STA 610 Homework 9

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Question 1.

(a) (4 points) The full conditional distribution of μ is

$$
\begin{aligned}
p(\mu|\cdot) &\propto p(\mu)\prod_{j=1}^{m} p(\theta_j|\mu,\tau^2) \\
&\propto \exp\left(-\frac{(\mu-\mu_0)^2}{2v_0} - \sum_{j=1}^{m}\frac{(\theta_j-\mu)^2}{2\tau^2}\right) \\
&\propto \exp\left(-\frac{1}{2}\left(\frac{1}{v_0}+\frac{m}{\tau^2}\right)\mu^2 + \left(\frac{\mu_0}{v_0}+\frac{\sum_{j=1}^{m}\theta_j}{\tau^2}\right)\mu\right) \\
&\sim N\left(\frac{\frac{\mu_0}{v_0}+\frac{\sum_{j=1}^{m}\theta_j}{\tau^2}}{\frac{1}{v_0}+\frac{m}{\tau^2}},\ \left(\frac{1}{v_0}+\frac{m}{\tau^2}\right)^{-1}\right).
\end{aligned}
$$

The full conditional distribution of τ^2 is

$$
\begin{aligned}
p(\tau^2|\cdot) &\propto p(\tau^2)\prod_{j=1}^{m} p(\theta_j|\mu,\tau^2) \\
&\propto (\tau^2)^{-\frac{\eta_0}{2}-1}\exp\left(-\frac{\eta_0\tau_0^2}{2\tau^2}\right)\cdot(\tau^2)^{-\frac{m}{2}}\exp\left(-\sum_{j=1}^{m}\frac{(\theta_j-\mu)^2}{2\tau^2}\right) \\
&\propto (\tau^2)^{-\frac{\eta_0+m}{2}-1}\exp\left(-\frac{\eta_0\tau_0^2+\sum_{j=1}^{m}(\theta_j-\mu)^2}{2\tau^2}\right) \\
&\sim \text{InverseGamma}\left(\frac{\eta_0+m}{2},\ \frac{\eta_0\tau_0^2+\sum_{j=1}^{m}(\theta_j-\mu)^2}{2}\right).
\end{aligned}
$$

The full conditional distribution of σ^2 is

$$\begin{aligned}
p(\sigma^2|\cdot) &\propto p(\sigma^2)\prod_{j=1}^{m}\prod_{i=1}^{n_j} p(y_{i,j}|\theta_j,\sigma^2) \\
&\propto (\sigma^2)^{-\frac{\nu_0}{2}-1}\exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right)\cdot(\sigma^2)^{-\frac{\sum_{j=1}^m n_j}{2}}\exp\left(-\sum_{j=1}^{m}\sum_{i=1}^{n_j}\frac{(y_{i,j}-\theta_j)^2}{2\sigma^2}\right) \\
&\propto (\sigma^2)^{-\frac{\nu_0+\sum_{j=1}^m n_j}{2}-1}\exp\left(-\frac{\nu_0\sigma_0^2+\sum_{j=1}^m\sum_{i=1}^{n_j}(y_{i,j}-\theta_j)^2}{2\sigma^2}\right) \\
&\sim \mathrm{InverseGamma}\left(\frac{\nu_0+\sum_{j=1}^m n_j}{2},\ \frac{\nu_0\sigma_0^2+\sum_{j=1}^m\sum_{i=1}^{n_j}(y_{i,j}-\theta_j)^2}{2}\right).
\end{aligned}$$

The full conditional distribution of θ_j is

$$\begin{aligned}
p(\theta_j|\cdot) &\propto p(\theta_j|\mu,\tau^2)\prod_{i=1}^{n_j} p(y_{i,j}|\theta_j,\sigma^2) \\
&\propto \exp\left(-\frac{(\theta_j-\mu)^2}{2\tau^2}-\sum_{i=1}^{n_j}\frac{(y_{i,j}-\theta_j)^2}{2\sigma^2}\right) \\
&\propto \exp\left(-\frac{1}{2}\left(\frac{1}{\tau^2}+\frac{n_j}{\sigma^2}\right)\theta_j^2+\left(\frac{\mu}{\tau^2}+\frac{\sum_{i=1}^{n_j} y_{i,j}}{\sigma^2}\right)\theta_j\right) \\
&\sim N\left(\frac{\frac{\mu}{\tau^2}+\frac{\sum_{i=1}^{n_j} y_{i,j}}{\sigma^2}}{\frac{1}{\tau^2}+\frac{n_j}{\sigma^2}},\ \left(\frac{1}{\tau^2}+\frac{n_j}{\sigma^2}\right)^{-1}\right).
\end{aligned}$$

(b) (4 points)

$$\mathbb{E}[\mu|\cdot] = \frac{\frac{\mu_0}{v_0}+\frac{\sum_{j=1}^m\theta_j}{\tau^2}}{\frac{1}{v_0}+\frac{m}{\tau^2}} = \frac{\frac{1}{v_0}}{\frac{1}{v_0}+\frac{m}{\tau^2}}\cdot\mu_0+\frac{\frac{m}{\tau^2}}{\frac{1}{v_0}+\frac{m}{\tau^2}}\cdot\frac{\sum_{j=1}^m\theta_j}{m}.$$

$$\mathbb{E}\left[\frac{1}{\tau^2}\Big|\cdot\right]^{-1} = \frac{\frac{\eta_0\tau_0^2+\sum_{j=1}^m(\theta_j-\mu)^2}{2}}{\frac{\eta_0+m}{2}} = \frac{\eta_0}{\eta_0+m}\cdot\tau^2+\frac{m}{\eta_0+m}\cdot\frac{\sum_{j=1}^m(\theta_j-\mu)^2}{m}.$$

$$\mathbb{E}[\tau^2|\cdot] = \frac{\frac{\eta_0\tau_0^2+\sum_{j=1}^m(\theta_j-\mu)^2}{2}}{\frac{\eta_0+m}{2}-1} = \frac{\eta_0}{\eta_0+m-2}\cdot\tau_0^2+\frac{m}{\eta_0+m-2}\cdot\frac{\sum_{j=1}^m(\theta_j-\mu)^2}{m}.$$

$$\mathbb{E}\left[\frac{1}{\sigma^2}\Big|\cdot\right]^{-1} = \frac{\frac{\nu_0\sigma_0^2+\sum_{j=1}^m\sum_{i=1}^{n_j}(y_{i,j}-\theta_j)^2}{2}}{\frac{\nu_0+\sum_{j=1}^m n_j}{2}} = \frac{\nu_0}{\nu_0+\sum_{j=1}^m n_j}\cdot\sigma_0^2+\frac{\sum_{j=1}^m n_j}{\nu_0+\sum_{j=1}^m n_j}\cdot\frac{\sum_{j=1}^m\sum_{i=1}^{n_j}(y_{i,j}-\theta_j)^2}{\sum_{j=1}^m n_j}.$$

$$\mathbb{E}[\sigma^2|\cdot] = \frac{\frac{\nu_0\sigma_0^2+\sum_{j=1}^m\sum_{i=1}^{n_j}(y_{i,j}-\theta_j)^2}{2}}{\frac{\nu_0+\sum_{j=1}^m n_j}{2}-1} = \frac{\nu_0}{\nu_0+\sum_{j=1}^m n_j-2}\cdot\sigma_0^2+\frac{\sum_{j=1}^m n_j}{\nu_0+\sum_{j=1}^m n_j-2}\cdot\frac{\sum_{j=1}^m\sum_{i=1}^{n_j}(y_{i,j}-\theta_j)^2}{\sum_{j=1}^m n_j}.$$

$$\mathbb{E}[\theta_j|\cdot] = \frac{\frac{\mu}{\tau^2}+\frac{\sum_{i=1}^{n_j} y_{i,j}}{\sigma^2}}{\frac{1}{\tau^2}+\frac{n_j}{\sigma^2}} = \frac{\frac{1}{\tau^2}}{\frac{1}{\tau^2}+\frac{n_j}{\sigma^2}}\cdot\mu+\frac{\frac{n_j}{\sigma^2}}{\frac{1}{\tau^2}+\frac{n_j}{\sigma^2}}\cdot\frac{\sum_{i=1}^{n_j} y_{i,j}}{n_j}.$$

(c) (2 points)

Algorithm 1 Gibbs Sampler

Initialize parameters $\mu^{(0)}, (\tau^2)^0, (\sigma^2)^{(0)}, \theta_1^{(0)}, \ldots, \theta_m^{(0)}$
for iteration $t = 1, 2, \ldots$ **do**
  Sample $\mu^{(t)} \sim p\left(\mu \;\middle|\; y, (\tau^2)^{(t-1)}, (\sigma^2)^{(t-1)}, \theta_{1:m}^{(t-1)}\right)$
  Sample $(\tau^2)^{(t)} \sim p\left(\tau^2 \;\middle|\; y, \mu^{(t)}, (\sigma^2)^{(t-1)}, \theta_{1:m}^{(t-1)}\right)$
  Sample $(\sigma^2)^{(t)} \sim p\left(\sigma^2 \;\middle|\; y, \mu^{(t)}, (\tau^2)^{(t)}, \theta_{1:m}^{(t-1)}\right)$
  for $j = 1, \ldots, m$ **do**
    Sample $\theta_j^{(t)} \sim p\left(\theta_j \;\middle|\; y, \mu^{(t)}, (\tau^2)^{(t)}, (\sigma^2)^{(t)}\right)$
  end for
end for