Seminar 3.Duality of Consumers Behavior

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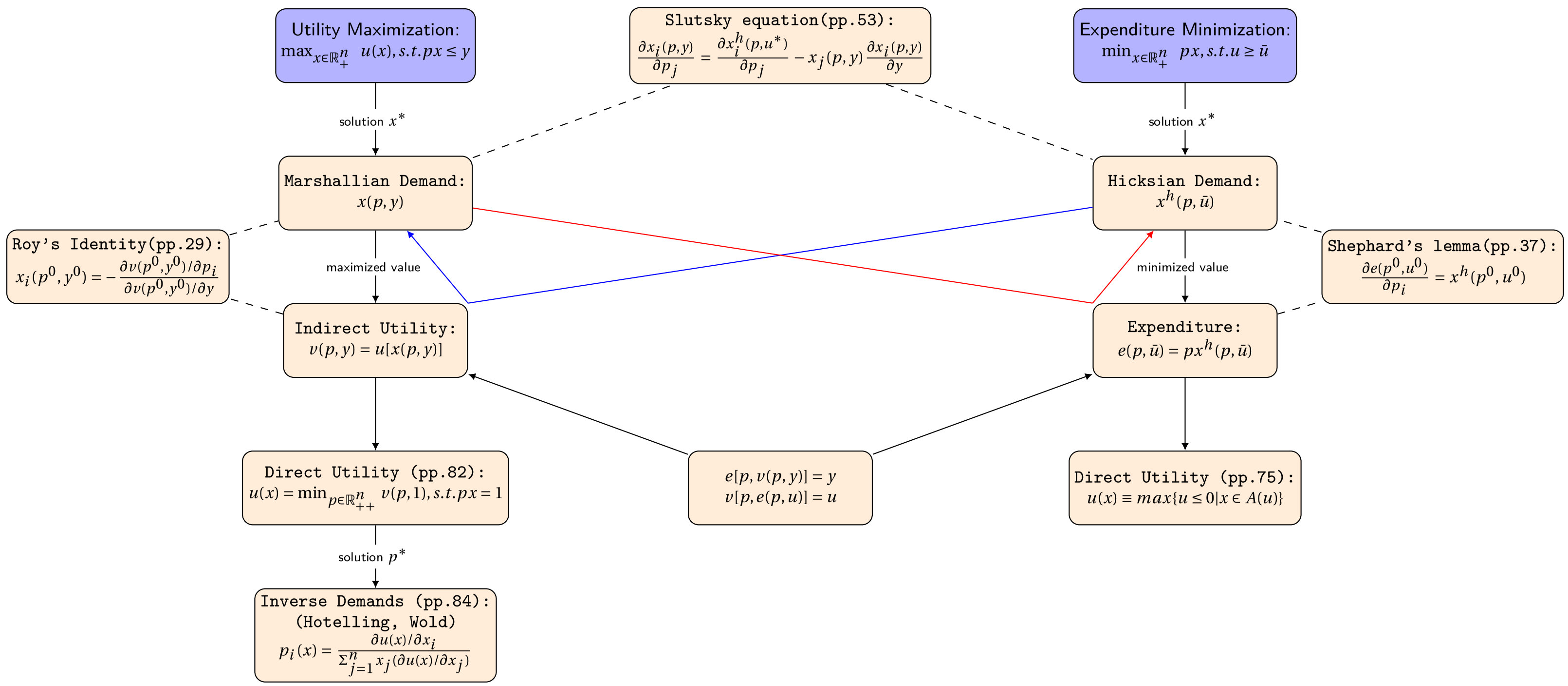
Consumption Duality

You will never lose your way with this Consumption Duality map!
All "derive this from that and verify some guy's equation"-like questions can be solved by finding the correct (shortest) route.



1 Jehle & Reny 2.3

Derive the consumers direct utility function if his indirect utility function has the form $v(p,y) = yp_1^{\alpha}p_2^{\beta}$ for negative α and β.

THEOREM 2.3 Duality Between Direct and Indirect Utility(Jehle & Reny pp.81)
Suppose that $u(x)$ is quasiconcave and differentiable on $\mathbb{R}^n_{++}$ with strictly positive partial derivatives there. Then for all $x \in \mathbb{R}^n_{++}$, $v(p, p\cdot x)$, the indirect utility function generated by $u(x)$, achieves a minimum in p on $\mathbb{R}^n_{++}$, and

$$u(x) = \min_{p\in\mathbb{R}^n_{++}} v(p,y), s.t. px = y$$

Let's call the solution p^*
Note that by **Theorem 1.6**(Jehle & Reny pp.29), $v(p,y)$ is homogeneous of degree zero in (p,y). We have $v(p, p\cdot x) = v(p/(p\cdot x), 1)$ whenever $p\cdot x > 0$. Thus the equation above can also be written as:

$$u(x) = \min_{p\in\mathbb{R}^n_{++}} v(p,1), s.t. px = 1$$

The solution $\hat{p} = p^*/p^*\cdot x = p^*/y$. We don't care about the difference between $\hat{p}$ and p^* because once you substitute them into $v(p, p\cdot x)$, you have the same result (homogeneity of degree zero).

$$u(x) = \min_{p\in\mathbb{R}^n_{++}} v(p,1) = p_1^{\alpha}p_2^{\beta}, s.t. px = 1$$

Lagrangian:

$$L = p_1^{\alpha}p_2^{\beta} + \lambda(1 - p_1x_1 - p_2x_2)$$

Note there should not be interior solution since

- $\frac{\partial v(p_1,p_2,1)}{\partial p_1} = \alpha p_1^{\alpha-1}p_2^{\beta}, \alpha,\beta < 0. \lim_{p_1\to 0}\frac{\partial v(p_1,p_2,1)}{\partial p_1} = -\infty$
- $\frac{\partial v(p_1,p_2,1)}{\partial p_2} = p_1^{\alpha}\beta p_2^{\beta-1}, \alpha,\beta < 0. \lim_{p_2\to 0}\frac{\partial v(p_1,p_2,1)}{\partial p_2} = -\infty$
- $v(p,1)$ is decreasing in p(this is always true for indirect utility function, see pp.29). For any $px < 1$, you can always decrease $v(p,1)$ by increasing p until $px = 1$.

FOCs.

$$\begin{cases} \frac{\partial L}{\partial p_1} = \alpha p_1^{\alpha-1}p_2^{\beta} - \lambda x_1 = 0 \\ \frac{\partial L}{\partial p_2} = p_1^{\alpha}\beta p_2^{\beta-1} - \lambda x_2 = 0 \\ p_1x_1 + p_2x_2 = 1 \end{cases}$$

Simplify:

$$\begin{cases} \alpha p_1^{\alpha-1} p_2^{\beta} = \lambda x_1 \\ \beta p_1^{\alpha} p_2^{\beta-1} = \lambda x_2 \\ p_1 x_1 + p_2 x_2 = 1 \end{cases} \tag{1}$$

Take the ratio between first and second condition to get:

$$\frac{x_1}{x_2} = \frac{\alpha}{\beta} \frac{p_2}{p_1}$$

Thus: $p_2 = \frac{\beta}{\alpha} \frac{x_1}{x_2} p_1$
Substitute p_2 with p_1 in the 3rd condition to get:

$$\begin{aligned} p_1 x_1 + \frac{\beta}{\alpha} \frac{x_1}{x_2} p_1 x_2 = 1 \\ p_1 (x_1 + \frac{\beta}{\alpha} x_1) = 1 \\ p_1^* = \frac{1}{x_1 (1 + \frac{\beta}{\alpha})} \\ \Rightarrow p_2^* = \frac{\beta}{\alpha} \frac{x_1}{x_2} p_1 = \frac{\beta}{\alpha} \frac{x_1}{x_2} \frac{1}{x_1 (1 + \frac{\beta}{\alpha})} = \frac{1}{x_2 (1 + \frac{\alpha}{\beta})} \end{aligned}$$

Substitute p_1^* and p_2^* into $v(p, 1)$ we get the minimized value, i.e. the direct utility function:

$$\begin{aligned} u(x_1 . x_2) &= [\frac{1}{x_1 (1 + \frac{\beta}{\alpha})}]^{\alpha} [\frac{1}{x_2 (1 + \frac{\alpha}{\beta})}]^{\beta} \\ &= A x_1^a x_2^b \end{aligned}$$

Where $A = [\frac{1}{1+\frac{\beta}{\alpha}}]^{\alpha} [\frac{1}{1+\frac{\alpha}{\beta}}]^{\beta}$,$a = -\alpha > 0$, $b = -\beta > 0$. The utility function is a Cobb-Douglas function.

As a cautious proof, you may want to check if $u(x)$ is quasiconcave and differentiable on $\mathbb{R}_{++}^n$ with strictly positive partial derivatives there, as assumed by Theorem 2.3.
In exam for this course, again, if the function is one- dimension, you should prove it; if it's a higher-dimension function, the proof is not required.

Like Jehle & Reny 1.51, you can actually transform $v(p_1, p_2, 1)$ into a function of only p_1 or p_1 using $p_1 x_1 + p_2 x_2 = 1$.

$$p_1 = \frac{1 - p_2 x_2}{x_1}$$

Substitute into $v(p_1, p_2, 1)$ to have:

$$v(p_1, p_2, 1) = [\frac{1 - p_2 x_2}{x_1}]^\alpha p_2^\beta$$

Since the question ask you to minimize $v(p_1, p_2, 1)$, if you solve $\frac{de(p_2)}{dp_2} = 0$ and get only one solution, it is the solution.

$$\begin{aligned}
\frac{de(p_2)}{dp_2} = \alpha(\frac{1 - p_2 x_2}{x_1})^{\alpha-1}(\frac{-x_2}{x_1})p_2^\beta + \frac{1 - p_2 x_2}{x_1}^\alpha \beta p_2^{\beta-1} = 0 \\
\alpha(\frac{1 - p_2 x_2}{x_1})^{\alpha-1}(\frac{x_2}{x_1})p_2^\beta = \frac{1 - p_2 x_2}{x_1}^\alpha \beta p_2^{\beta-1} \\
\alpha(\frac{x_1}{1 - p_2 x_2})(\frac{x_2}{x_1})p_2 = \beta \\
\alpha(\frac{x_2}{1 - p_2 x_2})p_2 = \beta \\
\alpha x_2 p_2 = \beta - \beta x_2 p_2 \\
(\alpha x_2 + \beta x_2)p_2 = \beta \\
p_2^* = \frac{\beta}{(\alpha + \beta)x_2}
\end{aligned}$$

You then solve p_1^* with the budget constraint

2 Jehle & Reny 2.5(a)

Consider the solution, $e(p, u) = u p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ at the end of Example 2.3. Derive the indirect utility function through the relation $e(p, v(p, y)) = y$ and verify Roy's identity.

3 Jehle & Reny 2.7

Derive the consumer's **inverse** demand functions, $p_1(x_1, x_2)$ and $p_2(x_1, x_2)$, when the utility function is of the Cobb-Douglas form, $u(x_1, x_2) = A x_1^\alpha x_2^{1-\alpha}$ for $0 < \alpha < 1$.