Seminar 2 - Expenditure function

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September 12, 2020

1 Jehle & Reny 1.38 - Properties of the Expenditure Function

Verify that the expenditure function obtained from the CES direct utility function in Example 1.3 (JR. pp.39) satisfies all the properties given in Theorem 1.7 (JR. pp.37).

Expenditure Function (JR. pp.35)

We define the expenditure function as theminimum-value function:

$$e(p,u) \equiv \min_{x \in \mathbb{R}^n_+} p \cdot x$$

Expenditure Function of CES direct utility function (JR. pp.39)

In Example 1.3 (JR. pp.39), we have a so called CES direct utility function:

$$u(x_1,x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}},\ where\ 0 \neq \rho < 1.$$

To derive the Expenditure Function, we need to solve the expenditure minimisation problem given some utility u. i.e.

$$\min_{x_1,x_2} p_1x_1 + p_2x_2 \ \ s.t.\ \ u(x_1,x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} = u,\ x_1 \leq 0,\ x_2 \leq 0$$

(Read JR. pp.40 to see how to minimize the expenditure with Lagrangian method.)

The solution to the expenditure-minimisation problem is called the consumers vector of **Hicksian demands**:

$$\begin{cases} x_1^h(p_1,p_2,u) = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_1^(r-1) \\ x_1^h(p_1,p_2,u) = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_2^(r-1) \end{cases} \tag{1}$$

Here $r \equiv \frac{\rho}{\rho-1}$.
Substitute the solution above (Equation 1) into our objective function $p_1x_1 + p_2x_2$ to obtain the Expenditure Function:

$$e(p_1,p_2,u)=p_1x_1^h(p_1,p_2,u)+p_2x_2^h(p_1,p_2,u)=u(p_1^r+p_2^r)^{\frac{1}{r}},r\equiv\frac{\rho}{\rho-1}$$

THEOREM 1.7 Properties of the Expenditure Function (JR. pp.37)

If $u(.)$ is continuous and strictly increasing, then $e(p,u)$ defined in (1.14) is

1. Zero when u takes on the lowest level of utility in $\mathscr{U}$,
2. Continuous on its domain $\mathbb{R}^n_{++}\times\mathscr{U}$,
3. For all $p\gg 0$, strictly increasing in u and unbounded above in u,
4. Increasing in p,
5. Homogeneous of degree 1 in p,
6. Concave in p.

If, in addition, $u(.)$ is strictly quasiconcave, we have

7. Shephards lemma: $e(p,u)$ is differentiable in p at (p^0,u^0) with $p^0\gg 0$, and
$$\frac{\partial e(p^0,u^0)}{\partial p_i}=x_i^h(p^0,u^0),\quad i=1,...,n.$$

Firstly we should know that the CES direct utility function $u(x_1,x_2)=(x_1^\rho+x_2^\rho)^{\frac{1}{\rho}},\ where\, 0\neq\rho<1.$ is continuous and strictly increasing (can be proved similarly as below), which is the prerequisite of Theorem 1.7.

1.1 $e(p,u)$ is zero when u takes on the lowest level of utility in $\mathscr{U}$

As usual, we assume non-negative consumption, i.e. $x\in R^n_+$. Since $u(.)$ is strictly increasing, $u_{min}=u(0,0)=0$.
When $u=0$, we have $e(p,0)=0\times(p_1^r+p_2^r)^{\frac{1}{r}}=0$.
"no consumption, no cost"

1.2 $e(p,u)$ is continuous on its domain $\mathbb{R}^n_{++}\times\mathscr{U}$,

For a ONE dimention function $f(x)$, if the derivative at $x=a$ ($f'(a)$) exists, it's differentiable at $x=a$.
If $f(x)$ is differentiable at $x=a$, it's also continuous at $x=a$.
For a higher dimension function, the existance of all partial/directional derivatives is not sufficient for its differentiability. Here are examples from Wikipedia: Differentiability in higher dimensions

Good news for your exam:
"There is no need to prove continuity (or concavity) of higher dimension functions (in your exam). I might ask you to prove continuity (or concavity) wrt one variable (a specific price, for example)." – Paolo

$e(p_1, p_2, u)$ is a 3-dimension function. If you really want to prove it's differentiable/continuous, you can either try to use the definition of differentiable/ continuous functions. You can also read theorem A2.21 (Jehle & Reny pp.602) and theorem 1.9 (point 3,Jehle & Reny pp.520).
For this sub-question, let's assume the prices are given (say, by the market). Therefore $e(p_1, p_2, u) = e(u)$ is one dimesional.
Obviously,

$$\frac{de(u)}{du} = \frac{du(p_1^r + p_2^r)^{\frac{1}{r}}}{du} = (p_1^r + p_2^r)^{\frac{1}{r}}$$

The derivetive exists for any $u \in \mathcal{U}$. $e(u)$ is differentiable and thus continuous.

1.3 For all $p \gg 0$, $e(p, u)$ is strictly increasing in u and unbounded above in u,

Since $\frac{de(u)}{du} = (p_1^r + p_2^r)^{\frac{1}{r}} > 0, \forall p \gg 0$, $e(p_1, p_2, u)$ is strictly increasing in u.
Given the positive prices, $\lim_{u\to\infty} e(u) = \lim_{u\to\infty} u(p_1^r + p_2^r)^{\frac{1}{r}} = +\infty$. It there is no upper boundary in u.

1.4 $e(p, u)$ is increasing in p,

$$\frac{\partial e(p_1, p_2, u)}{\partial p_1} = \frac{\partial u(p_1^r + p_2^r)^{\frac{1}{r}}}{\partial p_1} = u \cdot \frac{1}{r}(p_1^r + p_2^r)^{\frac{1}{r}-1} \cdot r p_1^{r-1} = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_1^{r-1} > 0$$

$$\frac{\partial e(p_1, p_2, u)}{\partial p_2} = \frac{\partial u(p_1^r + p_2^r)^{\frac{1}{r}}}{\partial p_2} = u \cdot \frac{1}{r}(p_1^r + p_2^r)^{\frac{1}{r}-1} \cdot r p_2^{r-1} = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_2^{r-1} > 0$$

Therefore $e(p, u)$ is increasing in p.

1.5 $e(p, u)$ is homogeneous of degree 1 in p,

Homogeneous functions (Jehle & Reny Definition A2.2 pp.561):
$f(x)$ is called homogeneous of degree k if $f(tx) \equiv t^k f(x),\ \forall t > 0$
You want to show $e(tp, u) = t^1 e(p, u),\ \forall t > 0$

For any $t > 0$,

$$
\begin{aligned}
e(u, tp) &= u[(tp_1)^r + (tp_2)^r]^{\frac{1}{r}} \\
&= u[t^r(p_1^r + p_2^r)]^{\frac{1}{r}} \\
&= tu(p_1^r + p_2^r)^{\frac{1}{r}} \\
&= t^1 e(u, p)
\end{aligned}
$$

Therefore $e(p, u)$ is homogeneous of degree 1 in p.

1.6 $e(p, u)$ is concave in p.

Concave functions (Jehle & Reny pp.534):
$f : D \to \mathbb{R}$ is a concave function if for all $x^1, x^2 \in D$,

$$f(x^t) \geq tf(x^1) + (1t)f(x^2), \ \forall t \in [0, 1]$$

Where $x^t \equiv tx^1 + (1-t)x^2$, $t \in [0, 1]$

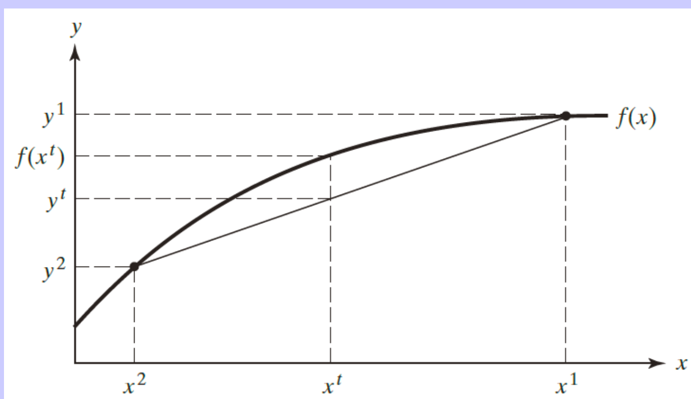

Figure 1: Concave functions

(Diminishing margin)

- Intuition: $e(p, u)$ is increasing **in** p but the marginal expenditure is diminishing.

For this specific function $e(p_1, p_2, u)$, you can follow the very smart proof on Jehle & Reny pp. 38-39 to show that for any given utility $\bar{u}$,

$$e[tp^1 + (1-t)p^2, \bar{u}] \geq te(p^1, \bar{u}) + (1-t)e(p^2, \bar{u})$$

Where $p^1 = \begin{pmatrix} p_1^1 \\ p_2^1 \end{pmatrix}$ and $p^2 = \begin{pmatrix} p_1^2 \\ p_2^2 \end{pmatrix}$ are two price sets. $t \in [0, 1]$

We can also prove it's concavity using derivatives.
For a one-dimension increasing function $f(x)$ with diminishing margin (concave towards x-axis), $f'(x) > 0$ and $f''(x) < 0$.
For a higher dimension function $e(p_1, p_2, u)$, it's natural to think about how the derivatives look like.

We call the second derivatives of a n-demension function f "**Hessian matrix**" ($H(x)$, Jehle & Reny pp.557):

$$H(x) = \begin{pmatrix} f_{11}(x) & f_{12}(x) & \cdots & f_{1n}(x) \\ f_{21}(x) & f_{22}(x) & \cdots & f_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(x) & f_{n2}(x) & \cdots & f_{nn}(x) \end{pmatrix} \tag{2}$$

We can use the following facts about $H(x)$ and concavity(More details and examples):

- **Theorem A2.4**(Jehle & Reny pp.559): If function f is twice continuously differentiable on a convex open set S, then f is concave $\iff$ the Hessian matrix $H(x)$ of f is negative semidefinite for all $x \in S$
- **Negative semidefinite**(Jehle & Reny pp.559): a $n \times n$ metrix H is negative semidefinite $\iff$ H's kth order principal minors are nonpositive for k odd and nonnegative for k even.
- The kth order principal minors of an $n \times n$ symmetric matrix H are the determinants of the $k \times k$ matrices obtained by deleting nk rows and the corresponding nk columns of H (where $k = 1, ..., n$).

The Hessian matrix of $e(p_1, p_2, u)$ (in p !) is:

$$H(p) = \begin{pmatrix} e_{p_1 p_1} & e_{p_1 p_2} \\ e_{p_2 p_1} & e_{p_2 p_2} \end{pmatrix} \tag{3}$$

$H(p)$ contains two 1st order ($k = 1, n-k = 2-1 = 1$) principal minors: $e_{p_1 p_1}$ and $e_{p_2 p_2}$, and one 2nd order ($k = 2, n-k = 2-2 = 0$) principal minors:

$$\begin{vmatrix} e_{p_1 p_1} & e_{p_1 p_2} \\ e_{p_2 p_1} & e_{p_2 p_2} \end{vmatrix} = e_{p_1 p_1} e_{p_2 p_2} - e_{p_1 p_2} e_{p_2 p_1} \tag{4}$$

We want to prove:

- $e_{p_1 p_1} \leq 0$ and $e_{p_2 p_2} \leq 0$
- $e_{p_1 p_1} e_{p_2 p_2} - e_{p_1 p_2} e_{p_2 p_1} \geq 0$

We already know $\frac{\partial e(p_1, p_2)}{\partial p_1} = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_1^{r-1}$, and $\frac{\partial e(p_1, p_2)}{\partial p_2} = u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_2^{r-1}$

$$
\begin{aligned}
\frac{\partial e(p_1,p_2)}{\partial p_1 \partial p_1} &= \frac{\partial u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_1^{r-1}}{\partial p_1} \\
&= u(\frac{1}{r}-1)(p_1^r+p_2^r)^{\frac{1}{r}-2} r p_1^{r-1} p_1^{r-1} + u(p_1^r+p_2^r)^{\frac{1}{r}-1}(r-1)p_1^{r-2} \\
&= u(p_1^r+p_2^r)^{\frac{1}{r}-1} p_1^{r-2}[(\frac{1}{r}-1)(p_1^r+p_2^r)^{-1} r p_1^r + (r-1)] \\
&= u(p_1^r+p_2^r)^{\frac{1}{r}-1} p_1^{r-2}(1-r)[\frac{p_1^r}{(p_1^r+p_2^r)} - 1] \\
&= u(p_1^r+p_2^r)^{\frac{1}{r}-2} p_1^{r-2}(r-1)p_2^r
\end{aligned}
$$

$$
\because \begin{cases} u(p_1^r+p_2^r)^{\frac{1}{r}-2} p_1^{r-2} p_2^r \geq 0 \\ r \equiv \frac{\rho}{\rho-1} = \frac{1}{1-\frac{1}{\rho}} < 1, \forall \rho < 1, \rho \neq 0 \Rightarrow r-1 \leq 0 \end{cases}
$$

$\therefore e_{p_1p_1} \leq 0$

Similarly, $e_{p_2p_2} = u(p_1^r+p_2^r)^{\frac{1}{r}-2} p_2^{r-2}(r-1)p_1^r \leq 0$ (note p_1 and p_2 are "symmetric").

$$
\begin{aligned}
\frac{\partial e(p_1,p_2)}{\partial p_1 \partial p_2} &= \frac{\partial u(p_1^r + p_2^r)^{\frac{1}{r}-1} p_1^{r-1}}{\partial p_2} \\
&= u(\frac{1}{r}-1)(p_1^r+p_2^r)^{\frac{1}{r}-2} r p_2^{r-1} p_1^{r-1} \\
&= u(1-r)(p_1^r+p_2^r)^{\frac{1}{r}-2} p_2^{r-1} p_1^{r-1}
\end{aligned}
$$

According to Young's theorem (Jehle & Reny pp.557), $\frac{\partial e(p_1,p_2)}{\partial p_1 \partial p_2} = \frac{\partial e(p_1,p_2)}{\partial p_2 \partial p_1}$

We have

$$
\begin{aligned}
e_{p_1p_1}e_{p_2p_2} - e_{p_1p_2}e_{p_2p_1} &= [u(p_1^r+p_2^r)^{\frac{1}{r}-2} p_1^{r-2}(r-1)p_2^r][u(p_1^r+p_2^r)^{\frac{1}{r}-2} p_2^{r-2}(r-1)p_1^r] \\
&\quad - [u(1-r)(p_1^r+p_2^r)^{\frac{1}{r}-2} p_2^{r-1} p_1^{r-1}]^2 \\
&= u^2(p_1^r+p_2^r)^{\frac{2}{r}-4}(p_1p_2)^{2r-2)}(r-1)^2 - u^2(1-r)^2(p_1^r+p_2^r)^{\frac{2}{r}-4}(p_2p_1)^{2r-2} \\
&= 0 \geq 0
\end{aligned}
$$

Therefore, $H(p)$ is negative semidefinite $\Rightarrow$ $e(p,u)$ is concave in p.

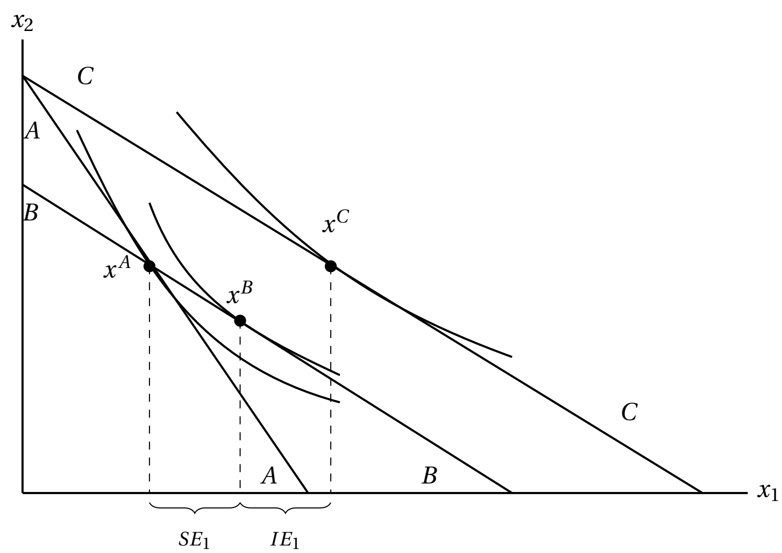

Figure 2: Slutsky decomposition

1.7 If $u(.)$ is also strictly quasiconcave, we have Shephards lemma: $e(p,u)$ is differentiable in p at (p^0,u^0) with $p^0 \gg 0$, and $\frac{\partial e(p^0,u^0)}{\partial p_i} = x_i^h(p^0,u^0), \quad i=1,...,n$.

2 Jehle & Reny 1.44 - Inferior and Normal goods

In a two-good case, show that if one good is inferior, the other good must be normal.

3 Jehle & Reny 1.51 - Substitues and Complements

Consider the utility function, $u(x_1,x_2) = (x_1)^{1/2} + (x_2)^{1/2}$.

a. Compute the demand functions, $x_i(p_1,p_2,y),\ i=1,2$.

b. Compute the substitution term in the Slutsky equation for the effects on x_1 of changes in p_2.

c. Classify x_1 and x_2 as (gross) complements or substitutes.