STA 610 Homework 3

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Question 1.

(a) (2 points)

$$\begin{aligned} p(\theta_j|\bar{y}_j, \mu, \sigma^2, \tau^2) &\propto p(\bar{y}_j|\theta_j, \sigma^2)p(\theta_j|\mu, \tau^2) \\ &\propto \exp\left(-\frac{n_j}{2\sigma^2}(\bar{y}_j - \theta_j)^2\right) \cdot \exp\left(-\frac{1}{2\tau^2}(\theta_j - \mu)^2\right) \\ &\propto \exp\left(-\frac{1}{2}\left(\frac{n_j}{\sigma^2} + \frac{1}{\tau^2}\right)\theta_j^2 + \left(\frac{n_j}{\sigma^2}\bar{y}_j + \frac{1}{\tau^2}\mu\right)\theta_j\right) \\ &\sim N\left(\left(\frac{n_j}{\sigma^2} + \frac{1}{\tau^2}\right)^{-1}\left(\frac{n_j}{\sigma^2}\bar{y}_j + \frac{1}{\tau^2}\mu\right),\ \left(\frac{n_j}{\sigma^2} + \frac{1}{\tau^2}\right)^{-1}\right). \end{aligned}$$

(b) (2 points) Since $\theta_j|\bar{y}_j, \mu, \sigma^2, \tau^2$ follows a normal distribution, obviously we have

$$\mathbb{P}(\theta_j \in \mathbb{E}[\theta_j|\bar{y}_j] \pm z_{1-\alpha/2} SD[\theta_j|\bar{y}_j]) = 1 - \alpha.$$

(c) (2 points) The width ratio is

$$r = \frac{SD[\theta_j|\bar{y}_j]}{\sigma/\sqrt{n_j}} = \sqrt{\frac{\frac{n_j}{\sigma^2}}{\frac{n_j}{\sigma^2} + \frac{1}{\tau^2}}}.$$

Recall from Slides 6 (NP) Page 4 that the shrinkage weight

$$w := \frac{\frac{1}{\tau^2}}{\frac{n_j}{\sigma^2} + \frac{1}{\tau^2}}.$$

Hence the width ratio

$$r = \sqrt{1 - w}.$$

The width of Bayesian confidence interval is shorter because additional information is obtained from the prior.