Dan Mark Orapa
2020 – 05453

Math 150.2 AY 2022-2023 2nd Semester
March 07, 2023
W2 Assessment 1

---

1. A population consists of the numbers $3, 6, 12, 18$. Consider all possible samples of size 2 which can be drawn with replacement from this population. Find the mean and standard deviation of the sampling distribution of the means.

   Solution. Let the population be X with mean and variance μ and σ^2, respectively. The sampling distribution is the mean of the realizations of the samples. We let T be the statistic of the random variables X_i. By the derivation of the mean and variance from the lecture, we have the following

$$\begin{aligned}\mathbb{E}[T] &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] \\ &= \mu \\ &= \frac{3+6+12+18}{4} = \boxed{\frac{39}{4}}.\end{aligned}$$

   Furthermore, the variance of the sampling distribution is equal to $\frac{\sigma^2}{n}$ which implies that

$$\begin{aligned}s^2 &= \frac{\sigma^2}{n} \\ &= \frac{1}{n}\cdot\mathbb{E}[(X-\mu)^2] \\ &= \frac{1}{n}\cdot\frac{(3-9.75)^2+(6-9.75)^2+(12-9.75)^2+(18-9.75)^2}{4} \\ &= \frac{1}{2}\cdot\frac{132.75}{4} \\ &= 16.59375\end{aligned}$$

   Therefore, the standard deviation of the sampling distribution is $\boxed{4.07354}$. □

2. Let $X_1, X_2, \ldots, X_n$ be random samples from a population with a uniform distribution on the interval $(0, \theta)$, and $Y_1, Y_2, \ldots, Y_n$ be the corresponding order statistics.

   (a) Find $f_{Y_1}(x)$.

   Solution. We are to find the pdf of the 1st order statistic. The process of finding the CDF of the specified order statistic and differentiating afterwards can be denoted by

$$f_{Y_k}(x) = \frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1}[1-F(x)]^{n-k}f(x)\mathbb{1}_{(0,\,\theta)}(x).$$

   Note that

$$F(x) = \int_0^x \frac{1}{\theta}\,dt = \frac{x}{\theta}$$

Therefore, we have

$$f_{Y_1}(x) = \frac{n!}{(n-1)!}\left(1-\frac{x}{\theta}\right)^{n-1}\left(\frac{1}{\theta}\right) = \boxed{\left(\frac{n}{\theta}\right)\left(1-\frac{x}{\theta}\right)^{n-1}\mathbb{1}_{(0,\,\theta)}(x)}.$$

□

(b) Is Y_1 an unbiased estimator of θ?

Solution. Given that the population follows uniform distribution, we conclude that the 1st order statistic is unbiased if $\mathbb{E}[Y_1] = \theta$.

i. The mean of the order statistic is

$$\begin{aligned}\mathbb{E}[Y_1] &= \int_0^\theta x f_{Y_1}(x)\,dx \\ &= \int_0^\theta x\left(\frac{n}{\theta}\right)\left(1-\frac{x}{\theta}\right)^{n-1}\,dx\end{aligned}$$

We let $u = 1 - \dfrac{x}{\theta}$, and it follows that $x = \theta(1-u)$ and $dx = -\theta du$. Moreover, the bounds of integration change to $x = 0 \to u = 1$ and $x = \theta \to u = 0$. Thus,

$$\begin{aligned}\mathbb{E}[Y_1] &= \int_0^1 \theta(1-u)\left(\frac{n}{\theta}\right)(u)^{n-1}\,\theta du \\ &= n\theta\int_0^1 (u^{n-1} - u^n)du \\ &= n\theta\left(\frac{u^n}{n} - \frac{u^{n+1}}{n+1}\right)\Bigg|_0^1 \\ &= n\theta\left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \frac{\theta}{n+1} \neq \theta\end{aligned}$$

ii. The second raw moment of the order statistic is

$$\begin{aligned}\mathbb{E}[Y_1^2] &= \int_0^\theta x^2\left(\frac{n}{\theta}\right)\left(1-\frac{x}{\theta}\right)^{n-1}\,dx \\ &= \int_0^1 \theta^2(1-u)^2\left(\frac{n}{\theta}\right)(u)^{n-1}\theta\,du \\ &= n\theta^2\int_0^1 u^{n-1}(1-2u+u^2)\,du \\ &= n\theta^2\int_0^1 (u^{n-1} - 2u^n + u^{n+1})\,du \\ &= n\theta^2\left(\frac{u^n}{n} - \frac{2u^{n+1}}{n+1} + \frac{u^{n+2}}{n+2}\right)\Bigg|_0^1 \\ &= \frac{2\theta^2}{(n+1)(n+2)}\end{aligned}$$

Then, the variance is equal to

$$\begin{aligned}\mathbb{E}[Y_1^2] - (\mathbb{E}[Y_1])^2 &= \frac{2\theta^2}{(n+1)(n+2)} - \left(\frac{\theta}{n+1}\right)^2 \\ &= \frac{2\theta^2(n+1) - \theta^2(n+2)}{(n+1)^2(n+2)} \\ &= \frac{\theta^2}{(n+1)^2(n+2)}\end{aligned}$$

Answer: $\boxed{\text{No.}}$ It is enough to show that the mean of the order statistic is not equal to θ. □