Seminar 2 - Expenditure function

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1 Jehle & Reny 1.38 - Properties of the Expenditure Function

Verify that the expenditure function obtained from the CES direct utility function in Example 1.3 (JR. pp.39) satisfies all the properties given in Theorem 1.7 (JR. pp.37).

Expenditure Function (JR. pp.35)

We define the expenditure function as the minimum-value function:

$$e(p, u) \equiv \min_{x \in \mathbb{R}_+^n} p \cdot x$$

Expenditure Function of CES direct utility function (JR. pp.39)

In Example 1.3 (JR. pp.39), we have a so called CES direct utility function:

$$u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}, \ where \ 0 \neq \rho < 1.$$

To derive the Expenditure Function, we need to solve the expenditure minimisation problem given some utility u. i.e.

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2 \ \ s.t. \ \ u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} = u, \ x_1 \leq 0, \ x_2 \leq 0$$

(Read JR. pp.40 to see how to minimize the expenditure with Lagrangian method.)

The solution to the expenditure-minimisation problem is called the consumers vector of **Hicksian demands**:

$$\begin{cases} x_1^h(p_1, p_2, u) = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_1^{(r-1)} \\ x_1^h(p_1, p_2, u) = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_2^{(r-1)} \end{cases} \tag{1}$$

Here $r \equiv \frac{\rho}{\rho-1}$.
Substitute the solution above (Equation 1) into our objective function $p_1 x_1 + p_2 x_2$ to obtain the Expenditure Function:

$$e(p_1, p_2, u) = p_1 x_1^h(p_1, p_2, u) + p_2 x_2^h(p_1, p_2, u) = u(p_1^r + p_2^r)^{\frac{1}{r}}, r \equiv \frac{\rho}{\rho - 1}$$

THEOREM 1.7 Properties of the Expenditure Function (JR. pp.37)

If $u(.)$ is continuous and strictly increasing, then $e(p, u)$ defined in (1.14) is

1. Zero when u takes on the lowest level of utility in $\mathscr{U}$,

2. Continuous on its domain $\mathbb{R}_{++}^n \times \mathscr{U}$,

3. For all $p \gg 0$, strictly increasing in u and unbounded above in u,

4. Increasing in p,

5. Homogeneous of degree 1 in p,

6. Concave in p.

If, in addition, $u(.)$ is strictly quasiconcave, we have

7. Shephards lemma: $e(p, u)$ is differentiable in p at (p^0, u^0) with $p^0 \gg 0$, and

$$\frac{\partial e(p^0, u^0)}{\partial p_i} = x_i^h(p^0, u^0), \quad i = 1, ..., n.$$

Firstly we should know that the CES direct utility function $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$, $where\, 0 \neq \rho < 1$. is continuous and strictly increasing (can be proved similarly as below), which is the prerequisite of Theorem 1.7.

1.1 $e(p, u)$ is zero when u takes on the lowest level of utility in $\mathscr{U}$

"no consumption, no cost"

As usual, we assume non-negative consumption, i.e. $x \in R_+^n$. Since $u(.)$ is strictly increasing, $u_{min} = u(0, 0) = 0$.
When $u = 0$, we have $e(p, 0) = 0 \times (p_1^r + p_2^r)^{\frac{1}{r}} = 0$.

1.2 $e(p, u)$ is continuous on its domain $\mathbb{R}_{++}^n \times \mathscr{U}$,

We want a function to be continuously differentiable because we can then take derivatives (to study its property) freely.
For a ONE dimention function $f(x)$, if the derivative at $x = a$ ($f'(a)$) exists, it's differentiable at $x = a$. If $f(x)$ is differentiable at $x = a$, it's also continuous at $x = a$.
For a higher dimension function, the existance of all partial/directional derivatives is not sufficient for its differentiability. Here are examples from Wikipedia:

Good news for your exam:
"There is no need to prove continuity (or concavity) of higher dimension functions (in your exam). I might ask you to prove continuity (or concavity) wrt one variable (a specific price, for example)." – Paolo

$e(p_1, p_2, u)$ is a 3-dimension function. If you really want to prove it's differentiable/continuous, you can either try to use the definition of differentiable/ continuous functions. You can also read theorem A2.21 (Jehle & Reny pp.602) and theorem 1.9 (point 3,Jehle & Reny pp.520).
For this sub-question, let's assume the prices are given (say, by the market). Therefore $e(p_1, p_2, u) = e(u)$ is one dimesional.
Obviously,

$$\frac{de(u)}{du} = \frac{du(p_1^r + p_2^r)^{\frac{1}{r}}}{du} = (p_1^r + p_2^r)^{\frac{1}{r}}$$

The derivetive exists for any $u \in \mathscr{U}$. $e(u)$ is differentiable and thus continuous.

1.3 For all $p \gg 0$, $e(p, u)$ is strictly increasing in u and unbounded above in u,

"more utility, more cost"

Since $\frac{de(u)}{du} = (p_1^r + p_2^r)^{\frac{1}{r}} > 0, \forall p \gg 0$, $e(p_1, p_2, u)$ is strictly increasing in u.
Given the positive prices, $\lim_{u \to \infty} e(u) = \lim_{u \to \infty} u(p_1^r + p_2^r)^{\frac{1}{r}} = +\infty$. It there is no upper boundary in u.

1.4 $e(p, u)$ is increasing in p,

$$\frac{\partial e(p_1, p_2, u)}{\partial p_1} = \frac{\partial u(p_1^r + p_2^r)^{\frac{1}{r}}}{\partial p_1} = u \cdot \frac{1}{r}(p_1^r + p_2^r)^{\frac{1}{r}-1} \cdot r p_1^{r-1} = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_1^{r-1} > 0$$

$$\frac{\partial e(p_1, p_2, u)}{\partial p_2} = \frac{\partial u(p_1^r + p_2^r)^{\frac{1}{r}}}{\partial p_2} = u \cdot \frac{1}{r}(p_1^r + p_2^r)^{\frac{1}{r}-1} \cdot r p_2^{r-1} = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_2^{r-1} > 0$$

Therefore $e(p, u)$ is increasing in p.

1.5 $e(p, u)$ is homogeneous of degree 1 in p,

"Inflation"
Homogeneous functions (Jehle & Reny Definition A2.2 pp.561):
$f(x)$ is called homogeneous of degree k if $f(tx) \equiv t^k f(x), \ \forall t > 0$

For any $t > 0$,

$$
\begin{aligned}
e(u, tp) &= u[(tp_1)^r + (tp_2)^r]^{\frac{1}{r}} \\
&= u[t^r (p_1^r + p_2^r)]^{\frac{1}{r}} \\
&= tu(p_1^r + p_2^r)^{\frac{1}{r}} \\
&= t^1 e(u, p)
\end{aligned}
$$

Therefore $e(p, u)$ is homogeneous of degree 1 in p.

1.6 $e(p, u)$ **is concave in** p.

- In exam one-dimension function is most common, which means you only need to calculate its second derivative to prove concavity.

- Before you prove it, think about the intuition of the curve (function). Diminishing marginal effects is very common in economics, which also implies concavity.

- Do NOT prove anything if the question doesn't require you. Just answer yes or no if the question simply asks "Is the function concave? Is it increasing? etc."

"Diminishing marginal cost (in p)"
Concave functions (Jehle & Reny pp.534):
$f : D \rightarrow \mathbb{R}$ is a concave function if for all $x^1, x^2 \in D$,

$$
f(x^t) \geq t f(x^1) + (1 \, t) f(x^2), \ \forall t \in [0, 1].
$$

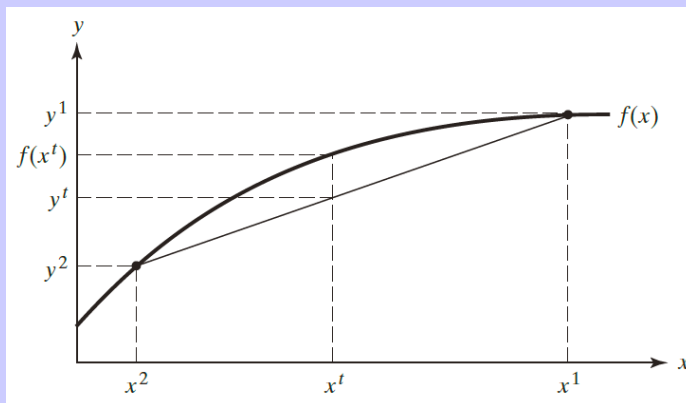Where $x^t \equiv tx^1 + (1 - t)x^2, t \in [0, 1].$

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Figure 1: Concave functions

- Intuition: $e(p, u)$ is increasing **in** p but the marginal expenditure is diminishing.

For this specific function $e(p_1, p_2, u)$, you can follow the very smart proof on Jehle & Reny pp. 38-39 to show that for any given utility $\bar{u}$,

$$e[tp^1 + (1-t)p^2, \bar{u}] \geq te(p^1, \bar{u}) + (1-t)e(p^2, \bar{u})$$

Where $p^1 = \begin{pmatrix} p_1^1 \\ p_2^1 \end{pmatrix}$ and $p^2 = \begin{pmatrix} p_1^2 \\ p_2^2 \end{pmatrix}$ are two price sets. $t \in [0, 1]$

We can also prove it's concavity using derivatives.

For a (twice differentiable) one-dimension function $f(x)$ with diminishing margin (concave towards x-axis), $f''(x) < 0$, i.e.

$$f(x) \; is \; concave \iff f''(x) < 0$$

Read the following creteria if you're interested.

For a higher dimension function $e(p_1, p_2, u)$, it's natural to think about how the derivatives look like.

We call the second derivatives of a n-demension function f "**Hessian matrix**" ($H(x)$, Jehle & Reny pp.557):

$$H(x) = \begin{pmatrix} f_{11}(x) & f_{12}(x) & \cdots & f_{1n}(x) \\ f_{21}(x) & f_{22}(x) & \cdots & f_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(x) & f_{n2}(x) & \cdots & f_{nn}(x) \end{pmatrix} \tag{2}$$

We can use the following facts about $H(x)$ and concavity(More details and examples):

- **Theorem A2.4**(Jehle & Reny pp.559): If function f is twice continuously differentiable on a convex open set S, then f is concave $\iff$ the Hessian matrix $H(x)$ of f is negative semidefinite for all $x \in S$

- **Negative semidefinite**(Jehle & Reny pp.559): a $n \times n$ metrix H is negative semidefinite $\iff$ H's kth order principal minors are nonpositive for k odd and nonnegative for k even.

The Hessian matrix of $e(p_1, p_2, u)$ (in p) is:

$$H(p) = \begin{pmatrix} e_{p_1 p_1} & e_{p_1 p_2} \\ e_{p_2 p_1} & e_{p_2 p_2} \end{pmatrix} \tag{3}$$

$H(p)$ contains two 1st order ($k = 1, n - k = 2 - 1 = 1$) principal minors: $e_{p_1 p_1}$ and $e_{p_2 p_2}$, and one 2nd order ($k = 2, n - k = 2 - 2 = 0$) principal minors:

$$\begin{vmatrix} e_{p_1 p_1} & e_{p_1 p_2} \\ e_{p_2 p_1} & e_{p_2 p_2} \end{vmatrix} = e_{p_1 p_1} e_{p_2 p_2} - e_{p_1 p_2} e_{p_2 p_1} \tag{4}$$

We want to prove:

We already know $\frac{\partial e(p_1, p_2)}{\partial p_1} = u(p_1^r + p_2^r)^{\frac{1}{r} - 1} p_1^{r-1}$, and $\frac{\partial e(p_1, p_2)}{\partial p_2} = u(p_1^r + p_2^r)^{\frac{1}{r} - 1} p_2^{r-1}$

$$
\begin{aligned}
\frac{\partial e(p_1, p_2)}{\partial p_1 \partial p_1} &= \frac{\partial u(p_1^r + p_2^r)^{\frac{1}{r} - 1} p_1^{r-1}}{\partial p_1} \\
&= u(\frac{1}{r} - 1)(p_1^r + p_2^r)^{\frac{1}{r} - 2} r p_1^{r-1} p_1^{r-1} + u(p_1^r + p_2^r)^{\frac{1}{r} - 1}(r-1) p_1^{r-2} \\
&= u(p_1^r + p_2^r)^{\frac{1}{r} - 1} p_1^{r-2}[(\frac{1}{r} - 1)(p_1^r + p_2^r)^{-1} r p_1^r + (r-1)] \\
&= u(p_1^r + p_2^r)^{\frac{1}{r} - 1} p_1^{r-2}(1-r)[\frac{p_1^r}{(p_1^r + p_2^r)} - 1] \\
&= u(p_1^r + p_2^r)^{\frac{1}{r} - 2} p_1^{r-2}(r-1) p_2^r
\end{aligned}
$$

$$\because \begin{cases} u(p_1^r + p_2^r)^{\frac{1}{r} - 2} p_1^{r-2} p_2^r \geq 0 \\ r \equiv \frac{\rho}{\rho - 1} = \frac{1}{1 - \frac{1}{\rho}} < 1, \forall \rho < 1, \rho \neq 0 \Rightarrow r - 1 \leq 0 \end{cases}$$

$\therefore e_{p_1 p_1} \leq 0$

Similarly, $e_{p_2 p_2} = u(p_1^r + p_2^r)^{\frac{1}{r} - 2} p_2^{r-2}(r-1) p_1^r \leq 0$ (note p_1 and p_2 are "symmetric").

$$\frac{\partial e(p_1, p_2)}{\partial p_1 \partial p_2} = \frac{\partial u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_1^{r-1}}{\partial p_2}$$

$$= u(\frac{1}{r} - 1)(p_1^r + p_2^r)^{\frac{1}{r}-2} r p_2^{r-1} p_1^{r-1}$$

$$= u(1-r)(p_1^r + p_2^r)^{\frac{1}{r}-2} p_2^{r-1} p_1^{r-1}$$

According to Young's theorem (Jehle & Reny pp.557), $\frac{\partial e(p_1,p_2)}{\partial p_1 \partial p_2} = \frac{\partial e(p_1,p_2)}{\partial p_2 \partial p_1}$
We have

$$e_{p_1 p_1} e_{p_2 p_2} - e_{p_1 p_2} e_{p_2 p_1} = [u(p_1^r + p_2^r)^{\frac{1}{r}-2} p_1^{r-2}(r-1)p_2^r][u(p_1^r + p_2^r)^{\frac{1}{r}-2} p_2^{r-2}(r-1)p_1^r]$$

$$- [u(1-r)(p_1^r + p_2^r)^{\frac{1}{r}-2} p_2^{r-1} p_1^{r-1}]^2$$

$$= u^2(p_1^r + p_2^r)^{\frac{2}{r}-4}(p_1 p_2)^{2r-2)}(r-1)^2 - u^2(1-r)^2(p_1^r + p_2^r)^{\frac{2}{r}-4}(p_2 p_1)^{2r-2}$$

$$= 0 \geq 0$$

Therefore, $H(p)$ is negative semidefinite $\Rightarrow$ $e(p, u)$ is concave in p.

1.7 If $u(.)$ is also strictly quasiconcave, we have Shephards lemma: $e(p, u)$ is differentiable in p at (p^0, u^0) with $p^0 \gg 0$, and $\frac{\partial e(p^0, u^0)}{\partial p_i} = x_i^h(p^0, u^0)$, $i = 1, ..., n$.

"Compensated demands"
Again, the proof for $e(p, u)$ is differentiable at (p^0, u^0) in p with $p^0 \gg 0$ is nor required by our course

CES utility function $u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$, $0 \neq \rho < 1$. has a similar form as the expenditure function $e(p_1, p_2, u) = u(p_1^r + p_2^r)^{\frac{1}{r}}$, $r \equiv \frac{\rho}{\rho-1}$. You can similarly prove it is concave ($\Rightarrow$ quasiconcave, see also Thorem A1.15 on Jehle & Reny pp. 541).

Shephards lemma states the fact that the Hicksian demands on x_i = the partial derivatives of the expenditure functions on p_i.
Hicksian demands are also called "compensated demands": given utility $\bar{u}$, when price p_i changed a little, how your demand will change to achieve the same utility level $\bar{u}$?

$$\frac{\partial e(p_1, p_2)}{\partial p_1} = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_1^{r-1}$$

$$\frac{\partial e(p_1, p_2)}{\partial p_2} = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_2^{r-1}$$

Compare these derivatives with the Hicksian demand in Equation 1.

2   Jehle & Reny 1.44 - Inferior and Normal goods

In a two-good case, show that if one good is inferior, the other good must be normal.

> The question is actually based on the assumption that $u(.)$ is strictly increasing. Therefore you will always use your budget up to maximize your utility ($p_1 x_1 + p_2 x_2 = y$). This is called "Budget Balancedness"(Theorem 1.10 on Jehle & Reny pp. 49).

Assume there are only 2 goods , $x_1, x_2 \in \mathbb{R}^2_+$, the price vector $p' = (p_1, p_2) \gg 0$, y is the income and the budget constraint is $p_1 x_1 + p_2 x_2 \le y$.
We always assume u(.) is strictly increasing, therefore we have budget balancedness to maximize the utility:$p_1 x_1 + p_2 x_2 = y$
If x_1 is inferior, it means the demand for x_1 will decrease when the income increases, i.e. $\frac{\partial x_1(p_1, p_2, y)}{\partial y} < 0$.
If x_2 is also inferior, what will happen when y increases? The left-hand side will decrease but the right-hand side will increase, which is impossible. Therefore x_2 must be normal.

> You can also take derivatives on both sides of the budget balancedness to avoid wordy argument:
>
> $$\frac{\partial(p_1 x_1 + p_2 x_2)}{\partial y} = \frac{dy}{dy}$$
> $$p_1 \frac{\partial x_1}{\partial y} + p_2 \frac{\partial x_2}{\partial y} = 1$$
> $$\frac{\partial x_2}{\partial y} = \frac{1}{p_2}(1 - p_1 \frac{\partial x_1}{\partial y}) > 0$$

3   Jehle & Reny 1.51 - Substitues and Complements

Consider the utility function, $u(x_1, x_2) = (x_1)^{1/2} + (x_2)^{1/2}$.

 a. Compute the demand functions, $x_i(p_1, p_2, y)$, $i = 1, 2$.

 b. Compute the substitution term in the Slutsky equation for the effects on x_1 of changes in p_2.

 c. Classify x_1 and x_2 as (gross) complements or substitutes.

3.1   Compute the demand functions, $x_i(p_1, p_2, y)$, $i = 1, 2$

We want to solve the utility maximization problem with some (x_1^*, x_2^*), i.e.

$$\max_{x_1, x_2} u(x_1, x_2) = x_1^{0.5} + x_2^{0.5} \quad s.t. \quad p_1 x_1 + p_2 x_2 = y$$

Lagrangean:

$$L = x_1^{0.5} + x_2^{0.5} + \lambda(y - p_1 x_1 - p_2 x_2)$$

FOCs:

$$\begin{cases} \frac{\partial L}{\partial x_1} &= 0.5x_1^{-0.5} - p_1\lambda = 0 \\ \frac{\partial L}{\partial x_2} &= 0.5x_2^{-0.5} - p_2\lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= y - p_1 x_1 - p_2 x_2 = 0 \end{cases}$$

Simplify:

$$\begin{cases} 0.5x_1^{-0.5} &= p_1\lambda \\ 0.5x_2^{-0.5} &= p_2\lambda \\ p_1 x_1 + p_2 x_2 &= y \end{cases} \tag{5}$$

Take the ratio between first and second condition to get:

$$[\frac{x_1}{x_2}]^{-0.5} = \frac{p_1}{p_2} \Rightarrow x_1 = (\frac{p_1}{p_2})^{-2} x_2$$

Substitute x_1 with x_2 in the 3rd condition to get:

$$p_1(\frac{p_1}{p_2})^{-2} x_2 + p_2 x_2 = y$$

$$\Rightarrow x_2 = \frac{y}{p_1(\frac{p_1}{p_2})^{-2} + p_2} = \frac{y}{p_1(\frac{p_2^2}{p_1^2}) + p_2} = \frac{p_1 y}{p_2^2 + p_1 p_2} = \frac{p_1}{p_2}\frac{y}{p_1 + p_2}$$

Thus:

$$x_1 = (\frac{p_1}{p_2})^{-2} x_2 = (\frac{p_1}{p_2})^{-2}\frac{p_1}{p_2}\frac{y}{p_1 + p_2} = \frac{p_2}{p_1}\frac{y}{p_1 + p_2}$$

3.2 Compute the substitution term in the Slutsky equation for the effects on x_1 of changes in p_2

THEOREM 1.11 Slutsky equation (Jehle & Reny pp.53)

Let $x(p, y)$ be the consumers Marshallian demand system. Let u^* be the level of utility the consumer achieves at prices p and income y. Then,

$$\underbrace{\frac{\partial x_i(p, y)}{\partial p_j}}_{TE} = \underbrace{\frac{\partial x_i^h(p, u^*)}{\partial p_j}}_{SE} \underbrace{- x_j(p, y) \frac{\partial x_i(p, y)}{\partial y}}_{IE}, \quad i, j = 1, 2, ..., n.$$

Slutsky equation decomposes the change of demand when price changes (TE) into 2 parts: substitution effects (SE) and income effects (IE).

The question asks you to compute SE, and you already have Marshallian demand function in question part (a), it's easier to use $SE = TE - IE$ directely

$$
\begin{aligned}
\frac{\partial x_1^h(p, u^*)}{\partial p_2} &= \frac{\partial x_1(p, y)}{\partial p_2} + x_2(p, y) \frac{\partial x_1(p, y)}{\partial y} \\
&= \frac{\partial(\frac{p_2}{p_1} \frac{y}{p_1 + p_2})}{\partial p_2} + (\frac{p_1}{p_2} \frac{y}{p_1 + p_2}) \frac{\partial(\frac{p_2}{p_1} \frac{y}{p_1 + p_2})}{\partial y} \\
&= \frac{1}{p_1} \frac{y}{p_1 + p_2} + \frac{p_2}{p_1} \frac{-y}{(p_1 + p_2)^2} + (\frac{p_1}{p_2} \frac{y}{p_1 + p_2})(\frac{p_2}{p_1} \frac{1}{p_1 + p_2}) \\
&= \frac{\frac{(p_1 + p_2)}{p_1} y}{(p_1 + p_2)^2} - \frac{\frac{p_2}{p_1} y}{(p_1 + p_2)^2} + \frac{y}{(p_1 + p_2)^2} \\
&= \frac{2y}{(p_1 + p_2)^2}
\end{aligned}
$$

3.3 Classify x_1 and x_2 as (gross) complements or substitutes

How to define complements and substitutes? If $\frac{\partial x_b}{\partial p_a} > 0$, are x_a, x_b substitutes?

Here is an example:

As fruits, apple and banana seems to be "substitutes", i.e. when apple price increases, we tend to buy fewer apples and more bananas($\frac{\partial x_b}{\partial p_a} > 0$).

However, since apples are very expensive, ony a few apples cost so much money that you have less money for bananas. As a result, you buy fewer apples and fewer bananas ($\frac{\partial x_b}{\partial p_a} < 0$).

Substitute effects may be "covered" by the income effects

To precisely describe "substitutes" we need to define:

- Gross substitutes: $\frac{\partial x_b}{\partial p_a} > 0$

- Net substitutes: $\frac{\partial x(\bar{u})_b^h}{\partial p_a} > 0$

Since $\frac{\partial x_1^h(p, u^*)}{\partial p_2} = \frac{2y}{(p_1 + p_2)^2} > 0$, we already know x_1, x_2 are net substitutes.

$$\frac{\partial x_1(p, y)}{\partial p_2} = \frac{\partial(\frac{p_2}{p_1} \frac{y}{p_1 + p_2})}{\partial p_2} = \frac{1}{p_1} \frac{y}{p_1 + p_2} > 0$$

$\Rightarrow x_1, x_2$ are also gross substitutes