COMP760: Geometry and Generative Models
Week 1: Geometry Primer I

Notes by: Joey Bose and Prakash Panangaden

1

METRIC SPACES

Please ignore remarks in this font.

We learn analysis for the first time over the real numbers $\mathbf{R}$ and we take as fundamental the idea of distance between two real numbers $d(x, y) = |x - y|$. The two most importance concepts that we learn in basic analysis are convergence of a sequence and continuity of a function. Both use the "usual" distance on $\mathbf{R}$ as defined above. However, one can abstract away from the variety of structures on $\mathbf{R}$ (it is an abelian group, a vector space, a ring ...) and keep the notion of distance as primitive. This leads to the concept of metric space; this is how the second course in analysis is usually taught. In this class we will introduce *point-set topology* which abstracts even further and treats convergence and continuity without *any quantitative notions*. First we recapitulate the theory of metric spaces.

1.1 Basics of metric spaces

Definition 1.1.1 *A **pseudometric space** is a set X equipped with a function $d : X \times X \longrightarrow \mathbf{R}^{\geq 0}$ satisfying:*

- *(reflexivity) $\forall x \in X, d(x, x) = 0$,*
- *(symmetry) $\forall x, y \in X, d(x, y) = d(y, x)$,*
- *(triangle inequality) $\forall x, y, z \in X, d(x, y) \leq d(x, z) + d(y, z)$.*

*The function d is called a **pseudometric**. If d satisfies the following additional property:*

$$\forall x, y \in X, d(x, y) = 0 \textit{ implies that } x = y$$

*we call d a **metric** and the pair (X, d) is called a **metric space**.*

Several metrics that we will study later on are actually pseudometrics but we will become sloppy as we get used to the situation and just say "metric" when it is clear from context whether we mean a real metric or a pseudometric. Strictly speaking, a metric space is the *pair* (X, d), but we will often just say "X is a metric space" when the d is clear from context. Note, however, we can define two different distance functions d and d' on the same set X. The pairs (X, d) and (X, d') are *different* metric spaces.

Exercise 1.1.2 *Given a pseudometric space (X, d) show that the binary relation $x \sim_d y$ given by $x \sim_d y$ if and only if $d(x, y) = 0$ is an equivalence relation. Show that there is a natural metric induced by d on the set of equivalence classes, denoted $X/\sim_d$; call this $d_\sim$. Show that $d_\sim$ is a proper metric.*

In view of Exercise 1.1.2 we see that we can always construct a "proper" metric from a pseudometric if we really need to.

Example 1.1.3 *The real numbers with the distance function $d(x, y) = |x - y|$ is a proper metric space. We can extend this to $\mathbf{R}^n$ by defining*

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}.$$

This is called the euclidean metric.

Example 1.1.4 *Consider the set $\mathcal{F}$ of real-valued integrable functions from $\mathbf{R}$ to $\mathbf{R}$. We define a* pseudo*metric on $\mathcal{F}$ by*

$$d(f, g) = \sqrt{\int_{-\infty}^{+\infty} |f(x) - g(x)|^2 \mathrm{d}x}.$$

Why is this not a proper metric? Because if f and g are different functions, but only differ on a negligible[1] *set of points, then the distance will be 0 even though f and g are different.*

Example 1.1.5 *Fix a finite set Σ, called the alphabet; elements of Σ are called* letters. *Let Σ^* be the set of finite strings or words constructed from Σ. We write $a, b, c, \ldots$ for letters and $u, v, x, y, z, \ldots$ for words. We write $x[n]$ for the letter that occurs at position n of the word x; we start the numbering of positions with 0. Now we define a metric on Σ^* as follows: let x and y be two words and let n be the* smallest *integer such that $x[n] \neq y[n]$. We set $d(x, y) = 2^{-n}$.*

Exercise 1.1.6 *Convince yourself that d as defined just above is a metric. Now prove the following properties of this metric:*

1. $$\forall x, y, z \in \Sigma^*, d(x, y) \leq \max(d(x, z), d(y, z)).$$

   A metric with this property is called an ultrametric. *Show that it*

If you have studied number theory the metric on the p-adic numbers is another such example.

[1] This needs to be made more precise and we will do so when we cover measure theory.

follows that one can only have isosceles triangles in this space.

2. *We call the set $\{x|d(x, x_0) < r\}$ the* open ball of radius r centred at x_0. *Show that two balls are either disjoint or one is contained inside the other.*

1.2 Convergence

Definition 1.2.1 *A **sequence** of elements in a metric space X is a countable family of elements indexed by $\mathbf{N}$. We write (x_n) for a sequence of elements.*

Definition 1.2.2 *A sequence of elements is said to **converge to** x if for every $\varepsilon > 0$ there is a natural number N such that for every $n \geq N$ we have $d(x_n, x) \leq \varepsilon$. We call x the **limit** of the sequence.*

Exercise 1.2.3 *Prove that in a proper metric space a sequence cannot converge to two* different points. *Of course this is not true in a pseudometric space.*

Not every sequence that looks like it "should converge" actually does converge. For example, if our metric space is $(0, 1)$ with the usual metric, the sequence $x_n = \frac{1}{n}$ wants to converge to 0 but zero is not part of the space so it has nothing to which it can converge. What do we mean by saying that a sequence "should converge"? The points are getting closer to each other.

Note: that I did not write the grammatically clumsy sentence "it has nothing to converge to".

Definition 1.2.4 *A sequence (x_n) is called a **Cauchy sequence** if $\forall \varepsilon > 0, \exists N \in \mathbf{N},$ such that $\forall n, m \geq N, d(x_n, x_m) \leq \varepsilon$.*

It can readily be seen that $(\frac{1}{n})$ is a Cauchy sequence.

Definition 1.2.5 *A metric space (X, d) is said to be **complete** if every Cauchy sequence converges to a point in X.*

The space $(0, 1)$ is not complete as we have seen, but it can be made complete by adding points at 0 and 1. In fact *every* metric space can be made complete by a systematic procedure. Essentially, one has to add new points corresponding to every possible Cauchy sequence. However, many of these Cauchy sequences will want to converge to the "same" point, so one has to define an equivalence relation on the Cauchy sequences and take the quotient of the resulting space. This process is called Cauchy completion. If one performs this operation on the rational numbers one obtains the reals.

Exercise 1.2.6 *Is the metric space of Example 1.1.5 complete? If "yes" prove it, if "no" then complete it and describe the new points that were added.*

1.3 Functions between metric spaces

In studying any kind of mathematical structure the functions that preserve the structure are fundamental.

Definition 1.3.1 *Let (X, d) and (Y, d') be metric spaces. A function $f : X \longrightarrow Y$ is called an* ***isometry*** *if $\forall u, v \in X, d(u, v) = d'(f(u), f(v))$. Usually an isometry is defined to be bijective, this then says that X and Y are basically the same metric space. However, we will not require this of an isometry. If an isometry is also a bijection we will call it an* ***isomorphism of metric spaces****.*

Isometries are very special and too rigid, they exclude many interesting maps. The concept that is well adapted to the study of metric spaces is the following.

Definition 1.3.2 *We say that $f(X, d) \longrightarrow (Y, d')$ is* ***nonexpansive*** *if $\forall u, v \in X, d'(f(u), f(v)) \leq d(x, y)$. We say that f is* ***k-Lipschitz****for a real number k if $\forall u, v \in X, d'(f(u), f(v)) \leq k \cdot d(x, y)$. Thus, nonexpansive is the same as 1-Lipschitz.*

The fundamental concept from analysis is the following.

Definition 1.3.3 *A function $f : (X, d_X) \longrightarrow (Y, d_Y)$ is said to be* ***continuous at*** *$x_0 \in X$ if*

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ such that } \forall x \in X, \text{ with } d_X(x, x_0) \leq \delta \text{ we have } d_Y(f(x), f(x_0)) \leq \varepsilon.$$

A function is ***continuous*** *if it is continuous at every point.*

Exercise 1.3.4 *Prove that any nonexpansive function is continuous. Give an example of a continuous function that is not nonexpansive.*

Exercise 1.3.5 *Suppose that f is a continuous function. Suppose that the sequence (x_n) converges to x. Show that the sequence $(f(x_n))$ converges to $f(x)$.*

A number of functions in optimization and machine learning are contractions.

Definition 1.3.6 *A function $f : (X, d) \longrightarrow (X, d)$ is called a* ***contraction*** *if there is a real number $\gamma \in (0, 1)$ such that $\forall x, y \in X, d(f(x), f(y)) \leq \gamma \cdot d(x, y)$.*

The fundamental theorem below is used in machine learning to justify the existence of optimal value functions.

Theorem 1.3.7 (Banach) *If $f : X \longrightarrow X$ is a contraction on the* complete metric space X*, there is a* unique *point x_∞ such that $f(x_\infty) = x_\infty$. Such a point is called a* ***fixed point*** *of f.*

Proof. Choose any point $x \in X$. Consider x and $f(x)$ we have $d(f(f(x)), f(x)) \leq \gamma d(f(x), x)$ since f is a contraction. By induction, $\forall k, d(f^{(k+1)}(x), f^{(k)}) \leq \gamma d(f^{(k)}, f^{(k-1)}(x)$. It is easy to see that the sequence $(f^{(n)}(x))$ is a Cauchy sequence. Thus the sequence converges to a point which we can call x_∞. Since f is continuous the sequence $(f(f^{(n)}(x))$ converges to $f(x_\infty)$. But this is the same sequence, just shifted by one, so it also converges to x_∞. Thus $f(x_\infty) = x_\infty$. Suppose there is another fixed point of f, say x_0. Then we must have $d(f(x_\infty), f(x_0)) \leq \gamma d(x_\infty, x_0)$. But $d(f(x_\infty), f(x_0)) = d(x_\infty, x_0)$ which means that $\gamma \geq 1$ which contradicts the fact that $\gamma \in (0, 1)$. ∎

1.4 Open, closed and bounded sets

We will reformulate the concept of continuity in a way that makes it amenable to generalization. The key concept is that of "open set". The notion of open interval is familiar from the study of the reals.

Definition 1.4.1 *An* ***open ball*** *of radius r centred at a point x_0 in a metric space (X, d) is the set $\{x | d(x, x_0) < r\}$. A* ***closed ball*** *of radius r centred at a point x_0 is the set $\{x | d(x, x_0) \leq r\}$. We will write $B(x, r)$ for an open ball centred at x with radius r.*

An open ball is a very special kind of set. The idea is that there are no points "on the edge"; in a closed ball the points at distance r are on the border. We will formalize the notion of "border" in a more general setting later.

Definition 1.4.2 *An* ***open set*** *U in (X, d) is a set with the property that every point u in U is contained in an open ball centred at u that is* contained in U.

Exercise 1.4.3 *Prove that an open ball is an open set.*

Open sets can be very irregular and even have sharp corners.

Example 1.4.4 *Consider* $\mathbf{R}^2$ *with the euclidean metric. The square* $\{(x,y)|0 < x < 1, 0 < y < 1\}$ *is an open set. It obviously is not a ball but no matter how close one is to a corner there is always space to squeeze in a sufficiently small open disc.*

One might think that closed sets should be defined analogously, but the most convenient definition is simply:

Definition 1.4.5 *A* ***closed set*** *is the complement of an open set.*

Why is closed a good name?

Exercise 1.4.6 *Show that if* C *is a closed set then a sequence in* C *that converges, must converge to a point in* C.

Closed sets have walls around them, you cannot sneak out "in the limit."

Another important class of sets are the bounded sets.

Definition 1.4.7 *A* ***bounded set*** *is one that is contained in a ball.*

It doesn't make any difference if we specify closed or open ball.

Clearly the union of any number, even uncountably many, open sets is open.

If it is not clear, make sure you convince yourself.

Exercise 1.4.8 *Prove the following.*

Proposition 1.4.9 *The intersection of finitely many open sets is also an open set.*

[Hint: it is very easy; don't look for a complicated answer.]

Dually, the intersection of an arbitrary collection of closed sets is closed and the union of a finite collection of closed sets is closed.

Now we come to the important concept of continuity in terms of open sets.

Proposition 1.4.10 *A function* f *is continuous if and only if the inverse image of an open set is open: if* O *is open then* $f^{-1}(O)$ *is open.*

Proof. Suppose that f is continuous and O is an open set. Let $u \in f^{-1}(O)$ be arbitrary. Now $f(u) \in O$ and, since O is open, there is a $\varepsilon > 0$ such that $B(f(u), \varepsilon) \subset O$. Since f is continuous, in particular, at u, there is a δ such that $\forall w, d(u, w) < \delta$ implies that $d(f(w), f(u)) < \varepsilon$. This means that for any $w \in B(u, \delta)$, $f(w) \in B(f(u), \varepsilon) \subset O$, so $w \in f^{-1}(O)$. Thus the open disc $B(u, \delta)) \subset f^{-1}(O)$. Thus $f^{-1}(O)$ is open.

I can call my r, ε if I want.

Now suppose that f satisfies the property that the inverse image of any open set is open. We will prove that it is continuous at z, where z is any point. Consider any $\varepsilon > 0$. The set $B(f(z), \varepsilon) = \{u \mid d(u, f(z)) < \varepsilon\}$ is an open set so, by the assumption on f, $A := f^{-1}(B(f(z), \varepsilon)$ is also an open set and clearly $z \in A$. Since A is open there is a $\delta > 0$ such that $B(z, \delta) \subset A$. Thus, for any $w \in B(z, \delta)$, we have that, $f(w) \in B(f(z), \varepsilon)$. Rewriting this, we have proved that for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any w if $d(w, z) < \delta$ then $d(f(w), f(z)) < \varepsilon$: in short f is continuous at z. ∎

The family of open sets is what determines which functions are continuous and which are not. It is completely possible to have two very different metrics with exactly the same open balls and hence the same open sets.

Exercise 1.4.11 *Give an example of two metrics that define the same open sets. I am sure you came up with a boring example. Come up with a more interesting example. If d is any metric on X we define a new metric by the formula*

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that this is a metric with exactly the same open balls. Note that $d(x, y) \leq 1$.

Definition 1.4.12 *We say two metrics are* ***equivalent*** *if they define the same open sets.*

We have defined metrics to take on real values. We can extend the concept to allow for infinite distances.

Definition 1.4.13 *An* ***extended metric*** *is a metric that can take on infinite values.*

The category of extended metric spaces is much better behaved that the category of ordinary metric spaces.

1.5 Dense sets and nowhere dense sets

We are always talking about sets in a fixed metric space (X, d).

Definition 1.5.1 *The **closure** of a set A is the smallest closed set containing A.*

Example 1.5.2 *The closure of the open ball $B(x, r)$ is the closed ball with the same center and radius written $C(x, r) = \{u|d(x, u) \leq r\}$.*

In metric spaces a good way of thinking of the closure is the set together with all the

Are all sets either open or closed? Certainly not, the interval $[a, b)$ is open on one side and closed on the other. It is neither open nor closed. There are some very interesting sets that are neither open nor closed. For example, the rational numbers are definitely not open and their complement is not open either so they are not closed. However, they have an interesting property: they are everywhere!

Definition 1.5.3 *A subset of X is said to be **dense** if its closure is X.*

The rationals are dense in the reals; every real is the limit of a sequence of rationals.

Exercise 1.5.4 (For the math whizzes who are bored.) *Show that the set $\{m + \sqrt{2}n|m, n \in \mathbf{Z}\}$ is dense in $\mathbf{R}$.*

Dually to the closure we define

Definition 1.5.5 *The **interior** of a set $A \subset X$ is the largest open set contained in it.*

Example 1.5.6 *The interior of $[0, 1)$ is $(0, 1)$ and its closure is $[0, 1]$. The closure of the rationals is all the reals but its interior is the empty set! There are only countably many rationals so we have a countable dense subset with empty interior.*

1.6 The Baire Category Theorem

This is one of the most important theorems in analysis yet its statement seems arcane and its proof seems easy.

Theorem 1.6.1 (Baire) *A complete metric space cannot be the union of countably many nowhere dense sets.*

Before we prove this we introduce some concepts that are important in their own right.

Definition 1.6.2 *If A is a set and x is a point we define*

$$d(x, A) = \inf_{a \in A} \{d(x, a)\}.$$

This is the distance from a point to a set. Of course it is not a metric. It is easy to see (?) that for a fixed A this defines a continuous function of x.

Definition 1.6.3 *The* ***diameter*** *of a set A is*

$$D(A) = \sup_{a,b \in A} d(a, b).$$

This sup need not be attained of course.

Theorem 1.6.4 *Let (X, d) be a* complete *metric space. Let C_n be a nonempty nested decreasing family of closed sets with the limit of the sequence $D(C_n)$ being 0. Then $C := \bigcap_n C_n$ contains exactly one point.*

Why not?

Proof. Now C cannot contain more than one point. Thus, we have to show that it is not empty. Choose a point x_n in C_n. Since the diameters are decreasing to 0, the sequence (x_n) must be Cauchy. Since X is complete there must be a limit, call it x. Now fix any m. All the x_n with $n \geq m$ must be in C_m and this subsequence has the same limit as the whole sequence, namely x, as we have only chopped off finitely many elements in the front. Since C_m is closed we have $x \in C_m$. But this holds for all the m so x is in all the C_m, hence in C. ∎

Proof. Of Baire's theorem. The readers can readily convince themselves that a set A is nowhere dense if and only if any open set U contains an open ball B such that $A \cap B = \emptyset$. Suppose we have a family $\{A_n\}$ of nowhere dense sets. Since A_1 is nowhere dense there is a ball B_1 of radius less than 1 disjoint from A_1. We now take a closed sphere D_1 with the same centre as B_1 and with radius $\frac{1}{2}$ of that of B_1. The interior of D_1 is of course open and so there must be a ball B_2 inside the interior of D_1 with radius less than $\frac{1}{2}$ that of B_1 and disjoint from A_2, since A_2 is also nowhere dense. Proceeding in this way we get a family of nested closed sets (D_n) each of which is non-empty and which has $D(D_n)$ going to 0. By Theorem 1.6.4 it follows that there is a limit, call it x, to the sequence and it is in all the $C_n's$. Thus it is disjoint from all the A_n so it is not in our union. ∎

2

BASIC TOPOLOGY

2.1 Basic ideas

Recall the first definition of continuity from metric spaces:

$f\colon (X, d) \longrightarrow (Y, d')$ is continuous at x_0 if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, d(x, x_0) < \delta \Rightarrow d'(f(x), f(x_0)) < \varepsilon$$

We then introduced the notion of *open ball* and then *open set* and proved that a function was continuous iff the inverse image of an open set is open. Thus we can take this as the *definition* of continuity. Notice that there is no explicit definition of ε or δ in this version. This suggests that we can make a definition that relies on open sets and not on metrics. But we defined open sets in terms of a metric! How do we get away from that? By *axiomatizing the notion of open set* and making it the central definition. This brings us to the realm of point-set topology. It is crucial that you understand that this is a proper generalization of metric space theory: there are topological spaces that *cannot* be defined in terms of any metrics. In fact topological spaces that can be defined by metrics are a *very* special class of spaces called *metrizable spaces.*

Definition 2.1.1 *A* ***topological space*** $(X, \mathcal{O})$ *is a set* X *equipped with a family of subsets, called* ***open sets,*** $\mathcal{O}$ *such that*

1. $X, \emptyset \in \mathcal{O}$
2. *If* $A, B \in \mathcal{O}$, *then* $A \cap B \in \mathcal{O}$
3. *If* $\{A_i \mid i \in I\}$ *is* any *family of sets in* $\mathcal{O}$ *(as large as you want), then* $\bigcup_i A_i \in \mathcal{O}$.

Now we can define continuous in this general setting.

Definition 2.1.2 *A function* $f : X \longrightarrow Y$ *between topological spaces* X *and* Y *is said to be continuous if the inverse image of an open set is open.*

How do we decide what are the open sets in a given situation? It depends on what *you* want to do with them. You get to choose the collection of open sets for some purpose of your own. The only constraint is that they conform to the requirements of the definition above. You should think of the choice this way: the open sets describe how well you can "see" a space. If there are lots of open sets you can see the details of the space. It is very rare that singleton sets are open. In the situation where the open sets do not include the individual points one should think that in some sense one cannot see the individual points of the space.

Here are some examples:

Example 2.1.3 *Indiscrete topology:* $\mathcal{O} = \{X, \emptyset\}$ *Here one has a very crude view of the space. One can only see the whole space and nothing of its internal structure.*

Example 2.1.4 *Discrete topology:* $\mathcal{O} = \mathcal{P}(X)$. *In this case* every *set is open and one can see everything. This is rarely a good description of a situation of interest.*

Example 2.1.5 *Sierpinski space:* $X = \{\top, \bot\}$, $\mathcal{O} = \{\emptyset, X, \{\top\}\}$. *Note that no metric can give such a topology.*

Example 2.1.6 *Real intervals:* $X = \mathbf{R}$, $U \in \mathcal{O}$ *if* $\forall x \in U \exists a, b \in \mathbf{R}, x \in (a, b) \subseteq U$. *We throw* $\emptyset$ *into* $\mathcal{O}$ *in order to satisfy the axioms.*

Example 2.1.7 *Infinite Sequences:* $\Sigma^\omega = \Sigma^* \cup \Sigma^\infty$ *ordered by prefix.* U *is open if* $\forall x \in U \exists w \in \Sigma^*. w \leq x$ *and* $\{y \mid w \leq y\} \subseteq U$.

Example 2.1.8 *Metric space: Let* (X, d) *a metric space. For* $r \in \mathbf{R}$, *the* balls *are the sets*

$$B_r(x) = \{u \mid d(x, u) < r\}$$

We define a topology as follows. $X, \emptyset$ *are open sets, and* U *is open if* $\forall x \in U \exists r \in \mathbf{R}^+. B_r(x) \subseteq U$. *We refer to this topology as the topology* induced by the metric. *Note that the same topology may be induced by different metrics.*

Example 2.1.9 *Cofinite topology:* $(\mathbf{N}, \mathrm{cof}(\mathbf{N}) \cap \{\emptyset\})$. *Here* $\mathrm{cof}(X)$ *means the collection of subsets of* X *such that the complement of the set is finite.*

2.1.1 Closed sets

Definition 2.1.10 *Let $(X, \mathcal{O})$ be a topological space. A* closed *set $A \subseteq X$ is the complement of an open set, i.e.*

$$A^c \in \mathcal{O}(X)$$

Remark 2.1.11 *$X, \emptyset$ are both open and closed. We call such sets "*clopen*" sets. The complement of a clopen set is clopen.*

Proposition 2.1.12 *The collection $\mathcal{C}$ of closed sets satisfies the following properties:*

- $X, \emptyset \in \mathcal{C}$
- $C_1, C_2 \in \mathcal{C} \Rightarrow C_1 \cup C_2 \in \mathcal{C}$
- $\{C_i \mid i \in I\}$ *is a family of closed sets* $\Rightarrow \bigcap_{i \in I} C_i \in \mathcal{C}$.

Hence the collection of clopen sets forms a boolean algebra.

Definition 2.1.13 *Given any $A \subseteq X$, the* closure *$\overline{A}$ of A is the smallest closed set containing A.*

$$\overline{A} = \bigcap_{C \in \mathcal{C}, A \subseteq C} C$$

The interior *A° of A is the largest open set contained in A:*

$$A^\circ = \bigcup_{U \in \mathcal{O}, U \subseteq A} U$$

Why are these things well defined?

Example 2.1.14 *Consider $\mathbf{R}$ with the usual topology.*

$\overline{((0,1])}^\circ = (0,1)$
$\overline{(0,1]} = [0,1]$
$\overline{\mathbf{Q}} = \mathbf{R}$
$\mathbf{Q}^\circ = \emptyset$.

Lemma 2.1.15 *The closure operation on sets satisfies:*

- $A \subseteq \overline{A}$

- $\overline{A} = \overline{\overline{A}}$
- $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$

We are thinking of open sets as capturing what we can see of a space. Let us say that we are looking at a point x, which means that we are considering open sets that contain x. Suppose that any such open set intersects a particular set A. This means that we cannot really see whether x is inside A or not, it is "right on the edge".

Definition 2.1.16 *Given a set A a* ***limit point*** *of A is a point x such that any open set containing x contains points of A other than x itself (if x is in A). We write A' for the set of limit points of A.*

Why do we have to have the awkward phrase "other than x itself"? It may happen that A contains a point that is by itself.

Definition 2.1.17 *Let A be a set in a topological space X. A point x is called an* ***isolated point*** *of X if there is an open set U such that $x \in U$ but no other point of A is in U.*

Here the open set U allows us to see that x is apart from the rest of A.

Definition 2.1.18

$$\tilde{A}\colon = \{x \mid \forall U \in \mathcal{O},\ x \in U \Rightarrow U \cap A \neq \emptyset\}$$

$\tilde{A}$ is exactly the set of limit points of A' together with A: $\tilde{A} = A \cup A'$.

Proposition 2.1.19 $\overline{A} = \tilde{A}$

Proof. Suppose $x \notin \tilde{A}$, i.e. $\exists U \in \mathcal{O}, x \in U \wedge U \cap A = \emptyset$. U^c is closed. $A \subseteq U^c$ and $x \notin U^c$ implies that $x \notin \overline{A}$ since $\overline{A}$ is the intersection of all closed sets that contain A.

Conversly, suppose $x \notin \overline{A}$. Then $\exists C \in \mathcal{C}, A \subseteq C, x \notin C$. $x \in C^c$ is open. $C^c \cap A = \emptyset$, so $x \notin \tilde{A}$. ∎

In view of this proof we can jettison the notation $\tilde{A}$ and just use $\overline{A}$.

The operation of forming the closure of a set adds the points "right next to the set". What points are "right on the edge" of a set?

Definition 2.1.20 *The **boundary** of a set A is written $\partial A := \overline{A} \cap \overline{A^c}$.*

The next proposition captures the basic relations between boundaries, interiors and closures.

Proposition 2.1.21 *For any subset A of a topological space $(X, \mathcal{T})$*

1. $\overline{A} = A \cup \partial A$
2. $A^\circ = A \setminus \partial A$
3. $X = A^\circ \cup \partial A \cup (A^c)^\circ$

Proof: DIY.

2.1.2 Bases and subbases

How can we construct topologies on a space? We would like to start from some collection of sets that we find interesting for some reason and construct a topology out of them. Can we always do this?

Definition 2.1.22 *A **base** or **basis** for a topology $(X, \mathcal{O})$ is a family of open sets $\mathcal{B}$ such that for every $U \in \mathcal{O}$ and for every $x \in U$ there is a set $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

Well clearly, we can take all of $\mathcal{O}$ as a base; the concept becomes interesting if we can define a topology more easily by giving an easy-to-describe base. For example, the open intervals form a base for the usual topology on $\mathbf{R}$.

How do we know whether some arbitrary collection of sets is the base for a topology?

Proposition 2.1.23 *Let X be a set and set $\mathcal{B}$ be a family of subsets such that:*

1. *each element x of X is contained in some $B \in \mathcal{B}$ and*
2. *if $x \in B_1 \cap B_2$ for some B_1, B_2 in $\mathcal{B}$ then there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.*

The collection of arbitrary unions of members of $\mathcal{B}$ forms a topology.

Proof. The empty set is the union of the empty collection and condition (1) says that X is the union of all the elements of $\mathcal{B}$. Suppose that $U_1 = \cup_{i \in I} B_i$ and $U_2 = \cup_{j \in J} B_j$ then

$$U_1 \cap U_2 = \bigcup_{i \in I, j \in J} B_i \cap B_j.$$

Now, for each $x \in B_i \cap B_j$ there is some B_x such that $x \in B_x \subset (B_i \cap B_j)$ so $B_i \cap B_j = \cup_{x \in B_i \cap B_j} B_x$. Thus

$$U_1 \cap U_2 = \bigcup_{i \in I, j \in J} \bigcup_{x \in B_i \cap B_j} B_x,$$

thus, $U_1 \cap U_2$ can be expressed as the union of members of $\mathcal{B}$. It is obvious that any union of sets that are themselves unions of sets of $\mathcal{B}$ is a union of sets of $\mathcal{B}$. ∎

As we noted above, the set of open intervals forms a base for the topology. This is a big relief: we can work with the basis sets most of the time and not have to worry about arbitrary open sets.

What if we have an *arbitrary* collection and we want to generate a topology from them? We need the intersection condition for a base otherwise we will not get a topology. However, we can construct the intersections first.

Proposition 2.1.24 *Let $\mathcal{S}$ be an arbitrary family of subsets of X. The collection $\mathcal{B}$ of all finite intersections of members of $\mathcal{S}$ is a base for a topology.*

Proof. Straightforward. ∎

If the resulting topology is $\mathcal{O}$ we say that $\mathcal{S}$ is a *subbase* for $\mathcal{O}$.

Example 2.1.25

- $\mathbf{R}$ *has a basis: the open intervals.*
- $\mathbf{R}^n$*: there is a basis of open balls.*
- $\Sigma^\omega = \Sigma^* \cup \Sigma^\infty$. $x \in \Sigma^*$, $x \uparrow = \{y \mid x \leq u\}$ *forms the base for the Scott topology.*

Definition 2.1.26 *A topological space with a countable base is called a* second countable *space.*

Example 2.1.27 *In* $\mathbf{R}$*, open intervals with rational end points forms a countable base; thus this is an example of a second countable space.*

Definition 2.1.28 *A set* $A \subseteq X$ *is* ***dense*** *if* $\overline{A} = X$*.*

Example 2.1.29 *The rationals are a dense subset of the reals.*

The finite streams are a dense subset of Σ^ω*.*

Perhaps this is clear, perhaps not. If not please prove it.

Note that a set is dense if every open set intersects it.

Definition 2.1.30 *A space is called* ***separable*** *if it has a countable dense subset.*

Theorem 2.1.31 *Every separable metric space is second countable.*

Proof. To show that we have a base for a topology, it suffices to show that every open set contains a basic open neighbourhood of each point. Suppose A is a countable dense subset of X. Take all open balls of the form

$$B(a,q) = \{x \mid a \in A, d(a,x) < q\}, q \in \mathbf{Q}$$

This is a countable family of open sets.

Let U be any open set with $x \in U$. The definition of being open in a metric space says $\exists r.B(x,r) \subseteq U$. Consider $B(x,r/3) \subseteq B(x,r) \subseteq U$. Being dense means that for any point x and any neighbourhood U of x, U must intersect the dense set. So $B(x,r/3)$ contains some $a \in A$. Choose $q \in \mathbf{Q}$ in $(r/3, 2r/3)$. Consider $B(a,q)$; $x \in B(a,q) \subseteq B(x,r) \subseteq U$. Thus every U can be expressed as a union of basic open sets. ∎

Proposition 2.1.32 *Every second countable topological space is separable.*

Proof. Let $\mathcal{B}$ be a countable base for a topological space X. Choose a point from every basic open set. Now any open set must contain a basic open set and hence must contain one of our chosen points. In view of the observation made above this shows that the set of points that we constructed is dense. ∎

Note that this is not just true for metric spaces.

2.1.3 Neighbourhood systems

Everything has been couched in terms of open sets (or closed sets) so far. It is often the case that in the spaces that we consider one point looks pretty similar to another. We can use a more local presentation of topology that is more convenient for most purposes.

Definition 2.1.33 *In a topological space X, a **neighbourhood** of a point x is a set A containing an open set U that contains x: $x \in U \subseteq A$. The set of neighbourhoods of x is called the **neighbourhood system** at x and is written $\mathcal{N}_x$.*

Often we mean to use neighbourhoods that are open, we will then say *open neighbourhood.* Indeed, some authors define neighbourhood to mean open neighbourhood.

Proposition 2.1.34 *The neighbourhood system $\mathcal{N}_x$ of a point x in a topological space X has the following properties:*

1. *if $A \in \mathcal{N}_x$ then $x \in A$,*
2. *if $A, B \in \mathcal{N}_x$ then $A \cap B \in \mathcal{N}_x$,*
3. *if $A \in \mathcal{N}_x$ then there is a $B \in \mathcal{N}_x$ such that $A \in \mathcal{N}_y$, for each $y \in B$,*
4. *if $A \in \mathcal{N}_x$ and $A \subset B$ then $B \in \mathcal{N}_x$.*
5. *A set U is open iff it contains a neighbourhood of each of its points.*

Conversely, if we have a family of collections of sets, one for each point of X and these families satisfy the conditions 1-4 and we use condition 5 to define the open sets we will get a topology and the topological neighbourhoods will be exactly the families that we started with.

The point of this fairly obvious proposition is that one can define topologies by defining neighbourhood systems instead of open sets. We will use whatever is more convenient.

Definition 2.1.35 *A **neighbourhood base** at a point x of a topological space X is a subcollection $\mathcal{B}_x$ of the neighbourhood system $\mathcal{N}_x$ such that each neighbourhood A of $\mathcal{N}_x$ contains some $B \in \mathcal{B}_x$.*

A neighbourhood base determines the neighbourhood system and thus a topology.

Definition 2.1.36 *A topological space is said to be **first countable** if there is a countable neighbourhood base for every point.*

Clearly second countable is a much stronger condition. Can you come up with an example of a first countable space that is not second countable?

2.1.4 Convergence of sequences

We can formulate convergence of sequences in purely topological terms.

Definition 2.1.37 *A sequence (x_n) in a topological space X **converges to a point** x if* every *neighbourhood of x (or open set containing x) contains all but finitely many of the x_n's.*

You should convince yourself that on metric spaces this yields exactly the usual definition.

Example 2.1.38 *Here is an example of wierdness that can happen in non-metric topologies. Recall the cofinite topology of Example 2.1.9. What is an open set? Any set that excludes only finitely many points. In this topology every sequence with infinitely many distinct points converges to* every point*. This is spectacularly different from the situation in metric spaces. By contrast, we can define a cocountable topology: the open sets are the complements of countable sets. Here only eventually constant sequences converge at all.*

3

BASIC TOPOLOGY PART 2

3.1 Continuity

Recall the definition of continuity.

Definition 3.1.1 *$f\colon X \to Y$ is* continuous *if for every open set $U \in \mathcal{O}(Y)$, $f^{-1}(U) \in \mathcal{O}(X)$.*

Many people think that the following is the definition of continuity; it is not!

Definition 3.1.2 *$f\colon X \to Y$ is called an* open map *if for all $U \in \mathcal{O}(X)$, $f(U) \in \mathcal{O}(Y)$.*

Continuous maps need not be open and open maps need not be continuous.

Example 3.1.3 *Consider the function $I : (\mathbf{R}, \mathsf{Usual}) \to (\mathbf{R}, \mathsf{Discrete})$ where I is the identity function, and we are referring to two* different *spaces with the same underlying set: the first one has the usual topology and the second has the discrete topology. Here I is open but certainly not continuous. If we consider the identity map in the other direction we have an example of a continuous map that is not open.*

Definition 3.1.4 *$f\colon X \to Y$ is called a* ***closed map*** *if for all $U \in \mathcal{C}(X)$, $f(U) \in \mathcal{C}(Y)$.*

Continuity can equally well be defined in terms of closed sets.

Proposition 3.1.5 *If $f\colon X \to Y$ is continuous and $C \in \mathcal{C}(Y)$, where $\mathcal{C}(Y)$ is the collection of closed sets, then $f^{-1}(C) \in \mathcal{C}(X)$.*

The identity function is always continuous:

$$I\colon (X, \mathcal{O}) \to (X, \mathcal{O})$$

Note: I mean when the spaces are the same as topological spaces.

$$I\colon (X,\mathcal{O}_1) \longrightarrow (X,\mathcal{O}_2).$$

I will be continuous if $\mathcal{O}_2$ is coarser than $\mathcal{O}_1$ (has fewer open sets). A constant function is *always* continous.

Here is an elementary but vital fact.

Proposition 3.1.6 *The composition of continuous functions is continuous.*

Though easy to prove (do it!) it is absolutely fundamental.

Definition 3.1.7 *$f\colon X \longrightarrow Y$ is called a **homeomorphism** if f is continuous, f is a bijection (so f^{-1} is a function), and f^{-1} is also continuous.*

Remark 3.1.8 *Homeomorphic spaces are the* same *topologically.*

The **support** of a continuous function $f : X \longrightarrow \mathbf{R}$ is the *closure* of the set of points where it does not vanish:

$$\mathsf{supp}(f) = \overline{\{x \in X | f(x) \neq 0\}}.$$

3.2 Subspaces, products and quotients

Here is a common way to make new topological spaces.

Definition 3.2.1 *If $(X,\mathcal{O})$ is a topological space and $A \subseteq X$ (A need not be open), A inherits a topology called the **subspace topology** where an open set has the form $A\bigcap U$; $U \in \mathcal{O}$.*

Example 3.2.2 $[0,1) \subseteq \mathbf{R}$. *In the inherited topology,* $[0,1/2)$ *is open but not in* $\mathbf{R}$.

Example 3.2.3 *Let $A \subseteq X$.* inc: $A \longrightarrow X$. *If A has the subspace topology,* inc *is continuous. It fact, the subspce topology is the coarsest topology making* inc *continuous.*

We say one topology is **coarser** than another if it has fewer open sets and **finer** if it has more open sets.

The following observations tell us that we can resrtict the domain or extend the range of a continuous function.

Proposition 3.2.4 *If $f\colon X \longrightarrow Y$ is continuous and $A \subseteq X$ then $f|_A\colon A \longrightarrow X$ is continuous. If $Y \subseteq Z$ and Y has the subspace topology, then $f\colon X \longrightarrow Z$ is continuous.*

Given a space X and an equivalence relation $\sim$ on it we can define the usual quotient space $X/\sim$. We define a topology on $X/\sim$ by endowing it with the finest topology (most open sets) that makes the quotient map $x \mapsto [x]$ continuous.

Now we define the very important notion of topological product.

Definition 3.2.5 *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. We define the* ***product space*** *to be the set $X \times Y$ equipped with a A basis for the topology on $X \times Y$ is given by $U \times V$ where $U \in \mathcal{O}(X)$, $V \in \mathcal{O}(Y)$.*

To verify that a function is continuous, it suffices to check that the inverse image of a basic open set is open. This makes it immediately clear that π_1 and π_2 are continuous.

For more general products we are guided by categorical considerations. Given an I-indexed family of topological spaces we define the product space $X = \Pi_{i\in I} X_i$ in the usual set-theoretic way. There are projection maps $\pi_i : X \longrightarrow X_i$ for every i. We define the topology on X to be the coarsest topology such that all the projection maps are continuous. What does this mean explicitly? Given any $U \subset X_i$ we want $\pi_i^{-1}(U)$ to be open in X. This means that any set of the form $X_1 \times X_2 \times \ldots \times X_{i-1} \times U \times X_{i+1} \times \ldots$ must be open. These are all that are needed to make the π_i's continuous, however, we need to satisfy the closure conditions. We therefore have to take these sets to be a subbase for the topology of X. Thus a base for the topology is the collection of all sets of the form $\Pi_i U_i$ where all the U_i are open in the respective X_i and *all but finitely many of them are* X_i. One needs to verify this this is indeed a basis, but that is easy. If this definition if dropped in one's lap it seems odd, but from the viewpoint of the projection functions being continuous it is entirely natural.

A related topology is the box topology. Here we take as base all sets of the form $\Pi_i U_i$ where the U_i are all open subsets of the corresponding X_i. For finite products this is the same as the product topology but for infinite products they are very different. The product topology as we have defined it has far nicer properties and plays a major role in mathematics; the box topology is just an unimportant curiosity.

3.3 Separation Properties

We would like to look at a variety of conditions that tell us how well the open sets allow us to see the points of a topological space.

Definition 3.3.1 *T_0: For every pair of points there is an open set containing one of the points but not the other.*

Definition 3.3.2 *T_1: Given a pair of points x, y, there are open sets U, V such that*

$$x \in U \setminus V, y \in V \setminus U$$

Example 3.3.3 *The Sierpinski space, $\{\emptyset, \{\top\}, S\}$, $S = \{\top, \bot\}$, is T_0 but not T_1*

Proposition 3.3.4 *A space is T_1 iff every singleton $\{x\}$ is a closed set.*

Proof. Suppose X is T_1. Consider $\{x\}$. Let $y \in \{x\}^c$ There is an open set U such that $y \in U$ and $x \notin U$. This means that $U \subseteq \{x\}^c$. So $\{x\}^c$ is open, and $\{x\}$ is closed.

Conversely, suppose all singletons are closed. Suppose $x \neq y$. $\{x\}^c$ is an open set containing y and not x, $\{y\}^c$ is an open set containing x but not y. ∎

Definition 3.3.5 *Given a topological space $(X, \mathcal{O})$ we define $x \lesssim y$ called the **specialization order** to mean that every open set containing x also contains y.*

In any T_1 space, the specialization order is trivial. For $\mathcal{S}$, this order gives $\bot \lesssim \top$. This is a complete lattice in fact a boolean algebra. Sierpinski space is both a topological space and a boolean algebra: a schizophrenic object. In the theory of ordered sets the specialization order plays an important role; clearly the relevant topology is not even T_1.

Definition 3.3.6 *T_2 (Hausdorff) Given a pair of points x, y there are disjoint open sets U, V such that $x \in U$ and $y \in V$.*

Example 3.3.7 $\mathbf{R}$ *with the cofinite topology is T_1 but not T_2*

The Hausdorff condition makes the concept of convergence more sane.

Proposition 3.3.8 *In a Hausdorff space every convergent sequence has a unique limit.*

Proof. Suppose $(x_i) \longrightarrow x$ and $y \neq x$. Then there are disjoint open neighbourhoods U, V of x, y respectively. All but finitely many of the x_i are in

U, so it cannot be true that all but finitely many are in V, so y is not a limit for (x_i). ∎

The next separation property is about points and closed sets.

Definition 3.3.9 *A topological space X is **regular** if $x \in X$, C is a closed subset of X and $x \notin C$, then $\exists U, V \in \mathcal{O}(X)$. $U \cap V = \emptyset$ and $x \in U$ and $C \subseteq V$. If a space is T_1 and regular we call it T_3.*

The reason that we have to add T_1 is because a space with the indiscrete topology is regular for silly reasons. Clearly a T_3 space is T_2 because in a T_1 space points are closed sets.

Example 3.3.10 *Any metric space is T_3.*

Example 3.3.11 $\mathbf{R}$ *with open intervals and* $\mathbf{Q}$ *as a subbase. Certainly T_2, but not T_3 because one cannot separate* 0 *and* $\mathbf{Q}^c$.

Definition 3.3.12 *A topological space X is **completely regular** if for any closed set A and point $x \notin A$ there is a continuous real-valued function f such that $f(x) = 0$ and $\forall a \in A, f(a) = 1$. A completely regular T_1 space is called a **Tychonoff** or $T_{3\frac{1}{2}}$ space.*

Again silly spaces can be completely regular for silly reasons. Some authors use "completely regular" to mean what we are calling a Tychonoff space.

Any $T_{3\frac{1}{2}}$ space is T_3. An example showing that the converse does not hold is complicated.

Definition 3.3.13 *A space is* normal *if given any two disjoint closed sets A, B there are disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$. A normal T_1 space is called T_4.*

Sierpinski space is normal because we cannot find two disjoint closed sets. All metric spaces are T_4.

Is there a $T_{4\frac{1}{2}}$ concept?

Theorem 3.3.14 (Urysohn) *Given a T_4 space X and two closed sets $C, D \subset X$ there is always a continuous function $f : X \longrightarrow \mathbf{R}$ such that $\forall x \in C, f(x) = 0$ and $\forall y \in D, f(y) = 1$.*

3.4 Compactness

The rôle of compactness in topology is to tame the infinite. If we are just counting cardinalities of sets then most sets that we encounter are infinite, indeed uncountable. Cardinality is a hopelessly crude classification of sets. Compact sets have many of the properties of finite sets, or more precisely, of bounded sets. In this section I use the notation A^c to stand for the complement of A.

Definition 3.4.1 *An* open cover *of a topological space* $(X, \mathcal{O}$ *is a family of open sets* $(U_\alpha \mid \alpha \in \mathcal{A})$ *such that*

$$X = \bigcup_{\alpha \in \mathcal{A}} U_\alpha.$$

If $A \subseteq X$ *is a subspace, then an open cover of* A *is a family of open sets* $(U_\alpha \mid \alpha \in \mathcal{A})$ *such that*

$$A \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha.$$

Definition 3.4.2 *A* subcover *of* $(U_\alpha \mid \alpha \in \mathcal{A})$ *is a subfamily that also covers* X *(or* $A \subseteq X$*).*

Here is the crucial definition.

Definition 3.4.3 $(X, \mathcal{O})$ *is* compact *if* every *open cover has a finite subcover.*

Thus *no matter how* you cover a space a finite subcover will suffice. Note that this is not just saying that the space has some finite cover by open sets: every space has that since X is an open set.

How does compactness relate to being closed?

Theorem 3.4.4 *A closed subspace of a compact space is compact.*

Proof. Suppose that X is some compact space and suppose $D \subset X$ is closed. Let $(U_\alpha \mid \alpha \in \mathcal{A})$ be an open cover of D. Then

$$\{U_\alpha \mid \alpha \in \mathcal{A}\} \cup D^c$$

is a cover of X. So it has a finite subcover $\{U_1, \ldots U_n\}$; this is clearly a finite subcover of D once we remove C^c if it occurs among the U_i. ∎

Theorem 3.4.5 *Let $f\colon X \longrightarrow Y$, f continuous, X compact; then $f(X)$ is compact.*

Proof. Suppose we have a cover $(U_\alpha) \mid \alpha \in \mathcal{A})$ of $f(X)$. Then $(f^{-1}(U_\alpha) \mid \alpha \in \mathcal{A})$ covers X. Since X is compact, a finite subcover of this covers X, so we have a finite collection of sets $\{f^{-1}(U_1), \ldots, f^{-1}(U_n)\}$ covers X. Then we have a finite subcover, namely $\{U_1, \ldots, U_n\}$ that covers $f(X)$. ∎

Definition 3.4.6 *A non-empty family of subsets of X, $\mathcal{F}$, is said to have the finite intersection property (fip) if every finite subfamily of $\mathcal{F}$ has non-empty intersection.*

Theorem 3.4.7 *A topological space $(X, \mathcal{O})$ is compact if and only if every family of closed sets with fip has non-empty intersection.*

Proof. $\Rightarrow$: suppose $\mathcal{K}$ is a family of closed sets with fip and assume

$$\bigcap_{K \in \mathcal{K}} K = \emptyset.$$

Note

$$\left(\bigcap_{K \in \mathcal{K}} K\right)^c = X,$$

i.e. $\bigcup_{K \in \mathcal{K}} K^c = X$.

Now $\{K^c \mid K \in \mathcal{K}\}$ is an open cover of X. There is a finite subcover $\{K_1^c, \ldots K_n^c\}$. Then $\bigcup_{i=1}^n K_i^c = X$ i.e. $\left(\bigcap_{i=1}^n K_i\right)^c = X$, and thus $\bigcap_{i=1}^n K_i = \emptyset$, but this contradicts the finite intersection assumption.

The converse direction is left as an exercise.

∎

I will state a simple and obvious facts without proof. Despite the fact that it is obvious it is useful because it makes it a lot easier to show that specific spaces are compact.

Proposition 3.4.8 *A space is compact if and only if every* basic *open cover has a finite subcover.*

Proposition 3.4.9 *In a metric space a compact set is always bounded.*

Proof. Suppose that $K \subset X$ is compact. Let x be any point in K. We cover K by the family of open balls $\{B_r(x) \mid r > 0\}$. Every point has to be at some distance from x so every point is in at least one of these balls. Since K is compact we have a finite collection $\{B_{r_1}(x), \ldots, B_{r_n}(x)\}$ that also covers K. One of these radii is the largest and must therefore contain K. Hence K is bounded. ∎

Theorem 3.4.10 *Heine-Borel A closed interval* $[a, b]$ *in* $\mathbf{R}$ *is compact.*

Proof. Let $\mathcal{U} = (U_\alpha \mid \alpha \in \mathcal{A})$ be an open cover of $[a, b]$ (assume $a < b$).

$$V = \{x \in [a, b] \mid [a, x] \text{ has a finite subcover by sets in } \mathcal{U}\}$$

$a \in V$, so V is not empty. Let $v = \sup(V)$. Then $\mathcal{U}$ contains some *open* set, call it U_0, such that $v \in U_0$, since v is $\sup(V)$, U_0 must intersect V, so some $z \in V \cap U_0$ since $z \in V.[a, z]$ has a finite subcover $\{U_1, \ldots, U_n\}$. But then $\{U_0, U_1, \ldots, U_n\}$ is a finite subcover of $[a, y]$ where $y \in U_0$ and is chosen so that $v < y$, i.e. v cannot be the sup of V, contradiction.

The only possibility is $v = b$. In that case, $[a, b]$ has a finite subcover. ∎

Notice that $(0, 1)$ is not compact. For example, $\{(0, 1 - 1/n) \mid n > 1\}$ is an open cover of $(0, 1)$, without any finite subcover.

How does compactness interact with the separation properties? Recall that every closed subset of a compact space is compact.

Proposition 3.4.11 *Every compact subset of a Hausdorff space is closed.*

Proof. Suppose $(X, \mathcal{O})$ is T_2 and $K \subseteq X$ is compact. Is K^c open? Let $x \in K^c$, let $y \in K$. So there are disjoint open sets U_y, V_y such that $x \in U_y$ and $y \in V_y$. Now V_y forms an open cover of K, so there is a finite subcover $\{V_1, \ldots, V_n\}$ of K. Now look at the corresponding Us. x is in each of them, so $x \in \bigcap_{i=1}^n U_i$: open. Clearly,

$$\left(\bigcap_{i=1}^n U_i\right) \cap \left(\bigcup_{i=1}^n U_i\right) = \emptyset$$

so $\bigcap_{i=1}^n U_i \cap K = \emptyset$, i.e. $\bigcap_{i=1}^n U_i \subseteq K^c$, so K^c is open; K is closed.

∎

Using a variation of the above argument one can show that compact Hausdorff spaces are T_4 or normal. Here is an interesting fact that makes a cute exercise for the reader.

Theorem 3.4.12 *If $f\colon X \longrightarrow Y$ continuous and bijective, X compact and Y Hausdorff, then f is a homeomorphism.*

An almost immediate consequence of this is that compact Hausdorff spaces are "on the edge." If you add any open sets the compactness property is lost and if you remove any open sets the Hausdorff property is lost.

3.4.1 Compactness of product spaces

For finite products we can argue from first principles to show that the product of compact spaces is compact.

The following lemma is called the *tube lemma.*

Lemma 3.4.13 *If X, Y are topological spaces and Y is compact, $x_0 \in X$ and U is an open set of $X \times Y$ containing $\{x_0\} \times Y$ then, there exists an open set W of X such that $x_0 \in W$ and $W \times Y \subseteq U$.*

Proof. Note that $\{x_0\} \times Y$ is homeomorphic to Y and hence compact. Let $\mathcal{U}$ be an open cover of $\{x_0\} \times Y$; we can take these sets to all lie inside U, for example by intersecting with U if necessary. A base for $X \times Y$ consists of sets of the form $V_i \times U_i$ where $V_i \in \mathcal{O}(X)$, $U_i \in \mathcal{O}(Y)$. Since $\{x_0\} \times Y$ is compact, finitely many of these sets suffice to cover $\{x_0\} \times Y$. Now, $W := V_1 \cap V_2 \cap ... \cap V_n$ is open and contains x_0. This family of sets actually covers all of $W \times Y$. Let (x, y) be a point in $W \times Y$. Consider the point (x_0, y), this belongs to some $V_i \times U_i$ so $y \in U_i$, but $x \in V_i$ because $x \in W$ hence it belongs to all sets of the form V_i, thus, $(x, y) \in V_i \times U_i$. Thus we have

$$W \times Y \subset \bigcup_{i=1}^{n} V_i \times U_i \subset U.$$

∎

Theorem 3.4.14 *If X, Y are both compact then, so is $X \times Y$.*

Proof. Let $\mathcal{U}$ be an open cover of $X \times Y$. For each $x_0 \in X$ the set $x_0 \times Y$ is compact so there is a finite family $\{U_1, \ldots, U_n\}$ of sets from $\mathcal{U}$ that cover it. The set $N = \cup_i U_i$ is an open set containing $\{x_0\} \times Y$, so, by the tube lemma, there is an open subset W of X with $W \times Y \subset N$. This set is covered by finitely many sets from $\mathcal{U}$, *viz* the $\{U_i\}$. We can construct such a W_x for every point $x \in X$. The sets W_x cover X, since X is compact we can find a finite subcover $\{W_1, \ldots, W_m\}$ of x. The tubes $W_i \times Y$ cover $X \times Y$ and each $W_i \times Y$ can be covered by finitely many sets from $\mathcal{U}$ so, we can always find a finite subcover of $\mathcal{U}$. ∎

The surprising fact is that *any* product of compact spaces is compact. First, a recap of the topology of $\prod_\alpha X_\alpha$. The projections

$$p_\beta : \left(\prod_{\alpha \in \mathcal{A}} X_\alpha\right) \to X_\beta$$

all have to be continuous. Let $U \in X_\beta$ open, what does $p_\beta^{-1}(U)$ look like?

$$X_1 \times ... \times X_{n-1} \times U \times X_{n+1} \times ...$$

(Warning: this is not formal, the components cannot be enumerated in general) These sets form a subbase for the product topology. What are "these sets?": All sets such that in *all except one* factor you have the entire space and in the remaining factor you have an open.

The basic open sets are finite intersections of these so, they have the form all of X_α for *all but finitely many* α. In the remaining components, you have open sets.

I will state a theorem without proof which will allow me to prove the theorem about compactness of arbitrary products. The proof that I am skipping is a bit more complicated than the proofs that I am showing you, and, more importantly, it uses the *axiom of choice*. Recall the definition of subbase. We define a *closed subbase* to be a family of closed sets whose complements form a subbase, similarly we define a *closed base* to be a family of closed sets whose complements form a base. One can get a closed base from a closed subbase by taking all finite unions.

Theorem 3.4.15 *A space is compact if every subbasic open cover has a finite subcover or equivalently if every class of subbasic closed sets with the finite intersection property has a non-empty intersection.*

Theorem 3.4.16 (Tychonoff) *If* $(X_\alpha \mid \alpha \in \mathcal{A})$ *is an arbitrary large family of compact spaces then* $\prod_{\alpha\in\mathcal{A}} X_\alpha$ *is also compact in the product topology.*

Proof. (Of theorem 3.4.16) Let $\{F_j\}$ be a family of closed subbasic sets. This means that each set F_j has the form $\prod\limits_{\alpha\in\mathcal{A}} F_\alpha^{(j)}$ with each $F_\alpha^{(j)}$ a closed subset of X_α and all but finitely many of them equal to the whole space. We assume that this family has the finite intersection property (fip). For a given fixed α the sets of the form $F_\alpha^{(j)}$ is a class of closed subsets with the fip. Since each X_α is assumed compact the intersection of all these sets is nonempty so it contains some point x_α. If we do this for each α and construct the point $(\ldots, x_\alpha, \ldots)$ we get a point in the intersection of all the F_j. Thus the product space is compact. ∎

3.4.2 Limit point compactness

Definition 3.4.17 *A space is* limit point compact *if every infinite subset has a limit point.*

Theorem 3.4.18 *A compact space is limit point compact.*

Proof. Given X compact, $A \subseteq X$. We will prove the contrapositive, *i.e.* that if A has no limit points it must be finite. Recall that $\overline{A} = A \cup \lim(A)$. If $\lim(A) = \emptyset$ them $A = \overline{A}$ in other words A is closed. Since A is a closed subset of a compact space it must be compact. For each $a \in A$ choose an open U_a with $a \in U_a$ and $U_a \cap (A \backslash \{a\}) = \emptyset$. This can be done because a is not a limit point of A. So $\{U_a \mid a \in A\}$ is an open cover of A so it has a finite subcover. Therefore, $U_{a_1} \cup U_{a_2} \cup ... \cup U_{a_n}$ covers A hence $A = \{a_1, a_2, ..., a_n\}$. ∎

There are examples of spaces that are limit point compact but not compact.

Definition 3.4.19 *A space is sequentially compact if every sequence has a convergent subsequence.*

In a metric space all these concepts coincide.

Recall the definition of first countable.

Definition 3.4.20 *A topological space is said to be first countable if every point has a countable base for its system of open neighbourhoods.*

Metric spaces are always first countable.

Theorem 3.4.21 *In a first-countable compact space every sequence has a convergent subsequence.*

The proof is omitted. Spaces with this property are said to be *sequentially compact.*

This gives some feeling for compactness: if you pack infinitely many points into a space they must pile up somewhere. In standard topology books you can find proofs that in metric spaces compactness is equivalent to sequential compactness. These proofs depend on the first countability property stated above. There are compact spaces for which the sequential compactness property does not hold.

In a compact metric space, for every open cover, there is an $r > 0$ such that every ball of radius r fits inside one of the sets in the open cover.

Definition 3.4.22 *An* ϵ-net *for* $\epsilon > 0$ *is a* finite *set of points* A *such that*

$$X = \bigcup_{a \in A} B_\epsilon(a).$$

Theorem 3.4.23 *Every compact metric space has an* ϵ*-net for every* ϵ*.*

3.4.3 Local compactness

Many spaces of interest are *not* compact: even the real line is not compact. However each "piece" in a suitable sense is compact.

Definition 3.4.24 *A topological space* X*, is said to be* ***locally compact*** *if* $\forall x \in X, \exists U_x \ni x$*, where* U_x *is open and the closure of* U_x *is compact.*

As a basic example the real line with its usual topology is locally compact, in fact $\mathbf{R}^n$ with the topology induced by the euclidean metric is compact for any n.

This works more generally than I am describing here.

Local compactness is much weaker than compactness. Remarkably, one can always extend a locally compact Hausdorff space to a compact Hausdorff space by just adding one point. Let X be a locally compact Hausdorff space. We add a new point which we write as ∞ and define a new space whose underlying set is $X^* = X \cup \{\infty\}$. The open sets of this space are all the open sets of X together with all sets of the form $X \setminus C \cup \{\infty\}$ where C is a closed and compact subset of X. The canonical embedding of X in X^* is both continuous and open. The space X^* is called the *one-point compactification* of X. The one-point compactification of the real line is the circle. To verify all this, one should show that the open sets I have described in X^* really do form a topology and that it yields a compact space and that X embeds as an open subset of X^*.

3.4.4 Paracompactness

Paracompactness is another property related to compactness: it is very important in manifold theory so it is worth reading this subsection carefully.

We first define certain special kinds of covers.

Definition 3.4.25 *An open cover $\{U_i | i \in I\}$ of a topological space X is said to be* ***locally finite*** *if $\forall x \in X, \exists$ an open neighbourhood U_x of x such that for only* finitely many *$i \in I$ does $U_x \cap U_i \neq \emptyset$.*

Now an arbitrary cover is certainly not going to be locally finite in general. However paracompactness tells us we can always find one, but unlike with compactness we do not take subcovers; we have to do something more subtle.

Definition 3.4.26 *Let X be a topological space and let $\mathcal{U} = \{U_i | i \in I\}$ be an open cover. A* ***refinement*** *of $\mathcal{U}$ is another open cover $\mathcal{V} = \{V_j | j \in J\}$ such that $\forall j \in J, \exists i \in I, V_j \subseteq U_i$.*

So each open set of $\mathcal{V}$ sits inside an open set of $\mathcal{U}$. It is crucial that $\mathcal{V}$ is also a cover.

Definition 3.4.27 *A topological space X is* ***paracompact*** *if every open cover has a refinement that is a locally finite cover.*

Obviously (?) every compact space is paracompact. Every locally compact, Hausdorff second countable space is paracompact. There are examples of spaces that are paracompact but not locally compact. Above we stated that every compact Hausdorff space is normal. In fact every paracompact Hausdorff space is normal. Every metric space is paracompact. The product of two paracompact spaces need not be paracompact.

Why should anyone care? This requires some more definitions.

Definition 3.4.28 *Let X be a topological space and let $\mathcal{U} = \{U_i | i \in I\}$ be an open cover of X. A* ***partition of unity subordinate to*** *$\mathcal{U}$ is a family $\{f_i | i \in I\}$ of continuous functions $f_i : X \longrightarrow [0, 1]$ such that:*

- *$\forall i \in I, f_i \subseteq U_i$,*
- *$\{\mathsf{supp}(f_i) | i \in I\}$ is a locally finite cover of X, and*
- *$\forall x \in X, \sum_{i \in I} f_i(x) = 1$.*

Notice that, because the supports of the f_i's form a locally finite cover, for each x, the sum above is nonzero for only finitely many i's; thus this sum is well defined.

Now the crucial theorem can be stated.

Theorem 3.4.29 *Suppose that X is a paracompact Hausdorff topological space, then every open cover has a subordinate partition of unity.*

This is a beautiful result and gives us a technical tool to "glue things together."

3.5 Connectedness

A space is connected if it does not fall apart into two disjoint open pieces.

Definition 3.5.1 *A topological space X is* ***connected*** *if it has no proper non-empty subset A that is both open and closed.*

Note that if A is open and closed then A^c is open so we have two disjoint open sets whose union gives X.

Proposition 3.5.2 *The continuous image of a connected space is connected.*

The maximal connected subsets are called *components* of a space.

Definition 3.5.3 *A space is said to be* ***totally disconnected*** *if the components are singletons.*

Here is a silly exercise: Show that a countable connected metric space is compact. (Why is it silly?)

A compact connected Hausdorff space is called a *continuum.*