STA 610 Homework 2

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Question 1.

a. (3 points) Letting $\bar{\mathbf{y}}$ denote the $m \times 1$ vector $(\bar{y}_1, \ldots, \bar{y}_m)^\top$ and $\mathbf{c}$ denote the $m \times 1$ all-c vector, we have

$$\begin{aligned}
\mathbb{E}\left[\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|^2 \;\middle|\; \boldsymbol{\theta} \right] &= \mathbb{E}\left[\|(1-\omega)\bar{\mathbf{y}} + \omega\mathbf{c} - \boldsymbol{\theta}\|^2 \;\middle|\; \boldsymbol{\theta} \right] \\
&= \mathbb{E}\left[\|(1-\omega)(\bar{\mathbf{y}} - \boldsymbol{\theta}) + \omega(\mathbf{c} - \boldsymbol{\theta})\|^2 \;\middle|\; \boldsymbol{\theta} \right] \\
&= (1-\omega)^2 \cdot \sum_{j=1}^m \mathbb{V}[\bar{y}_j | \boldsymbol{\theta}] + 0 + \omega^2 \sum_{j=1}^m (c - \theta_j)^2 \\
&= (1-\omega)^2 \frac{m\sigma^2}{n} + \omega^2 \sum_{j=1}^m (c - \theta_j)^2.
\end{aligned}$$

Taking derivatives w.r.t. ω and c, we have

$$\nabla_\omega = -\frac{2m\sigma^2}{n}(1-\omega) + 2\omega \sum_{j=1}^m (c - \theta_j)^2, \quad \nabla_c = 2\omega^2 \sum_{j=1}^m (c - \theta_j).$$

Letting both derivatives equal to zero, we obtain

$$c^* = \frac{1}{m}\sum_{j=1}^m \theta_j, \quad \omega^* = \frac{\frac{m\sigma^2}{n}}{\frac{m\sigma^2}{n} + \sum_{j=1}^m (c^* - \theta_j)^2}.$$

b. (2 points) Plugging in c^*, ω^*, we obtain

$$\begin{aligned}
\mathbb{E}\left[\|\widehat{\boldsymbol{\theta}}^* - \boldsymbol{\theta}\|^2 \;\middle|\; \boldsymbol{\theta} \right] &= \left(\frac{\sum_{j=1}^m (c^* - \theta_j)^2}{\frac{m\sigma^2}{n} + \sum_{j=1}^m (c^* - \theta_j)^2} \right)^2 \frac{m\sigma^2}{n} \\
&\quad + \left(\frac{\frac{m\sigma^2}{n}}{\frac{m\sigma^2}{n} + \sum_{j=1}^m (c^* - \theta_j)^2} \right)^2 \sum_{j=1}^m (c^* - \theta_j)^2 \\
&= \frac{\frac{m\sigma^2}{n} \cdot \sum_{j=1}^m (c^* - \theta_j)^2}{\frac{m\sigma^2}{n} + \sum_{j=1}^m (c^* - \theta_j)^2}.
\end{aligned}$$

Question 3.

a. (3 points) Similar to Question 1a, we have

$$\begin{aligned}
\mathbb{E}\left[(\widehat{\theta}-\theta)^2 \;\middle|\; \theta\right] &= \mathbb{E}\left[((1-\omega)(\bar{y}-\theta)+\omega(\mu-\theta))^2 \;\middle|\; \theta\right] \\
&= (1-\omega)^2\mathbb{V}[\bar{y}|\theta] + \omega^2(\mu-\theta)^2 \\
&= (1-\omega)^2\frac{\sigma^2}{n} + \omega^2(\mu-\theta)^2.
\end{aligned}$$

We therefore notice

$$\begin{aligned}
\mathbb{E}\left[(\widehat{\theta}-\theta)^2 \;\middle|\; \theta\right] \le \mathbb{V}[\bar{y}|\theta] \quad &\iff \quad \omega^2(\mu-\theta)^2 \le (1-(1-\omega)^2)\frac{\sigma^2}{n} \\
&\iff \quad \left((\mu-\theta)^2 + \frac{\sigma^2}{n}\right)\omega^2 \le \frac{2\sigma^2}{n}\omega \\
&\iff \quad \omega \le \frac{\frac{2\sigma^2}{n}}{(\mu-\theta)^2 + \frac{\sigma^2}{n}}.
\end{aligned}$$

b. (2 points) Plugging in $\omega^* = \frac{\frac{\sigma^2}{n}}{\frac{\sigma^2}{n}+\tau^2}$ from class, the condition becomes

$$\frac{\frac{\sigma^2}{n}}{\frac{\sigma^2}{n}+\tau^2} \le \frac{\frac{2\sigma^2}{n}}{(\mu-\theta)^2+\frac{\sigma^2}{n}},$$

which simplifies to

$$(\mu-\theta)^2 \le \frac{\sigma^2}{n} + 2\tau^2.$$