COMP 760 Week 6: Introduction to Hyperbolic Geometry

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Admin stuff week 7

Project Failure Modes

- Project scope is not well defined enough.
- The setting is not detailed enough. You haven’t written down the exact loss/formulation clearly enough.
- The minimum viable project is not defined yet. What does the simplest setting/result that demonstrates your idea?
- Literature survey is not detailed enough. Other people might have done something very similar or exactly the same thing.

Models of Hyperbolic Geometry

Hyperbolic Geometry

- Arose from breaking Euclid’s parallel postulate.
- Manifolds of constant negative curvature K
- At least 4 equivalent models of hyperbolic geometry: Klein Model, Poincare Disk Model, Poincare half-plane, and Lorentz Model.

Hyperbolic Geometry

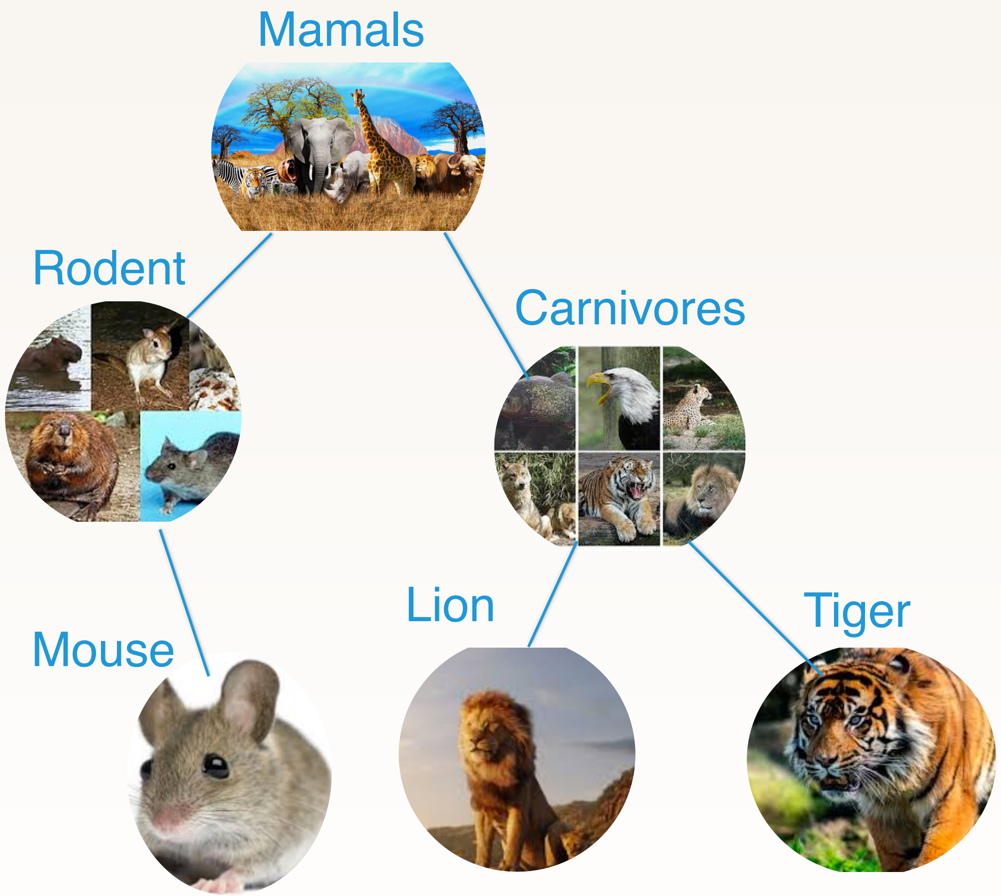
- Arose from breaking Euclid’s parallel postulate.

“If a line segment intersects two straight lines forming two interior angles on the same side that are less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.”

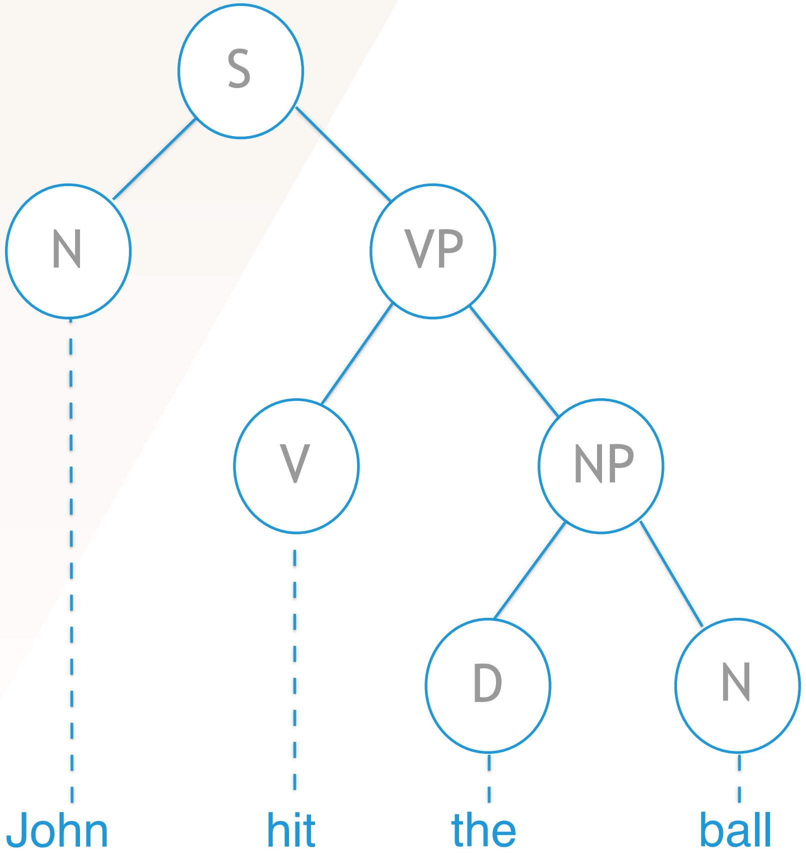
Hyperbolic Geometry

- Arose from breaking Euclid's parallel (5th) postulate.

Motivating Hyperbolic Geometry



Ontologies



Parse Trees

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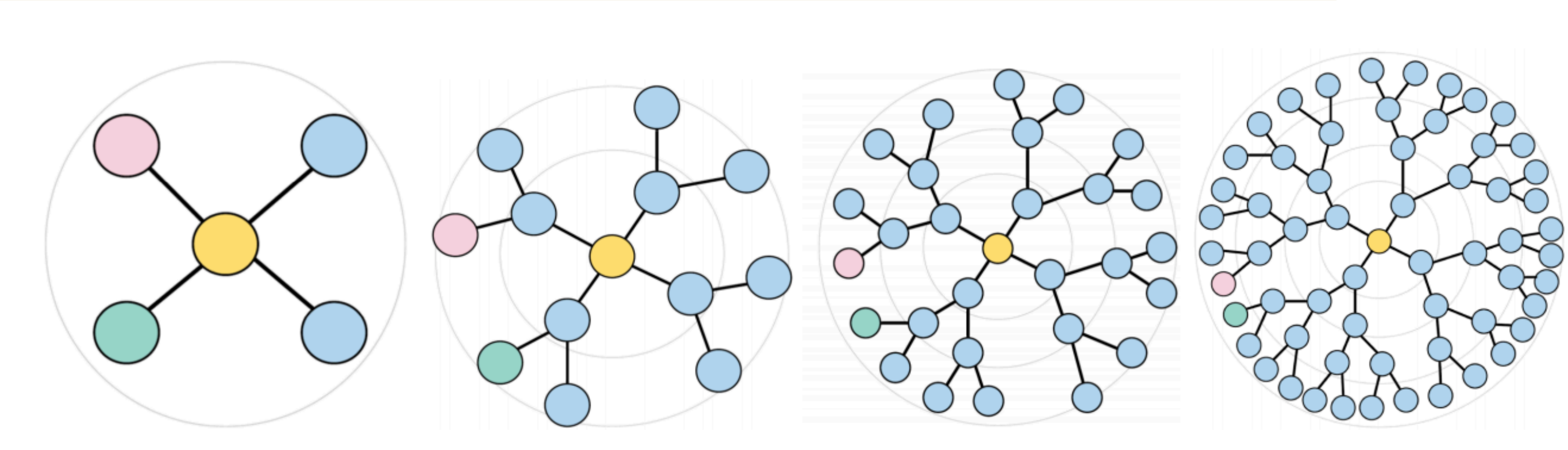
Embedding Trees in Euclidean space



We quickly run out of space! Node embedding distance does not respect graph distance!

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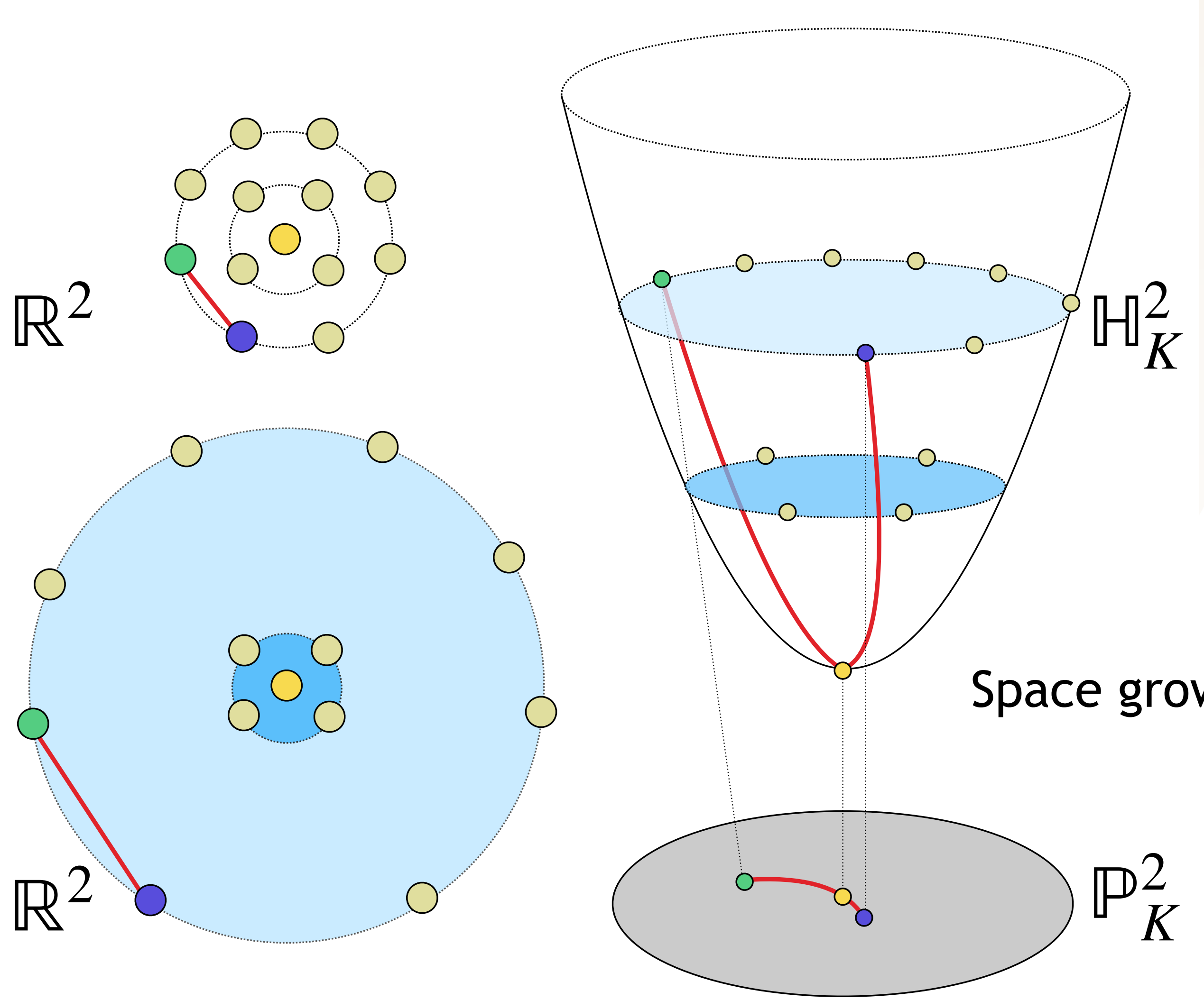
Figure credit: https://openreview.net/pdf?id=BJg73xHtvr

Embedding Trees in Euclidean space



Space grows exponentially!

Review: Smooth Manifolds

$\mathscr{M}$ – smooth

- d-dim topological space (paracompact, Haussdorf, and second countable).
- $\{(U_i, \phi_i)\}$ collection of charts that satisfy a compatibility condition.
- A smooth function f on $\mathscr{M}$ is the map $f \circ \phi^{-1} : \mathbb{R}^d \to \mathbb{R}$
- The set of smooth functions on $\mathscr{M}$ is denoted by $C^\infty(\mathscr{M})$

Poincare Ball Model

Poincare Ball

- Riemannian Manifold $\mathscr{B}_K^n$ is the open ball of radius $R = \frac{1}{\sqrt{|K|}}$, where $K < 0$.

$$\mathbb{P}_K^n = (\mathscr{B}_K^n, \mathfrak{g}_K)$$

- Metric Tensor

$$\mathfrak{g}_K(x) = (\lambda_x^K)^2 \mathfrak{g}^e(x) \qquad , \lambda_x^K = \frac{2}{1 + K||x||^2}$$

- Volume Form

$$d\text{Vol} = \sqrt{|G(x)|} = (\lambda_x^K)^n$$

Poincare Disk (2d)

- Disk $\{x \in \mathbb{R}^2 : |x| < 1\}$

- Metric:

$$ds^2 = \frac{4||x||^2}{(1-||x||^2)^2}$$

- Straight lines are circular arcs

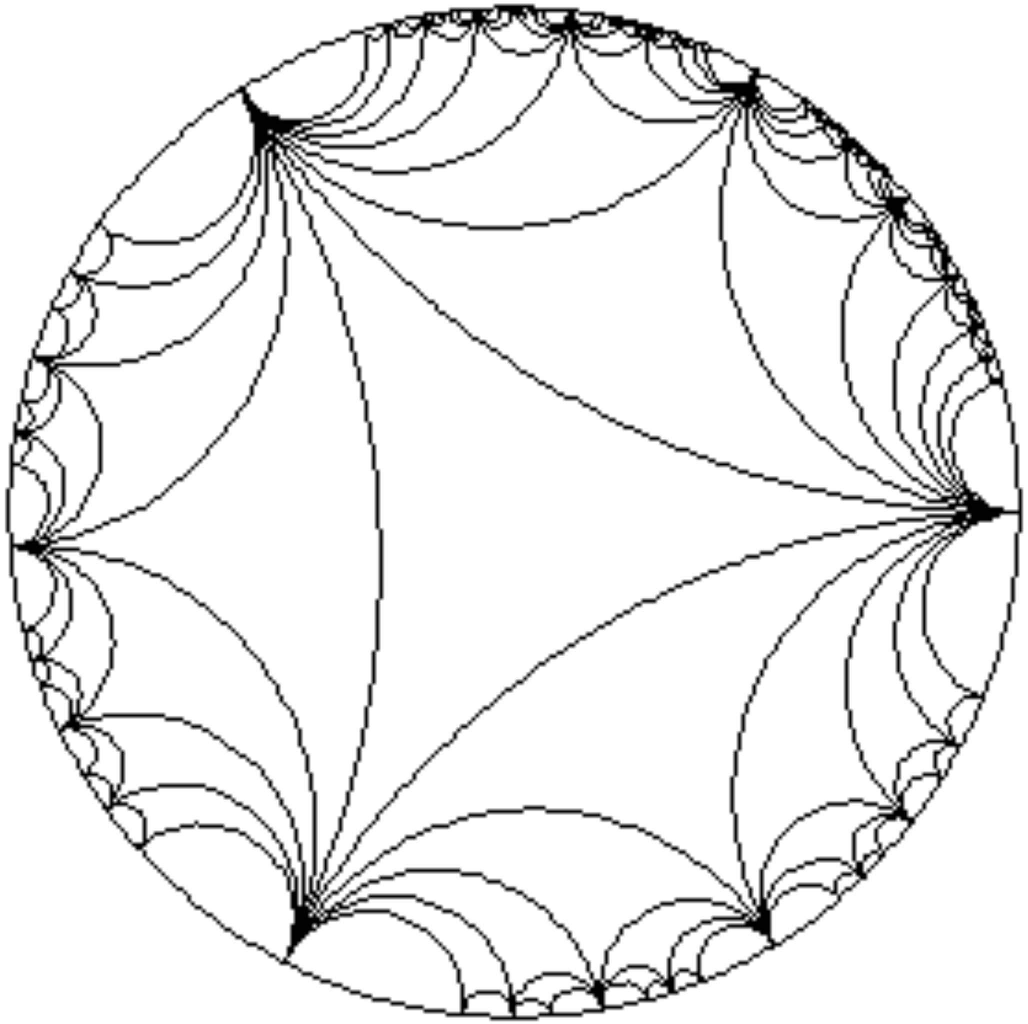


Fig credit: https://mathworld.wolfram.com/PoincareHyperbolicDisk.html

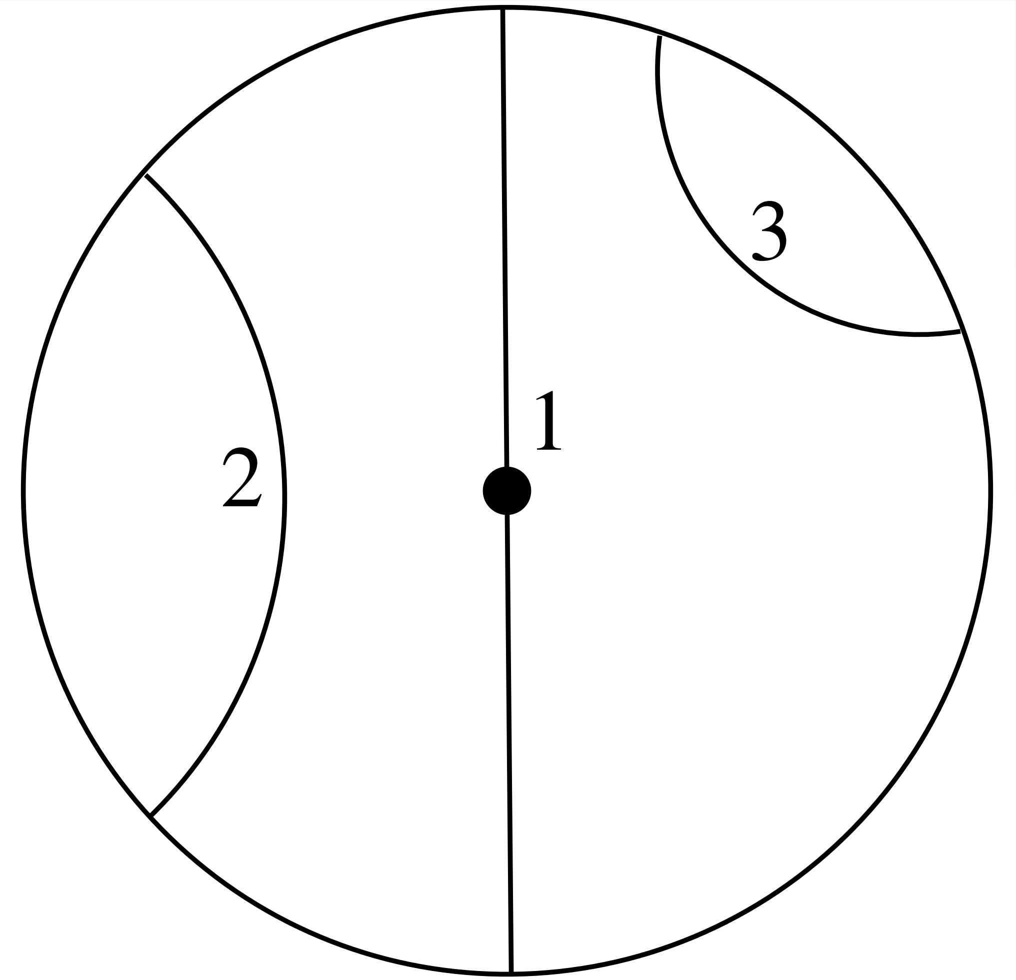
Mobius Addition

- From the theory of gyro vector spaces we can develop an analogue for addition in Euclidean spaces.

$$x \oplus_K y = \frac{(1 - 2K\langle x, y\rangle - K||y||_2^2)x(1 + K||x||_2^2)y}{1 - 2K\langle x, y\rangle + K^2||x||_2^2\ ||y||_2^2}$$

- Mobius addition is neither commutative nor associative.

Three Parallel Lines in the Poincare Disk



Exponential Map in the Poincare Model

$$\exp_x^K(v) = x \oplus_K \left(\tanh\left(\sqrt{-K} \frac{\lambda_x^K ||v|||_2}{2} \right) \frac{v}{\sqrt{-K}||v||_2} \right)$$

This is a cumbersome expression to work with.

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Gyration and Parallel Transport in the Poincare Model

$$\mathrm{gyr}[x, y]v = -(x \oplus_K y) \oplus_K (x \oplus_K (y \oplus_K v))$$

- Measures the extent to which Mobius addition $\oplus_K$ deviates from associativity for points $x, y, v \in \mathscr{B}^n_K$.

$$\mathrm{PT}^K_{x \to y}(v) = \frac{\lambda^K_x}{\lambda^K_y} \mathrm{gyr}[y, -x]v\,.$$

Distances in the Poincare Model and Numerical Instability

$$d_{\mathbb{P}}(x, y) = \frac{1}{\sqrt{-K}} \cosh^{-1}\left(1 - \frac{2K||x - y||_2^2}{(1 + K||x||_2^2)(1 + K||y||_2^2)}\right)$$

- Distance within the Poincaré ball model changes rapidly when the points are close to the boundary $||x|| \approx 1$ this leads to numerical instability.

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Lorentz Model

Hyperboloid Model

- Lorentz inner product

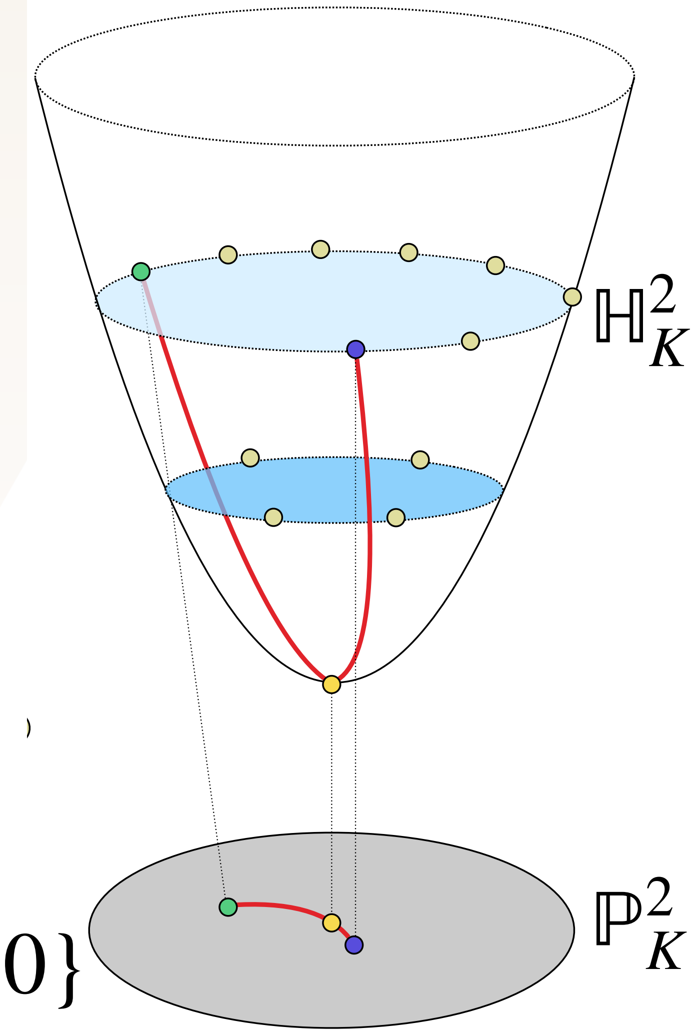
$$\langle \mathrm{x}, \mathrm{y} \rangle_{\mathscr{L}} := -x_0 y_0 + x_1 y_1 + \ldots + x_n y_n,$$

- Inner Product Type:

$$\langle \cdot\,, \cdot \rangle_{\mathscr{L}} : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$$

- pseudo-Riemannian manifold $(\mathbb{H}^n_K, g_{\mathbf{x}})$

$$\mathbb{H}^n_K := \{\mathrm{x} \in \mathbb{R}^{n+1} : \langle \mathrm{x} \,.\, \mathrm{x} \rangle_{\mathscr{L}} = 1/K,\ x_0 > 0,\ K < 0\}$$



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Tangent Spaces in the Hyperboloid Model

- The tangent space at point $\mathbf{p} \in \mathbb{H}_K^n$ can also be described as an embedded subspace of $\mathbb{R}^{1,n}$.

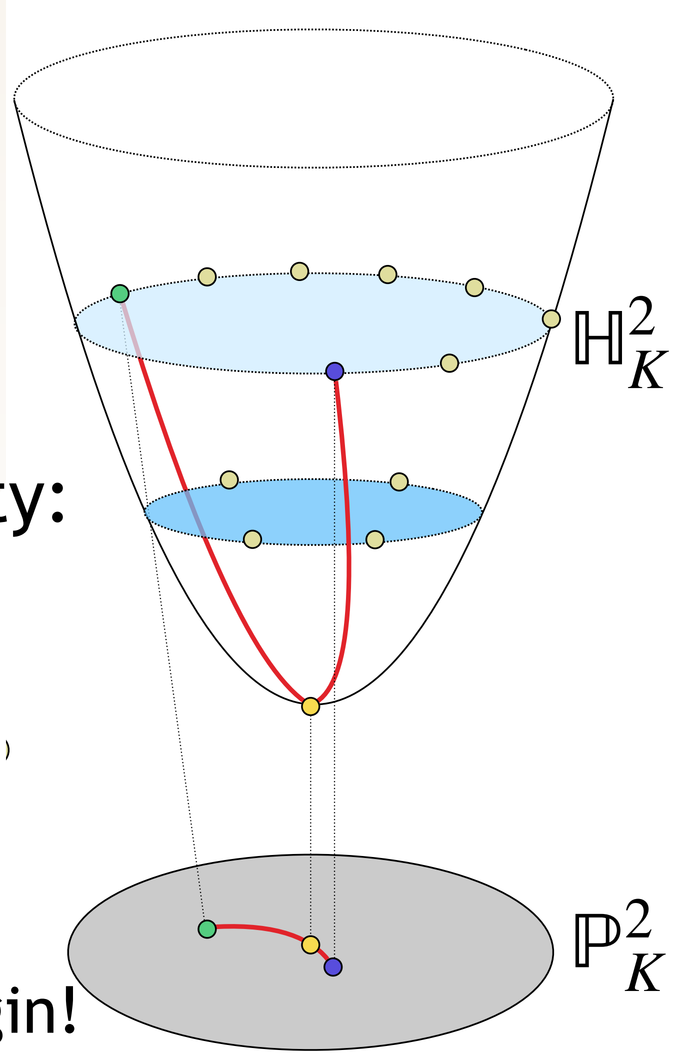
$$\mathcal{T}_{\mathbf{p}}\mathbb{H}_K^n := \{u : \langle u, \mathbf{p}\rangle_{\mathcal{L}} = 0\}$$

- Tangent Space at the origin have special property:

$$v \in \mathcal{T}_{\mathbf{o}}\mathbb{H}_K^n \qquad v_0 = 0$$

$$||\mathrm{v}||_{\mathcal{L}} := \sqrt{\langle \mathrm{v}, \mathrm{v}\rangle_{\mathcal{L}}} = ||\mathrm{v}||_2$$

Lorentz norm is the same as Euclidean norm at the origin!



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Lorentz Model Summary

$$\langle \mathrm{x}, \mathrm{y} \rangle_{\mathcal{L}} := -x_0 y_0 + x_1 y_1 + \ldots + x_n y_n,$$

Lorentz Inner Product

$$d(\mathrm{x}, \mathrm{y})_{\mathcal{L}} = \frac{1}{\sqrt{-K}} arccosh(-K \langle \mathrm{x}, \mathrm{y} \rangle_{\mathcal{L}})$$

Distance

$$\mathbb{H}^n_K := \{x \in \mathbb{R}^{n+1} : \langle \mathrm{x}, \mathrm{x} \rangle_{\mathcal{L}} = 1/K,\ x_0 > 0,\ K < 0\}$$

pseudo-Riemannian Manifold

$$\mathbb{R}^{n+1} \supset \mathbb{H}^n_K$$

Extrinsic view

Radius $R = \frac{1}{-K}$

Tangent Spaces in the Hyperboloid Model

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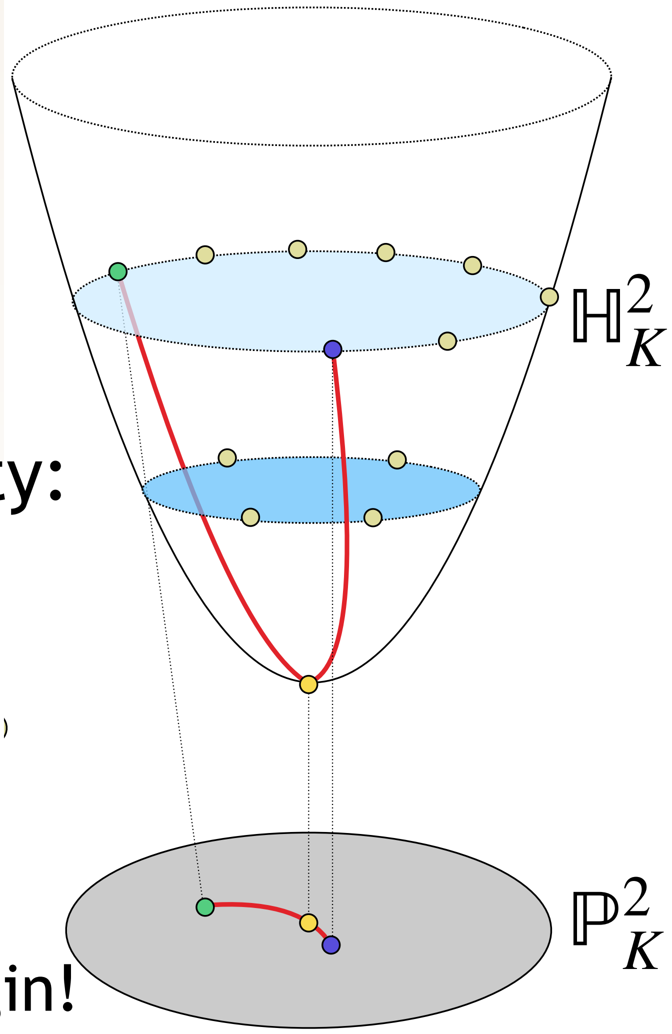

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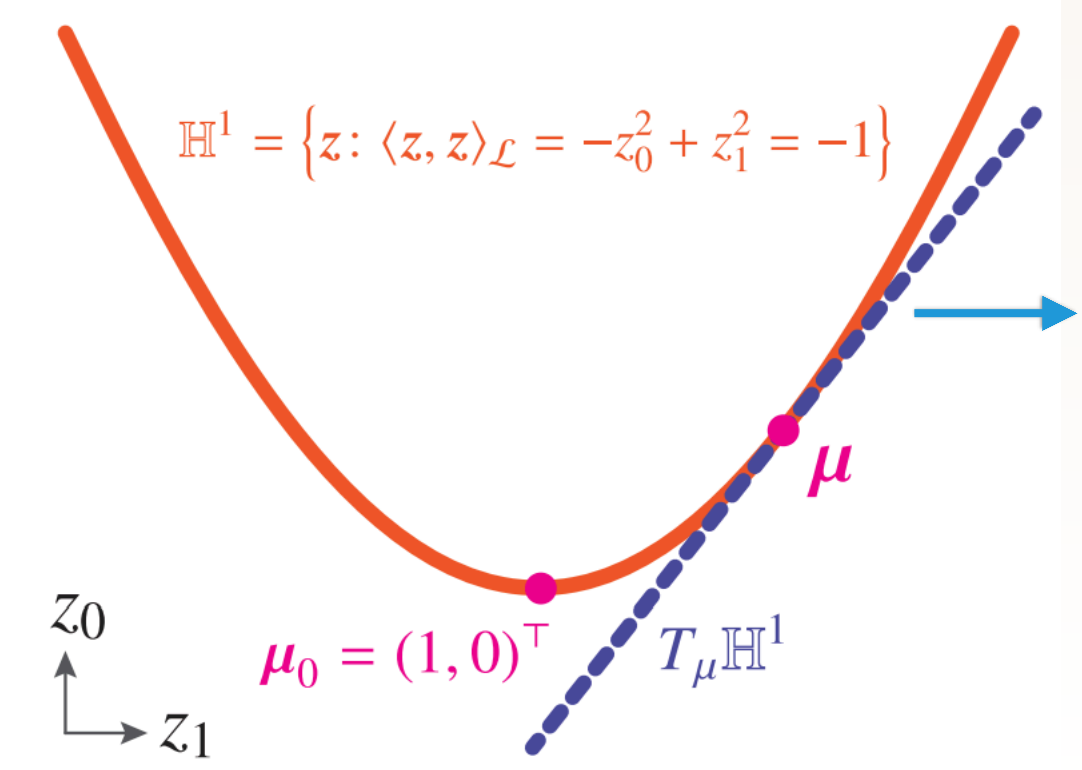
Tangent Spaces in the Hyperboloid Model

$$\mathcal{T}_{\mu}\mathbb{H}^n_K := \{u : \langle u, \mu \rangle_{\mathcal{L}} = 0\}$$



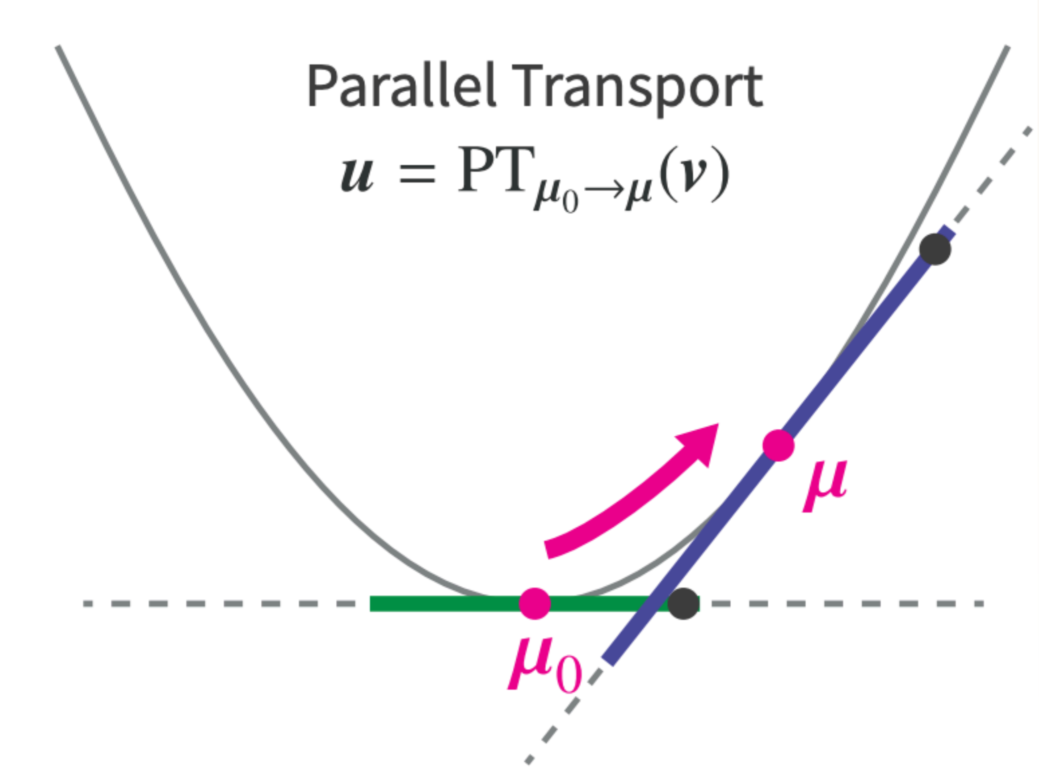
Tangent Space at μ

Euclidean space at μ

Fig credit: https://arxiv.org/pdf/1902.02992.pdf

Parallel Transport in the Lorentz Model

$$PT^K_{\mu_0 \to \mu}(v) = v + \frac{\langle \mathrm{y}, v \rangle_{\mathscr{L}}}{R^2 - \langle \mu_0, \mathrm{y} \rangle_{\mathscr{L}}}(\mu_0 + \mu)$$

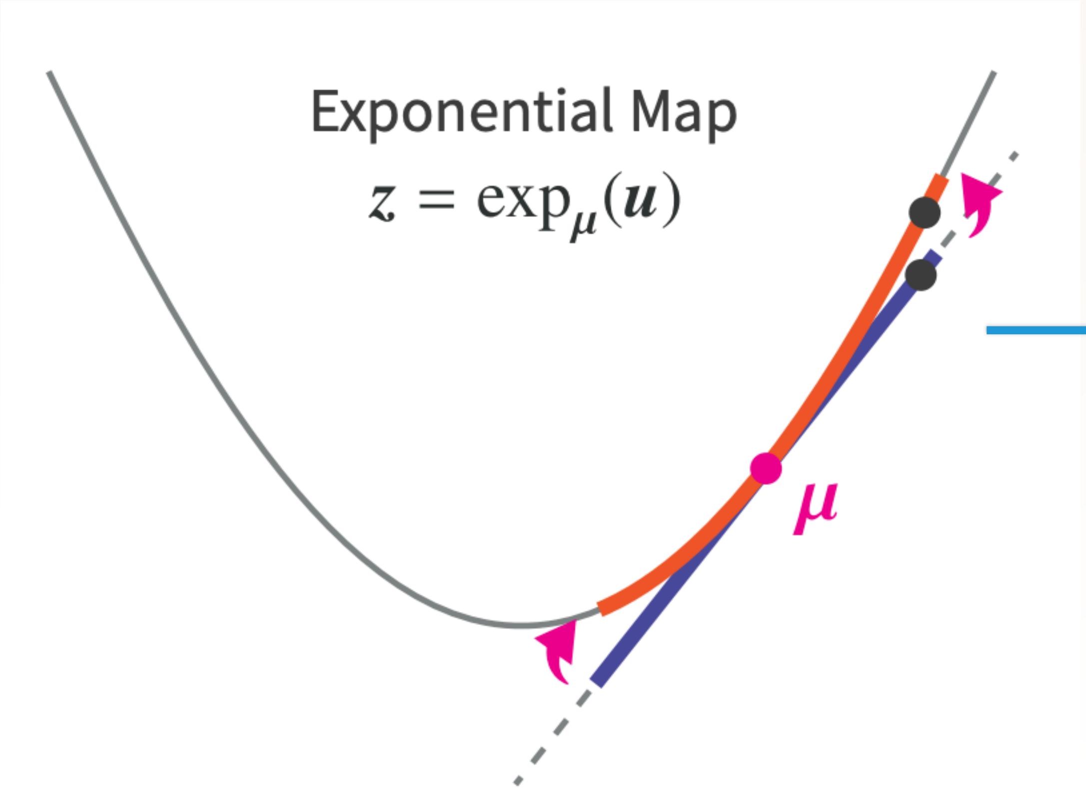


→ Moves a vector between tangent spaces

Fig credit: https://arxiv.org/pdf/1902.02992.pdf

Exponential Map in the Lorentz Model

$$exp_{\mu}^{K}(u) = \cosh\left(\frac{||u||_{\mathscr{L}}}{R}\right)\mu + \sinh\left(\frac{||u||_{\mathscr{L}}}{R}\right)\frac{Ru}{||u||_{\mathscr{L}}}$$



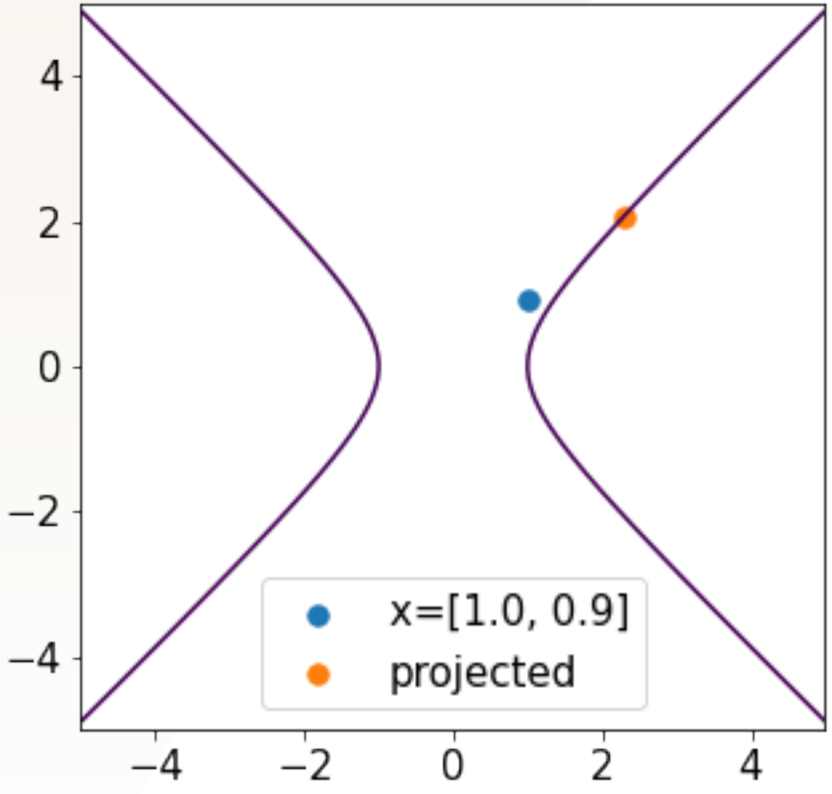
Maps a point back to the manifold

Fig credit: https://arxiv.org/pdf/1902.02992.pdf

Closest Point Projection in the Lorentz Model

$$\text{proj}_{\mathbb{H}^n_K}(x) = \frac{x}{\sqrt{-K}||x||_{\mathscr{L}}}$$

Closest-point projection of the point $(1.0, 0.9)$ onto the Hyperbolic manifold $\mathbb{H}^1_1$ in the Lorentz norm. This projection is clearly not the closest one in Euclidean distance.



Tangential Projection in the Lorentz Model

- We can derive the Tangential projection by any incremental change in x, denoted by dx,

$$d\langle x, x\rangle_{\mathscr{L}} = \boxed{2n_x} dx = 0$$

$$n_x = (-x_0, x_1, \ldots, x_d)$$

- Subtracting the normal contribution gives rise to the tangential projection $P_x = I - \frac{n_x n_x^\top}{||n_x||_2^2}$.

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Numerical Issues in the Lorentz Model

$$exp_{\mu}^{K}(u) = \cosh\left(\frac{||u||_{\mathscr{L}}}{R}\right)\mu + \sinh\left(\frac{||u||_{\mathscr{L}}}{R}\right)\frac{Ru}{||u||_{\mathscr{L}}}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \qquad \sinh x = \frac{e^{x} - e^{-x}}{2}$$

- For $x = 40$, $e^{38} = 2.35 \times 10^{17}$. Maximum value for float64 is $2^{63} - 1 = 9.22 \times 10^{18}$. We need to clamp higher values of x.

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