

Seminar 3.Duality of Consumers Behavior

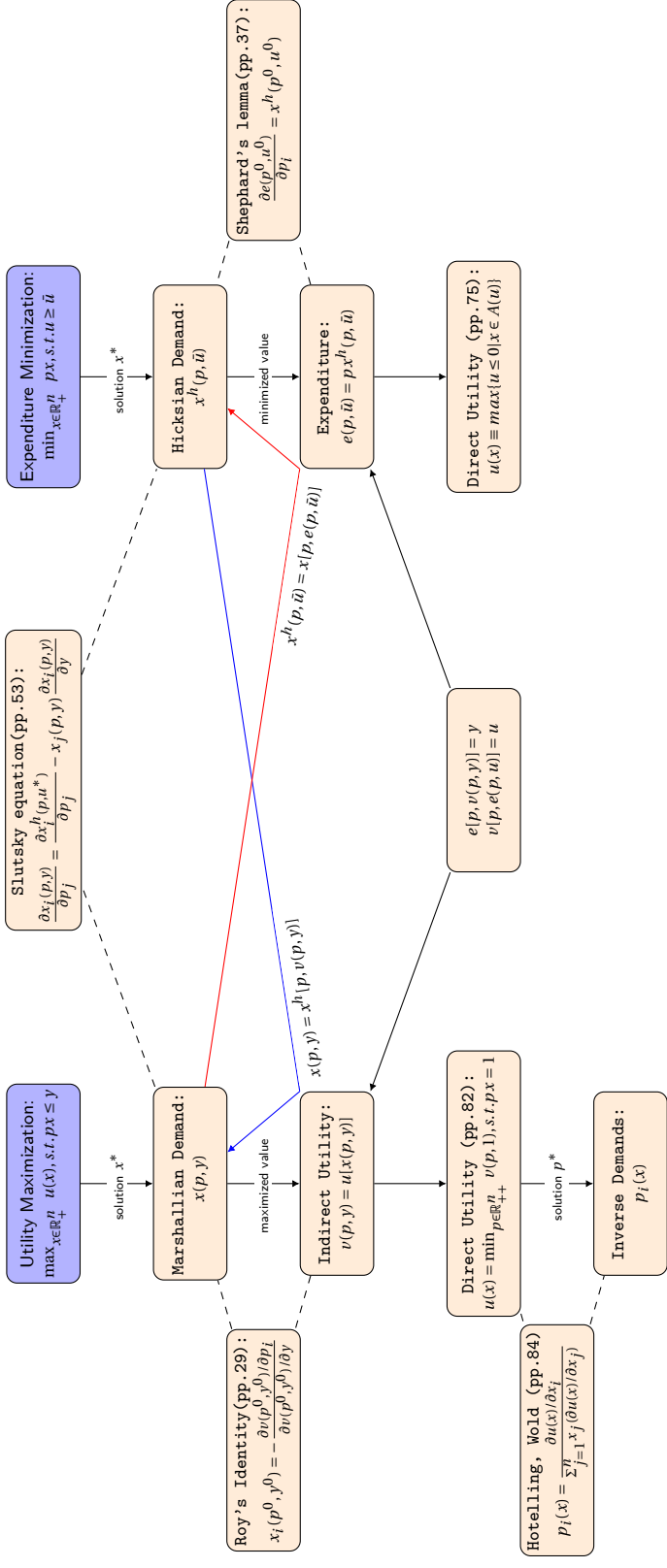
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Consumption Duality

You will never lose your way with this Consumption Duality map!

All "derive this from that and verify some guy's equation" -like questions can be solved by finding the correct (shortest) route.



1 Jehle & Reny 2.3

Derive the consumers direct utility function if his indirect utility function has the form $v(p, y) = yp_1^\alpha p_2^\beta$ for negative α and β .

THEOREM 2.3 Duality Between Direct and Indirect Utility(Jehle & Reny pp.81)

Suppose that $u(x)$ is quasiconcave and differentiable on \mathbb{R}_{++}^n with strictly positive partial derivatives there. Then for all $x \in \mathbb{R}_{++}^n$, $v(p, p \cdot x)$, the indirect utility function generated by $u(x)$, achieves a minimum in p on \mathbb{R}_{++}^n , and

$$u(x) = \min_{p \in \mathbb{R}_{++}^n} v(p, y), s.t. px = y$$

Let's call the solution p^*

Note that by **Theorem 1.6**(Jehle & Reny pp.29), $v(p, y)$ is homogeneous of degree zero in (p, y) . We have $v(p, p \cdot x) = v(p/(p \cdot x), 1)$ whenever $p \cdot x > 0$. Thus the equation above can also be written as:

$$u(x) = \min_{p \in \mathbb{R}_{++}^n} v(p, 1), s.t. px = 1$$

The solution $\hat{p} = p^* / p^* \cdot x = p^* / y$. We don't care about the difference between \hat{p} and p^* because once you substitute them into $v(p, p \cdot x)$, you have the same result (homogeneity of degree zero).

Besides, homogeneous of degree zero in (p, y) also indicates $\alpha + \beta = -1$ because:

$$v(tp, ty) = ty(tp_1^\alpha)(tp_2)^\beta = t^{1+\alpha+\beta} v(p, y)$$

$$u(x) = \min_{p \in \mathbb{R}_{++}^n} v(p, 1) = p_1^\alpha p_2^\beta, s.t. px = 1$$

Lagrangian:

$$L = p_1^\alpha p_2^\beta + \lambda(1 - p_1 x_1 - p_2 x_2)$$

Note there should not be corner solution since

- Since $\alpha, \beta < 0$, p can't be 0.
- You can also argue: $\lim_{p_i \rightarrow 0} \frac{\partial v(p_1, p_2, 1)}{\partial p_i} = -\infty, i = 1, 2$
- $v(p, 1)$ is decreasing in p (this is always true for indirect utility function, see pp.29). For any $px < 1$, you can always decrease $v(p, 1)$ by increasing p until $px = 1$.

FOCs.

$$\begin{cases} \frac{\partial L}{\partial p_1} = \alpha p_1^{\alpha-1} p_2^\beta - \lambda x_1 = 0 \\ \frac{\partial L}{\partial p_2} = p_1^\alpha \beta p_2^{\beta-1} - \lambda x_2 = 0 \\ p_1 x_1 + p_2 x_2 = 1 \end{cases}$$

Simplify:

$$\begin{cases} \alpha p_1^{\alpha-1} p_2^\beta = \lambda x_1 \\ \beta p_1^\alpha p_2^{\beta-1} = \lambda x_2 \\ p_1 x_1 + p_2 x_2 = 1 \end{cases} \quad (1)$$

Take the ratio between first and second condition to get:

$$\frac{x_1}{x_2} = \frac{\alpha}{\beta} \frac{p_2}{p_1}$$

Thus: $p_2 = \frac{\beta}{\alpha} \frac{x_1}{x_2} p_1$

Substitute p_2 with p_1 in the 3rd condition to get:

$$\begin{aligned} p_1 x_1 + \frac{\beta}{\alpha} \frac{x_1}{x_2} p_1 x_2 &= 1 \\ p_1 (x_1 + \frac{\beta}{\alpha} x_2) &= 1 \\ p_1^* &= \frac{1}{x_1 (1 + \frac{\beta}{\alpha})} \\ p_1^* &= \frac{\alpha}{x_1 (\alpha + \beta)} \\ p_1^* &= -\frac{\alpha}{x_1} \\ \Rightarrow p_2^* &= \frac{\beta}{\alpha} \frac{x_1}{x_2} p_1^* = \frac{\beta}{\alpha} \frac{x_1}{x_2} (-\frac{\alpha}{x_1}) = -\frac{\beta}{x_2} \end{aligned}$$

Substitute p_1^* and p_2^* into $v(p, 1)$ we get the minimized value, i.e. the direct utility function:

$$\begin{aligned} u(x_1, x_2) &= [-\frac{\alpha}{x_1}]^\alpha [-\frac{\beta}{x_2}]^\beta \\ &= A x_1^a x_2^b \end{aligned}$$

Where $A = [-\alpha]^\alpha [-\beta]^\beta$, $a = -\alpha > 0$, $b = -\beta > 0$. The utility function is a Cobb-Douglas function.

As a cautious proof, you may want to check if $u(x)$ is quasiconcave and differentiable on \mathbb{R}_{++}^n with strictly positive partial derivatives there, as assumed by Theorem 2.3.

In exam for this course, again, if the function is one- dimension, you should prove it; if it's a higher-dimension function, the proof is not required.

Alternative 1: You can use Roy's Identity(Jehle & Reny pp.29):

$$x_i(p^0, y^0) = - \frac{\partial v(p^0, y^0) / \partial p_i}{\partial v(p^0, y^0) / \partial y}$$

to get Marshallian demands directly (Note here $p_i^* = \frac{p_i}{y}$):

$$\begin{cases} x_1 = -\frac{\alpha}{p_1^*} \\ x_2 = -\frac{\beta}{p_1^*} \end{cases}$$

Then solve the Inverse demand from Marshallian demands:

$$\begin{cases} p_1^* &= -\frac{\alpha}{x_1} \\ p_2^* &= -\frac{\beta}{x_1} \end{cases}$$

Substitute into $v(p, 1)$ you'll have the same solution.

Alternative 2: Like Jehle & Reny 1.51, you can actually transform $v(p_1, p_2, 1)$ into a function of only p_1 or p_2 using $p_1 x_1 + p_2 x_2 = 1$.

$$p_1 = \frac{1 - p_2 x_2}{x_1}$$

Substitute into $v(p_1, p_2, 1)$ to have:

$$v(p_1, p_2, 1) = \left[\frac{1 - p_2 x_2}{x_1} \right]^\alpha p_2^\beta$$

Since the question ask you to minimize $v(p_1, p_2, 1)$, if you solve $\frac{de(p_2)}{dp_2} = 0$ and get only one solution, it is the solution.

$$\begin{aligned}
\frac{de(p_2)}{dp_2} &= \alpha \left(\frac{1-p_2x_2}{x_1} \right)^{\alpha-1} \left(\frac{-x_2}{x_1} \right) p_2^\beta + \frac{1-p_2x_2}{x_1}^\alpha \beta p_2^{\beta-1} = 0 \\
\alpha \left(\frac{1-p_2x_2}{x_1} \right)^{\alpha-1} \left(\frac{x_2}{x_1} \right) p_2^\beta &= \frac{1-p_2x_2}{x_1}^\alpha \beta p_2^{\beta-1} \\
\alpha \left(\frac{x_1}{1-p_2x_2} \right) \left(\frac{x_2}{x_1} \right) p_2 &= \beta \\
\alpha \left(\frac{x_2}{1-p_2x_2} \right) p_2 &= \beta \\
\alpha x_2 p_2 &= \beta - \beta x_2 p_2 \\
(\alpha x_2 + \beta x_2) p_2 &= \beta \\
p_2^* &= \frac{\beta}{(\alpha + \beta) x_2}
\end{aligned}$$

You then solve p_1^* with the budget constraint.

2 Jehle & Reny 2.5(a)

Consider the solution, $e(p, u) = u p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ at the end of Example 2.3. Derive the **indirect utility function** through the relation $e(p, v(p, y)) = y$ and verify Roy's identity.

Example 2.3 on Jehle & Reny pp.90 is a question from $x_i(p, y)$ to $e(p, u)$, where the Marshallian demand function is:

$$x_i(p_1, p_2, p_3, y) = \frac{\alpha_i y}{p_i}, \quad i = 1, 2, 3$$

$\alpha_i > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$

Check your map, the route is (note the expression below is only for the purpose of teaching and very informal):

$$x_i(p, y) \Rightarrow x^h(p, u) = x[p, e(p, u)] \Leftarrow \frac{\partial e(p, u)}{\partial p_i} = x^h(p, u)$$

$$\begin{aligned}
x[p, e(p, u)] &= \frac{\partial e(p, u)}{\partial p_i} \\
\frac{\alpha_i e(p, u)}{p_i} &= \frac{\partial e(p, u)}{\partial p_i} \\
\frac{\alpha_i}{p_i} &= \frac{1}{e(p, u)} \frac{\partial e(p, u)}{\partial p_i} \\
&= \frac{\partial \ln[e(p, u)]}{\partial p_i}
\end{aligned}$$

The rest part of the solution in the textbook is clear. Read page 91 if you're curious how we solve $e(p, u)$ out. It need "a little thought" as the textbook said :)

Indirect utility function:

We already know $e(p, v(p, y)) = y$.

Substitute $v(p, y)$ into $e(p, u) = y$ will solve the question directly:

$$\begin{aligned}
e(p, u) &= v(p, y) p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} = y \\
\Rightarrow v(p, y) &= \frac{y}{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}}
\end{aligned}$$

Verify Roy's identity:

Roy's Identity(Jehle & Reny pp.29):

$$x_i(p^0, y^0) = - \frac{\partial v(p^0, y^0) / \partial p_i}{\partial v(p^0, y^0) / \partial y}$$

Intuition: Your optimal consumption plan (Marshallian demand) is a trade-off between the importance of "commodity i " and "money (y)".

$$\begin{aligned}
\frac{\partial v(p, y)}{\partial p_1} &= \frac{\partial y p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}}{\partial p_1} = -\alpha_1 y p_1^{-\alpha_1-1} p_2^{-\alpha_2} p_3^{-\alpha_3} \\
\frac{\partial v(p, y)}{\partial p_2} &= -\alpha_2 y p_1^{-\alpha_1} p_2^{-\alpha_2-1} p_3^{-\alpha_3} \\
\frac{\partial v(p, y)}{\partial p_3} &= -\alpha_3 y p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3-1} \\
\frac{\partial v(p, y)}{\partial y} &= p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}
\end{aligned}$$

Therefore:

$$-\frac{\partial v(p, y) / \partial p_1}{\partial v(p, y) / \partial y} = -\frac{-\alpha_1 y p_1^{-\alpha_1-1} p_2^{-\alpha_2} p_3^{-\alpha_3}}{p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}} = \frac{\alpha_1 y}{p_1}$$

$$-\frac{\partial v(p, y)/\partial p_2}{\partial v(p, y)/\partial y} = -\frac{-\alpha_3 y p_1^{-\alpha_1} p_2^{-\alpha_2-1} p_3^{-\alpha_3}}{p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}} = \frac{\alpha_2 y}{p_2}$$

$$-\frac{\partial v(p, y)/\partial p_3}{\partial v(p, y)/\partial y} = -\frac{-\alpha_3 y p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3-1}}{p_1^{-\alpha_1} p_2^{-\alpha_2} p_3^{-\alpha_3}} = \frac{\alpha_3 y}{p_3}$$

Compare with the Marshallian demand!

3 Jehle & Reny 2.7

Derive the consumer's **inverse demand functions**, $p_1(x_1, x_2)$ and $p_2(x_1, x_2)$, when the **utility function** is of the Cobb-Douglas form, $u(x_1, x_2) = Ax_1^\alpha x_2^{1-\alpha}$ for $0 < \alpha < 1$.

The shortest route is using Hotelling, Wold (pp.84) directly.

$$p_i(x) = \frac{\partial u(x)/\partial x_i}{\sum_{j=1}^n x_j (\partial u(x)/\partial x_j)}$$

Intuition: the price reflects how important the commodity is.

The duality between direct and indirect utility functions showed by Hotelling, Wold makes it (hopefully) easier to solve $p_i^*(x)$

$$\begin{aligned} p_1(x_1, x_2) &= \frac{\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_1}{\sum_{j=1}^2 x_j (\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_j)} \\ &= \frac{\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_1}{x_1 \partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_1 + x_2 \partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_2} \\ &= \frac{A\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{x_1 A\alpha x_1^{\alpha-1} x_2^{1-\alpha} + x_2 A(1-\alpha) x_1^\alpha x_2^{-\alpha}} \\ &= \frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{\alpha x_1^\alpha x_2^{1-\alpha} + (1-\alpha) x_1^\alpha x_2^{1-\alpha}} \\ &= \frac{\alpha}{x_1} \end{aligned}$$

$$\begin{aligned}
p_2(x_1, x_2) &= \frac{\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_2}{\sum_{j=1}^2 x_j (\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_j)} \\
&= \frac{\partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_2}{x_1 \partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_1 + x_2 \partial(Ax_1^\alpha x_2^{1-\alpha})/\partial x_2} \\
&= \frac{A(1-\alpha)x_1^\alpha x_2^{-\alpha}}{x_1 A\alpha x_1^{\alpha-1} x_2^{1-\alpha} + x_2 A(1-\alpha)x_1^\alpha x_2^{-\alpha}} \\
&= \frac{(1-\alpha)x_1^\alpha x_2^{-\alpha}}{\alpha x_1^\alpha x_2^{1-\alpha} + (1-\alpha)x_1^\alpha x_2^{1-\alpha}} \\
&= \frac{1-\alpha}{x_2}
\end{aligned}$$

You can also try another route: maximize $u(x) \Rightarrow x(p, y) \Rightarrow p_i(x) = x^{-1}(x, 1)$
Use Lagrangian to maximize $u(x_1, x_2) = Ax_1^\alpha x_2^{1-\alpha}$ s.t. $p_1 x_1 + p_2 x_2 = 1$. The solution (Marshallian demands) is:

$$\begin{cases} x_1 = \frac{\alpha}{p_1} \\ x_2 = \frac{1-\alpha}{p_2} \end{cases}$$

The inverse of Marshallian demand function gives the inverse demand function

$$\begin{cases} p_1 = \frac{\alpha}{x_1} \\ p_2 = \frac{1-\alpha}{x_2} \end{cases}$$

Another example:

- You can also try to derive $p_i(x)$ from the Marshallian demand E.1 on pp. 32 and compare with the result derived from Hotelling-Wold identity on pp. 85. Both of them should be the same as E.5-E.6 on pp. 83, which is the solution for $v(p, 1)$ minimization problem.