THE METHOD OF STEEPEST DESCENT

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ABSTRACT. When solving a problem analytically is not possible, using iterative methods may be the best option. Among these methods include Newton's Method and The Method of Steepest Descent. In this particular problem, Newton's method can run in to complications so I will be using The Method of Steepest Descent. I will be calculating the minimum distance between two planetary orbits assuming the orbits lie on the same plane. However, each elliptical orbit will have a different tilt given by ϕ.

INTRODUCTION

Consider the differentiable function

$$f(x + \lambda d)$$

where d is the direction of steepest descent, and $\lambda \in (0, d)$ such that $d > 0$. Then $-\nabla f(x)^T d$ is the rate of decrease in the direction of d at a point x. If $||d|| = 1$, then for $x, y \in \mathbb{R}^n$

$$| < \nabla f(x), d > | \leq ||d|| \; ||\nabla f(x)|| = ||\nabla f(x)||.$$

If $d = \nabla f(x)/||\nabla f(x)||$,

$$< \nabla f(x), \frac{\nabla f(x)}{||\nabla f(x)||} >= \nabla f(x)^T \frac{\nabla f(x)}{||\nabla f(x)||} = ||\nabla f(x)||.$$

Therefore, $\nabla f(x)$ gives the direction of maximum increase and $-\nabla f(x)$ gives the direction of maximum decrease. In order to find the minimum distance between the two orbits, I will use $-\nabla f(x)$ with each iteration such that I am minimizing $f(x_k - \nabla f(x_k)\lambda)$ until $||\nabla f(x_k)|| < \epsilon$.

PROBLEM

A tilted elliptical orbit having $(0, 0)$ as one focal point is given by

$$\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} \cos(\phi_1) & \sin(\phi_1) \\ -\sin(\phi_1) & \cos(\phi_1) \end{bmatrix} \begin{bmatrix} \frac{P_1 - A_1}{2} + \frac{P_1 + A_1}{2}\cos(t) \\ \sqrt{P_1 A_1}\sin(t) \end{bmatrix}.$$

Think of the sun as being the focus $(0, 0)$. The tilt angle is given by ϕ_1, A_1, and P_1 are fixed parameters.

A second orbit is given by

$$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \cos(\phi_2) & \sin(\phi_2) \\ -\sin(\phi_2) & \cos(\phi_2) \end{bmatrix} \begin{bmatrix} \frac{P_2 - A_2}{2} + \frac{P_2 + A_2}{2}\cos(t) \\ \sqrt{P_2 A_2}\sin(t) \end{bmatrix}.$$

The objective is to find the minimum distance from a point (A_1, P_1, ϕ_1) on the first orbit to a point (A_2, P_2, ϕ_2) on the second orbit. For a measure of distance we can use the function

$$dis(t_1, t_2) = \frac{1}{2}[(x_1(t_1) - x_2(t_2))^2 + (y_1(t_1) - y_2(t_2))^2].$$

This is a function of two variables.
The objective is to minimize $dis(t_1, t_2)$. Use the parameters $(A_1, P_1, \phi_1) = (10, 2, \pi/8)$ and $(A_2, P_2, \phi_2) = (4, 1, -\pi/7)$.

SOLUTION

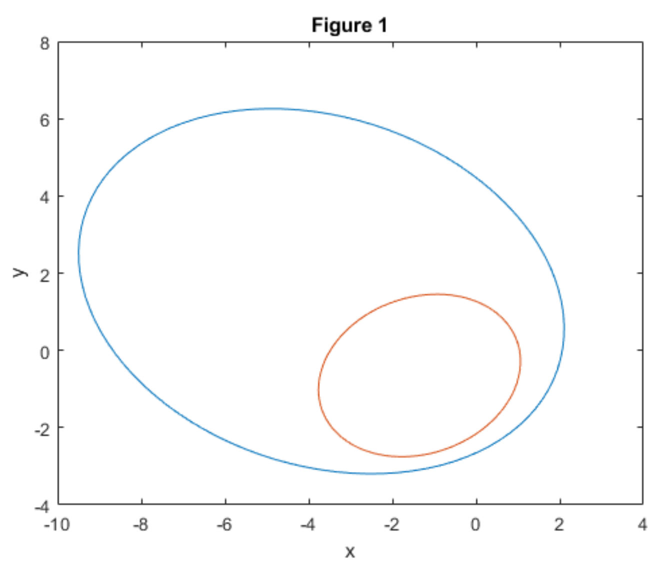
Before jumping in to The Method of Steepest descent, I wanted to have an idea of what these two orbits look like on a two dimensional plane. Consider Orbit 1 to have equation

$$\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} \cos(\phi_1) & \sin(\phi_1) \\ -\sin(\phi_1) & \cos(\phi_1) \end{bmatrix} \begin{bmatrix} \frac{P_1 - A_1}{2} + \frac{P_1 + A_1}{2}\cos(t) \\ \sqrt{P_1 A_1}\sin(t) \end{bmatrix}.$$

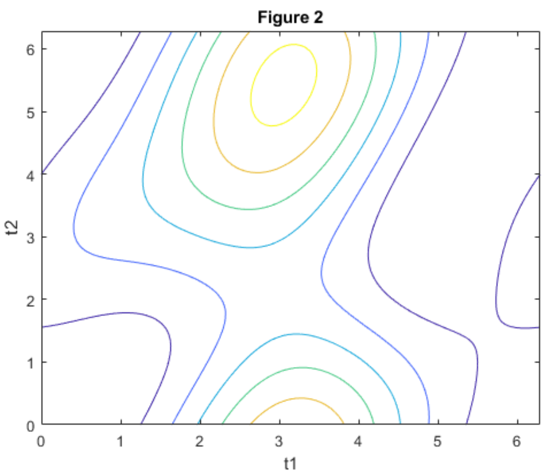
and orbit 2 to have equation

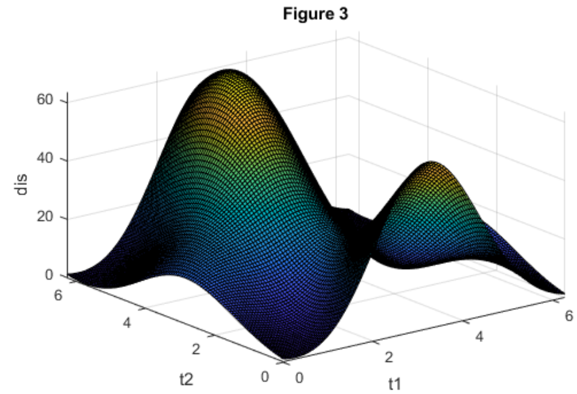
$$\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \cos(\phi_2) & \sin(\phi_2) \\ -\sin(\phi_2) & \cos(\phi_2) \end{bmatrix} \begin{bmatrix} \frac{P_2 - A_2}{2} + \frac{P_2 + A_2}{2}\cos(t) \\ \sqrt{P_2 A_2}\sin(t) \end{bmatrix}.$$

The following Figure, shows the two orbits on the same plot. Orbit 1 is the outer orbit and Orbit 2 is the inner orbit.



From the figure, we can see the minimum distance between the two orbits occurs near the point $(-1, -3)$. To gain a better understanding of which t values, $t1$ and $t2$, to use, I will next plot the contour graph of the distance between the two orbits.





From these two figures, we can hypothesize which t values that would give us a minimum distance between the two orbits. From Figure 2, it appears that $(4, 2)$ might provide a minimum distance, but Figure 3 shows that $(4, 2)$ may only be a local minimum. We can now determine a more accurate starting point so that when we use The Method of Steepest descent, we will need fewer iterations.

ALGORITHM

First, I initialized values for t_1 and t_2 by letting $t_1 = 4$ and $t_2 = 2$. These points are chosen based on data collected from Figure 2. Next I chose $\epsilon = .01$. When $||\nabla dis(t_1, t_2)|| < \epsilon, dis(t_1, t_2)$ is at a minimum for the given epsilon.

0.1. Finding $\nabla dis(t_1, t_2)$
The distance function is given by

$$dis(t_1, t_2) = \frac{1}{2}[(x_1(t_1) - x_2(t_2))^2 + (y_1(t_1) - y_2(t_2))^2].$$

Then by the chain rule,

$$\begin{aligned}\nabla dis(t_1, t_2) =<&(x_1(t_1) - x_2(t_2))x_1^{'}(t_1) + (y_1(t_1) - y_2(t_2))y_1^{'}(t_1),\\ &(x_1(t_1) - x_2(t_2))x_2^{'}(t_2) + (y_1(t_1) - y_2(t_2))y_2^{'}(t_2) >\end{aligned}$$

where

$$\begin{aligned}x_1^{'}(t_1) &= \cos(\phi_1)\frac{P_1 + A_1}{2}(-\sin(t_1)) + \sin(\phi_1)\sqrt{P_1 A_1}\cos(t_1),\\ y_1^{'}(t_1) &= -\sin(\phi_1)\frac{P_1 + A_1}{2}(-\sin(t_1)) + \cos(\phi_1)\sqrt{P_1 A_1}\cos(t_1),\\ x_2^{'}(t_2) &= \cos(\phi_2)\frac{P_2 + A_2}{2}(-\sin(t_2)) + \sin(\phi_2)\sqrt{P_2 A_2}\cos(t_2),\\ y_2^{'}(t_2) &= -\sin(\phi_2)\frac{P_2 + A_2}{2}(-\sin(t_2)) + \cos(\phi_2)\sqrt{P_2 A_2}\cos(t_2).\end{aligned}$$

If $||\nabla dis(t_1, t_2)|| < \epsilon$, then stop and we have obtained the minimum distance. Otherwise, let $d_k = -\nabla dis(t_1, t_2)$, and let λ be the optimal solution to the problem to minimize $dis(T_k + \lambda d_k)$ subject $\lambda \geq 0$ such that $T_k = [t_1, t_2]^T$. Once the best λ value is found that minimizes the distance with fixed T_k and d_k, we then change T_k and d_k by the equation $T_{k+1} = T_k + \lambda d_k$.

Note that we can use the T_{k+1} to find the value of d_{k+1}. Again, find the λ that minimizes $dis(T_{k+1} + \lambda d_{k+1})$. Continue doing this until $||\nabla dis(t_1, t_2)|| < \epsilon$.

0.2. Finding an appropriate λ

One notable aspect of this project was how to find a proper range of values for λ. The following illustrates my method for finding a proper range for λ:

$$0 \leq T_k + \lambda d_k \leq 2\pi, -T_k \leq \lambda d_k \leq 2\pi - T_k, \frac{-T_k}{d_k} \leq \lambda \leq \frac{2\pi - T_k}{d_k}.$$

To ensure the λ is always positive, it is enticing to use

$$\max(0, \frac{-T_k}{d_k}) \leq \lambda \leq \max(0, \frac{2\pi - T_k}{d_k}).$$

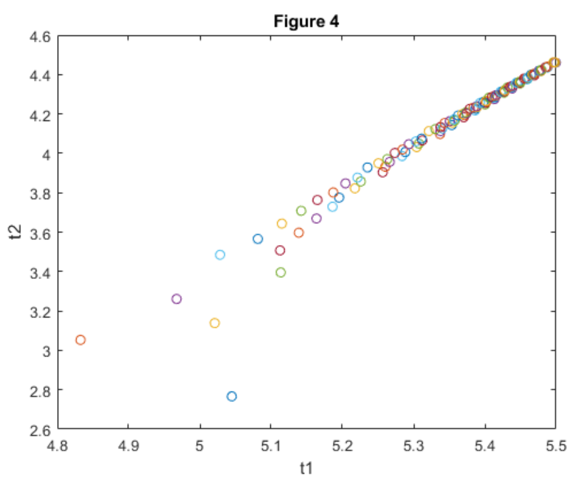
but $\lambda = 0$ may create difficulties. For this reason, I let λ be bound by

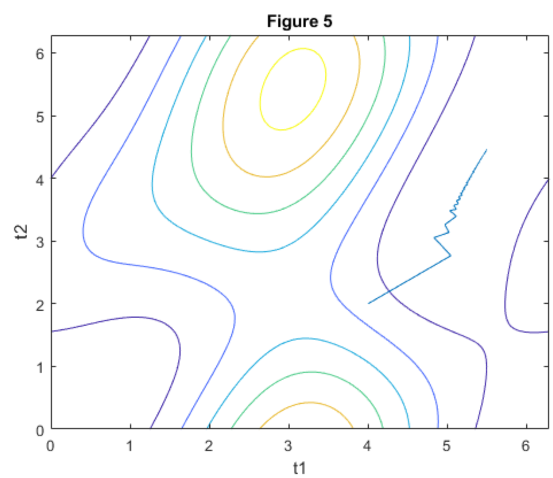
$$\max(.05, \frac{-T_k}{d_k}) \leq \lambda \leq \max(.05, \frac{2\pi - T_k}{d_k}).$$

Further issues by bounding λ in this manner is if d_k is negative, it will switch the signs of the inequality, but this is easily solved with programming tricks.

RESULTS

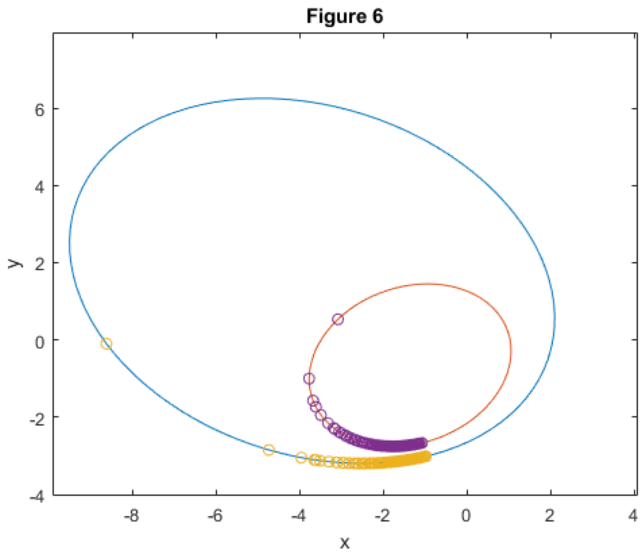
For $\epsilon = .01$, the minimum distance is $dis(t_1, t_2) = .0645$. This occurs when $t_1 = 5.4873$ and $t_2 = 4.4388$. The following graphs show how each iteration converges to these values.





I chose my initial starting point to be $t_1 = 4$ and $t_2 = 2$ and we can see how the t values converge to the final values.

Finally, Figure 6 shows at what point on both orbits that the minimum distance occurs.



For orbit 1, $x_1 = -1.0399$ and $y_1 = -3.2078$. For orbit 2, $x_2 = -1.1246$ and $y_2 = -2.6788$. Thus, using The Method of Steepest Descent, we have determined the minimum distance between the two orbits.