

## Seminar 3.Duality of Consumers Behavior

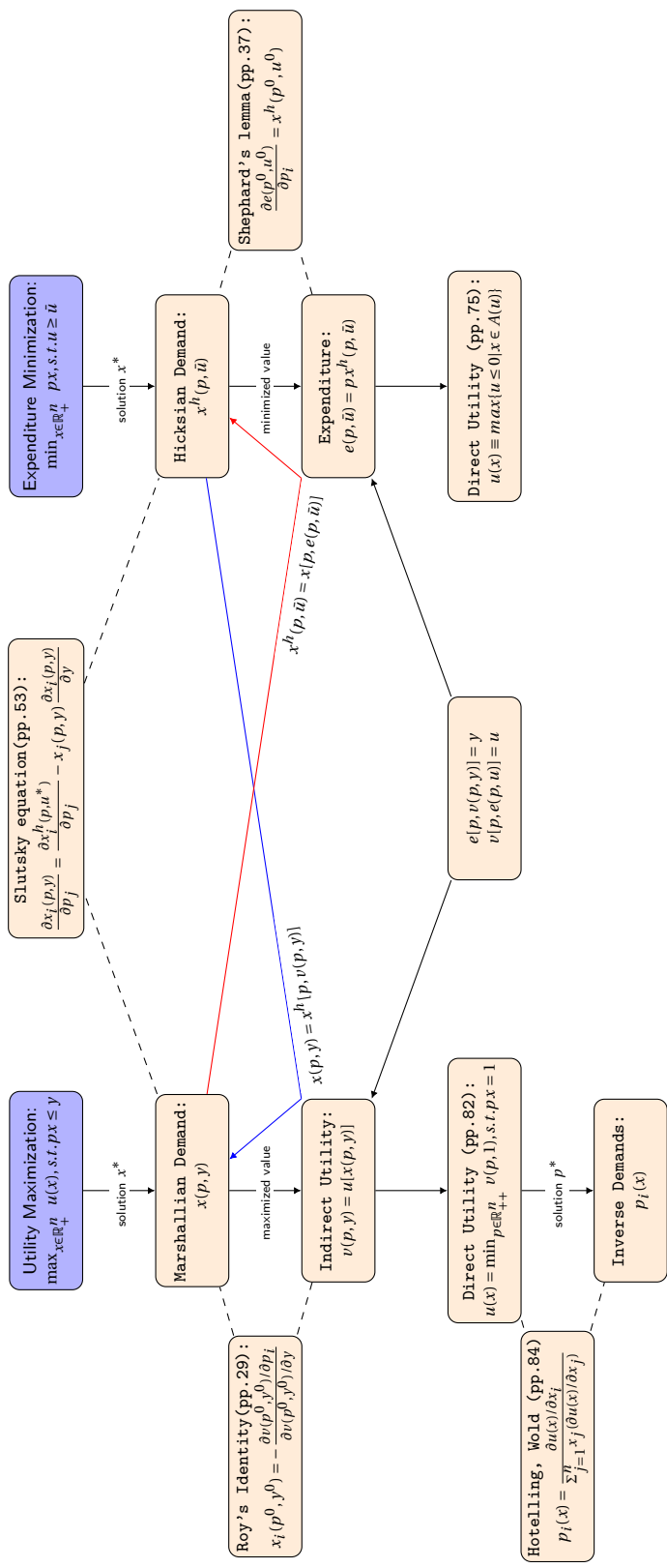
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# Consumption Duality

You will never lose your way with this Consumption Duality map!

All "derive this from that and verify some guy's equation"-like questions can be solved by finding the correct (shortest) route.



## 1 Jehle & Reny 2.3

Derive the consumers direct utility function if his indirect utility function has the form  $v(p, y) = yp_1^\alpha p_2^\beta$  for negative  $\alpha$  and  $\beta$ .

**THEOREM 2.3 Duality Between Direct and Indirect Utility**(Jehle & Reny pp.81)

Suppose that  $u(x)$  is quasiconcave and differentiable on  $\mathbb{R}_{++}^n$  with strictly positive partial derivatives there. Then for all  $x \in \mathbb{R}_{++}^n$ ,  $v(p, p \cdot x)$ , the indirect utility function generated by  $u(x)$ , achieves a minimum in  $p$  on  $\mathbb{R}_{++}^n$ , and

$$u(x) = \min_{p \in \mathbb{R}_{++}^n} v(p, y), \text{ s.t. } px = y$$

Let's call the solution  $p^*$

Note that by **Theorem 1.6**(Jehle & Reny pp.29),  $v(p, y)$  is homogeneous of degree zero in  $(p, y)$ . We have  $v(p, p \cdot x) = v(p/(p \cdot x), 1)$  whenever  $p \cdot x > 0$ . Thus the equation above can also be written as:

$$u(x) = \min_{p \in \mathbb{R}_{++}^n} v(p, 1), \text{ s.t. } px = 1$$

The solution  $\hat{p} = p^* / p^* \cdot x = p^* / y$ . We don't care about the difference between  $\hat{p}$  and  $p^*$  because once you substitute them into  $v(p, p \cdot x)$ , you have the same result (homogeneity of degree zero).

$$u(x) = \min_{p \in \mathbb{R}_{++}^n} v(p, 1) = p_1^\alpha p_2^\beta, \text{ s.t. } px = 1$$

Lagrangian:

$$L = p_1^\alpha p_2^\beta + \lambda(1 - p_1 x_1 - p_2 x_2)$$

Note there should not be interior solution since

- $\frac{\partial v(p_1, p_2, 1)}{\partial p_1} = \alpha p_1^{\alpha-1} p_2^\beta, \alpha, \beta < 0. \lim_{p_1 \rightarrow 0} \frac{\partial v(p_1, p_2, 1)}{\partial p_1} = -\infty$
- $\frac{\partial v(p_1, p_2, 1)}{\partial p_2} = p_1^\alpha \beta p_2^{\beta-1}, \alpha, \beta < 0. \lim_{p_2 \rightarrow 0} \frac{\partial v(p_1, p_2, 1)}{\partial p_2} = -\infty$
- $v(p, 1)$  is decreasing in  $p$ (this is always true for indirect utility function, see pp.29). For any  $px < 1$ , you can always decrease  $v(p, 1)$  by increasing  $p$  until  $px = 1$ .

FOCs.

$$\begin{cases} \frac{\partial L}{\partial p_1} = \alpha p_1^{\alpha-1} p_2^\beta - \lambda x_1 = 0 \\ \frac{\partial L}{\partial p_2} = p_1^\alpha \beta p_2^{\beta-1} - \lambda x_2 = 0 \\ p_1 x_1 + p_2 x_2 = 1 \end{cases}$$

Simplify:

$$\begin{cases} \alpha p_1^{\alpha-1} p_2^\beta = \lambda x_1 \\ \beta p_1^\alpha p_2^{\beta-1} = \lambda x_2 \\ p_1 x_1 + p_2 x_2 = 1 \end{cases} \quad (1)$$

Take the ratio between first and second condition to get:

$$\frac{x_1}{x_2} = \frac{\alpha}{\beta} \frac{p_2}{p_1}$$

Thus:  $p_2 = \frac{\beta}{\alpha} \frac{x_1}{x_2} p_1$

Substitute  $p_2$  with  $p_1$  in the 3rd condition to get:

$$\begin{aligned} p_1 x_1 + \frac{\beta}{\alpha} \frac{x_1}{x_2} p_1 x_2 &= 1 \\ p_1 (x_1 + \frac{\beta}{\alpha} x_2) &= 1 \\ p_1^* &= \frac{1}{x_1 (1 + \frac{\beta}{\alpha})} \\ \Rightarrow p_2^* &= \frac{\beta}{\alpha} \frac{x_1}{x_2} p_1^* = \frac{\beta}{\alpha} \frac{x_1}{x_2} \frac{1}{x_1 (1 + \frac{\beta}{\alpha})} = \frac{1}{x_2 (1 + \frac{\alpha}{\beta})} \end{aligned}$$

Substitute  $p_1^*$  and  $p_2^*$  into  $v(p, 1)$  we get the minimized value, i.e. the direct utility function:

$$\begin{aligned} u(x_1, x_2) &= \left[ \frac{1}{x_1 (1 + \frac{\beta}{\alpha})} \right]^\alpha \left[ \frac{1}{x_2 (1 + \frac{\alpha}{\beta})} \right]^\beta \\ &= A x_1^a x_2^b \end{aligned}$$

Where  $A = \left[ \frac{1}{1 + \frac{\beta}{\alpha}} \right]^\alpha \left[ \frac{1}{1 + \frac{\alpha}{\beta}} \right]^\beta$ ,  $a = -\alpha > 0$ ,  $b = -\beta > 0$ . The utility function is a Cobb-Douglas function.

As a cautious proof, you may want to check if  $u(x)$  is quasiconcave and differentiable on  $\mathbb{R}_{++}^n$  with strictly positive partial derivatives there, as assumed by Theorem 2.3.

In exam for this course, again, if the function is one- dimension, you should prove it; if it's a higher-dimension function, the proof is not required.

Like Jehle & Reny 1.51, you can actually transform  $v(p_1, p_2, 1)$  into a function of only  $p_1$  or  $p_2$  using  $p_1 x_1 + p_2 x_2 = 1$ .

$$p_1 = \frac{1 - p_2 x_2}{x_1}$$

Substitute into  $v(p_1, p_2, 1)$  to have:

$$v(p_1, p_2, 1) = \left[ \frac{1 - p_2 x_2}{x_1} \right]^\alpha p_2^\beta$$

Since the question ask you to minimize  $v(p_1, p_2, 1)$ , if you solve  $\frac{de(p_2)}{dp_2} = 0$  and get only one solution, it is the solution.

$$\begin{aligned} \frac{de(p_2)}{dp_2} &= \alpha \left( \frac{1 - p_2 x_2}{x_1} \right)^{\alpha-1} \left( \frac{-x_2}{x_1} \right) p_2^\beta + \frac{1 - p_2 x_2}{x_1}^\alpha \beta p_2^{\beta-1} = 0 \\ \alpha \left( \frac{1 - p_2 x_2}{x_1} \right)^{\alpha-1} \left( \frac{x_2}{x_1} \right) p_2^\beta &= \frac{1 - p_2 x_2}{x_1}^\alpha \beta p_2^{\beta-1} \\ \alpha \left( \frac{x_1}{1 - p_2 x_2} \right) \left( \frac{x_2}{x_1} \right) p_2 &= \beta \\ \alpha \left( \frac{x_2}{1 - p_2 x_2} \right) p_2 &= \beta \\ \alpha x_2 p_2 &= \beta - \beta x_2 p_2 \\ (\alpha x_2 + \beta x_2) p_2 &= \beta \\ p_2^* &= \frac{\beta}{(\alpha + \beta) x_2} \end{aligned}$$

You then solve  $p_1^*$  with the budget constraint.

## 2 Jehle & Reny 2.5(a)

Consider the solution,  $e(p, u) = u p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$  at the end of Example 2.3. Derive the **indirect utility function** through the relation  $e(p, v(p, y)) = y$  and verify Roy's identity.

Example 2.3 on Jehle & Reny pp.90 is a question from  $x_i(p, y)$  to  $e(p, u)$ , where the Marshallian demand function is:

$$x_i(p_1, p_2, p_3, y) = \frac{\alpha_i y}{p_i}, \quad i = 1, 2, 3$$

$\alpha_i > 0$  and  $\alpha_1 + \alpha_2 + \alpha_3 = 1$

Check your map, the route is (note the expression below is only for the purpose of teaching and very informal):

$$x_i(p, y) \Rightarrow x^h(p, u) = x[p, e(p, u)] \Leftarrow \frac{\partial e(p, u)}{\partial p_i} = x^h(p, u)$$

$$\begin{aligned}
x[p, e(p, u)] &= \frac{\partial e(p, u)}{\partial p_i} \\
\frac{\alpha_i e(p, u)}{p_i} &= \frac{\partial e(p, u)}{\partial p_i} \\
\frac{\alpha_i}{p_i} &= \frac{1}{e(p, u)} \frac{\partial e(p, u)}{\partial p_i} \\
&= \frac{\partial \ln[e(p, u)]}{\partial p_i}
\end{aligned}$$

The rest part of the solution in the textbook is clear. Read page 91 if you're curious how we solve  $e(p, u)$  out. It need "a little thought" as the textbook said :)

#### Indirect utility function:

We already know  $e(p, v(p, y)) = y$ .

Substitute  $v(p, y)$  into  $e(p, u) = y$  will solve the question directly:

$$\begin{aligned}
e(p, u) &= v(p, y) p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} = y \\
\Rightarrow v(p, y) &= \frac{y}{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}}
\end{aligned}$$

#### Verify Roy's identity:

Roy's Identity(Jehle & Reny pp.29):

$$x_i(p^0, y^0) = - \frac{\partial v(p^0, y^0) / \partial p_i}{\partial v(p^0, y^0) / \partial y}$$

**Intuition:** Your optimal consumption plan (Marshallian demand) is a trade-off between the importance of "commodity  $i$ " and "money ( $y$ )".

$$\begin{aligned}
\frac{\partial v(p, y)}{\partial p_1} &= \frac{\partial y p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}}{\partial p_1} = -\alpha_1 y p_1^{-\alpha_1-1} p_2^{-\alpha_2} p_3^{-\alpha_3} \\
\frac{\partial v(p, y)}{\partial p_2} &= -\alpha_2 y p_1^{-\alpha_1} p_2^{-\alpha_2-1} p_3^{-\alpha_3} \\
\frac{\partial v(p, y)}{\partial p_3} &= -\alpha_3 y p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3-1} \\
\frac{\partial v(p, y)}{\partial y} &= p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}
\end{aligned}$$

Therefore:

$$-\frac{\partial v(p, y) / \partial p_1}{\partial v(p, y) / \partial y} = -\frac{-\alpha_1 y p_1^{-\alpha_1-1} p_2^{-\alpha_2} p_3^{-\alpha_3}}{p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}} = \frac{\alpha_1 y}{p_1}$$

$$-\frac{\partial v(p, y)/\partial p_2}{\partial v(p, y)/\partial y} = -\frac{-\alpha_3 y p_1^{-\alpha_1} p_2^{-\alpha_2-1} p_3^{-\alpha_3}}{p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}} = \frac{\alpha_2 y}{p_1}$$

$$-\frac{\partial v(p, y)/\partial p_3}{\partial v(p, y)/\partial y} = -\frac{-\alpha_3 y p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3-1}}{p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}} = \frac{\alpha_3 y}{p_1}$$

Compare with the Marshallian demand!

### 3 Jehle & Reny 2.7

Derive the consumer's **inverse demand functions**,  $p_1(x_1, x_2)$  and  $p_2(x_1, x_2)$ , when the **utility function** is of the Cobb-Douglas form,  $u(x_1, x_2) = Ax_1^\alpha x_2^{1-\alpha}$  for  $0 < \alpha < 1$ .

The shortest route is using Hotelling, Wold (pp.84) directly.

$$p_i(x) = \frac{\partial u(x)/\partial x_i}{\sum_{j=1}^n x_j (\partial u(x)/\partial x_j)}$$

**Intuition:** the price reflects how important the commodity is.

The duality between direct and indirect utility functions showed by Hotelling, Wold makes it (hopefully) easier to solve  $p_i^*(x)$

$$\begin{aligned} p_1(x_1, x_2) &= \frac{\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_1}{\sum_{j=1}^2 x_j (\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_j)} \\ &= \frac{\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_1}{x_1 \partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_1 + x_2 \partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_2} \\ &= \frac{A\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{x_1 A\alpha x_1^{\alpha-1} x_2^{1-\alpha} + x_2 A(1-\alpha) x_1^\alpha x_2^{-\alpha}} \\ &= \frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{\alpha x_1^\alpha x_2^{1-\alpha} + (1-\alpha) x_1^\alpha x_2^{1-\alpha}} \\ &= \frac{\alpha}{x_1} \end{aligned}$$

$$\begin{aligned}
p_2(x_1, x_2) &= \frac{\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_2}{\sum_{j=1}^2 x_j (\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_j)} \\
&= \frac{\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_2}{x_1 \partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_1 + x_2 \partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_2} \\
&= \frac{A(1-\alpha)x_1^\alpha x_2^{-\alpha}}{x_1 A\alpha x_1^{\alpha-1} x_2^{1-\alpha} + x_2 A(1-\alpha)x_1^\alpha x_2^{-\alpha}} \\
&= \frac{(1-\alpha)x_1^\alpha x_2^{-\alpha}}{\alpha x_1^\alpha x_2^{1-\alpha} + (1-\alpha)x_1^\alpha x_2^{1-\alpha}} \\
&= \frac{1-\alpha}{x_2}
\end{aligned}$$

You can also try another route: maximize  $u(x) \Rightarrow x(p, y) \Rightarrow p_i(x) = x^{-1}(x, 1)$   
Use Lagrangian to maximize  $u(x_1, x_2) = Ax_1^\alpha x_2^{1-\alpha}$  s.t.  $p_1 x_1 + p_2 x_2 = 1$ . The solution (Marshallian demands) is:

$$\begin{cases} x_1 = \frac{\alpha}{p_1} \\ x_2 = \frac{1-\alpha}{p_2} \end{cases}$$

The inverse of Marshallian demand function gives the inverse demand function

$$\begin{cases} p_1 = \frac{\alpha}{x_1} \\ p_2 = \frac{1-\alpha}{x_2} \end{cases}$$

Another example:

- You can also try to derive  $p_i(x)$  from the Marshallian demand E.1 on pp. 32 and compare with the result derived from Hotelling-Wold identity on pp. 85. Both of them should be the same as E.5-E.6 on pp. 83, which is the solution for  $v(p, 1)$  minimization problem.