Seminar 4. Elasticity of Substitution

Xiaoguang Ling
xiaoguang.ling@econ.uio.no

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1 Substitution along an indifference curve

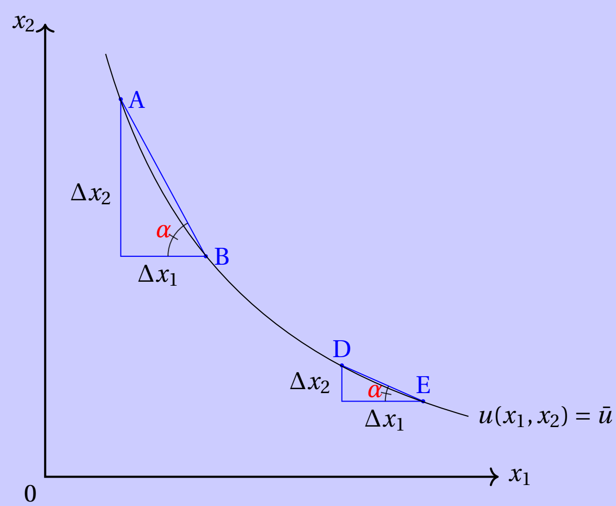

Figure 1: Marginal change and slope

When we move along $u(x_1, x_2) = \bar{u}$, we can observe the following facts:

- The small increase of x_1 (i.e. Δx_1) is always followed by some small decrease of x_2 (i.e. Δx_2). Angle α reflects how much you have to give up (substitute).
- $\alpha = -\frac{\Delta x_2}{\Delta x_1}$ depends on the relative amount of x_1 and x_2, i.e. $\frac{x_1}{x_2}$ (Note Δx_2 is negative here).

Intuition: To keep utility the same, you need to give up Δx_2 to consume Δx_1.

Two questions:

- How to calculate $\alpha = -\frac{\Delta x_2}{\Delta x_1}$?
- How to describe the relationship between α and $\frac{x_1}{x_2}$?

2 Marginal Rate of Substitution (MRS)

When x_1 increases one very small unit, we define $\angle\alpha$ as Marginal Rate of Substitution (MRS).

$$MRS = \alpha = -\frac{\Delta x_2}{\Delta x_1}$$

We can calculate α in the following 2 ways:

Method 1 (2-D thinking)
Similar to Jehle & Reny 1.27 in seminar 1, an indifference curve can be seen as the graph of a function $x_2 = f_(x_1)$ given some utility $\bar{u}$. $-\alpha$ is simply the slope (derivative), thus $\alpha = -\frac{dx_2}{dx_1}$.
An alternative way of thinking is: when line segment $A-B$ and $D-E$ are extremely short, α is simply the slope of the indifference curve.

Method 2 (3-D thinking)
Given $u(x_1, x_2)$ is a differentiable function (recall the hill-like 3-D graph I showed you). The small change of $u(x_1, x_2)$, i.e. $\Delta u(x_1, x_2)$, can always be attributed to the small change of x_1 and x_2. The "attribution" of each unit change of the two commodities is called "Marginal utility", $\frac{\partial u(x_1,x_2)}{\partial x_1}$ and $\frac{\partial u(x_1,x_2)}{\partial x_2}$:

$$\Delta u(x_1, x_2) = \frac{\partial u(x_1, x_2)}{\partial x_1}\Delta x_1 + \frac{\partial u(x_1, x_2)}{\partial x_2}\Delta x_2$$

(See also Total derivative.)
Now, let's keep $u(x_1, x_2) = \bar{u}$, i.e. $\Delta u(x_1, x_2) = 0$, we have

$$0 = \frac{\partial u(x_1, x_2)}{\partial x_1}\Delta x_1 + \frac{\partial u(x_1, x_2)}{\partial x_2}\Delta x_2$$

$$\alpha = -\frac{\Delta x_2}{\Delta x_1} = \frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2}$$

Intuition: The more important x_1 is, the more x_2 you'd like to give up.

Another virtue of the 3-D thinking is that it can be easily generalized to many-dimension problem. Given utility the same, **to consume more commodity i**, how much commodity j must you give up? This is called the **Marginal Rate of Substitution of good j for good i**:

$$MRS_{ij}(x) = \frac{\partial u(x)/\partial x_i}{\partial u(x)/\partial x_j}$$

Similarly, in the case of Production theory, given the quantity of production the same (along an isoquant), **to increase input i**, how much input j must be decreased? This is called the **Marginal Rate of Technical Substitution of input j for input i**:

$$MRTS_{ij}(x) = \frac{\partial f(x)/\partial x_i}{\partial f(x)/\partial x_j}$$

Here the word "Technical" refers to the technology $f(x)$.

3 Elasticity of Substitution

3.1 α and $\frac{x_1}{x_2}$

The relationship between α and $\frac{x_1}{x_2}$ also reflects the nature of the two commodities.
In Figure 2 and Figure 4, the change of $\frac{x_1}{x_2}$ can be expressed by $\angle\gamma$. When $\frac{x_1}{x_2}$ changes γ:

- in Figure 2, the "slope" MRS changes a lot.
- in Figure 4, the "slope" MRS changes a little.

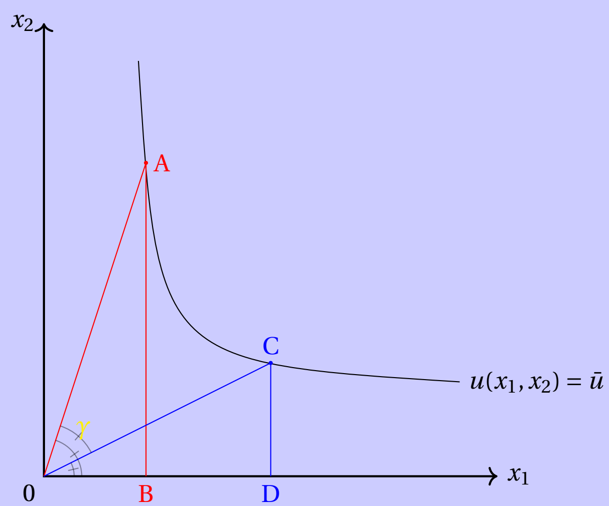

Figure 2: MRS changes much

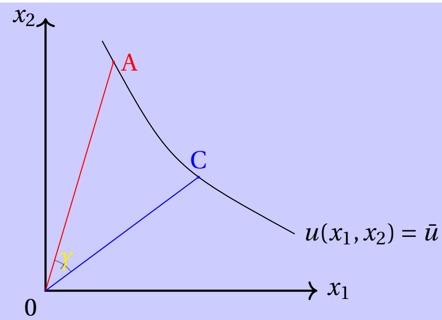

Figure 3: MRS changes a little

Now let's think about an extreme case. In Figure 3, when $\frac{x_1}{x_2}$ changes γ, MRS does not change ($MRS_{12}(x) = \frac{\partial u(x)/\partial x_1}{\partial u(x)/\partial x_2} = 0.5$). That is, MRS is independent of $\frac{x_1}{x_2}$.
You're never bored with x_1 comparing with x_2, no matter how much x_1 and x_2 you consumed. This can only happen when x_1 and x_2 are in nature the same commodity.
For example, x_1 is an apple, while x_2 is a pack of 2 apples. The quality of the apples are the same and the only difference is the amount per package. Since you can always substitute a 2-apple pack with 2 single apples, we call this condition as "**perfect substitution**".

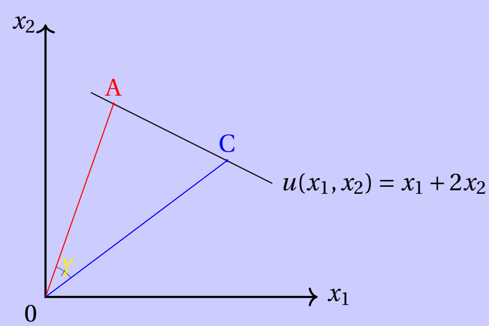

Figure 4: Perfect substitution

Another extreme example is Leontief preference. In Figure 5, when $\frac{x_1}{x_2}$ changes γ, MRS changes from $+\infty$ to 0. We already know with Leontief preference, you believe x_1 and x_2 are totally different and can only make sense with certain proportion, like bread and cheeze. There can be **no substitution** between the two.

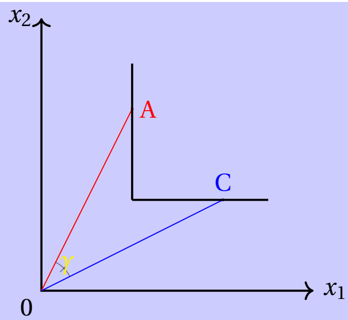

Figure 5: Leontief Preference

4 Elasticity of Substitution σ_{12}

We can use "**Elasticity**" to reflect the relationship between α and $\frac{x_1}{x_2}$.
$Elasticity = \frac{\Delta MRS/MRS}{d}$
DEFINITION: The Elasticity of Substitution (a 2-input case for DEFINITION 3.2 on Jehle & Reny pp. 129)

For a production function $f(x_1, x_2)$, the elasticity of substitution of input 2 for input 1 at the point $(x_1, x_2) \in \mathbb{R}^2_{++}$ is defined as:

$$\sigma_{12}(x_1, x_2) \equiv \left(\frac{d ln MRTS_{12}(\frac{x_2}{x_1})}{d ln(\frac{x_2}{x_1})}\right)^{-1}$$

Note both the numerator and the denominator are functions of $\frac{x_2}{x_1}$. If we define $\frac{x_2}{x_1} = r$, $\sigma_{12}(x_1, x_2)$ can be rewritten as:

$$\sigma_{12}(x_1, x_2) \equiv \left(\frac{d ln MRTS_{12}(r)}{d ln r}\right)^{-1}$$

An example from exam 2019 Q1(b)
Deb and Frank have the following utility functions