Seminar 3.Duality of Consumers Behavior
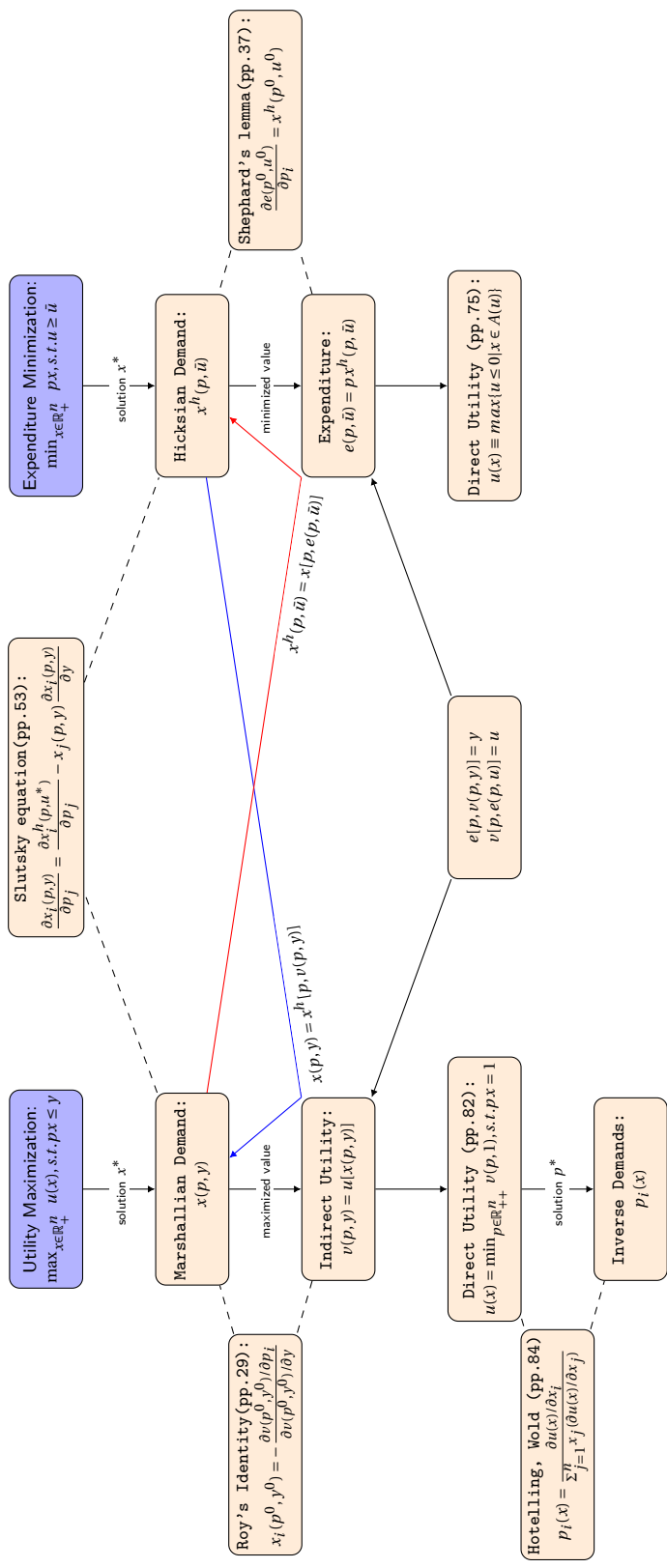
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Consumption Duality

You will never lose your way with this Consumption Duality map!
All "derive this from that and verify some guy's equation"-like questions can be solved by finding the correct (shortest) route.

Utility Maximization:
$\max_{x \in \mathbb{R}_+^n} u(x), s.t. px \le y$

$\xrightarrow{\text{solution } x^*}$

Marshallian Demand:
$x(p,y)$

$\xrightarrow{\text{maximized value}}$

Indirect Utility:
$v(p,y) = u[x(p,y)]$

Expenditure Minimization:
$\min_{x \in \mathbb{R}_+^n} px, s.t. u \ge \bar{u}$

$\xrightarrow{\text{solution } x^*}$

Hicksian Demand:
$x^h(p,\bar{u})$

$\xrightarrow{\text{minimized value}}$

Expenditure:
$e(p,\bar{u}) = p x^h(p,\bar{u})$

Shephard's lemma (pp.37):
$\frac{\partial e(p^0, u^0)}{\partial p_i} = x^h(p^0, u^0)$

Direct Utility (pp.75):
$u(x) = max\{u \le 0 | x \in A(u)\}$

Slutsky equation (pp.53):
$\frac{\partial x_i(p,y)}{\partial p_j} = \frac{\partial x_i^h(p,u^*)}{\partial p_j} - x_j(p,y) \frac{\partial x_i(p,y)}{\partial y}$

$x^h(p,\bar{u}) = x[p, e(p,\bar{u})]$

$x(p,y) = x^h[p, v(p,y)]$

$e[p, v(p,y)] = y$
$v[p, e(p,u)] = u$

Direct Utility (pp.82):
$u(x) = \min_{p \in \mathbb{R}_{++}^n} v(p,1), s.t. px = 1$

$\xrightarrow{\text{solution } p^*}$

Inverse Demands:
$p_i(x)$

Roy's Identity (pp.29):
$x_i(p^0, y^0) = -\frac{\partial v(p^0, y^0)/\partial p_i}{\partial v(p^0, y^0)/\partial y}$

Hotelling, Wold (pp.84)
$p_i(x) = \frac{\partial u(x)/\partial x_i}{\sum_{j=1}^n x_j (\partial u(x)/\partial x_j)}$

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1 Jehle & Reny 2.3

Derive the consumers direct utility function if his indirect utility function has the form $v(p, y) = y p_1^\alpha p_2^\beta$ for negative α and β.

> **THEOREM 2.3 Duality Between Direct and Indirect Utility** (Jehle & Reny pp.81)
>
> Suppose that $u(x)$ is quasiconcave and differentiable on $\mathbb{R}_{++}^n$ with strictly positive partial derivatives there. Then for all $x \in \mathbb{R}_{++}^n$, $v(p, p \cdot x)$, the indirect utility function generated by $u(x)$, achieves a minimum in p on $\mathbb{R}_{++}^n$, and
>
> $$u(x) = \min_{p \in \mathbb{R}_{++}^n} v(p, y), s.t. px = y$$
>
> Let's call the solution p^*
> Note that by **Theorem 1.6** (Jehle & Reny pp.29), $v(p, y)$ is homogeneous of degree zero in (p, y). We have $v(p, p \cdot x) = v(p/(p \cdot x), 1)$ whenever $p \cdot x > 0$. Thus the equation above can also be written as:
>
> $$u(x) = \min_{p \in \mathbb{R}_{++}^n} v(p, 1), s.t. px = 1$$
>
> The solution $\hat{p} = p^*/p^* \cdot x = p^*/y$. We don't care about the difference between $\hat{p}$ and p^* because once you substitute them into $v(p, p \cdot x)$, you have the same result (homogeneity of degree zero).

$$u(x) = \min_{p \in \mathbb{R}_{++}^n} v(p, 1) = p_1^\alpha p_2^\beta, s.t. px = 1$$

Lagrangian:

$$L = p_1^\alpha p_2^\beta + \lambda(1 - p_1 x_1 - p_2 x_2)$$

Note there should not be interior solution since

- $\frac{\partial v(p_1, p_2, 1)}{\partial p_1} = \alpha p_1^{\alpha-1} p_2^\beta, \alpha, \beta < 0. \lim_{p_1 \to 0} \frac{\partial v(p_1, p_2, 1)}{\partial p_1} = -\infty$

- $\frac{\partial v(p_1, p_2, 1)}{\partial p_2} = p_1^\alpha \beta p_2^{\beta-1}, \alpha, \beta < 0. \lim_{p_2 \to 0} \frac{\partial v(p_1, p_2, 1)}{\partial p_2} = -\infty$

- $v(p, 1)$ is decreasing in p (this is always true for indirect utility function, see pp.29). For any $px < 1$, you can always decrease $v(p, 1)$ by increasing p until $px = 1$.

FOCs.

$$\begin{cases} \frac{\partial L}{\partial p_1} = \alpha p_1^{\alpha-1} p_2^\beta - \lambda x_1 = 0 \\ \frac{\partial L}{\partial p_2} = p_1^\alpha \beta p_2^{\beta-1} - \lambda x_2 = 0 \\ p_1 x_1 + p_2 x_2 = 1 \end{cases}$$

Simplify:

$$\begin{cases} \alpha p_1^{\alpha-1} p_2^{\beta} = \lambda x_1 \\ \beta p_1^{\alpha} p_2^{\beta-1} = \lambda x_2 \\ p_1 x_1 + p_2 x_2 = 1 \end{cases} \qquad (1)$$

Take the ratio between first and second condition to get:

$$\frac{x_1}{x_2} = \frac{\alpha}{\beta} \frac{p_2}{p_1}$$

Thus: $p_2 = \frac{\beta}{\alpha} \frac{x_1}{x_2} p_1$
Substitute p_2 with p_1 in the 3rd condition to get:

$$p_1 x_1 + \frac{\beta}{\alpha} \frac{x_1}{x_2} p_1 x_2 = 1$$

$$p_1 (x_1 + \frac{\beta}{\alpha} x_1) = 1$$

$$p_1^* = \frac{1}{x_1 (1 + \frac{\beta}{\alpha})}$$

$$\Rightarrow p_2^* = \frac{\beta}{\alpha} \frac{x_1}{x_2} p_1 = \frac{\beta}{\alpha} \frac{x_1}{x_2} \frac{1}{x_1 (1 + \frac{\beta}{\alpha})} = \frac{1}{x_2 (1 + \frac{\alpha}{\beta})}$$

Substitute p_1^* and p_2^* into $v(p, 1)$ we get the minimized value, i.e. the direct utility function:

$$u(x_1.x_2) = [\frac{1}{x_1 (1 + \frac{\beta}{\alpha})}]^{\alpha} [\frac{1}{x_2 (1 + \frac{\alpha}{\beta})}]^{\beta}$$

$$= A x_1^a x_2^b$$

Where $A = [\frac{1}{1 + \frac{\beta}{\alpha}}]^{\alpha} [\frac{1}{1 + \frac{\alpha}{\beta}}]^{\beta}$, $a = -\alpha > 0$, $b = -\beta > 0$. The utility function is a Cobb-Douglas function.

As a cautious proof, you may want to check if $u(x)$ is quasiconcave and differentiable on $\mathbb{R}_{++}^n$ with strictly positive partial derivatives there, as assumed by Theorem 2.3.
In exam for this course, again, if the function is one- dimension, you should prove it; if it's a higher-dimension function, the proof is not required.

Like Jehle & Reny 1.51, you can actually transform $v(p_1, p_2, 1)$ into a function of only p_1 or p_1 using $p_1 x_1 + p_2 x_2 = 1$.

$$p_1 = \frac{1 - p_2 x_2}{x_1}$$

Substitute into $v(p_1, p_2, 1)$ to have:

$$v(p_1, p_2, 1) = [\frac{1 - p_2 x_2}{x_1}]^\alpha p_2^\beta$$

Since the question ask you to minimize $v(p_1, p_2, 1)$, if you solve $\frac{de(p_2)}{dp_2} = 0$ and get only one solution, it is the solution.

$$\frac{de(p_2)}{dp_2} = \alpha(\frac{1 - p_2 x_2}{x_1})^{\alpha-1}(\frac{-x_2}{x_1})p_2^\beta + \frac{1 - p_2 x_2}{x_1}^\alpha \beta p_2^{\beta-1} = 0$$

$$\alpha(\frac{1 - p_2 x_2}{x_1})^{\alpha-1}(\frac{x_2}{x_1})p_2^\beta = \frac{1 - p_2 x_2}{x_1}^\alpha \beta p_2^{\beta-1}$$

$$\alpha(\frac{x_1}{1 - p_2 x_2})(\frac{x_2}{x_1})p_2 = \beta$$

$$\alpha(\frac{x_2}{1 - p_2 x_2})p_2 = \beta$$

$$\alpha x_2 p_2 = \beta - \beta x_2 p_2$$

$$(\alpha x_2 + \beta x_2)p_2 = \beta$$

$$p_2^* = \frac{\beta}{(\alpha + \beta)x_2}$$

You then solve p_1^* with the budget constraint.

2   Jehle & Reny 2.5(a)

Consider the solution, $e(p, u) = u p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ at the end of Example 2.3. Derive the **indirect utility function** through the relation $e(p, v(p, y)) = y$ and verify Roy's identity.

Example 2.3 on Jehle & Reny pp.90 is a question from $x_i(p, y)$ to $e(p, u)$, where the Marshallian demand function is:

$$x_i(p_1, p_2, p_3, y) = \frac{\alpha_i y}{p_i}, \ i = 1, 2, 3$$

$\alpha_i > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$
Check your map, the route is (note the expression below is only for the purpose of teaching and very informal):

$$x_i(p, y) \Rightarrow x^h(p, u) = x[p, e(p, u)] \Leftarrow \frac{\partial e(p, u)}{\partial p_i} = x^h(p, u)$$

$$x[p, e(p,u)] = \frac{\partial e(p,u)}{\partial p_i}$$

$$\frac{\alpha_i e(p,u)}{p_i} = \frac{\partial e(p,u)}{\partial p_i}$$

$$\frac{\alpha_i}{p_i} = \frac{1}{e(p,u)} \frac{\partial e(p,u)}{\partial p_i}$$

$$= \frac{\partial ln[e(p,u)]}{\partial p_i}$$

The rest part of the solution in the textbook is clear. Read page 91 if you're curious how we solve $e(p,u)$ out. It need "a little thought" as the textbook said :)

Indirect utility function:

We already know $e(p, v(p,y)) = y$.

Substitute $v(p,y)$ into $e(p,u) = y$ will solve the question directly:

$$e(p,u) = v(p,y) p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} = y$$

$$\Rightarrow v(p,y) = \frac{y}{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}}$$

Verify Roy's identity:

Roy's Identity(Jehle & Reny pp.29):

$$x_i(p^0, y^0) = -\frac{\partial v(p^0, y^0)/\partial p_i}{\partial v(p^0, y^0)/\partial y}$$

Intuition:Your optimal consumption plan (Marshallian demand) is a trade-off between the importance of "comodity i" and "money (y)".

$$\frac{\partial v(p,y)}{\partial p_1} = \frac{\partial y p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}}{\partial p_1} = -\alpha_1 y p_1^{-\alpha_1-1} p_2^{-\alpha_2} p_3^{-\alpha_3}$$

$$\frac{\partial v(p,y)}{\partial p_2} = -\alpha_3 y p_1^{-\alpha_1} p_2^{-\alpha_2-1} p_3^{-\alpha_3}$$

$$\frac{\partial v(p,y)}{\partial p_3} = -\alpha_3 y p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3-1}$$

$$\frac{\partial v(p,y)}{\partial y} = p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}$$

Therefore:

$$-\frac{\partial v(p,y)/\partial p_1}{\partial v(p,y)/\partial y} = -\frac{-\alpha_1 y p_1^{-\alpha_1-1} p_2^{-\alpha_2} p_3^{-\alpha_3}}{p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}} = \frac{\alpha_1 y}{p_1}$$

$$-\frac{\partial v(p,y)/\partial p_2}{\partial v(p,y)/\partial y} = -\frac{-\alpha_3 y p_1^{-\alpha_1} p_2^{-\alpha_2-1} p_3^{-\alpha_3}}{p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}} = \frac{\alpha_2 y}{p_1}$$

$$-\frac{\partial v(p,y)/\partial p_3}{\partial v(p,y)/\partial y} = -\frac{-\alpha_3 y p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3-1}}{p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}} = \frac{\alpha_3 y}{p_1}$$

Compare with the Marshallian demand!

3 Jehle & Reny 2.7

Derive the consumer's **inverse demand functions**, $p_1(x_1, x_2)$ and $p_2(x_1, x_2)$, when the **utility function** is of the Cobb-Douglas form, $u(x_1, x_2) = A x_1^{\alpha} x_2^{1-\alpha}$ for $0 < \alpha < 1$.

> The shortest route is using Hotelling, Wold (pp.84) directly.
>
> $$p_i(x) = \frac{\partial u(x)/\partial x_i}{\Sigma_{j=1}^{n} x_j (\partial u(x)/\partial x_j)}$$
>
> **Intuition**: the price reflects how important the commodity is.
>
> The duality between direct and indirect utility functions showed by Hotelling, Wold makes it (hopefully) easier to solve $p_i^*(x)$

$$
\begin{aligned}
p_1(x_1, x_2) &= \frac{\partial(A x_1^{\alpha} x_2^{1-\alpha})/\partial x_1}{\Sigma_{j=1}^{2} x_j (\partial(A x_1^{\alpha} x_2^{1-\alpha})/\partial x_j)} \\[2mm]
&= \frac{\partial(A x_1^{\alpha} x_2^{1-\alpha})/\partial x_1}{x_1 \partial(A x_1^{\alpha} x_2^{1-\alpha})/\partial x_1 + x_2 \partial(A x_1^{\alpha} x_2^{1-\alpha})/\partial x_2} \\[2mm]
&= \frac{A \alpha x_1^{\alpha-1} x_2^{1-\alpha}}{x_1 A \alpha x_1^{\alpha-1} x_2^{1-\alpha} + x_2 A(1-\alpha) x_1^{\alpha} x_2^{-\alpha}} \\[2mm]
&= \frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{\alpha x_1^{\alpha} x_2^{1-\alpha} + (1-\alpha) x_1^{\alpha} x_2^{1-\alpha}} \\[2mm]
&= \frac{\alpha}{x_1}
\end{aligned}
$$

$$p_2(x_1, x_2) = \frac{\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_2}{\Sigma_{j=1}^2 x_j(\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_j)}$$

$$= \frac{\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_2}{x_1\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_1 + x_2\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_2}$$

$$= \frac{A(1-\alpha)x_1^\alpha x_2^{-\alpha}}{x_1 A\alpha x_1^{\alpha-1} x_2^{1-\alpha} + x_2 A(1-\alpha)x_1^\alpha x_2^{-\alpha}}$$

$$= \frac{(1-\alpha)x_1^\alpha x_2^{-\alpha}}{\alpha x_1^\alpha x_2^{1-\alpha} + (1-\alpha)x_1^\alpha x_2^{1-\alpha}}$$

$$= \frac{1-\alpha}{x_2}$$

You can also try another route: maximize $u(x) \Rightarrow x(p,y) \Rightarrow p_i(x) = x^{-1}(x,1)$
Use Lagrangian to maximize $u(x_1, x_2) = Ax_1^\alpha x_2^{1-\alpha}$ s.t. $p_1 x_2 + p_2 x_2 = 1$. The solution (Marshallian demands) is:

$$\begin{cases} x_1 = \frac{\alpha}{p_1} \\ x_2 = \frac{1-\alpha}{p_2} \end{cases}$$

The inverse of Marshallian demand function gives the inverse demand function

$$\begin{cases} p_1 = \frac{\alpha}{x_1} \\ p_2 = \frac{1-\alpha}{x_2} \end{cases}$$

Another example:

- You can also try to derive $p_i(x)$ from the Marshallian demand E.1 on pp. 32 and compare with the result E.5-E.6 on pp. 83, which is derived from Hotelling-Wold identity.