Seminar 5. Production Theory

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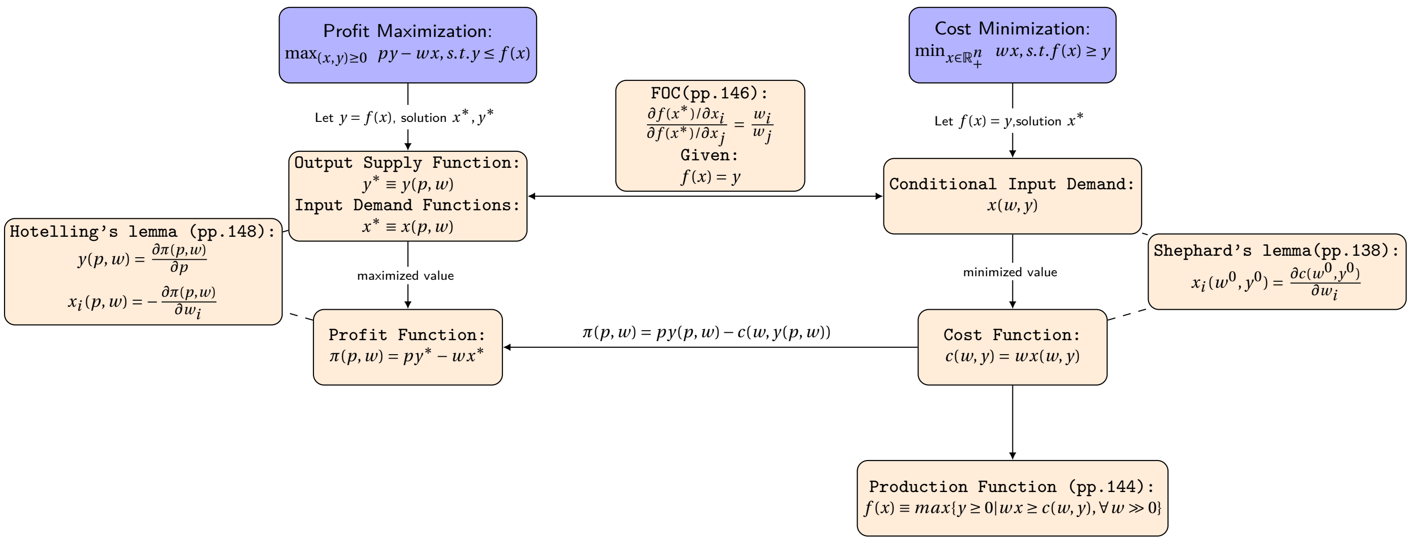
Production Duality

The dual relation between product maximization problem and cost minimization problem is simpler than consumers' theory.

- Given $y = f(x)$, profit function is only a function of x. That's why minimizing the cost will maximize the profit at the same time.



1 Jehle & Reny 3.35

Calculate the **cost function** and the **conditional input demands** for the linear production function, $y = \Sigma_{i=1}^{n} \alpha_i x_i$.

Production Function(Jehle & Reny pp.127)
We use a function $y = f(x)$ to denote y units of a certain commodity is produced using input x, where $x \in \mathbb{R}_{+}^{n}, y \in \mathbb{R}_{+}^{1}$

ASSUMPTION 3.1 Properties of the Production Function (Jehle & Reny pp.127)
The production function, $f : \mathbb{R}_{+}^{n} \to \mathbb{R}_{+}$, is continuous, strictly increasing, and strictly quasiconcave on $\mathbb{R}_{+}^{n}$, and $f(0) = 0$.

DEFINITION 3.5 The Cost Function (Jehle & Reny pp.136)
The cost function, defined for all input prices $w \gg 0$ and all output levels $y \in f(\mathbb{R}_{+}^{n})$ is the minimum-value function,

$$c(w, y) \equiv \min_{x \in \mathbb{R}_{+}^{n}} w \cdot x, \quad s.t. \quad f(x) \geq y.$$

The solution $x(w, y)$ is referred to as the firms **conditional input demand**, because it is conditional on the level of output y.

- **Conditional input demand** is similar to Hicksian demands for consumers.

Here the linear production function $y = \Sigma_{i=1}^{n} \alpha_i x_i$ is very similar to the "**perfect substitution**" preference in Seminar 4.

- The product can be produced by any input x_i, the only difference is that for each unit of input, different x_i produces different amount α_i of the output.
- The Marginal Rate of Technical Substitution of input x_j for input x_i is constant: $MRTS_{ij} = \frac{\partial f(x)/\partial x_i}{\partial f(x)/\partial x_j} = \frac{\alpha_i}{\alpha_j}$.

An example: an apple jam factory has 2 types of input, "single apple (x_1)" and "2-apple pack (x_2)".

- With a "single apple", the factory can produce a bottle of apple jam;
- with a "2-apple pack", 2 bottles.
- The production function is $y = 1 \cdot x_1 + 2 \cdot x_2$

How will the factory choose? Similarly to consumers' preference substitution preference, the factory will spend all money on the "cheapest per unit (of output)" input.

Denote the price for x_1 and x_2 as w_1, w_2

- If $\frac{w_1}{1} = \frac{w_2}{2}$, the factory doesn't care which to use;
- If $\frac{w_1}{1} < \frac{w_2}{2}$, single apple is cheaper;
- If $\frac{w_1}{1} > \frac{w_2}{2}$, 2-apple pack is cheaper.

Denote the price for input x_i as w_i. Define $\omega = min\{\frac{w_1}{\alpha_1}, \frac{w_2}{\alpha_2}, \dots, \frac{w_n}{\alpha_n}\}$

If ω is the price of only one input x_j, the firm will only use input x_j to minimize its cost.

- Thus $y = \alpha_j x_j$ can minimize the cost, and $x_j = \frac{y}{\alpha_j}$ is the conditional input demands.
- The cost function $c(w, y) = w_j \frac{y}{\alpha_j} = \omega y$.

If ω is the price of several inputs $x_m, m = 1, 2, \dots, M$, the firm can freely combine x_m to minimize its cost, as long as $\Sigma_{m=1}^{M} \alpha_m x_m = y$.

For cost funtion, let's assume $\frac{w_1}{\alpha_1} = \frac{w_2}{\alpha_2} = \dots = \frac{w_M}{\alpha_M} = \omega$, then ω is the price for 1 single apple (the input needed to produce 1 bottle of jam), for example. Again, to produce y bottles of jam, you need the same number (y) of single apples. The cost function is thus ωy

2 Another production & cost function example from exam 2019 Q1(b): HydroP

To produce electricity E, firm HydroP uses water W and a plant P as main inputs. It operates in a unique location, so that no further plants can be built. Without the plant the production is 0. With the plant, electricity can be produced according to the following production function:

$$E = \begin{cases} 0, & if \;\; W \leq \underline{W} \\ 4W, & if \;\; \underline{W} \leq W \leq \bar{W} \\ 3\bar{W}, & if \;\; \bar{W} \leq W \end{cases} \tag{1}$$

2.1 a) Function property

[5%] Is this production function: continuous? (strictly) increasing? (strictly) quasiconcave? increasing/decreasing/constant returns to scale?

When $W = \underline{W}$, the production function value is 0 and $4\underline{W}$. Therefore $\underline{W} = 0$.
But the guideline online treated "$W \leq \underline{W}$" as "$W < \underline{W}$", which is incorrect.

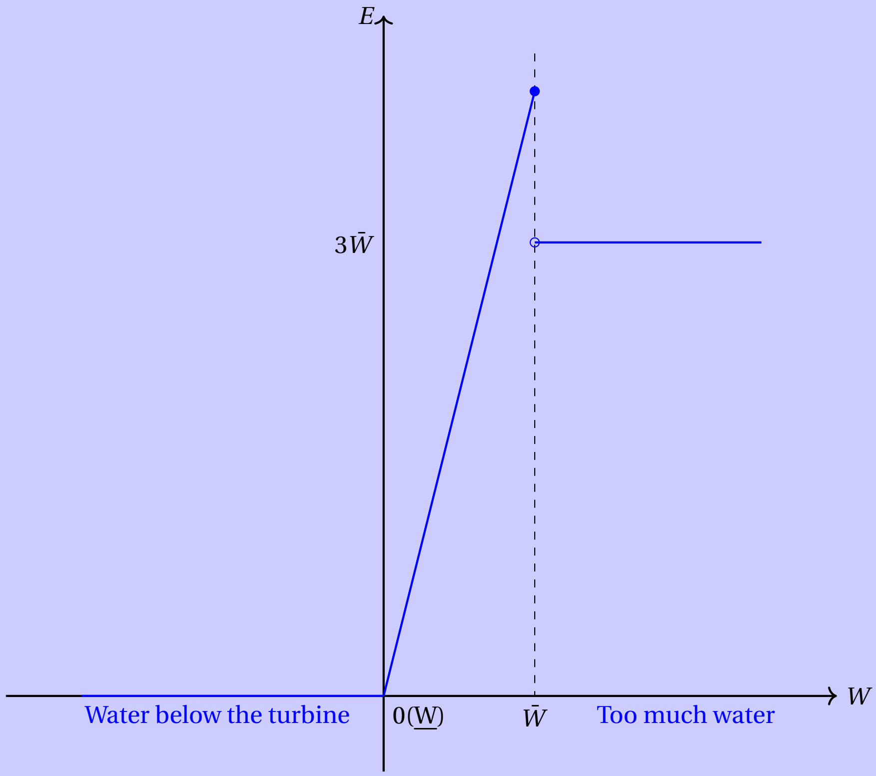


Figure 1: HydroP's Production function

Obviously, the production function is:

- not continuous at $(\bar{W}, 0)$
- not increasing in $(-\infty, 0)$ and $(\bar{W}, +\infty)$
- quasiconcave: $f(x^t) \geq min[f(x^1), f(x^2)]$ for all $t[0,1]$.
- not strictly quasiconcave: sometimes $f(x^t) = min[f(x^1), f(x^2)]$
- return to scale uncertain, "jump" around $(0,0)$ and $(\bar{W}, 0)$

Answer:
The production function is:

- not continuous
- not increasing and therefore not strictly increasing
- quasiconcave but not strictly quasiconcave
- return to scale uncertain, sometimes increase, sometimes constant or decrease.

- Draw sketch for unusual function!
- Don't be wordy, don't prove unnecessarily

2.2 b) Cost function

[7%] Determine the cost function for this firm (let the price of water be and let K be the cost of the plant). Is the cost function continuous?

Cost function: $c(w, y) \equiv \min_{x \in \mathbb{R}^n_+} w \cdot x, \;\; s.t. \;\; f(x) \geq y.$

- It is always the minimized value function
- It's a function of input price w and output amount y
- There can be fixed cost in the short run.

In this question, the only input is water, with a price of p_w. There is also a fixed cost K if you build the plant. The output is electricity E. We can write the cost function as $c(p_w, K, E)$

Now let's find the miminum cost with different output amount E on our sketch.

When $E = 0$, you will not build the plant to minimize your cost, $c(p_w, K, 0) = 0$

When $E \in (0, 4\bar{W}]$, you must build the plant with a fixed cost K. Besides, for any output E, you need $\frac{E}{4}$ water, which costs $p_w \frac{E}{4}$.

- Can you produce more than $4\bar{W}$? No, that's not allowed by your technology, the plant.
- Will you use water more than $\bar{W}$? No, that's not cost minimized. You always choose the most "economical" way to minimize the cost. Don't forget you're looking for the **cost function**

Many students lost points because they didn't know what is the "variable" of a cost function $c(w, y)$.

- A common mistake is: $c = \ldots, \;\; if \;\; W \in (0, 3\bar{W})$. Note W is input (water) amount, not a variable of cost function.
- Ask yourself, what is the input price w here? What is the product amount y here?

The cost function is:

$$c(p_w, K, E) = \begin{cases} 0, & if \;\; E = 0 \\ K + p_w \frac{E}{4}, & if \;\; 0 < E \leq 4\bar{W} \end{cases} \tag{2}$$

Again, note $\underline{W} = 0$. Compare the result with the online guideline.

- Based on the guideline's understanding of the question, is it possible to produce $E \in (0, 4\underline{W})$

2.3 c) Integrability

[5%] Can one recover the original production function from the cost function? Why not?

Observe function 2, can you guess what is the production function $f(x)$?
When $E = 0$, there is no cost and no output, $f(0) = 0$
When $E \in (0, 4\bar{W}]$, we know K is fixed cost, p_w is the price of your input, $\frac{E}{4}$ is the amount of water you used to generate electricity (output) E. Thus

$$f(\frac{E}{4}) = E \Rightarrow f(x) = 4x$$

- Can you know the production function is $f(x) = 3\bar{W}$ when $x > \bar{W}$? No, you can't imagine that from the cost function.
- Since cost function is always the minimized value function, the production function recovered from a cost function is only the "most efficient" part. Any non- cost -minimization technology can't be recovered.

No. Because when $x > \bar{W}$, the production is never cost minimizing, while cost function can only reflect the technology minimizing the cost.

3 Jehle & Reny 3.46

- Verify Theorem 3.7 for the profit function obtained in Example 3.5.
- Verify Theorem 3.8 for the associated output supply and input demand functions.

3.1 Verify Theorem 3.7

DEFINITION 3.7 The Profit Function (Jehle & Reny pp.148)

$$\pi(p, w) \equiv \max_{(x,y)\geq 0} py - wx, \quad s.t. y \leq f(x)$$

Note, $y \leq f(x)$ means "you can only decide to produce what's possible to be produced":

- Assume you want to set an optimal output y, forget input x for now;

- You can "waste", i.e., output $y < f(x)$ is possible. Input is not efficiently used for technology $f(x)$;
- You can't produce more than what your technology $f(x)$ allows, i.e. $y \not> f(x)$.

It's not easy to solve the maximization problem directly because there are 2 variable, y and x (again, y is **NOT** necessarily to be $f(x)$, but it can't exceed $f(x)$).
Since to waste will definitely reduce profit (you have some cost but don't produce anything), as a price reveiver, you will always avoid wasting by making fully use of your technology, i.e. $y = f(x)$. Then the profit maximization problem is transformed into:

$$\pi(p, w) = \max_{x \geq 0} pf(x) - wx$$

No constraint anymore! The function has only one variable, input x. FOC will (usually) solve the problem.
In **Example 3.5** (Jehle & Reny pp.151), the CES production function is:

$$y = (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}}$$

Where $\beta < 1$ and $0 \neq \rho < 1$. To obtain the profit function, we need to solve the maximization problem:

$$\max_{(x,y) \geq 0} py - wx, \ \ s.t. \ \ y \leq (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}}$$

Again, we don't waste, i.e. $y = (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}}$, the problem above is then:

$$\max_{x \geq 0} p(x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}} - w_1 x_1 - w_2 x_2$$

FOC are given in the textbook pp. 151.
The y^* solved is the **output supply function**:

$$y^* = (p\beta)^{-\frac{\beta}{\beta-1}} (w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \tag{3}$$

(You'll compare the output function with equation 9 derived from cost minimization.)
The x^* solved is the **input demand function**:

$$x_i^* = w_i^{\frac{1}{\rho-1}} (p\beta)^{\frac{-1}{\beta-1}} (w_1^r + w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}} \tag{4}$$

The profit function is thus:

$$\pi = py^* - wx^* = p^{-\frac{1}{\beta-1}} (w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \beta^{-\frac{\beta}{\beta-1}} (1-\beta) \tag{5}$$

(You'll compare the profit function with equation 10 derived from cost minimization.)

THEOREM 3.7 Properties of the Profit Function (Jehle & Reny pp.148)

If f satisfies Assumption 3.1, then for $p \geq 0$ and $w \geq 0$, the profit function $\pi(p,w)$, where well-defined, is continuous and

1. Increasing in p,
2. Decreasing in w,
3. Homogeneous of degree one in (p,w),
4. Convex in (p,w),
5. Differentiable in (p,w).
6. Moreover, under the additional assumption that f is strictly concave (Hotelling's lemma),

$$y(p,w) = \frac{\partial \pi(p,w)}{\partial p}, \ and \ \ x_i(p,w) = -\frac{\partial \pi(p,w)}{\partial w_i}. \quad i = 1,2,\ldots,n.$$

1.Increasing in p

For $\pi(p,w) = py^* - wx^* = p^{-\frac{1}{\beta-1}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}}\beta^{-\frac{\beta}{\beta-1}}(1-\beta)$,

$$\begin{aligned}
\frac{\partial \pi(p,w)}{\partial p} &= (-\frac{1}{\beta-1})p^{-\frac{1}{\beta-1}-1}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}}\beta^{-\frac{\beta}{\beta-1}}(1-\beta) \\
&= p^{\frac{\beta}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}}\beta^{-\frac{\beta}{\beta-1}} \\
&= (p\beta)^{\frac{\beta}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}}
\end{aligned}$$

When $0 < \beta < 1$, $\frac{\partial \pi(p,w)}{\partial p} \geq 0$

2.Decreasing in w

$$\begin{aligned}
\frac{\partial \pi(p,w)}{\partial w_i} &= p^{-\frac{1}{\beta-1}}[\frac{\beta}{r(\beta-1)}](w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}-1} r w_i^{r-1}\beta^{-\frac{\beta}{\beta-1}}(1-\beta) \\
&= p^{-\frac{1}{\beta-1}}[-\beta](w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}-1} w_i^{r-1}\beta^{-\frac{\beta}{\beta-1}} \\
&= p^{-\frac{1}{\beta-1}}[-1](w_1^r + w_2^r)^{\frac{\beta}{\frac{\rho}{\rho-1}(\beta-1)}-1} w_i^{\frac{\rho}{\rho-1}-1}\beta^{1-\frac{\beta}{\beta-1}} \\
&= -p^{-\frac{1}{\beta-1}}(w_1^r + w_2^r)^{\frac{\beta}{\frac{\rho}{\rho-1}(\beta-1)}-1} w_i^{\frac{1}{\rho-1}}\beta^{-\frac{1}{\beta-1}} \\
&= -(p\beta)^{-\frac{1}{\beta-1}}(w_1^r + w_2^r)^{\frac{(\rho-1)\beta}{\rho(\beta-1)}-1} w_i^{\frac{1}{\rho-1}} \\
&= -(p\beta)^{-\frac{1}{\beta-1}}(w_1^r + w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}} w_i^{\frac{1}{\rho-1}}
\end{aligned}$$

$i = 1, 2$.
When $0 < \beta < 1$, $\frac{\partial \pi(p,w)}{\partial w_i} \leq 0$

3.Homogeneous of degree one in (p, w)

$$\begin{aligned}
\pi(tp, tw) &= (tp)^{-\frac{1}{\beta-1}} [(tw_1)^r + (tw_2)^r]^{\frac{\beta}{r(\beta-1)}} \beta^{-\frac{\beta}{\beta-1}} (1-\beta) \\
&= t^{-\frac{1}{\beta-1}} p^{-\frac{1}{\beta-1}} t^{\frac{\beta}{(\beta-1)}} [(w_1)^r + (w_2)^r]^{\frac{\beta}{r(\beta-1)}} \beta^{-\frac{\beta}{\beta-1}} (1-\beta) \\
&= t^{-\frac{1}{\beta-1}+\frac{\beta}{(\beta-1)}} p^{-\frac{1}{\beta-1}} [(w_1)^r + (w_2)^r]^{\frac{\beta}{r(\beta-1)}} \beta^{-\frac{\beta}{\beta-1}} (1-\beta) \\
&= tp^{-\frac{1}{\beta-1}} [(w_1)^r + (w_2)^r]^{\frac{\beta}{r(\beta-1)}} \beta^{-\frac{\beta}{\beta-1}} (1-\beta) \\
&= t^1 \pi(p, w)
\end{aligned}$$

4.Convex in (p, w) Higher-dimension proof not required in exam
For 1-dimension function $f(x)$:

$$f(x) \ is \ convex \iff f''(x) \geq 0$$

Recall one-dimension condition:
Convex functions (Jehle & Reny pp.542):
$f : D \to \mathbb{R}$ is a convex function if for all $x^1, x^2 \in D$,

$$f(x^t) \leq tf(x^1) + (1-t)f(x^2), \ \forall t \in [0, 1].$$

Where $x^t \equiv tx^1 + (1-t)x^2, t \in [0, 1]$.

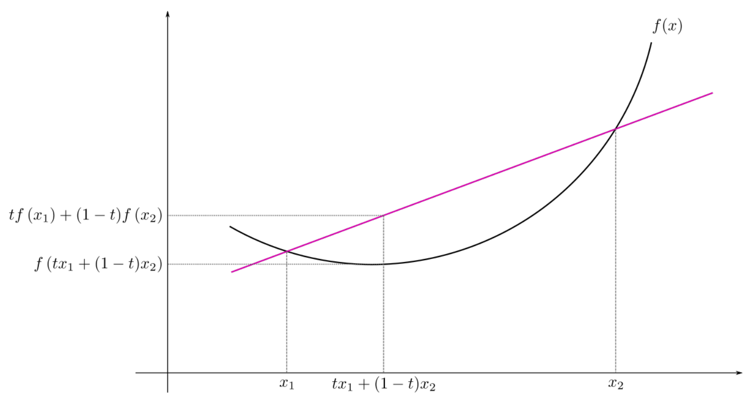


Figure 2: Convex function

5.Differentiable in (p, w)
Higher-dimension proof not required in exam.
Recall one-dimension condition: derivative exists at $x^0 \Rightarrow$ Differentiable at x^0.

6.Hotelling's lemma
We already calculated the derivatives. Compare them with equation 3 and 4.

3.2 Verify Theorem 3.8

THEOREM 3.8 Properties of Output Supply and Input Demand Functions (Jehle & Reny pp.149)
Suppose that f is a strictly concave production function satisfying Assumption 3.1 and that its associated profit function, $\pi(p,y)$, is twice continuously differentiable. Then, for all $p>0$ and $w \gg 0$ where it is well defined:

1. Homogeneity of degree zero:
$$y(tp,tw) = y(p,w), \forall t>0,$$
$$x_i(tp,tw) = x_i(p,w), \forall t>0 \; and \; i=1,\dots,n.$$
2. Own-price effects:
$$\frac{\partial y(p,w)}{\partial p} \geq 0,$$
$$\frac{\partial x_i(p,w)}{\partial w_i} \leq 0, \; \forall i=1,\dots,n.$$
3. The substitution matrix is symmetric and positive semidefinite.
$$\begin{pmatrix} \frac{\partial y(p,w)}{\partial p} & \frac{\partial y(p,w)}{\partial w_1} & \cdots & \frac{\partial y(p,w)}{\partial w_n} \\ \frac{-\partial x_1(p,w)}{\partial p} & \frac{-\partial x_1(p,w)}{\partial w_1} & \cdots & \frac{-\partial x_1(p,w)}{\partial w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-\partial x_n(p,w)}{\partial p} & \frac{-\partial x_n(p,w)}{\partial w_1} & \cdots & \frac{-\partial x_n(p,w)}{\partial w_n} \end{pmatrix} \tag{6}$$

Copy from Equation 3 and Equation 4,
Output supply function:
$$y(p,w) = (p\beta)^{-\frac{\beta}{\beta-1}} (w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}}$$

Input demand function:
$$x_i(p,w) = w_i^{\frac{1}{\rho-1}} (p\beta)^{\frac{-1}{\beta-1}} (w_1^r + w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}}$$

1.Homogeneity of degree zero

$$\begin{aligned} y(tp,tw) &= (tp\beta)^{-\frac{\beta}{\beta-1}} [(tw_1)^r + (tw_2)^r]^{\frac{\beta}{r(\beta-1)}} \\ &= t^{-\frac{\beta}{\beta-1}} (p\beta)^{-\frac{\beta}{\beta-1}} t^{\frac{\beta}{(\beta-1)}} (w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \\ &= t^0 (p\beta)^{-\frac{\beta}{\beta-1}} (w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \\ &= t^0 y(p,w) \end{aligned}$$

$$
\begin{aligned}
x_i(tp,tw) &= (tw_i)^{\frac{1}{\rho-1}}(tp\beta)^{\frac{-1}{\beta-1}}[(tw_1)^r+(tw_2)^r]^{\frac{\rho-\beta}{\rho(\beta-1)}}\\
&= t^{\frac{1}{\rho-1}}w_i^{\frac{1}{\rho-1}}t^{\frac{-1}{\beta-1}}(p\beta)^{\frac{-1}{\beta-1}}t^{r\frac{\rho-\beta}{\rho(\beta-1)}}(w_1^r+w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}}\\
&= t^{\frac{1}{\rho-1}+\frac{-1}{\beta-1}+\frac{\rho}{\rho-1}\frac{\rho-\beta}{\rho(\beta-1)}}w_i^{\frac{1}{\rho-1}}(p\beta)^{\frac{-1}{\beta-1}}(w_1^r+w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}}\\
&= t^0x_i(p,w)
\end{aligned}
$$

$i=1,2$.

2.Own-price effects

$$
\frac{\partial y(p,w)}{\partial p}=(-\frac{\beta}{\beta-1})p^{-\frac{\beta}{\beta-1}-1}(\beta)^{-\frac{\beta}{\beta-1}}(w_1^r+w_2^r)^{\frac{\beta}{r(\beta-1)}}
$$

When $\beta\in(0,1)$, $\frac{\partial y(p,w)}{\partial p}\geq 0$.

$$
\begin{aligned}
\frac{\partial x_i(p,w)}{\partial w_1} &= \frac{1}{\rho-1}w_1^{\frac{1}{\rho-1}-1}(p\beta)^{\frac{-1}{\beta-1}}(w_1^r+w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}}+w_1^{\frac{1}{\rho-1}}(p\beta)^{\frac{-1}{\beta-1}}\frac{\rho-\beta}{\rho(\beta-1)}(w_1^r+w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}-1}rw_1^{r-1}\\
&= \frac{1}{\rho-1}w_1^{\frac{1}{\rho-1}-1}(p\beta)^{\frac{-1}{\beta-1}}(w_1^r+w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}}+w_1^{\frac{1}{\rho-1}+r-1}(p\beta)^{\frac{-1}{\beta-1}}\frac{\rho-\beta}{\rho(\beta-1)}(w_1^r+w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}-1}\frac{\rho}{\rho-1}\\
&= \frac{1}{\rho-1}w_1^{\frac{1}{\rho-1}-1}(p\beta)^{\frac{-1}{\beta-1}}(w_1^r+w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}}+w_1^{\frac{1}{\rho-1}+r-1}(p\beta)^{\frac{-1}{\beta-1}}\frac{\rho-\beta}{(\rho-1)(\beta-1)}(w_1^r+w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}-1}\\
&= [\frac{1}{\rho-1}(w_1^r+w_2^r)+w_1^r\frac{\rho-\beta}{(\rho-1)(\beta-1)}]w_1^{\frac{1}{\rho-1}-1}(w_1^r+w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}-1}(p\beta)^{\frac{-1}{\beta-1}}\\
&= [(\frac{\beta-1}{(\rho-1)(\beta-1)}+\frac{\rho-\beta}{(\rho-1)(\beta-1)})w_1^r+\frac{1}{\rho-1}w_2^r]w_1^{\frac{1}{\rho-1}-1}(w_1^r+w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}-1}(p\beta)^{\frac{-1}{\beta-1}}\\
&= [\frac{1}{\beta-1}w_1^r+\frac{1}{\rho-1}w_2^r]w_1^{\frac{1}{\rho-1}-1}(w_1^r+w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}-1}(p\beta)^{\frac{-1}{\beta-1}}
\end{aligned}
$$

Similarly, we have

$$
\frac{\partial x_i(p,w)}{\partial w_2}=[\frac{1}{\beta-1}w_1^r+\frac{1}{\rho-1}w_2^r]w_2^{\frac{1}{\rho-1}-1}(w_1^r+w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}-1}(p\beta)^{\frac{-1}{\beta-1}}
$$

When $\beta\in(0,1),\rho<1$, $\frac{\partial y(p,w)}{\partial w_i}\leq 0$,$i=1,2$.

1. Assign some value to the parameters and variables to verify your calculation:
   - Let, for example, $p=2,\rho=-1,\beta=0.5,w_1=w_2=2$, then $r=\frac{\rho}{\rho-1}=0.5$, $\frac{\rho-\beta}{\rho(\beta-1)}=-3$
   - Calculate the first line of $\frac{\partial x_i(p,w)}{\partial w_1}$ and the simplified result in the last line, they should be the same ($-\frac{5}{256}$)

2. In exam, leave it to the last if the calculation is very heavy and you find the result seems to be wrong.
3. We will NOT repeatedly punish you for wrong results, i.e., if the wrong results in step 1 lead to mistakes in the rest steps, as long as your method is correct, you will get the point for the rest part.
For example, a question asks you to calculate Elasticity of Substitution and what substitution relationship the result implies.The correct answer is "$ES = +\infty$, perfect substitution".

- If you calculated it wrongly and have "$ES = 0$, no substitution", you will get points from "no substitution", the "correct" conclusion based on your wrong ES
- If you calculated it wrongly and have "$ES = 0$", but argue "$ES = 0$" implies "perfect substitution", then it's totally wrong and you get no point, even though the conclusion "perfect substitution" is the same as the solution.

3.Substitution matrix
Proof not really required in exam. But it may be helpful to understand it.

According to Hotelling's lemma,

$$y(p,w) = \frac{\partial \pi(p,w)}{\partial p}, \; and \;\; x_i(p,w) = -\frac{\partial \pi(p,w)}{\partial w_i}. \quad i = 1, 2, \ldots, n.$$

We can rewrite:

$$\frac{\partial y(p,w)}{\partial p} = \frac{\partial^2 \pi(p,w)}{\partial p^2}$$

$$\frac{\partial y(p,w)}{\partial w_i} = \frac{\partial^2 \pi(p,w)}{\partial p \partial w_i}$$

$$-\frac{\partial x_i(p,w)}{\partial p} = \frac{\partial^2 \pi(p,w)}{\partial w_i \partial p}$$

$$-\frac{\partial x_i(p,w)}{\partial w_j} = \frac{\partial^2 \pi(p,w)}{\partial w_i \partial w_j}$$

The Substitution matrix is therefore the Hessian Matrix (check seminar 2) of the Profit Function $\pi(p,w)$:

$$\begin{pmatrix} \frac{\partial^2 \pi(p,w)}{\partial p^2} & \frac{\partial^2 \pi(p,w)}{\partial p \partial w_1} & \cdots & \frac{\partial^2 \pi(p,w)}{\partial p \partial w_n} \\ \frac{\partial^2 \pi(p,w)}{\partial w_1 \partial p} & \frac{\partial^2 \pi(p,w)}{\partial w_1^2} & \cdots & \frac{\partial^2 \pi(p,w)}{\partial w_1 \partial w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \pi(p,w)}{\partial w_n \partial p} & \frac{\partial^2 \pi(p,w)}{\partial w_n \partial w_1} & \cdots & \frac{\partial^2 \pi(p,w)}{\partial w_n^2} \end{pmatrix} \tag{7}$$

According to Young's theorem (Jehle & Reny pp.557), $\frac{\partial^2 \pi(p,w)}{\partial p \partial w_i} = \frac{\partial^2 \pi(p,w)}{\partial w_i \partial p}$, the matrix must be symmetric.
As a convex function, Profit Function $\pi(p, w)$ must have a positive semidefinite Hessian Matrix.
To prove the matrix is positive semidenite is not required in exam, but you need to know the definition:
Positive semidefinite(Jehle & Reny pp.559): a $n \times n$ metrix A is positive semidefinite if for all vectors $z \in R^n$, $z^T Az \geq 0$
Assume matrix S is a Substitution matrix,

- Let $z = (1, 0, \dots, 0)^T$,
$$z^T Sz = \frac{\partial y(p, w)}{\partial p} \geq 0$$
- Let $z = (0, 1, \dots, 0)^T$,
$$z^T Sz = \frac{-\partial x_1(p, w)}{\partial w_1} \geq 0$$
- ...

You can find the diagonal elements of a positive semidenite matrix is always non-negative.
The conclusion above is the same as the so-called "Own-price effects" we have proved.

4 Jehle & Reny 3.49

1. Derive the **cost function** for the production function in Example 3.5 .
2. Solve $\max_y py - c(w, y)$
3. Compare its solution, $y(p, w)$, to the solution in (E.5). Check that $\pi(p, w) = py(p, w) - c(w, y(p, w))$.
4. Supposing that $\beta > 1$, confirm our conclusion that profits are minimised when the first-order conditions are satisfied by showing that marginal cost is decreasing at the solution.
5. Sketch your results.

4.1 Cost function

CES production function in Example 3.5 : $y = (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}}$, $\beta < 1$ and $0 \neq \rho < 1$
Cost function: $c(w, y) \equiv \min_{x \in \mathbb{R}^n_+} w \cdot x, \ \ s.t. \ \ f(x) \geq y$.

$$c(w, y) = \min_{x \in \mathbb{R}^n_+} w_1 x_1 + w_2 x_2, \ \ s.t. \ \ (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}} \geq y$$

No corner solution.

- Obviously x_1, x_2 can't both be 0 to produce $y > 0$.
- $\frac{\partial f(x)}{\partial x_i} = \beta(x_1^\rho + x_2^\rho)^{(\frac{\beta}{\rho})-1} x_i^{\rho-1}$. If $\rho \in (0,1), \beta > 0$, $\lim_{x_i \to 0} \frac{\partial f(x)}{\partial x_i} = +\infty$.
  - If $\rho \in (0,1), \beta < 0$, the production function doesn't make sense since $\lim_{x \to (0,0)} f(x) = +\infty$
  - If $\rho < 0$, the production function is not defined at $x_i = 0$
- $f(x) = y$ is binding: $f(x)$ is increasing in x, to reduce cost, we shouldn't produce more than required (y).

$$L = w_1 x_1 + w_2 x_2 + \lambda[y - (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}}]$$

FOC:

$$\begin{cases} \frac{\partial L}{\partial x_1} = w_1 - \lambda\beta(x_1^\rho + x_2^\rho)^{(\frac{\beta}{\rho})-1} x_1^{\rho-1} = 0 \\ \frac{\partial L}{\partial x_2} = w_2 - \lambda\beta(x_1^\rho + x_2^\rho)^{(\frac{\beta}{\rho})-1} x_2^{\rho-1} = 0 \\ y - (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}} = 0 \end{cases}$$

Simplify:

$$\begin{cases} w_1 = \lambda\beta(x_1^\rho + x_2^\rho)^{(\frac{\beta}{\rho})-1} x_1^{\rho-1} \\ w_2 = \lambda\beta(x_1^\rho + x_2^\rho)^{(\frac{\beta}{\rho})-1} x_2^{\rho-1} \\ (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}} = y \end{cases} \tag{8}$$

Taking the ratio between the first two gives:

$$\frac{w_1}{w_2} = (\frac{x_1}{x_2})^{\rho-1} \Rightarrow x_1 = (\frac{w_1}{w_2})^{\frac{1}{\rho-1}} x_2$$

Substituting in the third gives:

$$
\begin{aligned}
\{[(\frac{w_1}{w_2})^{\frac{1}{\rho-1}} x_2]^{\rho} + x_2^{\rho}\}^{\frac{\beta}{\rho}} &= y \\
[(\frac{w_1}{w_2})^{\frac{\rho}{\rho-1}} x_2^{\rho} + x_2^{\rho}]^{\frac{\beta}{\rho}} &= y \\
[(\frac{w_1}{w_2})^{\frac{\rho}{\rho-1}} + 1]^{\frac{\beta}{\rho}} x_2^{\beta} &= y \\
x_2 &= (\frac{y}{[(\frac{w_1}{w_2})^{\frac{\rho}{\rho-1}} + 1]^{\frac{\beta}{\rho}}})^{\frac{1}{\beta}} \\
&= y^{\frac{1}{\beta}} [\frac{1}{(\frac{w_1}{w_2})^{\frac{\rho}{\rho-1}} + 1}]^{\frac{1}{\rho}} \\
&= y^{\frac{1}{\beta}} [\frac{1}{(\frac{w_1}{w_2})^{\frac{\rho}{\rho-1}} + 1}]^{\frac{1}{\rho}} \\
&= y^{\frac{1}{\beta}} (\frac{w_2^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}})^{\frac{1}{\rho}} \\
\Rightarrow x_1 &= (\frac{w_1}{w_2})^{\frac{1}{\rho-1}} x_2 \\
&= y^{\frac{1}{\beta}} (\frac{w_1^{\frac{\rho}{\rho-1}}}{w_2^{\frac{\rho}{\rho-1}}} \frac{w_2^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}})^{\frac{1}{\rho}} \\
&= y^{\frac{1}{\beta}} (\frac{w_1^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}})^{\frac{1}{\rho}}
\end{aligned}
$$

Cost function:

$$
\begin{aligned}
c(w,y) = w_1 x_1 + w_2 x_2 &= w_1 y^{\frac{1}{\beta}} \Big(\frac{w_1^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}}\Big)^{\frac{1}{\rho}} + w_2 y^{\frac{1}{\beta}} \Big(\frac{w_2^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}}\Big)^{\frac{1}{\rho}} \\
&= w_1 y^{\frac{1}{\beta}} \frac{w_1^{\frac{1}{\rho-1}}}{(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{\frac{1}{\rho}}} + w_2 y^{\frac{1}{\beta}} \frac{w_2^{\frac{1}{\rho-1}}}{(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{\frac{1}{\rho}}} \\
&= y^{\frac{1}{\beta}} \Big[\frac{w_1^{(\frac{1}{\rho-1}+1)}}{(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{\frac{1}{\rho}}} + \frac{w_2^{(\frac{1}{\rho-1}+1)}}{(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{\frac{1}{\rho}}}\Big] \\
&= y^{\frac{1}{\beta}} \frac{w_1^r + w_1^r}{(w_1^r + w_2^r)^{\frac{1}{\rho}}} \\
&= y^{\frac{1}{\beta}} (w_1^r + w_2^r)^{1-\frac{1}{\rho}} \\
&= y^{\frac{1}{\beta}} (w_1^r + w_2^r)^{\frac{1}{r}}
\end{aligned}
$$

Where $r = \frac{\rho}{\rho-1}$

4.2 Solve $\max_y py - c(w,y)$

$$
py - c(w,y) = py - y^{\frac{1}{\beta}} (w_1^r + w_2^r)^{\frac{1}{r}}
$$

FOC:

$$
\frac{d(py - y^{\frac{1}{\beta}} (w_1^r + w_2^r)^{\frac{1}{r}})}{dy} = p - \frac{1}{\beta} y^{\frac{1}{\beta}-1} (w_1^r + w_2^r)^{\frac{1}{r}} = 0
$$

$$
\therefore y^{\frac{1-\beta}{\beta}} = p\beta (w_1^r + w_2^r)^{-\frac{1}{r}} \Rightarrow y = (p\beta)^{\frac{\beta}{1-\beta}} (w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \tag{9}
$$

4.3 Check $\pi(p,w) = py(p,w) - c(w, y(p,w))$

1.Output function
Compare the output function 9 with function 3, the results are the same.
1.Profit function

$$
\begin{aligned}
py(p,w) - c(w,y(p,w)) &= p[(p\beta)^{\frac{\beta}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}}] \\
&\quad - [(p\beta)^{\frac{\beta}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}}]^{\frac{1}{\beta}}(w_1^r + w_2^r)^{\frac{1}{r}} \\
&= p^{\frac{1}{1-\beta}}\beta^{\frac{\beta}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \\
&\quad - (p\beta)^{\frac{1}{1-\beta}}(w_1^r + w_2^r)^{\frac{1}{r(\beta-1)}}(w_1^r + w_2^r)^{\frac{1}{r}} \\
&= p^{\frac{1}{1-\beta}}\beta^{\frac{\beta}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} - (p\beta)^{\frac{1}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \qquad (10) \\
&= [p^{\frac{1}{1-\beta}}\beta^{\frac{\beta}{1-\beta}} - (p\beta)^{\frac{1}{1-\beta}}](w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \\
&= [p^{\frac{1}{1-\beta}}\beta^{\frac{\beta}{1-\beta}} - p^{\frac{1}{1-\beta}}\beta^{\frac{1}{1-\beta}}](w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \\
&= [\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}}]p^{\frac{1}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \\
&= [1-\beta^1]\beta^{\frac{\beta}{1-\beta}}p^{\frac{1}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \\
&= p^{\frac{1}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}}\beta^{\frac{\beta}{1-\beta}}(1-\beta^1)
\end{aligned}
$$

Compare the result 10 with the profit function 5 obtained from profit maximization problem.

Profit maximization $\iff$ Cost minimization

4.4 Marginal Cost and output

We already know the cost function given output y:

$$c(w,y) = y^{\frac{1}{\beta}}(w_1^r + w_2^r)^{\frac{1}{r}}$$

Forget that we already know **Profit maximization $\iff$ Cost minimization** for now.
To maximize our profit, we want instead the difference between py and $c(w,y)$(the least cost for every given y) to be as big as possible,i.e.:

$$\max_y py - c(w,y)$$

As a price receiver (competitive firm), how should we change y to achieve this?
FOC:

$$\frac{d(py - c(w,y))}{dy} = p - \frac{dc(w,y)}{dy} = 0$$

SOC:

$$\frac{d^2(py - c(w,y))}{dy^2} = -\frac{d^2 c(w,y)}{dy^2} \leq 0 \Rightarrow \frac{d^2 c(w,y)}{dy^2} \geq 0$$

Figure 3 is a very nice plot showing why FOC is not enough for "maximum poit":

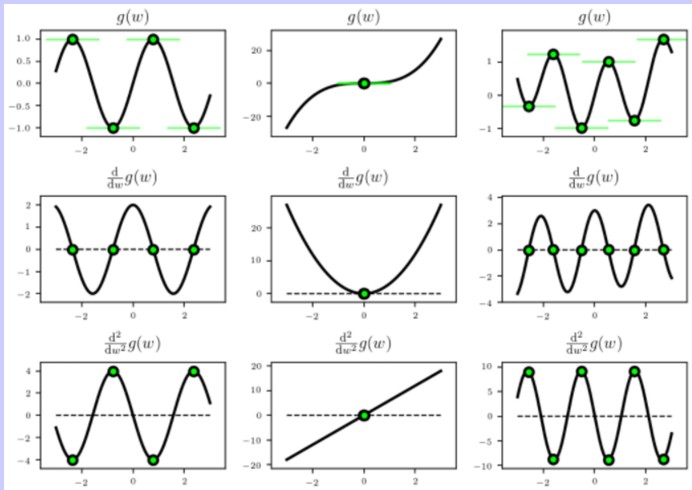


Figure 3: Second order condition

(The Figure is from Intuiting the condition by example)

By FOC, we choose the level y^* of output such that

$$\frac{dc(w,y)}{dy} = p$$

(Marginal cost = price). And,
By SOC, we also require

$$\frac{d^2 c(w,y)}{dy^2} \geq 0$$

(Marginal cost increasing in scale)

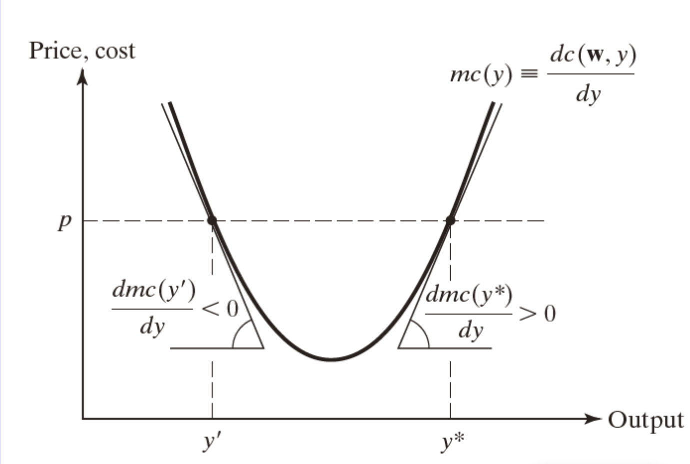


Figure 4: Marginal cost and price

Given $c(w, y) = y^{\frac{1}{\beta}}(w_1^r + w_2^r)^{\frac{1}{r}}$,

$$MC = \frac{dc(w, y)}{dy} = \frac{1}{\beta} y^{\frac{1}{\beta}-1}(w_1^r + w_2^r)^{\frac{1}{r}}$$

$$\frac{dMC}{dy} = \frac{1}{\beta}(\frac{1}{\beta} - 1) y^{\frac{1}{\beta}-2}(w_1^r + w_2^r)^{\frac{1}{r}}$$

If $\beta > 1$, $\frac{dMC}{dy} < 0$. The solution is therefore minimized profit, instead of maximized profit.

Intuitively, $MC = \frac{1}{\beta} y^{\frac{1}{\beta}-1}(w_1^r + w_2^r)^{\frac{1}{r}}$ is decreasing in y when $\beta > 1$, the more you produce, the less cost you need to pay for 1 more unit output. You will therefore continue to produce $+\infty$.

4.5 Sketch

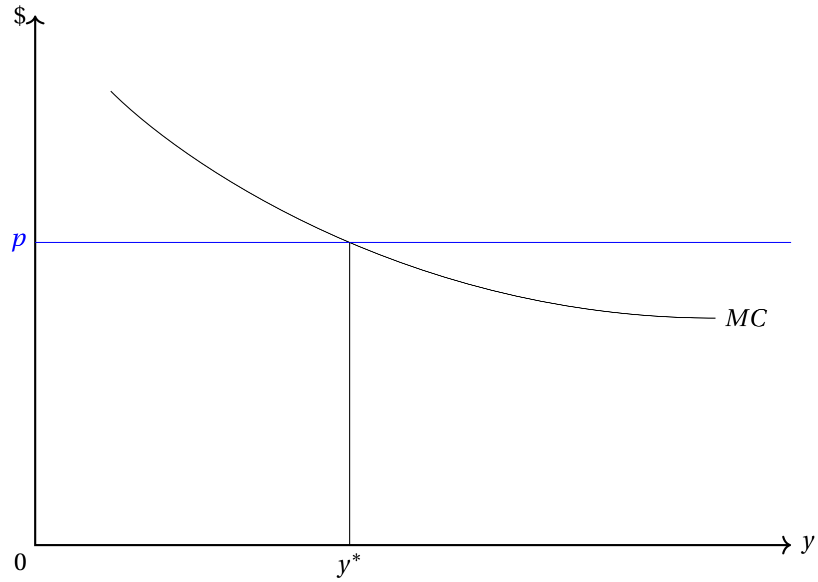


Figure 5: Decreasing MC and p

If you stop at y^*, you lose the most!