Two applications of the Lambert Growth Formula to the Original Master of Orion

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1 Introduction

The Lambert W function can be used to find a solution to the differential equation

$$y'(t) = \beta y(t - T).$$

One solution to this differential equation is

$$y(t) = \exp(\alpha T)$$

where

$$\alpha = W_0(\beta T)/T$$

and W_0 is the Lambert W function which is defined as follows.

Definition 1. *(Lambert W function) For every non-negative real number z there exists a unique non-negative real number x such that*

$$x \exp(x) = z.$$

Thus we can define a function $W_0 : [0, \infty) \to [0, \infty)$ by[1]

$$W_0(z) = x$$

if and only i

$$x \exp(x) = z.$$

Most people don't have the Lambert W function (also called Product-Log) on their calculator or computer. So, below are some ways to compute or estimate the growth rate without having that function available. If you want to get a very precise answer (15 digits of accuracy), Wolfram Alpha and Python seem to be the easiest way to get

[1] The notation $W_0 : [0, \infty) \to [0, \infty)$ means that W_0 accepts non-negative real numbers as input values and outputs a non-negative real number

it. If less accuracy is needed, we give some approximations that are within 2% of the correct answer which might be good enough for Master of Orion. (See also this nice website.) For higher accuracy approximations, the most common algorithms are the Newton-Raphson Method and Bisection Method. (The radius of convergence of Taylor Series of the Lambert W function seems to be 1, so that is not very practical).

For most of the methods below we calculate or estimate the growth rate assuming that $I/C = 0.2$ and $T = 30$.

We define the Lambert Growth Rate to be the real number α that satisfies the equation

$$\alpha \exp(\alpha T) = I/C, \quad \text{or, equivalently,} \quad \alpha = W_0(IT/C)/T.$$

METHODS FOR CALCULATING OR APPROXIMATING THE LAMBERT GROWTH RATE

- Use Wolfram Alpha to solve equation (2). You can literally type

  "solve a*exp(a*30) = 0.2"

  into the Wolfram Alpha input box. (The answer given is 0.0477468.)

- Use Wolfram Alpha to evaluate $W_0(T\frac{I}{C})/T$. You can type

  "W_0(30*0.2)/30"

  into the Wolfram Alpha input box.

- You can import the Lambert W function into Python as follows.

  ```
  from scipy.special import lambertw
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- In Javascript, you can use the Lambert W function by importing the math library and then using this code

  ```
  math.lambertW(x)
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- The Lambert W function can be approximated by

  $$w_1(x) = (.5391x - .4479)^{(1/2.9)}.$$

  If $1.6 \leq x \leq 22$, then w_1 is off by less that 2%. More precisely,

  $$|w_1(x)/W_0(x) - 1| < 0.02.$$

  We can use this approximation to estimate the growth rate

  $$W_0(30 * 0.2)/30 = W_0(6)/30 \approx (.5391 \times 6 - .4479)^{(1/2.9)}/30 \approx 0.047463321.$$

- If $0 \leq x \leq 2$, the Lambert W function can be approximated by

  $$w_2(x) = \frac{x}{(1+x)(1-0.109x)}.$$

  If $0 \leq x \leq 2$, then w_2 is off by less than 1%. More precisely,

  $$|w_2(x)/W_0(x) - 1| < 0.01.$$

- Once you have an estimate of the solution, you can use the Newton-Raphson Method to improve the accuracy. In order to solve, $a\exp(aT) = I/C$ we let $f(x) = x\exp(xT) - I/C$. If x is an estimate of the solution to $f(x) = 0$,

$$n(x) = x - f(x)/f'(x)$$

will often be a better approximation of the solution. For our problem

$$n(x) = x - \frac{xe^{Tx} - I/C}{(Tx+1)e^{Tx}}.$$

I ran a simulation using typical but random values for I, C, and T. Every time I generated random I, C, and T, I would use either w_1 or w_2 to estimate the Lambert growth rate $W_0(TI/C)/T$. Let $x = TI/C$. If $x < 1.8$, I use w_2 and I use w_1 otherwise. The average error using w_1 and w_2 was 0.0016. If I then applied n to the approximation, then the average error was 0.00017, about 10 times more accurate. When I applied n twice

$$W_0(x)/T \approx n(n(w_2(x)/T)) \text{ when } x < 1.8, \text{ and}$$

$$W_0(x)/T \approx n(n(w_1(x)/T)) \text{ when x} \geq 1.8,$$

the average error dropped to 0.000005 and the worst case error was 0.0001. If you want 16 digits of accuracy, you need to apply the function n five times.

Applying the Newton Raphson to estimate the Lambert growth rate for $I/C = 0.2$ and $T = 30$, gives

$$\begin{aligned} W_0(IT/C)/C = W_0(6)/30 &\approx w_1(6)/30 \approx 0.047463321 \\ W_0(6)/30 &\approx n(w_1(6)/30) \approx 0.047748535122 \\ W_0(6)/30 &\approx n(n(w_1(6)/30)) \approx 0.047746825925 \\ W_0(6)/30 &\approx 0.047746825863 \end{aligned}$$

- You can also use the Bisection Method to solve $a\exp(aT) = I/C$. For 16 digits of accuracy, you will need approximately 60 iterations, so that method is slower, but a bit simpler than Newton-Raphson.

2 Appendix 1 : A derivation of the growth formula

2.1 Lambert Growth Theorem

First we will give the definition of the Lambert W function and then we will prove the formula for the growth rate for a simplified version of Master of Orion. In order to clearly define the Lambert W function, it is helpful to first prove the following lemma.

Lemma 2. *For all non-negative real numbers z, there exists exactly one non-negative number x such that $x\exp(x) = z$.*

Proof. (This a slight modification of a Proof created by GPT!) We need to prove two things: existence and uniqueness.

Existence: Consider the function $f(x) = x\exp(x)$. We can see that $f(0) = 0$ and as x approaches infinity, $f(x)$ also approaches infinity. Thus, given any non-negative real number z, by the Intermediate Value Theorem, there exists at least one x in the interval $[0, \infty)$ such that $f(x) = z$.

Uniqueness: We now need to show that this x is unique.

Let's compute the derivative of $f(x) = x\exp(x)$. Using the product rule, we find that $f'(x) = \exp(x) + x\exp(x)$. For $x \geq 0$,

$$f'(x) = \exp(x) + x\exp(x) \geq 1 + 0 = 1.$$

Assume for contradiction that there exist x_1 and x_2 such that $x_1 < x_2$ and $f(x_1) = z = f(x_2)$. According to the Mean Value Theorem, there exists a c in the open interval (x_1, x_2) such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. But since $f(x_1) = f(x_2)$, the right-hand side of the equation becomes 0, contradicting the fact that $f'(c) \geq 1$. So there can be only one x such that $x\exp(x) = z$ for any non-negative real number z. ☐

Definition 3. *(Lambert W function) For every non-negative real number z there exists a unique non-negative real number x such that*

$$x\exp(x) = z.$$

Thus we can define a function $W_0 : [0, \infty) \to [0, \infty)$ by[2]

$$W_0(z) = x$$

if and only i

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The following theorem is inspired by the original Master of Orion (1993) computer game. Below the constant I represents the income of a "mature" planet, C is the cost of constructing a colony ship, T is the number of years between the construction year of a colony ship and the year that the colonized planet is "mature", and $b'(t)$ is the approximate income at time t. When the planet is mature, it starts producing new colony ships.

This theorem basically states that for a very simplified version of Master of Orion, income grows exponentially. More specifically, the total income is approximately proportional to $\exp(\alpha t)$ where t is the time in years,

$$\alpha = \frac{W_0\left(\frac{iT}{C}\right)}{T}, \quad \text{and} \tag{1}$$

[2]The notation $W_0 : [0, \infty) \to [0, \infty)$ means that W_0 accepts non-negative real numbers as input values and outputs a non-negative real number

$$\alpha \exp(\alpha T) = I/C. \tag{2}$$

Theorem 4. *(Lambert Growth Theorem) Choose any positive real constants I, C, and T and let*

$$\alpha = \frac{W_0\left(\frac{iT}{C}\right)}{T}$$

where W_0 is the Lambert W function. Then for all $B_0 \in \mathbb{R}$,

$$b(t) = B_0 \exp(\alpha t)$$

satisfies

$$b'(t) = \frac{I}{C}\, b(t-T).$$

Furthermore, α is the only real number that satisfies

$$\alpha \exp(\alpha T) = I/C.$$

Proof. (This proof is the result of a difficult collaboration between me and GPT.)

First, we find the derivative of the function $b(t) = B_0 \exp(\alpha t)$ with respect to t

$$b'(t) = B_0 \alpha \exp(\alpha t) = \alpha b(t).$$

Now with the given definition of $b(t)$, the following are equivalent:

$$\begin{aligned}
b'(t) &= \frac{I}{C}\, b(t-T) \\
\alpha B_0 \exp(\alpha t) &= \frac{I}{C} B_0 \exp(\alpha(t-T)) \\
\alpha &= \frac{I}{C} \exp(-\alpha T) \\
\alpha \exp(\alpha T) &= \frac{I}{C} \\
\alpha T \exp(\alpha T) &= \frac{IT}{C} \\
\alpha T &= W_0\left(\frac{IT}{C}\right) \\
\alpha &= \frac{W_0\left(\frac{IT}{C}\right)}{T}.
\end{aligned}$$

(We applied the definition of the Lambert W function. If $x = \alpha T$ is the unique solution to the equation $x \exp(x) = \frac{IT}{C}$, then $W_0(\frac{IT}{C}) = x$.)

Thus the function $b(t) = B_0 \exp(\alpha t)$ satisfies the given differential equation, and by the uniqueness of the solution to $x \exp(x) = y$ given by the Lambert W function, α is also the unique solution that satisfies $\alpha \exp(\alpha T) = I/C$. $\square$

2.2 Approximating the solution and the Lambert W function

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$$n(x) = x - \frac{xe^{Tx} - I/C}{(Tx+1)e^{Tx}}.$$

In the game Master of Orion, the economic growth rate in the early part of the game can be estimated by $\alpha = W_0(T\frac{I}{C})$ where I represents the income of a "mature" planet, C is the cost of constructing a colony ship, T is the number of years between the construction year of a colony ship and the year that the colonized planet is "mature". I ran a simulation using typical but random values where $10 \le I \le 200$, $200 \le C \le 575$, and $10 \le T \le 90$. Every time I generated random I, C, and T values, I would use either w_1 or w_2 to estimate the Lambert growth rate $W_0(TI/C)/T$. Let $x = TI/C$. If $x < 1.8$, I used w_2 and I used w_1 otherwise. The average error using w_1 and w_2 was 0.0016. If I then applied n to the approximation, then the average error was 0.00017, about 10 times more accurate. When I applied n twice the average error dropped to 0.000005 and the worst case error was 0.0001. If you want 16 digits of accuracy, you need to apply the function n five times.

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2.2.2 SUMMARY

In this section we presented several methods for approximating the Lambert Growth Rate

$$W_0(IT/C)/T.$$

If you don't have Wolfram Alpha or access to a programming environment that includes the Lambert W function, then one of the best methods for finding the solution is the Bisection Method. If $x = IT/C < 100$, then using w_1 or w_2 approximates the solution to within 2%. One iteration of Newton-Raphson typically reduces the error by a factor of 10. More iterations of Newton-Raphson significantly improve the approximation.