Bounding $\dfrac{\ln(b) - \ln(a)}{b - a}$

by GPT

April 25, 2023

Here is a nice simple proof generated by GPT that bounds the ratio.

Theorem 1. *If* $0 < a < b$ *and* $f(a, b) = \frac{\ln(b)-\ln(a)}{b-a}$*, then*

$$\frac{1}{a} < f(a, b) < \frac{1}{b},$$

Proof. (The following proof was generated by GPT and lightly edited by me[1].) To prove the inequality

$$\frac{1}{a} < f(a, b) < \frac{1}{b},$$

where $f(a, b) = \frac{\ln(b)-\ln(a)}{b-a}$ and $0 < a < b$, we can use calculus and the Mean Value Theorem.

First, let's define a function $g(x) = \ln(x)$, where $x > 0$. The derivative of $g(x)$ with respect to x is given by

$$g'(x) = \frac{d}{dx} \ln(x) = \frac{1}{x}.$$

Now, let's apply the Mean Value Theorem to the function $g(x)$ on the interval (a, b). According to the Mean Value Theorem, there exists a point c in the open interval (a, b) such that the derivative of $g(x)$ at c is equal to the average rate of change of $g(x)$ over the interval (a, b). Mathematically, this is expressed as

$$g'(c) = \frac{g(b) - g(a)}{b - a}.$$

Substituting the expressions for $g(x)$ and $g'(x)$, we get

$$\frac{1}{c} = \frac{\ln(b) - \ln(a)}{b - a}. \tag{1}$$

[1] I am "irchans" on Reddit

Since c is in the interval (a, b), we have $a < c < b$. Therefore,

$$\frac{1}{b} < \frac{1}{c} < \frac{1}{a}.$$

Substituting the expression for $\frac{1}{c}$ from equation (1), we get the desired inequalities

$$\frac{1}{b} < \frac{\ln(b) - \ln(a)}{b - a} < \frac{1}{a}.$$

This completes the proof that if $0 < a < b$, then $\frac{1}{a} < f(a, b) < \frac{1}{b}$, where $f(a, b) = \frac{\ln(b)-\ln(a)}{b-a}$. □

1 ArtEnt Post

The following function has come up twice in my research

$$f(a, b) = \frac{\ln(b) - \ln(a)}{b - a}$$

where $0 < a < b$.

Now, I had known that if a and b are close, then

$$f(a, b) \approx \frac{1}{\text{mean}(a, b)}$$

and

$$\frac{1}{b} < f(a, b) < \frac{1}{a}.$$

Let $m = (a + b)/2$ and $\Delta = (b - a)/2$.

Below GPT and I derive the following

$$\begin{aligned} 1/b < f(a, b) < 1/a, \\ f(a, b) = 1/m + O(\Delta^2/m^3), \text{ and} \\ f(a, b) = \frac{\tanh^{-1}(\Delta/m)}{\Delta} \end{aligned}$$

where $O(g(\Delta, m))$ is a function $h(\Delta, m)$ that obeys

$$|h(\Delta, m)| < cg(\Delta, m)$$

for some positive real constant c and for all Δ and m where $0 < \Delta < m$, and

$$\tanh^{-1}(y) = x$$

if and only if

$$\tanh(x) := \frac{e^x - e^{-x}}{e^x + e^{-x}} = y$$

for any real numbers x and y.

THE DERIVATION

At first I tried to use Taylor Series to bound $f(a, b)$, but it was a bit convoluted so I asked GPT. GPT created a much nicer, simpler proof. (See this PDF). GPT's key observation is that

$$f(a, b) = \frac{\ln(b) - \ln(a)}{b - a} = \frac{1}{b - a} \int_a^b \frac{dx}{x}$$

is the mean value of $1/x$ over the interval $[a, b]$. (In truth, I felt a bit dumb for not having noticed this. Lol.)

GPT's observation inspires a bit more analysis.

Let

$$z = \frac{x}{m} - 1, \text{ so } \ m(z + 1) = x.$$

If $x = a$, then

$$z = \frac{a}{m} - 1 = \frac{m - \Delta}{m} - 1 = -\frac{\Delta}{m}.$$

Similarly, if $x = b$,

$$z = \frac{b}{m} - 1 = \frac{m + \Delta}{m} - 1 = \frac{\Delta}{m}.$$

Appling these substitutions to the integral yields

$$\begin{aligned}
\int_{x=a}^{x=b} \frac{dx}{x} &= \int_{z=-\Delta/m}^{z=\Delta/m} \frac{m\ dz}{m(z+1)} \\
&= \int_{z=-\Delta/m}^{z=\Delta/m} \frac{dz}{z+1} \\
&= \int_{z=-\Delta/m}^{z=\Delta/m} (1 - z + z^2 - z^3 + \cdots) dz \\
&= \int_{z=-\Delta/m}^{z=\Delta/m} (1 + z^2 + z^4 + \cdots) dz \\
&= \int_{z=-\Delta/m}^{z=\Delta/m} \frac{dz}{1 - z^2} \\
&= \tanh^{-1}(z) \Big|_{z=-\Delta/m}^{z=\Delta/m} \\
&= \tanh^{-1}(\Delta/m) - \tanh^{-1}(-\Delta/m) \\
&= 2 \tanh^{-1}(\Delta/m).
\end{aligned}$$

(Above we twice applied the wonderful thumb rule

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

if $|x| < 1$. See idea #87 from the top 100 math ideas.)

So,

$$\frac{\ln(b) - \ln(a)}{b-a} = \frac{2\tanh^{-1}(\Delta/m)}{b-a} = \frac{\tanh^{-1}(\Delta/m)}{\Delta}.$$

Futhermore,

$$\tanh^{-1}(x) = x + x^3/3 + x^5/5 + \cdots,$$

so

$$\begin{aligned} \frac{\ln(b) - \ln(a)}{b-a} &= \frac{1}{\Delta}\left(\frac{\Delta}{m} + \frac{\Delta^3}{3m^3} + \frac{\Delta^5}{5m^5} + \cdots\right) \\ &= \frac{1}{m} + \frac{\Delta^2}{3m^3} + \frac{\Delta^4}{5m^5} + \cdots. \end{aligned}$$

This series gives some nice approximations for $f(a,b)$ when $\Delta/m < 1/2$.

EXAMPLE.

Let $a = 6/100$ and $b = 7/100$. Then $m = 13/200$, $\Delta = 1/200$,

$$\frac{\ln(b) - \ln(a)}{b-a} = \tanh^{-1}(\Delta/m)/\Delta \approx 15.415067982725830429,$$

$$1/m \approx 15.3846,$$

$$1/m + \Delta^2/(3m^3) \approx 15.41496, \text{ and}$$

$$1/m + \Delta^2/(3m^3) + \Delta^4/(5m^5) \approx 15.4150675.$$

Have a great day!