COMP 760 Week 6: Extrinsic view of Riemannian Geometry

By Joey Bose and Prakash Panangaden

Mila

Admin stuff week 6

Project Advice

- Start immediately. Don't wait till the last few weeks.
- Try to fail fast. If its empirical get a minimum working example asap. Usually Google Collab is good tool for this.
- Try to visualize results as much as possible. Plot loss curves, generated samples, ELBO.
- Think about as many sanity checks as possible. For example, are generated samples actually on the manifold?

Extrinsic View

Review: Smooth Manifolds

$\mathscr{M}$ – smooth

- d-dim topological space (paracompact, Haussdorf, and second countable).
- $\{(U_i, \phi_i)\}$ collection of charts that satisfy a compatibility condition.
- A smooth function f on $\mathscr{M}$ is the map $f \circ \phi^{-1} : \mathbb{R}^d \to \mathbb{R}$
- The set of smooth functions on $\mathscr{M}$ is denoted by $C^\infty(\mathscr{M})$

Review: Tangent Spaces

$$\boxed{D(fg) = f(x)D(g) + g(x)D(f)}$$ Derivation

- The set of all derivations at $x \in \mathcal{M}$ is called the tangent space $\mathcal{T}_x\mathcal{M}$

$$\mathcal{T}_x\mathbb{R}^d = \text{span}\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d}\}$$

Pushforward Maps (Differentials)

- If X is a vector field on $\mathscr{M}$ then we can define a new function point wise $X(f)(x) = X(x)(f)$ for some $f \in C^\infty(\mathscr{M})$.
- Now let $f : \mathscr{M} \to \mathscr{N}$, the differential df is a map $\mathscr{T}\mathscr{M} \to \mathscr{T}\mathscr{N}$. Note df is a co-vector.
- If f^{-1} exists the corresponding map df^{-1} is called the pullback.
- The pullback in coordinates satisfies $(df_x)^{-1} = (df^{-1})_{f(x)}$

Mila

Tangent Space Basis

- Let $\tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_d) = \phi(x)$ be local coordinates and $d\phi_x : \mathcal{T}_x\mathcal{M} \to \mathcal{T}_{\phi(x)}\mathbb{R}^d$ be an isomorphism.
- Then we can define basis vectors on $\tilde{E}_i$ on $\mathcal{T}_x\mathcal{M}$ as:

$$\tilde{E}_i = (d\phi_x)^{-1}\left(\frac{\partial}{\partial \tilde{x}_i}\right)$$

$$= (d\phi^{-1})_{\phi(x)}\left(\frac{\partial}{\partial \tilde{x}_i}\right)$$

Tangent Space Basis

- The tangent space $\mathcal{T}_x\mathcal{M}$ of $\mathcal{M}$ at x is spanned by $\{\tilde{E}_1, \cdots, \tilde{E}_d\}$.
- Any tangent vector V can be represented by $\sum_{i=1}^{d} \tilde{v}_i \tilde{E}_i$ for some coordinate-dependent coefficients $\tilde{v}_i$.

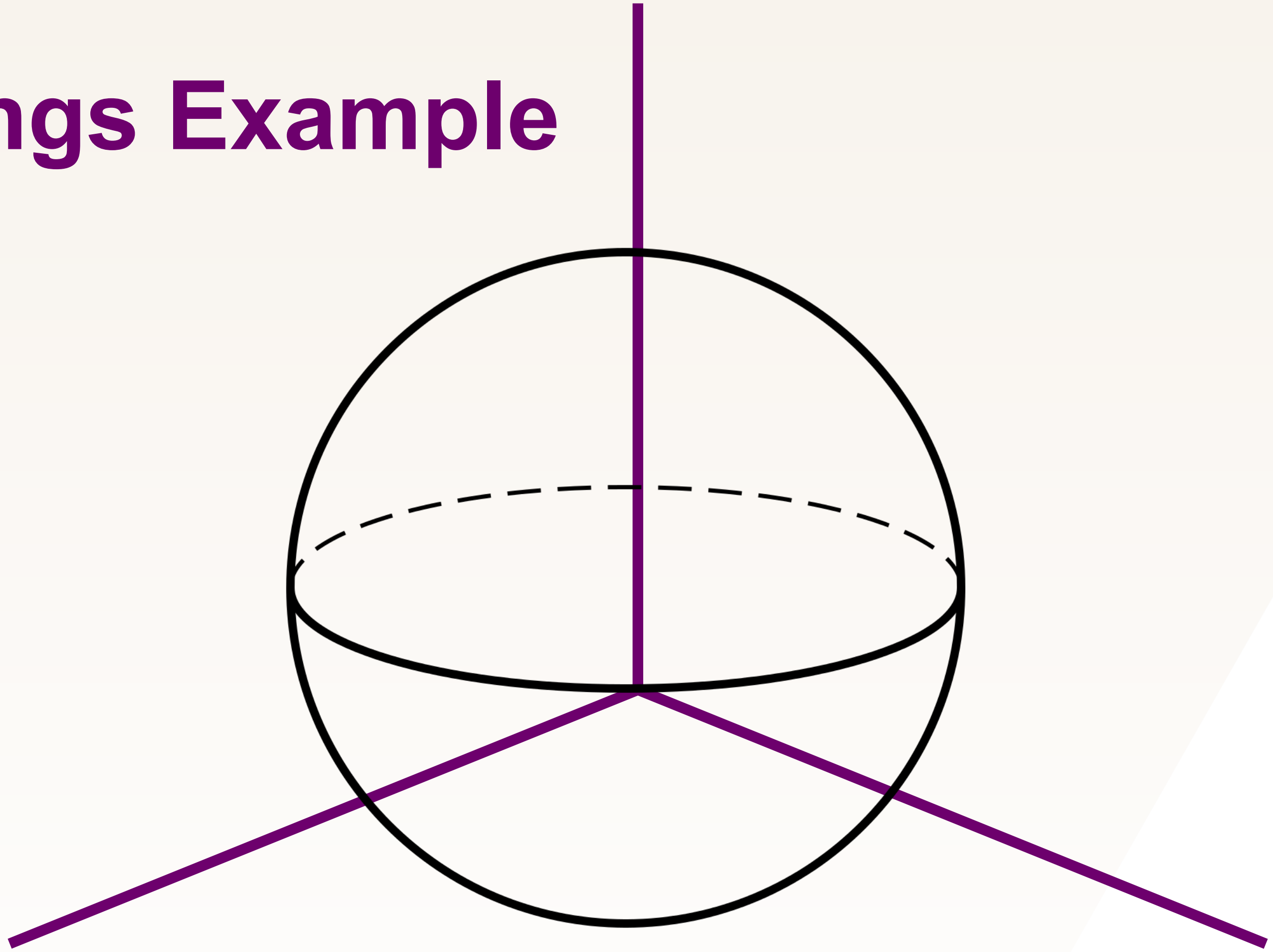
Embeddings

- A manifold is embedded in $\mathbb{R}^m$ if there is an inclusion map $\imath : \mathscr{M} \to \mathbb{R}^m$, where $m > d$.

$$\imath(x) = x \in \mathbb{R}^m \quad \forall x \in \mathscr{M}$$

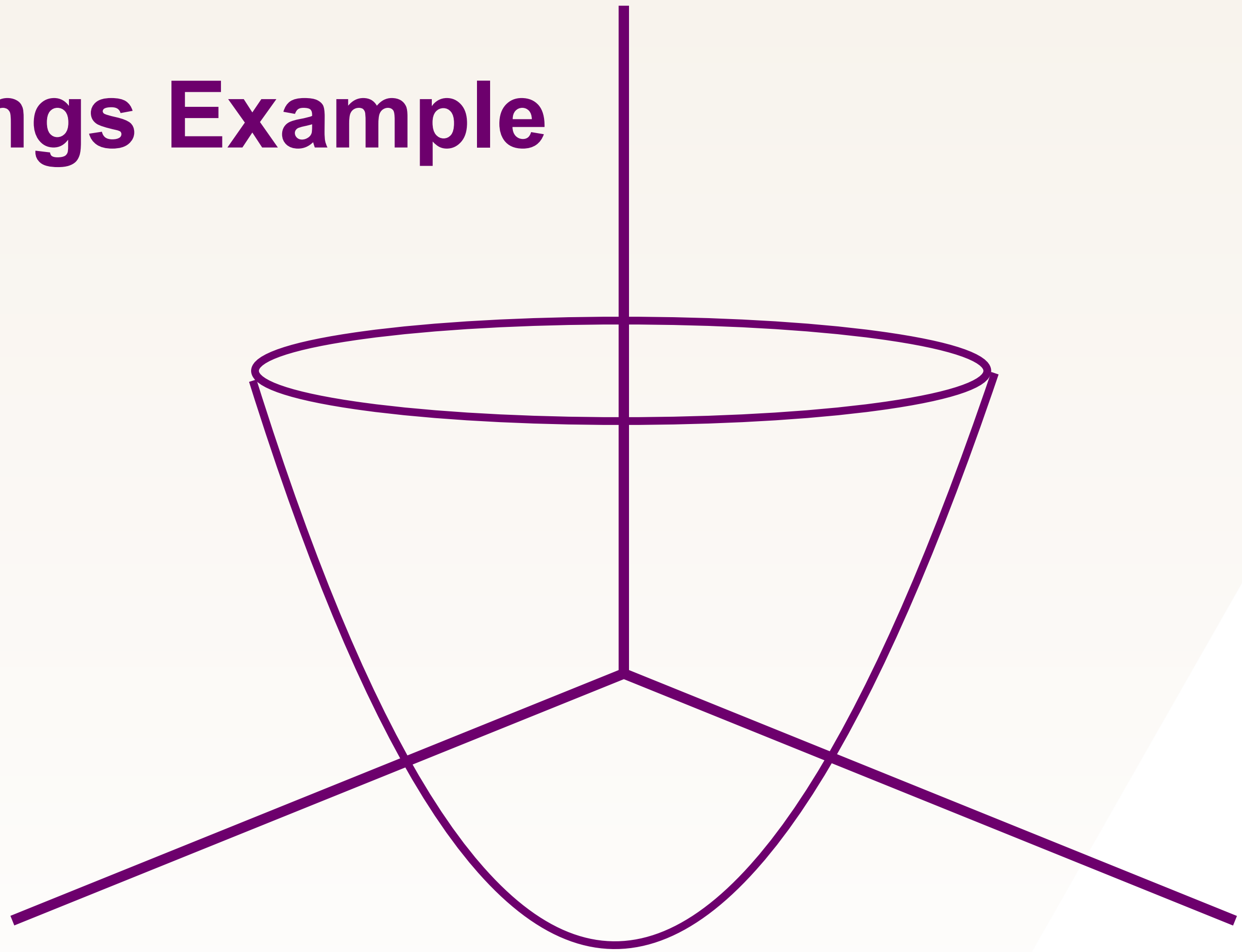
- The Nash Embedding Theorem guarantees that every smooth manifold can be embedded for a suitable m.

Embeddings Example



$\mathbb{S}^2$ sphere embedded in $\mathbb{R}^3$

Embeddings Example



$\mathbb{H}^2$ embedded in $\mathbb{R}^3$

Tangent Space Basis in Extrinsic Coordinates

$$\tilde{E}_i = (d\phi^{-1})_{\phi(x)}\left(\frac{\partial}{\partial \tilde{x}_i}\right)$$

$$= (d\iota^{-1})_{\iota(x)}(d\iota \circ \phi^{-1})_{\phi(x)}\left(\frac{\partial}{\partial \tilde{x}_i}\right)$$

$$= \sum_{j=1}^{m} \frac{\partial \phi_j^{-1}}{\partial \tilde{x}_i} \boxed{\frac{\partial}{\partial x_j}} \longrightarrow \text{Ambient Space basis}$$

A Tangent Vector in Extrinsic Coordinates

$$\sum_{i=1}^{d} \tilde{v}_i \tilde{E}_i = \sum_{i=1}^{d} \sum_{j=1}^{m} \tilde{v}_i \frac{\partial \phi_j^{-1}}{\partial \tilde{x}_i} \frac{\partial}{\partial x_j}$$

$$= \sum_{j=1}^{m} \bar{v}_j \frac{\partial}{\partial x_j}$$

Where $\bar{v}_j = \sum_{i=1}^{d} \tilde{v}_i \frac{\partial \phi_j^{-1}}{\partial \tilde{x}_i}$

Note that $\iota \circ (\phi^{-1})$ is a map from $\mathbb{R}^d \to \mathbb{R}^m$.

It takes $\phi(x) \to \iota(x)$ which is ϕ_j^{-1} for the j-th coordinate.

Mila

Riemannian Metric

$$g_x : \mathcal{T}_x\mathcal{M} \times \mathcal{T}_x\mathcal{M} \to \mathbb{R} \qquad \forall x \in \mathcal{M}$$

- Since g is an inner product, we also write $g(u, v) = \langle u, v \rangle_g$
- Euclidean metric: $\bar{g}$ for $\mathbb{R}^m$ is:

$$\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle = \delta_{ij}$$

Riemannian Metric Continued

- For any $U, V \in \mathcal{T}_x\mathcal{M}$ inner product with the Euclidean metric is:

$$\langle U, V \rangle_{\bar{g}} = \left\langle \sum_{i=1}^{m} \bar{u}_i \frac{\partial}{\partial x_i}, \sum_{j=1}^{m} \bar{v}_j \frac{\partial}{\partial x_j} \right\rangle_{\bar{g}} = \sum_{i=1}^{m} \bar{u}_i \bar{v}_i = \bar{u}^\top \bar{v}$$

Riemannian Metric in Local Coordinates

- Given a set of basis vectors (e.g. $\tilde{E}_i$) we can write this in matrix form:

$$\langle U, V \rangle_g = \sum_{i,j} \tilde{u}_i \tilde{v}_j \langle \tilde{E}_i, \tilde{E}_j \rangle_g = \sum_{i,j} \tilde{u}_i \tilde{v}_j g_{ij} = \tilde{u}^\top G \tilde{v}$$

Where we used the fact $g_{ij} := \langle \tilde{E}_i, \tilde{E}_j \rangle_g$

Riemannian Metric via the Inclusion Map

- Consider $\mathscr{M}$ embedded in a higher dimensional Euclidean space then $g = \imath^* \bar{g}$ (we pullback the metric).

$$g_x(u, v) = \bar{g}(d\imath_x(u), d\imath_x(v))$$

We push forward the tangent vectors to the ambient space.

Riemannian Metric via the Inclusion Map

- Expanding this out explicitly for g_{ij}:

$$g_{ij} = \left\langle d\imath_x(\tilde{E}_i), d\imath_x(\tilde{E}_j) \right\rangle_{\bar{g}}$$

$$= \left\langle \sum_{k=1}^{m} \frac{\partial \phi_k^{-1}}{\partial \tilde{x}_i} \frac{\partial}{\partial x_k}, \sum_{k'=1}^{m} \frac{\partial \phi_{k'}^{-1}}{\partial \tilde{x}_j} \frac{\partial}{\partial x_{k'}} \right\rangle_{\bar{g}}$$

$$= \sum_{k=1}^{m} \frac{\partial \phi_k^{-1}}{\partial \tilde{x}_i} \frac{\partial \phi_k^{-1}}{\partial \tilde{x}_j}$$

Riemannian Metric via the Inclusion Map

- If we let $\psi = \phi^{-1}$, then we can write:

$$G = \frac{d\psi}{d\tilde{x}}^{\top} \frac{d\psi}{d\tilde{x}}$$

Riemannian Gradient

- Similar to the definition of coordinate-free definition of Gradient in Euclidean spaces we define: $\nabla_g : f \in C^\infty(\mathscr{M}) \mapsto \nabla_g f \in \mathfrak{X}(\mathscr{M})$

$$\left\langle \nabla_g f, V \right\rangle_g = \boxed{V(f)} \quad \text{For any } V \in \mathfrak{X}(\mathscr{M})$$

Directional Derivative

Riemannian Gradient: Explicit Coordinates

$$\left\langle \nabla_g f, V \right\rangle_g = V(f)$$

V coordinates

$$\left\langle \nabla_g f, V \right\rangle_g = \sum_{i,j=1}^{d} \boxed{\tilde{u}_i} \boxed{\tilde{v}_j} g_{ij}$$

Gradient coordinates

Recall $V(f) = \sum_{j=1}^{d} \tilde{v}_j \frac{\partial}{\partial \tilde{x}_j} f \circ \phi^{-1}$

Mila

Riemannian Gradient: Explicit Coordinates

- Since v is arbitrary, this means for all j

$$\left\langle \nabla_g f, V \right\rangle_g = V(f)$$

$$\left\langle \nabla_g f, V \right\rangle_g = \sum_{i,j=1}^{d} \tilde{u}_i \tilde{v}_j g_{ij}$$

Components of G^{-1}

$$\sum_{i=1}^{d} \tilde{u}_i g_{ij} = \frac{\partial}{\partial \tilde{x}_j} f \circ \phi^{-1} \implies \tilde{u}_i = \sum_{j=1}^{d} \boxed{g^{ij}} \frac{\partial}{\partial \tilde{x}_j} f \circ \phi^{-1}.$$

Riemannian Divergence

Theorem: For any compactly supported $f \in \mathfrak{X}(\mathcal{M}), \int_{\mathcal{M}} \nabla_g \cdot f \, d\mu_g = 0.$

Divergence Theorem

Regular Vector Calculus

$$\int_{\Omega} \nabla \cdot X dA = \int_{\partial\Omega} n \cdot X dl$$

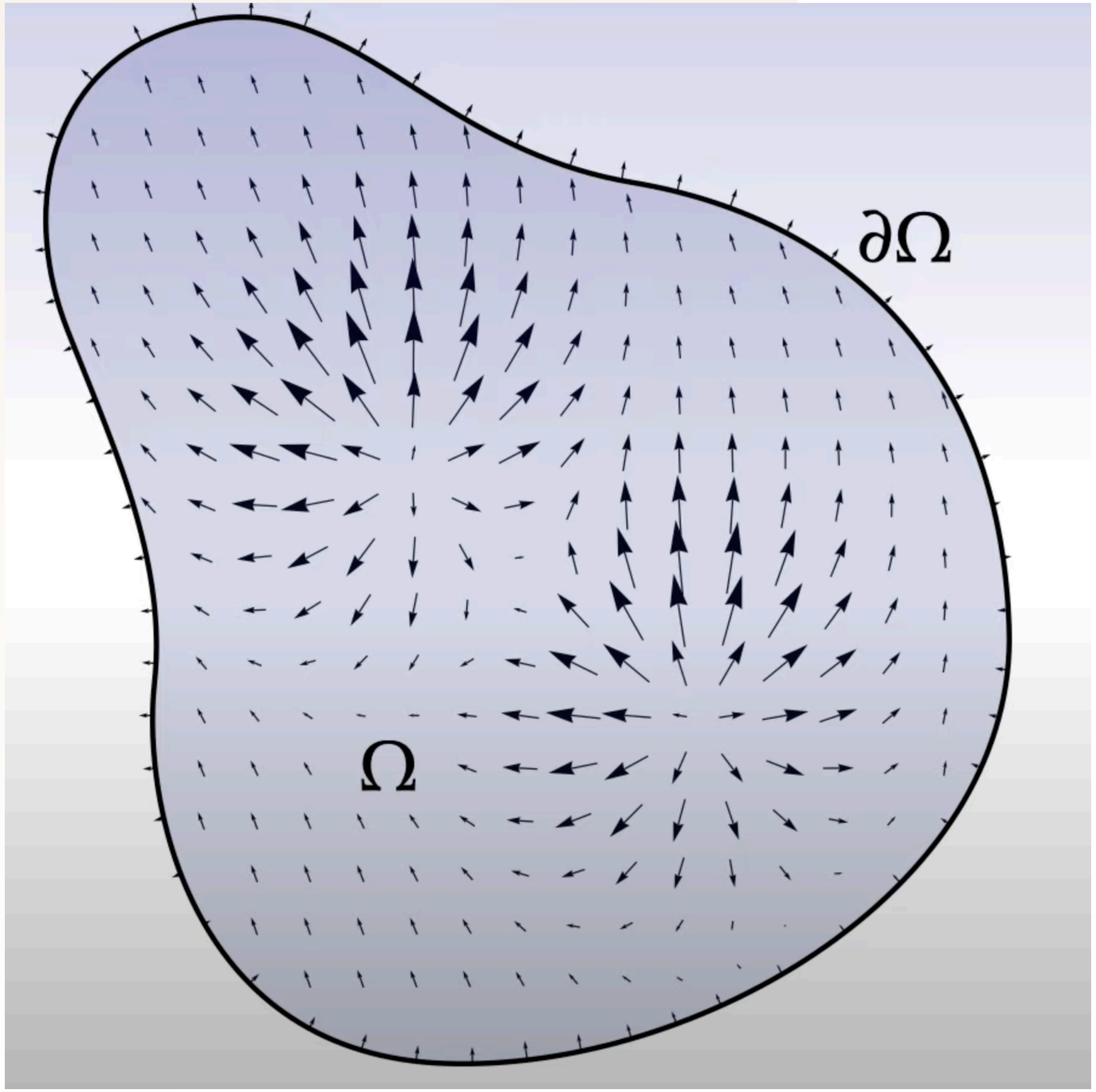


Fig credit: http://www.cs.cmu.edu/~kmcrane/Projects/DDG/paper.pdf

Divergence Theorem

Regular Vector Calculus

$$\int_{\Omega} \nabla \cdot X dA = \int_{\partial\Omega} n \cdot X dl$$

$$\int_{\Omega} d \star \alpha = \int_{\partial\Omega} \star \alpha$$

Integrating the normal component of a field.

Mila

Affine Connection

- An affine connection allows us to compare a vector field at nearby points and is denoted using the operator $\nabla : \mathfrak{X}(\mathscr{M}) \times \mathfrak{X}(\mathscr{M}) \to \mathfrak{X}(\mathscr{M})$.
- We use the notation $U, V \mapsto \nabla_U V$ for $U, V \in \mathfrak{X}(\mathscr{M})$

Linearity in U

Linearity in V

Product Rule

An affine connection satisfies the following properties.

Euclidean Connection

- If $U, V \in \mathfrak{X}(\mathbb{R}^m)$, the Euclidean connection $\bar{\nabla}$ is defined as

$$\bar{\nabla}_U V = \sum_{i=1}^{m} \sum_{j=1}^{m} \bar{u}_j \frac{\partial \bar{v}_i}{\partial x_j} \frac{\partial}{\partial x_i}$$

Tangential Connection

- Consider $\mathscr{M}$ embedded in a higher dimensional Euclidean space. The Tangential connection is:

$$\nabla_U^{\top} V = \boxed{P} \bar{\nabla}_{\bar{U}} \bar{V}$$

Tangential Projection

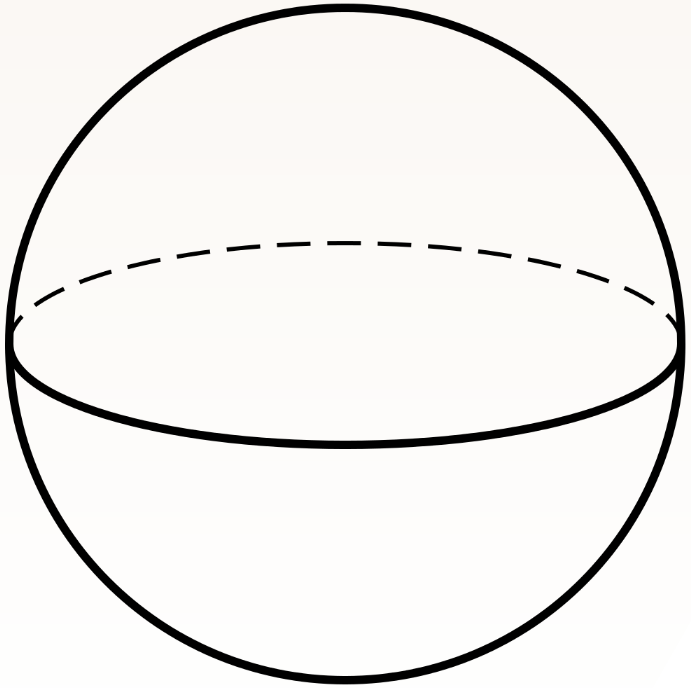
$$(PV)(x) = \sum_{j=1}^{m} (\boxed{P_x} \bar{v})_j \frac{\partial}{\partial x_j}$$

Orthogonal Projection Matrix onto the tangent space spanned by $\frac{\partial \phi}{\partial \tilde{x}_i}$

Spherical Geometry

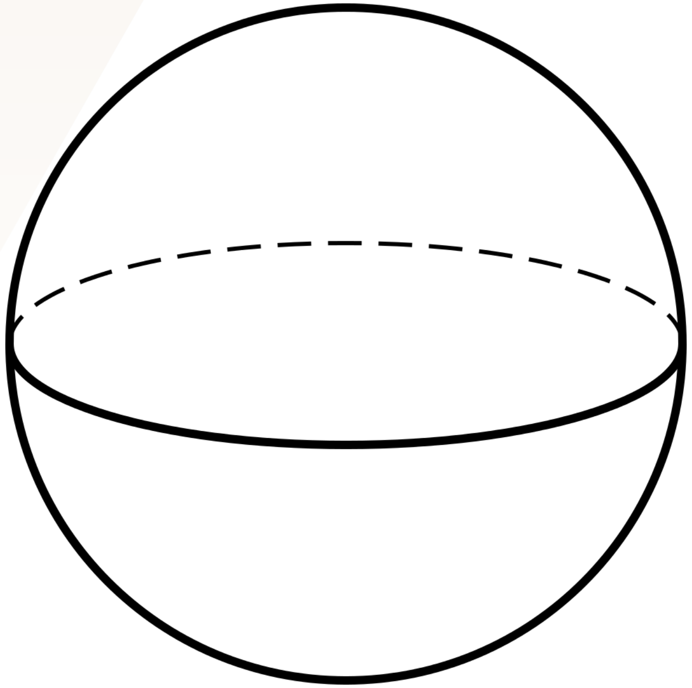
Spherical Geometry

- An d-sphere is $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : ||x||_2 = 1\}$



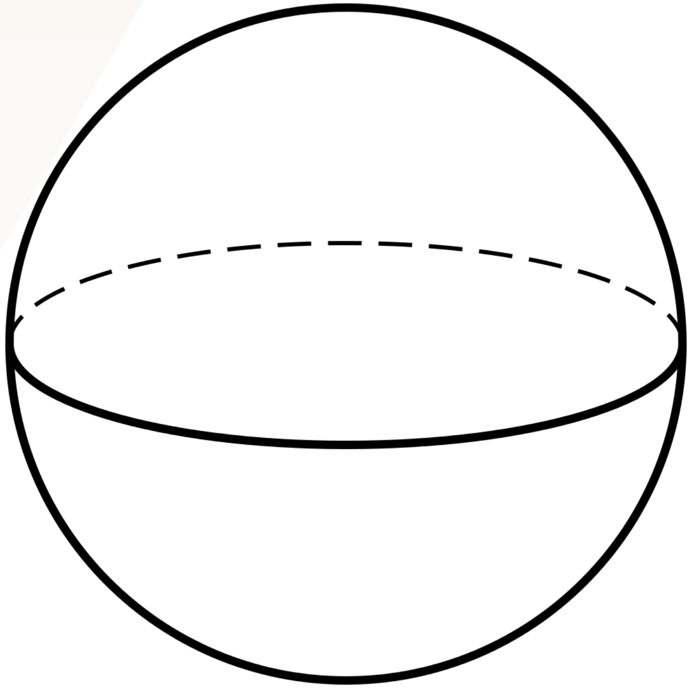
Spherical Geometry

- Alternatively, for $K > 0$, $\mathbb{S}^d_K = \{x \in \mathbb{R}^{d+1} \mid \langle x, x \rangle = 1/K\}$
- For $d = 2$ and $K = 1$we can use polar coordinates to represent any $x \in \mathbb{S}^2$. Let $\theta \in [0,\pi)$ and $\psi \in [0,2\pi)$ then
$r(\theta, \psi) = (\sin(\theta)\cos(\psi), \sin(\theta)\sin(\psi), \cos(\theta))$
- The volume form is then
$\sqrt{\det|G(\theta, \psi)|} = \sin(\theta)$



Distances on a Sphere

- For $K = 1$, $d(x, y) = \cos^{-1}(\langle x, y\rangle_2)$
- Intuitively, as $K \to 0$ the space gets flatter and since $\langle x, x\rangle_2 = 1/K$ for a point on the sphere we see that $\lim_{K\to 0^+} \langle x, x\rangle_2 = \infty$—i.e. all points go to infinity.



Tangential Connection: Example Sphere

- We can derive the Tangential projection by any incremental change in x, denoted by dx, will need to leave the norm $||x||_2$ unchanged.

$$d||x||_2^2 = 2x\,dx = 0$$

- This means x is normal to the tangential linear subspace and the orthogonal projection can be found by subtracting the normal component.

$$P_x = I - \frac{xx^\top}{||x||_2^2}$$

Sphere: Closest Point Projection

- The closest point onto a sphere is given by $\pi(x)$

$$\pi(x) = \frac{x}{||x||_2}$$

- One can verify this is the point on $\mathbb{S}^d$ that minimizes the Euclidean distance from $x \in \mathbb{R}^{d+1} \backslash \{0\}$.