Seminar 5. Production Theory

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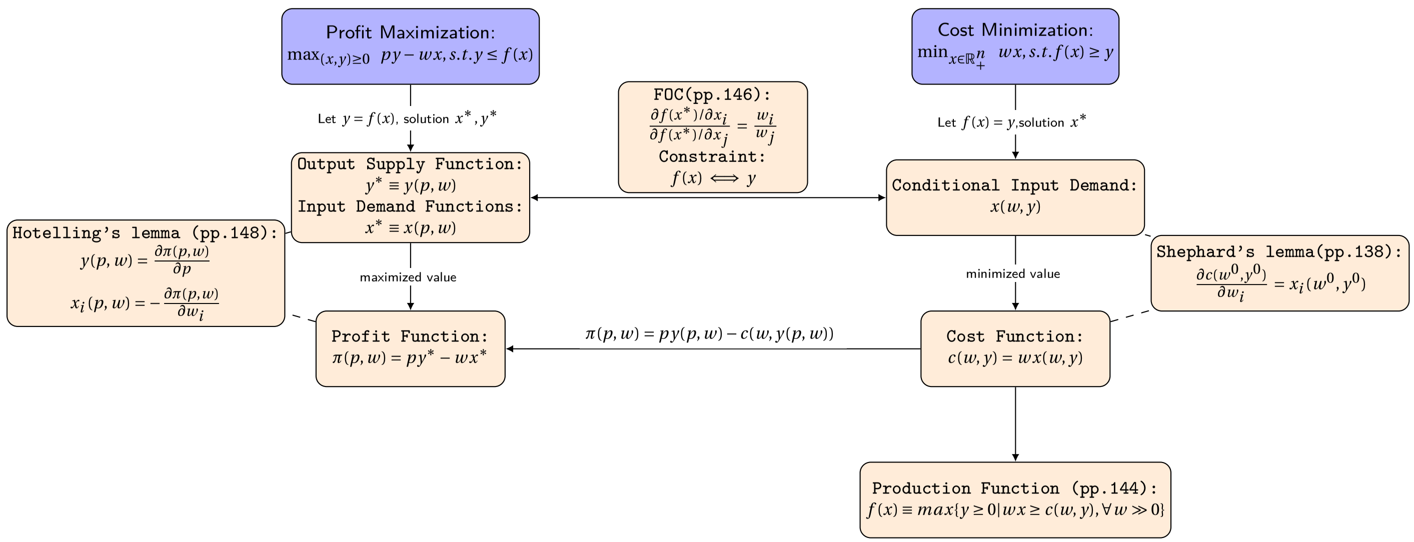
Production Duality



1 Jehle & Reny 3.35

Calculate the **cost function** and the **conditional input demands** for the linear production function, $y = \Sigma_{i=1}^{n} \alpha_i x_i$.

Production Function(Jehle & Reny pp.127)
We use a function $y = f(x)$ to denote y units of a certain commodity is produced using input x, where $x \in \mathbb{R}^n_+$, $y \in \mathbb{R}^1_+$

ASSUMPTION 3.1 Properties of the Production Function (Jehle & Reny pp.127)
The production function, $f : \mathbb{R}^n_+ \to \mathbb{R}_+$, is continuous, strictly increasing, and strictly quasiconcave on $\mathbb{R}^n_+$, and $f(0) = 0$.

DEFINITION 3.5 The Cost Function (Jehle & Reny pp.136)
The cost function, defined for all input prices $w \gg 0$ and all output levels $y \in f(\mathbb{R}^n_+)$ is the minimum-value function,

$$c(w, y) \equiv \min_{x \in \mathbb{R}^n_+} w \cdot x, \quad s.t. \ \ f(x) \geq y.$$

The solution $x(w, y)$ is referred to as the firms **conditional input demand**, because it is conditional on the level of output y.

- **Conditional input demand** is similar to Hicksian demands for consumers.

Here the linear production function $y = \Sigma_{i=1}^{n} \alpha_i x_i$ is very similar to the "**perfect substitution**" preference in Seminar 4.

- The product can be produced by any input x_i, the only difference is that for each unit of input, different x_i produces different amount α_i of the output.

- The Marginal Rate of Technical Substitution of input x_j for input x_i is $MRTS_{ij} = \frac{\partial f(x)/\partial x_i}{\partial f(x)/\partial x_j} = \frac{\alpha_i}{\alpha_j}$.

An example: an apple jam factory has 2 types of input, "single apple (x_1)" and "2-apple pack (x_2)".

- With a "single apple", the factory can produce a bottle of apple jam;

- with a "2-apple pack", 2 bottles.

- The production function is $y = 1 \cdot x_1 + 2 \cdot x_2$

How will the factory choose? Similarly to consumers' preference substitution preference, the factory will spend all money on the "cheapest per unit" input. Denote the price for x_1 and x_2 as w_1, w_2

- If $\frac{w_1}{1} = \frac{w_2}{2}$, the factory doesn't care which to use;
- If $\frac{w_1}{1} < \frac{w_2}{2}$, single apple is cheaper;
- If $\frac{w_1}{1} > \frac{w_2}{2}$, 2-apple pack is cheaper.

Denote the price for input x_i as w_i. Define $\omega = min\{\frac{w_1}{\alpha_1}, \frac{w_2}{\alpha_2}, \dots, \frac{w_n}{\alpha_n}\}$

If ω is the price of only one input x_j, the firm will only use input x_j to minimize its cost.

- Thus $y = \alpha_j x_j$ can minimize the cost, and $x_j = \frac{y}{\alpha_j}$ is the conditional input demands.
- The cost function $c(w, y) = w_j \frac{y}{\alpha_j} = \omega y$.

If ω is the price of several inputs $x_m, m = 1, 2, \dots, M$, the firm can freely combine x_m to minimize its cost, as long as $\Sigma_{m=1}^{M} \alpha_m x_m = y$.

For cost funtion, let's assume $\frac{w_1}{\alpha_1} = \frac{w_2}{\alpha_2} = \dots = \frac{w_M}{\alpha_M} = \omega$, then ω is the price for 1 single apple, for example. Again, to produce y bottles of jam, you need the same number of single apples. The cost function is thus ωy

2 Jehle & Reny 3.46

- Verify Theorem 3.7 for the profit function obtained in Example 3.5.
- Verify Theorem 3.8 for the associated output supply and input demand functions.

2.1 Verify Theorem 3.7

DEFINITION 3.7 The Profit Function (Jehle & Reny pp.148)

$$\pi(p, w) \equiv \max_{(x,y) \geq 0} py - wx, \ \ s.t. y \leq f(x)$$

Note, $y \leq f(x)$ means "you can only decide to produce what's possible to be produced":

- Assume you want to set an optimal output y, forget input x for now;
- You can "waste", i.e., output $y < f(x)$ is possible. Input is not efficiently used for technology $f(x)$;
- You can't produce more than what your technology $f(x)$ allows, i.e. $y \not> f(x)$.

It's not easy to solve the maximization problem directly because there are 2

variable, y and x (again, y is **NOT** necessarily to be $f(x)$, but it can't exceed $f(x)$).
Since to waste will definitely reduce profit (you have some cost but don't produce anything), you will always avoid wasting by making fully use of your technology, i.e. $y = f(x)$. Then the profit maximization problem is transformed into:

$$\pi(p, w) = \max_{x \geq 0} pf(x) - wx$$

No constraint anymore! The function has only one variable, input x. FOC will solve the problem.
In **Example 3.5** (Jehle & Reny pp.151), the CES production function is:

$$y = (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}}$$

Where $\beta < 1$ and $0 \neq \rho < 1$. To obtain the profit function, we need to solve the maximization problem:

$$\max_{(x,y)\geq 0} py - wx, \quad s.t. \quad y \leq (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}}$$

Again, we don't waste, i.e. $y = (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}}$, the problem above is then:

$$\max_{x \geq 0} p(x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}} - w_1 x_1 - w_2 x_2$$

FOC are given in the textbook pp. 151.
The y^* solved is the **output supply function**:

$$y^* = (p\beta)^{-\frac{\beta}{\beta-1}} (w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \tag{1}$$

(You'll compare the output function with equation 6 derived from cost minimization.)
The x^* solved is the **input demand function**:

$$x_i^* = w_i^{\frac{1}{\rho-1}} (p\beta)^{\frac{-1}{\beta-1}} (w_1^r + w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}} \tag{2}$$

The profit function is thus:

$$\pi = py^* - wx^* = p^{-\frac{1}{\beta-1}} (w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \beta^{-\frac{\beta}{\beta-1}} (1-\beta) \tag{3}$$

(You'll compare the profit function with equation 7 derived from cost minimization.)
THEOREM 3.7 Properties of the Profit Function (Jehle & Reny pp.148)
If f satisfies Assumption 3.1, then for $p \geq 0$ and $w \geq 0$, the profit function $\pi(p, w)$, where well-defined, is continuous and

1. Increasing in p,
2. Decreasing in w,
3. Homogeneous of degree one in (p,w),
4. Convex in (p,w),
5. Differentiable in (p,w).
6. Moreover, under the additional assumption that f is strictly concave (Hotelling's lemma),

$$y(p,w)=\frac{\partial\pi(p,w)}{\partial p},\ and\ \ x_i(p,w)=-\frac{\partial\pi(p,w)}{\partial w_i}.\quad i=1,2,\dots,n.$$

1.Increasing in p

For $\pi(p,w)=py^*-wx^*=p^{-\frac{1}{\beta-1}}(w_1^r+w_2^r)^{\frac{\beta}{r(\beta-1)}}\beta^{-\frac{\beta}{\beta-1}}(1-\beta)$,

$$\begin{aligned}\frac{\partial\pi(p,w)}{\partial p}&=(-\frac{1}{\beta-1})p^{-\frac{1}{\beta-1}-1}(w_1^r+w_2^r)^{\frac{\beta}{r(\beta-1)}}\beta^{-\frac{\beta}{\beta-1}}(1-\beta)\\&=p^{\frac{\beta}{1-\beta}}(w_1^r+w_2^r)^{\frac{\beta}{r(\beta-1)}}\beta^{-\frac{\beta}{\beta-1}}\\&=(p\beta)^{\frac{\beta}{1-\beta}}(w_1^r+w_2^r)^{\frac{\beta}{r(\beta-1)}}\end{aligned}$$

When $0<\beta<1$, $\frac{\partial\pi(p,w)}{\partial p}\geq 0$

2.Decreasing in w

$$\begin{aligned}\frac{\partial\pi(p,w)}{\partial w_i}&=p^{-\frac{1}{\beta-1}}[\frac{\beta}{r(\beta-1)}](w_1^r+w_2^r)^{\frac{\beta}{r(\beta-1)}-1}rw_i^{r-1}\beta^{-\frac{\beta}{\beta-1}}(1-\beta)\\&=p^{-\frac{1}{\beta-1}}[-\beta](w_1^r+w_2^r)^{\frac{\beta}{r(\beta-1)}-1}w_i^{r-1}\beta^{-\frac{\beta}{\beta-1}}\\&=p^{-\frac{1}{\beta-1}}[-1](w_1^r+w_2^r)^{\frac{\beta}{\frac{\rho}{\rho-1}(\beta-1)}-1}w_i^{\frac{\rho}{\rho-1}-1}\beta^{1-\frac{\beta}{\beta-1}}\\&=-p^{-\frac{1}{\beta-1}}(w_1^r+w_2^r)^{\frac{\beta}{\frac{\rho}{\rho-1}(\beta-1)}-1}w_i^{\frac{1}{\rho-1}}\beta^{-\frac{1}{\beta-1}}\\&=-(p\beta)^{-\frac{1}{\beta-1}}(w_1^r+w_2^r)^{\frac{(\rho-1)\beta}{\rho(\beta-1)}-1}w_i^{\frac{1}{\rho-1}}\\&=-(p\beta)^{-\frac{1}{\beta-1}}(w_1^r+w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}}w_i^{\frac{1}{\rho-1}}\end{aligned}$$

$i=1,2.$
When $0<\beta<1$, $\frac{\partial\pi(p,w)}{\partial w_i}\leq 0$

3.Homogeneous of degree one in (p,w)

$$
\begin{aligned}
\pi(tp, tw) &= (tp)^{-\frac{1}{\beta-1}} [(tw_1)^r + (tw_2)^r]^{\frac{\beta}{r(\beta-1)}} \beta^{-\frac{\beta}{\beta-1}} (1-\beta) \\
&= t^{-\frac{1}{\beta-1}} p^{-\frac{1}{\beta-1}} t^{\frac{\beta}{(\beta-1)}} [(w_1)^r + (w_2)^r]^{\frac{\beta}{r(\beta-1)}} \beta^{-\frac{\beta}{\beta-1}} (1-\beta) \\
&= t^{-\frac{1}{\beta-1}+\frac{\beta}{(\beta-1)}} p^{-\frac{1}{\beta-1}} [(w_1)^r + (w_2)^r]^{\frac{\beta}{r(\beta-1)}} \beta^{-\frac{\beta}{\beta-1}} (1-\beta) \\
&= tp^{-\frac{1}{\beta-1}} [(w_1)^r + (w_2)^r]^{\frac{\beta}{r(\beta-1)}} \beta^{-\frac{\beta}{\beta-1}} (1-\beta) \\
&= t^1 \pi(p, w)
\end{aligned}
$$

4.Convex in (p, w) Higher-dimension proof not required in exam
Recall one-dimension condition:
Convex functions (Jehle & Reny pp.542):
$f : D \to \mathbb{R}$ is a convex function if for all $x^1, x^2 \in D$,

$$f(x^t) \le tf(x^1) + (1-t)f(x^2), \ \forall t \in [0,1].$$

Where $x^t \equiv tx^1 + (1-t)x^2, t \in [0,1]$.

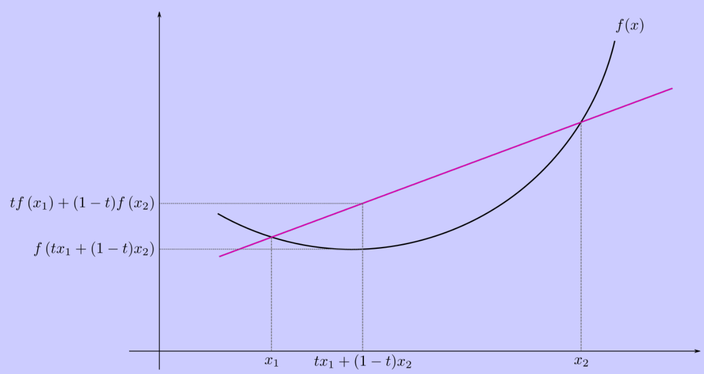


Figure 1: Convex function

5.Differentiable in (p, w) Higher-dimension proof not required in exam.
Recall one-dimension condition: derivative exists at $x^0 \Rightarrow$ Differentiable at x^0.

6.Hotelling's lemma
We already calculated the derivatives. Compare them with equation 1 and 2.

2.2 Verify Theorem 3.8

THEOREM 3.8 Properties of Output Supply and Input Demand Functions (Jehle & Reny pp.149)
Suppose that f is a strictly concave production function satisfying Assumption 3.1 and that its associated profit function, $\pi(p, y)$, is twice continuously differ-

entiable. Then, for all $p > 0$ and $w \gg 0$ where it is well defined:

1. Homogeneity of degree zero:

$$y(tp, tw) = y(p, w), \forall t > 0,$$

$$x_i(tp, tw) = x_i(p, w), \forall t > 0 \; and \; i = 1, \ldots, n.$$

2. Own-price effects:

$$\frac{\partial y(p, w)}{\partial p} \geq 0,$$

$$\frac{\partial x_i(p, w)}{\partial w_i} \leq 0, \; \forall i = 1, \ldots, n.$$

3. The substitution matrix is symmetric and positive semidefinite.

$$\begin{pmatrix} \frac{\partial y(p,w)}{\partial p} & \frac{\partial y(p,w)}{\partial w_1} & \cdots & \frac{\partial y(p,w)}{\partial w_n} \\ \frac{\partial x_1(p,w)}{\partial p} & \frac{\partial x_1(p,w)}{\partial w_1} & \cdots & \frac{\partial x_1(p,w)}{\partial w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n(p,w)}{\partial p} & \frac{\partial x_n(p,w)}{\partial w_1} & \cdots & \frac{\partial x_n(p,w)}{\partial w_n} \end{pmatrix} \tag{4}$$

Copy from Equation 1 and Equation 2,
Output supply function:

$$y(p, w) = (p\beta)^{-\frac{\beta}{\beta-1}} (w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}}$$

Input demand function:

$$x_i(p, w) = w_i^{\frac{1}{\rho-1}} (p\beta)^{\frac{-1}{\beta-1}} (w_1^r + w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}}$$

1.Homogeneity of degree zero

$$\begin{aligned} y(tp, tw) &= (tp\beta)^{-\frac{\beta}{\beta-1}} [(tw_1)^r + (tw_2)^r]^{\frac{\beta}{r(\beta-1)}} \\ &= t^{-\frac{\beta}{\beta-1}} (p\beta)^{-\frac{\beta}{\beta-1}} t^{\frac{\beta}{(\beta-1)}} (w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \\ &= t^0 (p\beta)^{-\frac{\beta}{\beta-1}} (w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \\ &= t^0 y(p, w) \end{aligned}$$

$$
\begin{aligned}
x_i(tp,tw) &= (tw_i)^{\frac{1}{\rho-1}}(tp\beta)^{\frac{-1}{\beta-1}}[(tw_1)^r+(tw_2)^r]^{\frac{\rho-\beta}{\rho(\beta-1)}} \\
&= t^{\frac{1}{\rho-1}} w_i^{\frac{1}{\rho-1}} t^{\frac{-1}{\beta-1}}(p\beta)^{\frac{-1}{\beta-1}} t^{r\frac{\rho-\beta}{\rho(\beta-1)}}(w_1^r+w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}} \\
&= t^{\frac{1}{\rho-1}+\frac{-1}{\beta-1}+\frac{\rho}{\rho-1}\frac{\rho-\beta}{\rho(\beta-1)}} w_i^{\frac{1}{\rho-1}}(p\beta)^{\frac{-1}{\beta-1}}(w_1^r+w_2^r)^{\frac{\rho-\beta}{\rho(\beta-1)}} \\
&= t^0 x_i(p,w)
\end{aligned}
$$

$i=1,2.$

2.Own-price effects

$$
\frac{\partial y(p,w)}{\partial p} = (-\frac{\beta}{\beta-1})p^{-\frac{\beta}{\beta-1}-1}(\beta)^{-\frac{\beta}{\beta-1}}(w_1^r+w_2^r)^{\frac{\beta}{r(\beta-1)}}
$$

When $\beta \in (0,1)$, $\frac{\partial y(p,w)}{\partial p} \geq 0$.

$$
\frac{\partial x_i(p,w)}{\partial w_i} =
$$

3.Substitution matrix

3 Jehle & Reny 3.49

1. Derive the **cost function** for the production function in Example 3.5 .
2. Solve $\max_y py - c(w,y)$
3. Compare its solution, $y(p,w)$, to the solution in (E.5). Check that $\pi(p,w) = py(p,w) - c(w,y(p,w))$.
4. Supposing that $\beta > 1$, confirm our conclusion that profits are minimised when the first-order conditions are satisfied by showing that marginal cost is decreasing at the solution.
5. Sketch your results.

3.1 Cost function

CES production function in Example 3.5 : $y = (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}}$, $\beta < 1$ and $0 \neq \rho < 1$
Cost function: $c(w,y) \equiv \min_{x \in \mathbb{R}_+^n} w \cdot x,\ \ s.t.\ \ f(x) \geq y.$

$$
c(w,y) = \min_{x \in \mathbb{R}_+^n} w_1 x_1 + w_2 x_2,\ \ s.t.\ \ (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}} \geq y
$$

No corner solution.

- Obviously x_1, x_2 can't both be 0 to produce $y > 0$.
- $\frac{\partial f(x)}{\partial x_i} = \beta(x_1^\rho + x_2^\rho)^{(\frac{\beta}{\rho})-1} x_i^{\rho-1}$. If $\rho \in (0,1), \beta > 0$, $\lim_{x_i \to 0} \frac{\partial f(x)}{\partial x_i} = +\infty$.
  - If $\rho \in (0,1), \beta < 0$, the production function doesn't make sense since $\lim_{x \to (0,0)} f(x) = +\infty$
  - If $\rho < 0$, the production function is not defined at $x_i = 0$
- $f(x) = y$ is binding: $f(x)$ is increasing in x, to reduce cost, we shouldn't produce more than required (y).

$$L = w_1 x_1 + w_2 x_2 + \lambda[y - (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}}]$$

FOC:

$$\begin{cases} \frac{\partial L}{\partial x_1} = w_1 - \lambda\beta(x_1^\rho + x_2^\rho)^{(\frac{\beta}{\rho})-1} x_1^{\rho-1} = 0 \\ \frac{\partial L}{\partial x_2} = w_2 - \lambda\beta(x_1^\rho + x_2^\rho)^{(\frac{\beta}{\rho})-1} x_2^{\rho-1} = 0 \\ y - (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}} = 0 \end{cases}$$

Simplify:

$$\begin{cases} w_1 = \lambda\beta(x_1^\rho + x_2^\rho)^{(\frac{\beta}{\rho})-1} x_1^{\rho-1} \\ w_2 = \lambda\beta(x_1^\rho + x_2^\rho)^{(\frac{\beta}{\rho})-1} x_2^{\rho-1} \\ (x_1^\rho + x_2^\rho)^{\frac{\beta}{\rho}} = y \end{cases} \tag{5}$$

Taking the ratio between the first two gives:

$$\frac{w_1}{w_2} = (\frac{x_1}{x_2})^{\rho-1} \Rightarrow x_1 = (\frac{w_1}{w_2})^{\frac{1}{\rho-1}} x_2$$

Substituting in the third gives:

$$
\begin{aligned}
\{[(\frac{w_1}{w_2})^{\frac{1}{\rho-1}} x_2]^\rho + x_2^\rho\}^{\frac{\beta}{\rho}} &= y \\
[(\frac{w_1}{w_2})^{\frac{\rho}{\rho-1}} x_2^\rho + x_2^\rho]^{\frac{\beta}{\rho}} &= y \\
[(\frac{w_1}{w_2})^{\frac{\rho}{\rho-1}} + 1]^{\frac{\beta}{\rho}} x_2^\beta &= y \\
x_2 &= (\frac{y}{[(\frac{w_1}{w_2})^{\frac{\rho}{\rho-1}} + 1]^{\frac{\beta}{\rho}}})^{\frac{1}{\beta}} \\
&= y^{\frac{1}{\beta}} [\frac{1}{(\frac{w_1}{w_2})^{\frac{\rho}{\rho-1}} + 1}]^{\frac{1}{\rho}} \\
&= y^{\frac{1}{\beta}} [\frac{1}{(\frac{w_1}{w_2})^{\frac{\rho}{\rho-1}} + 1}]^{\frac{1}{\rho}} \\
&= y^{\frac{1}{\beta}} (\frac{w_2^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}})^{\frac{1}{\rho}} \\
\Rightarrow x_1 &= (\frac{w_1}{w_2})^{\frac{1}{\rho-1}} x_2 \\
&= y^{\frac{1}{\beta}} (\frac{w_1^{\frac{\rho}{\rho-1}}}{w_2^{\frac{\rho}{\rho-1}}} \frac{w_2^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}})^{\frac{1}{\rho}} \\
&= y^{\frac{1}{\beta}} (\frac{w_1^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}})^{\frac{1}{\rho}}
\end{aligned}
$$

Cost function:

$$
\begin{aligned}
c(w,y) = w_1x_1 + w_2x_2 &= w_1 y^{\frac{1}{\beta}} \left(\frac{w_1^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}}\right)^{\frac{1}{\rho}} + w_2 y^{\frac{1}{\beta}} \left(\frac{w_2^{\frac{\rho}{\rho-1}}}{w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}}\right)^{\frac{1}{\rho}} \\
&= w_1 y^{\frac{1}{\beta}} \frac{w_1^{\frac{1}{\rho-1}}}{(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{\frac{1}{\rho}}} + w_2 y^{\frac{1}{\beta}} \frac{w_2^{\frac{1}{\rho-1}}}{(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{\frac{1}{\rho}}} \\
&= y^{\frac{1}{\beta}} \left[\frac{w_1^{(\frac{1}{\rho-1}+1)}}{(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{\frac{1}{\rho}}} + \frac{w_2^{(\frac{1}{\rho-1}+1)}}{(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{\frac{1}{\rho}}}\right] \\
&= y^{\frac{1}{\beta}} \frac{w_1^r + w_1^r}{(w_1^r + w_2^r)^{\frac{1}{\rho}}} \\
&= y^{\frac{1}{\beta}} (w_1^r + w_2^r)^{1-\frac{1}{\rho}} \\
&= y^{\frac{1}{\beta}} (w_1^r + w_2^r)^{\frac{1}{r}}
\end{aligned}
$$

Where $r = \frac{\rho}{\rho-1}$

3.2 Solve $\max_y py - c(w,y)$

$$py - c(w,y) = py - y^{\frac{1}{\beta}} (w_1^r + w_2^r)^{\frac{1}{r}}$$

FOC:

$$\frac{d(py - y^{\frac{1}{\beta}} (w_1^r + w_2^r)^{\frac{1}{r}})}{dy} = p - \frac{1}{\beta} y^{\frac{1}{\beta}-1} (w_1^r + w_2^r)^{\frac{1}{r}} = 0$$

$$\therefore y^{\frac{1-\beta}{\beta}} = p\beta(w_1^r + w_2^r)^{-\frac{1}{r}} \Rightarrow y = (p\beta)^{\frac{\beta}{1-\beta}} (w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \tag{6}$$

3.3 Check $\pi(p,w) = py(p,w) - c(w, y(p,w))$

1.output function
Compare the output function 6 with 1, the results are the same.
1.Profit function

$$
\begin{aligned}
py(p,w) - c(w,y(p,w)) &= p[(p\beta)^{\frac{\beta}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}}] \\
&\quad - [(p\beta)^{\frac{\beta}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}}]^{\frac{1}{\beta}}(w_1^r + w_2^r)^{\frac{1}{r}} \\
&= p^{\frac{1}{1-\beta}}\beta^{\frac{\beta}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \\
&\quad - (p\beta)^{\frac{1}{1-\beta}}(w_1^r + w_2^r)^{\frac{1}{r(\beta-1)}}(w_1^r + w_2^r)^{\frac{1}{r}} \\
&= p^{\frac{1}{1-\beta}}\beta^{\frac{\beta}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} - (p\beta)^{\frac{1}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \qquad (7) \\
&= [p^{\frac{1}{1-\beta}}\beta^{\frac{\beta}{1-\beta}} - (p\beta)^{\frac{1}{1-\beta}}](w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \\
&= [p^{\frac{1}{1-\beta}}\beta^{\frac{\beta}{1-\beta}} - p^{\frac{1}{1-\beta}}\beta^{\frac{1}{1-\beta}}](w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \\
&= [\beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}}]p^{\frac{1}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \\
&= [1-\beta^1]\beta^{\frac{\beta}{1-\beta}}p^{\frac{1}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}} \\
&= p^{\frac{1}{1-\beta}}(w_1^r + w_2^r)^{\frac{\beta}{r(\beta-1)}}\beta^{\frac{\beta}{1-\beta}}(1-\beta^1)
\end{aligned}
$$

Compare the result 7 with the profit function 3 obtained from profit maximization problem.

Profit maximization $\iff$ Cost minimization

3.4 Marginal Cost and output

We already know the cost function given output y:

$$c(w,y) = y^{\frac{1}{\beta}}(w_1^r + w_2^r)^{\frac{1}{r}}$$

Forget that we already know **Profit maximization $\iff$ Cost minimization** for now.
To maximize our profit, we want instead the difference between py and $c(w,y)$(the least cost for every given y) to be as big as possible,i.e.:

$$\max_y py - c(w,y)$$

As a price receiver (competitive firm), how should we change y to achieve this?
FOC:

$$\frac{d(py - c(w,y))}{dy} = p - \frac{dc(w,y)}{dy} = 0$$

SOC:

$$\frac{d^2(py - c(w,y))}{dy^2} = -\frac{d^2c(w,y)}{dy^2} \leq 0 \Rightarrow \frac{d^2c(w,y)}{dy^2} \geq 0$$

Figure 2 is a very nice plot showing why FOC is not enough for "maximum poit":

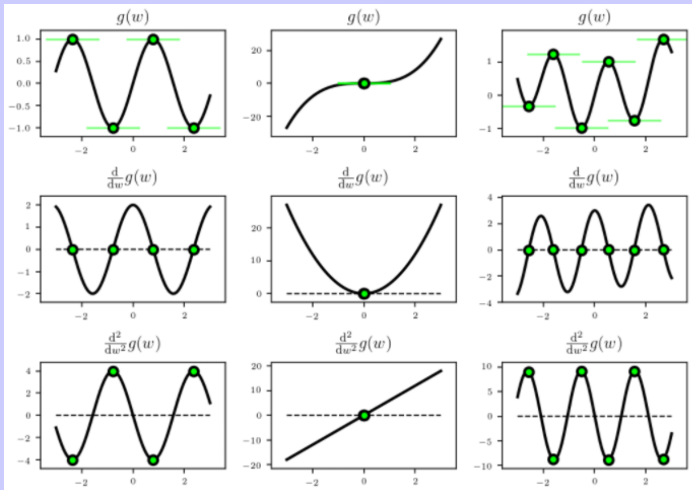


Figure 2: Second order condition

(The Figure is from Intuiting the condition by example)
We choose the level y^* of output such that

$$\frac{dc(w,y)}{dy} = p$$

(Marginal cost = price). And, $\frac{d^2c(w,y)}{dy^2} \geq 0$ (Marginal cost increasing in scale)

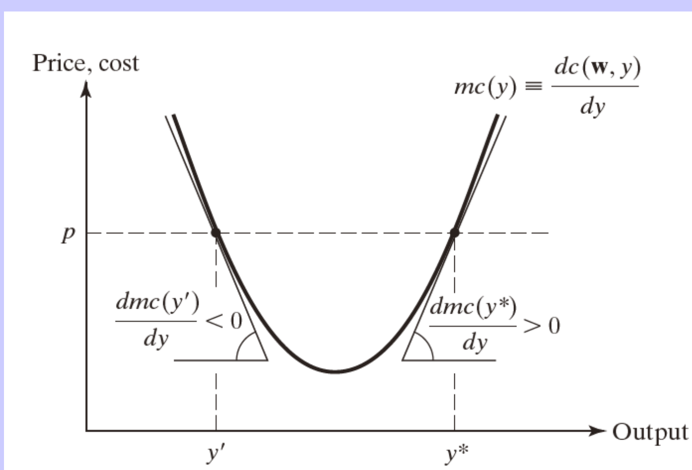


Figure 3: Marginal cost and price

Given $c(w, y) = y^{\frac{1}{\beta}}(w_1^r + w_2^r)^{\frac{1}{r}}$,

$$MC = \frac{dc(w, y)}{dy} = \frac{1}{\beta} y^{\frac{1}{\beta}-1}(w_1^r + w_2^r)^{\frac{1}{r}}$$

$$\frac{dMC}{dy} = \frac{1}{\beta}(\frac{1}{\beta} - 1) y^{\frac{1}{\beta}-2}(w_1^r + w_2^r)^{\frac{1}{r}}$$

If $\beta > 1$, $\frac{dMC}{dy} < 0$. The solution is therefore minimized profit, instead of maximized profit.

Intuitively, $MC = \frac{1}{\beta} y^{\frac{1}{\beta}-1}(w_1^r + w_2^r)^{\frac{1}{r}}$ is decreasing in y when $\beta > 1$, the more you produce, the less cost you need to pay for 1 more unit output. You will therefore continue to produce $+\infty$.

3.5 Sketch

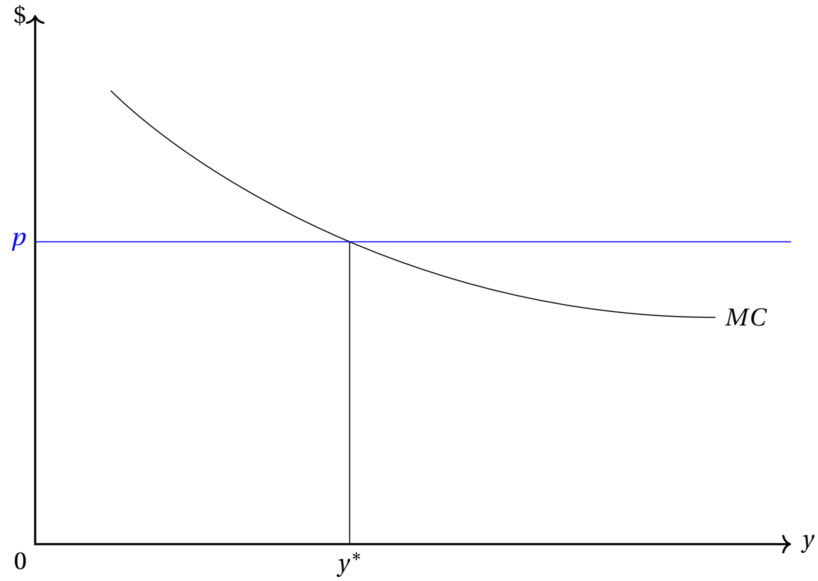


Figure 4: Decreasing MC and p

If you stop at y^*, you lose the most!