Seminar 12. Auctions and previous exam problems

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1 Jehle & Reny pp.484, exercise 9.2

Show in two ways that the symmetric equilibrium bidding strategy of a first-price auction with N symmetric bidders each with values distributed according to F, can be written as

$$\hat{b}(v) = v - \int_0^v \left(\frac{F(x)}{F(v)}\right)^{N-1} dx$$

For the first way, use our solution from the text and apply integration by parts. For the second way, use the fact that $F^{N-1}(r)(v - \hat{b}(r))$ is maximised in r when $r = v$ and then apply the envelope theorem to conclude that $d(F^{N-1}(v)(v - \hat{b}(v))/dv = F^{N-1}(v)$; now integrate both sides from 0 to v.

See lecture notes for the third lecture on the economics of information (on "Auctions and the revenue equivalence theorem"), pages 17 and 19.

2 Jehle & Reny pp.484, exercise 9.1 - Show that the bidding strategy in (9.5) is strictly increasing.

By exercise 9.1, the bid function can be written as:

$$\hat{b}(v_i) = v_i - \frac{\int_0^{v_i} [F(x)]^{N-1} dx}{[F(v_i)]^{N-1}}.$$

Then:

$$
\begin{aligned}
\frac{d}{dv_i}\left(\hat{b}(v_i)\right) &= \frac{d}{dv_i}\left(v_i - \frac{\int_0^{v_i}[F(x)]^{N-1}dx}{[F(v_i)]^{N-1}}\right) \\
&= 1 - \frac{[F(v_i)]^{N-1}}{[F(v_i)]^{N-1}} + \frac{\int_0^{v_i}[F(x)]^{N-1}dx}{[F(v_i)]^{2N-2}} \cdot \frac{d}{dv_i}\left([F(v_i)]^{N-1}\right) \\
&= \frac{\int_0^{v_i}[F(x)]^{N-1}dx}{[F(v_i)]^{2N-2}} \cdot (N-1)[F(v_i)]^{N-2} f(v_i) > 0 .
\end{aligned}
$$

3 Jehle & Reny pp.485, exercise 9.3

This exercise will guide you through the proof that the bidding function in (9.5) is in fact a symmetric equilibrium of the first-price auction.

(a)

Recall from (9.2) that

$$
\frac{du(r,v)}{dr} = (N-1)F^{N-2}(r)f(r)(v-\hat{b}(r)) - F^{N-1}(r)\hat{b}'(r).
$$

Using (9.3), show that

$$
\begin{aligned}
\frac{du(r,v)}{dr} &= (N-1)F^{N-2}(r)f(r)(v-\hat{b}(r)) - (N-1)F^{N-2}(r)f(r)(r-\hat{b}'(r)) \\
&= (N-1)F^{N-2}(r)f(r)(v-r)
\end{aligned}
$$

(b)

Use the result in part (a) to conclude that $du(r,v)/dr$ is positive when $r < v$ and negative when $r > v$, so that $u(r,v)$ is maximised when $r = v$.

4 Problems 2 of the exam in ECON4240, Spring 2005

Consider a strategic situation between an employer (E) and a worker (W). E can either accept (A) or reject (R) W. W can either become skilled (S) through education, or remain unskilled (U). W can be of two types; either he is inherently high ability (H) or he is inherently low ability (L). The players' payoffs depending on their actions and W's type is shown below.

H

	S	U
A	2, 3	-1, 2
R	0, 1	0, 0

L

	S	U
A	-1,-1	-3,-2
R	0, -3	0, 0

a) Rationalizable strategies & NE

For each of these games, determine the set of (pure) rationalizable strategies for each player, and the set of pure-strategy Nash equilibria.

In Game H: Only S and A are rationalizable. (S, A) is the unique NE.
In Game L. Only U and R are rationalizable. (U, R) is the unique NE.

b) Bayesian normal form

Assume next that only W knows his own type, while player E thinks that the two types of W are equally likely. Model this situation in an ex ante perspective by specifying the Bayesian normal form.

Denote the Worker's contingent choices are: If Game H, S and U; If Game L, S' and U'. The Baysian normal form is:

	SS'	SU'	US'	UU'
A	½, 1	-½, ½	-1, ½	-2, 0
R	0, -1	0, ½	0,-3/2	0, 0

c) Rationalizable strategy & BNE

For the Bayesian normal form found in part (b), determine the set of (pure) rationalizable strategies for each player, and the set of pure-strategy and/or mixedstrategy Nash equilibria.

	SS' p	SU' $1-p$	US'	UU'
A q	½, 1	-½, ½	-1, ½	-2, 0
R $1-q$	0, -1	0, ½	0,-3/2	0, 0

(1) Rationalizable strategy

For the Worker, US' is dominated by SS'; UU' is dominated by SU'. The rationalizable strategies are SS and SU'.

For the Employer, both A and R are rationalizable.

(2) Pure-strategy NE

$$(A, SS'), (R, SU')$$

(3) Mixed-strategy NE

Since US' and UU' are dominated, the Employer believes the Worker will choose them with probability (0,0).

The Employer believes the Worker chooses SS' and SU' with probability $(p, 1-p)$; The Worker believe the Employer chooses A and R with probability $(q, 1-q)$.

$$E(U_A^E) = p \times 0.5 + (1-p) \times (-0.5) = p - 0.5$$
$$E(U_R^E) = p \times 0 + (1-p) \times 0 = 0$$
$$E(U_A^E) = E(U_R^E) \Rightarrow p = 0.5$$

$$E(U_{SS'}^W) = q \times 1 + (1-q) \times (-1) = 2q - 1$$
$$E(U_{SU'}^W) = q \times 0.5 + (1-q) \times 0.5 = 0.5$$
$$E(U_{SS'}^W) = E(U_{SU'}^W) \Rightarrow 2q - 1 = 0.5 \Rightarrow \frac{3}{4}$$

The Mixed-strategy NE is: the Employer chooses A and R with probability $(0.5, 0.5)$; Worker chooses SS' and SU' with probability $(\frac{3}{4}, \frac{1}{4})$.

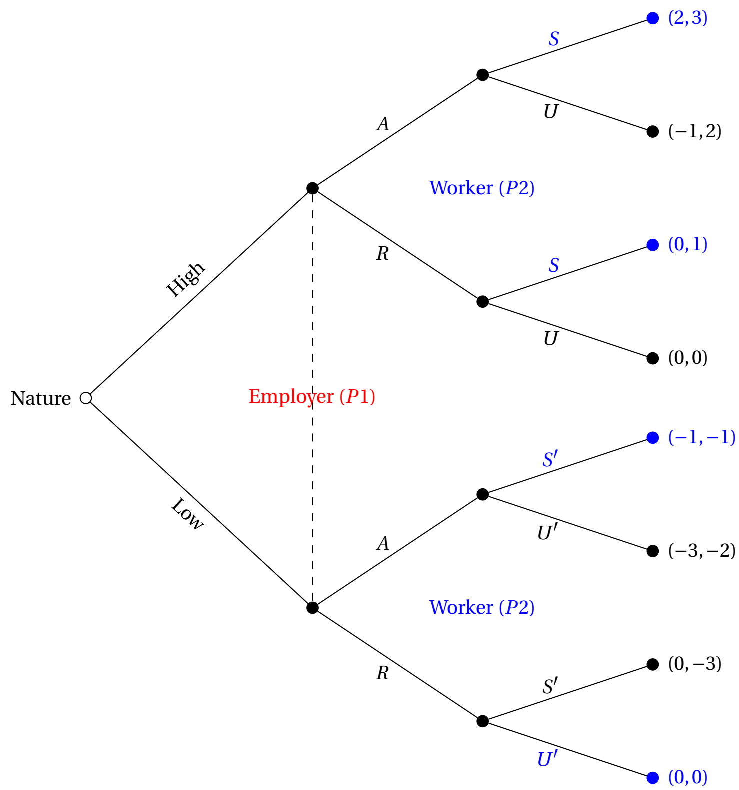
5 Problems 3 of the exam in ECON4240, Spring 2005

Problem 3 (20 %) Consider again the strategic situation between an employer (E) and a worker (W) described in Problem 2. Assume (as in parts b and c) of Problem 2) that only W knows his own type, while player E thinks that the two types of W are equally likely.

a) (Screening)

Assume now that E acts before W, and that E's choice of A or R can be observed by W before he makes his choice of S or U. Show that there is a unique subgame perfect Nash equilibrium.

Extensive Form:



The strategy of the $Worker$:

- If High:
  - when E chooses A, W chooses S
  - when E chooses R, W chooses S
- If Low:
  - when E chooses A, W chooses S'
  - when E chooses R, W chooses U'

For the $Employer$:

$$E(U_A^E) = \frac{1}{2} \times 2 + \frac{1}{2} \times (-1) = 0.5$$
$$E(U_R^E) = \frac{1}{2} \times 0 + \frac{1}{2} \times 0 = 0$$
$$\Rightarrow E(U_A^E) > E(U_R^E)$$

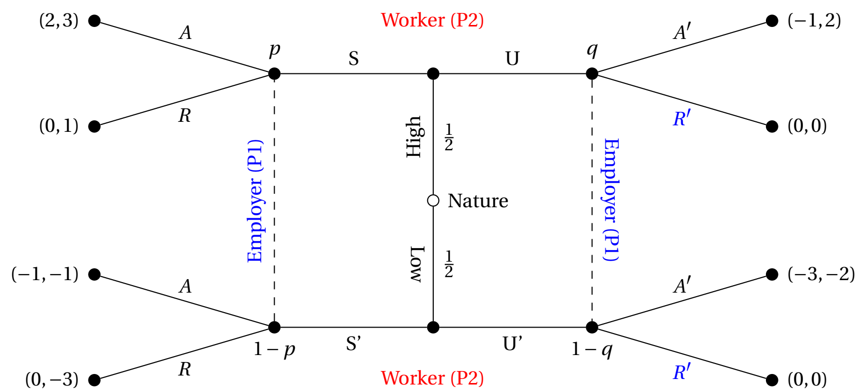
The strategy of the $Employer$ is to choose A.

Therefore there is a SPNE:
{(For High ability Worker:) S after A, S after R. (For Low ability Worker:) S' after A, U' after R ; (For Employer) A}

b) (Signaling)

Assume now that W acts before E, and that W's choice of S or U can be observed by E before she makes her choice of A or R. Show that there is a unique perfect Bayesian equilibrium.

Extensive Form:



The Worker has 4 possible strategies: SS', UU', SU', US'.

Denote the updated belief of the Worker :

- $Pr(\text{Strong}|P/P') = p$
- $Pr(\text{Weak}|U/U') = q$

(1) When the Employer believes the Worker chooses SS'

Then $p = Pr(\text{High}) = 0.5$,

$$E[U_A^E] = 0.5 \times 2 + 0.5 \times (-1) = 0.5$$
$$E[U_R^E] = 0.5 \times 0 + 0.5 \times 0 = 0$$
$$E[U_A^E] > E[U_R^E]$$

The BR of the Employer is to choose A.

If the Worker surprisingly chooses U/U', we can see that R' dominates A' for the Employer. Therefore, the Low ablility Worker will deviate from S' to U' to have a higher payoff (U' and R' leads to $(0,0)$)

$\Rightarrow SS'$ is not part of a PBE.

(2) When the Employer believes the Worker chooses UU'

Since R' dominates A', the Employer will always choose R'. For a High ability Worker, deviating from U to S will always lead to higher payoff (either 3 or 1), no matter how the Employer reacts.

$\Rightarrow UU'$ is not part of a PBE.

(3) When the Employer believes the Worker chooses SU'

Then $p = 1, q = 0$,

The BR of the Employer is to choose A after S/S' (payoff is be $(2,3)$) and R' after U/U'(payoff is be $(0,0)$).

If the High ability Worker eviates to U, the payoff is $(0,0)$, lower than $(2,3)$; If the Low ability Worker eviates to S, the payoff is $(-1,-1)$, lower than $(0,0)$;

There is no incentive for the Worker to deviate. $\Rightarrow SU'$ is part of a PBE.

(4) When the Employer believes the Worker chooses US'

Then $p = 0, q = 1$,

The BR of the Employer is to choose A after S/S' and R after U/U'.

For a Low ability Worker, deviating from S' to U' will increase the payoff from $(-1,-1)$ to $(0,0)$.

$\Rightarrow US'$ is not part of a PBE.

In conclusion, there is only one PBE:

$$\{SU', AR', p = 1, q = 0\}$$