Math 150.2 AY 2022-2023 2nd Semester
June 14, 2023
**Problem Set 2**

---

1. Let $X$ be a single observation from the density

$$f(x;\theta) = \theta x^{\theta-1}\mathbb{1}_{(0,1)}(x), \quad \theta > 0$$

(a) In testing $H_0 : \theta \leq 1$ vs. $H_1 : \theta > 1$, find the power and the size of the test given the following: Reject $H_0$ iff $X \leq 2/3$.

*Solution.* We define the parametric space $\Omega_0 := \{\theta : \theta \leq 1\}$. Solving for the power of the test, we have

$$\begin{aligned}\Pi_C(\theta) &= \mathbb{P}_\theta[X \leq 2/3] \\ &= \int_0^{2/3} \theta x^{\theta-1} dx \\ &= \left(\frac{2}{3}\right)^\theta\end{aligned}$$

and the size $\alpha$ of the test is

$$\begin{aligned}\alpha &= \sup_{\theta\in\Omega_0} \Pi_C(\theta) \\ &= 1\end{aligned}$$

since for such $\theta \in \Omega_0$, the power attains its maximum at $\theta = 0$. $\square$

(b) Find the GLRT of size $\alpha$ of $H_0 : \theta = 1$ vs. $H_1 : \theta \neq 1$.

*Solution.* First, we solve for the $\theta_{MLE}$ using methods of moments. That is,

$$\begin{aligned}L(\theta) &= \theta^n \prod_{i=1}^{n} x_i^{\theta-1} \\ \ell(\theta) &= n\ln(\theta) + (\theta-1)\sum_{i=1}^{n} \ln x_i \\ \ell'(\theta) &= \frac{n}{\theta} + \sum_{i=1}^{n} \ln x_i \\ \theta_{MLE} &= -\frac{n}{\sum_{i=1}^{n} \ln x_i}\end{aligned}$$

For this instance, given we only have a single observation, $\theta_{MLE} = -\dfrac{1}{\ln x}$. Moreover, we define $\Omega_0 := \{\theta : \theta = 1\}$. Hence,

$$\sup_{\theta\in\Omega_0} L(\theta; x_i) = f(x;1) = 1$$

Therefore,

$$\begin{aligned}\lambda &= \frac{\sup_{\theta\in\Omega_0} L(\theta; x_i)}{\sup_{\theta\in\Omega} L(\theta; x_i)}\\ &= \frac{1}{L(\theta_{MLE})}\\ &= \frac{1}{-(1/\ln x)x^{-1/\ln x-1}}\\ &= -\ln(x)x^{1/\ln x+1}\end{aligned}$$

and we reject $H_0$ if $\lambda \leq \lambda_0$ for some $\lambda_0 \in [0,1]$. □

2. Let $X_1, X_2, ..., X_n$ denote a random sample from a distribution that is $N(0,\theta)$, where the variance $\theta$ is an unknown positive number. Show that there exists a uniformly most powerful test of size $\alpha$ for testing the simple hypothesis $H_0 : \theta = \theta'$ where $\theta$ is a fixed positive number.

   *Solution.* Recall that the pdf of $N(0,\theta)$ is

$$f(x; 0, \theta) = \frac{1}{\sqrt{2\pi\theta}}\exp(-\frac{1}{2\theta}x^2)$$

   Solving for MLR, we have

$$\begin{aligned}\frac{L(\theta_a)}{L(\theta_b)} &= \frac{\left(\frac{1}{\sqrt{2\pi\theta_a}}\right)^n \exp\left(-\frac{1}{2\theta_a}\sum_{i=1}^n x_i^2\right)}{\left(\frac{1}{\sqrt{2\pi\theta_b}}\right)^n \exp\left(-\frac{1}{2\theta_b}\sum_{i=1}^n x_i^2\right)}\\ &= \left(\frac{\theta_b}{\theta_a}\right)^{n/2}\exp\left(\frac{1}{2}\sum x^2\left(\frac{1}{\theta_b}-\frac{1}{\theta_a}\right)\right) = M \quad \text{(say)}\end{aligned}$$

   and for every $\theta_a < \theta_b$, $M' < 0$. Hence, $M$ is a nonincreasing function of $T = \sum x^2$. Then there exists a MLR on $T$. Moreover,

$$\left(\frac{\theta_b}{\theta_a}\right)^{n/2}\exp\left(\frac{1}{2}\sum x^2\left(\frac{1}{\theta_b}-\frac{1}{\theta_a}\right)\right) \leq k' \longrightarrow \sum x^2 \leq 2\ln\left(\left(\frac{\theta_a}{\theta_b}\right)^{n/2} k'\right)\left(\frac{1}{\theta_b}-\frac{1}{\theta_a}\right)^{-1} = k^*$$

   and for every $k^*$ such that $\mathbb{P}_{\theta=\theta'}(T \leq k^*) = \alpha$, the test corresponding to $C := \{T \leq k^*\}$ is the UMPT of size $\alpha$ of $H_0 : \theta = \theta'$. □

3. A study recorded the growth in Standard & Poor's stock index following each election of a new president, given in the following table.

| Republicans | 22.4 | 24.0 | 38.0 | 45.7 | 21.2 | 17.9 | 38.2 | 33.7 | 23.8 |
|---|---|---|---|---|---|---|---|---|---|
| Democrats | 45.7 | 28.6 | 14.2 | 18.8 | 50.3 | 40.1 | 52.4 | | |

   Test, at a 10% significance level, if the election of a Republican president is not good for the stock market. Assume variances are not equal.

*Solution.* Let the means be $\mu_R$ and $\mu_D$ for the republican and democrats, respectively. We have the following hypotheses:

$$H_0 : \mu_R - \mu_D = 0 \quad \text{vs.} \quad H_1 : \mu_R - \mu_D < 0$$

We have the statistics $\overline{x}_R = 29.433$ and $\overline{x}_D = 35.7286$. at it follows that

$$s_R^2 = \frac{1}{8}\sum_{i=1}^{9}(x_i - \overline{x}_R)^2 = 93.0725$$
$$S_D^2 = \frac{1}{6}\sum_{i=1}^{7}(x_i - \overline{x}_D)^2 = 234.9457$$

Solving for $t-$statistic, we have

$$t = \frac{\overline{x}_R - \overline{x}_D}{\sqrt{s_R^2/n_R + s_D^2/n_D}} = -0.9501$$

Solving for $df$, we have

$$\begin{aligned} df &= \frac{(s_R^2/n_R + s_D^2/n_D)^2}{\dfrac{(s_R^2/n_R)^2}{n_R - 1} + \dfrac{(s_D^2/n_D)^2}{n_D - 1}} \\ &= 9.584 \approx 9 \end{aligned}$$

Hence, the critical value is $t_{0.1,9} = 1.383$. Since $t \not< -1.383$, we do not reject $H_0$. That is, there is no sufficient evidence to say that the election of a Republican president is not good for the stock market. □

4. Randomly pick eight integers from 0 to 500. Is there evidence that the standard deviation of the sample is different from 140?

*Solution.* We have the following hypotheses:

$$\begin{aligned} H_0 &: \sigma^2 - 140^2 = 0 \\ H_1 &: \sigma^2 - 140^2 \neq 0 \end{aligned}$$

Using MATLAB `randi` function[1], we got the following pseudorandom numbers: 479, 483, 78, 486, 479, 243, 400, and 71.

Solving for $s^2$, we have

$$\begin{aligned} n &= 8 \\ \overline{x} &= 339.875 \\ s^2 &= \frac{1}{7}\sum_{i=1}^{8}(x_i - 339.875)^2 = 33488.696 \end{aligned}$$

Hence, the $\chi^2-$statistic is

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = 11.96$$

Moreover, given $\alpha = 0.05$ and $df = 7$, we have the following critical values

$$\chi^2_{0.975,7} = 1.690$$
$$\chi^2_{0.025,7} = 16.013$$

and since $\chi^2 = 11.96 \in (1.690, 16.013)$, we do not reject $H_0$. That is, there is sufficient evidence that the standard deviation of the eight randomly generated numbers is not different from 140. $\square$

5. Let $X$ equal to the number of male children in a four-child family. At a 0.05 significance level, we test the null hypothesis that $X \sim Bi(4, 0.5)$. Among all the students taking Math 150.2, 50 came from families with 4 children. From the group, x = 0, 1, 2, 3 and 4 had counts of 3, 15, 11, 15, and 6, respectively.

   (a) Define the test statistic and critical region.

   *Solution.* We have the following hypotheses:

$$H_0 : X \sim Bi(4, 0.5)$$
$$H_1 : X \nsim Bi(4, 0.5)$$

   Recall that $X \sim f(x; 4, 0.5) = \binom{4}{x}\left(\frac{1}{16}\right)$. Then we have the following expected and sample values

| Expected | 50(1/16) | 50(1/4) | 50(3/8) | 50(1/4) | 50(1/16) |
|---|---|---|---|---|---|
| Sample | 3 | 15 | 11 | 15 | 6 |

   Hence, our $\chi^2-$ statistic is

$$\begin{aligned} Q = &\frac{((50/16)-3)^2}{50/16} + \frac{((50/4)-15)^2}{50/4} + \frac{((150/8)-11)^2}{150/8} + \frac{((50/4)-15)^2}{50/4} \\ &+ \frac{((50/16)-6)^2}{50/16} \\ = &\ 6.853 \end{aligned}$$

   Moreover, our critical value is $\chi^2_{0.05,4} = 9.488$. $\square$

   (b) Find the $p-$value of the test statistic.

   *Solution.* Given that $Q = 6.853$, we have $p \in (0.1, 0.9)$. $\square$

   (c) Give the conclusion to the test

   *Solution.* For the critical value, since $Q \ngtr 9.488$, we do not reject $H_0$. For the $p-$value, since $p \nless 0.05$, we also do not reject $H_0$. That is, $X \sim Bi(4, 0.5)$ $\square$

[1] A screenshot showing the generated numbers used