Seminar 4. Elasticity of Substitution

Xiaoguang Ling
xiaoguang.ling@econ.uio.no

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This review part is to help you to understand Elasticity of Substitution step by step. It contains the following key points:

1. The |slop| of indifference curve is **Marginal Rate of Substitution** (MRS);
2. MRS shows the relative importance of the two commodities;
3. The relationship between MRS_{12} and $\frac{x_2}{x_1}$ shows the **substitution relationship** between the two commodities;
4. The relationship between MRS_{12} and $\frac{x_2}{x_1}$ can be expressed by **Elasticity of Substitution** σ_{12}

After this part you will understand:

1. $\sigma_{12} = 0 \iff$ Leotief preference
2. $\sigma_{12} = constant \iff$ CES
3. $\sigma_{12} = +\infty \iff$ Perfect substitution

Note that I will use MRS instead of MRTS by default in this part, because we have spent more time on consumer theory and MRS can be easily generalized to MRTS.

1 Substitution along an indifference curve

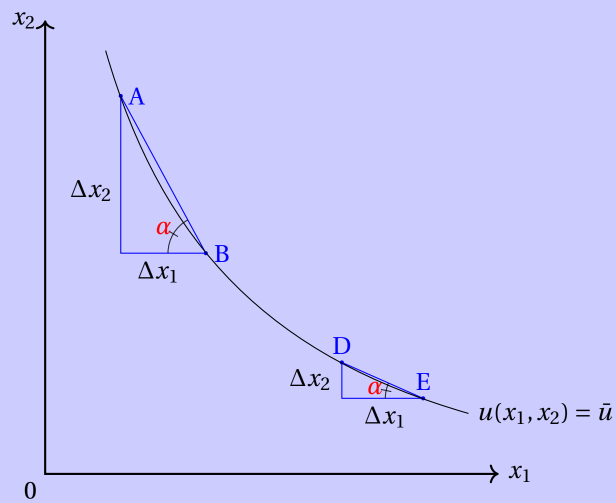

Figure 1: You have to give up Δx_2 to consume 1 more Δx_1

When we move along $u(x_1, x_2) = \bar{u}$, we can observe the following facts:

- The small increase of x_1 (i.e. Δx_1) is always followed by some small decrease of x_2 (i.e. Δx_2). Angle α reflects how much you have to give up (substitute).
- $\alpha = -\frac{\Delta x_2}{\Delta x_1}$ depends on the relative amount of x_1 and x_2, i.e. $\frac{x_2}{x_1}$ (Note Δx_2 is negative here).

Intuition: To keep utility the same, you need to give up Δx_2 to consume Δx_1.

Two questions:

- How to calculate $\alpha = -\frac{\Delta x_2}{\Delta x_1}$?
- How to describe the relationship between α and $\frac{x_2}{x_1}$?

2 Marginal Rate of Substitution (MRS)

When x_1 increases one very small unit, we define $\angle\alpha$ as Marginal Rate of Substitution (MRS).

$$MRS = \alpha = -\frac{\Delta x_2}{\Delta x_1}$$

We can calculate α in the following 2 ways:

Method 1 (2-D thinking)
When line segment $A-B$ and $D-E$ are extremely short, $-\alpha$ is simply the slope of the indifference curve.
Similar to Jehle & Reny 1.27 in seminar 1, an indifference curve can be seen as the graph of a function $x_2 = f_(x_1)$ given some utility $\bar{u}$. Thus $\alpha = -\frac{dx_2}{dx_1}$.

Method 2 (3-D thinking)
Given $u(x_1, x_2)$ is a differentiable function (recall the hill-like 3-D graph I showed you). The small change of $u(x_1, x_2)$, i.e. $\Delta u(x_1, x_2)$, can always be attributed to the small change of x_1 and x_2. "Marginal utility", $\frac{\partial u(x_1,x_2)}{\partial x_1}$ and $\frac{\partial u(x_1,x_2)}{\partial x_2}$, show how "effective" the two commodities are :

$$\Delta u(x_1, x_2) = \frac{\partial u(x_1, x_2)}{\partial x_1}\Delta x_1 + \frac{\partial u(x_1, x_2)}{\partial x_2}\Delta x_2$$

(See also Total derivative.)
Now, let's keep $u(x_1, x_2) = \bar{u}$, i.e. $\Delta u(x_1, x_2) = 0$, we have

$$0 = \frac{\partial u(x_1, x_2)}{\partial x_1}\Delta x_1 + \frac{\partial u(x_1, x_2)}{\partial x_2}\Delta x_2$$

$$\alpha = -\frac{\Delta x_2}{\Delta x_1} = \frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2}$$

Intuition: The more important x_1 is, the more x_2 you'd like to give up.

Another virtue of the 3-D thinking is that it can be easily generalized to many-dimension problems. Given utility the same, **to consume more commodity i**, how much commodity j must you give up? This is called the **Marginal Rate of Substitution of good j for good i**:

$$MRS_{ij}(x) = \frac{\partial u(x)/\partial x_i}{\partial u(x)/\partial x_j}$$

Similarly, in the case of Production theory, given the quantity of production the same (along an isoquant), **to increase input i**, how much input j must be decreased? This is called the **Marginal Rate of Technical Substitution of input j for input i**:

$$MRTS_{ij}(x) = \frac{\partial f(x)/\partial x_i}{\partial f(x)/\partial x_j}$$

Here the word "Technical" refers to the technology $f(x)$.

3 α and $\frac{x_2}{x_1}$

The relationship between α and $\frac{x_2}{x_1}$ also reflects the nature of the two commodities. Note that we can use angles to express $\frac{x_2}{x_1}$. For example, in Figure 2, $\angle AOB = \frac{AB}{OB} = \frac{x_2^1}{x_1^1}$. Similarly, $\angle COD = \frac{CD}{OD} = \frac{x_2^2}{x_1^2}$.

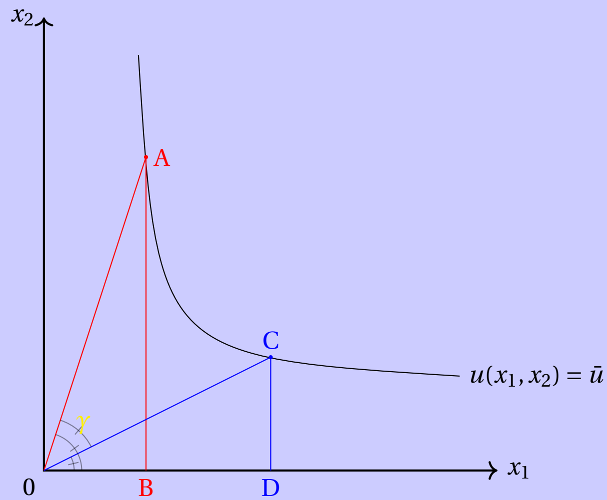

Figure 2: MRS changes much

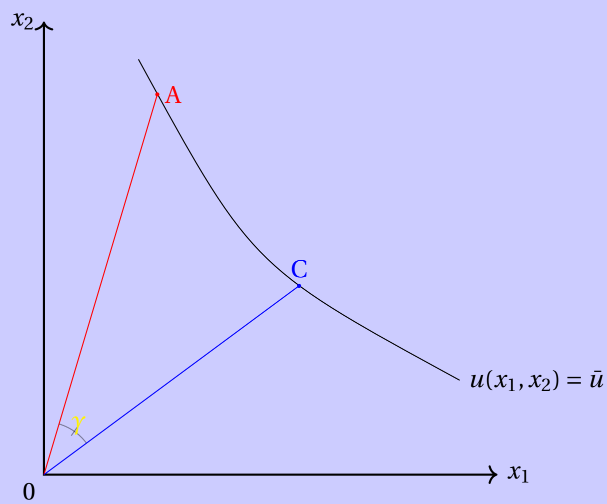

Figure 3: MRS changes a little

In Figure 2 and Figure 3, the change of $\frac{x_2}{x_1}$ can be expressed by $\angle\gamma$. When $\frac{x_2}{x_1}$ changes γ:

- in Figure 2, the "|slope|" MRS changes a lot.
- in Figure 3, the "|slope|" MRS changes a little.

Now let's think about an extreme case. In Figure 4, when $\frac{x_2}{x_1}$ changes γ, MRS does not change ($MRS_{12}(x) = \frac{\partial u(x)/\partial x_1}{\partial u(x)/\partial x_2} = 0.5$). That is, MRS is independent of $\frac{x_2}{x_1}$.

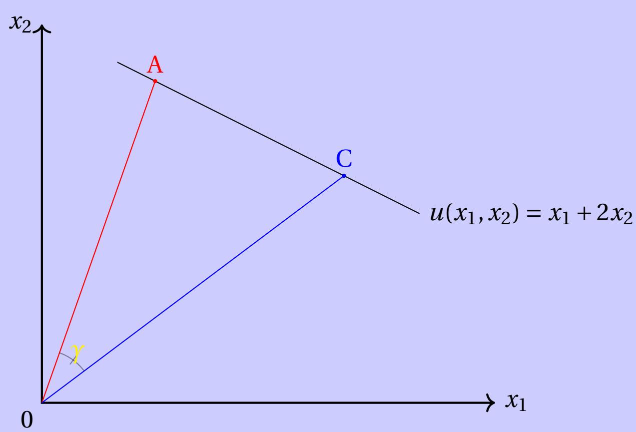

Figure 4: Perfect substitution

You're never bored with x_1 comparing with x_2, no matter how much x_1 and x_2 you consumed. This can only happen when x_1 and x_2 are in nature the same commodity.
For example, x_1 is an apple, while x_2 is a pack of 2 apples. The quality of the apples are the same and the only difference is the amount per package. Since you can always substitute a 2-apple pack with 2 single apples, we call this condition as "**Perfect Substitution**".

Another extreme example is Leontief preference. In Figure 5, when $\frac{x_2}{x_1}$ changes γ, MRS changes from $+\infty$ to 0.

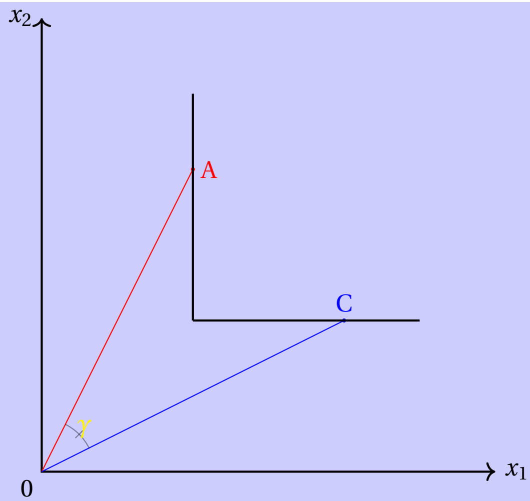

Figure 5: Leontief Preference

We already know with Leontief preference, you believe x_1 and x_2 are totally different and can only make sense with a certain proportion, like bread and cheeze. There can be **NO substitution** between the two.

4 Elasticity of Substitution σ_{12}

We can use "**Elasticity**" to describe the relationship between α and $\frac{x_2}{x_1}$.

$$
\begin{aligned}
Elasticity &= [\frac{1\%\ change\ of\ MRS}{1\%\ change\ of\ \frac{x_2}{x_1}}]^{-1} \\
&= [\frac{\frac{dMRTS}{MRTS}}{\frac{d(x_2/x_1}{x_2/x_1}}]^{-1} \\
&= [\frac{dln(MRTS)}{dln(x_2/x_1)}]^{-1}
\end{aligned}
$$

- Inside brackets: when $\frac{x_2}{x_1}$ changes 1 percent, how many percent MRS changes. E.g. 0 if perfect substitution, $+\infty$ if no substitution.
- Why inverse? We want it to be $+\infty$ when perfect substitution, and 0 when no substitution (more intuitive).

Similarly, for firms, we have:
DEFINITION: The Elasticity of Technical Substitution (a 2-input case for DEFINITION 3.2 on Jehle & Reny pp. 129)

For a production function $f(x_1, x_2)$, the elasticity of substitution of input 2 for input 1 at the point $(x_1, x_2) \in \mathbb{R}^2_{++}$ is defined as:

$$\sigma_{12}(x_1, x_2) \equiv \left(\frac{dlnMRTS_{12}(\frac{x_2}{x_1})}{dln(\frac{x_2}{x_1})}\right)^{-1}$$

Note both the numerator and the denominator are functions of $\frac{x_2}{x_1}$. If we define $\frac{x_2}{x_1} = r$, $\sigma_{12}(x_1, x_2)$ can be rewritten as:

$$\sigma_{12}(x_1, x_2) \equiv \left(\frac{dlnMRTS_{12}(r)}{dlnr}\right)^{-1}$$

An example from exam 2019 Q1(b)

Deb and Frank have the following utility functions $U_D = x_D + 2y_D$ and $U_F = x_F^{\alpha} y_F^{1-\alpha}$, with $\alpha \in (0, 1)$
b) [5%] What is the elasticity of substitution for Deb? What is the implication for the income and substitution effects?

Obviously, Deb has a "perfect substitution" preference exactly the same as Figure 4.

$$MRS_{xy} = \frac{\partial/\partial x}{\partial u/\partial y} = 0.5$$

$$\sigma_{xy} = \left[\frac{dln(MRTS)}{dln(x/y)}\right]^{-1} = \left[\frac{0}{dln(x/y)}\right]^{-1} = [0]^{-1} = +\infty$$

In this case, Deb believes x and y are the same product but with different amount per package.
As for income and substitution effects, the solution online is incorrect. Let's use apple as an example to make it clear:

Day 1

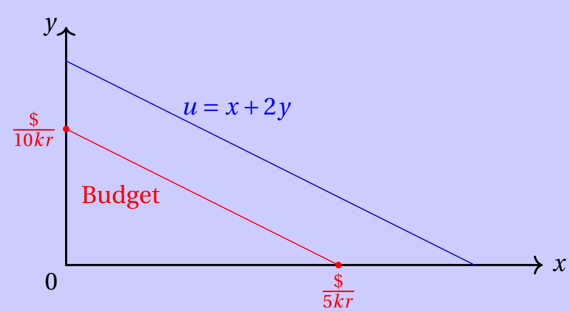

Figure 6: A fair deal

- x : an apple , 5 kr per package.
- y : a 2-apple pack, 10 kr per package.

It's fair! Deb will not mind to substitute 2 x with 1 y. Her demands lie on the budget frontier.

Day 2

- x : an apple , 5 kr per package.
- y : a 2-apple pack,5 kr per package. On Sale!.

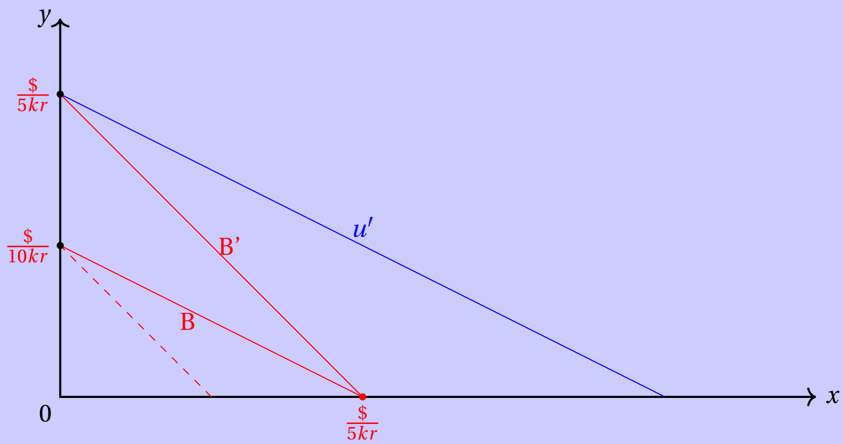

Figure 7: y is on sale

Deb will spend all income on y because it's cheaper per apple.
If Deb chose ($x = 0, y = \frac{\$}{10kr}$) in Day 1, when "relative price" changes (dash line), her demand will not change ⇒ no substitution effects, and income effects only exist for y:

$$y = \frac{\$}{10kr} \Longrightarrow y = \frac{\$}{5kr}$$

(Note in Day 1, indifference curve u is also on the budget frontier B)

Day 3

- x : an apple , 3 kr per package (also on sale but still more expensive).
- y : a 2-apple pack,5 kr per package. Still on sale.

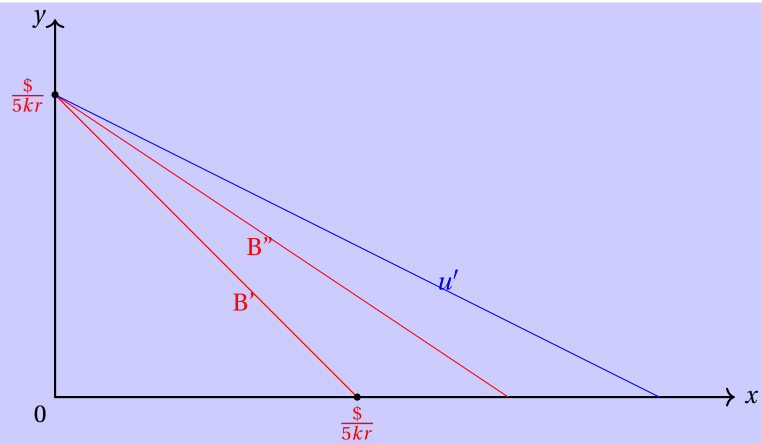

Figure 8: y is still cheaper

There is no substitution effect and income effect Demands unchange.

Day 4

- x : an apple , 2 kr per package (Cheaper!).
- y : a 2-apple pack,5 kr per package. Still on sale.

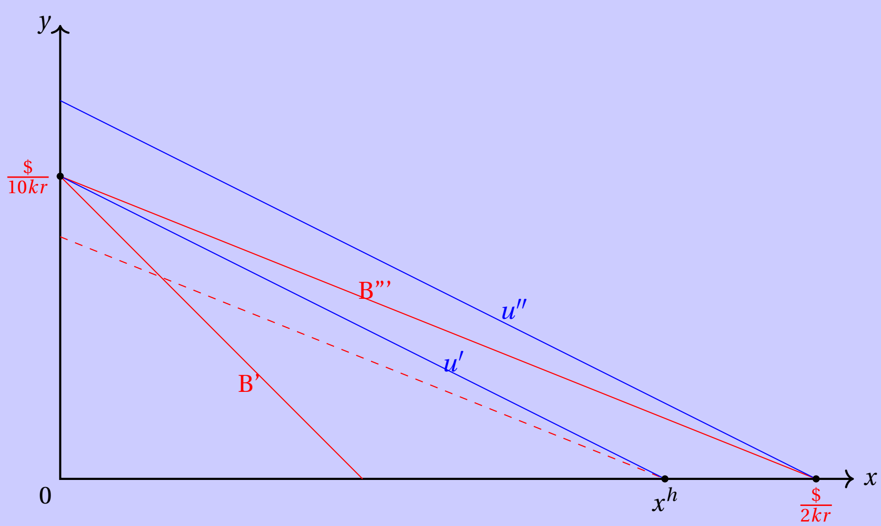

Figure 9: x is cheaper

If the relative price changed (red dash line),with the same utility, Deb's demand will change:

$$x = 0 \rightarrow x^h$$

$$y = \frac{\$}{10kr} \rightarrow y = 0$$

This is due to substitution effect. In this case, the income effects only exist for x:

$$x^h \rightarrow x = \frac{\$}{2kr}$$

Answer: The elasticity of substitution for Deb is infinite due to pefect substitution. Assume there is a cheaper commodity, like y in Day 2.

- When the cheaper is still cheaper(y in Day 3), no substitution effect, no income effect, no price effect.
- When the cheaper becomes the more expensive (y in Day 4), income effect exists only for the cheaper (x in Day 4).

Tips: Allocate your time according to the point of the questions.

- This is a 5% point question, then just answer it briefly.
- Use graphs can save your time.
- Use "common sense" to avoid proving (unless the question asks you to **calculate**), e.g. "straight" indifference curve ⇒ perfect substitution ⇒ $\sigma = +\infty$
- Just skip it if you find it difficult. This small sub-question is really hard to answer.

Question: What is the implication of Leontief preference's ES for the income and substitution effects?