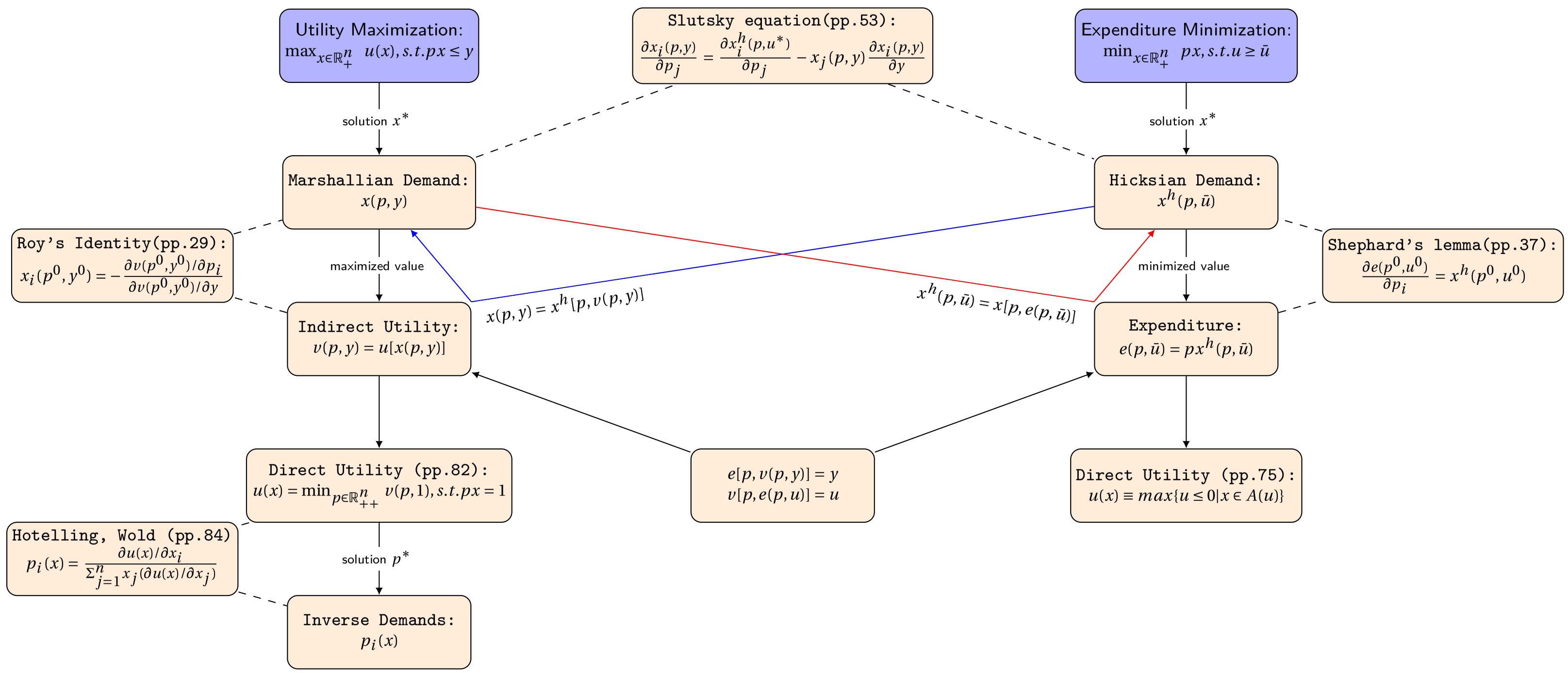
Seminar 3.Duality of Consumers Behavior

Xiaoguang Ling
xiaoguang.ling@econ.uio.no

September 20, 2020

Consumption Duality

You will never lose your way with this Consumption Duality map!
All "derive this from that and verify some guy's equation"-like questions can be solved by finding the correct (shortest) route.



1 Jehle & Reny 2.3

Derive the consumers direct utility function if his indirect utility function has the form $v(p, y) = yp_1^{\alpha} p_2^{\beta}$ for negative α and β.

THEOREM 2.3 Duality Between Direct and Indirect Utility(Jehle & Reny pp.81)
Suppose that $u(x)$ is quasiconcave and differentiable on $\mathbb{R}^n_{++}$ with strictly positive partial derivatives there. Then for all $x \in \mathbb{R}^n_{++}$, $v(p, p \cdot x)$, the indirect utility function generated by $u(x)$, achieves a minimum in p on $\mathbb{R}^n_{++}$, and

$$u(x) = \min_{p \in \mathbb{R}^n_{++}} v(p, y), s.t. px = y$$

Let's call the solution p^*
Note that by **Theorem 1.6**(Jehle & Reny pp.29), $v(p, y)$ is homogeneous of degree zero in (p, y). We have $v(p, p \cdot x) = v(p/(p \cdot x), 1)$ whenever $p \cdot x > 0$. Thus the equation above can also be written as:

$$u(x) = \min_{p \in \mathbb{R}^n_{++}} v(p, 1), s.t. px = 1$$

The solution $\hat{p} = p^* / p^* \cdot x = p^* / y$. We don't care about the difference between $\hat{p}$ and p^* because once you substitute them into $v(p, p \cdot x)$, you have the same result (homogeneity of degree zero).

$$u(x) = \min_{p \in \mathbb{R}^n_{++}} v(p, 1) = p_1^{\alpha} p_2^{\beta}, s.t. px = 1$$

Lagrangian:

$$L = p_1^{\alpha} p_2^{\beta} + \lambda(1 - p_1 x_1 - p_2 x_2)$$

Note there should not be interior solution since

- $\frac{\partial v(p_1, p_2, 1)}{\partial p_1} = \alpha p_1^{\alpha-1} p_2^{\beta}, \alpha, \beta < 0. \lim_{p_1 \to 0} \frac{\partial v(p_1, p_2, 1)}{\partial p_1} = -\infty$
- $\frac{\partial v(p_1, p_2, 1)}{\partial p_2} = p_1^{\alpha} \beta p_2^{\beta-1}, \alpha, \beta < 0. \lim_{p_2 \to 0} \frac{\partial v(p_1, p_2, 1)}{\partial p_2} = -\infty$
- $v(p, 1)$ is decreasing in p(this is always true for indirect utility function, see pp.29). For any $px < 1$, you can always decrease $v(p, 1)$ by increasing p until $px = 1$.

FOCs.

$$\begin{cases} \frac{\partial L}{\partial p_1} = \alpha p_1^{\alpha-1} p_2^{\beta} - \lambda x_1 = 0 \\ \frac{\partial L}{\partial p_2} = p_1^{\alpha} \beta p_2^{\beta-1} - \lambda x_2 = 0 \\ p_1 x_1 + p_2 x_2 = 1 \end{cases}$$

Simplify:

$$\begin{cases} \alpha p_1^{\alpha-1} p_2^{\beta} = \lambda x_1 \\ \beta p_1^{\alpha} p_2^{\beta-1} = \lambda x_2 \\ p_1 x_1 + p_2 x_2 = 1 \end{cases} \tag{1}$$

Take the ratio between first and second condition to get:

$$\frac{x_1}{x_2} = \frac{\alpha}{\beta}\frac{p_2}{p_1}$$

Thus: $p_2 = \frac{\beta}{\alpha}\frac{x_1}{x_2} p_1$
Substitute p_2 with p_1 in the 3rd condition to get:

$$\begin{aligned} p_1 x_1 + \frac{\beta}{\alpha}\frac{x_1}{x_2} p_1 x_2 = 1 \\ p_1(x_1 + \frac{\beta}{\alpha} x_1) = 1 \\ p_1^* = \frac{1}{x_1(1+\frac{\beta}{\alpha})} \\ \Rightarrow p_2^* = \frac{\beta}{\alpha}\frac{x_1}{x_2} p_1 = \frac{\beta}{\alpha}\frac{x_1}{x_2}\frac{1}{x_1(1+\frac{\beta}{\alpha})} = \frac{1}{x_2(1+\frac{\alpha}{\beta})} \end{aligned}$$

Substitute p_1^* and p_2^* into $v(p,1)$ we get the minimized value, i.e. the direct utility function:

$$\begin{aligned} u(x_1.x_2) &= [\frac{1}{x_1(1+\frac{\beta}{\alpha})}]^{\alpha}[\frac{1}{x_2(1+\frac{\alpha}{\beta})}]^{\beta} \\ &= A x_1^a x_2^b \end{aligned}$$

Where $A = [\frac{1}{1+\frac{\beta}{\alpha}}]^{\alpha}[\frac{1}{1+\frac{\alpha}{\beta}}]^{\beta}$,$a = -\alpha > 0$, $b = -\beta > 0$. The utility function is a Cobb-Douglas function.

As a cautious proof, you may want to check if $u(x)$ is quasiconcave and differentiable on $\mathbb{R}^n_{++}$ with strictly positive partial derivatives there, as assumed by Theorem 2.3.
In exam for this course, again, if the function is one- dimension, you should prove it; if it's a higher-dimension function, the proof is not required.

Like Jehle & Reny 1.51, you can actually transform $v(p_1, p_2, 1)$ into a function of only p_1 or p_1 using $p_1 x_1 + p_2 x_2 = 1$.

$$p_1 = \frac{1 - p_2 x_2}{x_1}$$

Substitute into $v(p_1, p_2, 1)$ to have:

$$v(p_1, p_2, 1) = [\frac{1 - p_2 x_2}{x_1}]^{\alpha} p_2^{\beta}$$

Since the question ask you to minimize $v(p_1, p_2, 1)$, if you solve $\frac{de(p_2)}{dp_2} = 0$ and get only one solution, it is the solution.

$$\begin{aligned}
\frac{de(p_2)}{dp_2} = \alpha(\frac{1 - p_2 x_2}{x_1})^{\alpha-1}(\frac{-x_2}{x_1})p_2^{\beta} + \frac{1 - p_2 x_2}{x_1}^{\alpha} \beta p_2^{\beta-1} &= 0 \\
\alpha(\frac{1 - p_2 x_2}{x_1})^{\alpha-1}(\frac{x_2}{x_1})p_2^{\beta} &= \frac{1 - p_2 x_2}{x_1}^{\alpha} \beta p_2^{\beta-1} \\
\alpha(\frac{x_1}{1 - p_2 x_2})(\frac{x_2}{x_1})p_2 &= \beta \\
\alpha(\frac{x_2}{1 - p_2 x_2})p_2 &= \beta \\
\alpha x_2 p_2 &= \beta - \beta x_2 p_2 \\
(\alpha x_2 + \beta x_2) p_2 &= \beta \\
p_2^* &= \frac{\beta}{(\alpha + \beta) x_2}
\end{aligned}$$

You then solve p_1^* with the budget constraint.

2 Jehle & Reny 2.5(a)

Consider the solution, $e(p, u) = u p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ at the end of Example 2.3. Derive the **indirect utility function** through the relation $e(p, v(p, y)) = y$ and verify Roy's identity.

Example 2.3 on Jehle & Reny pp.90 is a question from $x_i(p, y)$ to $e(p, u)$, where the Marshallian demand function is:

$$x_i(p_1, p_2, p_3, y) = \frac{\alpha_i y}{p_i}, \ i = 1, 2, 3$$

$\alpha_i > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$
Check your map, the route is (note the expression below is only for the purpose of teaching and very informal):

$$x_i(p, y) \Rightarrow x^h(p, u) = x[p, e(p, u)] \Leftarrow \frac{\partial e(p, u)}{\partial p_i} = x^h(p, u)$$

$$
\begin{aligned}
x[p, e(p,u)] &= \frac{\partial e(p,u)}{\partial p_i} \\
\frac{\alpha_i e(p,u)}{p_i} &= \frac{\partial e(p,u)}{\partial p_i} \\
\frac{\alpha_i}{p_i} &= \frac{1}{e(p,u)} \frac{\partial e(p,u)}{\partial p_i} \\
&= \frac{\partial ln[e(p,u)]}{\partial p_i}
\end{aligned}
$$

From now on, the textbook's solution is clear. Read page 91 if you're curious how we solve $e(p,u)$ out. It need "a little thought" as the textbook said :)

Indirect utility function:
We already know $e(p, v(p, y)) = y$.
Substitute $v(p, y)$ into $e(p, u) = y$ will solve the question directly:

$$
\begin{aligned}
e(p,u) = v(p,y) p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} = y \\
\Rightarrow v(p,y) = \frac{y}{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}}
\end{aligned}
$$

Verify Roy's identity:

Roy's Identity(Jehle & Reny pp.29):

$$
x_i(p^0, y^0) = -\frac{\partial v(p^0, y^0)/\partial p_i}{\partial v(p^0, y^0)/\partial y}
$$

Intuition:Your optimal consumption plan (Marshallian demand) is a trade- off between the importance of "comodity i" and "money (y)".

$$
\begin{aligned}
\frac{\partial v(p,y)}{\partial p_1} = \frac{\partial y p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}}{\partial p_1} &= -\alpha_1 y p_1^{-\alpha_1 - 1} p_2^{-\alpha_2} p_3^{-\alpha_3} \\
\frac{\partial v(p,y)}{\partial p_2} &= -\alpha_3 y p_1^{-\alpha_1} p_2^{-\alpha_2 - 1} p_3^{-\alpha_3} \\
\frac{\partial v(p,y)}{\partial p_3} &= -\alpha_3 y p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3 - 1} \\
\frac{\partial v(p,y)}{\partial y} &= p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}
\end{aligned}
$$

Therefore:

$$
-\frac{\partial v(p,y)/\partial p_1}{\partial v(p,y)/\partial y} = -\frac{-\alpha_1 y p_1^{-\alpha_1 - 1} p_2^{-\alpha_2} p_3^{-\alpha_3}}{p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}} = \frac{\alpha_1 y}{p_1}
$$

$$-\frac{\partial v(p,y)/\partial p_2}{\partial v(p,y)/\partial y} = -\frac{-\alpha_3 y p_1^{-\alpha_1} p_2^{-\alpha_2-1} p_3^{-\alpha_3}}{p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}} = \frac{\alpha_2 y}{p_1}$$

$$-\frac{\partial v(p,y)/\partial p_3}{\partial v(p,y)/\partial y} = -\frac{-\alpha_3 y p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3-1}}{p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}} = \frac{\alpha_3 y}{p_1}$$

Compare with the Marshallian demand!

3 Jehle & Reny 2.7

Derive the consumer's **inverse demand functions**, $p_1(x_1,x_2)$ and $p_2(x_1,x_2)$, when the **utility function** is of the Cobb-Douglas form, $u(x_1,x_2) = Ax_1^\alpha x_2^{1-\alpha}$ for $0 < \alpha < 1$.

The shortest route is using Hotelling, Wold (pp.84) directly.

$$p_i(x) = \frac{\partial u(x)/\partial x_i}{\Sigma_{j=1}^n x_j(\partial u(x)/\partial x_j)}$$

Intuition: the price reflects how important the commodity is.

The duality between direct and indirect utility functions showed by Hotelling, Wold makes it (hopefully) easier to solve $p_i^*(x)$

$$\begin{aligned}
p_1(x_1,x_2) &= \frac{\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_1}{\Sigma_{j=1}^2 x_j(\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_j)} \\
&= \frac{\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_1}{x_1\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_1 + x_2\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_2} \\
&= \frac{A\alpha x_1^{\alpha-1}x_2^{1-\alpha}}{x_1 A\alpha x_1^{\alpha-1}x_2^{1-\alpha} + x_2 A(1-\alpha)x_1^\alpha x_2^{-\alpha}} \\
&= \frac{\alpha x_1^{\alpha-1}x_2^{1-\alpha}}{\alpha x_1^\alpha x_2^{1-\alpha} + (1-\alpha)x_1^\alpha x_2^{1-\alpha}} \\
&= \frac{\alpha}{x_1}
\end{aligned}$$

$$
\begin{aligned}
p_2(x_1,x_2) &= \frac{\partial(Ax_1^{\alpha}x_2^{1-\alpha})/\partial x_2}{\Sigma_{j=1}^{2} x_j(\partial(Ax_1^{\alpha}x_2^{1-\alpha})/\partial x_j)} \\
&= \frac{\partial(Ax_1^{\alpha}x_2^{1-\alpha})/\partial x_2}{x_1\partial(Ax_1^{\alpha}x_2^{1-\alpha})/\partial x_1 + x_2\partial(Ax_1^{\alpha}x_2^{1-\alpha})/\partial x_2} \\
&= \frac{A(1-\alpha)x_1^{\alpha}x_2^{-\alpha}}{x_1 A\alpha x_1^{\alpha-1}x_2^{1-\alpha} + x_2 A(1-\alpha)x_1^{\alpha}x_2^{-\alpha}} \\
&= \frac{(1-\alpha)x_1^{\alpha}x_2^{-\alpha}}{\alpha x_1^{\alpha}x_2^{1-\alpha} + (1-\alpha)x_1^{\alpha}x_2^{1-\alpha}} \\
&= \frac{1-\alpha}{x_2}
\end{aligned}
$$

You can also try another route: maximize $u(x) \Rightarrow x(p,y) \Rightarrow p_i(x) = x^{-1}(x,1)$
Use Lagrangian to maximize $u(x_1,x_2) = Ax_1^{\alpha}x_2^{1-\alpha}$ s.t. $p_1x_2 + p_2x_2 = 1$. The solution (Marshallian demands) is:

$$
\begin{cases} x_1 = \frac{\alpha}{p_1} \\ x_1 = \frac{1-\alpha}{p_2} \end{cases}
$$

The inverse of Marshallian demand function gives the inverse demand function

$$
\begin{cases} p_1 = \frac{\alpha}{x_1} \\ p_1 = \frac{1-\alpha}{x_2} \end{cases}
$$

Another example:

- You can also try to derive $p_i(x)$ from the Marshallian demand E.1 on pp. 32 and compare with the results E.5-E.6 on pp. 83, which is derived from Hotelling-Wold identity.