COMP760: Geometry and Generative Models

Week 6: Spherical Geometry

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SPHERICAL GEOMETRY

To begin our treatment of Spherical geometry it is instructive to compare and contrast with the familiar Euclidean geometry. As a taste of how Spherical geometry differs recall the well-known fact in Euclidean geometry is that two lines that start off as parallel remain parallel when they are extended. In contrast, two lines that start off as parallel can intersect at a point on the sphere. For example, if two lines are parallel at the equator of a sphere embedded in $\mathbb{R}^3$ then these lines will intersect at either north or south poles. We now review some basic facts from Euclidean geometry.

1.1 Euclidean Geometry

This section should be largely familiar to most readers. Where there is value is in the unification of terminology and notation in the framework of Riemannian geometry. Specifically, once we elucidate the Riemannian metric for Euclidean spaces we immediately recover familiar notions of norms, inner-products, angles, and distances. Finally, we also cover the exponential and logarithmic maps and projection operators.

An n-dimensional Euclidean space $\mathbb{R}^d$ can equipped with the following Riemannian metric $\mathfrak{g}_e = I_d$. This means that the Euclidean inner product $\langle \cdot, \cdot \rangle$ is the standard dot product. Thus $\mathbb{R}^d$ can be turned into a Riemannian manifold as $\mathbb{E}^d = (\mathbb{R}^d, \mathfrak{g}_e)$, which is a flat space in the sense that the metric is the identity matrix everywhere.

Distances. The distance function between two vectors x, y in $\mathbb{E}^n$ is simply:

$$d(x, y) = ||x - y||_2 = \sqrt{(x_1 - y_1)^2 + \cdots + (x_d - y_d)^2} \tag{1.1}$$

We can also use this to define the squared distance:

$$\begin{aligned} ||x - y||_2^2 &= \langle x - y, x - y \rangle & (1.2) \\ &= ||x||_2^2 + ||y||_2^2 - 2||x||_2||y||_2 \cos^{-1}(\theta), & (1.3) \end{aligned}$$

where θ is the angle between the vectors x and y.

Exponential Map. In Riemannian geometry the exponential map takes a tangent vector at point x on the manifold $v \in \mathcal{T}_x\mathcal{M}$ and transports it along the unique geodesic which satisfies $\gamma_v(0) = x$ and $\gamma_v'(0) = v$ to the point $\exp_x(v) = \gamma_v(1)$—i.e. travelling a unit of time along the geodesic. In $\mathbb{E}^n$ the exponential map is trivially given by:

$$\exp_x(v) = x + v. \tag{1.4}$$

Logarithmic Map. The logarithmic map in some sense can be understood as the reverse of the exponential map. This is a very crude definition but it is serviceable for the current presentation. The logarithmic map is thus:

$$\log_x(v) = v - x. \tag{1.5}$$

1.2 Spherical Geometry

Spheres are defined as submanifolds in a Euclidean space of points with a unit Euclidean norm. Precisely, an d-sphere is $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : \|x\|_2 = 1\}$. Therefore the ambient space dimensionality of a d sphere is $m = d + 1$. Tori are products of 1-spheres (or circles); that is $\mathbb{T}^d = \Pi_{i=1}^d \mathbb{S}^1$. Naturally, we can embed a d-torus in a $m = 2d$-dimensional ambient space.

Spheres are also constant curvature manifolds which means an equivalent way of specifying the manifold is using the ambient space coordinate system as well as the curvature constant $K > 0$, $\mathbb{S}_K^d = \{x \in \mathbb{R}^{d+1} | \langle \mathbf{x}, \mathbf{x} \rangle = 1/K\}$. This viewpoint will aid us when we consider constant negative-curvature manifolds in later chapters. Lastly, from henceforth we will differentiate points on $\mathbb{S}_K^d$ using boldface in an effort to distinguish other vectors either in tangent or ambient spaces from points on the manifold.

Note: Constant negative curvature manifolds are called hyperbolic spaces

Parametrization in 2-dimension. When $d = 2$ we can use polar coordinates to parametrize $\mathbb{S}_K^2$. Without loss of generality, we will assume $K = 1$ which will also reduce notational clutter. In polar coordinates, we can represent any point $\mathbf{x} \in \mathbb{S}^2$ using two angles θ, ψ which are the polar and azimuth angles. The range of the polar angle is $\theta \in [0, \pi)$ and the range for the azimuth angle is $\psi \in [0, 2\pi)$. In ambient Cartesian coordinates $r(\theta, \psi) = (\sin(\theta)\cos(\psi), \sin(\theta)\sin(\psi), \cos(\theta))$. Finally, the volume form is given by $\sqrt{\det|G(\theta, \psi)|} = \sin(\theta)$.

Parametrization of Hyperspheres. For $\mathbb{S}^d$ we can use $d - 1$ angular coordinates $\psi_1, \ldots, \psi_{d-2} \in [0, \pi)$ and $\psi_{d-2} \in [0.2\pi)$ as outlined in Blumenson [1960]. The volume form for the hypersphere is correspondingly $\sqrt{\det|G(\psi)|} = \sin^{d-2}(\psi_1)\sin^{d-3}(\psi_2)\ldots\sin(\psi_{d-2})$.

Distances. The distance between two points on the hypersphere is given by:

Note: $\langle \cdot, \cdot \rangle_2$ refers to the Euclidean inner product.

$$d(\mathbf{x}, \mathbf{y}) = \frac{1}{\sqrt{K}} \cos^{-1}(K\langle \mathbf{x}, \mathbf{y} \rangle_2) \tag{1.6}$$

when $K = 1$ the equation reduces to $d(x, y) = \cos^{-1}(\langle \mathbf{x}, \mathbf{y} \rangle_2)$. As the curvature constant $K \to 0$ the space becomes "flatter". The intuition for this is to first recall that $\langle x, x \rangle_2 = 1/K$ for any $\mathbf{x} \in \mathbb{S}^d_K$. Taking the limit we see that $\lim_{K \to 0^+} \langle \mathbf{x}, \mathbf{x} \rangle_2 = \infty$, which is in line with the intuition that all points on the sphere go to infinity as the space gets flatter.

Exponential Map. The exponential map in $\mathbb{S}^d_K$ is:

$$\exp^K_{\mathbf{x}}(v) = \cos\left(\sqrt{K}||v||_2\right)\mathbf{x} + \sin\left(\sqrt{K}||v||_2\right)\frac{v}{\sqrt{K}||v||_2} \tag{1.7}$$

Note: The derivation for this can be found in Theorem A.8 in Skopek et al. [2019]

Logarithmic Map. The logarithmic map in $\mathbb{S}^d_K$ is.:

$$\log^K_{\mathbf{x}}(\mathbf{y}) = \frac{\cos^{-1}(\alpha)}{\sqrt{1-\alpha^2}}(\mathbf{y} - \alpha\mathbf{x}), \tag{1.8}$$

where $\alpha = K\langle \mathbf{x}, \mathbf{y} \rangle_2$.

Parallel Transport. The parallel transport between vectors in two tangent spaces $\mathcal{T}_{\mathbf{x}}\mathbb{S}^d_K$ and $\mathcal{T}_{\mathbf{y}}\mathbb{S}^d_K$ is given by:

$$\mathrm{PT}_{\mathbf{x}\to\mathbf{y}}(v) = v - \frac{K\langle \mathbf{y}, v \rangle_2}{1 + K\langle \mathbf{x}, \mathbf{y} \rangle_2}(\mathbf{x} + \mathbf{y}) \tag{1.9}$$

Tangential projection. Without loss of generality, we derive the orthogonal projection to the tangent space of spheres. The tangential projection of tori is just the same linear operator applied to d, $\mathbb{R}^2$ vectors independently.

To derive the tangential project, we note that any incremental change in $\mathbf{x}$, denoted by $\mathrm{d}\mathbf{x}$, will need to leave the norm $\|x\|_2$ unchanged. That is,

$$\mathrm{d}\,\|\mathbf{x}\|_2^2 = 2x\,\mathrm{d}\mathbf{x} = 0. \tag{1.10}$$

This means x is normal to the tangential linear subspace. We can find the orthogonal projection onto the tangent space by subtracting the normal component, via $P_{\mathbf{x}} = I - \frac{\mathbf{x}\mathbf{x}^\top}{\|\mathbf{x}\|_2^2}$.

Closest-point projection. The closest-point projection onto the sphere is $\pi(x) = \frac{x}{\|x\|_2}$. One can verify this is the point on $\mathbb{S}^d$ that minimizes the Euclidean distance from $x \in \mathbb{R}^{d+1} \setminus \{0\}$.

Bibliography

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