Seminar 7. Static and dynamic games

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October 18, 2020

We choose static game with complete information as our benchmark because it's the simplest game. More complex games need stricter conditions to locate the equilibria, i.e. refinement (rule out unrealistic NE).

- Be sure to know which type the game is before answering the questions!

Complete Information

Players have the symmetric information (no private information)
know clearly which "type" their opponents are.

Static (Watson Part II)

Players act simultaneously and independently.

- Rationality & Nash Equilibrium (NE)
- Pure/Mixed strategy
- Repeated game & trigger strategy

Dynamic (Watson Part III)

Empty threat: not every NE are realistic, refinement needed.

- Subgame Perfect Nash Equilibrium (SPNE)
- Imperfect innformation: examine every subgame by hand (Watson pp. 190)
- Perfect information: backward-induction method

Incomplete Information (Watson Part IV)

At least one player does not know which type his/her opponents are.

- Exogenous move: nature decides player's type.
- Players without private information can only "guess" (assign probability/belief to) the type of his/her opponent.

Static (Watson pp.327 - 377)

- Bayesian Nash Equilibrium (BNE)
- Bayesian normal form

Dynamic(Watson pp.378 - 406)

More information can be obtained from the opponent's behavior.
$\Rightarrow$ Probability (belief) can be adjusted dynamically (updated).

- Perfect Bayesian (Nash) Equilibrium (PBE)
- Screening: player **without** private information move first (e.g. Insurance scheme to screen risky clients; Contract to rule out low-capacity workers).
- Signaling: player **with** private information move first (High-risk clients pretend to be of low-risk).
- Pooling & Separating equilibrium.

Many students treated a dynamic game as static last year...
Narrative like "player 2 acts before/after player 1" is definitely dynamic. You must refine the NE properly!

- Correctly drawn extensive form for complex dynamic games may gain some points :)
- Do more exercises on screening and signaling games until you can solve it by yourself. It may take 20 points in exam.

1 Problem 1 - Simultaneous and sequential moves with complete information

You and a friend are in a restaurant, and the owner offers both of you an 8-slice pizza under the following condition. Each of you must **simultaneously** announce how many slices you would like; that is, each player $i \in \{1,2\}$ names his/her desired amount of pizza, $0 \leq s_i \leq 8$.

- If $s_1 + s_2 \leq 8$, then the players get their demands (and the owner eats any left-over slices).
- If $s_1 + s_2 > 8$, then the players get nothing.

Assume that you each care only about how much pizza you individually consume, preferring more pizza to less.

1.1 What is (are) each player's best response(s) for each of the possible demands for his/her opponent?

Best Response (Watson pp.54)
Suppose player i has a belief $\theta_i \in \Delta S_i$ about the strategies played by the other players. Player i's strategy $s_i \in S_i$ is a best response if $u_i(s_i, \theta_i) \geq u_i(s_i', \theta_i)$ for every $s_i' \in S_i$

Terms and notations(Watson pp.37):

- Belief: a player's assessment about the strategies of the others,i.e. probability distributions such as $(1/2, 1/6, 2/6)$ for 3 strategies U, M, D of the opponent.
- ΔS_i: the set of probability distributions over the strategies of all the players except player i .

In one word, the strategies bringing the highest payoff given the belief of the player.

The question wants you to find each palyer's BR for **each of the possible demands** for his/her opponent. We don't assume any belief for now.

Let's find BR together in the normal form:

Table 1: Normal form for pizza game

P2

		0	1	2	3	4	5	6	7	8
	0	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$	$(0,4)$	$(0,5)$	$(0,6)$	$(0,7)$	$(0,8)$
	1	$(1,0)$	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$	$(1,7)$	$(0,0)$
	2	$(2,0)$	$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$	$(0,0)$	$(0,0)$
	3	$(3,0)$	$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	$(0,0)$	$(0,0)$	$(0,0)$
P1	4	$(4,0)$	$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
	5	$(5,0)$	$(5,1)$	$(5,2)$	$(5,3)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
	6	$(6,0)$	$(6,1)$	$(6,2)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
	7	$(7,0)$	$(7,1)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
	8	$(8,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$

BR set if opponent chooses 0: $\{8\}$
BR set if opponent chooses 1: $\{7\}$
BR set if opponent chooses 2: $\{6\}$
BR set if opponent chooses 3: $\{5\}$
BR set if opponent chooses 4: $\{4\}$
BR set if opponent chooses 5: $\{3\}$
BR set if opponent chooses 6: $\{2\}$
BR set if opponent chooses 7: $\{1\}$
BR set if opponent chooses 8: $\{0,1,2,3,4,5,6,7,8\}$

Or:

BR set if opponent chooses n, $n \in [0,7]$: $\{8-n\}$
BR set if opponent chooses 8: $\{0,1,2,3,4,5,6,7,8\}$

1.2 Find all the pure-strategy Nash equilibria

A strategy profile $s \in S$ is a Nash equilibrium if and only if $s_i \in BR_i(s_{-i})$ for each player i. That is, $u_i(s_i, s_i) \geq u_i(s'_i, s_i)$ for every $s'_i \in S_i$ and each player i.

In one word, in a NE, the strategy is(note: strategy is a plan, not a choice) the BRs for every play.

- We can find NE based on the BR we found in question 1.1.
- Don't forget $(8,8)$. Intuition: no incentive to deviate from the state (initial strategy).

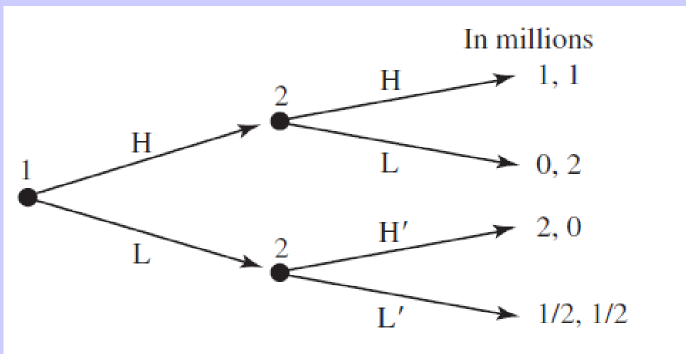
NE: $(0,8), (1,7), (2,6), (3,5), (4,4), (5,3), (6,2), (7,1), (8,0), (8,8)$

Reconsider the situation above, but assume now that **player 1 makes her demand before player 2** makes his demand. Player 2 observes player 1's demand before making his choice.

1.3 Explain what a strategy is for player 2 in this game with sequential moves.

Strategy: A strategy is a complete contingent plan for a player in the game. It describes what the player will do at each of his/her information sets (Watson pp.22).

Think about an easier question:



Let s_i denotes a single strategy of player i,
There are 2 strategies for player 1: $s_1 = H$ and $s_1 = L$.
There are 4 strategies for player 2 : $s_2 = HH'$, $s_2 = HL'$, $s_2 = LH'$, and $s_2 = LL'$.

- H and H' are the same choice for player 2, but under 2 different conditions (how player 1 moves).
- HH' means "when player 1 chooses H, player 2 chooses H; when player 1 chooses L, player 2 also chooses H".
- We use H' instead of H just to avoid ambiguity.

Player 2's strategy has a form of "$s_2 = [s_2(s_1 = H), s_2(s_1 = L)]$", where $s_2(s_1 = H)$ can be either H or L, and $s_2(s_1 = L)$ can be either H' or L'. There are $2 \times 2 = 4$ different strategies.

Now think, in the pizza problem

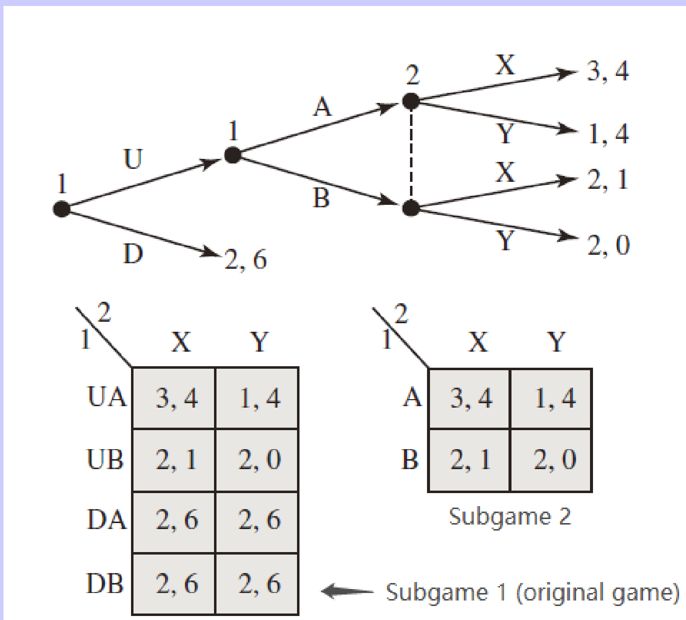
Both player 1 and player 2 have 9 different choices, and player 1 moves first. Player 2's strategy is:
$[s_2(s_1=0), s_2(s_1=1), s_2(s_1=2), s_2(s_1=3), s_2(s_1=4), s_2(s_1=5),$
$s_2(s_1=6), s_2(s_1=7), s_2(s_1=8)]$
where each $s_2(s_1=n)$ can take 9 values in $[0,8]$ (9^9 strategies in total).

Strategy Set/Space (Watson pp.23)
We use S_i to denote the strategy set/space for player i (a set comprising each of the possible strategies of player i in the game):

$$S_1 = \{H, L\}$$

$$S_1 = \{HH', HL', LH', LL'\}$$

Here is another example to show the problem more clearly.



In the original game, player 1 has 4 single strategies. The strategy space $S_1 = \{UA, UB, DA, DB\}$; while player 2 only has 2 strategies, $S_2 = \{X, Y\}$.

- The number of strategies depends on how many decisions (information set) player i has to make.

- Nodes linked by a dashed line is ONE information set (player 2 can't know if player 1 chooses A or B, therefore he/she can make one decision, X or Y.)

1.4 Find all the pure-strategy Nash equilibrium outcomes.

Subquesiton 1.4 and Subquesiton 1.5 are to help you understand why we need to refine NE to SPNE for a dynamic game.

A NE is a state in which nobody can improve his/her payoff by deviating from his/her own strategy.

To ahcieve a NE, player 2 must have his/her best response against s_1, and also make sure player 1 will not regret choosing s_1 (then deviate).

Actually, we already found all the static NE for in Table 1, what we need to do is to "rewrite" them in a dynamic way s.t. no one will deviate.

Take a NE $(2,6)$ in Table 1 for example.

- When player 1 chooses 2 first, the best response for player 2 is to choose 6;
- To make sure player 1 will not deviate from choosing 2, we must "eliminate" player 1's incentive to deviate, i.e. let $u_1(s_1 = 2, s_2) \geq u_1(s_1 \neq 2, s_2)$.
- A strategy s_2 such as $(4,8,6,6,5,4,3,2,1)$ achieves such a NE.
- Note the red 6 is the result (payoff) but note a "strategy". A strategy is a complete contingent plan. Here includes both the result player 2 wants (6) and the threat part ("if you choose less than 2, fine; if you choose more than 2, I'll let you have nothing").

We find $(2,(4,8,6,6,5,4,3,2,1))$ is a dynamic NE.

For all the NE (except $(8,8)$) found in Table 1, we can express them as:

$$s_1^* \in \{0,1,2,3,4,5,6,7,8\} \quad and$$

$$s_2^*(s_1) = 8 - s_1 \;\; if \;\; s_1 = s_1^*$$
$$s_2^*(s_1) > 8 - s_1 \;\; if \;\; s_1 > s_1^*$$
$$s_2^*(s_1) \in \{0,1,2,3,4,5,6,7,8\} \;\; if \;\; s_1 < s_1^*$$

The static NE $(8,8)$ is a little different to rewrite, but the idea is the same.

- When player 1 chooses 8 first, any natural number in $[0,1]$ is a best response of player 2. Thus 8 is one of the best responses of player 2.
- To make sure player 1 not deviating, player 2 must ruin player 1's incentive again. "If you don't choose 8, you get nothing either, then why deviate?"

$$s_1^* = 8 \quad and$$
$$s_2^*(s_1) > 8 - s_1 \;\; if \;\; s_1 \in \{1,2,3,4,5,6,7,8\}$$
$$s_2^*(s_1) \in \{0,1,2,3,4,5,6,7,8\} \;\; if \;\; s_1 = 0$$

To conclude, static NE is exactly the payoff result of dynamic NE. Dynamic NE are strategies that keep static NE stable.

(1) Type 1:
$s_1^* \in \{0,1,2,3,4,5,6,7,8\}$ and
$s_2^*(s_1) = 8 - s_1$ if $s_1 = s_1^*$,
$s_2^*(s_1) > 8 - s_1$ if $s_1 > s_1^*$,
$s_2^*(s_1) \in \{0,1,2,3,4,5,6,7,8\}$ if $s_1 < s_1^*$.

Example: $(4,(8,7,6,5,4,4,4,4,4))$
Here player 2 demands the pieces that are left if player 1 does not demand more than 4 pieces, but demands 4 pieces if player 1 demands more than 4 pieces. To demand 4 pieces is a best response for player 1, given that he will not get anything if demands more than 4 pieces. It is a best response for player 2, given that player 1 demands 4 pieces, as his strategy specifies.

(2) Type 2:
$s_1^* = 8$ and
$s_2^*(s_1) > 8 - s_1$ if $s_1 \in \{1,2,3,4,5,6,7,8\}$,
$s_2^*(s_1) \in \{0,1,2,3,4,5,6,7,8\}$ if $s_1 = 0$.

Example: $(8, (8, 8, 8, 8, 8, 8, 8, 8, 8))$
Here player 2 demands all the 8 pieces independently of what player 1 demands. To demand all the 8 pieces is a best response for player 1, given that he will not get anything anyway. It is a best response for player 2, given that player 1 demands all the 8 pieces, as his strategy specifies.

1.5 Find all the pure-strategy subgame perfect equilibria.

The NE found in Subquestion 1.5 sound good, but is player 2's threat like "if you choose more than 2, I'll let you have nothing" valid?

- If player 2 chooses 7, and the game is played only once, what will player 1 do to maximize utility (BR)?
- "To give in and accept the 1 piece pizza left by player 1" is BR for player 2.

NE is an ex ante view of the game. But in a dynamic game, players need to think "if something already happened, what should I do", which is a ex post view.

Sequential rationality (Watson pp.186): An optimal strategy for a player should maximize his or her expected payoff, conditional on every information set at which this player has the move. That is, player is strategy should specify an optimal action from each of player is information sets, even those that player i does not believe (ex ante) will be reached in the game.

SPNE (Watson pp.189) A strategy profile is called a subgame perfect Nash equilibrium (SPNE) if it specifies a Nash equilibrium in every subgame of the original game.

- SPE is a NE
- For games of perfect information, backward induction yields subgame perfect equilibria(Watson pp.191).
- For games of imperfect information, we need to check every subgame (Read Watson pp.190).

Here is an example of extensive form when player 1 chooses 3 and 7 (I didn't draw the whole extensive form because it takes too much space).

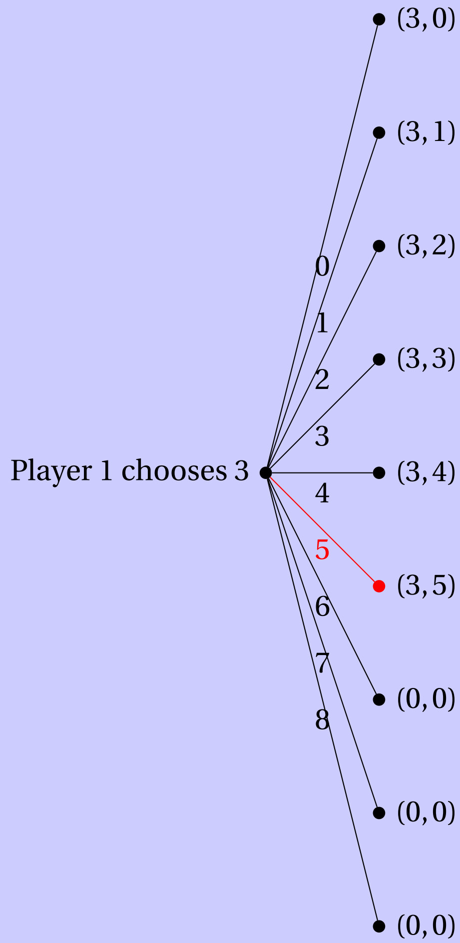


Figure 1: When player 1 chooses 3

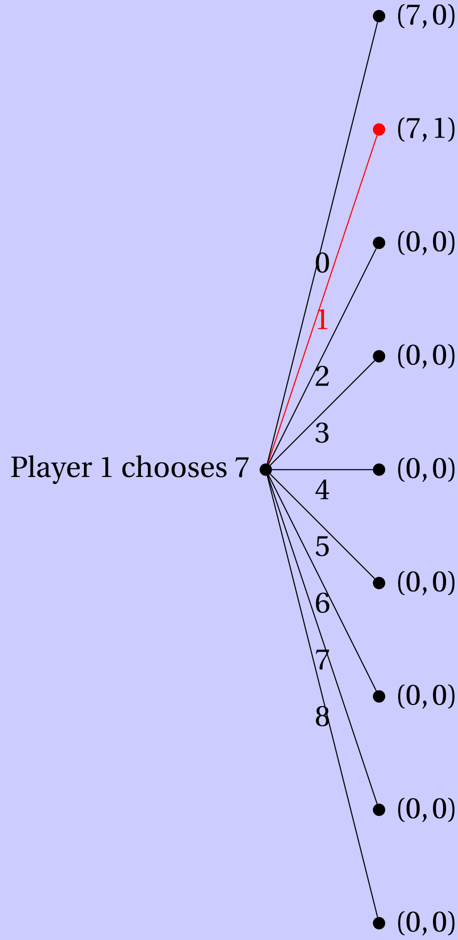


Table 2: When player 1 chooses 7

With backward induction, we can find when player 1 chooses 7, player 2 will accept 1. Therefore player 1 will never choose less than 7.

Then how about player 1 chooses 8? In this case, player 2 get nothing, no matter what he/she chooses. But on the other hand, player 2 can freely (there is nothing to lose, anyway) choose any pieces more than 0 to punish greedy player 2. As a result, we have SPNE:

$$(7,(8,7,6,5,4,3,2,1,n)),\ n \in \{1,2,3,4,5,6,7,8\}$$

However, if player 2 doesn't decide to punish player 1 (maybe they don't want the restaurant owner to have the pizza back), there can still be a SPNE:

$$(8,(8,7,6,5,4,3,2,1,0))$$

The two types of SPNE results from the fact that player 2's payoff is 0 (and then indifferent to any choices) whenever player 1 chooses 8.

(1) Type 1:
$s_1^* = 7$ and
$s_2^*(s_1) = 8 - s_1$ if $s_1 \in \{0,1,2,3,4,5,6,7\}$,
$s_2^*(s_1) > 8 - s_1$ if $s_1 = 8$.

Example: $(7,(8,7,6,5,4,3,2,1,1))$
Here player 2 demands the pieces that are left if player 1 demands less that all the 8 pieces, but demands 1 piece if player 1 demands all 8 pieces. This is a best response for player 2, not only if player 1 demands 7 pieces, as his strategy specifies, but also for all other choices that player 1 might do. To demand 7 pieces is a best response for player 1, given that he will not get anything if he demands all the 8 pieces.

(2) Type 2:
$s_1^* = 8$ and
$s_2^*(s_1) = 8 - s_1$ if $s_1 \in \{0,1,2,3,4,5,6,7,8\}$
That is: $(8,(8,7,6,5,4,3,2,1,0))$

Here player 2 requires the pieces that are left. This is a best response for player 2, not only if player 1 demands all the 8 pieces, as his strategy specifies, but also for all other choices that player 1 might do. To demand all the 8 pieces is a best response for player 1.

2 Problem 2 - Best response sets

Exercise 6.4
For the game of Figure 6.2 (Watson pp.55), determine the following best-response sets.

FIGURE 6.2 (Watson pp. 55)
An example of best response.

1 \ 2	L	C	R
U	2, 6	0, 4	4, 4
M	3, 3	0, 0	1, 5
D	1, 1	3, 5	2, 3

(a) $BR_1(\theta_2)$ **for** $\theta_2 = (1/6, 1/3, 1/2)$

$$U_U^1(\theta_2) = \frac{1}{6} \times 2 + \frac{1}{3} \times 0 + \frac{1}{2} \times 4 = \frac{14}{6}$$

$$U_M^1(\theta_2) = \frac{1}{6} \times 3 + \frac{1}{3} \times 0 + \frac{1}{2} \times 1 = 1$$

$$U_D^1(\theta_2) = \frac{1}{6} \times 1 + \frac{1}{3} \times 3 + \frac{1}{2} \times 5 = \frac{14}{6}$$

$BR_1(\theta_2) = \{U\}$

(b) $BR_2(\theta_1)$ **for** $\theta_1 = (1/6, 1/3, 1/2)$

$$U_L^2(\theta_1) = \frac{1}{6} \times 6 + \frac{1}{3} \times 3 + \frac{1}{2} \times 1 = 2.5$$

$$U_C^2(\theta_1) = \frac{1}{6} \times 4 + \frac{1}{3} \times 0 + \frac{1}{2} \times 5 = 2.5 + \frac{2}{3}$$

$$U_R^2(\theta_1) = \frac{1}{6} \times 4 + \frac{1}{3} \times 5 + \frac{1}{2} \times 3 = 2.5 + \frac{4}{3}$$

$BR_2(\theta_1) = \{R\}$
You just keep doing this...

(a) $\{U\}$
(b) $\{R\}$
(c) $\{U\}$
(d) $\{U, D\}$

(e) $\{L, R\}$

3 Problem 3 - BR functions, NE, rationalizable strategies

(Watson Exercise 9.6)
Consider a game in which, simultaneously, player 1 selects any real number x and player 2 selects any real number y. The payoffs are given by:

$$u_1(x,y) = 2x - x^2 + 2xy$$
$$u_2(x,y) = 10y - 2xy - y^2$$

3.1 Calculate and graph each player's best-response function as a function of the opposing players pure strategy.

Given any y, how can player i maximize his/her utility? FOC and SOC!

For player 1: FOC:

$$\frac{du_1(x,y)}{dx} = \frac{d2x - x^2 + 2xy}{dx} = 2 - 2x + 2y = 0 \Rightarrow x = 1 + y$$

SOC:

$$\frac{d^2u_1(x,y)}{dx^2} = -2 < 0$$

Therefore $BR_1(y) = 1 + y$

For player 2: FOC:

$$\frac{du_2(x,y)}{dy} = \frac{d10y - 2xy - y^2}{dy} = 10 - 2x - 2y = 0 \Rightarrow y = 5 - x$$

SOC:

$$\frac{d^2u_2(x,y)}{dy^2} = -2 < 0$$

Therefore $BR_2(x) = 5 - x$

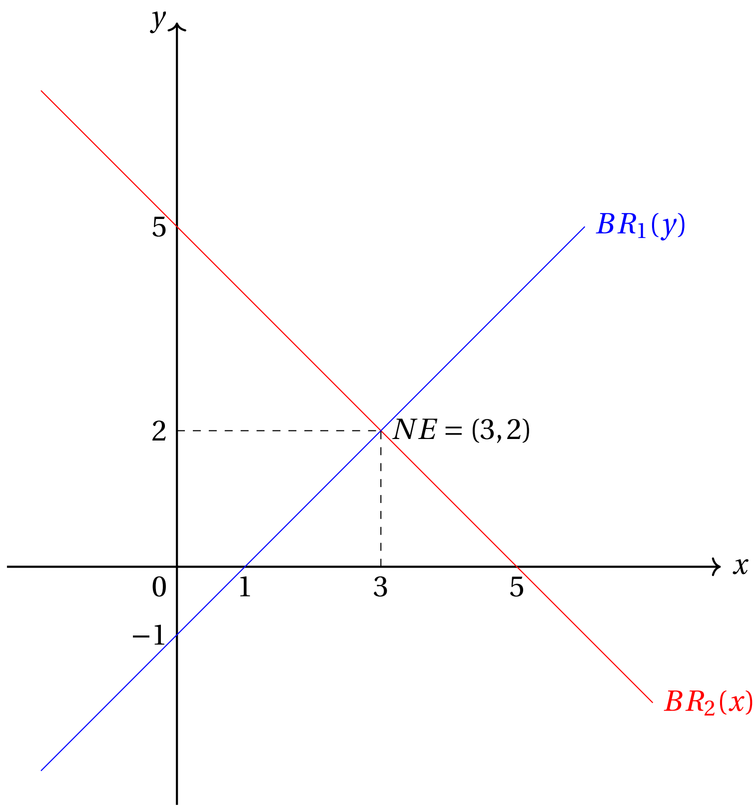


Figure 2: $BR_1(y)$ and $BR_2(x)$

3.2 Find and report the Nash equilibria of the game.

$NE(x^*, y^*)$ is a point s.t. $BR_1(y^*) = x^*$ and $BR_2(x^*) = y^*$, thus

$$x^* = 1 + y^*$$
$$y^* = 5 - x^*$$

(2 functions, 2 variables, we can solve x^*, y^*)
$\Rightarrow x^* = 3, y^* = 2 \Rightarrow NE = (3,2)$

3.3 Determine the rationalizable strategy profiles for this game.

rationalizable strategies(Watson pp.70): The set of strategies that survive iterated dominance is therefore called the rationalizable

strategies.
Dominance(Watson pp.50): A pure strategy s_i of player i is dominated if there is a strategy (pure or mixed) $\sigma_i \in \Delta S_i$ such that $u_i(\sigma_i, s_i) > u_i(s_i, s_i)$, for all strategy profiles $s_i \in S_i$ of the other players.

Since $BR_1(y) = 1 + y$, ll the numbers in $(-\infty, +\infty)$ can be the best response of player 1 sometimes;
Since $BR_2(x) = 5 - x$ All the numbers in $(-\infty, +\infty)$ can be the best response of player 2 sometimes too.
Therefore there is no dominated strategy (a number that can never be BR) for the two players
The set of rationalizable strategies is $(-\infty, +\infty) \times (-\infty, +\infty) = (-\infty, +\infty)$.

Note the symbol "×" denotes the Cartesian product here. For example, if $S_1 = A, b, S_2 = X, Y$,then

$$S = S_1 \times S_2 = (A, X), (A, Y), (B, X), (B, Y)$$

4 Problem 4 - True or False?

For each of the statements, if true, try to explain why, and if false, provide a counter-example.

(a) In a finite extensive-form game of perfect information, there always exists a subgame perfect Nash equilibrium.

True(Watson pp.188). A subgame-perfect Nash equilibrium can be constructed by using backward induction.

A more detailed explanation is on Jehle & Reny pp.317 : every finite extensive-form game possesses at least one NE.

(b) In a finite extensive-form game of perfect information, there always exists a unique subgame perfect Nash equilibrium.

False. The pizza game is a complex counter-example. Here is another:

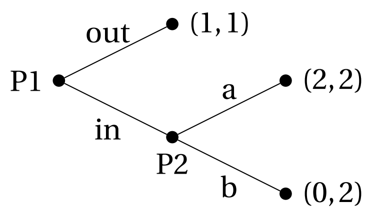


Figure 3: Counter-example

When player 2 is **indifferent**(recall the pizza game) to a and b, there can be many SPNE:

$$(out, (b, b)), (out, (a, b)), (in, (b, a)), (in, (a, a))$$

5 Problem 5- Firm-union bargaining

A firm's output is $L(100-L)$ when it uses $L \leq 50$ units of labor, and 2500 when it uses $L \geq 50$ units of labor. The price of output is 1. A union that represents workers presents a wage demand (a nonnegative number w), which the firm either accepts or rejects. If the firm accepts the demand, it chooses the number L of workers to employ (which you should take to be a continuous variable, not an integer); if it rejects the demand, no production takes place ($L = 0$). The firm's preferences are represented by his/her profits, the union's preferences are represented by the value of wL.

5.1 Formulate this situation as an extensive game with perfect information.

Players: $N = \{U, F\}$.
Strategies: U chooses a wage w from the set of non-negative number; F chooses a function that to any non-negative wage w determines a non-negative employment $L(w)$.
Payoffs: The union's payoff is $wL(w)$; the firm's payoff is $L(w)(100 - L(w)) - wL(w)$.
No need to have a specific accept/reject decision at $L(w) = 0$ is in effect a rejection by the firm of the demand w, giving both a payoff of 0.

5.2 Find the subgame perfect equilibrium (equilibria?) of the game.

Maximizing the firm's payoff yields $L(w) = \frac{100-w}{2}$ for $w \leq 100$ and $L(w) = 0$ otherwise.
The union's best response to this strategy is setting $w = 50$.

5.3 Is there an outcome of the game that both parties prefer to any subgame perfect equilibrium outcome?

The subgame-perfect equilibrium outcome is $w = 50$ and $L = 25$, yielding a payoff of 1250 for the union and 625 for the firm. Joint surplus is maximized for $L = 50$, yielding a maximized total surplus of 2500. At this employment level, any wage w between 25 and 37.5 would lead to a Pareto-improvement.

5.4 Find a Nash equilibrium for which the outcome differs from any subgame perfect equilbrium outcome.

Consider $L(w) = \frac{100-w}{2}$ for $w \leq 20$ and $L(w) = 0$ otherwise. Then the union's best response is $w = 20$, leading to the employment $L = 40$ and the payoffs 800 for the union and 1600 for the firm. This is a Nash equilibrium, but it is not subgame-perfect.