# Seminar 5. Production Theory

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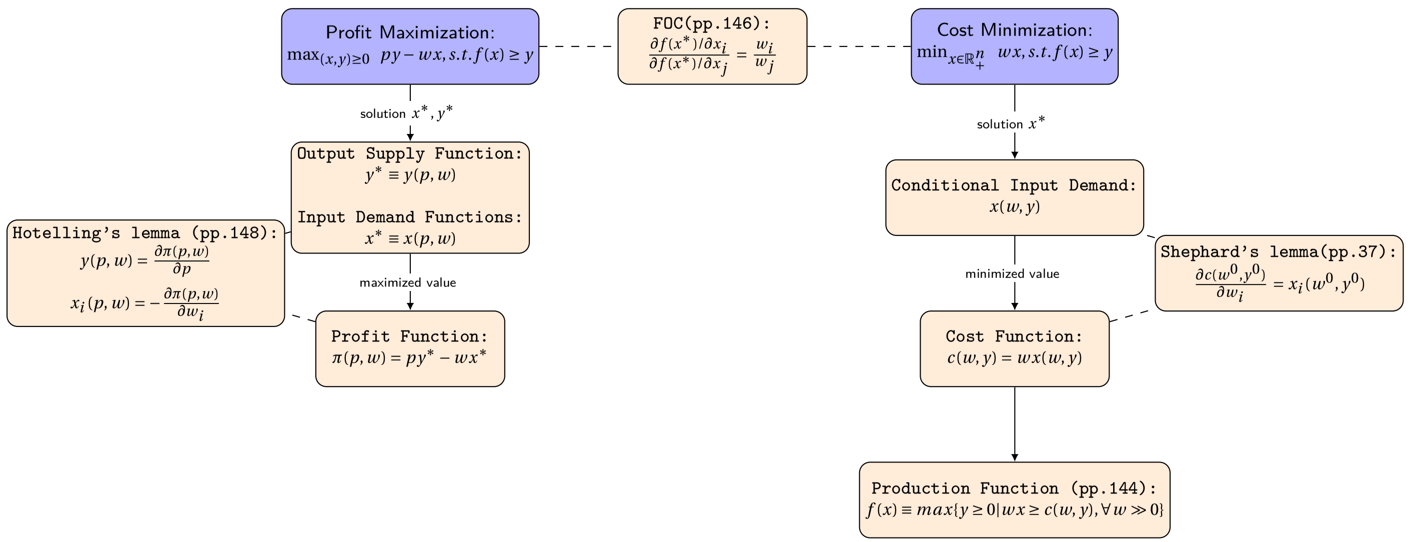
# Production Duality



# 1 Jehle & Reny 3.35

Calculate the **cost function** and the **conditional input demands** for the linear production function, $y = \Sigma_{i=1}^{n} \alpha_i x_i$.

**Production Function**(Jehle & Reny pp.127)
We use a function $y = f(x)$ to denote $y$ units of a certain commodity is produced using input $x$, where $x \in \mathbb{R}_+^n, y \in \mathbb{R}_+^1$

**ASSUMPTION 3.1 Properties of the Production Function** (Jehle & Reny pp.127)
The production function, $f : \mathbb{R}_+^n \to \mathbb{R}_+$ , is continuous, strictly increasing, and strictly quasiconcave on $\mathbb{R}_+^n$, and $f(0) = 0$.

**DEFINITION 3.5 The Cost Function** (Jehle & Reny pp.136)
The cost function, defined for all input prices $w \gg 0$ and all output levels $y \in f(\mathbb{R}_+^n)$ is the minimum-value function,

$$c(w, y) \equiv \min_{x \in \mathbb{R}_+^n} w \cdot x, \quad s.t. \quad f(x) \geq y.$$

The solution $x(w, y)$ is referred to as the firms **conditional input demand**, because it is conditional on the level of output $y$.

- **Conditional input demand** is similar to Hicksian demands for consumers.

Here the linear production function $y = \Sigma_{i=1}^{n} \alpha_i x_i$ is very similar to the "**perfect substitution**" preference in Seminar 4.

- The product can be produced by any input $x_i$, the only difference is that for each unit of input, different $x_i$ produces different amount $\alpha_i$ of the output.

- The Marginal Rate of Technical Substitution of input $x_j$ for input $x_i$ is $MRTS_{ij} = \frac{\partial f(x)/\partial x_i}{\partial f(x)/\partial x_j} = \frac{\alpha_i}{\alpha_j}$.

An example: an apple jam factory has 2 types of input, "single apple ($x_1$)" and "2-apple pack ($x_2$)".

- With a "single apple", the factory can produce a bottle of apple jam;

- with a "2-apple pack", 2 bottles.

- The production function is $y = 1 \cdot x_1 + 2 \cdot x_2$

How will the factory choose? Similarly to consumers' preference substitution preference, the factory will spend all money on the "cheapest per unit" input. Denote the price for $x_1$ and $x_2$ as $w_1$, $w_2$

- If $\frac{w_1}{1} = \frac{w_2}{2}$, the factory doesn't care which to use;
- If $\frac{w_1}{1} < \frac{w_2}{2}$, single apple is cheaper;
- If $\frac{w_1}{1} > \frac{w_2}{2}$, 2-apple pack is cheaper.

Denote the price for input $x_i$ as $w_i$. Define $\omega = min\{\frac{w_1}{\alpha_1}, \frac{w_2}{\alpha_2}, \dots, \frac{w_n}{\alpha_n}\}$

If $\omega$ is the price of only one input $x_j$, the firm will only use input $x_j$ to minimize its cost.

- Thus $y = \alpha_j x_j$ can minimize the cost, and $x_j = \frac{y}{\alpha_j}$ is the conditional input demands.
- The cost function $c(w, y) = w_j \frac{y}{\alpha_j} = \omega y$.

If $\omega$ is the price of several inputs $x_m, m = 1, 2, \dots, M$, the firm can freely combine $x_m$ to minimize its cost, as long as $\Sigma_{m=1}^{M} \alpha_m x_m = y$.

For cost funtion, let's assume $\frac{w_1}{\alpha_1} = \frac{w_2}{\alpha_2} = \dots = \frac{w_M}{\alpha_M} = \omega$, then $\omega$ is the price for 1 single apple, for example. Again, to produce $y$ bottles of jam, you need the same number of single apples. The cost function is thus $\omega y$

# 2 Jehle & Reny 3.46

- Verify Theorem 3.7 for the profit function obtained in Example 3.5.
- Verify Theorem 3.8 for the associated output supply and input demand functions.

## 2.1 Verify Theorem 3.7

**DEFINITION 3.7 The Profit Function** (Jehle & Reny pp.148)

$$\pi(p, w) \equiv \max_{(x,y) \geq 0} py - wx, \;\; s.t. y \leq f(x)$$

**Note,** $y \leq f(x)$ means "you can only decide to produce what's possible to be produced":

- Assume you want to set an optimal output $y$, forget input $x$ for now;
- You can "waste", i.e., output $y < f(x)$ is possible. Input is not efficiently used for technology $f(x)$;
- You can't produce more than what your technology $f(x)$ allows, i.e. $y \not> f(x)$.

It's not easy to solve the maximization problem directly because there are 2

variable, $y$ and $x$ (again, $y$ is **NOT** necessarily to be $f(x)$, but it can't exceed $f(x)$).
Since to waste will definitely reduce profit (you have some cost but don't produce anything), you will always avoid wasting by making fully use of your technology, i.e. $y = f(x)$. Then the profit maximization problem is transformed into:

$$\pi(p, w) = \max_{x \geq 0} pf(x) - wx$$

No constraint anymore! The function has only one variable, input $x$. FOC will solve the problem.
In **Example 3.5** (Jehle & Reny pp.151), the CES production function is:

$$y = (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}}$$

Where $\beta < 1$ and $0 \neq \rho < 1$. To obtain the profit function, we need to solve the maximization problem:

$$\max_{(x,y)\geq 0} py - wx, \;\; s.t. \;\; y \leq (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}}$$

Again, we don't waste, i.e. $y = (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}}$, the problem above is then:

$$\max_{x \geq 0} p(x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}} - w_1 x_1 - w_2 x_2$$

FOC are given in the textbook pp. 151.
The $y^*$ solved is the **output supply function**:

$$y^* = (p\beta)^{-\beta/(\beta-1)}(w_1^r + w_2^r)^{\beta/r(\beta-1)}$$

The $x^*$ solved is the **input demand function**:

$$x_i^* = w_i^r (p\beta)^{-1/(\beta-1)}(w_1^r + w_2^r)^{(\rho-\beta)/\rho(\beta-1)}$$

The profit function is thus

$$\pi = py^* - wx^* = p^{-1/(\beta-1)}(w_1^r + w_2^r)^{\beta/r(\beta-1)}\beta^{-\beta/(\beta-1)}(1-\beta)$$

**THEOREM 3.7 Properties of the Profit Function** (Jehle & Reny pp.148)
If $f$ satisfies Assumption 3.1, then for $p \geq 0$ and $w \geq 0$, the profit function $\pi(p, w)$, where well-defined, is continuous and

1. Increasing in $p$,
2. Decreasing in $w$,
3. Homogeneous of degree one in $(p, w)$,

4. Convex in $(p, w)$,

5. Differentiable in $(p, w)$.

6. Moreover, under the additional assumption that $f$ is strictly concave (Hotelling's lemma),

$$y(p,w) = \frac{\partial \pi(p,w)}{\partial p}, \ and \ \ x_i(p,w) = -\frac{\partial \pi(p,w)}{\partial w_i}. \quad i = 1,2,\ldots,n.$$

### 2.2 Verify Theorem 3.8

**THEOREM 3.8 Properties of Output Supply and Input Demand Functions** (Jehle & Reny pp.149)
Suppose that $f$ is a strictly concave production function satisfying Assumption 3.1 and that its associated profit function, $\pi(p, y)$, is twice continuously differentiable. Then, for all $p > 0$ and $w \gg 0$ where it is well defined:

1. Homogeneity of degree zero:

$$y(tp, tw) = y(p, w), \forall t > 0,$$

$$x_i(tp, tw) = x_i(p, w), \forall t > 0 \ and \ i = 1, \ldots, n.$$

2. Own-price effects:

$$\frac{\partial y(p,w)}{\partial p} \geq 0,$$

$$\frac{\partial x_i(p,w)}{\partial w_i} \leq 0, \ \forall i = 1, \ldots, n.$$

3. The substitution matrix is symmetric and positive semidefinite.

$$\begin{pmatrix} \frac{\partial y(p,w)}{\partial p} & \frac{\partial y(p,w)}{\partial w_1} & \cdots & \frac{\partial y(p,w)}{\partial w_n} \\ \frac{\partial x_1(p,w)}{\partial p} & \frac{\partial x_1(p,w)}{\partial w_1} & \cdots & \frac{\partial x_1(p,w)}{\partial w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n(p,w)}{\partial p} & \frac{\partial x_n(p,w)}{\partial w_1} & \cdots & \frac{\partial x_n(p,w)}{\partial w_n} \end{pmatrix} \tag{1}$$

## 3 Jehle & Reny 3.49

1. Derive the **cost function** for the production function in Example 3.5 .

2. Solve $\max_y py - c(w, y)$

3. Compare its solution, $y(p, w)$, to the solution in (E.5). Check that $\pi(p, w) = py(p, w) - c(w, y(p, w))$.

4. Supposing that $\beta > 1$, confirm our conclusion that profits are minimised when the first-order conditions are satisfied by showing that marginal cost is decreasing at the solution.

5. Sketch your results.

### 3.1 Cost function

CES production function in Example 3.5 : $y = (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}}$, $\beta < 1$ and $0 \neq \rho < 1$
Cost function: $c(w, y) \equiv \min_{x \in \mathbb{R}_+^n} w \cdot x,\ \ s.t.\ \ f(x) \geq y$.

$$c(w, y) = \min_{x \in \mathbb{R}_+^n} w_1 x_1 + w_2 x_2,\ \ s.t.\ \ (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}} \geq y$$

No corner solution.

- Obviously $x_1, x_2$ can't both be 0 to produce $y > 0$.
- $\frac{\partial f(x)}{\partial x_i} = \beta(x_1^\rho + x_2^\rho)^{(\frac{\beta}{\rho})-1} x_i^{\rho-1}$. If $\rho \in (0,1), \beta > 0$, $\lim_{x_i \to 0} \frac{\partial f(x)}{\partial x_i} = +\infty$.
  - If $\rho \in (0,1), \beta < 0$, the production function doesn't make sense since $\lim_{x \to (0,0)} f(x) = +\infty$
  - If $\rho < 0$, the production function is not defined at $x_i = 0$
- $f(x) = y$ is binding: $f(x)$ is increasing in $x$, to reduce cost, we shouldn't produce more than required $(y)$.

$$L = w_1 x_1 + w_2 x_2 + \lambda[y - (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}}]$$

FOC:

$$\begin{cases} \frac{\partial L}{\partial x_1} = w_1 - \lambda\beta(x_1^\rho + x_2^\rho)^{(\frac{\beta}{\rho})-1} x_1^{\rho-1} = 0 \\ \frac{\partial L}{\partial x_2} = w_2 - \lambda\beta(x_1^\rho + x_2^\rho)^{(\frac{\beta}{\rho})-1} x_2^{\rho-1} = 0 \\ y - (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}} = 0 \end{cases}$$

Simplify:

$$\begin{cases} w_1 = \lambda\beta(x_1^\rho + x_2^\rho)^{(\frac{\beta}{\rho})-1} x_1^{\rho-1} \\ w_2 = \lambda\beta(x_1^\rho + x_2^\rho)^{(\frac{\beta}{\rho})-1} x_2^{\rho-1} \\ (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}} = y \end{cases} \tag{2}$$

Taking the ratio between the first two gives:

$$\frac{w_1}{w_2} = (\frac{x_1}{x_2})^{\rho-1} \Rightarrow x_1 = (\frac{w_1}{w_2})^{\frac{1}{\rho-1}} x_2$$

Substituting in the third gives:

$$
\begin{aligned}
\{[(\frac{w_1}{w_2})^{\frac{1}{\rho-1}} x_2]^\rho + x_2^\rho\}^{\frac{\beta}{\rho}} &= y \\
[(\frac{w_1}{w_2})^{\frac{\rho}{\rho-1}} x_2^\rho + x_2^\rho]^{\frac{\beta}{\rho}} &= y \\
[(\frac{w_1}{w_2})^{\frac{\rho}{\rho-1}} + 1]^{\frac{\beta}{\rho}} x_2^\beta &= y \\
x_2 &= (\frac{y}{[(\frac{w_1}{w_2})^{\frac{\rho}{\rho-1}} + 1]^{\frac{\beta}{\rho}}})^{\frac{1}{\beta}} \\
&= y^{\frac{1}{\beta}} [\frac{1}{(\frac{w_1}{w_2})^{\frac{\rho}{\rho-1}} + 1}]^{\frac{1}{\rho}} \\
&= y^{\frac{1}{\beta}} [\frac{1}{(\frac{w_1}{w_2})^{\frac{\rho}{\rho-1}} + 1}]^{\frac{1}{\rho}} \\
&= y^{\frac{1}{\beta}} (\frac{w_2^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}})^{\frac{1}{\rho}} \\
\Rightarrow x_1 &= (\frac{w_1}{w_2})^{\frac{1}{\rho-1}} x_2 \\
&= y^{\frac{1}{\beta}} (\frac{w_1^{\frac{\rho}{\rho-1}}}{w_2^{\frac{\rho}{\rho-1}}} \frac{w_2^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}})^{\frac{1}{\rho}} \\
&= y^{\frac{1}{\beta}} (\frac{w_1^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}})^{\frac{1}{\rho}}
\end{aligned}
$$

Cost function:

$$
\begin{aligned}
c(w,y) = w_1x_1 + w_2x_2 &= w_1 y^{\frac{1}{\beta}} \left(\frac{w_1^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}}\right)^{\frac{1}{\rho}} + w_2 y^{\frac{1}{\beta}} \left(\frac{w_2^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}}\right)^{\frac{1}{\rho}} \\
&= w_1 y^{\frac{1}{\beta}} \frac{w_1^{\frac{1}{\rho-1}}}{(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{\frac{1}{\rho}}} + w_2 y^{\frac{1}{\beta}} \frac{w_2^{\frac{1}{\rho-1}}}{(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{\frac{1}{\rho}}} \\
&= y^{\frac{1}{\beta}} \left[\frac{w_1^{(\frac{1}{\rho-1}+1)}}{(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{\frac{1}{\rho}}} + \frac{w_2^{(\frac{1}{\rho-1}+1)}}{(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{\frac{1}{\rho}}}\right] \\
&= y^{\frac{1}{\beta}} \frac{w_1^r + w_1^r}{(w_1^r + w_2^r)^{\frac{1}{\rho}}} \\
&= y^{\frac{1}{\beta}} (w_1^r + w_2^r)^{1-\frac{1}{\rho}} \\
&= y^{\frac{1}{\beta}} (w_1^r + w_2^r)^{\frac{1}{r}}
\end{aligned}
$$

Where $r = \frac{\rho}{\rho-1}$

### 3.2 $\max_y py - c(w,y)$

### 3.3 $\pi(p,w) = py(p,w) - c(w, y(p,w))$

### 3.4 FOC

### 3.5 Sketch