The First Banker's Problem: Buying a Better Interest Rate*

irchans on Reddit†

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Contents

*In the future, I hope to write about the second and third banker's problems. The second banker's problem is very similar to the first, but the payment comes from interest rather than principal. In the third banker's problem, the saver can buy two interest rate increases.

†About thirty years ago, my nick name on IRC was Hans.

1 Buying a better interest rate

Consider the following game. You have a bank account that is compounded continuously with an interest rate of r_1. Your banker offers you the following deal. At any time in the future, if your account balance is greater than c dollars, you can pay c dollars from the account to increase your interest rate to r_2.[1] You can only use the funds in the account to pay for this interest rate increase and you can never add or subtract any money until retirement which is very far in the future. When should you accept the banker's offer? Let's call this game "the first banker's game".[2] (We assume that c, r_1, and r_2 are positive real numbers and $1 > r_2 > r_1 > 0$.)

At first you might be tempted to say that you should buy the interest rate increase as early as possible, but it turns out that that is a bad idea. If you have exactly c dollars in the account and you buy the rate increase, then you will have zero dollars in the account and you are stuck with zero dollars in the account until you retire. Similarly, if you have $c + \$0.01$ in the account and you buy the interest rate increase, then you will only have one cent in the account after the purchase, and it will take a long time to grow that one cent into a large amount of money.

On the other hand, it is probably wrong to wait until the last microsecond before you retire to buy an interest rate increase because the increased amount of interest that you receive is unlikely to be larger than the cost c.

2 Optimal Purchase Time

The answer to the first banker's problem is that you should take the deal at time

$$t_{\text{buy}} = \frac{1}{r_1} \ln\left(\frac{c\, r_2}{B_0(r_2 - r_1)}\right) \tag{1}$$

where B_0 is the amount of money in the account at time zero, and $\ln(x)$ is the natural log.

$$\ln(x) = \frac{\log_{10}(x)}{\log_{10}(e)}.$$

[1] This game is similar to purchasing a growth technology in the game Master of Orion. For example, if you buy the "Improved Industrial Tech 9" technology early in the game, the cost of factories is reduced for the remainder of the game. Reducing the cost of factories increases your economy's growth rate.

[2] The second and third banker's problem are briefly described in the first foot note on page 1 of this paper

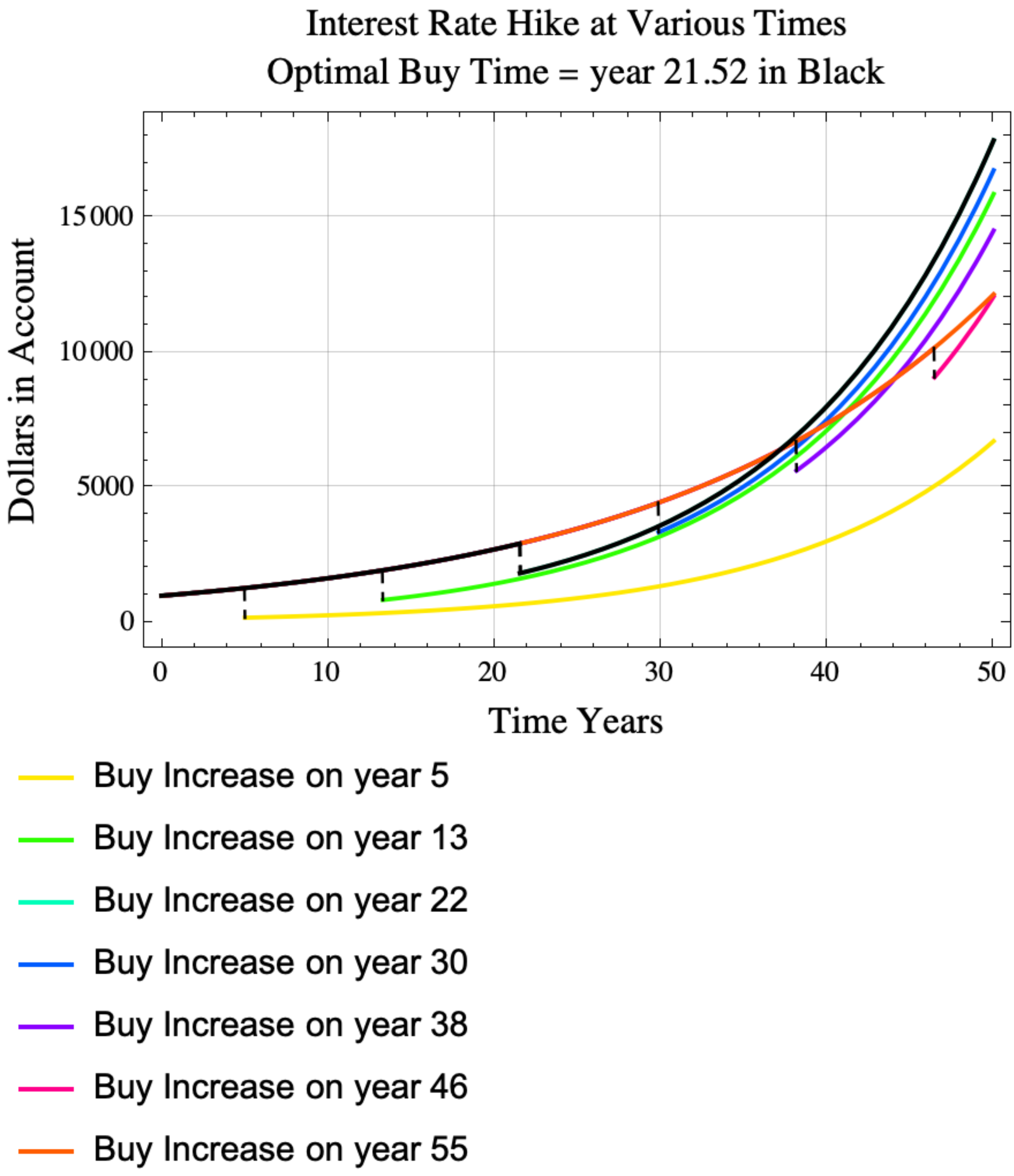
3 Example

For example, if

- you initially have \$1000 in the account,
- the cost of increasing the interest rate is \$1,100,
- $r_1 = 0.05 = 5\%$, and
- $r_2 = 0.08 = 8\%$,

then the the correct time to buy the interest rate increase is

$$\begin{aligned} t_{\text{buy}} &= \frac{1}{r_1} \ln\left(\frac{c\, r_2}{B_0(r_2 - r_1)}\right) \\ &= \frac{1}{0.05} \ln\left(\frac{1100 \cdot 0.08}{1000(0.08 - 0.05)}\right) \\ &= \frac{1}{0.05} \ln\left(\frac{88}{1000(0.03)}\right) \\ &= 20 \ln\left(\frac{88}{30}\right) \\ &\approx 21.5228 \text{ years.} \end{aligned}$$



In the diagram above, the black line shows the results of paying $1,100 at year 21.5228, the optimal time to invest. The red line shows what happens if the player does not invest before year 50. The yellow line shows the result if he invests on year 5.

4 Interest Income immediately before the optimal purchase time

The income from interest just before the optimal purchase time is

$$\text{income immediately before purchase} = \frac{c\, r_1 r_2}{r_2 - r_1}. \tag{2}$$

Notice that this income does not depend on the initial amount in the bank account B_0.

This formula can be derived from the equation (1). Before purchasing the interest rate hike, the amount of money in the account is

$$y(t) = B_0 \exp(r_1 t).$$

The interest income at time t before purchasing is

$$y'(t) = B_0 r_1 \exp(r_1 t).$$

The optimal purchase time is

$$t_{\text{buy}} = \frac{1}{r_1} \ln\left(\frac{c\, r_2}{B_0(r_2 - r_1)}\right).$$

The interest income just before the purchase is

$$\begin{aligned}
\text{income immediately before purchase} &= B_0 r_1 \exp(r_1\, t_{\text{buy}}) \\
&= B_0 r_1 \exp\left(r_1\, \frac{1}{r_1} \ln\left(\frac{c\, r_2}{B_0(r_2 - r_1)}\right)\right) \\
&= B_0 r_1 \exp\left(\ln\left(\frac{c\, r_2}{B_0(r_2 - r_1)}\right)\right) \\
&= \frac{c\, r_1 r_2}{r_2 - r_1}.
\end{aligned}$$

Applying equation (2) to the example in the previous section, we get

$$\begin{aligned}
\text{income immediately before purchase} &= \frac{c\, r_1 r_2}{r_2 - r_1} \\
&= \frac{\$1100 \cdot 0.05 \cdot 0.08}{0.08 - 0.05} \\
&= \frac{\$55 \cdot 0.08}{0.03} \approx \$146.67 \text{ per year.}
\end{aligned}$$

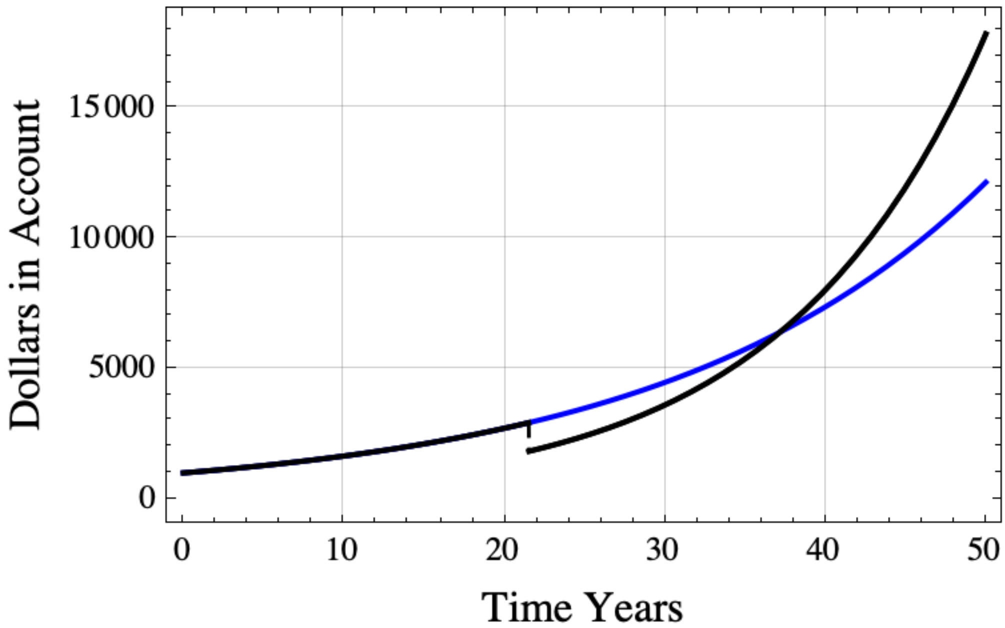
5 Recovery Time

If you do purchase the interest rate hike at the optimal time, how many years will you need to wait until the optimal strategy surpasses the never buy strategy? The answer is

$$t_{\text{surpass}} = t_{buy} + \frac{\ln(r_2) - \ln(r_1)}{r_2 - r_1}.$$

For the example,

$$t_{\text{surpass}} \approx 21.5228 + \frac{\ln(0.08) - \ln(0.05)}{0.08 - 0.05} \approx 37.1868.$$



The black line shows the results of buying the interest rate at the optimal time. The blue line shows what happens if you never buy the interest rate hike and just continue to get 5% interest.

The number of years needed to catch up is between $1/r_2$ years and $1/r_1$ years. GPT wrote a nice proof proof of this fact. (See section 7.4.)

If you buy the interest rate hike at the optimal time, then you will maximize the account balance at all times $t > 1/r_1$ years later. (Mathematically, for every $t > t_{\text{buy}} + 1/r_1$, the strategy of purchasing the interest rate upgrade at time t_{buy} results in an account balance at time t that exceeds the account balance at time t using any other strategy. See Theorem 5.)

Lastly,

$$\frac{\ln(r_2) - \ln(r_1)}{r_2 - r_1} = \frac{1}{m} \frac{\tanh^{-1}(\Delta/m)}{(\Delta/m)} \approx 1/m$$

where $\Delta = \frac{r_2 - r_1}{2}$ and m is the mean of r_1 and r_2 (i.e. $m = \frac{r_1 + r_2}{2}$).[3] (See section 7.4, this old fashion http insecure blog post, or this PDF for details.)

6 Proof of optimal purchase time formula

In this section we make following assumptions

- The cost of the interest rate upgrade is $c > 0$,
- At the time of purchase, interest rate increases from α to β where $\beta > \alpha > 0$,
- $$T = \frac{1}{\alpha} \ln\left(\frac{c\beta}{\beta - \alpha}\right),$$
- $$\Delta = \frac{\ln(\beta) - \ln(\alpha)}{\beta - \alpha},$$
- The function $g : \mathbb{R}^2 \to \mathbb{R}$ where[4] $g(t, t_1)$ represents the balance in the account at time t if the interest rate increase is purchased at time t_1 is defined by

$$\begin{aligned} g(t, t_1) &= \exp(\alpha t) \quad \text{when} \quad t < t_1, \\ g(t_1, t_1) &= \exp(\alpha t_1) - c, \quad \text{and} \\ g(t, t_1) &= (\exp(\alpha t_1) - c) \exp(\beta(t - t_1)) \quad \text{when} \quad t > t_1. \end{aligned}$$

In this section, we will prove that the best long term strategy is to purchase the interest rate upgrade at time T. Specifically,

$$g(t, T) > g(t, \tau)$$

for any $t > T + \frac{\ln(\beta) - \ln(\alpha)}{\beta - \alpha} = T + \Delta$ and any real number $\tau \neq T$. (See Theorem 5.)

[3] You can use Simpson's rule to get a better approximation than $1/m$. The Simpson's rule approximation is $\frac{\ln(r_2) - \ln(r_1)}{r_2 - r_1} \approx \frac{1}{6}\left(\frac{1}{r_1} + \frac{4}{m} + \frac{1}{r_2}\right)$.

[4] The notation $g : \mathbb{R}^2 \to \mathbb{R}$ indicates that g takes two real numbers as inputs and outputs one real number.

First we prove some lemmas.

Lemma 1.

$$\exp(\alpha T) = \frac{c\beta}{\beta - \alpha}.$$

Proof.

$$\begin{aligned}
\exp(\alpha T) &= \exp\left(\alpha \frac{1}{\alpha} \ln\left(\frac{c\beta}{\beta - \alpha}\right)\right) \\
&= \exp\left(\ln\left(\frac{c\beta}{\beta - \alpha}\right)\right) \\
&= \frac{c\beta}{\beta - \alpha}.
\end{aligned}$$

□

Lemma 2.

$$g(T, T) = \exp(\alpha T) - c = \frac{\alpha}{\beta} \exp(\alpha T).$$

Proof. Applying Lemma 1 twice gives

$$\begin{aligned}
g(T, T) &:= \exp(\alpha T) - c \\
&= \frac{c\beta}{\beta - \alpha} - c \\
&= \frac{c\beta - c(\beta - \alpha)}{\beta - \alpha} \\
&= \frac{c\alpha}{\beta - \alpha} \\
&= \frac{\alpha}{\beta} \frac{c\beta}{\beta - \alpha} \\
&= \frac{\alpha}{\beta} \exp(\alpha T).
\end{aligned}$$

□

Lemma 3. $g(T+\Delta, T) = \exp(\alpha(T+\Delta))$.

Proof.

$$\begin{aligned}
\frac{\ln(\beta) - \ln(\alpha)}{\beta - \alpha} &= \Delta \\
\frac{\ln(\beta/\alpha)}{\beta - \alpha} &= \Delta \\
\ln(\beta/\alpha) &= \Delta(\beta - \alpha) \\
\Delta\alpha &= \ln(\alpha/\beta) + \Delta\beta \\
\exp(\Delta\alpha) &= \frac{\alpha}{\beta}\exp(\Delta\beta) \\
\exp(\alpha T)\exp(\Delta\alpha) &= \frac{\alpha}{\beta}\exp(\alpha T)\exp(\Delta\beta) \\
\exp(\alpha(T+\Delta)) &= (\exp(\alpha T) - c)\exp(\Delta\beta) \quad \text{by Lemma 2} \\
\exp(\alpha(T+\Delta)) &= g(T+\Delta, T).
\end{aligned}$$

□

Lemma 4. *Fix any $t \in \mathbb{R}$. Let $h : (-\infty, t] \to \mathbb{R}$ be defined by*

$$h(\tau) := g(t, \tau)/\exp(\beta t).$$

Then for all $\tau \leq t$,

1. *$h'(\tau) > 0$ if and only if $\tau < T$,*
2. *$h'(\tau) < 0$ if and only if $\tau > T$,*
3. *if $\tau \neq T$ and $t \geq T$, then*
$$g(t, \tau) < g(t, T),$$
4. *if $t \geq T$, then*
$$\max_{\tau \leq t} g(t, \tau) = g(t, T).$$

Proof. Assume $\tau \leq t$. By the definitions of g and h,

$$\begin{aligned}
h(\tau) &= g(\tau, \tau) \cdot \exp(\beta(t - \tau))/\exp(\beta t) \\
&= (\exp(\alpha\tau) - c)\exp(-\beta\tau) \\
&= \exp((\alpha - \beta)\tau) - c\exp(-\beta\tau).
\end{aligned}$$

The following are equivalent

$$\begin{aligned}
0 &< h'(\tau) \\
0 &< (\alpha-\beta)\exp((\alpha-\beta)\tau) + c\beta\exp(-\beta\tau) \\
0 &< (\alpha-\beta)\exp(\alpha\tau) + c\beta \\
(\beta-\alpha)\exp(\alpha\tau) &< c\beta \\
\exp(\alpha\tau) &< c\beta/(\beta-\alpha) \\
\alpha\tau &< \ln(c\beta/(\beta-\alpha)) \\
\tau &< \frac{1}{\alpha}\ln(\frac{c\beta}{\beta-\alpha}) = T
\end{aligned}$$

proving part 1 of the Lemma.

By reversing the inequalities, we get $0 > h'(\tau)$ if and only if $\tau > T$ proving part 2 of the Lemma.

To prove part 3, assume $t \geq T$ and $\tau \neq T$.

Case 1: $\tau < T$. In this case part 1 implies h is strictly increasing on the interval (τ, T), so $h(\tau) < h(T)$.

Case 2: $T < \tau$. In this case part 2 implies h is strictly decreasing on the interval (T, τ), so $h(\tau) < h(T)$.

In either case,

$$\begin{aligned}
h(\tau) &< h(T) \\
g(t,\tau)/\exp(\beta t) &< g(t,T)/\exp(\beta t) \\
g(t,\tau) &< g(t,T)
\end{aligned}$$

proving part 3.

To prove part 4, assume $\tau \leq t$ and $t \geq T$. If $\tau \neq T$, then $g(t,\tau) < g(t,T)$ by part 3. If $\tau = T$, then $g(t,\tau) = g(t,T)$. Thus

$$\max_{\tau \leq t} g(t,\tau) = g(t,T).$$

$\square$

Theorem 5. *If*

$$t > T + \frac{\ln(\beta) - \ln(\alpha)}{\beta - \alpha} = T + \Delta,$$

then the following hold:

1. *For any real number $\tau \neq T$,*

$$g(t, \tau) < g(t, T). \tag{3}$$

2. $$g(t, T) = \max_{t_1 \in \mathbb{R}} g(t, t_1). \tag{4}$$

Proof. Let $t^* = T + \Delta$ and fix any $t > t^*$. Now for any real number $\tau \neq T$, one of the following two cases must hold.

Case 1: $\tau \leq t$ and $\tau \neq T$. Then inequality (3) holds by Lemma 4.3.

Case 2: $\tau > t$. In this case,

$$\begin{aligned}
g(t, \tau) &= \exp(\alpha t) \\
&= \exp(\alpha(t^* + t - t^*)) \\
&= \exp(\alpha t^*) \exp(\alpha(t - t^*)) \\
&= g(t^*, T) \exp(\alpha(t - t^*)) \quad \text{by Lemma 3} \\
&< g(t^*, T) \exp(\beta(t - t^*)) \\
&= g(T, T) \exp(\beta(t^* - T)) \exp(\beta(t - t^*)) \\
&= g(T, T) \exp(\beta(t - T)) \\
&= g(t, T).
\end{aligned} \tag{5}$$

In either case, inequality (3) holds proving part 1.

To prove part 2, note that if $\tau \neq T$, then $g(t, \tau) < g(t, T)$ by part 1. If $\tau = T$, then $g(t, \tau) = g(t, T)$. Thus

$$\max_{\tau \in \mathbb{R}} g(t, \tau) = g(t, T).$$

$\square$

7 Approximating $(\ln(b) - \ln(a))/(b - a)$

Let
$$f(a, b) = \frac{\ln(b) - \ln(a)}{b - a}$$
where $0 < a < b$.

Now, I had known that if a and b are close, then
$$f(a, b) \approx \frac{1}{\text{mean}(a, b)}$$
and
$$\frac{1}{b} < f(a, b) < \frac{1}{a}.$$
But last week, with a little help from GPT, I got some better approximations and bounds on $f(a, b)$.

Let $m = (a + b)/2$ and $\Delta = (b - a)/2$.

Below GPT and I derive the following three facts:

1. $$1/b < f(a, b) < 1/a,$$

2. $$1/m < f(a, b) < 1/m + \frac{\Delta^2}{3m}\left(\frac{1}{m^2 - \Delta^2}\right), \text{ and}$$

3. $$\begin{aligned} f(a, b) &= \frac{\tanh^{-1}(\Delta/m)}{\Delta} \\ &= \frac{1}{\Delta}\left(\frac{\Delta}{m} + \frac{\Delta^3}{3m^3} + \frac{\Delta^5}{5m^5} + \cdots\right) \end{aligned}$$

where
$$\tanh^{-1}(y) = x$$
if and only if
$$\tanh(x) := \frac{e^x - e^{-x}}{e^x + e^{-x}} = y$$
for any real numbers x and y.

(Alternatively,
$$\tanh^{-1}(x) = \frac{1}{2}\ln(x + 1) - \frac{1}{2}\ln(1 - x)$$
where $\ln(x)$ is the natural log and $|x| < 1$.)

7.1 Derivation

At first I tried to use Taylor series to bound $f(a, b)$, but it was a bit convoluted, so I asked GPT. GPT created a much nicer, simpler proof. (See this Section 7.4). GPT's key observation was that

$$f(a, b) = \frac{\ln(b) - \ln(a)}{b - a} = \frac{1}{b - a} \int_a^b \frac{dx}{x}$$

is the mean value of $1/x$ over the interval $[a, b]$. (In truth, I felt a bit dumb for not having noticed this. Lol.)

GPT's observation inspires a bit more analysis.

Let

$$z = \frac{x}{m} - 1, \text{ so } m(z + 1) = x.$$

If $x = a$, then

$$z = \frac{a}{m} - 1 = \frac{m - \Delta}{m} - 1 = -\frac{\Delta}{m}.$$

Similarly, if $x = b$,

$$z = \frac{b}{m} - 1 = \frac{m + \Delta}{m} - 1 = \frac{\Delta}{m}.$$

Applying these substitutions to the integral yields

$$\begin{aligned}
\int_{x=a}^{x=b} \frac{dx}{x} &= \int_{z=-\Delta/m}^{z=\Delta/m} \frac{m\,dz}{m(z+1)} \\
&= \int_{z=-\Delta/m}^{z=\Delta/m} \frac{dz}{z+1} \\
&= \int_{z=-\Delta/m}^{z=\Delta/m} (1 - z + z^2 - z^3 + \cdots)dz \\
&= \int_{z=-\Delta/m}^{z=\Delta/m} (1 + z^2 + z^4 + \cdots)dz \\
&= \int_{z=-\Delta/m}^{z=\Delta/m} \frac{dz}{1 - z^2} \\
&= \tanh^{-1}(z)\Big|_{z=-\Delta/m}^{z=\Delta/m} \\
&= \tanh^{-1}(\Delta/m) - \tanh^{-1}(-\Delta/m) \\
&= 2\tanh^{-1}(\Delta/m).
\end{aligned}$$

(Above we twice applied the wonderful thumb rule

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots$$

if $|x| < 1$. See idea #87 from the top 100 math ideas.)

So,

$$\frac{\ln(b)-\ln(a)}{b-a} = \frac{2\tanh^{-1}(\Delta/m)}{b-a} = \frac{\tanh^{-1}(\Delta/m)}{\Delta}.$$

Furthermore,

$$\tanh^{-1}(x) = x + x^3/3 + x^5/5 + \cdots,$$

so

$$\begin{aligned} f(a,b) = \frac{\ln(b)-\ln(a)}{b-a} &= \frac{1}{\Delta}\left(\frac{\Delta}{m} + \frac{\Delta^3}{3m^3} + \frac{\Delta^5}{5m^5} + \cdots\right) \\ &= \frac{1}{m} + \frac{\Delta^2}{3m^3} + \frac{\Delta^4}{5m^5} + \frac{\Delta^6}{7m^7} + \cdots \end{aligned}$$

This series gives us some nice approximations of $f(a,b)$ when $\Delta/m < 1/2$. We can also bound the error of the approximation

$$f(a,b) \approx 1/m$$

as follows

$$\begin{aligned} \frac{1}{m} < \frac{\ln(b)-\ln(a)}{b-a} &= \frac{1}{m} + \frac{\Delta^2}{3m^3} + \frac{\Delta^4}{5m^5} + \frac{\Delta^6}{7m^7} + \cdots \\ &< \frac{1}{m} + \frac{\Delta^2}{3m^3} + \frac{\Delta^4}{3m^5} + \frac{\Delta^6}{3m^7} + \cdots \\ &= \frac{1}{m} + \frac{\Delta^2}{3m^3}\left(1 + \frac{\Delta^2}{m^2} + \frac{\Delta^4}{m^4} + \cdots\right) \\ &= \frac{1}{m} + \frac{\Delta^2}{3m^3}\left(\frac{1}{1-\frac{\Delta^2}{m^2}}\right) \\ &= \frac{1}{m} + \frac{\Delta^2}{3m}\left(\frac{1}{m^2-\Delta^2}\right). \end{aligned}$$

7.2 Example

Let $a = 6/100$ and $b = 7/100$. Then $m = 13/200$, $\Delta = 1/200$,

$$\frac{\ln(b)-\ln(a)}{b-a} = \tanh^{-1}(\Delta/m)/\Delta \approx 15.415067982725830429,$$

$$1/m \approx 15.3846,$$

$$1/m + \Delta^2/(3m^3) \approx 15.41496,$$

$$1/m + \Delta^2/(3m^3) + \Delta^4/(5m^5) \approx 15.4150675, \text{ and}$$

$$1/m + \frac{\Delta^2}{3m}\left(\frac{1}{m^2-\Delta^2}\right) \approx 15.41514.$$

7.3 Applying Simpson's rule

We can also use Simpson's rule to approximate $\ln(b) - \ln(a)$. The error formula for Simpson's rule is

$$\int_a^b g(x)\,dx = \frac{\Delta}{3}[g(a) + 4g(m) + g(b)] - \frac{\Delta^5}{90}g^{(4)}(\xi)$$

for some ξ in the interval (a, b). Setting $g(x) = 1/x$,

$$I = \ln(b) - \ln(a), \quad \text{and} \quad h(a,b) = \frac{\Delta}{3}\left(\frac{1}{a} + \frac{4}{m} + \frac{1}{b}\right)$$

with $m = (a+b)/2$ and $\Delta = (b-a)/2$ gives

$$\begin{aligned}
\ln(b) - \ln(a) &= \frac{\Delta}{3}\left(\frac{1}{a} + \frac{4}{m} + \frac{1}{b}\right) - \frac{\Delta^5}{90}\frac{24}{\xi^5} \\
I &= h(a,b) - \frac{4\Delta^5}{15\xi^5} \\
I - h(a,b) &= -\frac{4\Delta^5}{15\xi^5} \\
-\frac{4\Delta^5}{15a^5} < I - h(a,b) &< -\frac{4\Delta^5}{15b^5}.
\end{aligned}$$

Now we divide by $2\Delta = b - a$ to conclude that

$$\begin{aligned}
\frac{\ln(b) - \ln(a)}{b-a} &= \frac{1}{6}\left(\frac{1}{a} + \frac{4}{m} + \frac{1}{b}\right) + error \\
&\approx \frac{1}{6}\left(\frac{1}{a} + \frac{4}{m} + \frac{1}{b}\right) - \frac{2\Delta^4}{15m^5}
\end{aligned}$$

where

$$-\frac{2\Delta^4}{15a^5} < error < -\frac{2\Delta^4}{15b^5}.$$

7.4 GPT bound

Here is a nice simple proof generated by GPT that bounds the ratio.

Theorem 6. *If* $0 < a < b$ *and* $f(a,b) = \frac{\ln(b)-\ln(a)}{b-a}$, *then*

$$\frac{1}{a} < f(a,b) < \frac{1}{b},$$

Proof. (The following proof was generated by GPT and lightly edited by me.) To prove the inequality

$$\frac{1}{a} < f(a,b) < \frac{1}{b},$$

where $f(a,b) = \frac{\ln(b)-\ln(a)}{b-a}$ and $0 < a < b$, we can use calculus and the Mean Value Theorem.

First, let's define a function $g(x) = \ln(x)$ where $x > 0$. The derivative of $g(x)$ with respect to x is given by

$$g'(x) = \frac{d}{dx}\ln(x) = \frac{1}{x}.$$

Now, let's apply the Mean Value Theorem to the function $g(x)$ on the interval (a,b). According to the Mean Value Theorem, there exists a point c in the open interval (a,b) such that the derivative of $g(x)$ at c is equal to the average rate of change of $g(x)$ over the interval (a,b). Mathematically, this is expressed as

$$g'(c) = \frac{g(b)-g(a)}{b-a}.$$

Substituting the expressions for $g(x)$ and $g'(x)$, we get

$$\frac{1}{c} = \frac{\ln(b)-\ln(a)}{b-a}. \tag{6}$$

Since c is in the interval (a,b), we have $a < c < b$. Therefore,

$$\frac{1}{b} < \frac{1}{c} < \frac{1}{a}.$$

Substituting the expression for $\frac{1}{c}$ from equation (6), we get the desired inequalities

$$\frac{1}{b} < \frac{\ln(b)-\ln(a)}{b-a} < \frac{1}{a}.$$

This completes the proof that if $0 < a < b$, then $\frac{1}{a} < f(a,b) < \frac{1}{b}$, where $f(a,b) = \frac{\ln(b)-\ln(a)}{b-a}$. □

8 Acknowledgements

Thanks to GPT, StackEdit(https://stackedit.io/), and Reddit users modnar_hajile and charming_science_360.

9 Appendix - Python Simulation Code

```python
#!/usr/bin/env python3
import numpy as np

def assert_condition(condition, message):
    if not condition:
        raise ValueError(message)

# B0 - initial balance
def simBank1(B0, r1, r2, c, tbuy, tend, bPrint):
    assert_condition(B0 > 0, "failed assertion")
    assert_condition(r1 > 0, "failed assertion")
    assert_condition(r2 > r1, "failed assertion")
    assert_condition(c > 0, "failed assertion")
    assert_condition(tend > 0, "failed assertion")

    if B0 * np.exp(r1 * tbuy) <= c:
        print(f"Insufficient funds available to buy the rate increase at time {tbuy}")
        raise ValueError("InsufficientFunds")

    vOut = []
    balance = B0

    for t in range(1,tend + 1):
        if bPrint:
            print(f"t={t} balance={balance}")

        if tbuy - t >= 1:
            balance = balance * np.exp(r1)
        elif 0 <= tbuy - t < 1:
            bal1 = balance * np.exp(r1 * (tbuy - t))
            bal2 = bal1 - c
            balance = bal2 * np.exp(r2 * (t + 1 - tbuy))

            if bPrint:
                print([bal1, bal2, balance])
        else:
            balance = balance * np.exp(r2)

        vOut.append(balance)

    return vOut

# Example usage:
result = simBank1(1000, .05, .08, 1100, 50, 40, False)
print(result)
result = simBank1(1000, .05, .08, 1100, 21.5228, 40, False)
print(result)
```

Listing 1: Python Simulation Of First Banker's Problem