Seminar 6. Walrasian Equilibrium in a Barter Economy

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1 Jehle & Reny 5.4 - Excess demand function and GE

Derive the excess demand function $z(p)$ for the economy in Example 5.1. Verify that it satisfies Walras' law.

Suppose we have a good-exchange economy,

- I is the set of all the individuals (consumers) in the economy,
- The prices of all n commodities is expressed by a vector $p = (p_1, p_2, \ldots, p_n)$,
- Every consumer has some endowments in the form of commodities expressed by a vector $e^i = (e_1^i, e_2^i, \ldots, e_n^i)$,
- $p \cdot e^i$ is the income of consumer i,

Assume (Assumption 5.1 on pp.203) that every consumer has a utility function u^i, which is continuous, strongly increasing, and strictly quasiconcave on $\mathbb{R}_+^n$.

- By solving consumer i's' utility maximization problem, consumer i's Marshallian demand function is $x^i(p, p \cdot e^i) = (x_1^i, x_2^i, \ldots, x_n^i)$

General Equilibrium: When demand equal to supply in **every market** (market for every commodity), we would say that the system of markets is in General Equilibrium.

We use **Excess Demand** to describe "demand equal to supply".

DEFINITION 5.4 Aggregate Excess Demand (Jehle & Reny pp.204)
The aggregate excess demand function for good k is the real-valued function,

$$z_k(p) \equiv \Sigma_{i \in I} x_k^i(p, p \cdot e^i) - \Sigma_{i \in I} e_k^i$$

Where,

- $\Sigma_{i\in I} x_k^i(p, p\cdot e^i)$ is the summation of all consumers' Marshallian demand for commodity k,

- $\Sigma_{i\in I} e_k^i$ is the total amount of commodity k in this economy.

When $z_k(p) > 0$, the aggregate demand for good k exceeds the aggregate endowment of good k and so there is excess demand for good k. When $z_k(p) < 0$, there is excess supply of good k. That's why $z_k(p)$ is called "Excess Demand" for k.

The **aggregate excess demand function** is a vector-valued function,

$$z_(p) \equiv [z_1(p), z_2(p), \ldots, z_n(p)]$$

When $\exists\ p^* \in \mathbb{R}^n_{++}$ s.t. $z(p^*) = 0$, we say Walrasian Equilibrium (WE) exists. A WE in a barter economy includes a price vector p^* and an allocation (e.g. Marshallian demand) vector $x(p^*, p^* \cdot e)$.

THEOREM 5.2 Properties of Aggregate Excess Demand Functions (pp.204)
If for each consumer i, u^i satisfies Assumption 5.1, then for all $p \gg 0$,

1. Continuity: $z(.)$ is continuous at p.

2. Homogeneity: $z(\lambda p) = z(p) \quad \forall \lambda > 0$.

3. Walras' law: $p \cdot z(p) = 0$.

THEOREM 5.5 Existence of Walrasian Equilibrium
If each consumer's utility function satisfies Assumption 5.1, and $\Sigma_{i=1}^I e^i \gg 0$, then there exists at least one price vector, $p^* \gg 0$, such that $z(p^*)$.

Example 5.1 on pp.211
In a simple two-person economy, consumers 1 and 2 have identical CES utility functions,

$$u^i(x_1, x_2) = x_1^\rho + x_2^\rho, \quad i = 1, 2$$

where $\rho \in (0, 1)$.
The initial endowments are $e^1 = (1, 0), e^2 = (0, 1)$.

Does WE exist?
Yes. The requirements of Theorem 5.5 are satisfied.

- $\Sigma_{i=1}^2 e^i = (1, 0) + (0, 1) = (1, 1) \gg 0$

- $u^i(x_1, x_2) = x_1^\rho + x_2^\rho$ is strongly increasing and strictly quasiconcave on $\mathbb{R}^n_+$ when $\rho \in (0, 1)$

How to find WE?
We let $z(p) = 0$ to find p.

How to find WEA?
By substituting p^* and $y^* = p^* e$ into $x(p, y)$.

1.1 Excess demand function $z(p)$

From Example 1.11 on pp.26, we know the Marshallian demands of consumer i for commodity 1 and commodity 2 are:

$$x_1^i(p, y^i) = \frac{p_1^{r-1} y^i}{p_1^r + p_2^r},$$
$$x_2^i(p, y^i) = \frac{p_2^{r-1} y^i}{p_1^r + p_2^r}.$$

where $r = \frac{\rho}{\rho - 1}$, $i = 1, 2$.
Given any price vector $p = (p_1, p_2)$, and initial endowment $e^1 = (1, 0), e^2 = (0, 1)$, we know the income of the two consumers are

$$y^1 = p(1, 0)' = p_1$$
$$y^2 = p(0, 1)' = p_2$$

According to Deffinition 5.4, we have aggregated excess demand for commodity 1:

$$\begin{aligned}
z_1(p) &= \Sigma_{i=1}^2 x_1^i(p, p \cdot e^i) - \Sigma_{i=1}^2 e_1^i \\
&= [x_1^1(p, p_1) + x_1^2(p, p_2)] - (e_1^1 + e_1^2) \\
&= (\frac{p_1^{r-1} p_1}{p_1^r + p_2^r} + \frac{p_1^{r-1} p_2}{p_1^r + p_2^r}) - (1 + 0) \\
&= \frac{p_1^{r-1}(p_1 + p_2)}{p_1^r + p_2^r} - 1
\end{aligned}$$

Similarly, the aggregated excess demand for commodity 2 is:

$$\begin{aligned}
z_2(p) &= \Sigma_{i=1}^2 x_2^i(p, p \cdot e^i) - \Sigma_{i=1}^2 e_2^i \\
&= [x_2^1(p, p_1) + x_2^2(p, p_2)] - (e_2^1 + e_2^2) \\
&= (\frac{p_2^{r-1} p_1}{p_1^r + p_2^r} + \frac{p_2^{r-1} p_2}{p_1^r + p_2^r}) - (1 + 0) \\
&= \frac{p_2^{r-1}(p_1 + p_2)}{p_1^r + p_2^r} - 1
\end{aligned}$$

Thus, the **Aggregated Excess Demand Function** is vector:

$$z(p) = (z_1(p), z_2(p)) = (\frac{p_1^{r-1}(p_1+p_2)}{p_1^r+p_2^r} - 1, \frac{p_2^{r-1}(p_1+p_2)}{p_1^r+p_2^r} - 1)$$

Note Aggregated Excess Demand Function $z(p)$ is a vector, and each element corresponds with one commodity.

1.2 Walras' law

- Walras' law: $p \cdot z(p) = 0$.

$$\begin{aligned}
p \cdot z(p) &= (p_1, p_2)(z_1(p), z_2(p))' \\
&= p_1[\frac{p_1^{r-1}(p_1+p_2)}{p_1^r+p_2^r} - 1] + p_2[\frac{p_2^{r-1}(p_1+p_2)}{p_1^r+p_2^r} - 1] \\
&= [\frac{p_1^r(p_1+p_2)}{p_1^r+p_2^r} - p_1] + [\frac{p_2^r(p_1+p_2)}{p_1^r+p_2^r} - p_2] \\
&= \frac{(p_1^r+p_2^r)(p_1+p_2)}{p_1^r+p_2^r} - p_1 - p_2 \\
&= 0
\end{aligned}$$

2 Jehle & Reny 5.5 - WEA and Edgeworth box

In Example 5.1, calculate the consumers' Walrasian equilibrium allocations and illustrate in an Edgeworth box. Sketch in the contract curve and identify the core.

2.1 WEA

We already have $z(p)$. Now let $z(p) = 0$ to find p^*.
When $(z_1(p), z_2(p)) = (0, 0)$, we have: (Note I omitted star below for simplicity, but you should know only p^* leads to $z(p) = 0$)

$$\frac{p_1^{r-1}(p_1+p_2)}{p_1^r+p_2^r} = 1, \quad \frac{p_2^{r-1}(p_1+p_2)}{p_1^r+p_2^r} = 1$$

For the first commodity:

$$\frac{p_1^{-r}}{p_1^{-r}} \frac{p_1^{r-1}(p_1+p_2)}{p_1^r+p_2^r} = 1$$
$$\frac{p_1^{-1}(p_1+p_2)}{(p_1/p_1)^r+(p_2/p_1)^r} = 1$$
$$\frac{1+p_2/p_1}{1+(p_2/p_1)^r} = 1$$
$$1+\frac{p_2}{p_1} = 1+(\frac{p_2}{p_1})^r$$
$$(\frac{p_2}{p_1})^{r-1} = 1$$

We know $r-1 = \frac{\rho}{\rho-1} - 1 = \frac{1}{\rho-1} \neq 0$, $p \gg 0$. Then $\frac{p_2}{p_1} = 1$
Similarly, for the second commodity, we have $\frac{p_1}{p_2} = 1$
To conclude, the WE price p^* is $(p_1^*.p_2^*)$ $s.t.$ $p_1^* = p_2^*$. Let's just denote the $p_1^* = p_2^* = a$, the demands under the price p^* are:

$$x_1^i(p,a) = \frac{a^{r-1}a}{a^r+a^r} = 0.5,$$
$$x_2^i(p,a) = \frac{a^{r-1}a}{a^r+a^r} = 0.5.$$

$i = 1,2$. The WEA is thus $x^* = ((0.5,0.5),(0.5,0.5))$

- Only relative price $\frac{p_1}{p_2}$ matters, since you can always "rescale" the prices;
- To describe WE, you need to denote both p^* and WEA.

2.2 Edgeworth box

Contract curve The curve that links the two consumers' indifference curves' tangent point.
Core Given some endowment e, the core of the economy is the set of all feasible allocations that are not against ("blocked") by any consumers (a formal definition is on pp.200-201).

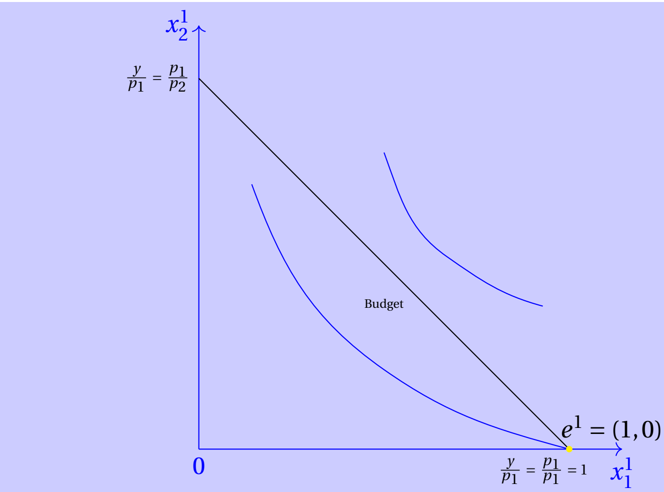

Figure 1: Consumer 1

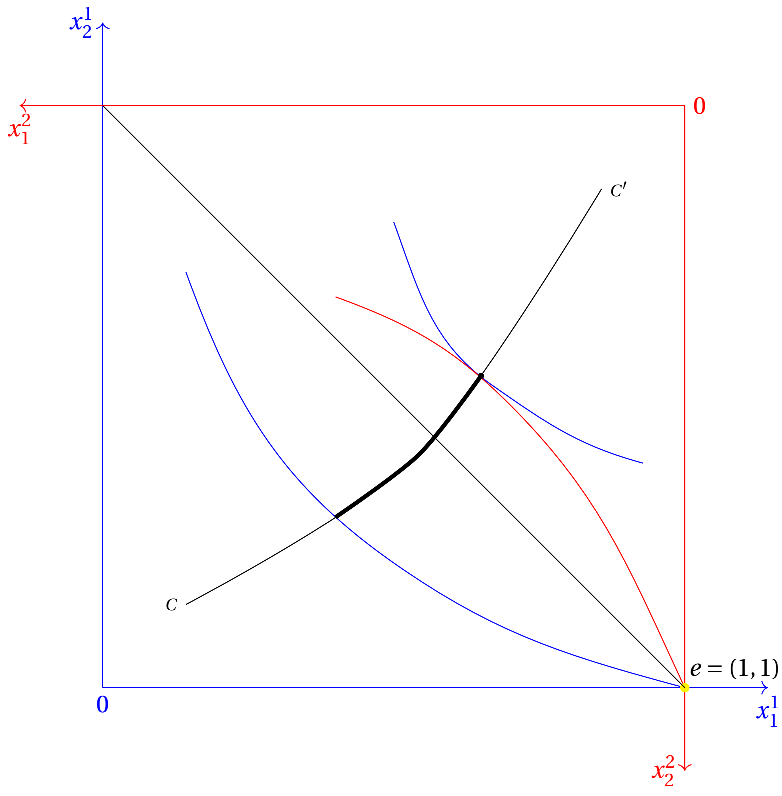

Figure 2: The core

3 Jehle & Reny 5.11 - Pareto-efficient allocations and WEA

Consider a two-consumer, two-good exchange economy. Utility functions and endowments are

$$u^1(x_1, x_2) = (x_1 x_2)^2 \quad and \quad e^1 = (18, 4)$$
$$u^2(x_1, x_2) = ln(x_1) + 2ln(x_2) \quad and \quad e^2 = (3, 6)$$

1. Characterise the set of Pareto-efficient allocations as completely as possible.
2. Characterise the core of this economy.
3. Find a Walrasian equilibrium and compute the WEA.
4. Verify that the WEA you found in part (c) is in the core.

3.1 Pareto-efficient allocations

Pareto-efficient allocations

3.2 Core

3.3 WEA

3.4 WEA is in the core