Seminar 10. Incomplete Information in Dynamic Games

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November 19, 2020

1 Problem 1 - Screening and signaling

Consider again the strategic situation described in Problem 1 of the exercise set for the nineth seminar,

Game 1

	L	R
U	0,0	4,2
D	2,6	0,8

Game 2

	L	R
U'	0,2	0,0
D'	2,0	2,2

where only player 1 knows which game is being played, while player 2 thinks that the two games are **equally likely**.

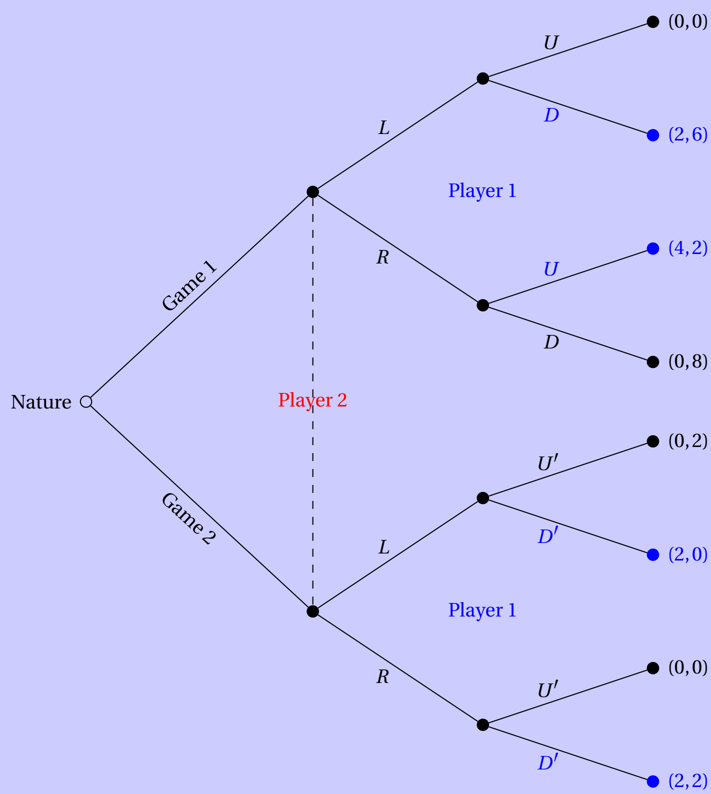
(a) Screening

Assume now that **player 2 acts before player 1**, and that 2's choice can be observed by 1 before he makes his choice. Show that there is a unique subgame perfect Nash equilibrium.

Screening: Player **without** private information moves first $\Rightarrow$ there is nothing to infer (since player 1 knows everything, and player 2 has no chance to infer anything)

- Player 1 has private information: contingent strategy
- Player 2 acts before player 1: can only choose between L and R

Extensive form:



Player 1 acts contingently; Player 2 acts according to the expected payoff based on some belief $(\frac{1}{2}, \frac{1}{2})$;

- Use backward induction method to firstly determin how player 1 will react (in blue) and then calculate player 2's expected payoff.
- We can also find when Game 2 is played, for player 1, D' dominates U'

The contingent strategy for player 1 is easy to express, since there is no incomplete information.

The strategy of player 1:

- If Game 1 is played:
  - when player 2 chooses L, player 1 chooses D
  - when player 2 chooses R, player 1 chooses U
- If Game 2 is played:
  - when player 2 chooses L, player 1 chooses D'

- when player 2 chooses R, player 1 chooses D'

For player 2:

$$E(U_L^2) = \frac{1}{2} \times 6 + \frac{1}{2} \times 0 = 3$$
$$E(U_R^2) = \frac{1}{2} \times 2 + \frac{1}{2} \times 2 = 2$$
$$\Rightarrow E(U_L^2) > E(U_R^2)$$

The strategy of player 2 is to choose L.

Therefore there is only one SPNE:
{(For player 1) In Game 1: D after L, U after R. In Game 2, D' after L, D' after R ; (For player 2) L}

Compare the question with the 2019 exam problem 2(e).

Don't get tricked when the question says "U acts before I" but "U can choose between the four strategies CC', CE', EC' and EE' "

In the question, U's strategy is like an agreement, which is passive and can only be activated by the choice of I; while only I has contingent strategy.

(b) Signaling

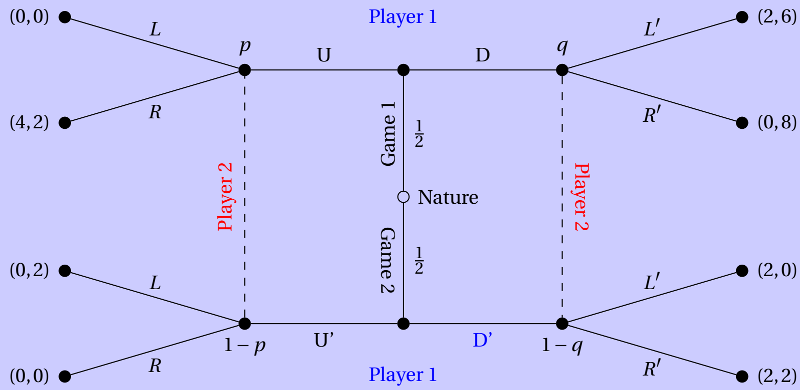
Assume now that player 1 acts before player 2, and that 1's choice can be observed by 2 before she makes her choice. Show that there is a unique separating perfect Bayesian equilibrium.

Signaling: Player **with** private information moves first $\Rightarrow$ Player 2 can observe and infer the "type" of player 1.

- Both players have contingent strategy (Geir prefers not to distinguish player 2's contingent strategy in notation)
- Player 2 observes U/U' or D/D' $\Rightarrow$ updates belief (p&q) $\Rightarrow$ choose between L and R

where player 2's updated belief is:

- $Pr(\text{Game1}|U/U') = p$
- $Pr(\text{Game1}|D/D') = q$



Note that for player 1, when Game 2 is played, D' dominates U'. Therefore player 1 can only have 2 possible strategies: UD' (if separating) and DD'(if pooling).

The question wants you to show there is a unique separating PBE. So you only need to show UD' can be part of a PBE.

(1) Separating PBE

If player 2 believes UD' is the strategy of player 2, then

$$p = 1, q = 0$$

and the BR of player 2 is RR'.

Will player 1 deviate from U to D if Game 1 is played? No, if player 1 deviates to D, according to player 2's belief (player 2 believes Game 2 is played and thus chooses R'), the payoff is $(0,8)$, lower than $(4,2)$ if not deviate.

This is therefore a unique separating PBN:

$$\{UD', RR', p = 1, q = 0\}$$

(don't forget the belief supporting the PBN)

(2) Pooling PBE

There actually can also be a pooling PBE.

If player 2 believes DD' is the strategy of player 2, then

$$q = Pr(\text{Game 1}) = \frac{1}{2}$$

You can apply the conclusion $q = Pr(\text{Game 1})$ in a pooling PBE directly, but this is acutally consistent with Bayes' rule:

$$Pr(\text{Game1}|D/D') = \frac{Pr(D/D'|\text{Game1})Pr(\text{Game1})}{Pr(D/D')}$$
$$= \frac{1 \times \frac{1}{2}}{1}$$

Therefore, when observing D/D', player 2's expected payoff is:

$$E[U^2_{L'}] = \frac{1}{2} \times 6 + \frac{1}{2} \times 0 = 3$$
$$E[U^2_{R'}] = \frac{1}{2} \times 8 + \frac{1}{2} \times 2 = 5$$

Player 2's BR is R'.

Will player 1 deviate from D to U if Game 1 is played? It depends on what player 2 will do if the "surprise" shows up.

If player 2 chooses R after U, player 1 will deviate; if player 2 chooses L after U, there is no incentive to deviate.

Then under what condition will player 2 chooses L after U? Expected payoff!

$$E[U^2_L] = p \times 0 + (1-p) \times 2 = 2(1-p)$$
$$E[U^2_R] = p \times 2 + \frac{1}{2} \times 0 = 2p$$

$$E[U^2_L] \geq E[U^2_R] \Rightarrow 2(1-p) \geq 2p \Rightarrow p \leq \frac{1}{2}$$

To support the pooling PBE, we must have $p \leq \frac{1}{2}$. Note here since U is not expected by player 2, we don't need to think about Bayes' rule.

The pooling PBE is:

$$\{DD', LR', p \leq \tfrac{1}{2}, q = \tfrac{1}{2}\}$$

However, this PBE is not realistic. Since player 2 knows clearly that if Game 2 is played, U' will never be chosen by player 1. Therefore $1 - p = 0, p = 1 \geq \frac{1}{2}$.

We can say that the PBE here is consistent with the definition of PBE, but unrealistic. This is a defect of PBE. In exam you'll often be asked about which PBE (either pooling or separating) is unrealistic/unlikely to happen.

The following is how to answer the question in exam:

(1) Draw the extensive form first

Important. Be careful about the order of payoff, who is player 1 and who is player 2.

From the extensive form we can see D' dominates U'. Therefore there can only be one separating PBE where player 1's strategy is UD'

(2) Denote the updated belief of player 2:

- $Pr(\text{Game1}|U/U') = p$
- $Pr(\text{Game1}|D/D') = q$

For the separating PBE, $p = 1, q = 0$. The BR of player 2 is RR'. Therefore the PBE is

$$\{UD', RR', p = 1, q = 0\}$$

2 Problem 2 - Sequential moves and incomplete information; Perfect Bayesian equilibrium

Consider the situation of Problem 1 of the eighth seminar, but assume now in addition that the pizza comes in 5 different sizes, each with x slices, where $x \in \{4, 6, 8, 10, 12\}$. **Player 1 observes** x before making her demand, while players 2 only observes player 1's demand, but not x, before having to make his own demand. Before observing player 1's demand, player 2 thinks that the 5 different pizza sizes are equally likely, but he may infer something from her demand.

(a) Explain what a strategy is for player 1 in this game of incomplete information.

Player 1's strategy is contingent on the size of pizza (x), i.e. $s_1(x)$, where $x \in \{4, 6, 8, 10, 12\}$.

(b) Perfect Bayesian equilibrium

Show that the following strategy for player 1 can be part of a perfect Bayesian equilibrium: $s_1(4) = 2$, $s_1(6) = 3$, $s_1(8) = 4$, $s_1(10) = 5$, $s_1(12) = 11$. Specify both player 2's strategy and player 2's beliefs.

The strategy can only be part of a separating PBE. In this case, player 2 believes that player 1 sends signals by taking half the pizza, i.e. he thinks the size is 4, 6, 8, 10, 12 once he observes player 1 asked for 2, 3, 4, 5, 11.

Denote player 2's belief (probability on sizes 4, 6, 8, 10, 12) as $\mu(x)$, and the BR of player 2 as s_2^*, we have:

	$\mu(x)$ for $x =$					
s_1	4,	6,	8,	10	12	s_2^*
2	(1,	0,	0,	0,	0)	2
3	(0,	1,	0,	0,	0)	3
4	(0,	0,	1,	0,	0)	4
5	(0,	0,	0,	1,	0)	5
11	(0,	0,	0,	0,	1)	1

What if player 1 surprisingly asked for, for example, 6 slices? As long as the payoff is lower than PBE result, there is no deviation. The following belief can make sure so:

	$\mu(x)$ for $x =$					
s_1	4,	6,	8,	10	12	s_2^*
0	(1,	0,	0,	0,	0)	4
1	(1,	0,	0,	0,	0)	3
2	(1,	0,	0,	0,	0)	2
3	(0,	1,	0,	0,	0)	3
4	(0,	0,	1,	0,	0)	4
5	(0,	0,	0,	1,	0)	5
6	(0,	0,	0,	0,	1)	6
7	(0,	0,	0,	0,	1)	5
8	(0,	0,	0,	0,	1)	4
9	(0,	0,	0,	0,	1)	3
10	(0,	0,	0,	0,	1)	2
11	(0,	0,	0,	0,	1)	1
12	(0,	0,	0,	0,	1)	≥ 1

For example, if the size $x = 10$, on the equilibrium path, player 1 should ask for 5. But if player 1 surprisingly asks for 6, according to player 2's belief (player 2 believes the size is 12), player 2 asks for 6 and the payoff is $(0, 0)$. There is no incentive to deviate.

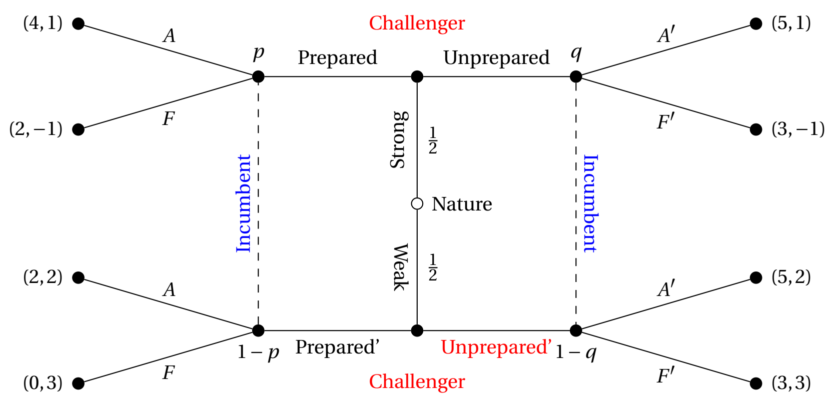
(c) Are there other perfect Bayesian equilibria in this game?

Here is another example of PBE: player 2 thinks that player 1 sends signal by taking $x-1$ slices of pizza; if player 2 thinks player 1 surprisingly took the whole pizza, he will ruin the pizza to punish player 1. There is no incentive to deviate.

	$\mu(x)$ for $x =$					
s_1	4,	6,	8,	10	12	s_2^*
0	(1,	0,	0,	0,	0)	4
1	(1,	0,	0,	0,	0)	3
2	(1,	0,	0,	0,	0)	2
3	(1,	0,	0,	0,	0)	1
4	(1,	0,	0,	0,	0)	≥ 1
5	(0,	1,	0,	0,	0)	1
6	(0,	1,	0,	0,	0)	≥ 1
7	(0,	0,	1,	0,	0)	1
8	(0,	0,	1,	0,	0)	≥ 1
9	(0,	0,	0,	1,	0)	1
10	(0,	0,	0,	1,	0)	≥ 1
11	(0,	0,	0,	0,	1)	1
12	(0,	0,	0,	0,	1)	≥ 1

3 Problem 3 - Challenging an incumbent

Consider a market where there is an incumbent firm and a challenger. The challenger is *strong* with probability $\frac{1}{2}$ and *weak* with probability $\frac{1}{2}$; it knows its type, but the incumbent does not. The challenger may either *prepare* itself for battle or remain *unprepared*. The incumbent observes the challenger's preparedness, but not its type, and chooses whether to *fight* (F) or *acquiesce* (A). The extensive form and the payoffs are given by the following figure. The challenger's payoff is listed first, the incumbent's second.



(a) What are the (pure) strategies for the challenger?

The challenger's strategy set: $\{PP', PU', UP', UU'\}$.

Contingent on the type. For example, PU' means "Prepared if I'm strong, Unprepared if I'm weak".

(b) Why is there no perfect Bayesian equilibrium where the weak challenger chooses *Prepared′* ?

Prepared′ is dominated by Unprepared′. It is always better for the weak challenger to choose U', independently of what the incumbent does.

Is Prepared dominated by Unprepared (for strong challenger)? No! If the Incumbent has (contingent) strategy AF', Prepared has better payoff.

(c) Separating

Show that there is a perfect Bayesian equilibrium where the strong challenger chooses *Prepared* and the weak challenger chooses *Unprepared′*.
Denote the updated belief of player 2:

- $Pr(\text{Strong}|P/P') = p$
- $Pr(\text{Weak}|U/U') = q$

For the separating PBE, $p = 1, q = 0$. The BR of the incumbent is AF'. Therefore the PBE is

$$\{PU', AF', p = 1, q = 0\}$$

Will the challenger deviate?

(d) Pooling

Show that there is a perfect Bayesian equilibrium where the strong challenger chooses *Unprepared* and the weak challenger chooses *Unprepared′*. What do we call such an equilibrium?
For the pooling PBE, $q = Pr(\text{Strong}) = \frac{1}{2}$ according to Bayes' rule.

$$E[U^I_{A'}] = \frac{1}{2} \times 1 + \frac{1}{2} \times 2 = 1.5$$

$$E[U^I_{F'}] = \frac{1}{2} \times (-1) + \frac{1}{2} \times 3 = 1$$

The BR of player 2 is A'. For a strong challenger, choosing U will lead to payoff result $(5, 1)$. Higher than deviating to P no matter what the incumbent does. There can be 2 pooling PBE:

(1) When the challenger surprisingly chooses P/P', the incumbent chooses A:

$$E[U^I_A] = p \times 1 + (1 - p) \times 2 = p + 2(1 - p) = 2 - p$$

$$E[U^I_F] = p \times (-1) + \frac{1}{2} \times 3 = -p + 3(1-p) = 3 - 4p$$

$$E[U^I_A] \geq E[U^I_F] \Rightarrow 2 - p \geq 3 - 4p \Rightarrow p \geq \frac{1}{3}$$

The PBE is

$$\{UU', AA', p \geq \frac{1}{3}, q = \frac{1}{2}\}$$

(2) When the challenger surprisingly chooses P/P', the incumbent chooses F:

$$E[U^I_A] \leq E[U^I_F] \Rightarrow 2 - p \leq 3 - 4p \Rightarrow p \leq \frac{1}{3}$$

The PBE is

$$\{UU', FA', p \leq \frac{1}{3}, q = \frac{1}{2}\}$$

Which pooling PBE is more likely to happen? Why?