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Assessment 3: Powerful Tests

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1. Let X be a single observation from the density $f(x;\theta) = 2\theta x + 1 - \theta$, $x \in (0,1)$ where $\theta \in [-1,1]$.

   (a) Find the most powerful test of size α of $H_0 : \theta = 0$ vs $H_1 : \theta = 1$.

   Solution. Suppose we let $\alpha = \mathbb{P}_{\theta_0}[\text{reject } H_0] = \mathbb{P}_{\theta_0}[x \in C]$ for a single realization x of X. Furthermore, suppose

   $$\lambda = \frac{L_0}{L_1} = \frac{f(x;\theta_0)}{f(x;\theta_1)} = \frac{1}{2x} \leq k$$

   for some positive k. That is, $x \geq 1/(2k)$. Then we solve for α.

   $$\begin{aligned}\mathbb{P}_{\theta_0}[x \geq 1/(2k)] &= \int_{2k}^{1} 2\theta_0 x + 1 - \theta_0 dx \\ &= 1 - \frac{1}{2k} \\ &= \alpha\end{aligned}$$

   and that $1/(2k) = 1 - \alpha$. Therefore, we reject H_0 if $x \geq 1 - \alpha$ for some size α is the most powerful test. □

   (b) To test $H_0 : \theta \leq 0$ vs $H_1 : \theta > 1$, the following was used: Reject H_0 if X exceeds 1/2. Find the power and the size of the test.

   Solution. Let $\Omega_0 := \{\theta : \theta \leq 0\}$. The power is given by

   $$\begin{aligned}\Pi_C(\theta) &= \mathbb{P}_\theta[X > 1/2] \\ &= \int_{1/2}^{1} 2\theta x + 1 - \theta dx \\ &= x(\theta x + 1 - \theta)|_{1/2}^{1} \\ &= \frac{1}{4} - \frac{\theta}{4}\end{aligned}$$

   and $\alpha = \sup_{\theta \in \Omega_0} \Pi_C(\theta) = \Pi_C(-1) = 0.5$. □

   (c) Is there a uniformly most powerful test of size α of $H_0 : \theta \leq 0$ vs $H_1 : \theta > 0$?

   Solution. Solving for the MLR, we have

   $$\begin{aligned}\frac{L(\theta')}{L(\theta'')} &= \frac{2\theta' x + 1 - \theta'}{2\theta]]x + 1 - \theta''} \\ &= \frac{\theta'(2x-1)+1}{\theta''(2x-1)+1} = M\end{aligned}$$

   and for $\theta' < \theta''$, M is nonincreasing function of $T = 2x - 1$. Therefore, by *ii* of Theorem 06, we conclude that

Reject H_0 if $2x - 1 \geq k$ for some k.

$\square$