訊號與系統
SIGNAL AND SYSTEM

Lecture 2

梁　勝　富
成功大學 資訊工程系
sfliang@ncku.edu.tw
Office:資訊系館 12F 65C06, Tel: Ext. 62549

Lab:神經運算與腦機介面實驗室
(3F 65301, Tel: 62530-2301)
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2.2 IMPULSE RESPONSE

The impulse response to an LTI (linear, time-invariant) system is the output of the system to a unit impulse function.

Assume $$y(t) = \mathbf{T}\, x(t)$$

$\Longrightarrow$ $$h(t) = \mathbf{T}\, \delta(t)$$

where $\mathbf{T}$ is an operator transforming $x(t)$ into $y(t)$.

$$y(t) = \mathbf{T}x(t) = \mathbf{T}\underbrace{\left\{ \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau \right\}}_{x(t)}$$

$$= \int_{-\infty}^{\infty} x(\tau)\mathbf{T}\{\delta(t-\tau)\}\,d\tau = \underbrace{\int_{-\infty}^{\infty} x(\tau)\boxed{h(t-\tau)}\,d\tau}_{\text{convolution integral}}$$

Recall

- Time-Invariant Systems

-In a time-invariant system, a time shift (advance or delay) in the input signal leads to the time shift in the output signal.

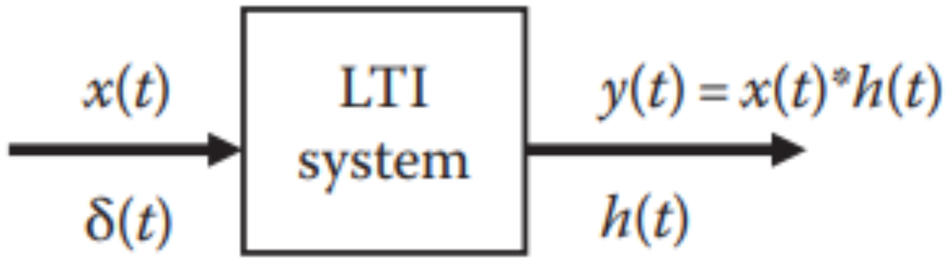
For a continuous-time system, the system is time-invariant when

$$T\{x(t-\tau)\} = y(t-\tau)$$

2.3 CONVOLUTION INTEGRAL

The convolution of two signals $x(t)$ and $h(t)$ is usually written in terms of the operator * as

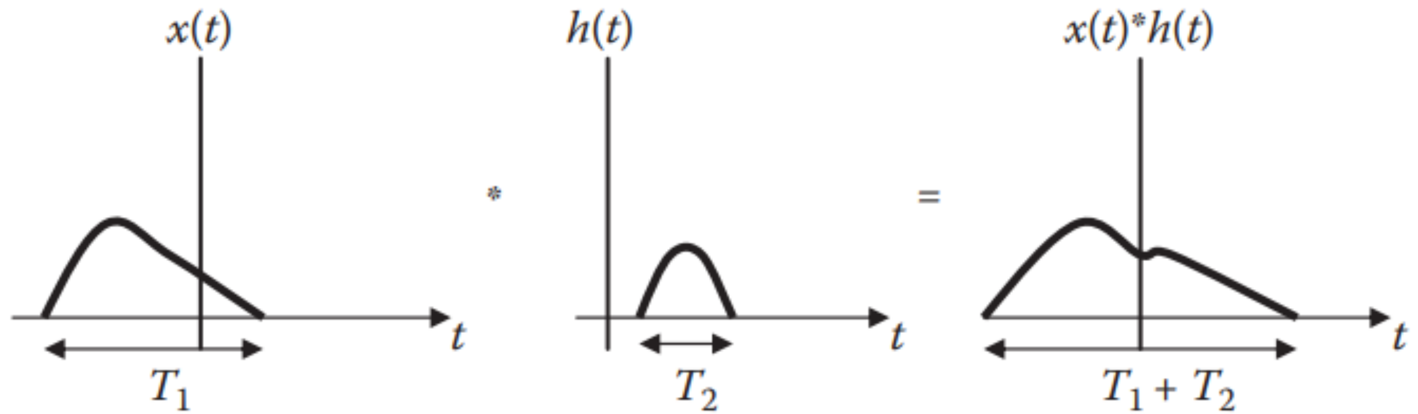
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$



Assume $x(t) = 0$ for $t < 0$, and the system is causal, $h(t) = 0$ for $t < 0$

$$y(t) = x(t) * h(t) = \int_{0}^{t} x(\tau)h(t-\tau)d\tau$$

- If the durations of $x(t)$ and $h(t)$ are T_1 and T_2, respectively, then the duration of $y(t) = x(t)*h(t)$ is $T_1 + T_2$ 週期相加
- If the areas under $x(t)$ and $h(t)$ are A_1 and A_2, respectively, then the area under $y(t) = x(t)*h(t)$ is A_1A_2. 面積相乘



<proof>

Let $y(t)$ be the convolution of $x(t)$ with $h(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

The area under $y(t)$ is

$$\int_{-\infty}^{\infty} y(t) dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau dt$$

$$\boxed{\int_{-\infty}^{\infty} y(t) dt = \underbrace{\int_{-\infty}^{\infty} x(\tau)}_{\text{Area of x}} \underbrace{\int_{-\infty}^{\infty} h(t-\tau) dt}_{\text{Area of h}} \boxed{d\tau}}$$

https://web.eecs.utk.edu/~mjr/WebAppendices/D-ConvProperties.PDF

If

$$x_1(t) * x_2(t) = y(t)$$

then

$$x_1(t+t_1) * x_2(t) = y(t+t_1)$$

$$x_1(t+t_1) * x_2(t+t_2) = y(t+t_1+t_2)$$ ⇒ shift相加

$$y(at) = a x_1(at) * x_2(at), \quad a > 0 \text{ (time scaling)}$$ ⇒ 週期a倍

$$y(-t) = x_1(-t) * x_2(-t) \quad \text{(time reversal)}$$

<proof>

$$y(at) = ax_1(at) * x_2(at), \qquad a > 0 \text{ (time scaling)}$$

Let $y(t) = x(t) * h(t)$ and $z(t) = x(at) * h(at)$, $a > 0$. Then

$$z(t) = \int_{-\infty}^{\infty} x(a\tau) h(a(t-\tau)) d\tau .$$

Making the change of variable, $\lambda = a\tau \Rightarrow d\tau = d\lambda / a$, for $a > 0$ we get

$$z(t) = \frac{1}{a} \int_{-\infty}^{\infty} x(\lambda) h(at - \lambda) d\lambda .$$

Since

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

it follows that $z(t) = (1/a) y(at)$ and $(1/a) y(at) = x(at) * h(at)$.

Quick guess

$$\text{Area } y(t) = \int_{-\infty}^{\infty} y(t)\, dt$$

$$\text{Area } y(at) = \int_{-\infty}^{\infty} y(at)\, dt$$

$$\text{let } \tau = at \quad d\tau = a\, dt$$

變數變換

$$\Rightarrow \text{Area } y(\tau) = \int_{-\infty}^{\infty} y(\tau) \cdot \frac{1}{a}\, d\tau$$

$$= \frac{1}{a} y(t)_{area} \Rightarrow y(at)_{area} = \frac{1}{a} y(t)_{area}$$

$$\text{Area } x_1(at) * x_2(at)$$

$$= \frac{1}{a} x_1(t)_{area} \times \frac{1}{a} x_2(t)_{area}$$

$$= \frac{1}{a^2} x_1(t)_{area} \times x_2(t)_{area}$$

差a倍

$$\Rightarrow y(at)_{area} = a\, x_1(t) * x_2(t)_{area}$$

Example 2.2

The input $x(t)$ and the impulse response $h(t)$ of an LTI system are given by $x(t) = u(t)$ and $h(t) = e^{-3t}u(t)$. Find the output response.

↳ 0以下無值

Solution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} u(\tau)e^{-3(t-\tau)}u(t-\tau)d\tau$$

$$= e^{-3t}\int_{0}^{t} e^{3\tau}d\tau = \frac{e^{-3t}}{3}e^{3\tau}\Big|_{0}^{t}$$

$$= \frac{1}{3}(1-e^{-3t}), \quad t > 0$$

$$u(\tau) = \begin{cases} 1, & \tau > 0 \\ 0, & \tau < 0 \end{cases}$$

$$u(t-\tau) = \begin{cases} 1, & t-\tau > 0 \\ 0, & t-\tau < 0 \end{cases} = \begin{cases} 1, & \tau < t \\ 0, & \tau > t \end{cases}$$

$$\Rightarrow \; u(\tau)u(t-\tau) = \begin{cases} 1, & \boxed{0 < \tau < t} \\ 0, & \text{otherwise} \end{cases}$$

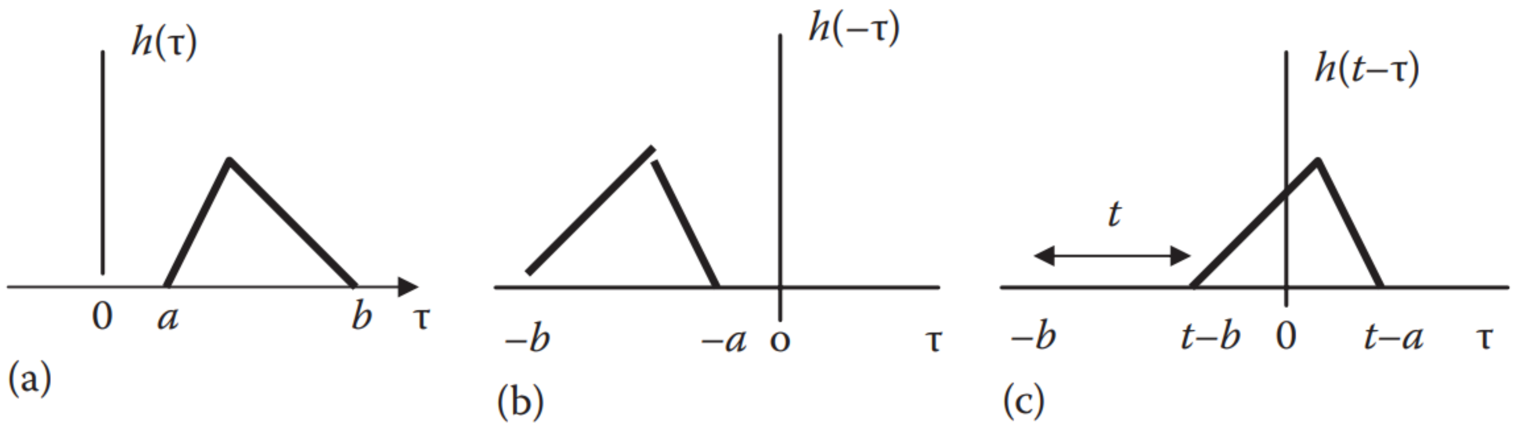
2.4 GRAPHICAL CONVOLUTION

Graphical method usually involves four steps:

1. *Folding*: Take the mirror image of $h(\tau)$ about the ordinate (or vertical) axis to obtain $h(-\tau)$
2. *Shifting*: Displace or shift $h(-\tau)$ by t to obtain $h(t-\tau)$ $-(\tau-t)$
   ↳ 右移t (直接背)
3. *Multiplication*: Multiply $h(t-\tau)$ and $x(\tau)$ together
4. *Integration*: For a given t, integrate the product $h(t-\tau)x(\tau)$ over $0 < \tau < t$ to get $y(t)$ at t

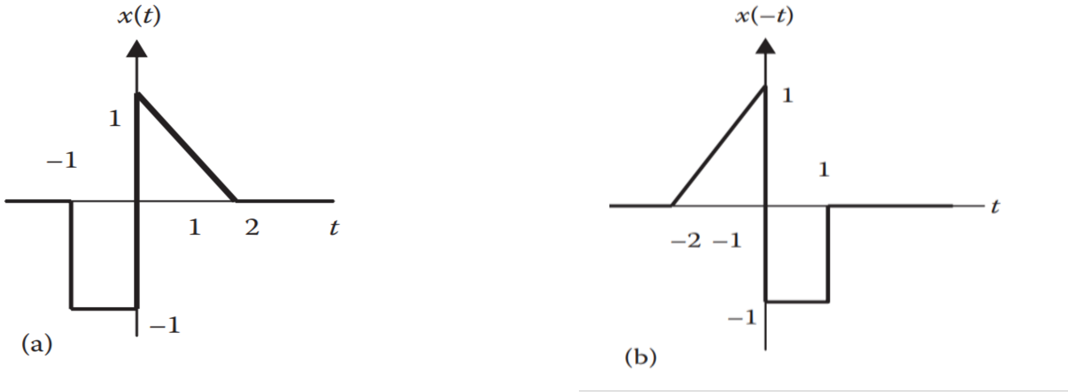
The folding operation in step 1 is the reason for the term *convolution*. The function $h(t-\tau)$ scans or slides over $x(\tau)$.

$$h'(\tau) = h(-\tau) \quad h'(\tau - t) = h(t-\tau)$$

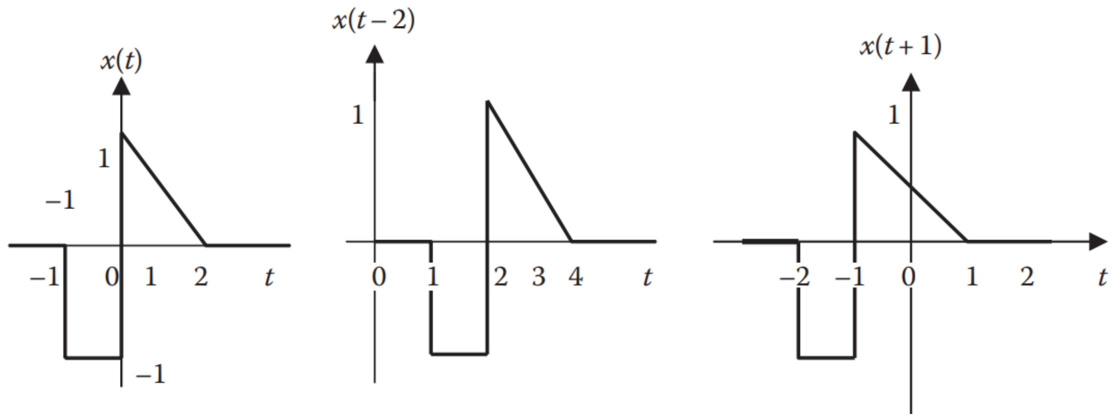


Recall

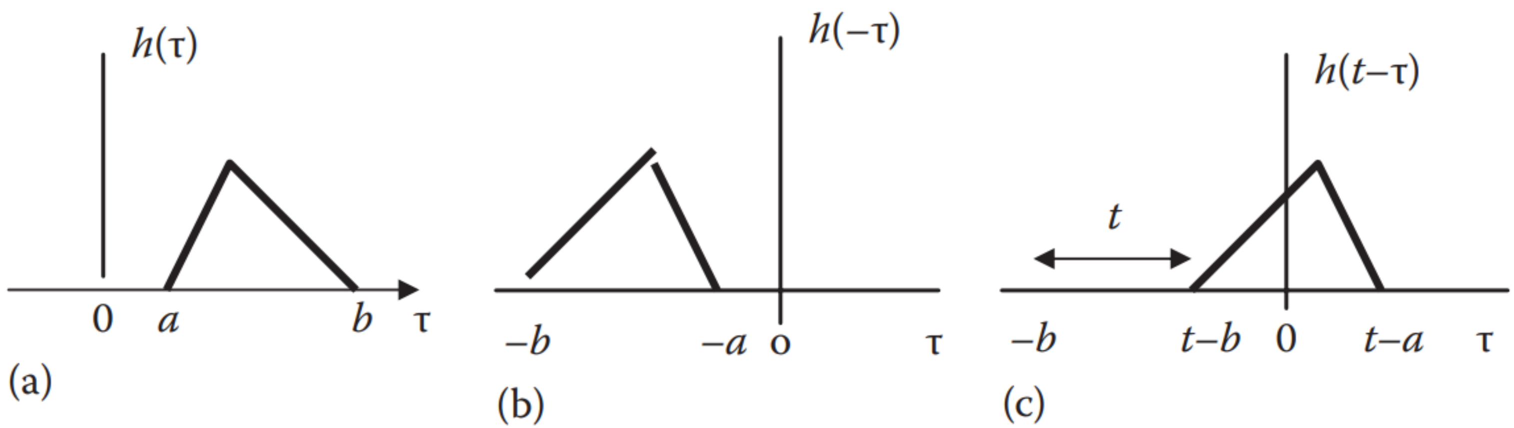
- Time Reversal



- Time Shifting

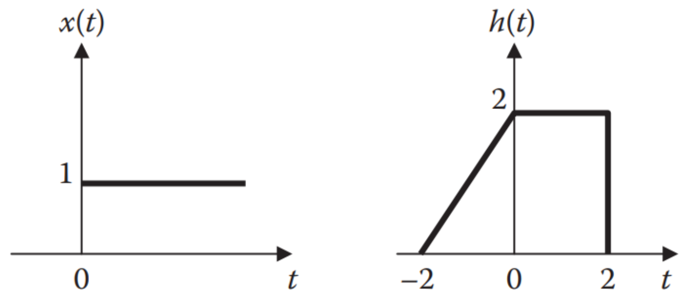


$$h'(\tau) = h(-\tau) \qquad h(t-\tau) = h(-(\tau - t)) = h'(\tau - t)$$



Example 2.3

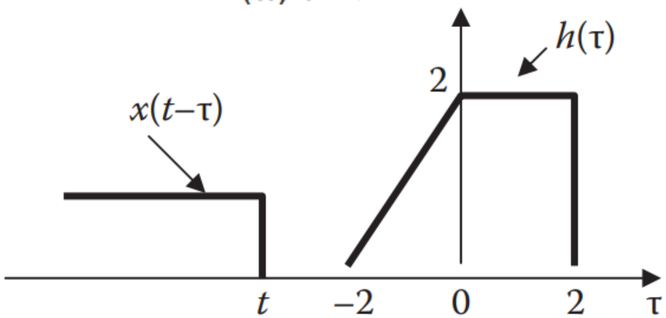
Obtain $x(t)*h(t)$



Solution

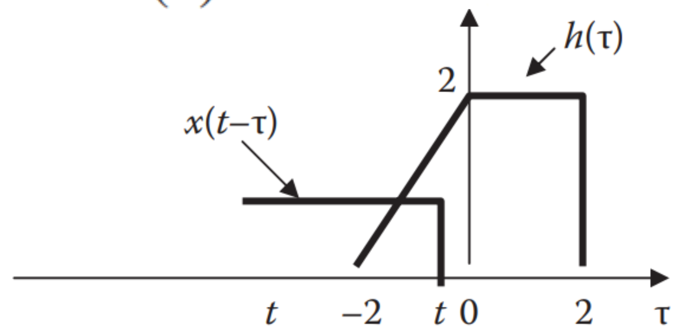
圖解

(a) $t < -2$



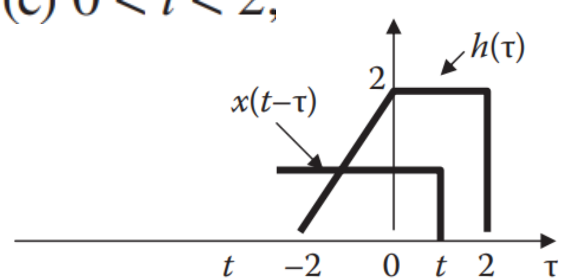
(a)

(b) $-2 < t < 0$



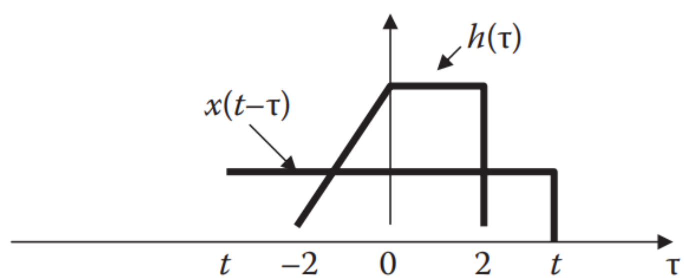
(b)

(c) $0 < t < 2$,



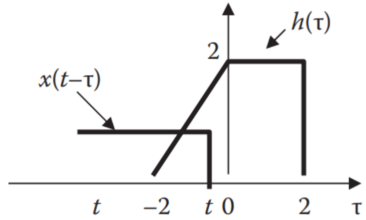
(c)

(d) $t > 2$.



(d)

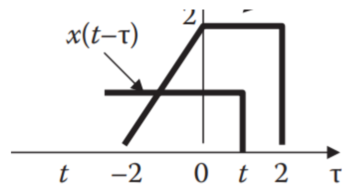
For $-2 < t < 0$



$$y(t) = \int_{-2}^{t} h(\tau)x(t-\tau)d\tau \; \boxed{= \int_{-2}^{t} (2+\tau)(1)d\tau} = 2\tau + \frac{\tau^2}{2}\Bigg|_{-2}^{t}$$

$$= 0.5t^2 + 2t + 2, \quad -2 < t < 0$$

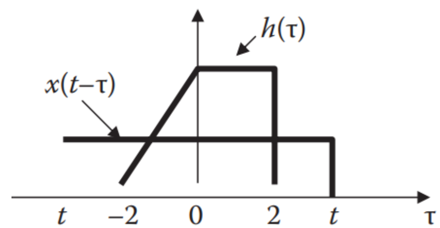
For $0 < t < 2$



$$y(t) = \int_{-2}^{0} (2+\tau)(1)d\tau + \int_{0}^{t} (2)(1)d\tau$$

$$= \left(2\tau + \frac{\tau^2}{2}\right)\Bigg|_{-2}^{0} + 2\tau\Big|_{0}^{t} = 4 - 2 + 2t = 2(t+1), \quad 0 < t < 2$$

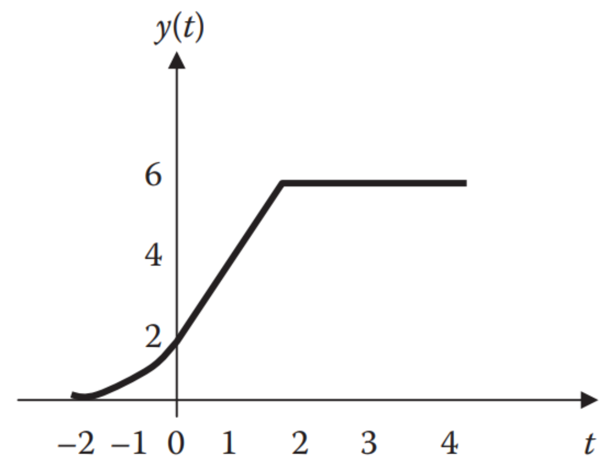
For $2 < t$



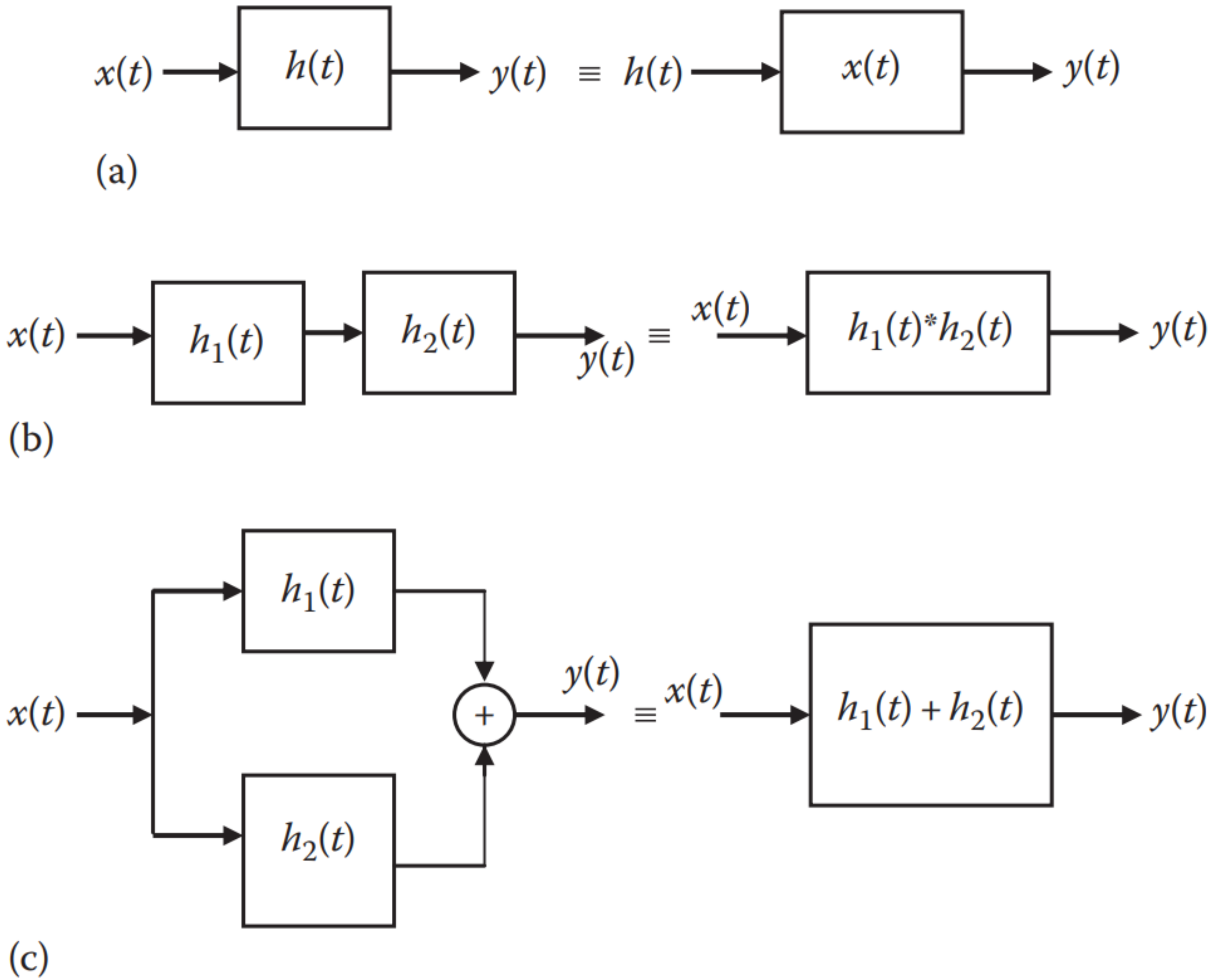
$$y(t) = \int_{-2}^{0} (2+\tau)(1)d\tau + \int_{0}^{2} (2)(1)d\tau$$

$$= \left(2\tau + \frac{\tau^2}{2}\right)\Bigg|_{-2}^{0} + 2\tau\Big|_{0}^{2} = 4 - 2 + 4 = 6, \quad t > 2$$

$$y(t) = \begin{cases} 0.5t^2 + 2t + 2, & -2 \le t \le 0 \\ 2(t+1), & 0 \le t \le 2 \\ 6, & t > 2 \\ 0, & \text{otherwise} \end{cases}$$

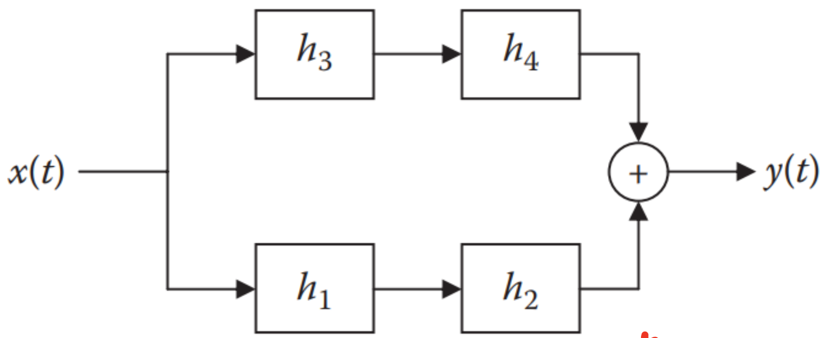


2.5 BLOCK DIAGRAM REPRESENTATION



Example 2.5

Find the impulse response of the system



$$h_1(t) = 3\delta(t)$$

$$h_2(t) = 2e^{-t}u(t)$$

$$h_3(t) = 4e^{-2t}u(t)$$

$$h_4(t) = e^{-3t}u(t)$$

Solution

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$$h(t) = h_1(t) * h_2(t) + h_3(t) * h_4(t)$$

$$h_1(t) * h_2(t) = 3\delta(t) * \left\{2e^{-t}u(t)\right\} = 6e^{-t}u(t)$$

$$h_3(t) * h_4(t) = \int_0^t 4e^{-2\tau}e^{-3(t-\tau)}d\tau = 4e^{-3t}\int_0^t e^{(3-2)\tau}d\tau$$

$$= 4e^{-3t}e^{\tau}\Big|_0^t = 4(e^{-2t} - e^{-3t}), \quad t > 0$$

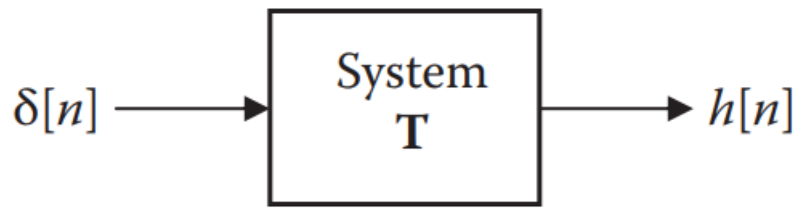
$$= 4(e^{-2t} - e^{-3t})u(t)$$

$$\Rightarrow \quad h(t) = 6e^{-t}u(t) + 4(e^{-2t} - e^{-3t})u(t)$$

2.6 DISCRETE-TIME CONVOLUTION

The impulse response $h[n]$ of a discrete-time LTI system is the response of the system when the input is $\delta[t]$.

$$\boxed{h[n] = T\delta[n]}$$



$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$y[n] = Tx[n] = T\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, T\delta[n-k] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

convolution sum

If both $x[n]$ and $h[n]$ are causal, $x[n]$ and $h[n]$ are zero for all integers n < 0,

$$y[n] = \sum_{k=0}^{n} h[k]x[n-k], \quad n \geq 0$$ 常用

The convolution of an M-point sequence with an N-point sequence produces an $(M + N - 1)$-point sequence.

The convolution sum requires the following steps:

1. The signal $h[k]$ is time-reversed to get $h[-k]$ and then shifted by n to form $h[n-k]$ or $h[-(k-n)]$, which should be regarded as a function of k with parameter n.
2. For a fixed value of n, multiply $x[k]$ and $h[n-k]$ for all values of k.
3. The product $x[k]h[n-k]$ is summed over all k to produce a single value of $y[n]$.
4. Repeat steps 1–3 for various values of n to produce the entire output $y[n]$.

Example 2.6

Let $r[n]$ be the convolution of two unit step sequences. Find $r[n]$

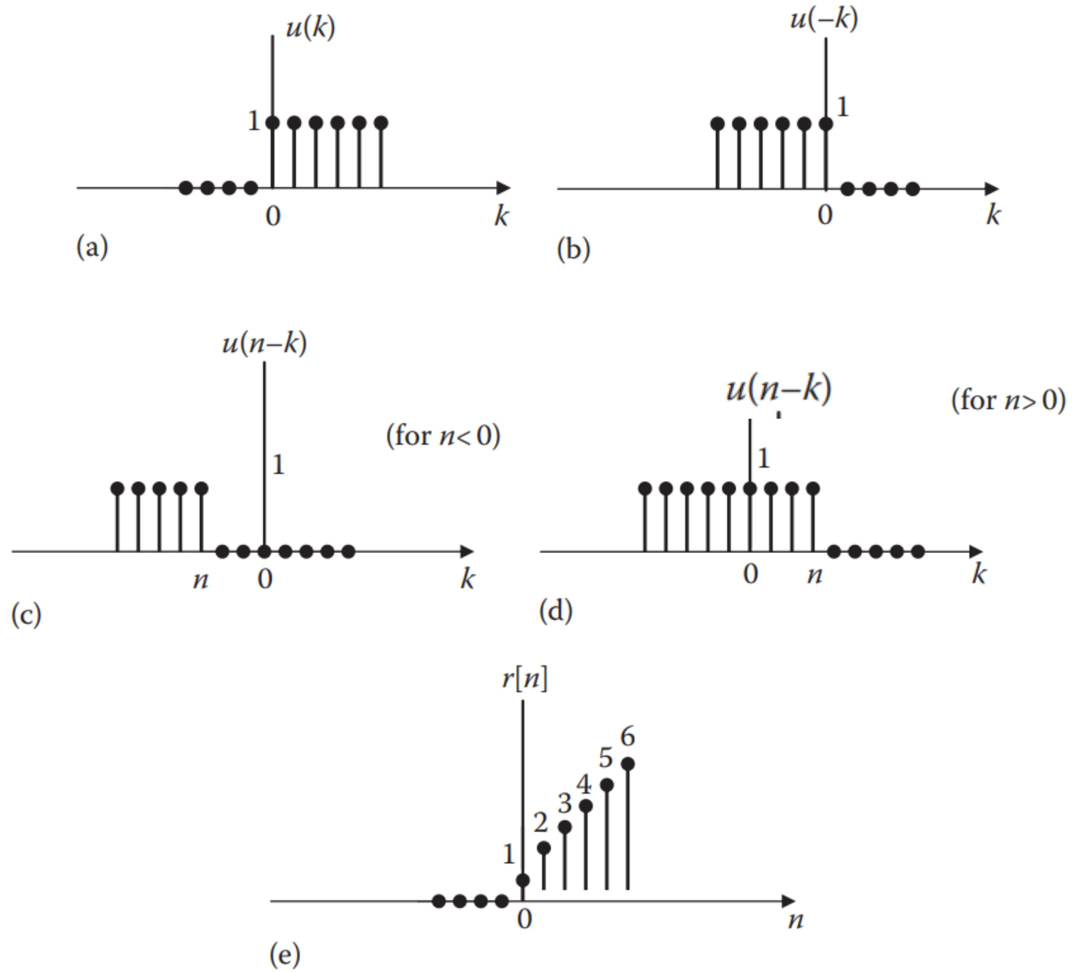
$$r[n] = u[n] * u[n]$$

Solution

$$r[n] = u[n] * u[n] = \sum_{k=-\infty}^{\infty} u[k]u[n-k]$$

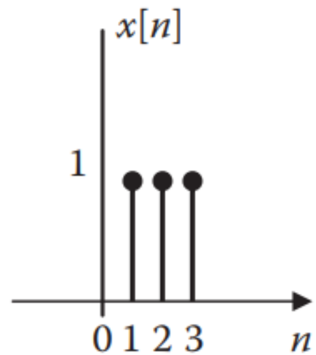
$$= \sum_{k=0}^{n} u[k]u[n-k] = \sum_{k=0}^{n} (1) = n+1$$

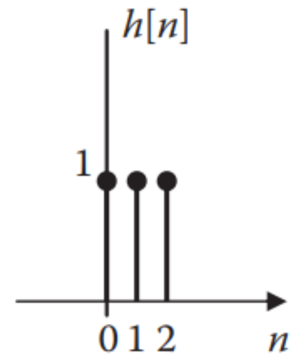
$$= (n+1)u[n]$$



Example 2.7

Find $y[n] = x[n]*h[n]$





Solution

(1) (a) Analytically

$$x[n] * \delta[n-1] = \sum_{k=-\infty}^{\infty} x[k]\delta[(n-1)-k] = x[n-1]$$

$$x[n] = \delta[n-1] + \delta[n-2] + \delta[n-3]$$

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

$$\boxed{y[n] = x[n] * h[n] = x[n] * \{\delta[n] + \delta[n-1] + \delta[n-2]\}}$$

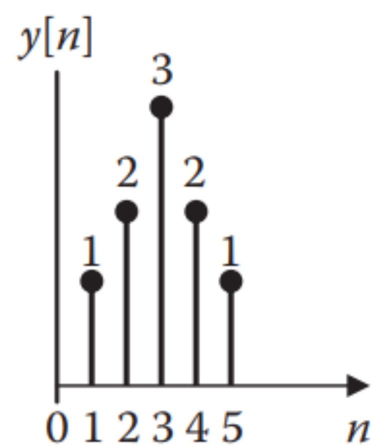
$$= x[n] + x[n-1] + x[n-2]$$

$$= \delta[n-1] + \delta[n-2] + \delta[n-3]$$

$$+\delta[n-2] + \delta[n-3] + \delta[n-4]$$

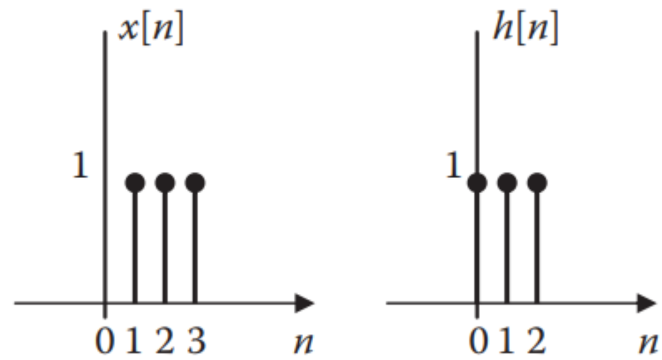
$$+\delta[n-3] + \delta[n-4] + \delta[n-5]$$

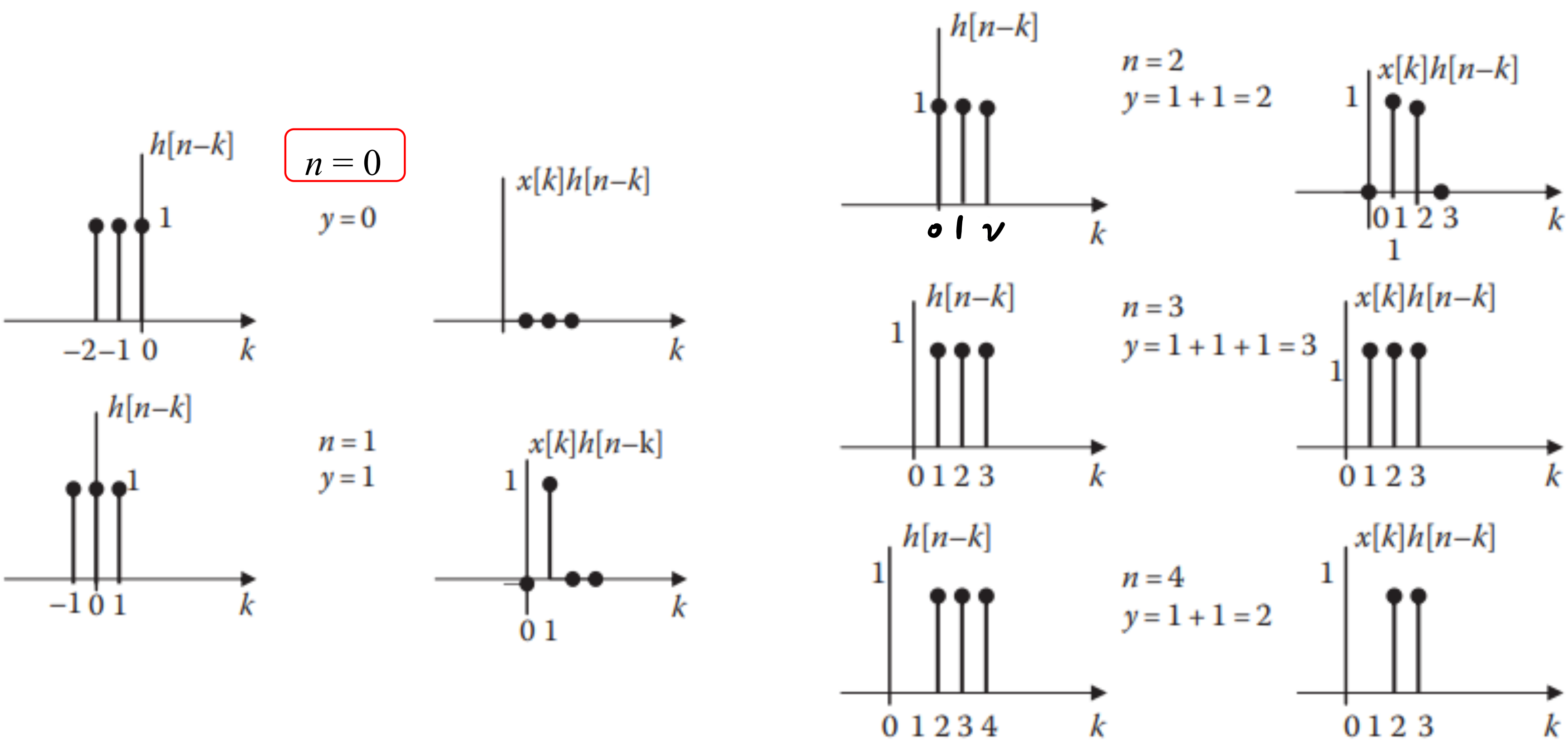
$$= \delta[n-1] + 2\delta[n-2] + 3\delta[n-3] + 2\delta[n-4] + \delta[n-5]$$



Solution

(b) graphically





$$n = 5, y = 1$$

Matlab implementation

$$y = conv(x, h)$$

- The length of y = length(x)+length(h)-1.

Convolution

- Ex. x=[1 2 3]; h= [2 3 4], y=?

```
x=[1 2 3];
h=[2 3 4];
y=conv(x,h);
```

⇨

y= [2 7 16 17 12];

length = 3+3-1 = 5

Y(0)
    1 2 3
4 3 2

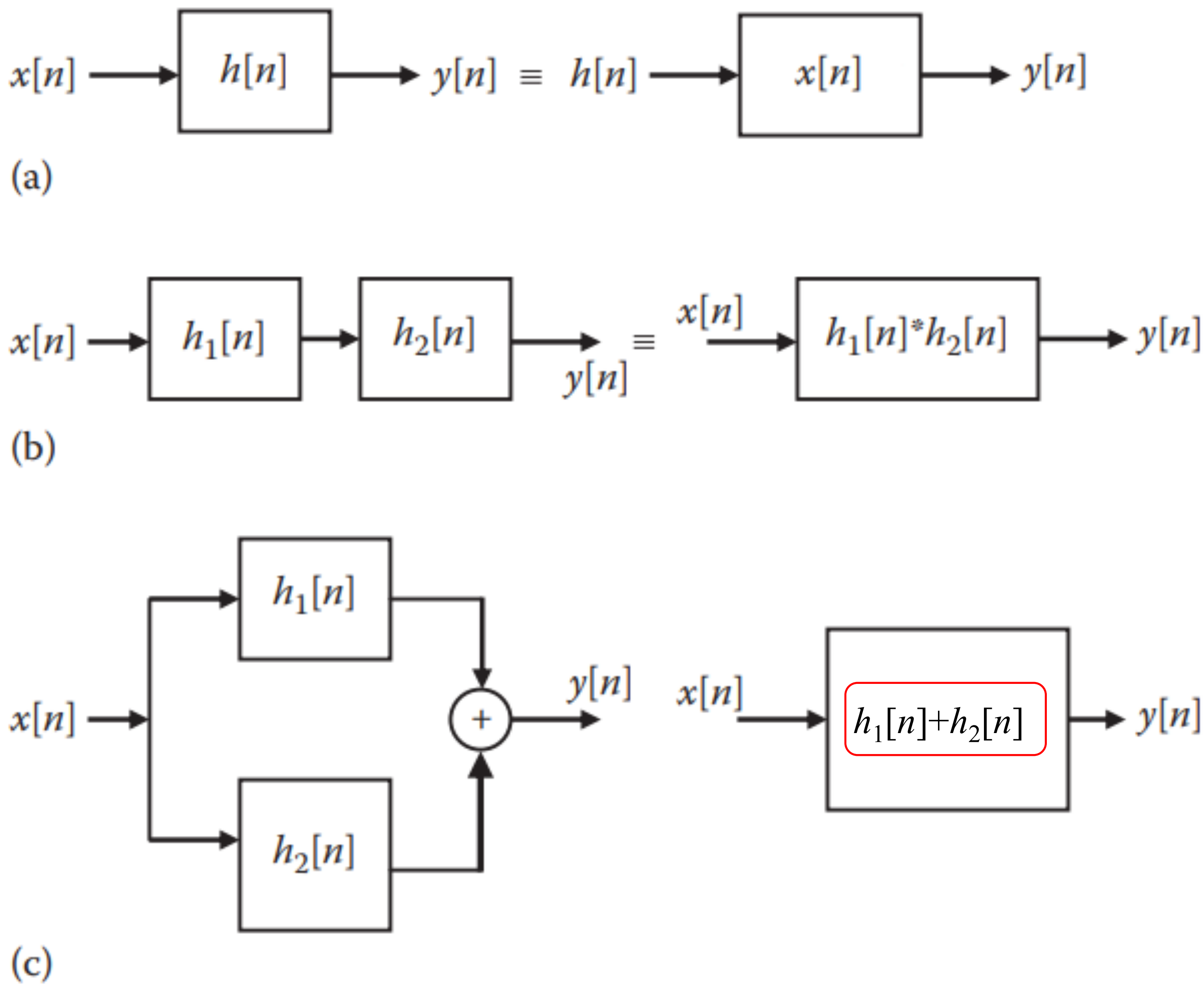
Y(1)
  1 2 3
4 3 2

Y(2)
1 2 3
4 3 2

Y(3)
1 2 3
  4 3 2

Y(4)
1 2 3
    4 3 2

2.7 BLOCK DIAGRAM REALIZATION



串聯 ⇒ conv

並聯 ⇒ +

Example 2.9

Let $x[n] = \overset{\text{input}}{\{3, 0, 2, 6\}}$ and $y[n] = \overset{\text{output}}{\{\boxed{6, 12, 25, 20, 38, 42}\}}$. Find $h[n]$

Solution

Method 1 (*Long division*):

We regard the given sequences $x[n]$ and $y[n]$ as coefficients of the following polynomials in descending order.

(1)長除法 ⇒ linear, time invariant才能用

$$x(z) = 3z^3 + 0z^2 + 2z + 6, \qquad y(z) = 6z^5 + 12z^4 + 25z^3 + 20z^2 + 38z + 42$$

$$\begin{array}{r} 2z^2 + 4z + 7 \\ 3z^3 + 0z^2 + 2z + 6 \overline{\big)\, 6z^5 + 12z^4 + 25z^3 + 20z^2 + 38z + 42} \\ \underline{6z^5 + 0z^4 + 4z^3 + 12z^2} \\ 12z^4 + 21z^3 + 8z^2 + 38z + 42 \\ \underline{12z^4 + 0z^3 + 8z^2 + 24z} \\ 21z^3 + 0z^2 + 14z + 42 \\ \underline{21z^3 + 0z^2 + 14z + 42} \\ 0 \end{array}$$

$h(z) = 2z^2 + 4z + 7$ or $h[n] = \{2, 4, 7\}$.

Why?

$$
\begin{array}{r}
2z^2 + 4z + 7 \\
3z^3 + 0z^2 + 2z + 6 \overline{\smash{\big)}\, 6z^5 + 12z^4 + 25z^3 + 20z^2 + 38z + 42} \\
\underline{6z^5 + 0z^4 + 4z^3 + 12z^2} \\
12z^4 + 21z^3 + 8z^2 + 38z + 42 \\
\underline{12z^4 + 0z^3 + 8z^2 + 24z} \\
21z^3 + 0z^2 + 14z + 42 \\
\underline{21z^3 + 0z^2 + 14z + 42} \\
0
\end{array}
$$

$$
\begin{aligned}
y[0] &= h[0]\,x[0] \\
y[1] &= h[0]\,x[1] + h[1]\,x[0] \\
y[2] &= h[0]\,x[2] + h[1]\,x[1] + h[2]\,x[0] \\
y[3] &= h[0]\,x[3] + h[1]\,x[2] + h[2]\,x[1] \\
y[4] &= h[1]\,x[3] + h[2]\,x[2] \\
y[5] &= h[2]\,x[3]
\end{aligned}
$$

Solution $x[n] = \{3, 0, 2, 6\}$ and $y[n] = \{6, 12, 25, 20, 38, 42\}$

(II) 麻煩

Method 2 (recursive algorithm):

By definition,

$$y[n] = x[n] * h[n] = \sum_{k=0}^{n} h[k]x[n-k]$$

$$\begin{aligned}
y[\overset{n}{0}] &= h[\overset{k}{0}]\, x[\overset{n-k}{0}] \\
y[1] &= h[0]\, x[1] + h[1]\, x[0] \\
y[2] &= h[0]\, x[2] + h[1]\, x[1] + h[2]\, x[0] \\
y[3] &= h[0]\, x[3] + h[1]\, x[2] + h[2]\, x[1] \\
y[4] &= h[1]\, x[3] + h[2]\, x[2] \\
y[5] &= h[2]\, x[3]
\end{aligned}$$

$$y[n] = h[n]x[0] + \sum_{k=0}^{n-1} h[k]x[n-k] \qquad h[n] = \frac{1}{x[0]}\left[y[n] - \sum_{k=0}^{n-1} h[k]x[n-k]\right]$$

$$y[0] = x[0]h[0] \rightarrow h[0] = y[0]/x[0] = 6/3 = 2$$

$$h[1] = \frac{1}{x[0]}\left[y[1] - \sum_{k=0}^{0} h[k]x[1-k]\right] = \frac{1}{x[0]}\left[y[1] - h[0]x[1]\right] = \frac{1}{3}[12 - 2\times 0] = 4$$

$$h[2] = \frac{1}{x[0]}\left[y[2] - \sum_{k=0}^{1} h[k]x[2-k]\right] = \frac{1}{x[0]}\left[y[2] - h[0]x[2] - h[1]x[1]\right]$$

$$= \frac{1}{3}[25 - 2\times 2 - 4\times 0] = 7$$

```
x = [3 0 2 6 ]
y = [6 12 26 20 38 42]
h = deconv(y, x)
```

Example 2.10

Use MATLAB to find the convolution of the sequences:

$$x_1[n] = \{0.2, 1.4, 2.6, 5.1, 3.4, 8.4\}$$

$$x_2[n] = \{1.0, 4.2, 3.7, 0.8, 3.9\}$$

Solution

$$x_1 = [0.2,\ 1.4,\ 2.6,\ 5.1,\ 3.4,\ 8.4]$$

$$x_2 = [1.0,\ 4.2,\ 3.7,\ 0.8,\ 3.9]$$

$$y = \text{conv}(x_1, x_2)$$

$$y = 0.2000\ 2.2400\ 9.2200\ 21.3600\ 36.3400\ 49.0900\ 62.0800\ 53.6900\ 19.9800\ 32.7600$$

Example 2.11

A system is represented by its impulse response:

$$h(t) = \frac{1}{4}\left(e^{-2t} - e^{-t}\right)$$

Find and plot the response when the input is $x(t) = \cos(t)\, u(t)$.

Solution

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau$$

Let T, the step size or sampling period, be small and let $t = kT$ and $\tau = nT$, the convolution integral becomes a convolution summation which can be expressed as

$$y(kT) \approx T\sum_{n=0}^{kT} x(nT)h((k-n)T) = T\sum_{n=0}^{k} x[n]h[k-n] \qquad (2.29)$$

This approximates a rectangular rule integration. Equation 2.29 can be written as

$$y(k) \approx T\sum_{n=0}^{k} x[n]h[k-n] \qquad (2.30)$$

To use conv function (discrete-time convolution) for continuous-time convolution

```
T = 0.1; % sampling period
t = 0:T:10;
x = cos(t); % calculates x(t)
h = 0.25*(exp(-2*t) - exp(-t)); % calculates h(t)
y = T*conv(x,h); %this contains Lx + Lh - 1
t0 = (0:200)*T
plot(t0,y) % or use this plot(t,y(1:101))
xlabel('Time (s)')
ylabel('Response y(t)')
```