

Seminar 12. Auctions and previous exam problems

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The following review part is based on Jehle & Reny pp.428-432

First-price sealed-bid auction

There are N bidders in an auction, and any bidder i has a private value v_i for the auctioned good. Bidder i can submit any sealed bid b_i . We define the bid that can maximize the bidder's payoff (best response) as $b_i^* = \hat{b}(v_i)$. We assume

- Private value v_i is independent among the bidders; the CDF of v_i is $F(x)$
- Best response bidding function $\hat{b}(\cdot)$ is strictly increasing and identical for all bidders (we want to find a symmetric NE).

What is the BR $\hat{b}(v_i)$ for bidder i ?

If bidder i with $v_i = r$ submits a price higher than bidder 1's bid, we must have:

$$\hat{b}(v_1) < \hat{b}(r) \iff v_1 < r, i \neq 1$$

Therefore bidder i can guess how likely he/she defeats bidder 1,

$$Pr\{\text{bidder } i \text{ defeats bidder 1}\} = Pr\{v_1 < r\} = F(r)$$

and similarly,

$$Pr\{\text{bidder } i \text{ defeats all the other } N-1 \text{ bidders}\} = F^{(N-1)}(r)$$

Here is another interesting way to understand function $F^{(N-1)}(x)$:

Since “bidder i defeats all the other $N-1$ bidders” \iff “bidder i defeats the bidder the second-highest price”, if we denote the second-highest price as θ , we can also write the probability above as

$$Pr\{\theta < r\} = F^{(N-1)}(r) \quad (1)$$

We find that $F^{(N-1)}(x)$ is actually the CDF of θ ! We will use the conclusion later in equation 2.

The maximized expected utility function of bidder i is:

$$\begin{aligned} u(r) &= Pr\{\text{bidder } i \text{ defeats all the other } N-1 \text{ bidders}\} \times (r - \hat{b}(r)) \\ &\quad + Pr\{\text{bidder } i \text{ lose the auction}\} \times 0 \\ &= F^{(N-1)}(r)(r - \hat{b}(r)) + (1 - F^{(N-1)}(r)) \times 0 \\ &= F^{(N-1)}(r)(r - \hat{b}(r)) \end{aligned}$$

(To calculate $\hat{b}(\cdot)$) Imagine the real private value of bidder i is actually v , not r , but the bidder reacted (by mistake) as if his/her private value was r , i.e. bid at $\hat{b}(r)$ instead of $\hat{b}(v)$. When the bidder realized the mistake, the price had already been submitted...

The expected utility function due to the mistake is (eq. 9.1, Jehle & Reny pp.430):

$$u(r, v) = F^{(N-1)}(r)(v - \hat{b}(r))$$

We know only when the wrong private value r is luckily the same as the real private value v , will the expected payoff be maximized (since $\hat{b}(v)$ instead of $\hat{b}(r)$ is the best response). That is to say, $r = v$ solves $\max_r u(r, v)$.

The FOC of $\max_r u(r, v)$ is (eq. 9.2, Jehle & Reny pp.430):

$$\frac{dF^{(N-1)}(r)(v - \hat{b}(r))}{dr} = (N-1)F^{(N-2)}(r)f(r)(v - \hat{b}(r)) - F^{(N-1)}(r)\hat{b}'(r) = 0$$

Since $r = v$ solves $\max_r u(r, v)$, it must solve the FOC above, i.e. (eq. 9.3, Jehle & Reny pp.430)

$$(N-1)F(v)^{N-2}f(v)(v - \hat{b}(v)) - F^{(N-1)}(v)\hat{b}'(v) = 0$$

Which gives,

$$(N-1)F(v)^{N-2}f(v)\hat{b}(v) + F^{(N-1)}(v)\hat{b}'(v) = (N-1)F(v)^{N-2}vf(v)$$

(If we treat v as a variable:)
$$\frac{dF^{(N-1)}(v)\hat{b}(v)}{dv} = (N-1)F(v)^{N-2}vf(v)$$

$$F^{(N-1)}(v)\hat{b}(v) = \int_0^v (N-1)F^{(N-2)}(x)xf(x)dx + C$$

(C is constant)

Since a bidder with $v = 0$ must bid 0 to maximize the expected payoff, we know $\hat{b}(0) = 0$ in the NE.

$$\begin{aligned} F(0)^{N-1} \hat{b}(0) &= \int_0^0 (N-1)F^{(N-2)}(x)xf(x)dx + C \\ 0 &= 0 + C \end{aligned}$$

Thus,

$$\begin{aligned} F^{(N-1)}(v) \hat{b}(v) &= \int_0^v (N-1)F^{(N-2)}(x)xf(x)dx \\ \Rightarrow \hat{b}(v) &= \frac{\int_0^v (N-1)F^{(N-2)}(x)xf(x)dx}{F^{(N-1)}(v)} \\ &= \frac{\int_0^v x dF^{(N-1)}(x)}{F^{(N-1)}(v)} \\ &= \frac{1}{F^{(N-1)}(v)} \int_0^v x dF^{(N-1)}(x) \quad (eq.9.5) \\ &= \int_0^v x d \frac{F^{(N-1)}(x)}{F^{(N-1)}(v)} \end{aligned}$$

$\hat{b}(v)$ as the conditional expectation

Recall that in Seminar 11, we already know: for a random variable $\theta \in [a, b] \subset \mathbb{R}_1$ and $m \in (a, b)$

$$E(\theta|\theta \leq m) = \int_a^m x dF(x|\theta \leq m) = \int_a^m x d \frac{F(x)}{F(m)}$$

where $F(x) = Pr(\theta \leq x)$ is the CDF of θ .

$\hat{b}(v)$ is actually an expectation of some variable θ conditional on $\theta \leq v$, and the variable θ 's CDF is $Pr(\theta \leq x) = F^{(N-1)}(x)$. In equation 1 we already showed θ is actually the second highest price.

In conclusion, the the BR of any bidder i with private value v , is the expectation of the second highest price conditional on v is higher than the second highest price (i.e. bidder i wins):

$$\hat{b}(v) = E(\theta|\theta \leq v) \quad (2)$$

1 Jehle & Reny pp.484, exercise 9.2

Show in two ways that the symmetric equilibrium bidding strategy of a first-price auction with N symmetric bidders each with values distributed according to F , can be written as

$$\hat{b}(v) = v - \int_0^v \left(\frac{F(x)}{F(v)} \right)^{N-1} dx$$

For the first way, use our solution from the text and apply integration by parts. For the second way, use the fact that $F^{N-1}(r)(v - \hat{b}(r))$ is maximised in r when $r = v$ and then apply the envelope theorem to conclude that $d(F^{N-1}(v)(v - \hat{b}(v)))/dv = F^{N-1}(v)$; now integrate both sides from 0 to v .

Method 1: Integration by parts

Integration by parts:

$$\int_a^b u(x) \cdot v'(x) dx = [u(x) \cdot v(x)]_a^b - \int_a^b u'(x) \cdot v(x) dx$$

We start from the equation above equation (9.5) on Jehle & Reny pp.431, let's call it equation 3,

$$\hat{b}(v) = \frac{(N-1)}{F^{(N-1)}(v)} \int_0^v x f(x) F^{(N-2)}(x) dx \quad (3)$$

Denote,

$$\begin{cases} u(x) = x \\ v(x) = \frac{F^{(N-1)}(x)}{(N-1)} \end{cases} \Rightarrow \begin{cases} u'(x) = 1 \\ v'(x) = F^{(N-2)}(x) f(x) \end{cases}$$

Equation 3 can be rewritten as,

$$\begin{aligned} \hat{b}(v) &= \frac{(N-1)}{F^{(N-1)}(v)} \int_0^v x f(x) F^{(N-2)}(x) dx \\ &= \frac{(N-1)}{F^{(N-1)}(v)} \int_0^v u(x) \cdot v'(x) dx \\ \text{(Integration by parts:)} \quad &= \frac{(N-1)}{F^{(N-1)}(v)} \left\{ [u(x) \cdot v(x)]_0^v - \int_0^v u'(x) \cdot v(x) dx \right\} \\ &= \frac{(N-1)}{F^{(N-1)}(v)} \left\{ \left[x \cdot \frac{F^{(N-1)}(x)}{(N-1)} \right]_0^v - \int_0^v 1 \cdot \frac{F^{(N-1)}(x)}{(N-1)} dx \right\} \\ &= \frac{(N-1)}{F^{(N-1)}(v)} \left\{ v \cdot \frac{F^{(N-1)}(v)}{(N-1)} - \frac{1}{N-1} \int_0^v F^{(N-1)}(x) dx \right\} \\ &= v - \int_0^v \left[\frac{F(x)}{F(v)} \right]^{(N-1)} dx \end{aligned}$$

Method 2: Envelop theorem

Envelop theorem (unconstraint case):

$f(x; a)$ is a function of x with a as parameter. Given any a , let x^* be the solution maximizing or minimizing object function $f(x, a)$, i.e. $\max_x f(x, a) = f(x^*, a)$, then

$$\frac{df(x^*, a)}{da} = \frac{f(x, a)}{da} \Big|_{x=x^*}$$

We start from equation (9.1) on Jehle & Reny pp.430 (the one called as “expected utility function due to the mistake” in the review part),

$$u(r, v) = F^{(N-1)}(r)(v - \hat{b}(r))$$

We know $r^* = v$ maximizes $u(r, v)$, i.e.

$$\max_r u(r, v) = u(r^*, v) = F^{(N-1)}(v)(v - \hat{b}(v)) \quad (4)$$

On the other hand, by envelop theorem,

$$\begin{aligned} \frac{du(r^*, v)}{dv} &= \frac{f(r, v)}{dv} \Big|_{r=v} \\ &= F^{(N-1)}(r) \Big|_{r=v} \\ &= F^{(N-1)}(v) \end{aligned}$$

we find another way to express $u(r^*, v)$:

$$u(r^*, v) = \int_0^v F^{(N-1)}(x) dx \quad (5)$$

Therefore, by equation 4 and equation 5,

$$\begin{aligned} F^{(N-1)}(v)(v - \hat{b}(v)) &= \int_0^v F^{(N-1)}(x) dx \\ \hat{b}(v) &= v - \int_0^v \left[\frac{F(x)}{F(v)} \right]^{(N-1)} dx \end{aligned}$$

2 Jehle & Reny pp.484, exercise 9.1 - Show that the bidding strategy in (9.5) is strictly increasing.

By exercise 9.1 we know,

$$\hat{b}(v) = v - \frac{\int_0^v F^{N-1}(x) dx}{F^{N-1}(v)}.$$

Then:

$$\begin{aligned}
\frac{d}{dv}(\hat{b}(v)) &= \frac{d}{dv} \left(v - \frac{\int_0^v F^{N-1}(x) dx}{F^{N-1}(v)} \right) \\
&= 1 - \frac{\frac{\int_0^v F^{N-1}(x) dx}{dv} \cdot [F^{N-1}(v)] - [\int_0^v [F^{N-1}(x)] dx] \cdot \frac{F^{N-1}(v)}{dv}}{[F^{N-1}(v)]^2} \\
&= 1 - \frac{F^{N-1}(v) \cdot [F^{N-1}(v)] - [\int_0^v [F^{N-1}(x)] dx] \cdot [(N-1)F^{N-2}(v)f(v)]}{F^{2N-2}(v)} \\
&= 1 - 1 + \frac{[\int_0^v [F^{N-1}(x)] dx] \cdot [(N-1)f(v)]}{F^N(v)} \\
&= \frac{[\int_0^v [F^{N-1}(x)] dx] \cdot [(N-1)f(v)]}{F^N(v)}
\end{aligned}$$

If we assume $F(x)$ is strictly increasing and $N > 1$, then $\hat{b}(v) > 0$.

3 Jehle & Reny pp.485, exercise 9.3

This exercise will guide you through the proof that the bidding function in (9.5) is in fact a symmetric equilibrium of the first-price auction.

(a) Another way to write the derivative

Recall from (9.2) that

$$\frac{du(r, v)}{dr} = (N-1)F^{N-2}(r)f(r)(v - \hat{b}(r)) - F^{N-1}(r)\hat{b}'(r).$$

Using (9.3), show that

$$\begin{aligned}
\frac{du(r, v)}{dr} &= (N-1)F^{N-2}(r)f(r)(v - \hat{b}(r)) - (N-1)F^{N-2}(r)f(r)(r - \hat{b}'(r)) \\
&= (N-1)F^{N-2}(r)f(r)(v - r)
\end{aligned}$$

Equation 9.3 on Jehle & Reny pp.430 is,

$$(N-1)F(v)^{N-2}f(v)\hat{b}(v) + F^{(N-1)}(v)\hat{b}'(v) = (N-1)F(v)^{N-2}vf(v)$$

Equation 9.3 holds for any v , since v is the variable. It will also hold if we take $v = r$, i.e.

$$(N-1)F^{N-2}(r)f(r)\hat{b}(r) + F^{(N-1)}(r)\hat{b}'(r) = (N-1)F^{N-2}(r)rf(r)$$

which yields,

$$\begin{aligned}
F^{(N-1)}(r)\hat{b}'(r) &= (N-1)F^{N-2}(r)rf(r) - (N-1)F^{N-2}(r)f(r)\hat{b}(r) \\
&= (N-1)F^{N-2}(r)f(r)[r - \hat{b}(r)]
\end{aligned}$$

Substituting the result above into equation 9.2 yields,

$$\begin{aligned}\frac{du(r, v)}{dr} &= (N-1)F^{N-2}(r)f(r)(v - \hat{b}(r)) - F^{N-1}(r)\hat{b}'(r) \\ &= (N-1)F^{N-2}(r)f(r)(v - \hat{b}(r)) - (N-1)F^{N-2}(r)f(r)[r - \hat{b}(r)] \\ &= (N-1)F^{N-2}(r)f(r)(v - r)\end{aligned}$$

(b) Minimum or maximum?

Use the result in part (a) to conclude that $du(r, v)/dr$ is positive when $r < v$ and negative when $r > v$, so that $u(r, v)$ is maximised when $r = v$.

$$\frac{du(r, v)}{dr} = (N-1)F^{N-2}(r)f(r)(v - r)$$

Obviously, if we assume $F(x)$ is strictly increasing and $N > 1$, $\frac{du(r, v)}{dr} > 0$ when $r < v$ and $\frac{du(r, v)}{dr} < 0$ when $r > v$. Therefore $r = v$ maximizes $u(r, v)$.

4 Problems 2 of the exam in ECON4240, Spring 2005

Consider a strategic situation between an employer (E) and a worker (W). E can either accept (A) or reject (R) W . W can either become skilled (S) through education, or remain unskilled (U). W can be of two types; either he is inherently high ability (H) or he is inherently low ability (L). The players' payoffs depending on their actions and W 's type is shown below.

		H				L	
		S	U			S	U
E	A	2, 3	-1, 2	E	A	-1, -1	-3, -2
	R	0, 1	0, 0		R	0, -3	0, 0

a) Rationalizable strategies & NE

For each of these games, determine the set of (pure) rationalizable strategies for each player, and the set of pure-strategy Nash equilibria.

In Game H : Only S and A are rationalizable. (S, A) is the unique NE.

In Game L : Only U and R are rationalizable. (U, R) is the unique NE.

b) Bayesian normal form

Assume next that only W knows his own type, while player E thinks that the two types of W are equally likely. Model this situation in an ex ante perspective by specifying the Bayesian normal form.

Denote the Worker's contingent choices are: If Game H , S and U ; If Game L , S' and U' . The Bayesian normal form is:

	SS'	SU'	US'	UU'
A	$\frac{1}{2}, 1$	$-\frac{1}{2}, \frac{1}{2}$	$-1, \frac{1}{2}$	$-2, 0$
R	$0, -1$	$0, \frac{1}{2}$	$0, -\frac{3}{2}$	$0, 0$

c) Rationalizable strategy & BNE

For the Bayesian normal form found in part (b), determine the set of (pure) rationalizable strategies for each player, and the set of pure-strategy and/or mixed-strategy Nash equilibria.

	SS' _p	SU' _{1-p}	US'	UU'
A _q	$\frac{1}{2}, 1$	$-\frac{1}{2}, \frac{1}{2}$	$-1, \frac{1}{2}$	$-2, 0$
R _{1-q}	$0, -1$	$0, \frac{1}{2}$	$0, -\frac{3}{2}$	$0, 0$

(1) Rationalizable strategy

For the Worker, US' is dominated by SS' ; UU' is dominated by SU' . The rationalizable strategies are SS and SU' .

For the Employer, both A and R are rationalizable.

(2) Pure-strategy NE

$$(A, SS'), (R, SU')$$

(3) Mixed-strategy NE

Since US' and UU' are dominated, the Employer believes the Worker will choose them with probability $(0,0)$.

The Employer believes the Worker chooses SS' and SU' with probability $(p, 1-p)$; The Worker believe the Employer chooses A and R with probability $(q, 1-q)$.

$$E(U_A^E) = p \times 0.5 + (1 - p) \times (-0.5) = p - 0.5$$

$$E(U_R^E) = p \times 0 + (1 - p) \times 0 = 0$$

$$E(U_A^E) = E(U_R^E) \Rightarrow p = 0.5$$

$$E(U_{SS'}^W) = q \times 1 + (1 - q) \times (-1) = 2q - 1$$

$$E(U_{SU'}^W) = q \times 0.5 + (1 - q) \times 0.5 = 0.5$$

$$E(U_{SS'}^W) = E(U_{SU'}^W) \Rightarrow 2q - 1 = 0.5 \Rightarrow \frac{3}{4}$$

The Mixed-strategy NE is: the Employer chooses A and R with probability $(0.5, 0.5)$; Worker chooses SS' and SU' with probability $(\frac{3}{4}, \frac{1}{4})$.

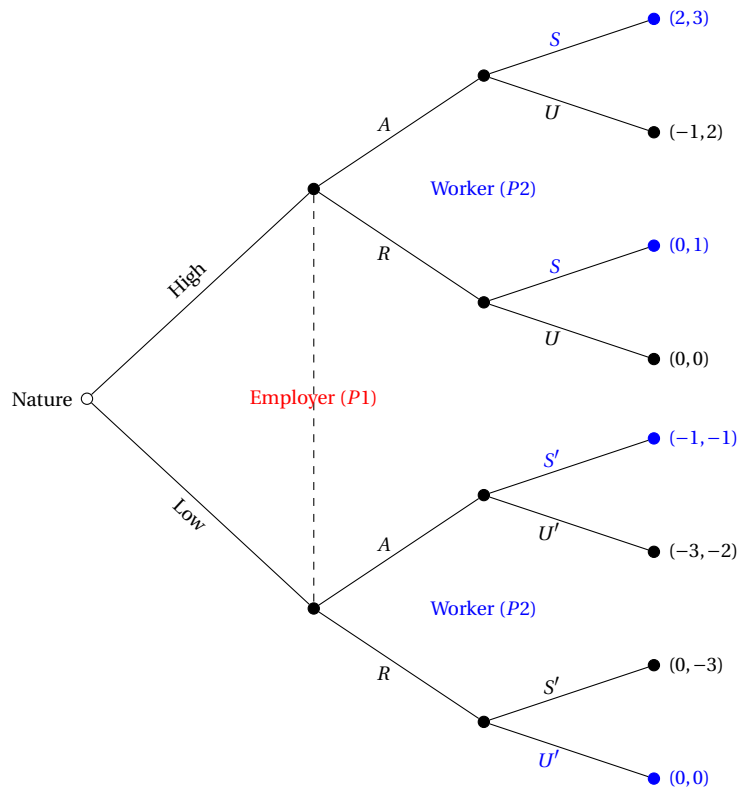
5 Problems 3 of the exam in ECON4240, Spring 2005

Problem 3 (20 %) Consider again the strategic situation between an employer (E) and a worker (W) described in Problem 2. Assume (as in parts b and c) of Problem 2) that only W knows his own type, while player E thinks that the two types of W are equally likely.

a) (Screening)

Assume now that E acts before W , and that E 's choice of A or R can be observed by W before he makes his choice of S or U . Show that there is a unique subgame perfect Nash equilibrium.

Extensive Form:



The strategy of the *Worker*:

- If High:
 - when E chooses A , W chooses S
 - when E chooses R , W chooses S
- If Low:
 - when E chooses A , W chooses S'
 - when E chooses R , W chooses U'

For the *Employer*:

$$\begin{aligned}
 E(U_A^E) &= \frac{1}{2} \times 2 + \frac{1}{2} \times (-1) = 0.5 \\
 E(U_R^E) &= \frac{1}{2} \times 0 + \frac{1}{2} \times 0 = 0 \\
 &\Rightarrow E(U_A^E) > E(U_R^E)
 \end{aligned}$$

The strategy of the *Employer* is to choose A .

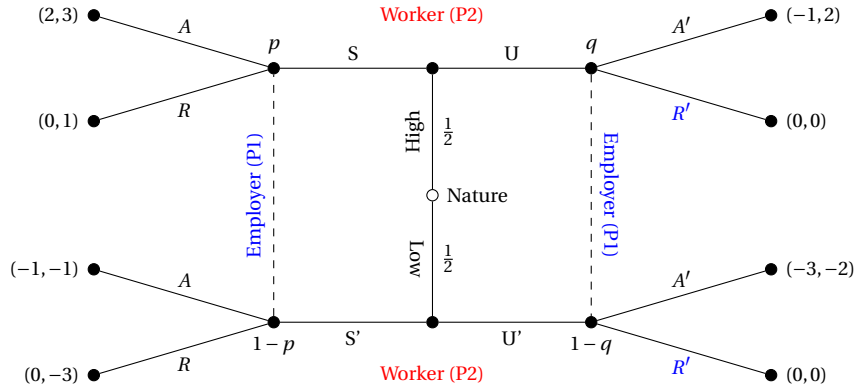
Therefore there is a SPNE:

{(For High ability Worker:) S after A , S after R . (For Low ability Worker:) S' after A , U' after R ; (For Employer) A }

b) (Signaling)

Assume now that W acts before E , and that W 's choice of S or U can be observed by E before she makes her choice of A or R . Show that there is a unique perfect Bayesian equilibrium.

Extensive Form:



The Worker has 4 possible strategies: SS', UU', SU', US' .

Denote the updated belief of the Worker :

- $Pr(\text{Strong}|P/P') = p$
- $Pr(\text{Weak}|U/U') = q$

(1) When the Employer believes the Worker chooses SS'

Then $p = Pr(\text{High}) = 0.5$,

$$E[U_A^E] = 0.5 \times 2 + 0.5 \times (-1) = 0.5$$

$$E[U_R^E] = 0.5 \times 0 + 0.5 \times 0 = 0$$

$$E[U_A^E] > E[U_R^E]$$

The BR of the Employer is to choose A .

If the Worker surprisingly chooses U/U' , we can see that R' dominates A' for the Employer. Therefore, the Low ability Worker will deviate from S' to U' to have a higher payoff (U' and R' leads to (0, 0))

$\Rightarrow SS'$ is not part of a PBE.

(2) When the Employer believes the Worker chooses UU'

Since R' dominates A' , the Employer will always choose R' . For a High ability Worker, deviating from U to S will always lead to higher payoff (either 3 or 1), no matter how the Employer reacts.

$\Rightarrow UU'$ is not part of a PBE.

(3) When the Employer believes the Worker chooses SU'

Then $p = 1, q = 0$,

The BR of the Employer is to choose A after S/S' (payoff is be $(2,3)$) and R' after U/U' (payoff is be $(0,0)$).

If the High ability Worker eviates to U , the payoff is $(0,0)$, lower than $(2,3)$; If the Low ability Worker eviates to S , the payoff is $(-1,-1)$, lower than $(0,0)$;

There is no incentive for the Worker to deviate. $\Rightarrow SU'$ is part of a PBE.

(4) When the Employer believes the Worker chooses US'

Then $p = 0, q = 1$,

The BR of the Employer is to choose A after S/S' and R after U/U' .

For a Low ability Worker, deviating from S' to U' will increase the payoff from $(-1,-1)$ to $(0,0)$.

$\Rightarrow US'$ is not part of a PBE.

In conclusion, there is only one PBE:

$$\{SU', AR', p = 1, q = 0\}$$