6.1 (a) Show that $X(\Omega)$ is periodic with period 2π, that is, $X(\Omega + 2\pi) = X(\Omega)$.

(b) Specifically show that $X(\Omega) = \dfrac{1}{1 - ae^{-j\Omega}}$ is periodic.

(a)

給定 $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$

則 $X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\Omega + 2\pi)n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \cdot \underline{e^{-jn2\pi}}$

但 $e^{-jn2\pi} = \cos(2n\pi) - j\sin(2n\pi) = 1 - j0 = 1$

$\therefore X(\Omega + 2\pi) = X(\Omega) \Rightarrow$ periodic ※

(b)

$X(\Omega + 2\pi) = \dfrac{1}{1 - ae^{-j(\Omega + 2\pi)}} = \dfrac{1}{1 - a^{-j\Omega} e^{-j2\pi}}$

但 $e^{-j2\pi} = \cos(2\pi) - j\sin(2\pi) = 1 - j0 = 1$

$\therefore X(\Omega + 2\pi) = \dfrac{1}{1 - a^{-j\Omega}} = X(\Omega) \Rightarrow$ periodic ※

6.10 The DTFT of a signal $x[n]$ is

$$X(\Omega) = \frac{2}{3 + e^{-j\Omega}}$$

4. Frequency-shifting (modulation)	$e^{j\Omega_0 n} x[n]$	$X(\Omega - \Omega_0)$

Find the DTFT of the following signals:

(a) $y[n] = x[-n]$

(b) $z[n] = nx[n]$

(c) $w[n] = x[n] + x[n-1]$

(d) $v[n] = x[n]\cos(n\pi)$

(a) $x[-n] \Rightarrow X(-\Omega)$

$\therefore Y(\Omega) = X(-\Omega) = \dfrac{2}{3 + e^{j\Omega}}$ ※

(c) $x[n - n_0] \Rightarrow e^{-j\Omega n_0} X(\Omega)$

$\therefore W(\Omega) = \dfrac{2}{3 + e^{-j\Omega}} + e^{-j\Omega} \dfrac{2}{3 + e^{-j\Omega}}$

$= \dfrac{2(1 + e^{-j\Omega})}{3 + e^{-j\Omega}}$ ※

(b) $nx[n] \Rightarrow j \dfrac{dX(\Omega)}{d\Omega}$

$\therefore Z(\Omega) = j \dfrac{dX(\Omega)}{d\Omega} = j \dfrac{0 - 2 \cdot (-j) e^{-j\Omega}}{(3 + e^{-j\Omega})^2}$

$= \dfrac{-2e^{-j\Omega}}{(3 + e^{-j\Omega})^2}$ ※

(d) $x[n]\cos(n\pi) = x[n] \frac{1}{2} (e^{j\pi n} + je^{-j\pi n})$

$\therefore V(\Omega) = \frac{1}{2} X(\Omega + \pi) + \frac{1}{2} X(\Omega - \pi)$

$= \dfrac{1}{3 + e^{-j(\Omega + \pi)}} + \dfrac{1}{3 + e^{-j(\Omega - \pi)}}$

但 $e^{j\pi} = \cos\pi + j\sin\pi = -1$, $e^{-j\pi} = \cos(-\pi) + j\sin(-\pi) = -1$

$= \dfrac{2}{3 - e^{-j\Omega}}$ ※

6.13 Determine the signal $x[n]$ corresponding to each of the following Fourier transforms:

(a) $X(\Omega) = 1 + 3e^{-j2\Omega} - 2e^{j4\Omega} + e^{j5\Omega}$

(b) $X(\Omega) = \dfrac{e^{-j\Omega} - \frac{1}{2}}{1 - \frac{1}{2}e^{-j\Omega}}$

(c) $X(\Omega) = 3\pi[\delta(\Omega-2) + \delta(\Omega+2)]$ $\quad \Omega_0 = 2$

$\cos(\Omega_0 n)$	$\pi \sum_{k=-\infty}^{\infty} \{\delta(\Omega - \Omega_0 - 2\pi k) + \delta(\Omega + \Omega_0 - 2\pi k)\}$
$\sin(\Omega_0 n)$	$\frac{\pi}{j} \sum_{k=-\infty}^{\infty} \{\delta(\Omega - \Omega_0 - 2\pi k) - \delta(\Omega + \Omega_0 - 2\pi k)\}$

(a)

$$x[n] = \delta[n] + 3\delta[n-2] - 2\delta[n+4] + \delta[n+5] \ \#$$

(b)

$$X(\Omega) = \frac{e^{-j\Omega}}{1 - \frac{1}{2}e^{-j\Omega}} - \frac{1}{2}\,\frac{1}{1 - \frac{1}{2}e^{-j\Omega}}$$

$$= \left(\tfrac{1}{2}\right)^{n-1} u[n-1] - \left(\tfrac{1}{2}\right)^{n+1} u[n] \ \#$$

$$F\{e^{j2n}\} = 2\pi\delta(\Omega-2)$$

$$F\{e^{-j2n}\} = 2\pi\delta(\Omega+2)$$

(c) $X(\Omega) = \frac{3}{2} \cdot 2\pi[\delta(\Omega-2) + \delta(\Omega+2)]$

$$\Rightarrow x[n] = \frac{3}{2}\left(e^{j2n} + e^{-j2n}\right) = 3\cos(2n) \ \#$$

6.18 Find the convolution $y[n] = h[n] * x[n]$ of the following pairs of signals:

(a) $x[n] = \left(\frac{1}{4}\right)^n u[n], \quad h[n] = 1$

(b) $x[n] = \left(\frac{1}{3}\right)^n u[n], \quad h[n] = \delta[n] + \delta[n-1]$

(c) $x[n] = \left(\frac{1}{2}\right)^n u[n], \quad h[n] = \left(\frac{1}{3}\right)^n u[n]$

(a)

$$X(\Omega) = \frac{1}{1 - \frac{1}{4}e^{-j\Omega}} \qquad H(\Omega) = \cancel{\frac{1}{1 - e^{-j\Omega}}}\ 2\pi\delta(\Omega)$$

$$Y(\Omega) = X(\Omega)H(\Omega) = \frac{A}{1 - \frac{1}{4}e^{-j\Omega}} + \frac{B}{1 - e^{-j\Omega}} = \frac{2\pi\delta(\Omega)}{1 - \frac{1}{4}e^{-j\Omega}}$$

$$A = -\frac{1}{3}, \quad B = \frac{4}{3}$$

$$\Rightarrow y[n] = -\frac{1}{3}\left(\frac{1}{4}\right)^n u[n] + \frac{4}{3}u[n] \ \# = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2\pi\delta(\Omega)}{1 - \frac{1}{4}e^{-j\Omega}} e^{j\Omega n}\, d\Omega$$

$$= \left.\frac{e^{j\Omega n}}{1 - \frac{1}{4}e^{-j\Omega}}\right|_{\Omega=0} = \frac{4}{3}$$

(b)

$$X(\Omega)=\frac{1}{1-\frac{1}{3}e^{-j\Omega}} \quad H(\Omega)=1+e^{-j\Omega}$$

$$Y(\Omega)=X(\Omega)H(\Omega)=\frac{1}{1-\frac{1}{3}e^{-j\Omega}}+\frac{e^{-j\Omega}}{1-\frac{1}{3}e^{-j\Omega}}$$

$$\Rightarrow y[n]=\left(\frac{1}{3}\right)^n u[n]+\left(\frac{1}{3}\right)^{n-1}u[n-1] \#$$

(c)

$$X(\Omega)=\frac{1}{1-\frac{1}{2}e^{-j\Omega}} \quad H(\Omega)=\frac{1}{1-\frac{1}{3}e^{-j\Omega}}$$

$$Y(\Omega)=X(\Omega)H(\Omega)=\frac{A}{1-\frac{1}{2}e^{-j\Omega}}+\frac{B}{1-\frac{1}{3}e^{-j\Omega}}$$

$A=3 \quad B=-2$

$$\Rightarrow y[n]=3\left(\frac{1}{2}\right)^n u[n]-2\left(\frac{1}{3}\right)^n u[n] \#$$

6.20 Prove the following DFT properties:

(a) $X[0]=\sum_{n=0}^{N-1} x[n]$

(b) $X[N/2]=\sum_{n=0}^{N-1} (-1)^n x[n]$

(a)

$$X[k]=\sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N}$$

當 $k=0$, $X[0]=\sum_{n=0}^{N-1} x[n]e^{-j0}=\sum_{n=0}^{N-1} x[n] \#$

(b)

當 $k=\frac{N}{2}$, $X[\frac{N}{2}]=\sum_{n=0}^{N-1} x[n]e^{-jn\pi}$, 但 $e^{-jn\pi}=\cos(n\pi)-j\sin(n\pi)=(-1)^n$

$$=\sum_{n=0}^{N-1}(-1)^n x[n] \#$$

6.21 Find the DFT of the following sequences:

(a) $x[n] = \{0, 1, 2, 3\}$

(b) $y[n] = \{1, 1, -1, -1, 1, 1, -1, -1\}$

$$X[k]=\sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N}$$

(a) $N=4,\ k=0,1,2,3$

$$X[k]=x[0]+x[1]e^{-j\pi k/2}+x[2]e^{-j\pi k}+x[3]e^{-j\pi 3k/2}$$

$$=e^{-j\pi k/2}+2e^{-j\pi k}+3e^{-j\pi 3k/2},\quad k=0,1,2,3$$

$$=[6,-2+j2,-2,-2-j2]_{\#}$$

(b) $N=8,\ k=0\sim 7$

$$Y[k]=y[0]+y[1]e^{-j\pi k/4}+y[2]e^{-j\pi k/2}+y[3]e^{-j\pi 3k/4}+y[4]e^{-j\pi k}+y[5]e^{-j\pi 5k/4}+y[6]e^{-j\pi 3k/2}+y[7]e^{-j\pi 7k/4}$$

$$=1+e^{-j\pi k/4}-e^{-j\pi k/2}-e^{-j\pi 3k/4}+e^{-j\pi k}+e^{-j\pi 5k/4}-e^{-j\pi 3k/2}-e^{-j\pi 7k/4},\quad k=0\sim 7$$

$$=[0,0,4-j4,0,0,0,4+j4,0]_{\#}$$

6.23 Show that

(a) $x[n]\cos\left(\dfrac{2\pi km}{N}\right)\longleftrightarrow\dfrac{1}{2}\left[X(\overset{k-m}{m-k})+X(\overset{k+m}{m+k})\right]$

(b) $x[n]\sin\left(\dfrac{2\pi km}{N}\right)\longleftrightarrow\dfrac{1}{2j}\left[X(\overset{k-m}{m-k})-X(\overset{k+m}{m+k})\right]$

$$X(k)=\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N},\quad x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X(k)e^{j2\pi kn/N}$$

(a) ① $$x[n]\cos\left(\frac{2\pi km}{N}\right)=x[n]\cdot\frac{1}{2}\left(e^{j\frac{2\pi km}{N}}+e^{-j\frac{2\pi km}{N}}\right)$$

$$=\frac{1}{2}\left(x[n]e^{j\frac{2\pi km}{N}}+x[n]e^{-j\frac{2\pi km}{N}}\right)\xrightarrow{\text{DFT}}\frac{1}{2}[X(k-m)+X(k+m)]$$

②

$$\frac{1}{2}[X(k-m)+X(k+m)]\xrightarrow{\text{IDFT}}\frac{1}{2}\left(x[n]e^{-j\frac{2\pi km}{N}}+x[n]e^{j\frac{2\pi km}{N}}\right)=x[n]\cos\frac{2\pi km}{N}$$

由①②得證#

(b)

① $$x[n]\sin\left(\frac{2\pi km}{N}\right)=x[n]\cdot\frac{1}{2j}\left(e^{j\frac{2\pi km}{N}}-e^{-j\frac{2\pi km}{N}}\right)$$

$$=\frac{1}{2j}\left(x[n]e^{j\frac{2\pi km}{N}}-x[n]e^{-j\frac{2\pi km}{N}}\right)\xrightarrow{\text{DFT}}\frac{1}{2j}[X(k-m)-X(k+m)]$$

②

$$\frac{1}{2j}[X(k-m)-X(k+m)]\xrightarrow{\text{IDFT}}\frac{1}{2j}\left(x[n]e^{j\frac{2\pi km}{N}}-x[n]e^{-j\frac{2\pi km}{N}}\right)=x[n]\sin\frac{2\pi km}{N}$$

由①②得證#

6.25 The DFT of a signal $x[n]$ is

$$X(0)=1,\quad X(1)=1+j2,\quad X(2)=1-j,\quad X(3)=1+j,\quad X(4)=1-j2$$

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X(k)e^{j2\pi nk/N}$$

Compute $x[n]$.

$N=5$

$$x[0]=\frac{1}{5}\sum_{k=0}^{4}X(k)e^{j0}=\frac{1}{5}\left(X(0)+\cdots+X(4)\right)=1$$

$$x[1]=\frac{1}{5}\sum_{k=0}^{4}X(k)e^{j2\pi k/5}\doteq -0.5257$$

$$x[2]=\frac{1}{5}\sum_{k=0}^{4}X(k)e^{j4\pi k/5}\doteq -0.8507$$

$$x[3]=\frac{1}{5}\sum_{k=0}^{4}X(k)e^{j6\pi k/5}\doteq 0.8507$$

$$x[4]=\frac{1}{5}\sum_{k=0}^{4}X(k)e^{j8\pi k/5}\doteq 0.5257$$ #

6.25 $x[0] = 1$, $x[1] = -0.5257$, $x[2] = -0.8507$, $x[3] = 0.8507$, $x[4] = 0.5257$

6.28 Use MATLAB to compute the FFT of the following signals. For each signal, plot $|X(k)|$.

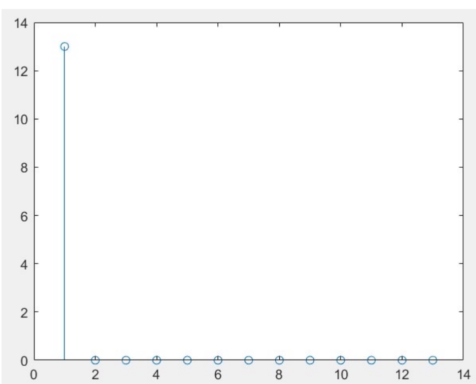
(a) $x[n] = 1,\ 0 \le n \le 12$

(b) $x[n] = n,\ 0 \le n \le 10$

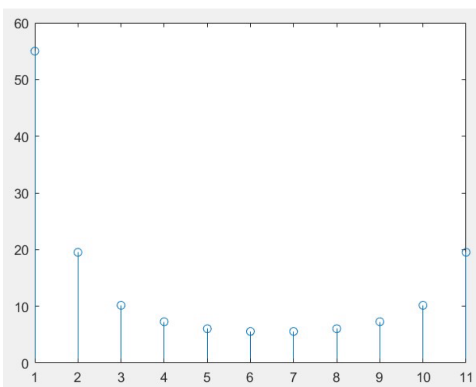
(c) $x[n]=\begin{cases}1, & n=0\\ 1/n, & n=1,2,\ldots,10\\ 0, & \text{otherwise}\end{cases}$

(d) $x[n] = n(0.8)^n,\quad 0 \le n \le 10$

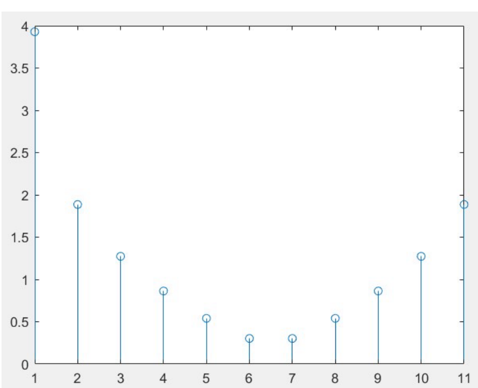
(a)



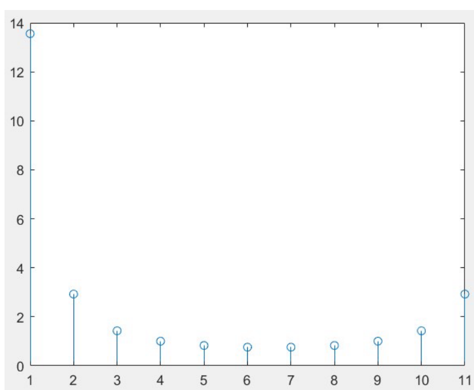
(b)



(c)



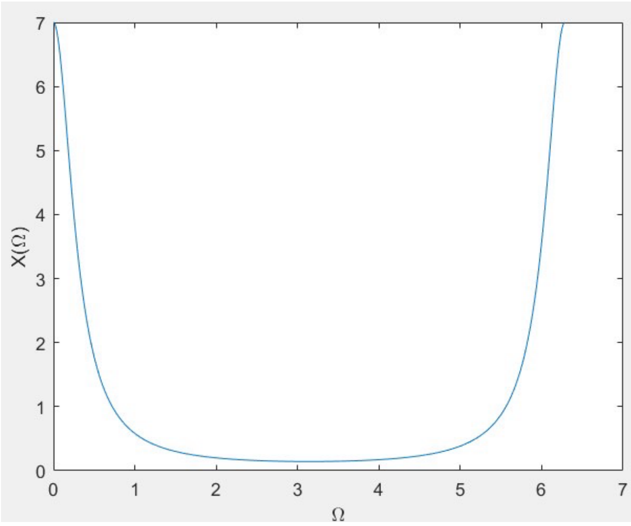
(d)



6.29 In Example 6.3, the DTFT of the signal $x[n] = a^{|n|}$ is

$$X(\Omega) = \frac{1-a^2}{1-2a\cos\Omega + a^2}$$

For $a = 0.75$ and $0 < \Omega < 2\pi$, plot $|X(\Omega)|$.



6.30 Use MATLAB to find the DFT of the discrete signal

$$x[n] = \{1, 2, 0, -1, -2, 1, 5, 4\}$$

```
>> HW5_30
  Columns 1 through 3

  10.0000 + 0.0000i   7.2426 + 7.8284i  -6.0000 + 0.0000i

  Columns 4 through 6

  -1.2426 - 2.1716i  -2.0000 + 0.0000i  -1.2426 + 2.1716i

  Columns 7 through 8

  -6.0000 + 0.0000i   7.2426 - 7.8284i
```