Numerical Analysis
ODE Systems

---

1. Given the third-order ODE $y''' - y' = t$
   (a) Show that $y_{exact}(t) = (e^t + e^{-t} - t^2)/2 - 1$ is an exact solution of the ODE.
   (b) Convert the given to a first-order system of ODEs.
   (c) Modify Runge-Kutta code provided so that it can be used to solve the first-order system given $y(0) = y'(0) = y''(0)$. Plot the two curves, then approximate the solution for $y(t)$ and the exact solution y_{exact} in one figure on the interval $[0, 10]$.

2. Consider the IVP

$$y' = t - y, \quad y(0) = 0$$

   (a) Determine the explicit (exact) solution $y_{exact}(t)$ to the IVP.
   (b) Find an approximation to the solution of the IVP using Euler's method y_{euler}. Make a loglog plot of the error $|y_{exact} - y_{euler}|$ of the Euler's method at $t = 1$ as a function of $h = 0.1 \times 2^{-k}$ for $k \in [0, 5]$.

3. The population of two species, a prey denoted by y_1 and a predator denoted by y_2, can be modeled by the autonomous, nonlinear DE

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_1(\alpha_1 - \beta_1 y_2) \\ y_2(-\alpha_2 + \beta_2 y_1) \end{bmatrix}$$

   used by Volterra in 1926 to describe fish and shark populations and earlier by Lotka to describe oscillations in chemical reactions. The parameters α_1 and α_2 are the natural birth and death rates in isolation of prey and predators, respectively, and the parameters β_1 and β_2 determine the effect of interactions between the two populations, where the probability of interaction is proportional to the product of the populations.

   (a) In one figure, plot the two populations as a function of time using a built-in Matlab ODE solver with the following parameter values: $\alpha_1 = 1$, $\alpha_2 = 0.5$, $\beta_1 = 0.1$, $\beta_2 = 0.02$, $y_1(0) = 100$, and $y_2(0) = 10$.
   (b) In another figure, plot the trajectory of the point $(y_1(t), y_2(t))$ in the plane as a function of time and give a physical interpretation of the behavior you observe.
   (c) Vary the initial populations. Can you find nonzero initial populations such that either of the populations eventually become extinct?