Model-based Survey Weighting Using Logistic Regression

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Abstract. Equivalent weights from regression models provide a bridge between design-based and model-based survey inference. We introduce a closed-form construction of logistic regression equivalent weights, defined by matching the weighted sample mean of the outcome to the population mean predicted under the logistic model. Although logistic weighting is nonlinear, a first-order expansion shows that outcome dependence is negligible, allowing the weights to be treated as functions of covariates alone. We establish asymptotic properties of these weights, including consistency, asymptotic linearity, and a plug-in variance estimator. The resulting estimators preserve efficiency relative to design-based alternatives, with population estimation properties compatible with those of linear weighting, but with logistic weighting showing the strongest preservation of effective sample size (ESS) after raking. Simulation evidence confirms that the asymptotic results hold well in finite samples of the scale typical in survey applications. An application to the Fragile Families and Child Wellbeing Study illustrates how logistic weighting modifies prevalence estimates while remaining consistent with simulation results. Together, these results extend regression-based weighting theory to the logistic case and provide new asymptotic justification for its use in survey practice.

1 Introduction

Survey weights are central to making sample data representative of target populations. Constructing base weights or adjusting them involves multiple steps and subjective choices, often relying on auxiliary data to align estimates with census counts or other benchmarks. Yet, survey data frequently face challenges such as missingness and nonresponse, which complicate these procedures and introduce instability.

A substantial body of literature addresses nonresponse adjustment, design- and model-based inference, and small area estimation (Chapman et al., 1986; Little, 1986; Bethlehem et al., 1996; Chu and Goldman, 1997; Lu and Gelman, 2003; Little, 2015; Rao and Molina, 2015; Haziza and Beaumont, 2017; Skinner and Wakefield, 2017; Chen et al., 2017; Liu et al., 2023). At the same time, concerns remain regarding the construction of base weights themselves, where methods are often ad hoc, hindered by the limited availability of detailed and confidential census data (Carlson, 2008), and lacking systematic comparison. Moreover, the role of weights in regression remains controversial: they do not always reduce bias, and adjustments such as raking can yield extreme weights even after trimming (Little and Rubin, 1987; Deville and Särndal, 1992; Miller, 2011).

Traditional approaches to constructing base weights follow design-based guidelines (Chu and Goldman, 1997; Valliant et al., 2013; Valliant and Dever, 2018). Model-based alternatives have been proposed, including regression-based constructions (Gelman, 2007), but their implementation and advantages over design-based methods remain underexplored. In particular, while linear regression has been considered for equivalent weights, logistic regression has not been fully developed as a basis for constructing weights, despite its natural appeal for binary or categorical sampling indicators.

This paper develops a model-based framework that constructs base weights via logistic regression, building on and extending Gelman (2007). We compare design-based, linear, and logistic weighting through both simulation and empirical analysis, providing practical guidance for researchers confronting challenges in weight construction. Beyond empirical evidence, we establish asymptotic properties: Theorem 1 and Corollary 1 show that the effective sample size ratio converges to a population limit, and Theorem 2 provides an asymptotic linear expansion with a consistent plug-in variance formula. These results demonstrate that logistic weights yield stable large-sample properties and offer a theoretical benchmark for interpreting finite-sample performance.

1.1 Target Study Population and Design

The target survey population consists of live births occurring in large U.S. cities with populations over 200,000 between 1998 and 2000. This focus is motivated by the process of constructing base weights for the Future of Families and Child Wellbeing Study (FFCWS). The FFCWS sample follows a stratified multistage design with 4,898 children, oversampling births to unmarried mothers at a ratio of 3 to 1, with the inclusion of a large number of Black, Hispanic, and low-income families. Follow-up interviews were conducted across seven waves, when children were approximately ages 1, 3, 5, 9, 15, and 22. In constructing the FFCWS weights, four demographic variables were used for poststratification, and geographic information was incorporated to estimate population birth counts using the Centers for Disease Control and Prevention (CDC) annual natality data. These variables are mother's marital status, race/ethnicity, age, education, and city of birth (Carlson,2008). To mirror the FFCWS study, we generate data and construct base weights in our simulation analysis that replicate the characteristics of the FFCWS design.

In large national surveys, stratified multistage cluster sampling (SMCS) is a common approach because it balances logistical challenges with the need to obtain sufficiently precise estimates for key subgroups. In this context, clusters, also referred to as primary sampling units (PSUs), are often used to simplify fieldwork. For example, the stratum is the city and the PSU is the hospital for cities selected with certainty in the FFCWS study. A simpler alternative is stratified simple random sampling (SRS), in which each unit in the population is assigned a group label and SRS is conducted within each group. These groups typically correspond to demographic strata or geographic regions.

In our simulation study, we choose to construct design-based weights using stratified simple random sampling (SRS) rather than stratified multistage cluster sampling (SMCS). While SMCS is widely used, intra-cluster correlation reduces efficiency by limiting the independent information gained from additional sampled units within the same cluster. This leads to higher standard errors and lower overall precision. Moreover, we expect minimal differences in weight estimation between SMCS and SRS once raking is applied. For our experiments, we draw samples of size $n = 3000$ from a finite population of size $N = 1{,}000{,}000$, chosen to mirror realistic survey conditions where the sample represents only a small fraction of the population.

1.2 Design-based Weighting

For a finite population and a probability sample, each unit i has a known selection probability $\pi_i > 0$, and the *base weight* is defined as

$$w_i = \frac{1}{\pi_i}.$$

A probability sample is realized under the following conditions: the set of all possible samples that can be selected from the finite population is well-defined under the sampling design, and each possible sample is associated with a known probability of selection. Moreover, every unit in the target population has a nonzero probability of being selected, with selection occurring via a random mechanism (Valliant and Dever, 2018). In our simulation study, the selection probability π_i is determined based on factors such as city of birth and demographic characteristics. These factors are used to form cells, where units within the same cell share identical probabilities of selection.

We then apply two standard adjustments: (i) nonresponse adjustment, where weights are multiplied by the inverse of the weighted response rate within adjustment cells (e.g., within each city) to account for survey nonrespondents; and (ii) raking, where weights are adjusted so that weighted sample counts align with known population margins. Nonresponse adjustments renormalize the weights to sum to the total population size, while raking aligns marginal totals without requiring full cross-classification, thereby avoiding the problem of empty cells. Given the extensive literature on nonresponse adjustments, we do not explore them in detail here. For recent applications to the FFCWS data, including a state-of-the-art two-stage approach using an optimally balanced Gaussian process, see (Vegetabile et al, 2020), refer to Lee and Gelman (2024).

Table 1 summarizes base weights from a representative simulation.

Table 1: Summary of base weights (simulation example)

	Min	1st Quantile	Median	Mean	3rd Quantile	Max
Value	2.86	121.40	260.80	333.33	492.49	1396.96

1.3 Estimating Population Mean from a Sample

Let X denote the poststratification variables whose joint distribution in the population is known, along with the survey response of interest y that we aim to estimate for the population. The possible strata of X form poststratification cells, denoted as s, with population sizes N_s and sample sizes n_s. The total population size is $N = \sum_{s=1}^{S} N_s$ and the total sample size is $n = \sum_{s=1}^{S} n_s$.

The population mean of y is defined as the weighted average of the stratum-specific means:

$$\theta = \frac{\sum_{s=1}^{S} N_s \, \theta_s}{N},$$

where θ_s is the mean of y within stratum s. The corresponding sample-based poststratified estimator is

$$\hat{\theta}^{PS} = \frac{\sum_{s=1}^{S} N_s \, \hat{\theta}_s}{N}.$$

When estimating the population mean from sample survey data, one typically uses the Horvitz–Thompson (HT) or Hájek estimator:

$$\bar{y}_{HT} = \frac{\sum_{i=1}^{n} w_i y_i}{N}, \qquad \bar{y}_{H} = \frac{\sum_{i=1}^{n} w_i y_i}{\hat{N}},$$

where $\hat{N} = \sum_{i=1}^{n} w_i$. In our context, N is known and equals $\hat{N}$, so the HT and Hájek estimators coincide and both serve as unbiased estimators of the population mean (Horvitz and Thompson, 1952; Hájek, 1971).

Importantly, the poststratified estimator $\hat{\theta}^{PS}$ is mathematically equivalent to the Hájek estimator when estimating the population mean. We will leverage this relationship to estimate equivalent unit weights under model-based approaches.

2 Model-based Weighting Methods

2.1 Linear Regression Weight

Let X denote the $n \times k$ design matrix of auxiliary variables for the samples, and let X^{pop} denote the $S \times k$ design matrix for the population poststratification cells. Here, n is the sample size, S is the number of poststratification cells, and k is the number of auxiliary variables (e.g., demographic or geographic characteristics) used for weighting. If the survey response of interest y has a linear relationship with the raking variables, then

$$\hat{\theta}_s = X_s^{pop}\hat{\beta}, \qquad \hat{\beta} = (X^\top X)^{-1}X^\top y,$$

where $\hat{\beta}$ are estimated regression coefficients. The poststratified estimate of the population mean is

$$\hat{\theta}^{PS} = \frac{1}{N}\sum_{s=1}^{S} N_s\big(X_s^{pop}\hat{\beta}\big) = \frac{1}{N}(N^{pop})^\top X^{pop}(X^\top X)^{-1}X^\top y.$$

Define $\hat{w}$ as the equivalent unit weights such that

$$\hat{\theta}^{PS} = \frac{1}{n}\sum_{i=1}^{n}\hat{w}_i y_i, \qquad \sum_{i=1}^{n}\hat{w}_i = n.$$

Solving for $\hat{w}$ gives

$$\hat{w} = \frac{n}{N}(N^{pop})^\top X^{pop}(X^\top X)^{-1}X^\top.$$

The outcome y cancels out, so weights depend only on auxiliary variables. This is an advantage, as demographic and geographic information is typically available with minimal missingness. The formula produces S unique equivalent unit weights, one for each poststratification cell. For linear unit weights, renormalization to sum to n is unnecessary unless negative weights occur. Moreoever, multiplying the weights by any constant does not affect the estimation of the population mean. Thus, the weights can be renormalized to sum to the known population size, ensuring that $\sum_{i=1}^{n} \hat{w}_i = \hat{N}$. After this renormalization, the population estimate is

$$\hat{\theta}^{PS} = \frac{\sum_{i=1}^{n} \hat{w}_i y_i}{\hat{N}},$$

,which corresponds to a form of the Hájek estimator.

Table 2: Summary of linear equivalent weights (simulation example)

	Min	1st Quantile	Median	Mean	3rd Quantile	Max
Value	0.0022	0.15	0.82	1.00	1.52	7.00

2.2 Logistic Regression Weight

Let $y = (y_1, \ldots, y_n)$ be a binary response and X the $n \times k$ matrix of weighting variables. The logistic regression model is

$$\Pr(y_i = 1 \mid X_i) = p_i, \qquad \operatorname{logit}(p_i) = X_i^{\top}\beta,$$

with $\hat{p} = \sigma(X\hat{\beta})$. For population cells X_s^{pop} $(s = 1, \ldots, S)$, define equivalent unit weights $\hat{w}$ by requiring

$$\frac{1}{n}\sum_{i=1}^{n} \hat{w}_i y_i = \frac{1}{N}\sum_{s=1}^{S} N_s \sigma\left(X_s^{pop}\hat{\beta}\right). \tag{1}$$

In survey practice it is standard to assume weights depend only on X, not on y. Logistic regression is nonlinear, so we assess the impact of this assumption by a first-order Taylor expansion.

Differentiating (1) with respect to y yields

$$(\nabla\hat{w})y + \hat{w} = \left(\frac{n}{N}(N^{pop})^{\top}\operatorname{diag}\Big[\sigma(X^{pop}\hat{\beta}) \odot (1 - \sigma(X^{pop}\hat{\beta}))\Big] X^{pop}\left(\frac{d\hat{\beta}}{dy}\right)^{\top}\right)^{\top}, \tag{2}$$

where $\odot$ is the Hadamard (elementwise) product and

$$\frac{d\hat{\beta}}{dy} = (X^{\top}WX)^{-1}X^{\top}, \qquad W = \operatorname{diag}(\hat{p} \odot (1 - \hat{p})).$$

The term $\frac{d\hat{\beta}}{dy}$ on the right-hand side, derived using implicit differentiation, is expressed as

$$\frac{d\hat{\beta}}{dy} = (X^t W X)^{-1} X^t,$$

where $X^t W X$ is the Hessian matrix of the log-likelihood function in logistic regression, capturing the curvature of the likelihood function around the maximum likelihood estimate (MLE). The entries of the diagonal matrix W represent the conditional variance of the predicted probabilities for the sample units.

Substituting this expression back into the right-hand side shows how the first-order correction term $(\nabla\hat{w})y$ depends on the curvature of the likelihood surface. In practice, however, our empirical results demonstrate that this derivative term is negligible, which justifies treating the equivalent weights as functions of the auxiliary variables alone.

Note that the left-hand side can be written as

$$(\nabla\hat{w})y + \hat{w} = J(y)y + \hat{w}(y),$$

where $J(y)$ is the $n \times n$ Jacobian matrix with entries $[J(y)]_{ij} = \frac{\partial \hat{w}_i(y)}{\partial y_j}$.

Worked illustration of the Taylor approximation. To make the derivative term transparent, we expand the estimating equation one unit at a time. Let $\star$ denote the left-hand side of the equation at observed y. For example,

$$\star = \hat{w}(1,1,0,\ldots,1) + \frac{dw}{dy_1}(1,1,0,\ldots,1) + \frac{dw}{dy_2}(1,1,0,\ldots,1) + \cdots + \frac{dw}{dy_n}(1,1,0,\ldots,1)$$

Substituting $y_i = 1$ by $y_i = 0$ for each unit gives:

$$\hat{w}(0,1,0,\ldots,1) + \frac{dw}{dy_2}(0,1,0,\ldots,1) + \cdots + \frac{dw}{dy_n}(0,1,0,\ldots,1) \tag{1}$$

$$\hat{w}(1,0,0,\ldots,1) + \frac{dw}{dy_1}(1,0,0,\ldots,1) + \cdots + \frac{dw}{dy_n}(1,0,0,\ldots,1) \tag{2}$$

$$\vdots$$

$$\hat{w}(1,1,0,\ldots,0) + \frac{dw}{dy_1}(1,1,0,\ldots,0) + \frac{dw}{dy_2}(1,1,0,\ldots,0) + \cdots + \frac{dw}{dy_n}(1,1,0,\ldots,0) \tag{n_1}$$

and the Taylor approximation gives

$$(\star) - (1) \approx \frac{dw}{dy_1}, \quad (\star) - (2) \approx \frac{dw}{dy_2}, \quad \ldots, \quad (\star) - (n_1) \approx \frac{dw}{dy_{n_1}}$$

so that

$$(\nabla \hat{w})y \;\approx\; (\star) - (1) + (\star) - (2) + \cdots + (\star) - (n_1).$$

Empirical finding (derivative is negligible). Across simulations and the FFCWS application, the derivative term $\Delta(\hat{w})(y) = (\nabla \hat{w})y$ is extremely small: values range from about -1.4×10^{-4} to 4.2×10^{-5}, with mean -4.0×10^{-6} and median -2.4×10^{-6}. In a separate check, perturbing a single outcome changed $(\nabla \hat{w})y + \hat{w}$ by at most 0.02 (median 0.002). These findings confirm that outcome perturbations have negligible effect on the weights, justifying the assumption that weights depend only on X.

Closed-form expression for logistic weights. Since the derivative contribution $(\nabla \hat{w})y$ is empirically negligible, we assume that the equivalent weights depend only on the auxiliary weighting variables. Accordingly, we set $(\nabla \hat{w})y = 0$ in the estimating equation. This yields a closed-form expression for the logistic equivalent weights that depend only on the auxiliary variables X, consistent with survey practice. The resulting logistic weights are then renormalized to sum to n, regardless of whether negative weights are present. Importantly, the formula produces S unique equivalent weights, one for each poststratification cell.

$$\hat{w}(\hat{\beta}) = \left(\frac{n}{N} (N^{\text{pop}})^\top W^{\text{pop}}(\hat{\beta}) X^{\text{pop}} \left((X^\top W(\hat{\beta}) X)^{-1} X^\top \right)^\top \right)^\top,$$

Table 3 summarizes the distribution of logistic equivalent weights from our simulations. While the weights are generally well-behaved, a small fraction attain extreme values. To mitigate their influence, we trim weights above the 95th percentile by replacing them with the 95th percentile value, a standard adjustment in survey practice (DevilleSarnal1992, Valliant2013).

Table 3: Summary of logistic equivalent weights (simulation example)

	Min	1st Quantile	Median	Mean	3rd Quantile	Max
Value	0.0024	0.12	0.83	1.00	1.53	7.01

2.3 Negative Weights in Model-based Methods

Negative weights are not uncommon when constructing weights using model-based approaches. They arise when real-world surveys fail to accurately represent the target population due to oversampling or undersampling of particular demographic groups, or because of limited sample sizes. This is expected, as one of the motivations for constructing weights is precisely to correct for imbalances in representativeness.

In most cases, linear equivalent unit weights are positive when samples are randomly drawn from the population. However, in our simulation study designed to mimic the FFCWS data, we encountered negative weights for both linear and logistic equivalent unit weights. To address this, we replace negative weights with the minimum value of the positive weights, an adjustment that has negligible impact on the total weight count across demographic cells once weights are renormalized. Ideally, negative weights would be adjusted within the same cells to preserve cell-specific totals, but in practice this is rarely feasible, as cells with negative weights often do not overlap with cells containing positive weights. Consequently, adjusting for negative weights results in only minor discrepancies in total counts relative to population benchmarks.

2.4 Missingness and Nonresponse

In practice, missing data in demographic and geographic variables used for survey weighting is rare, but it should be assessed and addressed before constructing weights. When auxiliary variables are missing, constructing base weights becomes more challenging. For example, city-level birth counts are not included in the CDC natality data, which only provides county-level totals. However, population birth counts can still be estimated at the city level by combining sample information with the CDC natality file. In our application, we linked birth occurrence location with maternal residence to construct city-level estimates.

For nonresponse adjustment, non-respondents are excluded from the weight construction process. Design-based methods require an explicit nonresponse adjustment step, whereas model-based methods inherently account for population totals by incorporating auxiliary information directly into the estimation. For example, the linear weighting method, when free of negative weights, preserves both the sample count and the total population count, since rescaling does not affect the estimate. By contrast, the logistic weighting method also aligns the estimated weights with known population totals, but in practice additional trimming and renormalization are often required to ensure exact agreement because of the nonlinearity of the logistic model.

3 Asymptotic Properties

Lemma 1 (Uniform constant envelope for logistic equivalent weights with categorical X)**.** *Let X consist entirely of categorical variables encoded as dummies, so that each X_i takes values in a finite set $\mathcal{X}$ and $\sup_{x\in\mathcal{X}}\|x\| < \infty$. Define the closed-form logistic equivalent weights (as a function of β)*

$$\hat{w}(\beta) = \left(\frac{n}{N}(N^{\text{pop}})^{\top} W^{\text{pop}}(\beta) X^{\text{pop}} \left((X^{\top}W(\beta)X)^{-1}X^{\top}\right)^{\top}\right)^{\top},$$

where $W(\beta) = \operatorname{diag}\big(\sigma(X\beta)\odot(1-\sigma(X\beta))\big)$ and $X^{\text{pop}}, N^{\text{pop}}$ are fixed finite matrices/vectors. Assume there exists a neighborhood $\mathcal{N}$ of β_0 and a constant $c > 0$ such that the weighted information matrix $H(\beta) := X^{\top}W(\beta)X$ satisfies $\lambda_{\min}(H(\beta)) \geq c$ for all $\beta \in \mathcal{N}$. Then there exists $C < \infty$ such that, for all $\beta \in \mathcal{N}$ and all i,

$$|\hat{w}_i(\beta)| \leq C.$$

Lemma 2 (Continuity of logistic equivalent weights in β)**.** *Under the same assumptions as Lemma 1, the map $\beta \mapsto \hat{w}(\beta)$ is continuous on $\mathcal{N}$ (coordinatewise and in any matrix norm).*

Theorem 1 (Plug-in LLN for logistic equivalent weights with categorical X)**.** *Let $\{X_i\}_{i=1}^n$ be i.i.d. with law F. Let $\hat{w}_i(\beta)$ denote the closed-form logistic equivalent weights defined in Lemma 1. Assume:*

1. *(Envelope) The assumptions of Lemma 1 hold, so there exists a constant $C < \infty$ such that $|\hat{w}_i(\beta)| \leq C$ for all $\beta \in \mathcal{N}$ and all i (hence $g(x) \equiv C$ is an integrable envelope with $\mathbb{E}g < \infty$ and $\mathbb{E}g^2 < \infty$);*

2. *(Continuity) The assumptions of Lemma 2 hold, so $\beta \mapsto \hat{w}(\beta)$ is continuous on the neighborhood $\mathcal{N}$ of β_0;*

3. *(Compactness) $\mathcal{N}$ is compact and $\beta_0 \in \operatorname{int}(\mathcal{N})$.*

4. $\hat{\beta} \xrightarrow{p} \beta_0$*;*

Define

$$\bar{w}_n(\beta) = \frac{1}{n}\sum_{i=1}^{n} \hat{w}_i(\beta), \qquad \overline{w^2}_n(\beta) = \frac{1}{n}\sum_{i=1}^{n} \hat{w}_i(\beta)^2.$$

Then

$$\bar{w}_n(\hat{\beta}) \xrightarrow{p} \mathbb{E}[\hat{w}_1(\beta_0)], \qquad \overline{w^2}_n(\hat{\beta}) \xrightarrow{p} \mathbb{E}[\hat{w}_1(\beta_0)^2].$$

Corollary 1 (ESS Limit)**.** *Let* $\mathrm{ESS}_n = \left(\sum_{i=1}^n w(X_i;\hat{\beta})\right)^2 / \sum_{i=1}^n w(X_i;\hat{\beta})^2$*. Under the assumptions of Theorem 1,*

$$\frac{\mathrm{ESS}_n}{n} = \frac{\bar{w}_n(\hat{\beta})^2}{\overline{w^2}_n(\hat{\beta})} \xrightarrow{p} \frac{\left(\mathbb{E}[w(X;\beta_0)]\right)^2}{\mathbb{E}[w(X;\beta_0)^2]}.$$

Proof. Apply Theorem 1 to $\bar{w}_n(\hat{\beta})$ and $\overline{w^2}_n(\hat{\beta})$, then invoke the continuous mapping theorem. □

Remark 1 (Scale invariance). If $\tilde{w} = c\,w$ for any constant $c > 0$ (e.g., normalization), then ESS_n/n is unchanged since both numerator and denominator are multiplied by c^2.

Lemma 3 (Differentiability and uniform derivative envelope)**.** *Under the assumptions of Lemmas 1–2, the map* $\beta \mapsto \hat{w}(\beta)$ *is continuously differentiable (*C^1*) on* $\mathcal{N}$*. Moreover, there exists a finite constant* C' *(depending only on* $X, X^{\mathrm{pop}}, N^{\mathrm{pop}}, c$*) such that*

$$\sup_{\beta\in\mathcal{N}} \|\nabla_\beta \hat{w}_i(\beta)\| \;\le\; C' \qquad \textit{for all } i.$$

Theorem 2 (Asymptotic linearity of $\hat{\theta} = \frac{1}{n}\sum_{i=1}^n \hat{w}_i(\hat{\beta})Y_i$)**.** *Let* $\hat{w}_i(\beta)$ *be the logistic equivalent weights. Assume:*

(i) (Constant envelope for weights) Lemma 1 holds: $|\hat{w}_i(\beta)| \le C$ *for all* $\beta \in \mathcal{N}$ *and all* i*.*

(ii) (C^1 *and uniform derivative envelope) Lemma 3 holds:* $\beta \mapsto \hat{w}(\beta)$ *is* C^1 *on* $\mathcal{N}$ *and there exists* $C' < \infty$ *such that* $\sup_{\beta\in\mathcal{N}} \|\nabla_\beta \hat{w}_i(\beta)\| \le C'$ *for all* i*.*

(iii) (Logistic MLE regularity) The logistic model is correctly specified at β_0 *with* $p_0(x) = \sigma(x^\top\beta_0)$*;* β_0 *is an interior point of the parameter space;* X *has* $\mathbb{E}\|X\|^2 < \infty$*; and the Fisher information* $\mathcal{I}(\beta_0) = \mathbb{E}[XX^\top p_0(X)\{1 - p_0(X)\}]$ *is positive definite.*

Define $g(\beta) = \mathbb{E}[\hat{w}_1(\beta)\, Y_1]$ *and*

$$A \;:=\; \nabla_\beta g(\beta)\big|_{\beta=\beta_0} \;=\; \mathbb{E}\big[\nabla_\beta \hat{w}_1(\beta_0)\, Y_1\big] \quad \textit{(under correct logit spec.,} \; = \mathbb{E}[\nabla_\beta \hat{w}_1(\beta_0)\, p_0(X_1)]\textit{)}.$$

Then

$$\sqrt{n}\,\big(\hat{\theta} - \theta_0\big) \;=\; \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \underbrace{\Big\{\hat{w}_i(\beta_0)\, Y_i - \mathbb{E}[\hat{w}_1(\beta_0) Y_1]\Big\}}_{\psi_1(Z_i)} + A^\top \sqrt{n}\,(\hat{\beta} - \beta_0) \;+\; o_p(1),$$

so that

$$\sqrt{n}\,(\hat{\theta} - \theta_0) \;\Rightarrow\; \mathcal{N}\big(0,\; V\big), \qquad V = \mathbb{E}\big[\psi(Z)^2\big],$$

with influence function

$$\psi(Z) = \big\{\hat{w}(\beta_0)\, Y - \mathbb{E}[\hat{w}(\beta_0) Y]\big\} \;+\; A^\top \mathcal{I}^{-1} X\,\{Y - p_0(X)\}.$$

A consistent plug-in variance is

$$\widehat{V} = \frac{1}{n} \sum_{i=1}^{n} \hat{\psi}_i^2, \quad \hat{\psi}_i = \hat{w}_i(\hat{\beta})\, Y_i - \bar{m}_n \;+\; \hat{A}^\top \hat{\mathcal{I}}^{-1} X_i\,\{Y_i - \hat{p}_i\},$$

where

$$\bar{m}_n = \frac{1}{n} \sum_{j=1}^{n} \hat{w}_j(\hat{\beta})\, Y_j, \quad \hat{p}_i = \sigma(X_i^\top \hat{\beta}), \quad \hat{\mathcal{I}} = \frac{1}{n} \sum_{j=1}^{n} X_j X_j^\top\, \hat{p}_j(1 - \hat{p}_j), \quad \hat{A} = \frac{1}{n} \sum_{j=1}^{n} \nabla_\beta \hat{w}_j(\hat{\beta})\, \hat{p}_j.$$

4 Results

4.1 Simulation Study

We set the sample size at $n = 3000$, drawn from a finite population of size $N = 1{,}000{,}000$. This choice reflects typical survey conditions where the sample is small relative to the population, yet large enough for asymptotic approximations to provide meaningful guidance. Thus, the simulation study serves as a finite-sample check of the asymptotic properties established in Section 3.

We begin by comparing weighting methods based on their accuracy in reproducing population counts. Table 4 reports the estimated population counts by education. The linear model produces estimates that, on average, are closest to the true population distribution, whereas the design-based and logistic models exhibit larger deviations in certain categories.

Table 4: Population counts by education before raking

Education	< 8 grade	Some HS	HS	Some College	College+	Total
Population	97,409	188,417	300,059	189,532	224,583	1,000,000
Design-based	8,733	179,550	534,609	163,598	113,510	1,000,000
Linear model	90,234	192,799	320,665	188,261	208,041	1,000,000
Logistic model	90,290	190,412	315,998	191,782	211,517	1,000,000

Next, we evaluate methods on their performance in estimate population means of a binary outcome. Table 5 presents the estimates by education before raking. The linear model produces estimates that most closely align with the true population rates, while the design-based and logistic models show larger deviations, particularly for some categories.

Table 5: Population means of a binary outcome by education after raking

Education	< 8 grade	Some HS	HS	Some College	College+
Population rate	0.47	0.43	0.40	0.23	0.35
Design-raked	0.33	0.40	0.34	0.18	0.28
Linear model	0.34	0.45	0.41	0.22	0.34
Logistic model	0.34	0.45	0.40	0.22	0.34

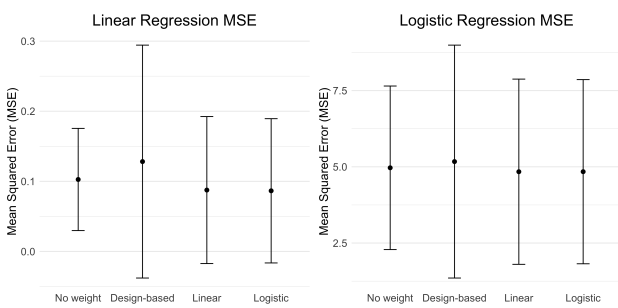
Finally, Table 6 reports results by education after raking. Raking improves alignment with the true population rates, and the design-based, linear, and logistic models now yield broadly similar results. Nevertheless, in certain education categories the estimated population means still deviate from the benchmark, indicating that some residual bias remains even after adjustment.

Table 6: Population means of a binary outcome by education after raking

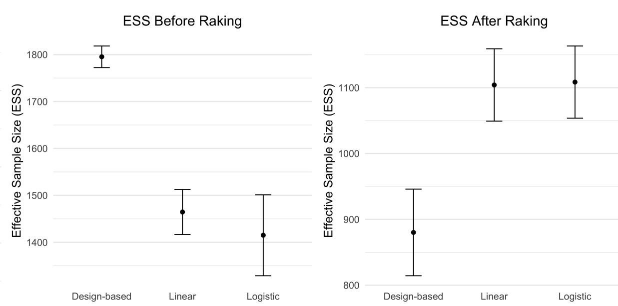
Education	< 8 grade	Some HS	HS	Some College	College+
Population rate	0.47	0.43	0.40	0.24	0.35
Design-raked	0.33	0.45	0.41	0.23	0.38
Linear model	0.35	0.46	0.42	0.23	0.36
Logistic model	0.36	0.46	0.42	0.23	0.36

We also examine the pooled mean squared error (MSE) of regression coefficients across 300 indepndent resamples, as shown in Figure 1a. Both model-based approaches—linear and logistic weighting—achieve lower MSE compared to the design-based and unweighted regressions, highlighting their relative stability in regression modeling. Additionally, Figure 1b presents the effective sample size (ESS) of the weights across 300 replications. The design-based method starts out with the highest ESS before raking, followed by linear and logistic weights. After raking, however, the model-based approaches, particularly logistic weighting, preserve ESS more effectively, whereas the design-based weights experience a more pronounced reduction.

$$\mathrm{ESS} = \frac{\left(\sum_{i=1}^{n} w_i\right)^2}{\sum_{i=1}^{n} w_i^2}.$$



(a) MSE across resamples.



(b) ESS across resamples ($n = 3000$).

To further connect the finite-sample evidence with the asymptotic theory, we provide two additional checks. To evaluate Theorem 1 and Corollary 1, We compare the empirical ESS ratio $\bar{w}_n(\hat{\beta})^2/\overline{w_n^2}(\hat{\beta})$ with the population target $\mathbb{E}[w]^2/\mathbb{E}[w^2]$ using 300 replications. In practice we compute the effective sample size ratio using sample proportions,

$$\frac{\left(\sum_{s=1}^{S} \hat{\pi}_s w(x_s;\beta_0)\right)^2}{\sum_{s=1}^{S} \hat{\pi}_s\, w(x_s;\beta_0)^2}, \qquad \hat{\pi}_s = \frac{n_s}{n},$$

as a finite-sample analogue of the population quantity

$$\frac{\mathbb{E}[w(X;\beta_0)]^2}{\mathbb{E}[w(X;\beta_0)^2]} = \frac{\left(\sum_{s=1}^{S} \pi_s w(x_s;\beta_0)\right)^2}{\sum_{s=1}^{S} \pi_s\, w(x_s;\beta_0)^2}, \qquad \pi_s = \frac{N_s}{N}.$$

Since $\hat{\pi}_s \to \pi_s$ as $n \to \infty$, the sample-based ratio converges to the population limit in Corollary 1. Table 7 reports the mean and interquartile range of two finite-sample analogues of the

ESS ratio across 300 replications. The first row ("empirical ESS ratio") is computed directly from the individual weights as $\bar{w}_n(\hat{\beta})^2/\overline{w_n^2}(\hat{\beta})$. The second row ("cell-based ESS ratio") evaluates the population-style formula using sample proportions $\hat{\pi}$ in place of π. Both are close in scale, with the cell-based version slightly larger on average, reflecting finite-sample variability around the population limit in Corollary 1.

Table 7: Finite-sample ESS/n ratios: mean and IQR

	Mean	IQR [Q1, Q3]
Empirical ESS ratio ($\bar{w}_n(\hat{\beta})^2/\overline{w_n^2}(\hat{\beta})$)	0.38	[0.37, 0.39]
Cell-based ESS ratio (with $\hat{\pi}_s$)	0.37	[0.36, 0.38]

To evaluate the finite-sample performance of Theorem 2, we computed the centered and scaled estimator $\sqrt{n}(\hat{\theta} - \theta_0)$ across 300 replications. For asymptotic variance estimation, we use the unnormalized raw equivalent weights when computing the gradient term $\hat{A}$, while the normalized weights are used in the influence function $\hat{\psi}_i$. This normalization ensures that $\hat{\theta}$ is expressed on the $1/n$ scale required in Theorem 2. The distribution of the centered and scaled estimator closely follows the Gaussian limit, as shown in the Q–Q plot in Figure 2, with only mild deviations in the tails. The plug-in variance estimator $\hat{V}$ from Theorem 2 was stable across replications, with mean 0.55 and interquartile range 0.53 to 0.57, and its values aligned closely with the empirical variance of 0.54. Taken together, the variance comparison and the Q–Q plot provide strong evidence that the asymptotic linear representation in Theorem 2 offers a reliable approximation even in finite samples.

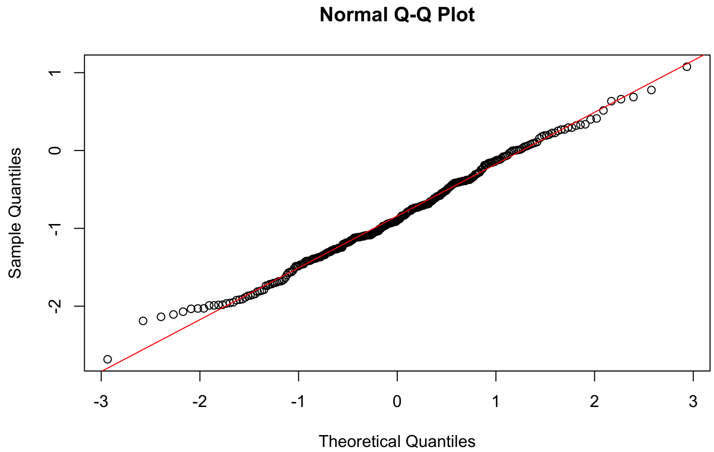


Figure 2: Normal Q–Q plot of the centered and scaled estimator $\sqrt{n}(\hat{\theta} - \theta_0)$ across 300 replications.

	Value
Empirical variance	0.54
Mean $\hat{V}$	0.55
IQR of $\hat{V}$	(0.53, 0.57)

Table 8: Comparison of empirical and plug-in variance estimates.

4.2 Applications to FFCWS Study

We now apply our methods to the FFCWS study, using $n = 3,442$ sampled units representing a population of 1,131,308 individuals. Both model-based methods produce negative weights, which we address by replacing them with the smallest positive weight. For the logistic model, the father's interview status is used as the outcome variable since it has no missing values. When raking the model-based weights, we use the same four demographic variables as in the national FFCWS weights, excluding the city variable. We refer to the national FFCWS weights as the design-based weights, which are already raked in the FFCWS data.

Analyzing population counts of live births by education level (Table 9), we find that both the linear and logistic models provide similar approximations to population totals, though some deviations appear across categories. Effective sample size (ESS) comparisons in Table 10 show that the logistic model yields the highest ESS, both before and after raking, followed by the linear model. After raking, both model-based approaches preserve ESS more effectively than the design-based method, with the logistic model showing the minimal reduction and thus the strongest preservation of ESS.

Table 9: Population counts of live births by education before raking

Education	<8 grade	Some HS	HS	Some College	College+	Total
Design-based	111,324	211,988	340,211	214,319	253,467	1,131,309
Linear model	73,232	267,083	367,476	242,049	181,469	1,131,309
Logistic model	60,214	207,301	316,517	258,784	288,491	1,131,309

Table 10: Effective Sample Size (ESS) for FFCWS Weights ($n = 3442$)

	Design-based	Linear	Logit	Linear-raked	Logit-raked
ESS	527	1273	1435	1082	1301

Using the logistic equivalent weights, the estimated prevalence of CPS contact in the FFCWS is $\hat{\theta} = 0.80$ and $\hat{V} = 0.42$, which yields a 95% confidence interval [0.78, 0.82] based on the plug-in asymptotic variance from Theorem 2. This provides a direct illustration of our theoretical results in an empirical setting: the plug-in variance formula can be applied to a single dataset to obtain standard errors and confidence intervals, even without resampling or replication.

To assess the accuracy of estimated population means, we evaluate Child Protective Services (CPS) contact prevalence across cities (Figure 3). Considering the expected national prevalence of CPS contact at age 18 ranges from 0.3 to 0.6 across cities, the weighting methods generally lower the unweighted estimates, pulling them closer to the national range (Jung et al, 2025). The design-based and linear weighting methods produce similar adjustments but with noticeable differences in cities like 9, 14, and 15. The logistic weighting method shows noticeable differences in cities like 2, 3, 15 and 16. Overall, all methods move the unweighted estimates toward the national range, but the magnitude of adjustment varies by city. This shows that the choice of weighting method primarily matters for certain subgroups (cities), rather than uniformly, making some weighting methods more sensitive than others to city-level demographic imbalances. The design-based and linear approaches yield more consistent corrections across cities, while logistic weighting exhibits greater sensitivity in specific locations. This highlights a trade-off between stability and flexibility: logistic weighting may better capture city-level heterogeneity but introduces more variability.

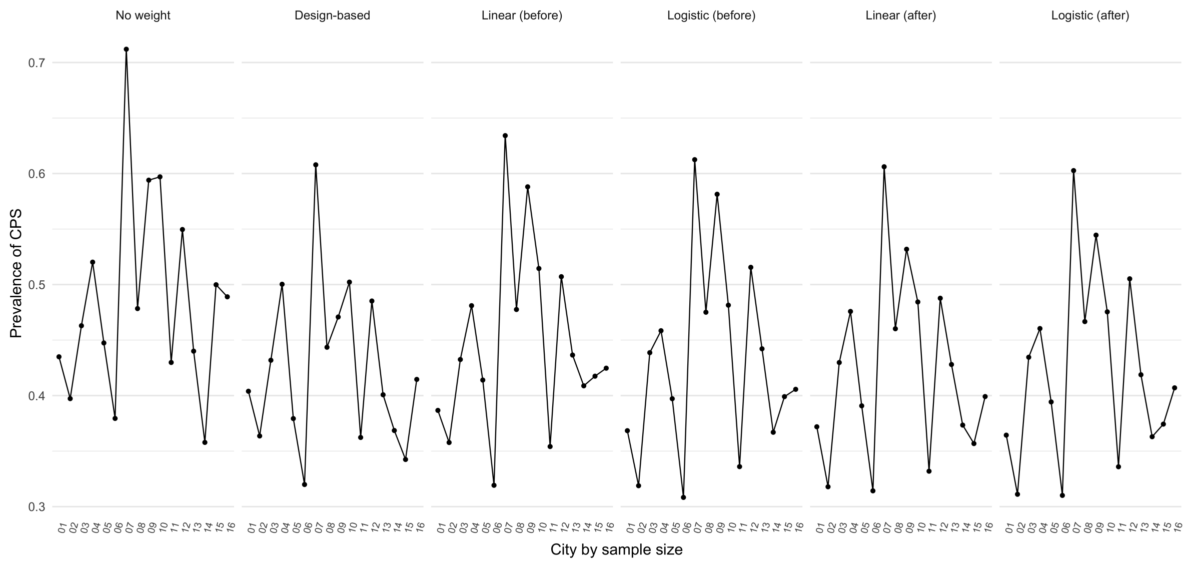


Figure 3: Prevalence of Child Protective Services (CPS) contact by city before and after raking

5 Discussion

5.1 Key Findings and Limitations

Our study provides a detailed comparison of model-based and design-based weighting methods, evaluating their performance in terms of accuracy, efficiency, and effective sample size (ESS) retention. The simulation study, based on drawing $n = 3000$ units from a population of size $N = 1{,}000{,}000$, demonstrates that both model-based approaches—linear and logistic weighting—outperform design-based weighting in regression analyses, achieving lower mean squared error (MSE) of regression coefficients.

A key distinction emerges in ESS preservation. Before raking, the design-based method starts with the largest ESS. After raking, however, model-based methods, particularly logistic weighting, retain a larger fraction of their efficiency, while design-based weights experience a more pronounced reduction. This finding aligns with our asymptotic theory, which predicts stability of model-based weights under finite-sample conditions.

In the application to the FFCWS study, logistic weighting consistently produced the highest ESS, both before and after raking, followed by linear weighting. While all weighting methods shifted unweighted prevalence estimates of Child Protective Services (CPS) contact toward the expected national range, the magnitude of adjustment varied across cities. Linear weighting provided more stable and consistent adjustments across subgroups, whereas logistic weighting displayed greater sensitivity to city-level heterogeneity. This highlights an important trade-off: linear weighting offers greater stability, while logistic weighting may better capture subgroup-specific imbalances at the cost of higher variability.

Overall, our results highlight the superiority of model-based weighting approaches over the traditional design-based method, both in regression performance and in ESS retention. Moreover, the asymptotic properties we establish—consistency of ESS and asymptotic normality of the weighted estimator with a plug-in variance formula—are supported by simulation evidence, confirming their practical relevance. To our knowledge, this is the first work to derive model-based survey weights using logistic regression and to establish their asymptotic distribution, providing both methodological innovation and theoretical justification.

Several limitations warrant discussion. First, although logistic weighting preserves ESS more effectively, it occasionally produces extreme or negative weights, requiring trimming and renormalization. Second, raking did not uniformly improve estimates across categories, underscoring that post-hoc adjustments may not systematically enhance performance. Third, our model-based approach does not leverage auxiliary population information nor outcome population information, which may limit its efficiency gains relative to methods that incorporate richer calibration targets. This limitation may also help explain the biased results we observed in estimating certain population statistics using logistic weights, as our simulations were designed to reflect realistic conditions by generating binary outcomes that depend on both weighting and non-weighting variables. Finally, while our asymptotic results provide theoretical justification for the model-based methods, the finite-sample performance remains sensitive to design choices, such as the definition of poststratification cells and the prevalence of binary outcomes.

5.2 Future Research

Our proposed approach does not rely on prior information about the population; weights are constructed solely from observed auxiliary variables without incorporating population margins as priors. Future research could therefore explore methods that extend beyond this framework by integrating population information more explicitly.

One promising direction is Bayesian hierarchical models, where population margins could be incorporated as priors while maintaining computational efficiency. Hierarchical Bayesian models that embed population priors directly may enhance the robustness of weight estimates and improve representativeness of the weighted sample. At the same time, the complexity of such models, particularly with numerous demographic and geographic categories, raises challenges in terms of computation and convergence.

Another avenue involves non-parametric methods such as Gaussian Processes (GPs), which offer flexibility in capturing nonlinear relationships between covariates and weights. An open question is how to incorporate population information in these models without losing predictive accuracy or overparameterizing. Possible strategies include kernel methods or structured priors that connect observed data to known population totals.

Finally, resampling techniques such as bootstrapping could be further explored to quantify uncertainty in weight estimates while maintaining alignment with population characteristics. This line of research may yield more robust methods for addressing variability and bias in survey weighting.

Appendix: Proofs

Proof of Lemma 1

Proof. The logistic variance satisfies $\sigma(t)\{1-\sigma(t)\} \le 1/4$ for all t, so $\|W(\beta)\|_{\mathrm{op}} \le 1/4$. The matrices X^{pop} and N^{pop} are fixed, hence have finite operator norms. Because $H(\beta)$ is symmetric positive definite with $\lambda_{\min}(H(\beta)) \ge c$, its spectral norm satisfies

$$\|H(\beta)^{-1}\| \;=\; \frac{1}{\lambda_{\min}(H(\beta))} \;\le\; \frac{1}{c}, \tag{3}$$

for all $\beta \in \mathcal{N}$. Since $X_i \in \mathcal{X}$ takes values in a finite set of dummy vectors, $\|X\|$ and $\|X^\top\|$ are uniformly bounded. Thus in the general form one might write

$$|\hat{w}_i(\beta)| \;\le\; C\big(1 + \|X_i\|^{m'}\big),$$

but because $\|X_i\|$ is uniformly bounded this polynomial term can be absorbed into a constant, yielding the sharper bound

$$|\hat{w}_i(\beta)| \;\le\; C',$$

for some finite C' depending only on $(X, X^{\text{pop}}, N^{\text{pop}}, c)$. □

Proof of Lemma 2

Proof. The logistic map $\sigma(t) = 1/(1+e^{-t})$ is infinitely differentiable on $\mathbb{R}$, hence continuous. Therefore $\beta \mapsto \sigma(X^{\text{pop}}\beta)$ and $\beta \mapsto W(\beta)$ are continuous. Matrix multiplication and the Hadamard product are continuous operations, so $\beta \mapsto X^\top W(\beta) X = H(\beta)$ is continuous. Now, because $\lambda_{\min}(H(\beta)) \ge c > 0$ for all $\beta \in \mathcal{N}$, the set $\{H(\beta) : \beta \in \mathcal{N}\}$ consists entirely of nonsingular matrices bounded away from singularity. On this set, the matrix inversion map $A \mapsto A^{-1}$ is continuous. Therefore $\beta \mapsto H(\beta)^{-1}$ is continuous on $\mathcal{N}$. Finally, composing continuous maps and multiplying on the left and right by the fixed matrices X^{pop} and $(N^{\text{pop}})^\top$ preserves continuity. Hence $\beta \mapsto \hat{w}(\beta)$ is continuous on $\mathcal{N}$. □

Proof of Theorem 1

Proof. Write

$$\bar{w}_n(\hat{\beta}) - \mathbb{E}[\hat{w}_1(\beta_0)] = \underbrace{\Big(\bar{w}_n(\hat{\beta}) - \mathbb{E}[\hat{w}_1(\hat{\beta})]\Big)}_{(A)} + \underbrace{\Big(\mathbb{E}[\hat{w}_1(\hat{\beta})] - \mathbb{E}[\hat{w}_1(\beta_0)]\Big)}_{(B)}.$$

(A) Uniform LLN. Consider the class $\mathcal{F} = \{\hat{w}(\cdot;\beta) : \beta \in \mathcal{N}\}$. By Lemma 2, $\beta \mapsto \hat{w}(\beta)$ is continuous; by Lemma 1, $|\hat{w}_i(\beta)| \le C$ for all $\beta \in \mathcal{N}$, so $g \equiv C$ is an integrable envelope. With continuity and compact $\mathcal{N}$, $\mathcal{F}$ is totally bounded in $L_1(F)$ and, with envelope g, is Glivenko–Cantelli. By the uniform weak law of large numbers (Wald, 1949; van der Vaart, 1998),

$$\sup_{\beta \in \mathcal{N}} \Big| \bar{w}_n(\beta) - \mathbb{E}[\hat{w}_1(\beta)] \Big| \xrightarrow{p} 0.$$

Since $\hat{\beta} \xrightarrow{p} \beta_0 \in \text{int}(\mathcal{N})$, we have $(A) \xrightarrow{p} 0$.

(B) Dominated convergence. By Lemma 2, $\hat{w}_1(\hat{\beta}) \to \hat{w}_1(\beta_0)$ almost surely as $\hat{\beta} \to \beta_0$, and by Lemma 1, $|\hat{w}_1(\hat{\beta})| \leq g(X_1) \equiv C$ with $\mathbb{E}[g] < \infty$. Therefore, by dominated convergence, $\mathbb{E}[\hat{w}_1(\hat{\beta})] \to \mathbb{E}[\hat{w}_1(\beta_0)]$, so $(B) \to 0$.

Combining (A) and (B) gives $\bar{w}_n(\hat{\beta}) \xrightarrow{p} \mathbb{E}[\hat{w}_1(\beta_0)]$.

The argument for $\overline{w^2}_n(\hat{\beta})$ is identical, applied to the class $\{\hat{w}(\cdot;\beta)^2 : \beta \in \mathcal{N}\}$ with envelope $g^2 \equiv C^2$ (and $\mathbb{E}[g^2] < \infty$), yielding $\overline{w^2}_n(\hat{\beta}) \xrightarrow{p} \mathbb{E}[\hat{w}_1(\beta_0)^2]$. □

Proof of Lemma 3

Proof. By Lemmas 1 and 2, $\hat{w}(\beta)$ is bounded and continuous on $\mathcal{N}$. Since the logistic map σ is C^∞, we have $W(\beta) = \operatorname{diag}(\sigma(X\beta) \odot (1 - \sigma(X\beta))) \in C^1$ and $H(\beta) = X^\top W(\beta) X \in C^1$. On $\mathcal{N}$, $\lambda_{\min}(H(\beta)) \geq c > 0$, hence the inversion map is C^1 with

$$D(A^{-1})[\Delta] = -\,A^{-1} \Delta A^{-1} \qquad \text{(Fréchet derivative)}. \tag{4}$$

Step 1: Bound $D(H^{-1})$. For a direction $h \in \mathbb{R}^p$,

$$DH(\beta)[h] \;=\; X^\top \, DW(\beta)[h] \, X,$$

and, writing $p_i = \sigma(x_i^\top \beta)$,

$$\big(DW(\beta)[h]\big)_{ii} \;=\; p_i(1-p_i)(1-2p_i)\,(x_i^\top h).$$

Hence $\|DW(\beta)[h]\| \leq \frac{1}{4}\,\|X\|\,\|h\|$, and by submultiplicativity

$$\|DH(\beta)[h]\| \;\leq\; C_X\,\|h\|, \qquad C_X := \tfrac{1}{4}\,\|X^\top\|\,\|X\|^2. \tag{5}$$

Using (4),

$$D\big(H(\beta)^{-1}\big)[h] \;=\; -\,H(\beta)^{-1}\,DH(\beta)[h]\,H(\beta)^{-1},$$

so

$$\|D(H(\beta)^{-1})[h]\| \;\leq\; \|H(\beta)^{-1}\|^2\,\|DH(\beta)[h]\| \;\leq\; \|H(\beta)^{-1}\|^2\,C_X\,\|h\|, \tag{6}$$

with

$$\|H(\beta)^{-1}\| \;=\; \frac{1}{\lambda_{\min}(H(\beta))} \;\le\; \frac{1}{c}, \tag{7}$$

and combining (6)–(7) yields

$$\|D(H(\beta)^{-1})[h]\| \;\le\; c^{-2} C_X \,\|h\|. \tag{8}$$

Step 2: Bound $D\hat{w}(\beta)[h]$. Write $\hat{w}(\beta) = \frac{n}{N}\, T(\beta)^\top S(\beta)$ with

$$T(\beta) := H(\beta)^{-1} X^\top, \qquad S(\beta) := X^{\mathrm{pop}\top} W^{\mathrm{pop}}(\beta) N^{\mathrm{pop}}.$$

By product rule,

$$D\hat{w}(\beta)[h] = \tfrac{n}{N}\Big(DT(\beta)[h]^\top\, S(\beta) \;+\; T(\beta)^\top\, DS(\beta)[h]\Big).$$

Using (8),

$$\|DT(\beta)[h]\| \;\le\; c^{-2} C_X \|X^\top\|\, \|h\|, \qquad \|T(\beta)\| \;\le\; \|H(\beta)^{-1}\|\, \|X^\top\| \;\le\; c^{-1}\|X^\top\|.$$

Since $X^{\mathrm{pop}}, N^{\mathrm{pop}}$ are fixed and $\sigma(u)(1-\sigma(u)) \le \frac{1}{4}$, there exist finite constants C_1, C_2 (depending only on $X^{\mathrm{pop}}, N^{\mathrm{pop}}$ and logistic bounds) such that

$$\|S(\beta)\| \le C_1, \qquad \|DS(\beta)[h]\| \le C_2\, \|h\|.$$

Therefore, by submultiplicativity,

$$\|D\hat{w}(\beta)[h]\| \;\le\; \frac{n}{N}\Big(\big(c^{-2} C_X \|X^\top\|\big) C_1 + \big(c^{-1}\|X^\top\|\big) C_2\Big)\, \|h\| \;=:\; C'\, \|h\|. \tag{9}$$

Step 3: From directional to gradient bound (each coordinate). For the i-th coordinate $f(\beta) = \hat{w}_i(\beta)$ (a scalar),

$$|D\hat{w}_i(\beta)[h]| \;\le\; \|D\hat{w}(\beta)[h]\| \;\le\; C'\, \|h\|.$$

Taking $\sup_{\|h\|=1}$ gives

$$\|\nabla_\beta \hat{w}_i(\beta)\| = \sup_{\|h\|=1} |D\hat{w}_i(\beta)[h]| \;\le\; C' \qquad \text{for all } \beta \in \mathcal{N} \text{ and all } i,$$

which is the desired uniform derivative envelope. Finally, since all components used are C^1 and $H(\beta)^{-1}$ is C^1 on $\mathcal{N}$, it follows that $\beta \mapsto \hat{w}(\beta)$ is C^1 as well (part (i)). This completes the proof of the lemma. □

Proof of Theorem 2

Proof. Write

$$\hat{\theta} \;=\; P_n\{\hat{w}(\hat{\beta})Y\}, \qquad \theta_0 \;=\; P\{\hat{w}(\beta_0)Y\},$$

hence

$$\hat{\theta}-\theta_0 = \underbrace{\{P_n - P\}[\hat{w}(\beta_0)Y]}_{(I)} \;+\; \underbrace{\Big(P\,\hat{w}(\hat{\beta})Y - P\,\hat{w}(\beta_0)Y\Big)}_{(II)} \;+\; \underbrace{\Big\{(P_n\hat{w}(\hat{\beta})Y - P\hat{w}(\hat{\beta})Y) - (P_n\hat{w}(\beta_0)Y - P\hat{w}(\beta_0)Y)\Big\}}_{(III)}.$$

(I) By Lemma 1, $\hat{w}(\beta_0)Y$ has an integrable constant envelope; the i.i.d. CLT gives

$$\sqrt{n}(I) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\big\{\hat{w}_i(\beta_0)Y_i - P[\hat{w}(\beta_0)Y]\big\} \;\Rightarrow\; \mathcal{N}\big(0, \mathbb{V}[\hat{w}(\beta_0)Y]\big).$$

(II) By Lemma 3, $\beta \mapsto \hat{w}(\beta)$ is C^1 on $\mathcal{N}$; therefore $g(\beta) = P[\hat{w}(\beta)Y]$ is differentiable at β_0 with gradient $A = P[\nabla_\beta \hat{w}(\beta_0)Y]$. A first-order expansion yields $g(\hat{\beta}) - g(\beta_0) = A^\top(\hat{\beta}-\beta_0) + o_p(\|\hat{\beta}-\beta_0\|)$. By (iii) and the influence function representation of the MLE in logistic regression (Hampel et al., 1986; van der Vaart, 1998, Thm. 5.23),

$$\begin{aligned}\sqrt{n}(II) &= A^\top\sqrt{n}(\hat{\beta}-\beta_0) + o_p(1)\\ &= A^\top\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathcal{I}^{-1}X_i\{Y_i - p_0(X_i)\}\right) + o_p(1).\end{aligned}$$

(III) By Lemma 1, the class $\mathcal{F} := \{\hat{w}(\cdot;\beta)Y : \beta \in \mathcal{N}\}$ has a constant envelope. By Lemma 3, $\beta \mapsto \hat{w}(\beta)$ is C^1 on $\mathcal{N}$ with $\sup_{\beta\in\mathcal{N}}\|\nabla_\beta \hat{w}_i(\beta)\| \le C'$, so the parametrization $\beta \mapsto \hat{w}(\cdot;\beta)Y$ is

Lipschitz in β, uniformly on $\mathcal{N}$. Restricting to a compact neighborhood of β_0, the class is finite-dimensional, uniformly bounded, and $L^2(P)$-Lipschitz in β, and is therefore P-Donsker. Since we also have $\|\hat{w}(\cdot;\hat{\beta}) - \hat{w}(\cdot;\beta_0)\|_{L^2(P)} \xrightarrow{p} 0$, by (van der Vaart, 1998, Lemma 19.24),

$$\sqrt{n}\,(III) = \sqrt{n}\,(P_n - P)\big[\hat{w}(\hat{\beta})Y - \hat{w}(\beta_0)Y\big] = o_p(1).$$

Combining (I)–(III) gives the stated linear expansion and asymptotic variance. Consistency of $\widehat{V}$ follows by LLN and continuous mapping. □

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