COMP 760 Week 5: Integration on Manifolds

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Admin stuff week 5

Project Proposal

- Short two page document formatted in latex, similar to a regular paper.
- Should include a minimum viable product. What is the minimal thing you commit to doing by the end of the semester.
- Short review of related work. This could include potential baselines and other methods that put your work into context.
- Some nice to haves if you had more time or could push this forward for a real submission.

Differential Forms

Review: Vector Spaces

$$\mathscr{V} = (V, + , \cdot\ )$$

- $+ : V \times V \to V$ "addition"
- $\cdot : \mathbb{R} \times V \to V$ "scalar multiplication"
- Addition satisfies: "Commutativity, Associativity, Identity element, Inverse element.
- Scalar Multiplication satisfies: "Identity element, Distributivity with vector and field addition , Compatibility with field multiplication

Review: Inner Product

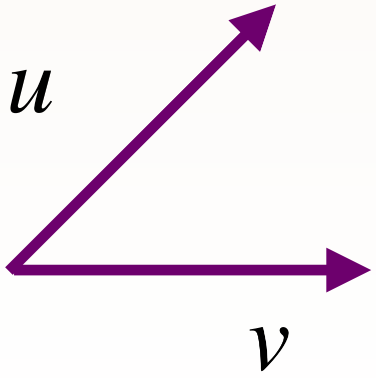
$$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$$

- Symmetry: $\langle x, y \rangle = \langle y, x \rangle$
- Linearity: $\langle ax, y \rangle = a\langle x, y \rangle$ and $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
- Positivity: $\langle x, x \rangle > 0,\ x \neq 0$ and $\langle x, x \rangle = 0,\ x = 0$

Review: Span of a Vector Space

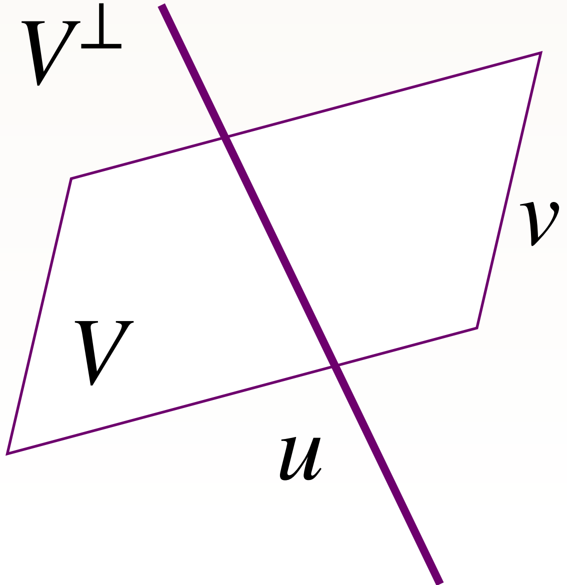
$$\mathrm{span}(\{v_1, \ldots, v_n\}) := \left\{ x \in V \mid x = \sum_{i=1}^{k} a_i v_i \quad a_i \in \mathbb{R} \right\}$$

- The span of a vector space forms a linear subspace



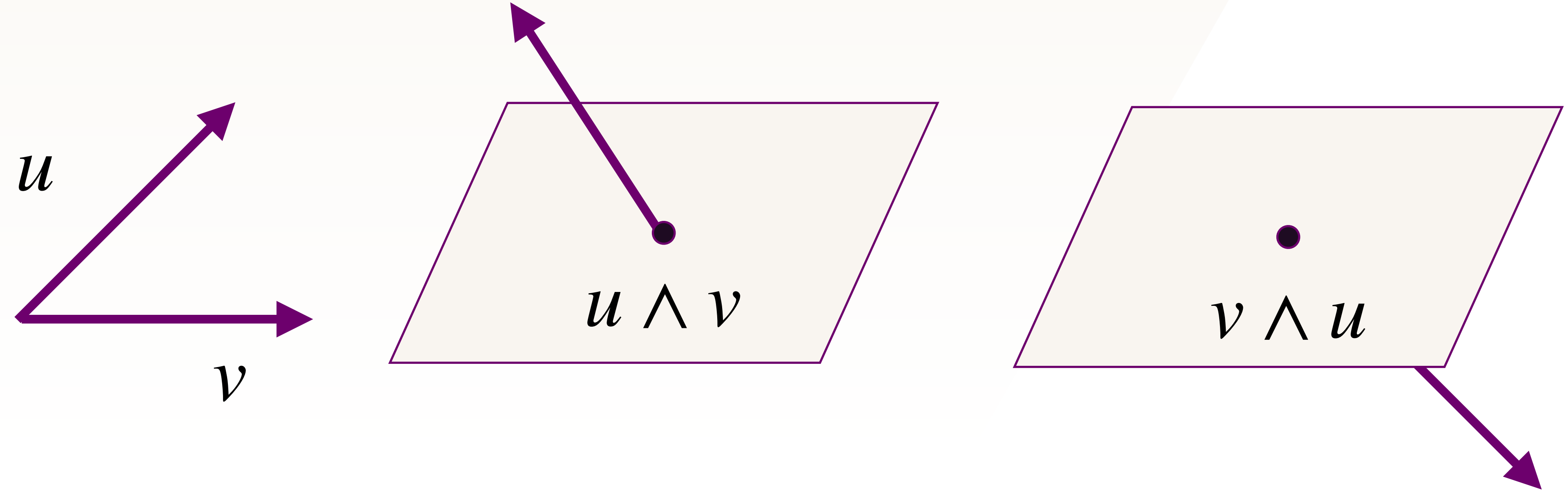
Review: Orthogonal Complement

- $V := \mathrm{span}(\{u, v\})$
- $V^\perp := \{x \in \mathbb{R}^n \,|\, \langle x, w \rangle = 0 \,\forall w \in V\}$
- We need to define an inner product first.



Wedge Product $\wedge$

- $u \wedge v = -v \wedge u$ and $u \wedge u = 0$
- We have a sense of orientation and it is anti-symmetric
- Output of a wedge product is k-vector



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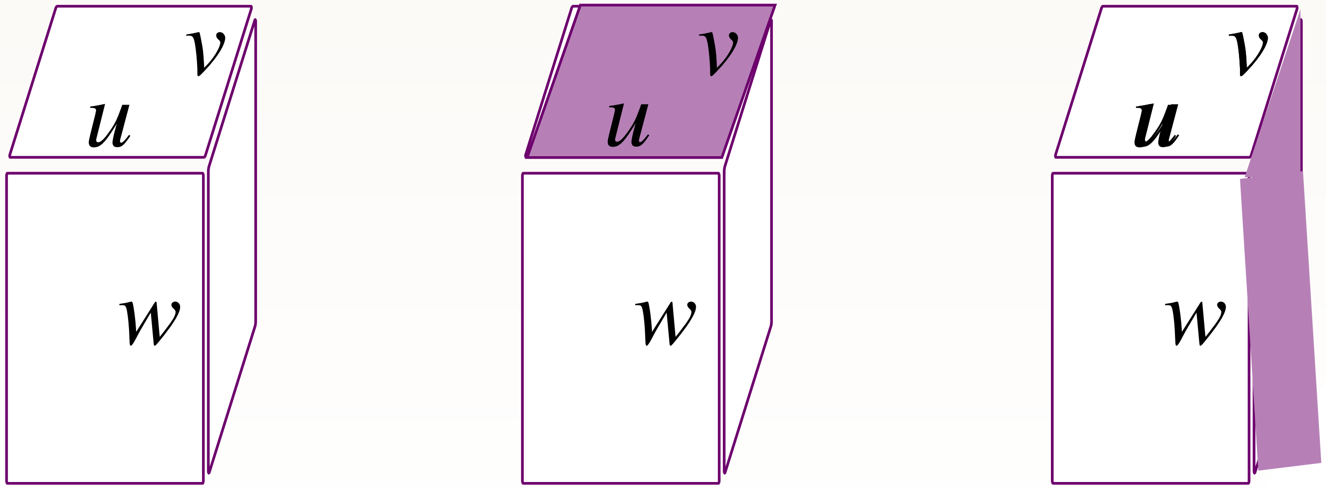
Wedge Product $\wedge$ Rules

- Associativity
- Distributivity: $u \wedge v_1 + u \wedge v_2 = u \wedge (v_1 + v_2)$
- Output of a wedge product is k-vector



$u \wedge v \wedge w \qquad (u \wedge v) \wedge w \qquad u \wedge (v \wedge w)$

Wedge Product $\wedge$ Rules

- Associativity
- Distributivity: $u \wedge v_1 + u \wedge v_2 = u \wedge (v_1 + v_2)$
- Output of a wedge product is k-vector
- Skew-Commutative: $\omega \wedge \phi = (-1)^{kl} \phi \wedge \omega$

k-vectors

- You can think of them as having only a direction and magnitude.
- 2-vectors are parallelograms with a direction.
- Two k-vectors are the same if the have both the same direction and magnitude.
- 0-vectors are just normal scalar and don't have a direction.

Hodge Star $\star$

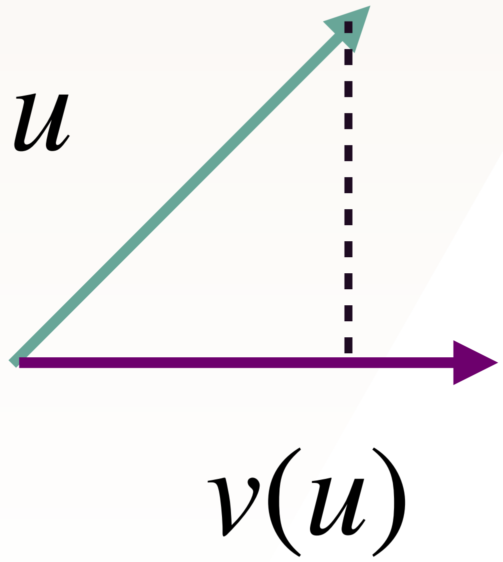
- Analogy: similar to orthogonal complement

- $k \to (n-k)$ form
- Convention: $z \wedge \star z$ is positively oriented.

k-forms

- Applying $\wedge$ to vectors gives us k-vectors.
- Applying $\wedge$ to co-vectors gives us k-forms.

vectors get “measured”



Co-vector “measure” vectors

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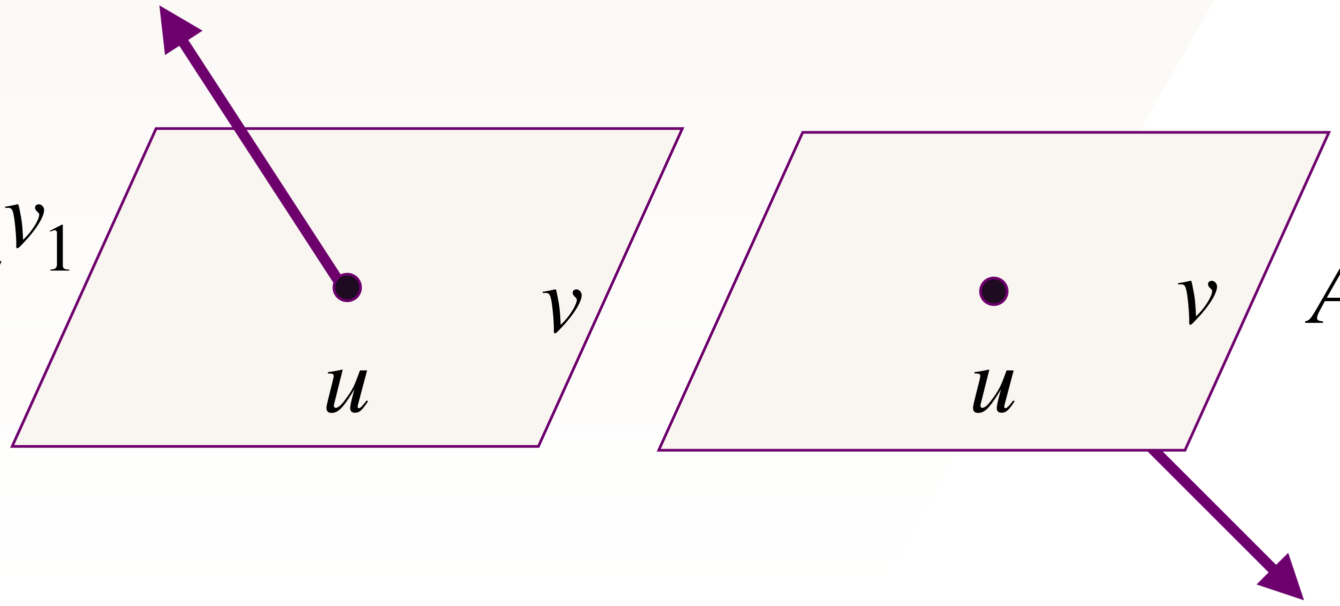
k-forms

- k-forms are used to "measure" k-vectors.
- This will be multi-linear (linear in each slot).
- Determinant: We should think of this as signed volume (e.g. cross product in 3d)

$$A = u_1 v_2 - u_2 v_1$$



$$A = v_1 u_2 - v_2 u_1$$

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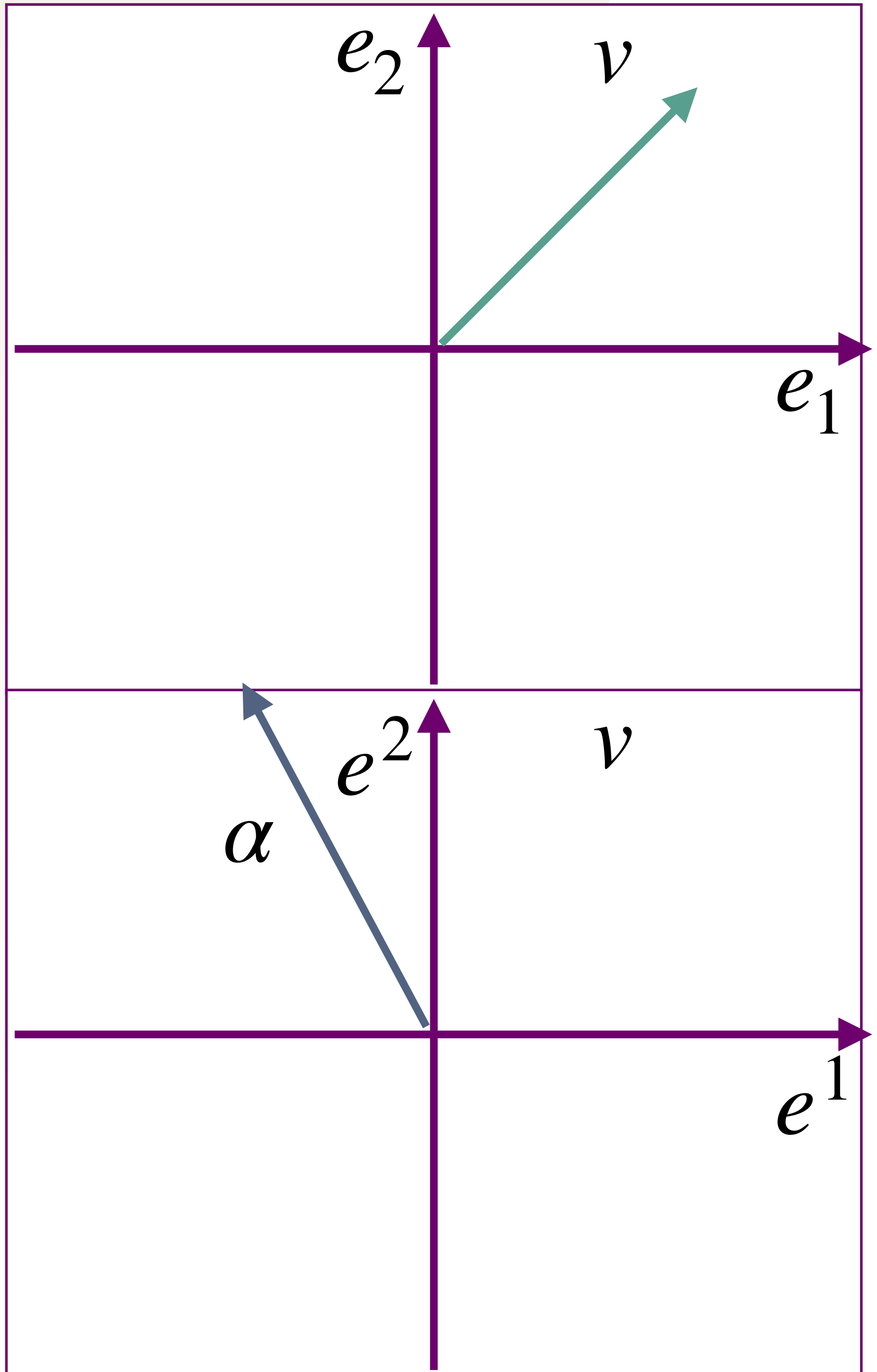
1-form Example

- Vector v and 1-form α

$$v = 2e_1 + 2e_2$$

$$\alpha = -2e^1 + 3e^2$$



1-form Example

- What is $\alpha(v)$?

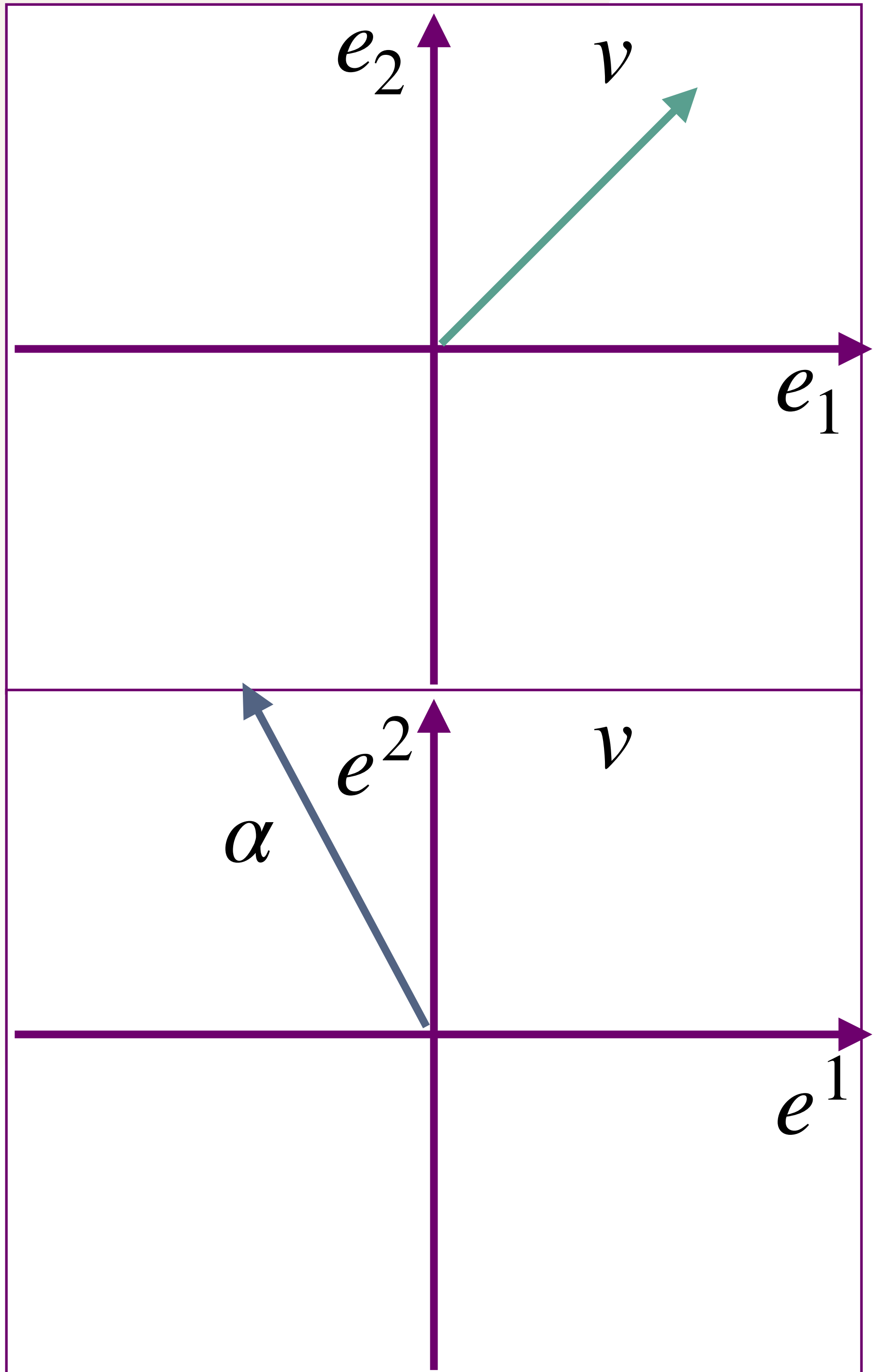
$$\alpha(v) = (-2e^1 + 3e^2)(2e_1 + 2e_2)$$

$$\alpha(v) = (-2e^1 + 3e^2)(2e_1 + 2e_2)$$

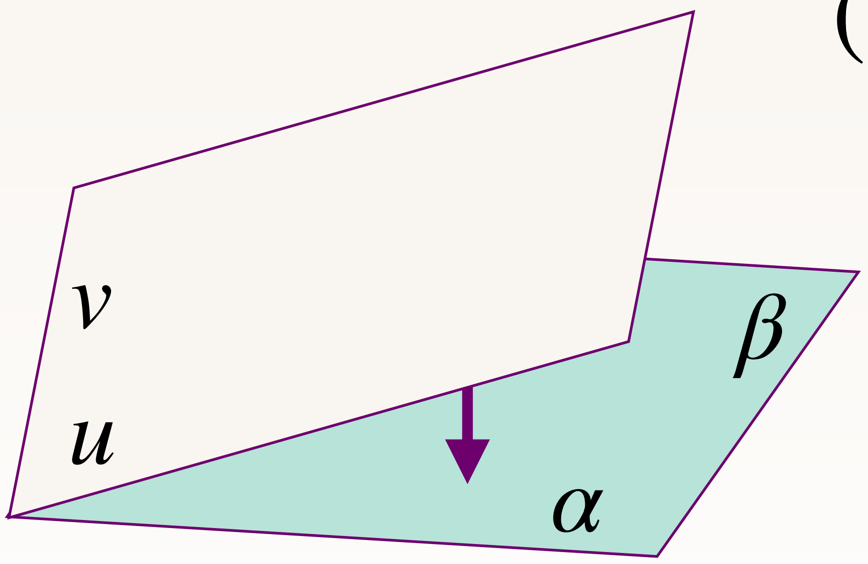
$$= -2e^1(2e_1 + 2e_2) + 3e^2(2e_1 + 2e_2)$$

$$= -4e^1(e_1) - 4e^1(e_2) + 6e^2(e_1) + 6e^2(e_2)$$

$$= -4 + 6 = 2$$



Measurements of 2-vectors



$$(\alpha \wedge \beta)(u, v) := \alpha(u)\beta(v) - \alpha(v)\beta(u)$$

Area of the projection of one parallelogram onto another

Differential Forms

- Recall a vector field can be thought of as an assignment of a vector to every point in space.
- A differential k-form is an assignment of a k-form to every point in space.

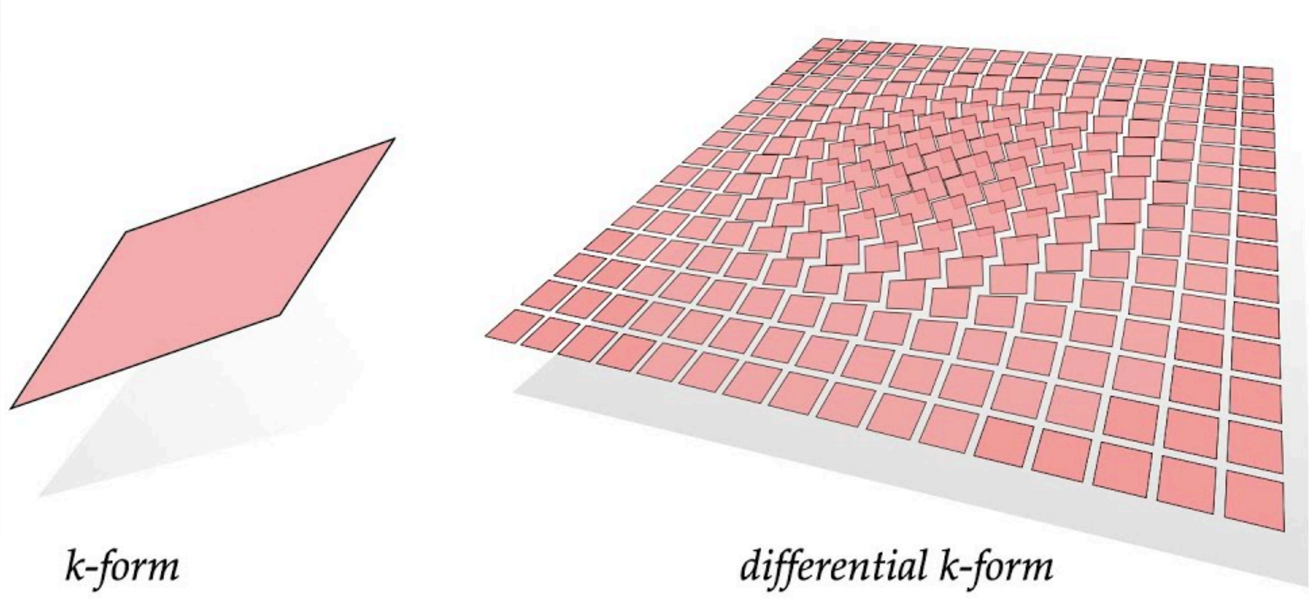


Fig credit: http://www.cs.cmu.edu/~kmcrane/Projects/DDG/paper.pdf

Differential Forms

- Differential 1-forms can be used to "measure" a vector field.
- Differential 2-forms can be used to "measure" a 2-vector field.
- Most operations on differential k-forms are point wise.

$$(\star\alpha)_p := \star\,(\alpha_p)$$

$$(\alpha \wedge \beta)_p := (\alpha_p) \wedge (\beta_p)$$

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Differential Forms in Coordinates

Vector Fields

$$v = v_1 \frac{\partial}{\partial x_1} + \ldots + v_n \frac{\partial}{\partial x_n}$$

Differential 1-forms

$$\alpha = \alpha_1 dx^1 + \ldots + \alpha_n dx^n$$

$$dx^i \left(\frac{\partial}{\partial x_j} \right) = \delta^i_j$$

Relationship between the basis

Example: Differential Forms

Consider the differential forms

$$\alpha := x dx \qquad \beta := (1 - x)dx + (1 - y)dy$$

Whats the wedge product here?

$$\alpha \wedge \beta =$$

Example: Differential Forms

Consider the differential forms

$$\alpha := xdx \qquad\qquad \beta := (1-x)dx + (1-y)dy$$

Whats the wedge product here?

$$\begin{aligned}
\alpha \wedge \beta &= (xdx) \wedge ((1-x)dx + (1-y)dy) \\
&= (xdx) \wedge ((1-x)dx) + (xdx) \wedge ((1-y)dy) \\
&= x(1-x)dx \wedge dx + x(1-y)dx \wedge dy \\
&= (x - xy)dx \wedge dy
\end{aligned}$$

Top Forms

- In n-dimensions any positive multiple of $dx^1 \wedge \ldots \wedge dx^n$ is called a top form.
- Any two top forms are related by $w = cw'$, where c is a positive constant.
- A choice of a top form is called a volume form. You can only define volume forms on manifolds that are orientable.
- Any k-form with $k > n$ will automatically be 0 due to antisymmetry.

Exterior Derivative

- Ω^k: The space of all differential k-forms
- Unique linear map $d : \Omega^k \to \Omega^{k+1}$
- For $k = 0$ this is the regular differential from vector calculus

$$d\phi(X) = D_X\phi$$

Exterior derivative applied to $\phi(X)$ is the directional derivative

Exterior Derivative Rules

- Ω^k: The space of all differential k-forms
- Unique linear map $d : \Omega^k \rightarrow \Omega^{k+1}$
- Product Rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$
- Exactness: $d \circ d = 0$

Differential of a function

- Unique 1-form such that applying to any vector field gives directional derivatives along those directions.

$$d\phi(X) = D_X\phi$$

- In coordinates:

$$d\phi := \frac{\partial \phi}{\partial x^1}dx^1 + \ldots + \frac{\partial \phi}{\partial x^n dx^n}$$

- The Gradient depends on a choice of inner product, the differential does not.

Exterior Derivative Example

- Let $\alpha := udx$, $\beta = vdy$, and $\gamma := wdz$ be differential 1 forms on $\mathbb{R}^n$, where $u, v, w : \mathbb{R}^n \to \mathbb{R}$ are 0-forms. Also, let $\omega := \alpha \wedge \beta$.

$$d(\omega \wedge \gamma) = d\omega \wedge \gamma + (-1)^2 \omega \wedge (d\gamma)$$

Recursively evaluate:

$$d\omega = (d\alpha) \wedge \beta + (-1)^1 \alpha \wedge (d\beta)$$

$$d\alpha = (du) \wedge dx + (-1)^0 u(ddx) = (du) \wedge dx$$

$$d\beta = (dv) \wedge dy + (-1)^0 v(ddy) = (dv) \wedge dy$$

$$d\gamma = (dw) \wedge dz + (-1)^0 w(ddz) = (dw) \wedge dz$$

Exterior Derivative Example

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We stop at the base case of 0-forms.

$$d\omega = (d\alpha) \wedge \beta + (-1)^1 \alpha \wedge (d\beta)$$

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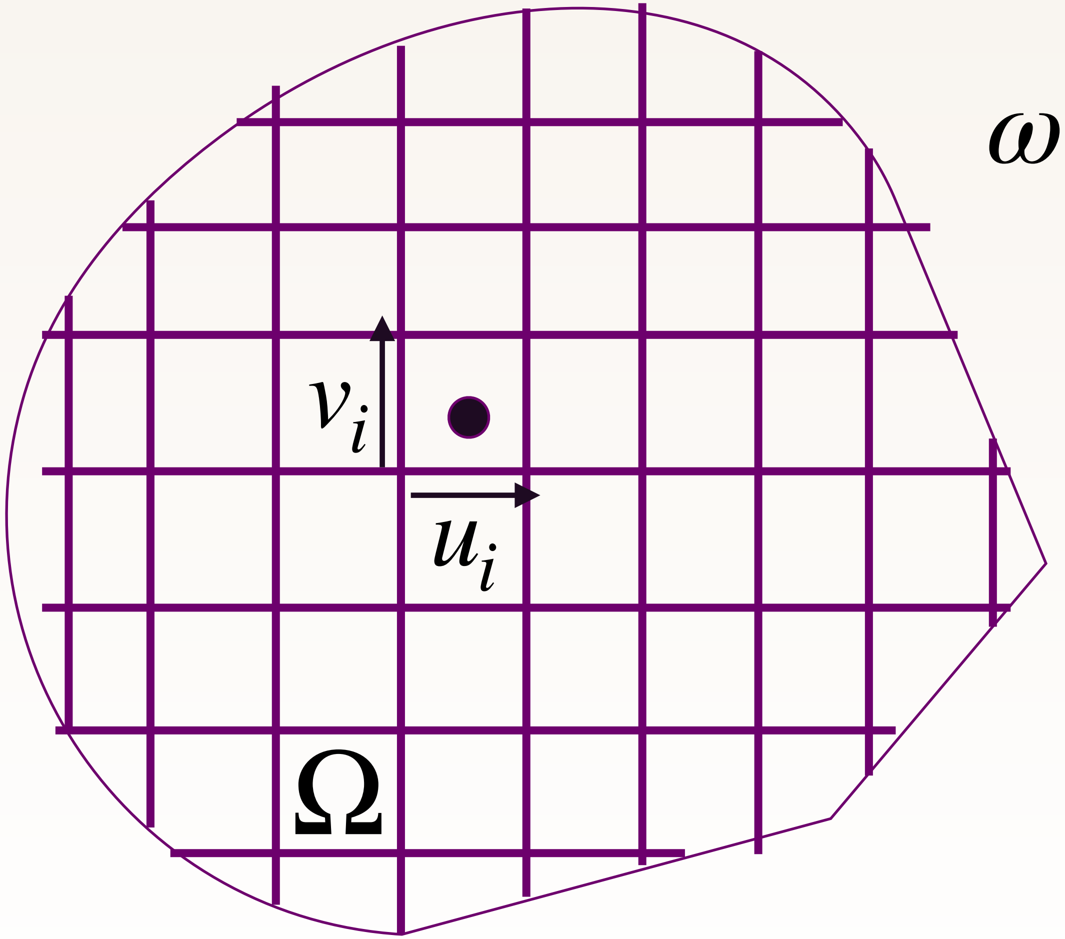
Integration of Differential Forms



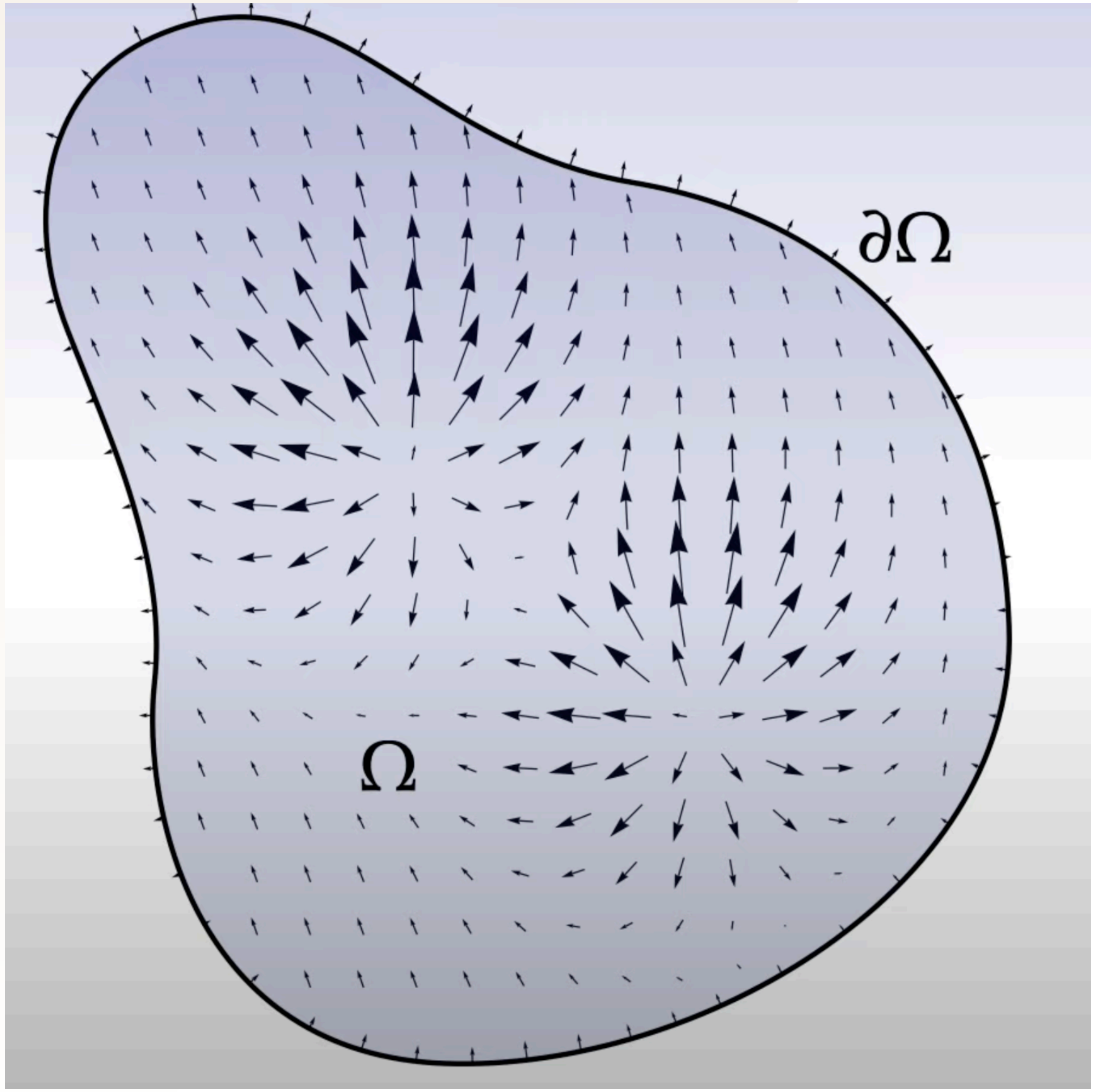
ω is a differential 2-form

$$\sum_i \omega_{p_i}(u_i, v_i) \Rightarrow \int_\Omega \omega$$

Divergence Theorem

Regular Vector Calculus

$$\int_{\Omega} \nabla \cdot X dA = \int_{\partial\Omega} n \cdot X dl$$



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Divergence Theorem

Regular Vector Calculus

$$\int_{\Omega} \nabla \cdot X dA = \int_{\partial\Omega} n \cdot X dl$$

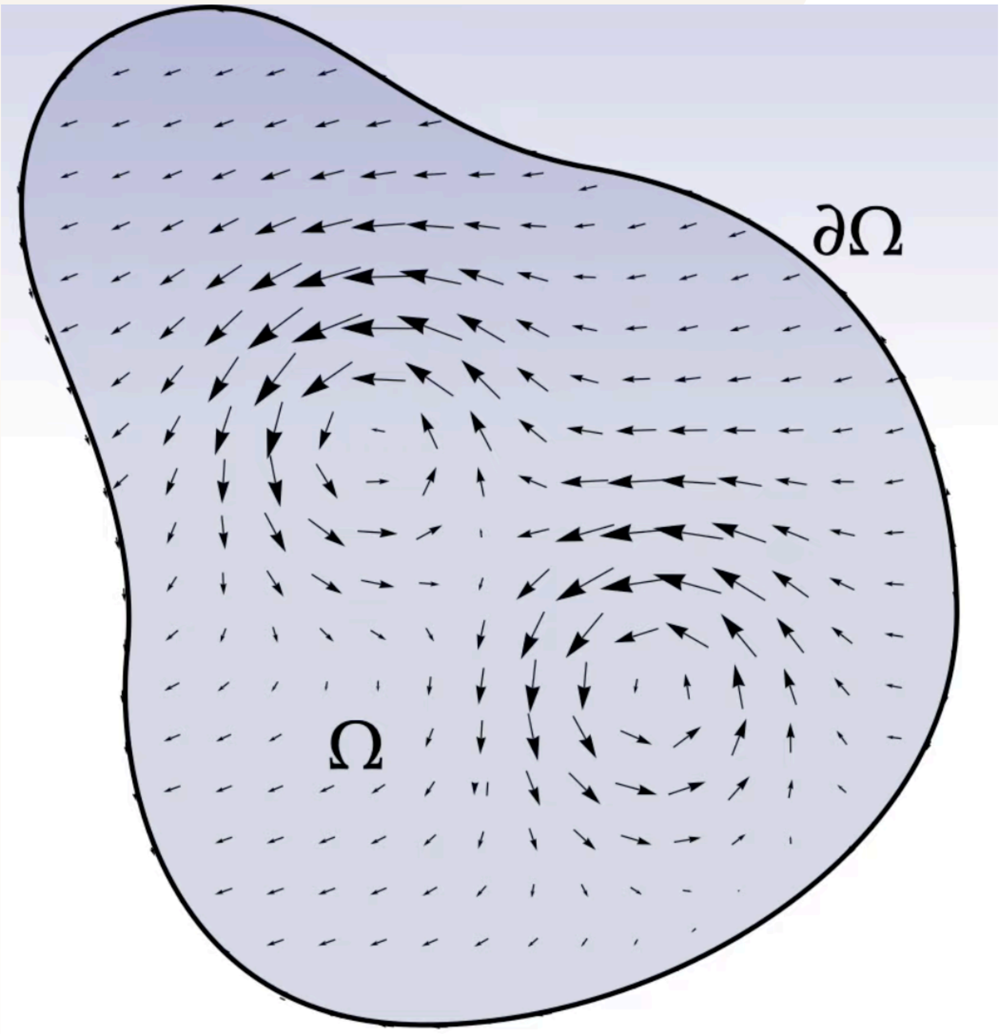
$$\int_{\Omega} d \star \alpha = \int_{\partial\Omega} \star \alpha$$

Integrating the normal component of a field.

Green's Theorem

Regular Vector Calculus

$$\int_{\Omega} \nabla \times X dA = \int_{\partial\Omega} t \cdot X dl$$



Green’s Theorem

Regular Vector Calculus

$$\int_{\Omega} \nabla \times X dA = \int_{\partial\Omega} t \cdot X dl$$

$$\int_{\Omega} d\alpha = \int_{\partial\Omega} \alpha$$

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Stokes Theorem

Fundamental Theorem of Calculus

$$\int_a^b f'(x)dx = f(b) - f(a)$$

$$\int_{\Omega} \underbrace{d\alpha}_{n\text{ form}} = \int_{\partial\Omega} \underbrace{\alpha}_{n-1\text{ form}}$$

Understanding Exactness using Stoke's theorem

Exactness: $d \circ d = 0$

$$\int_{\Omega} dd\phi = \int_{\partial\Omega} d\phi$$

$$= \int_{\partial\partial\Omega} \phi$$

$$= 0$$

Boundary of a Boundary is empty

Geometric picture of Determinants

- Let ω be a volume form on a vector space V with basis vectors $e = (e_1, \ldots, e_n)$
- Let ϕ be an endomorphism (a $T^1_1 V$) tensor.

$$\det \phi := \frac{\omega(\phi(e_1), \ldots, \phi(e_n))}{\omega(e_1, \ldots, e_n)}$$

- The determinant is independent of the choice of top form as well the basis. Why?

Geometric picture of Determinants

Interchanging Rows

$$\det B = -\det A$$

$$\det \phi := \frac{\omega(\phi(e_1), \ldots, \phi(e_n))}{\omega(e_1, \ldots, e_n)}$$

$$\det \phi := -\frac{\omega(\phi(e_2), \phi(e_1,), \ldots, \phi(e_n))}{\omega(e_1, \ldots, e_n)}$$

Anti-symmetry!

Geometric picture of Determinants

Multiplying Rows

$$\det B = k^n \det A$$

$$\det \phi := \frac{\omega(\phi(e_1), \ldots, \phi(e_n))}{\omega(e_1, \ldots, e_n)}$$

$$\det \phi := \frac{\omega(k\phi(e_2), k\phi(e_1,), \ldots, k\phi(e_n))}{\omega(e_1, \ldots, e_n)}$$

$$\det \phi := k^n \frac{\omega(\phi(e_2), \phi(e_1,), \ldots, \phi(e_n))}{\omega(e_1, \ldots, e_n)}$$

Geometric picture of Determinants

Composition

$$\det(\phi \circ \psi) = \det(\phi)\det(\psi)$$

$$\det(\phi \circ \psi) := \frac{\omega(\phi \circ \psi(e_1), \ldots, \phi \circ \psi(e_n))}{\omega(e_1, \ldots, e_n)}$$

$$\det(\phi \circ \psi) := \frac{\omega(\phi(e_1), \ldots, \phi(e_n))}{\omega(e_1, \ldots, e_n)} \cdot \frac{\omega(\psi(e_1), \ldots, \psi(e_n))}{\omega(e_1, \ldots, e_n)}$$

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Distributions on Manifolds

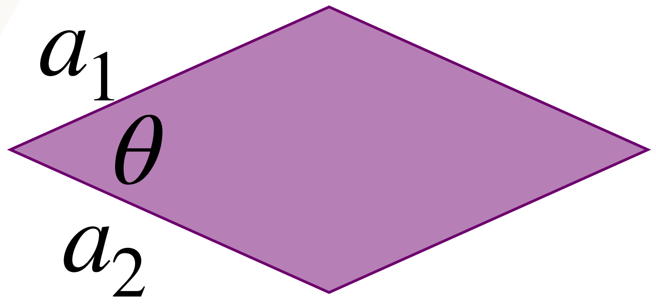
Intuitive Picture in Euclidean space

- An infinitesimal volume form is a parallelopiped. In 2 -dimensions this area is:

$$A = ||a_1||||a_2||\sin\theta$$

$$= ||a_1||||a_2||\sqrt{1 - \cos^2\theta}$$

$$= \sqrt{||a_1||_2^2||a_2||_2^2 - \langle a_1, a_2\rangle_2^2}$$



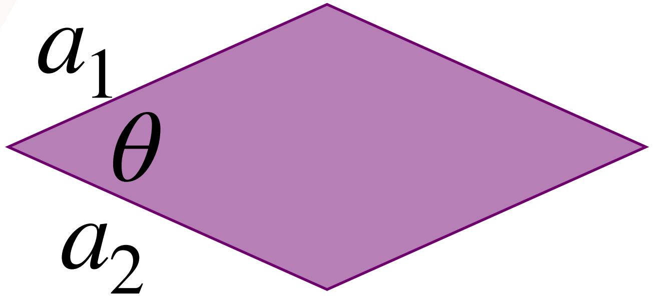
Intuitive Picture in Euclidean space

- An infinitesimal volume form is a parallelopiped. In 2 -dimensions this area is:

$$A = \sqrt{||a_1||_2^2 ||a_2||_2^2 - \langle a_1, a_2 \rangle_2^2}$$

$$A = \sqrt{\det \boxed{A^T A}}$$

Grammian matrix: guaranteed to be square



Riemannian Volume Forms

- We will follow a similar strategy but induce the volume form at every tangent space using the metric.
- At a point we can define the basis vectors of the tangent space $\tilde{E} = \{\tilde{E}_1, \ldots, \tilde{E}_n\}$.

$$g_{i,j} = \langle e_i, e_j \rangle_g$$

- Matrix representation of the Riemannian metric is $G = \tilde{E}^T \tilde{E}$

Riemannian Volume Forms

- Define a probability space $(\Omega, \mathscr{B}(\Omega), \mathrm{Pr})$
- A random point on a Riemannian manifold $\mathscr{M}$ is a Borel measurable function from $\Omega \rightarrow \mathscr{M}$
- This allows us to induce a probability measure on the manifold itself.
- We will use volume forms to define densities and integrate.

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Induced Volume Form

- The induced volume form on a tangent space at point $x \in \mathscr{M}$ is then:

$$d\text{Vol} = \sqrt{\det |G|}\, dx$$

- We can define the density in a chart (U, ψ) such that $\tilde{x} = \psi(x)$. We can link the pdf on $\mathscr{M}$ to $\mathbb{R}^n$ via:

$$\rho_x(y) = p_x(y)\sqrt{|G(y)|}$$