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Bounding (ln(b)-ln(a))/(b-a) with a little help from GPT

April 25, 2023 in Math by hundalhh | No comments (edit)

The following function has come up twice in my research

$$f(a,b) = \frac{\ln(b)-\ln(a)}{b-a}$$

where $0 < a < b$.

Now, I had known that if a and b are close, then

$$f(a,b) \approx \frac{1}{\mathrm{mean}(a,b)}$$

and

$$\frac{1}{b} < f(a,b) < \frac{1}{a}.$$

But last week, with a little help from GPT, I got some better approximations and bounds on $f(a,b)$.

Let $m = (a+b)/2$ and $\Delta = (b-a)/2$.

Below GPT and I derive the following

- $$1/b < f(a,b) < 1/a,$$
- $$1/m < f(a,b) < 1/m + \frac{\Delta^2}{3m}\left(\frac{1}{m^2-\Delta^2}\right),$$
- $$-\frac{2\Delta^4}{15a^5} < f(a,b) - \frac{1}{6}\left(\frac{1}{a}+\frac{4}{m}+\frac{1}{b}\right) < -\frac{2\Delta^4}{15b^5}, \text{ and}$$
- $$\begin{aligned} f(a,b) &= \frac{\tanh^{-1}(\Delta/m)}{\Delta} \\ &= \frac{1}{\Delta}\left(\frac{\Delta}{m}+\frac{\Delta^3}{3m^3}+\frac{\Delta^5}{5m^5}+\cdots\right) \end{aligned}$$

where

$$\tanh^{-1}(y) = x$$

if and only if

$$\tanh(x) := \frac{e^x-e^{-x}}{e^x+e^{-x}} = y$$

for any real numbers x and y.

(Alternatively,

$$\tanh^{-1}(x) = \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(1-x)$$

where $\ln(x)$ is the natural log and $|x| < 1$.)

THE DERIVATION

At first I tried to use Taylor Series to bound $f(a,b)$, but it was a bit convoluted so I asked GPT. GPT created a much nicer, simpler proof. (See this PDF). GPT's key observation was that

$$f(a,b) = \frac{\ln(b)-\ln(a)}{b-a} = \frac{1}{b-a}\int_a^b \frac{dx}{x}$$

is the mean value of $1/x$ over the interval $[a,b]$. (In truth, I felt a bit dumb for not having noticed this. Lol.)

GPT's observation inspires a bit more analysis.

Let

$$z = \frac{x}{m} - 1, \text{ so } m(z+1) = x.$$

If $x = a$, then

$$z = \frac{a}{m} - 1 = \frac{m-\Delta}{m} - 1 = -\frac{\Delta}{m}.$$

Similarly, if $x = b$,

$$z = \frac{b}{m} - 1 = \frac{m+\Delta}{m} - 1 = \frac{\Delta}{m}.$$

Applying these substitutions to the integral yields

$$\begin{aligned} \int_{x=a}^{x=b} \frac{dx}{x} &= \int_{z=-\Delta/m}^{z=\Delta/m} \frac{m\,dz}{m(z+1)} \\ &= \int_{z=-\Delta/m}^{z=\Delta/m} \frac{dz}{z+1} \\ &= \int_{z=-\Delta/m}^{z=\Delta/m} (1 - z + z^2 - z^3 + \cdots)dz \\ &= \int_{z=-\Delta/m}^{z=\Delta/m} (1 + z^2 + z^4 + \cdots)dz \\ &= \int_{z=-\Delta/m}^{z=\Delta/m} \frac{dz}{1-z^2} \\ &= \tanh^{-1}(z)\Big|_{z=-\Delta/m}^{z=\Delta/m} \\ &= \tanh^{-1}(\Delta/m) - \tanh^{-1}(-\Delta/m) \\ &= 2\tanh^{-1}(\Delta/m). \end{aligned}$$

(Above we twice applied the wonderful thumb rule

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

if $|x| < 1$. See idea #87 from the top 100 math ideas.)

So,

$$\frac{\ln(b)-\ln(a)}{b-a} = \frac{2\tanh^{-1}(\Delta/m)}{b-a} = \frac{\tanh^{-1}(\Delta/m)}{\Delta}.$$

Furthermore,

$$\tanh^{-1}(x) = x + x^3/3 + x^5/5 + \cdots,$$

so

$$\begin{aligned} f(a,b) = \frac{\ln(b)-\ln(a)}{b-a} &= \frac{1}{\Delta}\left(\frac{\Delta}{m}+\frac{\Delta^3}{3m^3}+\frac{\Delta^5}{5m^5}+\cdots\right) \\ &= \frac{1}{m}+\frac{\Delta^2}{3m^3}+\frac{\Delta^4}{5m^5}+\frac{\Delta^6}{7m^7}+\cdots \end{aligned}$$

This series gives us some nice approximations of $f(a,b)$ when $\Delta/m < 1/2$. We can also bound the error of the approximation

$$f(a,b) \approx 1/m$$

as follows

$$\begin{aligned} \frac{1}{m} < \frac{\ln(b)-\ln(a)}{b-a} &= \frac{1}{m}+\frac{\Delta^2}{3m^3}+\frac{\Delta^4}{5m^5}+\frac{\Delta^6}{7m^7}+\cdots \\ &< \frac{1}{m}+\frac{\Delta^2}{3m^3}+\frac{\Delta^4}{3m^5}+\frac{\Delta^6}{3m^7}+\cdots \\ &= \frac{1}{m}+\frac{\Delta^2}{3m^3}(1+\frac{\Delta^2}{m^2}+\frac{\Delta^4}{m^4}+\cdots) \\ &= \frac{1}{m}+\frac{\Delta^2}{3m^3}\left(\frac{1}{1-\frac{\Delta^2}{m^2}}\right) \\ &= \frac{1}{m}+\frac{\Delta^2}{3m}\left(\frac{1}{m^2-\Delta^2}\right). \end{aligned}$$

EXAMPLE.

Let $a = 6/100$ and $b = 7/100$. Then $m = 13/200$, $\Delta = 1/200$,

- $$\frac{\ln(b)-\ln(a)}{b-a} = \tanh^{-1}(\Delta/m)/\Delta \approx 15.415067982725830429,$$
- $$1/m \approx 15.3846,$$
- $$1/m + \Delta^2/(3m^3) \approx 15.41496, \text{ and}$$
- $$1/m + \Delta^2/(3m^3) + \Delta^4/(5m^5) \approx 15.4150675.$$
- $$1/m + \frac{\Delta^2}{3m}\left(\frac{1}{m^2-\Delta^2}\right) \approx 15.41514$$

APPLYING SIMPSON'S RULE

We can also use Simpson's rule to approximate $\ln(b)-\ln(a)$. The error formula for Simpson's rule is

$$\int_a^b g(x)\,dx = \frac{\Delta}{3}[g(a)+4g(m)+g(b)] - \frac{\Delta^5}{90}g^{(4)}(\xi)$$

for some ξ in the interval (a,b). Setting $g(x) = 1/x$,

$$I = \ln(b)-\ln(a), \quad \text{and} \quad h(a,b) = \frac{\Delta}{3}\left(\frac{1}{a}+\frac{4}{m}+\frac{1}{b}\right)$$

with $m = (a+b)/2$ and $\Delta = (b-a)/2$ gives

$$\begin{aligned} \ln(b)-\ln(a) &= \frac{\Delta}{3}\left(\frac{1}{a}+\frac{4}{m}+\frac{1}{b}\right) - \frac{\Delta^5}{90}\frac{24}{\xi^5} \\ I &= h(a,b) - \frac{4\Delta^5}{15\xi^5} \\ I - h(a,b) &= -\frac{4\Delta^5}{15\xi^5} \\ -\frac{4\Delta^5}{15a^5} < I - h(a,b) &< -\frac{4\Delta^5}{15b^5}. \end{aligned}$$

Now we divide by $2\Delta = b - a$ to conclude that

$$\begin{aligned} \frac{\ln(b)-\ln(a)}{b-a} &= \frac{1}{6}\left(\frac{1}{a}+\frac{8}{a+b}+\frac{1}{b}\right) + error \\ &= \frac{1}{6}\left(\frac{1}{a}+\frac{4}{m}+\frac{1}{b}\right) + error \\ &\approx \frac{1}{6}\left(\frac{1}{a}+\frac{4}{m}+\frac{1}{b}\right) - \frac{2\Delta^4}{15m^5} \end{aligned}$$

where

$$-\frac{2\Delta^4}{15a^5} < error < -\frac{2\Delta^4}{15b^5}.$$

(Thanks to GPT and StackEdit(https://stackedit.io/).)

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