Seminar 9.Incomplete information in Static Games

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October 29, 2020

1 Problem 1 - Bayesian normal form and NE

Consider the following two normal form games. Assume that **only player** 1 **knows** which game is being played, while player 2 thinks that the two games are **equally likely**.

Game 1

	L	R
U	0,0	4,2
D	2,6	0,8

Game 2

	L	R
U'	0,2	0,0
D'	2,0	2,2

(a) Bayesian normal form

Model this situation in an *ex ante* perspective by specifying the Bayesian normal form.

> Who has contingent strategies?
>
> - Player 1 has private information (Game 1 or Game 2 is played), therefore has $2 \times 2 = 4$ types of contingent strategies,
>
> - Player 2 doesn't know either which game is played (incomplete) or how player 1 moves (static), thus can only choose L or R.

	L	R
UU'	0,1	2,1
UD'	1,0	3,2
DU'	1,4	0,4
DD'	2,3	1,5

We calculate the payoff according to player 2's belief. For example, the payoff for (UU', L) is (0,1):

$$U^1 = 0.5 \times 0 + 0.5 \times 0 = 0$$

$$U^2 = 0.5 \times 0 + 0.5 \times 2 = 1$$

Note: the question is slightly different from those in Watson textbook.

In the textbook, it often says the nature moves with probability (Pr_1, Pr_2), i.e. both of the two players have expected payoff. Here only player 2 has to guess, while player 1 doesn't need to.

Although the question can be solved in the same way, you may wonder if it's reasonable to calculate player 1's "expected payoff" according to player 2's belief.

The payoff in the Bayesian noraml form is actually player 2's imagination.

- Player 2 believes player 1 has the expected payoff $U^1 = 0.5 \times \cdots + 0.5 \times \cdots$, and player 1 knows what player 2 believes.

- In the view of player 1, there is actually no expected payoff, since the which game to play is determined for him/her.

We calculate player 1's "expected payoff" according to player 2's belief for 2 reasons:

- Player 2 believes player 1 has this payoff, and acts according to it. This will in return affect palyer 1's behavior.

- Even though player 1's "expected payoff" is calcualted by $U^1 = 0.5 \times \cdots + 0.5 \times \cdots$, what it reflects is actually his/her peference given which game is played. For example:

    - We know if Game 1 is played and player 2 chooses L, then $D \succsim U$ for player 1.
    - In the Bayesian normal form, this preference is reflected by: if player 2 chooses L, $DU' \succsim UU'$, $DD' \succsim UD'$, i.e. no matter U' or D' is chosen in Game 2, once Game 1 is played and player 2 chooses L, $D \succsim U$.

    (Actually player 1's preference can be preserved by any belief $(0 < Pr < 1)$ of player 2.)

Anyway, the method is the same, simply calculate the expected payoff according to the belief for both of the two players.

(b) NE

For the Bayesian normal form found in part (a), determine a NE. Is there more than one NE? (UD', R) is the unique NE.

> DU' is dominated by $DD' \Rightarrow L$ is dominated by $R \Rightarrow UD'$ dominates the rest 2 strategies. Dominated strategies can't be BR and should be assigned with probability 0.

2   Problem 2 - Bayesian normal form

Consider the following two variants of a battle-of-the-sexes game. The game at the top – variant (i) – is of the usual kind where both players wish to meet each other, while the one at the bottom – variant (ii) – has the unusual feature that 1 wishes to avoid 2.

Variant (i)

| | | q | 1-q |
		O	M
p	O	$3,1$	$0,0$
1-p	M	$0,0$	$1,3$

P2 across top, P1 down left side.

Variant (ii)

| | | q | 1-q |
		O	M
p	O'	$0,1$	$3,0$
1-p	M'	$1,0$	$0,3$

(a) NE - static & complete information

For each of these games, determine the set of (pure) rationalizable strategies for each player, and the set of pure-strategy and/or mixed-strategy NE.

All strategies are rationalizable.

> Recall: Rationalizable strategies(Watson pp.70): The set of strategies that survive iterated dominance is therefore called the rationalizable strategies.
> Here all strategies can be BR (= UD for 2-player games).

Variant (i) has 2 pure NE (O, O), (M, M), and a mixed NE $\left(\left(\frac{3}{4}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{3}{4}\right)\right)$.

To derive the mixed-strategy NE, let p be the probability of P_1 choosing O and q be the probability of P_2 choosing O.

$$U_O^1 = 3q$$

$$U_M^1 = 1 - q$$

$$U_O^1 = U_M^1 \Rightarrow 3q = 1 - q \Rightarrow q = \tfrac{1}{4}$$

$$U_O^2 = p$$

$$U_M^2 = 3(1 - q)$$

$$U_O^2 = U_M^2 \Rightarrow p = 3(1 - p) \Rightarrow p = \tfrac{3}{4})$$

Variant (ii) has no pure NE but a mixed NE $\left(\left(\tfrac{3}{4}, \tfrac{1}{4}\right), \left(\tfrac{3}{4}, \tfrac{1}{4}\right)\right)$.

To derive the mixed-strategy NE in variant (ii), let p the probability of 1 choosing O' and q being the probability of 2 choosing O.

$$U_O^1 = 3(1 - q)$$

$$U_M^1 = q$$

$$U_O^1 = U_M^1 \Rightarrow 3(1 - q) = q \Rightarrow q = \tfrac{3}{4}$$

$$U_O^2 = p$$

$$U_M^2 = 3(1 - p)$$

$$U_O^2 = U_M^2 \Rightarrow p = 3(1 - p) \Rightarrow p = \tfrac{3}{4}$$

> Recall: why players are indifferent between O and M in a Mixed strategy?

(b) Bayesian normal form

Assume next that only player 1 knows which game is being played, while player 2 thinks that the two games are equally likely. Model this situation in an *ex ante* perspective by specifying the Bayesian normal form.

> Think: Who has private information and therefore contingent strategies?
>
> Payoff is calculated similarly to Question 1. For example, for (OO', O) is $(\tfrac{3}{2}, 1)$:
>
> $$U^1 = 0.5 \times 3 + 0.5 \times 0 = \tfrac{3}{2}$$
>
> $$U^2 = 0.5 \times 1 + 0.5 \times 1 = 1$$

	O	M
OO'	$\tfrac{3}{2}, 1$	$\tfrac{3}{2}, 0$
OM'	$2, \tfrac{1}{2}$	$0, \tfrac{3}{2}$
MO'	$0, \tfrac{1}{2}$	$2, \tfrac{3}{2}$
MM'	$\tfrac{1}{2}, 0$	$\tfrac{1}{2}, 3$

(c) Rationalizable strategies and NE

For the Bayesian normal form found in part (b), determine the set of (pure) rationalizable strategies for each player, and the set of pure-strategy and/or mixed-strategy NE.

MM' is dominated by OO'. All the other strategies are rationalizable.

NE: (MO', M), $\left(\left(\frac{1}{2}, \frac{1}{2}, 0, 0\right), \left(\frac{3}{4}, \frac{1}{4}\right)\right)$, and $\left(\left(\frac{1}{2}, 0, \frac{1}{2}, 0\right), \left(\frac{1}{4}, \frac{3}{4}\right)\right)$.

For mixed stategy, P_1 chooses strategies among (note MM' is dominated and can't be chosen):

$$\begin{cases} OO' \\ OM' \\ MO' \end{cases}$$

There are 4 possible combinations:

$$(1) \begin{cases} OO' \\ OM' \end{cases} \quad (2) \begin{cases} OM' \\ MO' \end{cases} \quad (3) \begin{cases} OO' \\ MO' \end{cases} \quad (4) \begin{cases} OO' \\ OM' \\ MO' \end{cases}$$

Let q being the probability of P_2 choosing O.

$$U^1_{OO'} = \tfrac{3}{2}q + \tfrac{3}{2}(1-q) = \tfrac{3}{2}$$
$$U^1_{OM'} = 2q + 0 \times (1-q) = 2q$$
$$U^1_{MO'} = 0 \times q + 2(1-q) = 2(1-q)$$

(1) If P_1 mixes OO' and OM' with probability $(p, 1-p)$

For P_1:

$$U^1_{OO'} = U^1_{OM'} \Rightarrow \tfrac{3}{2} = 2q \Rightarrow q = \tfrac{3}{4}$$

For P_2:

$$U^2_O = p + \tfrac{1}{2}(1-p)$$
$$U^2_M = 0 \times p + \tfrac{3}{2}(1-p) = \tfrac{3}{2}(1-p)$$
$$U^2_O = U^2_M \Rightarrow p + \tfrac{1}{2}(1-p) = \tfrac{3}{2}(1-p) \Rightarrow p = \tfrac{1}{2}$$

The NE is : $\left[\left(\frac{1}{2}, \frac{1}{2}, 0, 0\right), \left(\frac{3}{4}, \frac{1}{4}\right)\right]$

(2) If P_1 mixes OM' and MO' with probability $(p, 1-p)$

For P_2, facing with P_1's mixed strategy, M dominates O (therefore $q = 1$). The mixed strategy brokes down.

(3) If P_1 mixes OO' and MO' with probability $(p, 1-p)$

For P_1:

$$U^1_{OO'} = U^1_{MO'} \Rightarrow \tfrac{3}{2} = 2(1-q) \Rightarrow q = \tfrac{1}{4}$$

For P_2:

$$U^2_O = p + \tfrac{1}{2}(1-p)$$
$$U^2_M = 0 \times p + \tfrac{3}{2}(1-p) = \tfrac{3}{2}(1-p)$$
$$U^2_O = U^2_M \Rightarrow p + \tfrac{1}{2}(1-p) = \tfrac{3}{2}(1-p) \Rightarrow p = \tfrac{1}{2}$$

The NE is :$[\left(\tfrac{1}{2}, 0, \tfrac{1}{2}, 0\right), \left(\tfrac{1}{4}, \tfrac{3}{4}\right)]$

(4) If P_1 mixes OO', OM' and MO' with probability $(m, n, 1-m-n)$

There must be $U^1_{OO'} = U^1_{OM'} = U^1_{MO'}$.
From (1) and (3) we already know P_1's belief (i.e. P_2 chooses O with $Pr = q$ and M with $Pr = 1-q$) supporting $U^1_{OO'} = U^1_{OM'}$ is different from his/her belief supporting $U^1_{OO'} = U^1_{MO'}$. Therefore there is no such a brief that supports $U^1_{OO'} = U^1_{OM'} = U^1_{MO'}$.

3 Problem 3 - Rationalizability in incomplete information games

(Watson Exercise 26.3, solution is on pp. 469-470)

Suppose that nature selects A with probability 1/2 and B with probability 1/2. If nature selects A, then players 1 and 2 interact according to matrix "A." If nature selects B, then the players interact according to matrix "B." These matrices are pictured here. Suppose that, before the players select their actions, player 1 observes nature's choice. That is, **player 1 knows from which matrix the payoffs are drawn**, and player 1 can condition his or her decision on this knowledge. Player 2 does not know which matrix is being played when he or she selects between L and R.
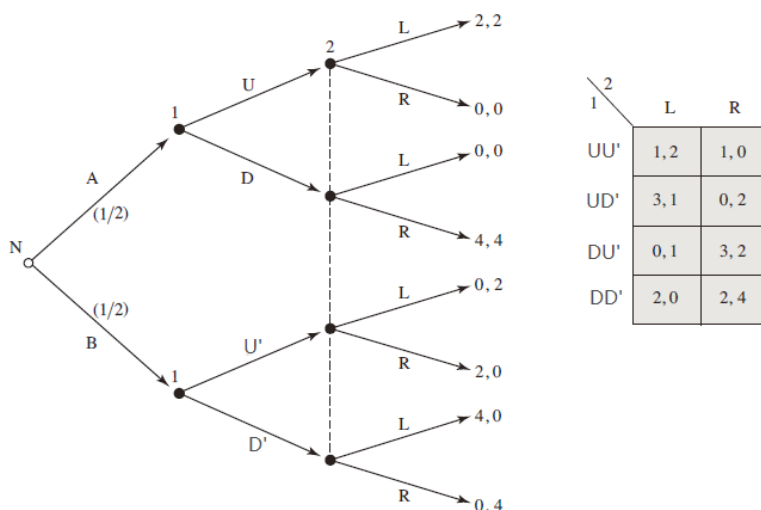
	L	R
U	2, 2	0, 0
D	0, 0	4, 4

A

	L	R
U'	0, 2	2, 0
D'	4, 0	0, 4

B

(a) Representation, rationalizability and NE

(1) Draw the extensive-form and Bayesian normal form.

	L	R
UU'	1, 2	1, 0
UD'	3, 1	0, 2
DU'	0, 1	3, 2
DD'	2, 0	2, 4

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(2) Compute the set of rationalizable strategies and NE

UU' is dominated by DD', then L is dominated by R. In the end, only (DU', R) survives iterated dominance method.

Therefore the set of rationalizable strategies is $\{(DU', R)\}$. The only NE is (DU', R)

(b) A three-player interpretation

Consider a three-player interpretation of this strategic setting in which each of player 1's types is modeled as a separate player. That is, the game is played by players 1A, 1B, and 2. Assume that player 1A's payoff is zero whenever nature chooses B; likewise, player 1B's payoff is zero whenever nature selects A. Depict this version of the game in the **extensive form** (remember that payoff vectors consist of three numbers) and in the **normal form**. Compute the set of **rationalizable strategies and find the NE**.

Extensive form:

Normal form:
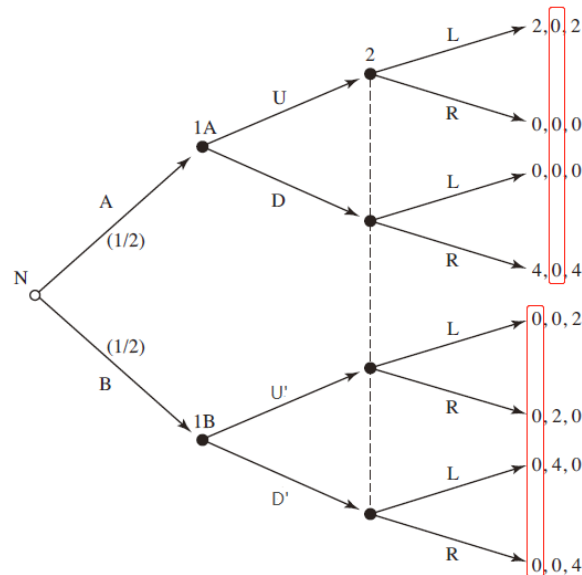
Although there are 3 players, player 2 still doesn't know who he/she is playing with. As a result, the payoff is in player 2's imagination, and player $1A,1B$ know how player 2 thinks clearly (an *ex ante* perspective).

You can also draw the normal form the way on Geir's slides (lecture part 5, pp.5), i.e. draw the matrix for player $1B$ and player 2, and player $1A$ joins as another player.

- We use the textbook's way because it makes the next question more clearly,

- Note also the difference between the textbook and Geir's slides:

    - The textbook calculates expected payoff for all the 3 players, It's more intuitive and can get the correct Bayesian NE.
    - The slides pp.5 is an *ex post* perspective, and the payoff for player $1A$ and $1B$ is not expected payoff but what they can really get. The result should be the same.
    - More about *ex post* and *ex ante* perspective can be found here.

Rationalizable strategies

For player $1A$ **and** $1B$**:**
When player 2 choose L, D' dominates U', and U dominates D. $((U,D')$ survives iterated dominance)
When player 2 choose R, D dominates U, and U' dominates D'. $((D,U')$ survives iterated dominance)
As a result, all the strategies for $1A$ and $1B$ can be BR.

For Player 2**:**
Neither L nor R is dominated by the other action. Therefore both of the two actions are rationalizable.

In conclusion: the rationalizable strategy set is: $\{(U,D),(U',D'),(L,R)\}$.

Find the NE

By backward induction, if player 2 chooses L, the payoff is 1 (by (U, D', L)); if player 2 chooses R, the payoff is 2 (by (D, U', R)). Thus player 2 will choose R and the NE is (D, U', R).

> Note although player 2 will only choose R by backward induction, R doesn't dominate L. Dominance means "always better no matter how the opponents act". Obviously, under some conditions L yields more payoff than R.

(c) 2-player view & 3-player view

Explain why the predictions of parts (a) and (b) are the same in regard to equilibrium but different in regard to rationalizability. (Hint: The answer has to do with the scope of the players' beliefs.)

> In the Bayesian normal form (in part(a)), DD' strictly dominates UU'. That means that DD' is better than UU' independently of what beliefs player 1 has about the choice of player 2.
> This is true because player 1 is one person and has a coherently belief.
>
> If we use the view of 3 players, then palyer 1 are 2 different people, then (DD', L) means both player $1A$ and player $1B$ think player 2 will choose L, and (DD', R) means both player $1A$ and $1B$ think player 2 will choose R.
>
> What if player $1A$ and player $1B$ have different belief? For example, player $1A$ believes player 2 chooses L, then U is prefered; and at the same time, player $1B$ believes player 2 chooses R, then U' is prefered.
>
> In this case, (U,U') is chosen separetely, not dominated!
>
> This belief separation doesn't affect NE because both player $1A$ and player $1B$ must agree that by backward induction, player 2 will choose R.

In part (b), the beliefs of players $1A$ and $1B$ do not have to coincide, so they may end up making U and U' as rational choices based on their separated beliefs. While in NE, the beliefs of player $1A$ and $1B$ are the same (by backward induction).

4 Problem 4 - Differentiated duopoly

(Watson Exercise 26.5, solution is on pp. 470.)

Consider a differentiated duopoly market in which firms compete by selecting prices and produce to fill orders. Let p_1 be the price chosen by firm 1 and let p_2 be the price of firm 2. Let q_1 and q_2 denote the quantities demanded (and produced) by the two firms. Suppose that the demand for firm 1 is given by $q_1 = 22 - 2p_1 + p_2$, and the demand for firm 2 is given by $q_2 = 22 - 2p_2 + p_1$. Firm 1 produces at a constant marginal cost of 10 and no fixed cost. Firm 2 produces at a constant marginal cost of c and no fixed cost. The payoffs are the firms' individual profits.

(a) Payoff functions

The firms' strategies are their prices. Represent the normal form by writing the firms' payoff functions.

$$\begin{aligned}
U^1 &= p_1 q_1 - 10 q_1 \\
&= p_1(22 - 2p_1 + p_2) - 10(22 - 2p_1 + p_2) \\
&= 22P_1 - 2p_1^2 + p_1 p_2 - 220 + 20p_1 - 10p_2 \\
&= -2p_1^2 + p_1 p_2 + 42p_1 - 10p_2 - 220
\end{aligned}$$

$$\begin{aligned}
U^2 &= p_2 q_2 - c q_2 \\
&= p_1(22 - 2p_2 + p_1) - c(22 - 2p_2 + p_1) \\
&= 22P_2 - 2p_2^2 + p_1 p_2 - 22c + 2cp_2 - cp_1 \\
&= -2p_2^2 + p_1 p_2 - cp_1 + (22 + 2c)p_2 - 22c
\end{aligned}$$

(b) Calculate the firms' best-response functions.

FOC:

$$\frac{\partial U^1}{\partial p_1} = 0 \Rightarrow -4p_1^* + 42 + p_2 = 0 \Rightarrow p_1^* = \frac{42 + p_2}{4}$$

SOC:

$$\frac{\partial^2 U^1}{\partial p_1^2} = -4 < 0$$

FOC:

$$\frac{\partial U^2}{\partial p_2} = 0 \Rightarrow -4p_2^* + 22 + 2c + p_1 = 0 \Rightarrow p_2^* = \frac{22 + 2c + p_1}{4}$$

SOC:

$$\frac{\partial^2 U^2}{\partial p_2^2} = -4 < 0$$

> For payoff functions, we use FOC & SOC to find the BR functions. Be clear about what is variable an what is parameter. See also seminar 7 problem 3.

(c) Complete information

Suppose that $c = 10$ so the firms are identical (the game is symmetric). Calculate the NE prices.

When $c = 10$, we have:

$$p_1^* = \frac{42 + p_2}{4}$$
$$p_2^* = \frac{42 + p_1}{4}$$

NE is a price s.t. both of the two firms are applying BR and there is no incentive to devicate. That is, $p_1^* = \frac{42 + p_2^*}{4} = \frac{42 + \frac{42 + p_1}{4}}{4}$ (or $p_2^* = \frac{42 + p_1^*}{4}$).
Therefore $p_1^* = p_2^* = 14$.

(d) Incomplete information

Now suppose that firm 1 does not know firm 2's marginal cost c. With probability 1/2 nature picks $c = 14$, and with probability 1/2 nature picks $c = 6$. **Firm 2 knows** its own cost (that is, it observes nature's move), but firm 1 only knows that firm 2's marginal cost is either 6 or 14 (with equal probabilities). Calculate the best-response functions of player 1 and the two types ($c = 6$ and $c = 14$) of player 2 and calculate the Bayesian NE quantities.

> With different MC, firm 2 can set different (contingent) price, while firm 1 can only set one price according to its expected payoff function.
>
> (Compare this with the extensive from games in this seminar. Always be clear about who has contingent strategies!)

For player 2:

(1) **When** $c = c^H = 14$

(I omitted the superscript ^{H} of p_2 and c for simpliciy.)

$$U^{2H} = -2p_2^2 + p_1 p_2 - cp_1 + (22 + 2c)p_2 - 22c \qquad \text{(the result from question a.)}$$
$$= -2p_2^2 + p_1 p_2 - 14p_1 + 50p_2 - 22 \times 14$$

FOC:

$$\frac{\partial U^{2H}}{\partial p_2} = 0 \Rightarrow -4p_2^{H*} + 50 + p_1 = 0 \Rightarrow p_2^{H*} = \frac{50 + p_1}{4}$$

SOC:

$$\frac{\partial^2 U^{2H}}{\partial p_2^2} = -4 < 0$$

(2)When $c = c^L = 6$

(I omitted the superscript ^{L} of p_2 and c for simpliciy.)

$$\begin{aligned} U^{2L} &= -2p_2^2 + p_1 p_2 - c p_1 + (22 + 2c)p_2 - 22c \qquad \text{(the result from question a.)} \\ &= -2p_2^2 + p_1 p_2 - 6p_1 + 34p_2 - 22 \times 6 \end{aligned}$$

FOC:

$$\frac{\partial U^{2L}}{\partial p_2} = 0 \Rightarrow -4p_2^{L*} + 34 + p_1 = 0 \Rightarrow p_2^{L*} = \frac{34 + p_1}{4}$$

SOC:

$$\frac{\partial^2 U^{2L}}{\partial p_2^2} = -4 < 0$$

For player 1:

$$\begin{aligned} E(U^1) &= 0.5U^1(p_1, p_2^H) + 0.5U^1(p_1, p_2^L) \\ &= 0.5(-2p_1^2 + p_1 p_2^H + 42p_1 - 10p_2^H - 220) + 0.5(-2p_1^2 + p_1 p_2^L + 42p_1 - 10p_2^L - 220) \\ &= -2p_1^2 + 42p_1 + (p_1 - 10)(0.5p_2^H + 0.5p_2^L) - 220 \end{aligned}$$

FOC:

$$\frac{\partial U^1}{\partial p_1} = 0 \Rightarrow -4p_1^* + 42 + (0.5p_2^H + 0.5p_2^L) = 0 \Rightarrow p_1^* = \frac{42 + 0.5(p_2^H + p_2^L)}{4}$$

SOC:

$$\frac{\partial^2 U^1}{\partial p_1^2} = -4 < 0$$

NE: Denote the NE as $(p_1^*, p_2^{*H}, p_2^{*L})$, we have

$$p_1^* = \frac{42 + 0.5(p_2^{*H} + p_2^{*L})}{4} = \frac{42 + 0.5(\frac{50+p_1}{4} + \frac{34+p_1}{4})}{4}$$

Therefore:

$$\begin{cases} p_1^* = 14 \\ p_2^{*H} = 16 \\ p_2^{*L} = 12 \end{cases}$$