Multivariable Differentiation

Explained with Bugs

Say we have a bug, 🐌, that depends on three other bugs; 🐛, 🐜 and 🐝 (by now, we've established that the snail is the main function bug). We will define this relationship as follows;

$$🐌 = 3🐛^3 - 2🐜^2 + 4🐝$$

Let's also define 🐜 and 🐝 as functions of 🐛.

$$🐜 = 🐛^2$$

$$🐝 = -🐜$$

If you saw the last episode on *Ordinary and Partial Derivatives*, you might've noticed that 🐜 here has a different relationship with 🐛—the ant and caterpillar colonies had some discussions since last time. This is some important lore in the Explained with Bugs universe, and is definitely not so I can do less math.

Suppose 🐛 is nudged a little, maybe it ate a leaf and got bigger... or smaller. Perhaps this place has negative matter. Nevertheless, since 🐛 changed, 🐌, which depends on 🐛, must also change by $\frac{\partial 🐌}{\partial 🐛}$. We use a partial derivative here because we only want what 🐛 did to 🐌, not what every bug did to it. For the entire sequence of bugs, we will be following a path from 🐛 to 🐌, as 🐛 is the bug we are differentiating with respect to. A good way to view this is a dependency tree;

$$🐛 \to 🐌$$

$$🐛 \to 🐜 \to 🐌$$

$$🐛 \to 🐜 \to 🐝 \to 🐌$$

Continuing the **chain** (thus the name of this rule I secretly just explained, the **chain rule**), 🐜 also changes due to 🐛 by $\frac{\mathrm{d}🐜}{\mathrm{d}🐛}$. We use an ordinary derivative here because the only bug 🐜 depends on is 🐛, so it doesn't matter if we take a partial or ordinary derivative. Following the trend, 🐝 also changes due to 🐛, but because it relies on 🐜, which itself changes due to 🐛. Here, 🐝 changes by $\frac{\mathrm{d}🐝}{\mathrm{d}🐜}\frac{\mathrm{d}🐜}{\mathrm{d}🐛}$. We use ordinary derivatives here for the same reason as before.

You may think, 🐌 depends on all of these bugs, so it should also be changed due to 🐜 and 🐝. Indeed, we can do what we did with the 🐛 term, and change

🐌 by $\frac{\partial 🐌}{\partial 🐜}$ and $\frac{\partial 🐌}{\partial 🐝}$. These are partial derivatives for aforementioned reason in the last paragraph. We must multiply each $\frac{\partial 🐌}{\partial x}$ by how the bug placed in x's place changes with respect to 🐛, as those bugs change indirectly when 🐛 changes.

Now that we've got every term resolved, we can construct our derivative;

$$\frac{\mathrm{d}🐌}{\mathrm{d}🐛} = \frac{\partial 🐌}{\partial 🐛} + \frac{\partial 🐌}{\partial 🐜}\frac{\mathrm{d}🐜}{\mathrm{d}🐛} + \frac{\partial 🐌}{\partial 🐝}\frac{\mathrm{d}🐝}{\mathrm{d}🐛}$$

Also, notice how $\frac{\mathrm{d}🐝}{\mathrm{d}🐜}\frac{\mathrm{d}🐜}{\mathrm{d}🐛}$ neatly simplifies to $\frac{\mathrm{d}🐝}{\mathrm{d}🐛}$, like how a pair of fractions with a matching numerator and denominator cancel out. There are many cases in which these fraction-like behaviours will occur, the beauty of this notation.

Let us get to solving these terms now.

$$\frac{\partial 🐌}{\partial 🐛} = 9🐛^2$$

$$\frac{\partial 🐌}{\partial 🐜} = -4🐜$$

$$\frac{\partial 🐌}{\partial 🐝} = 4$$

Here, I simply used the power rule, mentioned in the previous episode. $3🐛^3$, $-2🐜^2$ and $4🐝$ become their respective terms above. We can use this same rule to find the rest of our terms.

$$\frac{\mathrm{d}🐜}{\mathrm{d}🐛} = 2🐛$$

$$\frac{\mathrm{d}🐝}{\mathrm{d}🐛} = -2🐛$$

🐝 is defined as $-🐜$, so we can just take the negative derivative of 🐜 with respect to 🐛 for the derivative of 🐝 with respect to 🐛. You'll notice this would be the same result as solving $\frac{\mathrm{d}🐝}{\mathrm{d}🐜}\frac{\mathrm{d}🐜}{\mathrm{d}🐛}$, as it evaluates to $-1 \times 2🐛$. Substituting these for the terms in our main derivative, we get;

$$\frac{\mathrm{d}🐌}{\mathrm{d}🐛} = 9🐛^2 - 8🐛🐜 - 8🐛$$

Here, we can substitute 🐜 for its definition to simplify the equation;

$$\frac{\mathrm{d}🐌}{\mathrm{d}🐛} = 9🐛^2 - 8🐛(🐛^2) - 8🐛$$

$$= 9🐛^2 - 8🐛^3 - 8🐛$$

Great, we have solved the derivative! If you're feeling extra special, or you're just very cool like me, you'll transform this cubic equation into its standard form.

$$\frac{\mathrm{d}🐌}{\mathrm{d}🐛} = -8🐛^3 + 9🐛^2 - 8🐛$$

You may also factor the result as follows, but beware that this will officially make you a *nerd*;

$$\frac{\mathrm{d}🐌}{\mathrm{d}🐛} = -🐛(8🐛^2 - 9🐛 + 8)$$