct flights

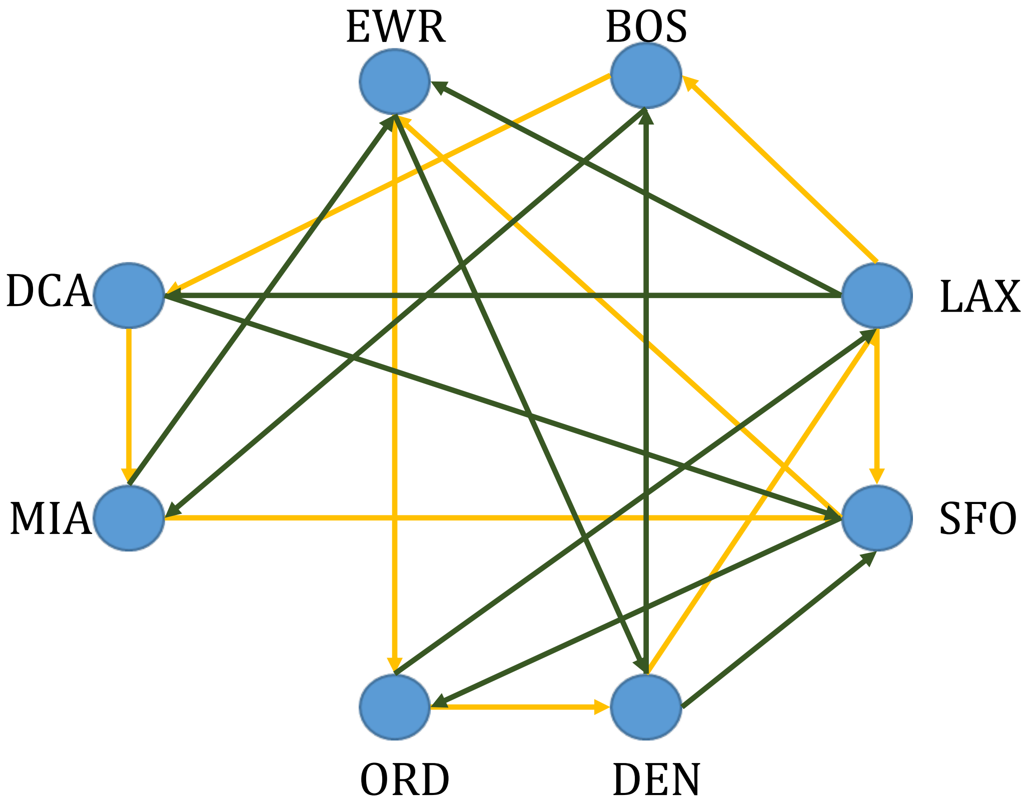
$R \circ R = R^2$: one-stop flights

$R \circ R \circ R = R^3$: two-stop flights

In general: $(a, b) \in R^k$ iff there is a sequence of k flights from a to b.

Our little airline

$$A = \{\mathrm{EWR}, \mathrm{BOS}, \mathrm{DCA}, \mathrm{LAX}, \mathrm{SFO}, \mathrm{ORD}, \mathrm{DEN}, \mathrm{MIA}\}$$
$$FLIGHTS = \{(\mathrm{EWR}, \mathrm{ORD}), (\mathrm{BOS}, \mathrm{DCA}), (\mathrm{LAX}, \mathrm{SFO}), (\mathrm{DEN}, \mathrm{LAX}), (\mathrm{DCA}, \mathrm{MIA}), (\mathrm{SFO}, \mathrm{EWR}), (\mathrm{ORD}, \mathrm{DEN}), (\mathrm{LAX}, \mathrm{BOS}), (\mathrm{MIA}, \mathrm{SFO})\}$$



$\boldsymbol{R}$

$\boldsymbol{R \circ R}$

Composing Relations

Suppose A consists of n cities and that one can fly (directly or indirectly) from a to b

Then there is a sequence of k flights where $1 \leq k \leq n$. (Why not $n - 1$?)

In other words, $(a, b) \in R \cup R^2 \cup R^3 \cup \cdots \cup R^n$

Theorem: For any relation R over a set A, $|A| = n$,

$$R^+ = R \cup R^2 \cup R^3 \cup \cdots \cup R^n$$

Corollary: If R is reflexive then $R^+ = R^n$

since $R \subseteq R^2 \subseteq R^3 \cdots \subseteq R^n$