The Second Banker's Problem: Buying a Better Interest Rate Again*

irchans† on Reddit

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Contents

*In the future, I hope to write about the third banker's problems. In the third banker's problem, the saver can buy two interest rate increases.

†About thirty years ago, my nick name on IRC was Hans.

1 Second Banker's Problem

Consider the following game. You have a bank account that is compounded continuously with an interest rate of r_1. Your banker offers you the following deal. At any time in the future, you can pay c dollars using only the interest from the account to increase your interest rate to r_2. If your balance is b at the time of the upgrade, then your balance will remain at b for $c/(r_1 b)$ years which is the time required to pay c dollars from interest alone.[1] After the $c/(r_1 b)$ years, the account grows with interest rate r_2. Other than paying for the interest upgrade, you can never add or subtract any money until retirement which is very far in the future. When should you accept the banker's offer? Let's call this game "the second banker's problem". (We assume that c, r_1, and r_2 are positive real numbers and $1 > r_2 > r_1 > 0$.)

For the first banker's problem, the balance on the account is immediately reduced by c dollars and the interest rate is immediately increased to r_2. See this old style http blog or this PDF for an analysis of the first banker's problem. The third banker's problem allows the saver to buy two separate interest rate increases.

For the second banker's problem, you might be tempted to say that you should buy the interest rate increase as early as possible, but that might be bad if $c/(r_1 b)$ is a large number of years. On the other hand, there is no reason to buy the interest rate upgrade if the time to retirement is less than $c/(r_1 b)$ years.

2 Optimal Purchase Time

Perhaps surprisingly, the optimal time to purchase the interest rate hike is the same for the first and the second banker's problems. For either problem, you should take the deal at time

$$t_{\text{buy}} = \frac{1}{r_1} \ln\left(\frac{c\, r_2}{B_0(r_2 - r_1)}\right) \tag{1}$$

where B_0 is the amount of money in the account at time zero, and $\ln(x)$ is the natural log.

$$\ln(x) = \frac{\log_{10}(x)}{\log_{10}(e)}.$$

I was a bit surprised to find that the optimal purchase time t_{buy} for the second banker's problem is the same as the optimal purchase time for the first banker's problem.

The amount of money in the account at the optimal purchase time is

$$b_{\text{opt}} = \frac{c\, r_2}{r_2 - r_1}.$$

[1] This game is similar to purchasing a growth technology in the game Master of Orion (Original 1993 version). For example, if you buy the "Improved Industrial Tech 9" technology early in the game, the cost of factories is reduced for the remainder of the game. Reducing the cost of factories increases your economy's growth rate.

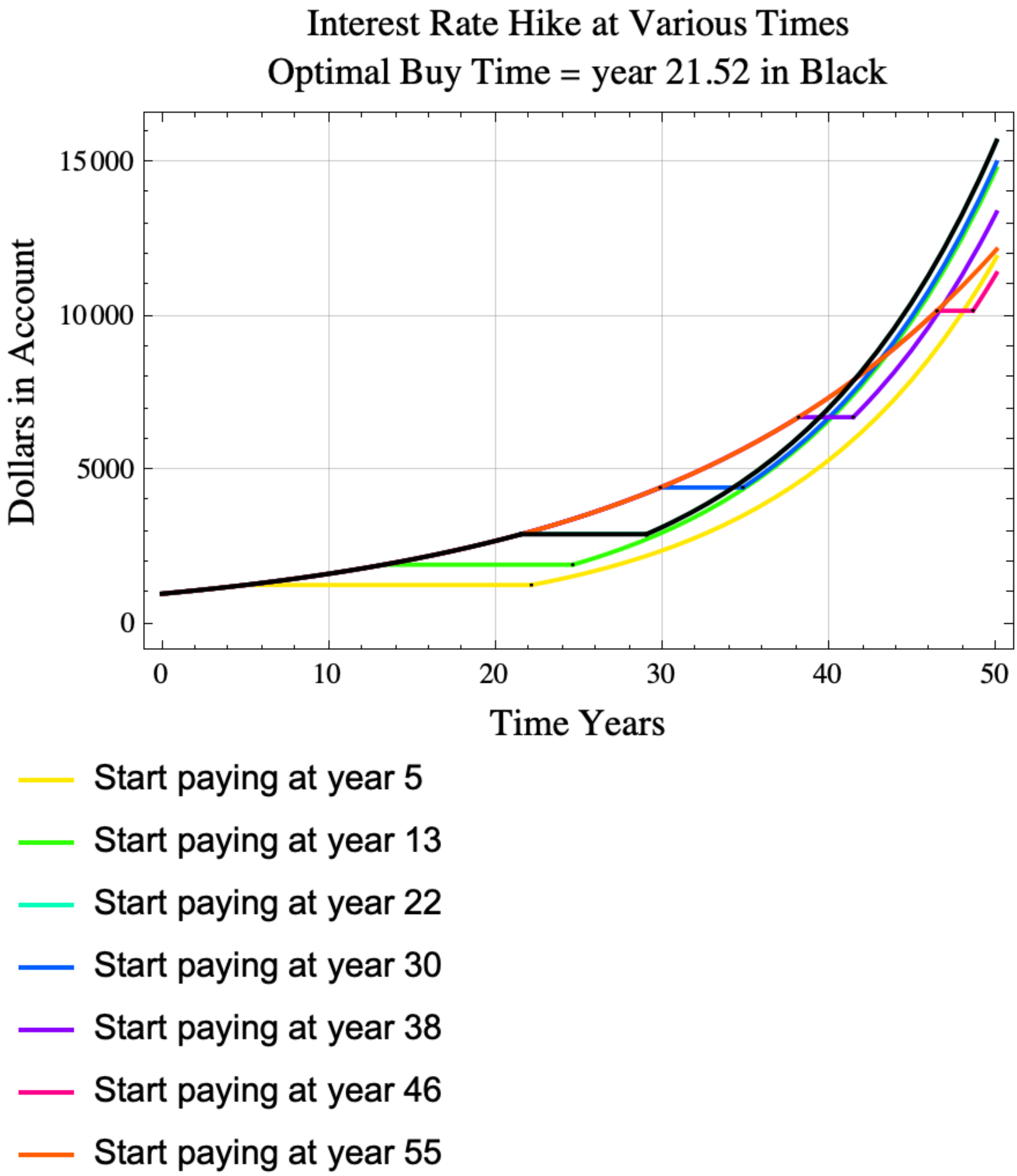
3 Example

For example, if

- you initially have \$1000 in the account,
- the cost of increasing the interest rate is \$1,100,
- $r_1 = 0.05 = 5\%$, and
- $r_2 = 0.08 = 8\%$,

then the the correct time to buy the interest rate increase is

$$\begin{aligned}
t_{\text{buy}} &= \frac{1}{r_1} \ln\left(\frac{c\, r_2}{B_0(r_2 - r_1)}\right) \\
&= \frac{1}{0.05} \ln\left(\frac{1100 \cdot 0.08}{1000(0.08 - 0.05)}\right) \\
&= \frac{1}{0.05} \ln\left(\frac{88}{1000(0.03)}\right) \\
&= 20 \ln\left(\frac{88}{30}\right) \\
&\approx 21.5228 \text{ years.}
\end{aligned}$$



In the diagram above, the black line shows the results of paying \$1,100 from interest starting at year 21.5228, the optimal time to start paying the investment. The orange line shows what happens if the player does not invest before year 50. The yellow line shows the result if she or he starts paying for the investment on year 5.

4 Interest Income immediately before the optimal purchase time for the Second Panker's problem

As with the first banker's problem, the income from interest just before the optimal purchase time is

$$\text{income immediately before purchase} = \frac{c\, r_1 r_2}{r_2 - r_1}. \tag{2}$$

Notice that this income does not depend on the initial amount in the bank account B_0.

Applying equation (2) to the example in the previous section, we get

$$\begin{aligned}\text{interest income immediately before purchase} &= \frac{c\, r_1 r_2}{r_2 - r_1}\\ &= \frac{\$1100 \cdot 0.05 \cdot 0.08}{0.08 - 0.05}\\ &= \frac{\$55 \cdot 0.08}{0.03} \approx \$146.67 \text{ per year.}\end{aligned}$$

Remark 1. *Note that the interest income can also be expressed by*

$$\text{interest income immediately before purchase} = c/t_{\text{pay}} \tag{3}$$

where

$$t_{\text{pay}} = \frac{1}{r_1} - \frac{1}{r_2} = \frac{c}{B_0 r_1 \exp(r_1 t_{\text{buy}})} \tag{4}$$

is the amount of time it takes to pay c dollars from interest starting at the optimal time t_{buy}.

If we apply formulas (3) and (4) to the example, we get

$$t_{\text{pay}} = \frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{0.05} - \frac{1}{0.08} = 20 - 12.5 = 7.5 \text{ years, and}$$

$$\text{interest income immediately before purchase} = c/t_{\text{pay}} = 1100/7.5 \approx \$146.67/\text{year}.$$

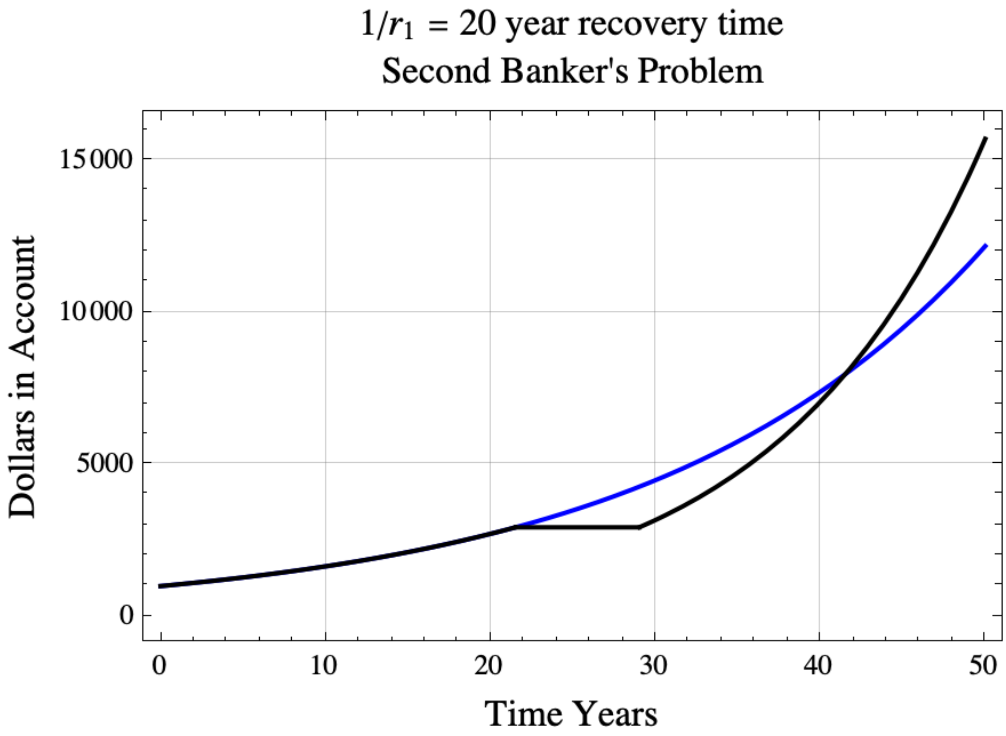
5 Recovery Time

If you do purchase the interest rate hike at the optimal time, how many years will you need to wait until the optimal strategy surpasses the never buy strategy. The recovery time for the second banker's problem is almost, but not quite the same as the first banker's problem. For the second banker's problem,

$$t_{\text{surpass}} = t_{buy} + 1/r_1.$$

For the example,

$$t_{\text{surpass}} \approx 21.5228 + \frac{1}{0.05} = 41.5228.$$



The black line shows the results of buying the interest rate increase at the optimal time. The blue line shows what happens if you never buy the interest rate hike and just continue to get 5% interest.

If you buy the interest rate hike at the optimal time, then you will maximize the account balance at all times $t > 1/r_1$ years later. (Mathematically, for every $t > t_{\text{buy}} + 1/r_1$, the strategy of purchasing the interest rate upgrade at time t_{buy} results in an account balance at time t that exceeds the account balance at time t using any other strategy. See Theorem 8.)

6 Proof of optimal purchase time formula for the Second Banker's problem

In this section, we state and prove the optimal purchase time assuming that the account balance is one dollar at time zero. This makes the math a bit simpler. In the next section, we state and prove the optimal purchase time in the more general case where the account balance is B_0 at time zero.

For the remainder of the paper, we make following assumptions:

- The cost of the interest rate upgrade is $c > 0$.
- The original interest rate is α and the bank account contains $\exp(\alpha t)$ dollars at any time t before paying for the interest rate upgrade.[2]
- If the saver chooses to start paying for the interest rate upgrade at time t_1, then the amount of money in the account remains at $\exp(\alpha t_1)$ dollars for $c/(\alpha \exp(\alpha t_1))$ years. After that time has elapsed, the money in the account grows with an interest rate of β. We will assume that $1 > \beta > \alpha > 0$.
- $$T = \frac{1}{\alpha} \ln\left(\frac{c\beta}{\beta - \alpha}\right),$$
- $\Delta : \mathbb{R} \to \mathbb{R}$ is defined[3] by $\Delta(t) = c/(\alpha \exp(\alpha t))$,
- The function $g : \mathbb{R}^2 \to \mathbb{R}$ where[4] $g(t, t_1)$ represents the balance in the account at time t if the interest rate increase payment begins at time t_1. The function g is defined by

$$\begin{aligned}
&g(t, t_1) = \exp(\alpha t) \quad \text{when} \quad t < t_1, \\
&g(t, t_1) = \exp(\alpha t_1) \quad \text{if} \quad t_1 \le t \le t_1 + \Delta(t_1), \quad \text{and} \\
&g(t, t_1) = \exp(\alpha t_1) \exp(\beta(t - t_1 - \Delta(t_1))) \quad \text{when} \quad t_1 + \Delta(t_1) < t.
\end{aligned}$$

[2] In this section, we changed the variable names. The optimal time to buy the interest rate hike t_{buy} is replaced by T. The interest rates r_1 and r_2 are replaced by α and β respectively. The amount of time required to pay the c dollars if the payments are made from interest only starting at time t is represented by $\Delta(t)$. We have also set the starting balance B_0 to \$1 to make the math a little simpler. In the next section, we let B_0 be any positive real number.

[3] The notation $\Delta : \mathbb{R} \to \mathbb{R}$ indicates that Δ takes one real number as input and outputs one real number.

[4] The notation $g : \mathbb{R}^2 \to \mathbb{R}$ indicates that g takes two real numbers as inputs and outputs one real number.

We begin with some lemmas.

Lemma 2.

$$\exp(\alpha T) = \frac{c\beta}{\beta - \alpha}.$$

Proof. By the definition of T,

$$\begin{aligned}
\exp(\alpha T) &= \exp\left(\alpha \frac{1}{\alpha} \ln\left(\frac{c\beta}{\beta - \alpha}\right)\right) \\
&= \exp\left(\ln\left(\frac{c\beta}{\beta - \alpha}\right)\right) \\
&= \frac{c\beta}{\beta - \alpha}.
\end{aligned}$$

□

Lemma 3.

1.

$$\Delta(T) = \frac{\beta - \alpha}{\alpha\beta} = \frac{1}{\alpha} - \frac{1}{\beta},$$

2. $1/\alpha - \Delta(T) = 1/\beta$, *and*

3. $\Delta(T) < 1/\alpha$.

Proof. By the definition of Δ and the previous lemma,

$$\Delta(T) = \frac{c}{\alpha \exp(\alpha T)} = \frac{c}{\alpha \frac{c\beta}{\beta - \alpha}} = \frac{\beta - \alpha}{\alpha\beta} = \frac{1}{\alpha} - \frac{1}{\beta} < 1/\alpha$$

proving parts 1 and 3. Part 2 follows from part 1 with a little algebra. □

Lemma 4. $g(T + 1/\alpha, T) = \exp(\alpha(T + 1/\alpha)) = \exp(\alpha T + 1)$.

Proof. By the definition of g and Lemma 3.3,

$$\begin{aligned}
g(T + 1/\alpha, T) &= \exp(\beta(T + 1/\alpha - T - \Delta(T))) \exp(\alpha T) \\
&= \exp(\beta(1/\alpha - \Delta(T))) \exp(\alpha T) \\
&= \exp(\beta(1/\beta)) \exp(\alpha T) \quad \text{by Lemma 3.2} \\
&= \exp(1) \exp(\alpha T) \\
&= \exp(\alpha T + 1) \\
&= \exp(\alpha(T + 1/\alpha)).
\end{aligned}$$

□

Lemma 5. *If $\tau + \Delta(\tau) < t_1 < t_2$, then*

$$g(t_2, \tau) = \exp(\beta(t_2 - t_1)) \cdot g(t_1, \tau)$$

Proof. By the definition of g,

$$\begin{aligned}
g(t_2, \tau) &= \exp(\alpha\tau) \cdot \exp(\beta(t_2 - \tau - \Delta(\tau))) \\
&= \exp(\alpha\tau) \cdot \exp(\beta(t_2 - t_1 + t_1 - \tau - \Delta(\tau))) \\
&= \exp(\alpha\tau) \cdot \exp(\beta(t_1 - \tau - \Delta(\tau))) \cdot \exp(\beta(t_2 - t_1)) \\
&= g(t_1, \tau) \cdot \exp(\beta(t_2 - t_1)).
\end{aligned}$$

□

Lemma 6. *If $t > T + 1/\alpha$, then*

$$\exp(\alpha t) < g(t, T).$$

Proof.

$$\begin{aligned}
\exp(\alpha t) &= \exp(\alpha(t - T - 1/\alpha + T + 1/\alpha) \\
&= \exp(\alpha(t - T - 1/\alpha)) \cdot \exp(\alpha(T + 1/\alpha)) \\
&= \exp(\alpha(t - T - 1/\alpha)) \cdot g(T + 1/\alpha, T) \quad \text{by Lemma 4} \\
&< \exp(\beta(t - T - 1/\alpha)) \cdot g(T + 1/\alpha, T) \\
&= g(t, T) \quad \text{by Lemmas 3.3 and 5.}
\end{aligned}$$

□

Lemma 7. *If $h : \mathbb{R} \to \mathbb{R}$ is defined by*

$$h(\tau) = \frac{g(\tau + \Delta(\tau), \tau)}{\exp(\beta(\tau + \Delta(\tau)))}$$

and $\tau \neq T$, then $h(\tau) < h(T)$.

Proof. By the definitions of h and g,

$$\begin{aligned}
h(\tau) &= \frac{g(\tau + \Delta(\tau), \tau)}{\exp(\beta(\tau + \Delta(\tau)))} \\
&= \exp(\alpha\tau) \exp(-\beta(\tau + \Delta(\tau))) \\
&= \exp(\alpha\tau - \beta(\tau + \Delta(\tau))).
\end{aligned}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$\begin{aligned}
f(\tau) &= \alpha\tau - \beta(\tau + \Delta(\tau)) \\
&= (\alpha - \beta)\tau - \frac{c\beta}{\alpha \exp(\alpha\tau)} \\
&= (\alpha - \beta)\tau - \frac{c\beta}{\alpha} \exp(-\alpha\tau).
\end{aligned}$$

For any real number t, the following are equivalent

$$\begin{aligned}
0 &< f'(t) \\
0 &< \alpha - \beta + c\beta \exp(-\alpha t) \\
\beta - \alpha &< c\beta \exp(-\alpha t) \\
\frac{\beta - \alpha}{c\beta} &< \exp(-\alpha t) \\
\frac{c\beta}{\beta - \alpha} &> \exp(\alpha t) \\
\exp(\alpha T) &> \exp(\alpha t) \quad \text{by Lemma 2} \\
T &> t.
\end{aligned}$$

In summary, for all $t < T$, $f'(t) > 0$, so

$$\text{for all } t < T, \quad f(t) < f(T) \quad \text{hence} \quad h(t) < h(T). \tag{5}$$

By reversing the inequalities in the argument above, we can infer that for all $t > T$, $f'(t) < 0$, so

$$\text{for all } t > T, \quad f(T) > f(t) \quad \text{hence} \quad h(T) > h(t). \tag{6}$$

We conclude from (5) and (6) that for all real numbers $t \neq T$,

$$h(t) < h(T).$$

□

Theorem 8. *For any $t > T + 1/\alpha$ the following two statements hold:*

1. *for every real number $\tau \neq T$,*

$$g(t, \tau) < g(t, T), \quad \text{and}$$

2. $\max_{\tau \in \mathbb{R}} g(t, \tau) = g(t, T)$.

Proof. First assume that $\tau \neq T$.

Case 1: $t < \tau$. By Lemma 6 and the definition of g,

$$g(t, \tau) = \exp(\alpha t) < g(t, T).$$

Case 2: $\tau \leq t \leq \tau + \Delta(\tau)$. By the definition of g and Lemma 6,

$$g(t, \tau) = \exp(\alpha \tau) \leq \exp(\alpha t) < g(t, T).$$

Case 3: $\tau + \Delta(\tau) < t$.

$$\begin{aligned}
g(t,\tau) &= \exp(\alpha\tau) \cdot \exp(\beta(t - \tau - \Delta(\tau)) \\
&= g(\tau + \Delta(\tau), \tau) \cdot \exp(\beta(t - \tau - \Delta(\tau)) \\
&= \frac{g(\tau + \Delta(\tau), \tau)}{\exp(\beta(\tau + \Delta(\tau)))} \cdot \exp(\beta t) \\
&< \frac{g(T + \Delta(T), T)}{\exp(\beta(T + \Delta(T)))} \cdot \exp(\beta t) \quad \text{by Lemma 7} \\
&= g(T + \Delta(T), T) \cdot \exp(\beta(t - T - \Delta(T))) \\
&= g(t, T) \quad \text{by Lemma 3.3 and because } t > T + 1/\alpha.
\end{aligned}$$

We have proven in all three cases that $g(t,\tau) < g(t,T)$ which proves part 1 of the theorem. Part 2 follows from Part 1.

□

The previous theorem shows that using the optimal strategy of buying the investment rate increase at time

$$T = \frac{1}{r_1} \ln\left(\frac{c\, r_2}{r_2 - r_1}\right)$$

creates a larger balance than any other strategy for all $t > T + 1/r_1$ if the balance of the account is one dollar at time zero.

7 Proof of the optimal purchase time in the more general case

Let B_0 be any positive real number. Now we define two functions, $\delta(t)$ and $G(t, t_1)$ as follows.

$\delta : \mathbb{R} \to \mathbb{R}$ is defined by

$$\delta(t) = c/(\alpha B_0 \exp(\alpha t)).$$

The function $G : \mathbb{R}^2 \to \mathbb{R}$ where $G(t, t_1)$ represents the balance in the account at time t if the interest rate increase payment begins at time t_1. The function G is defined by

$$\begin{aligned}
&G(t, t_1) = B_0 \exp(\alpha t) \quad \text{when} \quad t < t_1, \\
&G(t, t_1) = B_0 \exp(\alpha t_1) \quad \text{if} \quad t_1 \le t \le t_1 + \delta(t_1), \quad \text{and} \\
&G(t, t_1) = B_0 \exp(\alpha t_1) \exp(\beta(t - t_1 - \delta(t_1))) \quad \text{when} \quad t_1 + \delta(t_1) < t.
\end{aligned}$$

Note that $G(0, t_1) = B_0$ if $t_1 \ge 0$.

Theorem 9. *(time shift theorem):*

$$G(t, t_1) = g(t + \ln(B_0)/\alpha, t_1 + \ln(B_0)/\alpha).$$

Proof. Let $d = \ln(B_0)/\alpha$. Note that for any real number t,

$$\exp(\alpha(t + d)) = \exp(\alpha(t + \ln(B_0)/\alpha)) = B_0 \exp(\alpha t), \text{ so}$$

$$\delta(t) = \frac{c\beta}{\alpha B_0 \exp(\alpha t)} = \frac{c\beta}{\alpha \exp(\alpha(t + \ln(B_0)/\alpha))} = \Delta(t + \ln(B_0)/\alpha) = \Delta(t + d).$$

Now we divide the proof up into three cases.

Case 1: $t < t_1$.

$$\begin{aligned} G(t, t_1) &= B_0 \exp(\alpha t) \\ &= \exp(\alpha(t + d)) \\ &= g(t + d, t_1 + d). \end{aligned}$$

Case 2: $t_1 \le t \le t_1 + \delta(t_1)$. In this case note that

$$t_1 + d \le t + d$$

and

$$\begin{aligned} t + d &\le t_1 + \delta(t_1) + d \\ t + d &\le t_1 + d + \Delta(t_1 + d), \end{aligned}$$

so

$$\begin{aligned} G(t, t_1) &= \beta_0 \exp(\alpha t_1) \\ &= \exp(\alpha(t_1 + d)) \\ &= g(t + d, t_1 + d). \end{aligned}$$

Case 3: $t_1 + \delta(t_1) < t$. Note that in this case

$$\begin{aligned} t_1 + \delta(t_1) + d &< t + d \\ t_1 + d + \Delta(t_1 + d) &< d, \end{aligned}$$

so

$$\begin{aligned} G(t, t_1) &= B_0 \exp(\alpha t_1) \exp(\beta(t - t_1 - \delta(t_1)) \\ &= \exp(\alpha(t_1 + d)) \exp(\beta((t + d) - (t_1 + d) - \Delta(t_1 + d)) \\ &= g(t + d, t_1 + d). \end{aligned}$$

We have proven that in all three cases

$$G(t, t_1) = g(t + \ln(B_0)/\alpha, t_1 + \ln(B_0)/\alpha).$$

.

$\square$

Corollary 10. *Let*

$$t_{\text{buy}} = \frac{1}{\alpha} \ln \left(\frac{c\beta}{B_0(\beta - \alpha)} \right).$$

For all $t > t_{\text{buy}} + 1/\alpha$, the following two statements hold:

1. *for any $\tau \neq t_{\text{buy}}$,* $\quad G(t, t_{\text{buy}}) > G(t, \tau)$*, and*
2. $\max_{\tau \in \mathbb{R}} G(t, \tau) = G(t, t_{\text{buy}})$.

Proof. Let $d = \ln(B_0)/\alpha$. Note that

$$T - d = \frac{1}{\alpha} \ln \left(\frac{c\beta}{\beta - \alpha} \right) - \ln(B_0)/\alpha = \frac{1}{\alpha} \ln \left(\frac{c\beta}{B_0(\beta - \alpha)} \right) = t_{\text{buy}}.$$

Theorem 8 implies that for any $t > T + 1/\alpha - d = t_{\text{buy}} + 1/\alpha$ and every real number $\tau \neq T - d = t_{\text{buy}}$,

$$g(t + d, \tau + d) < g(t + d, T).$$

The time shift theorem states that for any real numbers t and t_1,

$$G(t, t_1) = g(t + d, t_1 + d).$$

So for any $t > t_{\text{buy}} + 1/\alpha$ and every real number $\tau \neq t_{\text{buy}}$,

$$\begin{aligned} g(t + d, \tau + d) &< g(t + d, T) \\ G(t, \tau) &< G(t, T - d) \\ G(t, \tau) &< G(t, t_{\text{buy}}). \end{aligned}$$

proving part 1. Part 2 of the corollary follows from part 1. □

This corollary shows that using the optimal strategy of buying the investment rate increase at time

$$t_{\text{buy}} = \frac{1}{r_1} \ln \left(\frac{c\, r_2}{B_0(r_2 - r_1)} \right)$$

creates a larger balance than any other strategy at any time $t > t_{\text{buy}} + 1/r_1$.

8 Acknowledgements

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9 Appendix - Python Simulation Code

```python
#!/usr/bin/env python3
# -*- coding: utf-8 -*-
#   SECOND BANKER'S PROBLEM

import numpy as np
import matplotlib.pyplot as plt

def balance(t, t1, r1, r2, c, b0):
    tdone = t1 + c / (b0 * r1 * np.exp(r1 * t1))
    constbal = b0 * np.exp(r1 * t1)

    if t < t1:
        return b0 * np.exp(r1 * t)
    elif t < tdone:
        return constbal
    else:
        return constbal * np.exp(r2 * (t - tdone))

b0 = 1000
r1 = 0.05
r2 = 0.08
c = 1100
tbuy = 1 / r1 * np.log(c * r2 / (b0 * (r2 - r1)))
tsurpass = tbuy + 1 / r1
ttest = int(np.round(tsurpass + 1))

print("optimal buy time =", tbuy)
print("balance at year", ttest, "using optimal strategy", balance(ttest,
    tbuy, r1, r2, c, b0))

data = [[t1, balance(ttest, t1, r1, r2, c, b0)] for t1 in range(1, 41)]
print("\nYear of Purchase and Balance at  t=" + str(ttest))
for row in data:
    print(row)

t1_values = np.arange(1, 41)
balance_values = [balance(ttest, t1, r1, r2, c, b0) for t1 in t1_values]

plt.plot(t1_values, balance_values)
plt.scatter(tbuy, balance(ttest, tbuy, r1, r2, c, b0), color='red', label
    ='Optimal Buy Time')
plt.xlabel('Year of Purchase')
plt.ylabel('Balance at t='+str(ttest) )
plt.grid(True)
plt.legend()
plt.show()
```

Listing 1: Python Simulation Of Second Banker's Problem