訊號與系統
SIGNAL AND SYSTEM

Lecture 3

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2024

Laplace Transform

- Laplace transform is **a frequency-domain** representation that makes analysis and design of **linear systems** simpler.
- Laplace transform is powerful for providing us in one single operation the complete response, that is, the steady-state plus transient.
- It allows us to convert **ordinary differential** equations into **algebraic** equations, which are easier to manipulate and solve.
- It converts convolution into a simple multiplication.
- We can apply Laplace transform to generate the **transfer function** representation of a continuous-time LTI system.

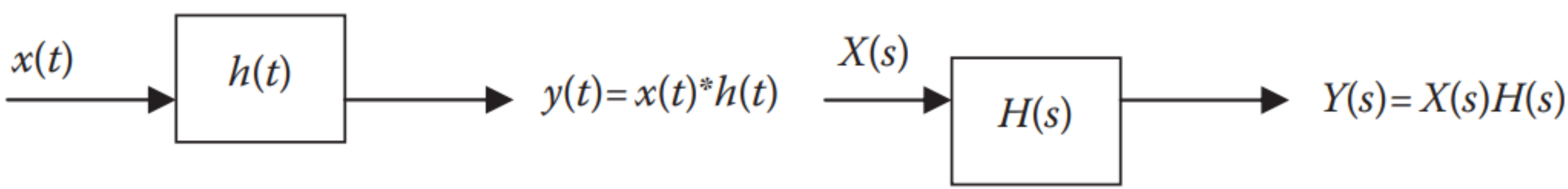
TRANSFER FUNCTION

The transfer function $H(s)$ is defined as the ratio of the output response $Y(s)$ to the input excitation $X(s)$, assuming all initial conditions are zero.

$$\boxed{H(s) = \frac{Y(s)}{X(s)}}$$

$$y(t) = x(t) * h(t)$$

$$Y(s) = X(s)H(s)$$



APPLICATIONS

- Integro-Differential Equations

Use the Laplace transform to solve the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{-t}u(t) \qquad y(0) = 1, \quad \frac{dy(0)}{dt} = 0.$$

Solution

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 3\left[sY(s) - y(0)\right] - 4Y(s) = \frac{1}{s+1}$$

$$\left(s^2 + 3s - 4\right)Y(s) = s + 3 + \frac{1}{s+1} = \frac{s^2 + 4s + 4}{s+1}$$

$$Y(s) = \frac{s^2 + 4s + 4}{(s^2 + 3s - 4)(s+1)} = \frac{s^2 + 4s + 4}{(s-1)(s+1)(s+4)}$$

- Circuit Analysis

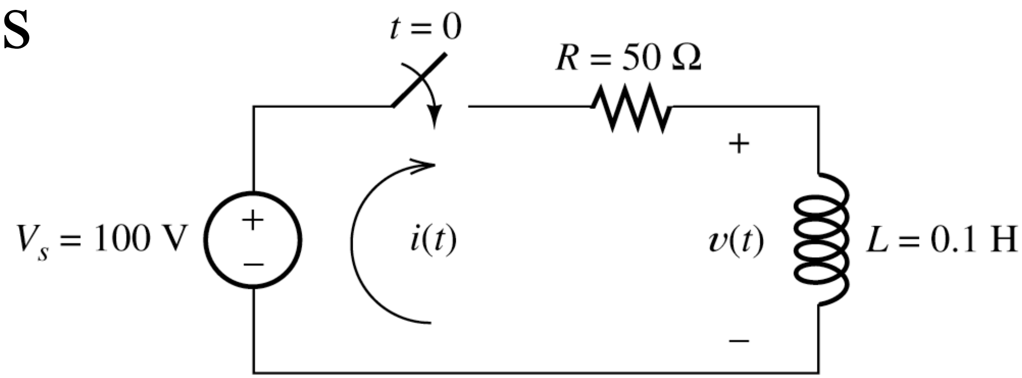


Figure 4.7 The circuit analyzed in Example 4.2.

$$Ri(t) + L\frac{di}{dt} = V_s$$

$$RI(s) + L[sI(s) - i(0)] = V_s(s) = \frac{V_s}{s}$$

$$I(s) = \frac{V_s}{s(R + Ls)} = \frac{V_s}{L}\frac{1}{s(s + R/L)}$$

$$= \frac{V_s}{L}(\frac{A}{s} + \frac{B}{s+R/L})$$

$$As + A\left({}^{R}/_{L}\right) + Bs = 1$$

$$B = -A \qquad A = {}^{L}/_{R}$$

$$I(s) = \frac{V_s}{L}\left(\frac{{}^{L}/_{R}}{s} + \frac{-{}^{L}/_{R}}{s + {}^{R}/_{L}}\right)$$

$$= \frac{V_s}{R}\left(\frac{1}{s} - \frac{1}{s + {}^{R}/_{L}}\right)$$

$$i(t) = 2 - 2e^{-({}^{R}/_{L})t}$$

3.2 DEFINITION OF LAPLACE TRANSFORM

The Laplace transform of a signal $x(t)$ is the integration of the product of x(t) and e^{-st} over the interval from 0 to +∞ (commonly)

Bilateral

$$\mathcal{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Unilateral (more commonly)

$$\mathcal{L}[x(t)] = X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

s is the complex frequency given by $s = \sigma + j\omega$

Inverse Laplace transform

$$\mathcal{L}^{-1}[X(s)] = x(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} X(s)e^{st}ds$$

where the integration is performed along a straight line ($\sigma_1 + j\omega,\ -\infty < \omega < \infty$)

- Laplace transformable
  The integral must converge.

A signal $x(t)$ is Laplace transformable if the integral exists

$$\left| \int_{0^-}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt \right| < \infty$$

$$\left| \int_{0^-}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt \right| \le \int_{0^-}^{\infty} \left| x(t) e^{-(\sigma + j\omega)t} dt \right| \le \int_{0^-}^{\infty} \left| x(t) \right| \left| e^{-(\sigma + j\omega)t} dt \right| < \infty$$

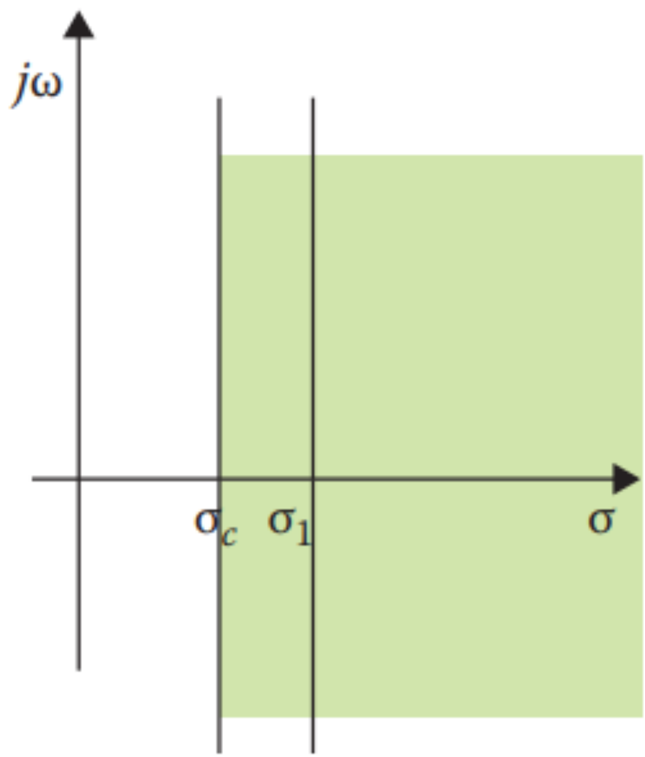
Since $|e^{j\omega t}| = 1$ for any value of t,

$$\int_{0^-}^{\infty} \left| x(t) \right| e^{-\sigma t} dt < \infty$$ for some real value of $\sigma = \sigma_c$.

- Region of Convergence (ROC)

The range of *s* for which the Laplace transform converges.

The region of convergence (ROC) for Laplace transform is $\mathrm{Re}(s) = \sigma > \sigma_c$



Example 3.1

Find the Laplace transform of the following functions and establish the ROC for each case.

(a) $e^{-5t}u(t)$
(b) $\delta(t)$

$$\boxed{\mathcal{L}[e^{-at}u(t)] = \frac{1}{s+a}}$$

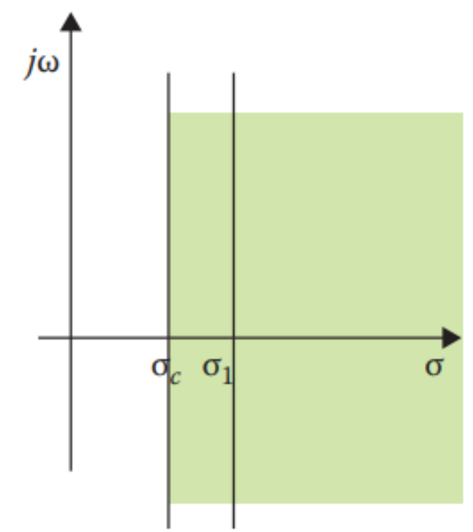
Solution

(a)

$$\mathcal{L}[e^{-5t}u(t)] = \int_{0^-}^{\infty} e^{-5t}e^{-st}dt = -\frac{1}{s+5}e^{-(s+5)t}\Big|_{0^-}^{\infty} = \frac{1}{s+5}$$

The ROC is obtained from

$$\left|e^{-(s+5)t}\right| = \left|e^{-(\sigma+5)t}\right|\left|e^{-j\omega t}\right| = \left|e^{-(\sigma+5)t}\right| < \infty$$



which is valid when σ + 5 > 0 or σ > −5= σ_c.

Example 3.1

Solution

(b)

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = e^{-0} = 1 \quad \text{for all } s$$

The transform does not depend on *s* and hence the region of convergence is the entire *s* plane.

Example 3.2

Find the Laplace transform of $x(t)$ = sin ωt $u(t)$.

Solution

$$X(s) = \mathcal{L}[\sin\omega t] = \int_0^\infty (\sin\omega t)e^{-st}dt = \int_0^\infty \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)e^{-st}dt$$

$$= \frac{1}{2j}\int_0^\infty \left(e^{-(s-j\omega)t} - e^{-(s+j\omega)t}\right)dt$$

$$= \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) = \frac{\omega}{s^2+\omega^2}$$

$$\boxed{\mathcal{L}[e^{-at}u(t)] = \frac{1}{s+a}}$$

3.3 PROPERTIES OF THE LAPLACE TRANSFORM

- Linearity

$$\boxed{\mathcal{L}\left[a_1x_1(t)+a_2x_2(t)\right]=a_1X_1(s)+a_2X_2(s)}$$

$$\mathcal{L}[a_1x_1(t)+a_2x_2(t)]=\int_0^{\infty}[a_1x_1(t)+a_2x_2(t)\,e^{-st}dt$$

$$=a_1\int_0^{\infty}x_1(t)e^{-st}dt+a_2\int_0^{\infty}x_2(t)\,e^{-st}dt$$

$$=a_1X_1(s)+a_2X_2(s)$$

- Scaling

$$\boxed{\mathcal{L}[x(at)] = \frac{1}{a} X\left(\frac{s}{a}\right), \quad a > 0}$$

Let $\lambda = at$, $d\lambda = a\,dt$,

$$\mathcal{L}[x(at)] = \int_0^\infty x(at) e^{-st} dt = \frac{1}{a}\int_0^\infty x(\lambda) e^{-\lambda\left(\frac{s}{a}\right)} d\lambda$$

$$= \frac{1}{a} X\left(\frac{s}{a}\right), \quad a > 0$$

- Time Shifting

$$\boxed{\mathcal{L}\left[x(t-a)u(t-a)\right] = e^{-\alpha s}X(s)}$$

$$\mathcal{L}[x(t-a)u(t-a)] = \int_0^\infty x(t-a)u(t-a)e^{-st}dt, \quad a \geq 0$$

$$= \int_a^\infty x(t-a)(1)e^{-st}dt$$

substituting $\lambda = t - a$, $d\lambda = dt$, and $t = \lambda + a$. As $t \to a$, $\lambda \to 0$, and as $t \to \infty$, $\lambda \to \infty$

$$\mathcal{L}[x(t-a)u(t-a)] = \int_0^\infty x(\lambda)e^{-s(\lambda+a)}d\lambda$$

$$= e^{-as}\int_0^\infty x(\lambda)e^{-s\lambda}d\lambda$$

$$= e^{-as}X(s)$$

- Frequency Shifting $\boxed{\mathcal{L}\left[e^{-at}x(t)u(t)\right] = X(s+a)}$

$$\mathcal{L}\left[e^{-at}x(t)u(t)\right] = \int_0^{\infty} e^{-at}x(t)e^{-st}dt = \int_0^{\infty} x(t)e^{-(s+a)t}dt = X(s+a)$$

- Time Differentiation $\boxed{\mathcal{L}\left[x'(t)\right] = sX(s) - x(0^-)}$

$$\frac{d}{dt}x(t)e^{-st} = \frac{dx}{dt}e^{-st} + x(t)(\text{-s } e^{-st})$$

$$\frac{dx}{dt}e^{-st} = \frac{d}{dt}x(t)e^{-st} - x(t)(\text{-s } e^{-st})$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{dt}e^{-st}dt = \int_0^{\infty} (\frac{d}{dt}x(t)e^{-st})dt - \int_0^{\infty} x(t)(-\text{s } e^{-st})dt$$

$$= x(t)e^{-st}\Big|_{0^-}^{\infty} - \int_0^{\infty} x(t)\left[-se^{-st}\right]dt \quad = 0 - x(0^-) + s\int_{0^-}^{\infty} x(t)e^{-st}dt \quad = sX(s) - x(0^-)$$

- Time Convolution $\boxed{\mathcal{L}[x(t)*h(t)] = X(s)H(s)}$

$$\mathcal{L}\left[x(t)*h(t)\right] = \int_0^\infty \left[\int_0^\infty h(\tau)x(t-\tau)d\tau\right]e^{-st}dt$$

$$\mathcal{L}\left[x(t)*h(t)\right] = \int_0^\infty h(\tau)\left[\int_0^\infty x(t-\tau)e^{-st}dt\right]d\tau$$

$$= \int_0^\infty h(\tau)\left[\int_0^\infty x(\lambda)\boxed{e^{-s(\tau+\lambda)}d}\lambda\right]d\tau$$

$$= \int_0^\infty h\left(\boxed{\tau)e^{-s\tau}d\tau}\right)\int_0^\infty x(\lambda)e^{-s\lambda}d\lambda = H(s)X(s)$$

+

- Laplace transform of $u(t)$?

$$\boxed{\mathcal{L}[e^{-at}u(t)] = \frac{1}{s+a}}$$

Let $a = 0$, we get $L[u(t)] = 1/s$

- Laplace transform of $r(t)$?

$r(t) = u(t)*u(\mathrm{t})$, $t>0$

$\Rightarrow$

$$L[t] = \frac{1}{s}\frac{1}{s} = \frac{1}{s^2}$$, $t>0$

$r(t)$ or $t \cdot u(t)$

↳确保0以下為0

- Time Integration

$$\mathcal{L}\left[\int_0^t x(t)\,dt\right] = \frac{1}{s}X(s)$$

$$\mathcal{L}\left[\int_0^t x(\lambda)d\lambda\right] = \mathcal{L}\left[\int_0^\infty x(\lambda)u(t-\lambda)d\lambda\right] = \mathcal{L}\left[u(t) * x(t)\right] = \frac{1}{s}X(s)$$

$$\mathcal{L}\left[\int_0^t x(t)dt\right] = \frac{1}{s}X(s)$$

- Frequency Differentiation

Let $X(s)$ be the Laplace transform of the signal $x(t)$. Then Laplace transform of the frequency differentiation is given as

$$\boxed{\mathcal{L}\left[tx(t)\right] = -\frac{dX(s)}{ds}}$$ 對s微分

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt$$

Differentiating both sides, we get

$$\frac{dX(s)}{ds} = \int_{0}^{\infty} x(t)(-te^{-st})dt = \int_{0^-}^{\infty} \left(-tx(t)\right)e^{-st}dt = \mathcal{L}\left[-tx(t)\right]$$

The frequency differentiation property is given as

$$\mathcal{L}\left[tx(t)\right] = -\frac{dX(s)}{ds}$$

$\Longrightarrow$ 推廣 $$\mathcal{L}\left[t^n x(t)\right] = (-1)^n \frac{d^n X(s)}{ds^n}$$

3.4 THE INVERSE LAPLACE TRANSFORM

Inverse Laplace transform

$$\mathcal{L}^{-1}[X(s)] = x(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} X(s)e^{st}\,ds$$

$$X(s) = \frac{N(s)}{D(s)}$$

$N(s)$ is the numerator polynomial
$D(s)$ is the denominator polynomial

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0}$$

$$X(s) = \frac{N(s)}{D(s)} = k\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

roots of $N(s) = 0$ are called the *zeros* of $X(s)$

roots of $D(s) = 0$ are the *poles* of $X(s)$,

where $k = b_m/a_n$.

- Simple Poles

$$X(s) = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$X(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \cdots + \frac{k_n}{s+p_n}$$

$$(s+p_1)X(s) = k_1 + \frac{(s+p_1)k_2}{s+p_2} + \cdots + \frac{(s+p_1)k_n}{s+p_n}$$

$$(s+p_1)X(s)\Big|_{s=-p_1} = k_1 \qquad \boxed{k_i = (s+p_i)X(s)\Big|_{s=-p_i}}$$

$$\mathcal{L}^{-1}[k/(s+a)] = ke^{-at}u(t) \qquad \boxed{x(t) = \left(k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \cdots + k_n e^{-p_n t}\right)u(t)}$$

- Repeated Poles

$$X(s) = \frac{k_n}{(s+p)^n} + \frac{k_{n-1}}{(s+p)^{n-1}} + \cdots + \frac{k_2}{(s+p)^2} + \frac{k_1}{s+p} + X_1(s)$$

recall

$$k_n = (s+p)^n X(s)\Big|_{s=-p}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^n}\right] = \frac{t^{n-1}e^{-at}}{(n-1)!}$$

$$\boxed{\mathcal{L}\left[tx(t)\right] = -\frac{dX(s)}{ds}}$$

$$k_{n-1} = \frac{d}{ds}\left[(s+p)^n X(s)\right]\Big|_{s=-p}$$

$$\boxed{x(t) = \left(k_1e^{-pt} + k_2te^{-pt} + \frac{k_3}{2!}t^3e^{-pt} + \cdots + \frac{k_n}{(n-1)!}t^{n-1}e^{-pt}\right)u(t) + x_1(t)}$$

$$k_{n-2} = \frac{1}{2!}\frac{d^2}{ds^2}\left[(s+p)^n X(s)\right]\Big|_{s=-p}$$

$$k_{n-m} = \frac{1}{m!}\frac{d^m}{ds^m}\left[(s+p)^n X(s)\right]\Big|_{s=-p}$$

- Complex Poles

$$X(s) = \frac{A_1 s + A_2}{s^2 + as + b} + X_1(s)$$

$$\boxed{\mathcal{L}\left[e^{-at}x(t)u(t)\right] = X(s+a)}$$

$$s^2 + as + b = s^2 + 2\alpha s + \alpha^2 + \beta^2 = (s+\alpha)^2 + \beta^2$$

$$u(t)\cos(\omega_o t) \Leftrightarrow \frac{s}{s^2 + \omega_o^2}$$

$$A_1 s + A_2 = A_1(s+\alpha) + B_1\beta$$

$$u(t)\sin(\omega_o t) \Leftrightarrow \frac{\omega_o}{s^2 + \omega_o^2}$$

$$X(s) = \frac{A_1(s+\alpha)}{(s+\alpha)^2 + \beta^2} + \frac{B_1\beta}{(s+\alpha)^2 + \beta^2} + X_1(s)$$

$$\boxed{x(t) = \left(A_1 e^{-\alpha t}\cos\beta t + B_1 e^{-\alpha t}\sin\beta t\right)u(t) + x_1(t)}$$

The sine and cosine terms can be combined if desired.

$$A\cos\theta + B\sin\theta = R\cos(\theta - \phi) \qquad R = \sqrt{A^2 + B^2}$$

$$\boxed{\cos\left(x \pm y\right) = \cos x \cos y \mp \sin x \sin y} \qquad \phi = \tan^{-1}(B/A)$$

Example 3.8

Find the inverse Laplace transform of

$$X(s) = 1 + \frac{2}{s} + \frac{4}{s-1} - \frac{3s}{s^2+9}$$

Solution

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}(1) + \mathcal{L}^{-1}\left(\frac{2}{s}\right) + \mathcal{L}^{-1}\left(\frac{4}{s-1}\right) - 3\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) \\ &= \delta(t) + (2 + 4e^t - 3\cos 3t)u(t) \end{aligned}$$

Example 3.9

Obtain $h(t)$ given that

$$H(s) = \frac{4}{(s+1)(s+3)}$$

Solution

Residue method

$$H(s) = \frac{4}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$$A = (s+1)H(s)\big|_{s=-1} = \left.\frac{4}{(s+3)}\right|_{s=-1} = \frac{4}{(2)} = 2$$

$$B = (s+3)H(s)\big|_{s=-3} = \left.\frac{4}{(s+1)}\right|_{s=-3} = \frac{4}{(-2)} = -2$$

Algebraic method

$$4 = A(s+3) + B(s+1)$$

$$H(s) = \frac{4}{(s+1)(s+3)} = \frac{2}{s+1} - \frac{2}{s+3}$$

$$h(t) = \left(2e^{-t} - 2e^{-3t}\right)u(t)$$

Example 3.10

Determine $x(t)$ given that

$$X(s) = \frac{2s^2 + 4s + 1}{(s+1)(s+2)^3}$$

Solution

$$X(s) = \frac{2s^2 + 4s + 1}{(s+1)(s+2)^3} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3}$$

$$A = F(s)(s+1)\big|_{s=-1} = \frac{2-4+1}{-1+2} = -1$$

$$D = F(s)(s+2)^3\big|_{s=-2} = \frac{8-8+1}{-2+1} = -1$$

$$C = \frac{d}{ds}\left[(s+2)^3 X(s)\right]\Big|_{s=-2} = \frac{d}{ds}\left[\frac{2s^2+4s+1}{(s+1)}\right]\Bigg|_{s=-2}$$

$$= \frac{(s+1)(4s+4) - (2s^2+4s+1)\times 1}{(s+1)^2}\Bigg|_{s=-2} = \frac{2s^2+4s+3}{(s+1)^2}\Bigg|_{s=-2}$$

$$= \frac{8-8+3}{1} = 3$$

$$B = \frac{1}{2}\frac{d^2}{ds^2}\left[(s+2)^3 X(s)\right]\Big|_{s=-2} = \frac{1}{2}\frac{d}{ds}\left[\frac{2s^2+4s+3}{(s+1)^2}\right]\Bigg|_{s=-2}$$

$$= \frac{(s+1)^2(4s+4) - (2s^2+4s+3)\times 2(s+1)}{2(s+1)^4}\Bigg|_{s=-2} = \frac{-2}{2(s+1)^3}\Bigg|_{s=-2}$$

$$= \frac{-2}{-2} = 1$$

Example 3.11

Find the inverse transform of

$$G(s) = \frac{\boxed{s+1}}{(s+2)(s^2+2s+5)}$$

Solution

$G(s)$ has a pair of complex poles at $s^2 + 2s + 5 = 0$ or $s = \boxed{-1 \pm} j2$. We let

$$G(s) = \frac{s+1}{(s+2)(s^2+2s+5)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+2s+5}$$

$$A = G(s)(s+2)\big|_{s=-2} = \frac{-1}{5}$$

$$s+1 = A(s^2+2s+5) + B(s^2+2s) + C(s+2)$$

$$s^2 : 0 = A + B \rightarrow B = -A = \frac{1}{5}$$

$$s^1 : 1 = 2A + 2B + C = 0 + C \rightarrow C = 1$$

$$s^0 : 1 = 5A + 2C = -1 + 2 = 1$$

$\Rightarrow$

$$G(s) = \frac{-1/5}{s+2} + \frac{1/5 \cdot s + 1}{(s+1)^2 + 2^2} = \frac{-1/5}{s+2} + \frac{1/5(s+1)}{(s+1)^2+2^2} + \frac{4/5}{(s+1)^2+2^2}$$

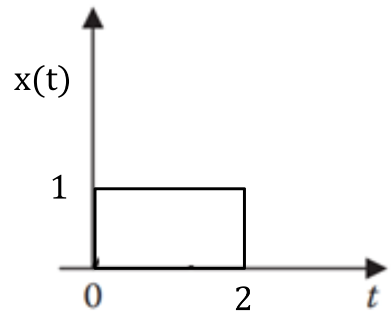
$$\frac{4/5}{(s+1)^2+2^2} = \frac{2 * 0.4}{(s+1)^2+2^2}$$

$$g(t) = (-0.2e^{-2t} + 0.2e^{-t}\cos(2t) + 0.4e^{-t}\sin(2t))u(t)$$

Example 3.12

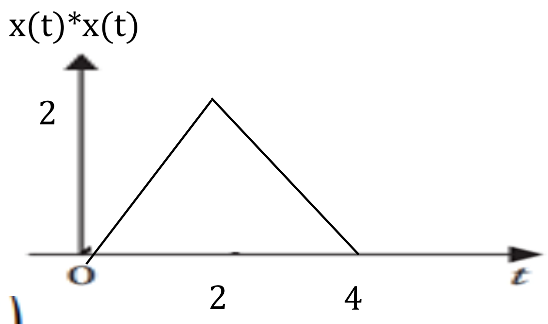
Consider the rectangular pulse or gate function $x(t) = u(t) - u(t-2)$. Obtain $y(t) = x(t)*x(t)$, that is, the convolution of the rectangular pulse with itself.

Solution



The Laplace transform of $x(t)$ is

$$X(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$



Using the convolution property,

$$Y(s) = X(s)X(s) = X^2(s) = \frac{1}{s^2}\left(1 - 2e^{-2s} + e^{-4s}\right)$$

Taking the inverse Laplace transform of each term,

$$y(t) = x(t) * x(t) = tu(t) - 2(t-2) + (t-4)u(t-4)$$

$$= r(t) - 2r(t-2) + r(t-4)$$

Example 3.13

Determine the inverse Laplace transform of

$$X(s) = \frac{se^{-2s} + e^{-3s}}{s(s^2 + 5s + 4)}$$

Solution

Let $X(s) = X_1(s)e^{-2s} + X_2(s)e^{-3s}$

$$X_1(s) = \frac{s}{s(s^2+5s+4)} = \frac{1}{(s+1)(s+4)} = \frac{A}{s+1} + \frac{B}{s+4}$$

$$A = \left.\frac{1}{s+4}\right|_{s=-1} = \frac{1}{-1+4} = \frac{1}{3} \qquad B = \left.\frac{1}{s+1}\right|_{s=-4} = \frac{1}{-4+1} = -\frac{1}{3}$$

$$X_2(s) = \frac{1}{s(s^2+5s+4)} = \frac{1}{s(s+1)(s+4)} = \frac{C}{s} + \frac{D}{s+1} + \frac{E}{s+4}$$

$$C = \left.\frac{1}{(s+1)(s+4)}\right|_{s=0} = \frac{1}{(1)(4)} = \frac{1}{4} \qquad D = \left.\frac{1}{s(s+4)}\right|_{s=-1} = \frac{1}{(-1)(3)} = -\frac{1}{3}$$

$$E = \left.\frac{1}{s(s+1)}\right|_{s=-4} = \frac{1}{(-4)(-3)} = \frac{1}{12}$$

$$X_1(s) = \frac{1}{3}\left[\frac{1}{s+1} - \frac{1}{s+4}\right]$$

$$x_1(t) = \frac{1}{3}\left(e^{-t} - e^{-4t}\right)u(t)$$

$$X_2(s) = \frac{1/4}{s} - \frac{1/3}{s+1} + \frac{1/12}{s+4}$$

$$x_2(t) = \left[\frac{1}{4} - \frac{1}{3}e^{-t} + \frac{1}{12}e^{-4t}\right]u(t)$$

$$x(t) = x_1(t-2)u(t-2) + x_2(t-3)u(t-3)$$

$$= \frac{1}{3}\left(e^{-(t-2)} - e^{-4(t-2)}\right)u(t-2) + \left[\frac{1}{4} - \frac{1}{3}e^{-(t-3)} + \frac{1}{12}e^{-4(t-3)}\right]u(t-3)$$

Example 3.14

Obtain $g(t)$ given that $G(s)=\dfrac{s^3+5s^2+10}{s^2+3s+2}$

Solution

$$
\begin{array}{r|l}
 & s+2 \\
\hline
s^2+3s+2 & s^3+5s^2+0s+10 \\
 & \underline{s^3+3s^2+2s} \\
 & 2s^2-2s+10 \\
 & \underline{2s^2+6s+4} \\
 & -8s+6
\end{array}
$$

$\Rightarrow$

$$G(s)=s+2+\frac{-8s+6}{s^2+3s+2}$$

$$\text{Let } Y(s)=\frac{-8s+6}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2}$$

$$A=\left.\frac{-8s+6}{s+2}\right|_{s=-1}=\frac{8+6}{-1+2}=14$$

$$B=\left.\frac{-8s+6}{s+1}\right|_{s=-2}=\frac{16+6}{-2+1}=-22$$

$$G(s)=s+2+\frac{14}{s+1}-\frac{22}{s+2}$$

$$g(t)=\frac{d}{dt}\delta(t)+2\delta(t)+\left(14e^{-t}-22e^{-2t}\right)u(t)$$

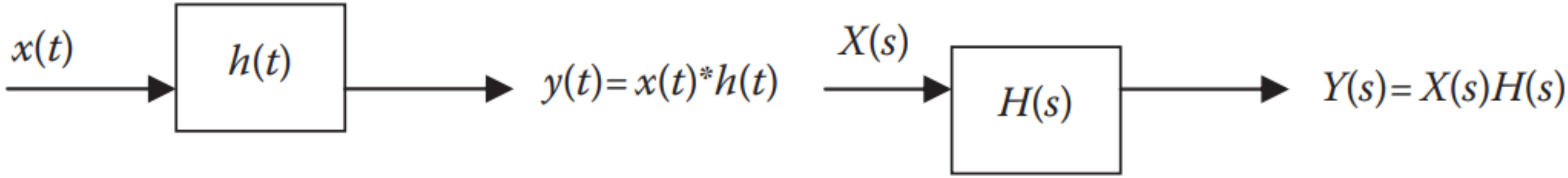
3.5 TRANSFER FUNCTION

The transfer function $H(s)$ is defined as the ratio of the output response $Y(s)$ to the input excitation $X(s)$, assuming all initial conditions are zero.

$$\boxed{H(s) = \frac{Y(s)}{X(s)}}$$

$$y(t) = x(t) * h(t)$$

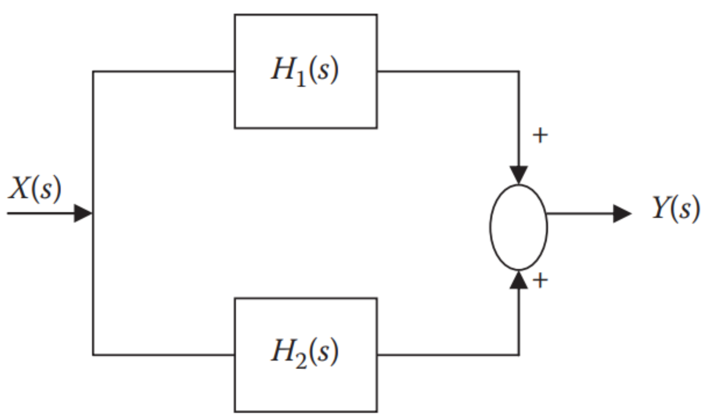
$$Y(s) = X(s)H(s)$$



- Cascade connection

$$Y(s) = H_1(s)H_2(s)X(s) \qquad H(s) = \frac{Y(s)}{X(s)} = H_1(s)H_2(s)$$

- Parallel interconnection

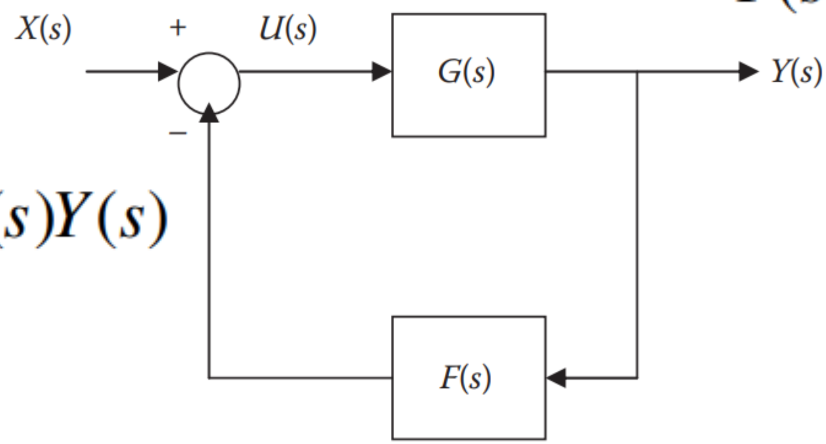


$$Y(s) = \left[H_1(s) + H_2(s) \right] X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = H_1(s) + H_2(s)$$

- Feedback interconnection

$$Y(s) = G(s)U(s)$$



$$U(s) = X(s) - F(s)Y(s)$$

$$Y(s) = G(s)(X(s) - F(s)Y(s))$$

$$Y(s) + G(s)F(s)Y(s) = G(s)X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + F(s)G(s)}$$

Example 3.15

The output of a linear system is $y(t) = 10e^{-t}\cos 4t\, u(t)$, when the input is $x(t) = e^{-t}u(t)$.
Find the transfer function of the system.

Solution

$$X(s) = \frac{1}{s+1} \quad \text{and} \quad Y(s) = \frac{10(s+1)}{(s+1)^2 + 4^2}$$

$\Rightarrow$

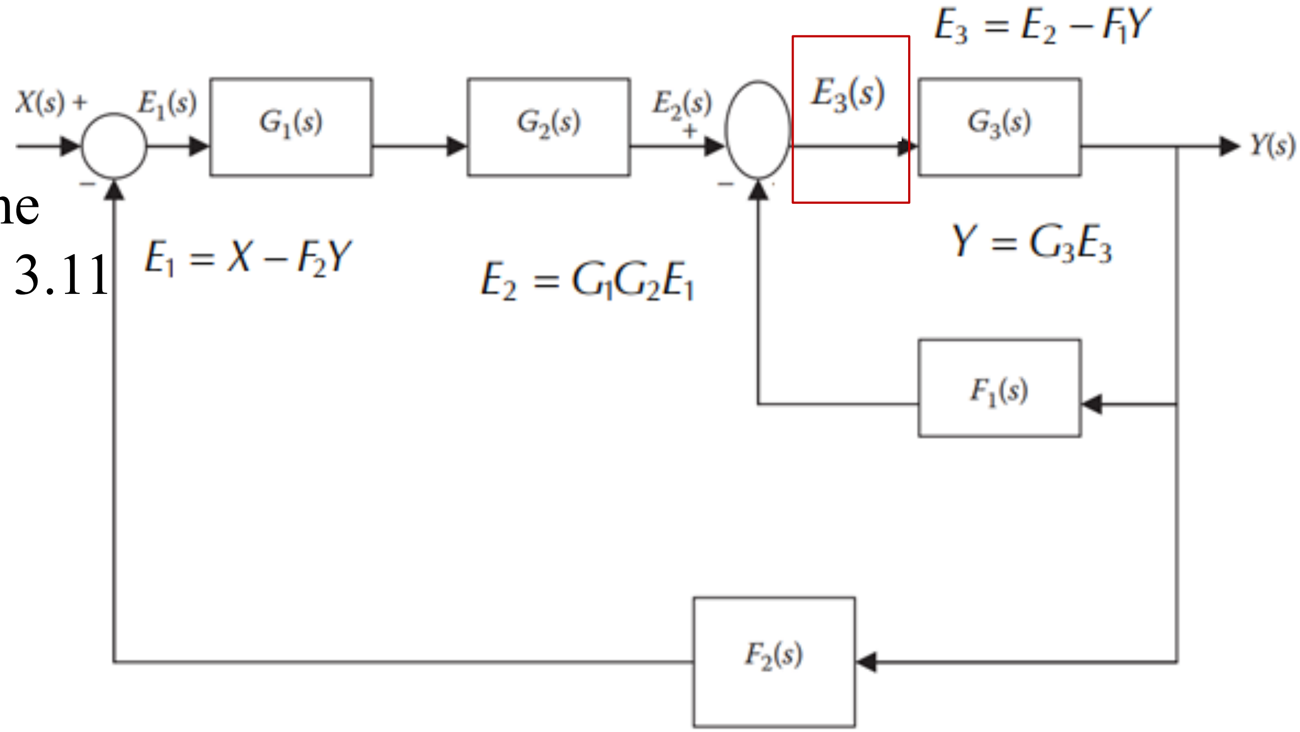
$$H(s) = \frac{Y(s)}{X(s)} = \frac{10(s+1)^2}{(s+1)^2 + 16} = \frac{10(s^2 + 2s + 1)}{s^2 + 2s + 17}$$

$$= 10 - 40\boxed{\frac{4}{s^2 + 2s + 17}} = \boxed{\frac{4}{(s+1)^2 + 4^2}}$$

$$h(t) = 10\delta(t) - 40e^{-t}\sin 4t\ u(t)$$

Example 3.16

Find the transfer function for the feedback system shown in Fig. 3.11



Solution

$$\frac{Y}{G_3} = E_2 - F_1 Y = G_1 G_2 E_1 - F_1 Y = G_1 G_2 [X - F_2 Y] - F_1 Y$$

$$Y = G_1 G_2 G_3 X - G_1 G_2 G_3 F_2 Y - F_1 G_3 Y$$

$$H = \frac{Y}{X} = \frac{G_1 G_2 G_3}{1 + F_1 G_3 + F_2 G_1 G_2 G_3}$$

3.6 APPLICATIONS

- Integro-Differential Equations

Example 3.17

Use the Laplace transform to solve the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{-t}u(t) \qquad y(0) = 1, \quad \frac{dy(0)}{dt} = 0.$$

Solution

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 3\left[sY(s) - y(0)\right] - 4Y(s) = \frac{1}{s+1}$$

$$\left(s^2 + 3s - 4\right)Y(s) = s + 3 + \frac{1}{s+1} = \frac{s^2 + 4s + 4}{s+1}$$

$$Y(s) = \frac{s^2 + 4s + 4}{(s^2 + 3s - 4)(s+1)} = \frac{s^2 + 4s + 4}{(s-1)(s+1)(s+4)}$$

$$Y(s) = \frac{s^2 + 4s + 4}{(s^2 + 3s - 4)(s+1)} = \frac{s^2 + 4s + 4}{(s-1)(s+1)(s+4)} = \frac{A}{(s-1)} + \frac{B}{(s+1)} + \frac{C}{(s+4)}$$

$$A = (s-1)Y(s)\Big|_{s=1} = \frac{1+4+4}{2(5)} = \boxed{\frac{9}{10}}$$

$$B = (s+1)Y(s)\Big|_{s=-1} = \frac{1-4+4}{(-2)(3)} = \frac{1}{-6}$$

$$C = (s+4)Y(s)\Big|_{s=-4} = \frac{16-16+4}{(-5)(-3)} = \frac{4}{15}$$

$$Y(s) = \frac{\boxed{9/10}}{(s-1)} - \frac{1/6}{(s+1)} + \frac{4/15}{(s+4)}$$

$$y(t) = \left(\boxed{\frac{9}{10} e^{t}} - \frac{1}{6} e^{-t} + \frac{4}{15} e^{-4t} \right) u(t)$$

Example 3.18

Solve the integro-differential equation

$$\frac{dv}{dt} + 2v + 5\int_0^t v(\lambda)d\lambda = 4u(t)$$

with $v(0) = -1$ and determine $v(t)$ for $t > 0$.

Solution

$$[sV(s) - v(0)] + 2V(s) + \frac{5}{s}V(s) = \frac{4}{s}$$

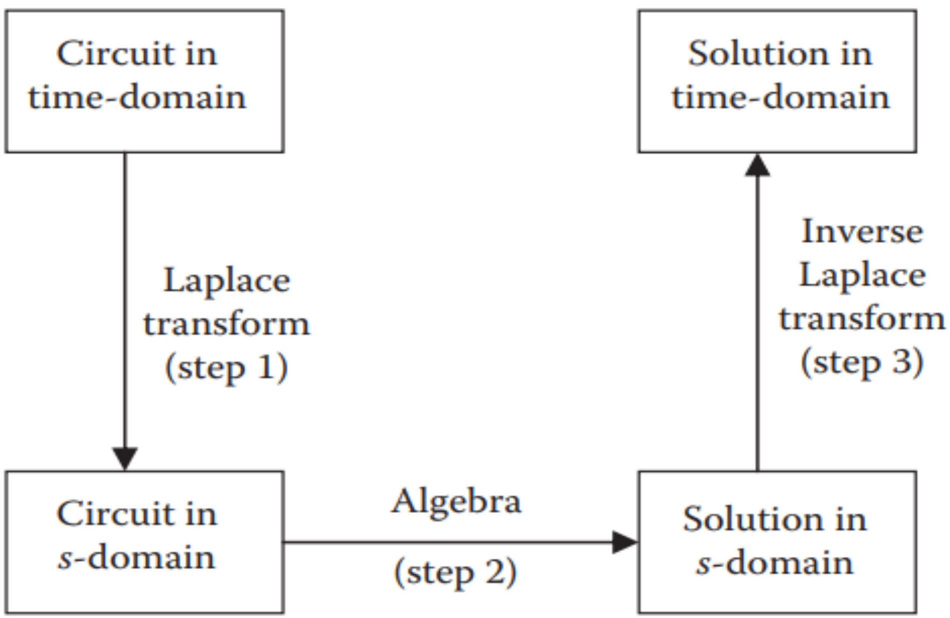
$$[sV(s) + 1] + 2V(s) + \frac{5}{s}V(s) = \frac{4}{s} \rightarrow V(s) = \frac{4 - s}{s^2 + 2s + 5}$$

$$V(s) = \frac{-(s+1) + 5}{(s+1)^2 + 2^2} = \frac{-(s+1)}{(s+1)^2 + 2^2} + \frac{5}{2}\frac{2}{(s+1)^2 + 2^2}$$

$$v(t) = (-e^{-t}\cos 2t + 2.5e^{-t}\sin 2t)u(t)$$

- Circuit Analysis

1. Laplace transform the circuit from the time-domain to the frequency domain (or s-domain).
2. The circuit in s-domain is solved using circuit analysis techniques (such as voltage division, current division, nodal analysis, mesh analysis, source transformation, and superposition) and we obtain the desired quantity $X(s)$.
3. Obtain the inverse Laplace transform of $X(s)$ to get the desired solution $x(t)$ in the time domain.



- Resistor

$$v(t) = Ri(t)$$

$$V(s) = RI(s)$$

- Inductor

$$v(t) = L\frac{di}{dt}$$

$$V(s) = L\left[sI(s) - i(0^-)\right] = sLI(s) - Li(0^-)$$

- Capacitor

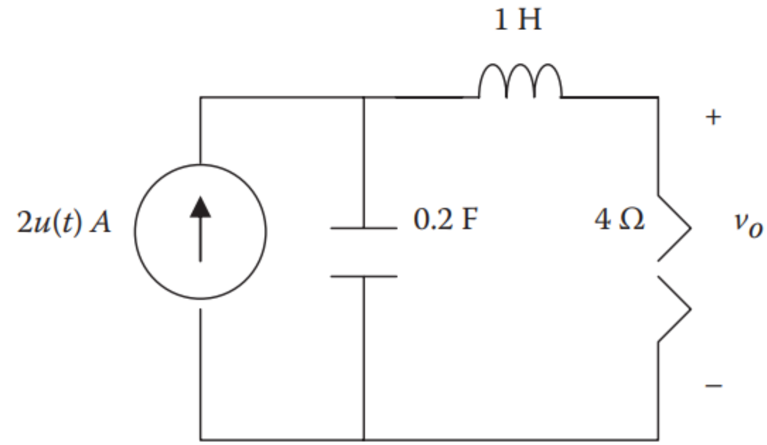
$$i(t) = C\frac{dv(t)}{dt}$$

$$I(s) = C\left[sV(0s) - v(0^-)\right] = sCV(s) - Cv(0^-)$$

$$V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}$$

Example 3.19

Find $v_o(t)$ in the circuit in Figure 3.14.

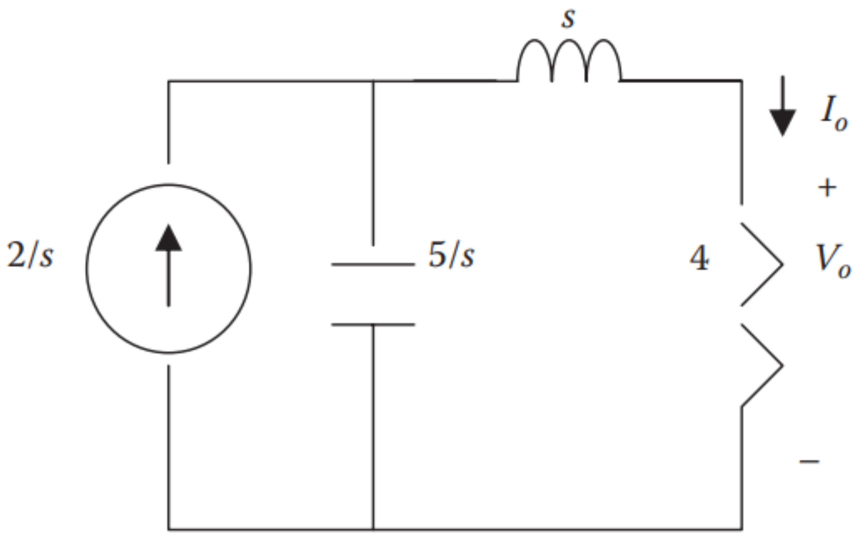


Solution

$$2u(t) \Rightarrow \frac{2}{s}$$

$$1\,\text{H} \Rightarrow sL = s$$

$$0.2\,\text{F} \Rightarrow \frac{1}{sC} = \frac{1}{0.2s} = \frac{5}{s}$$



$$I_o = \frac{5/s}{5/s + s + 4} \times \frac{2}{s} = \frac{5}{s^2 + 4s + 5} \times \frac{2}{s}$$

$$V_o = 4I_o = \frac{40}{s(s^2 + 4s + 5)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 5}$$

$$40 = A(s^2 + 4s + 5) + Bs^2 + Cs$$

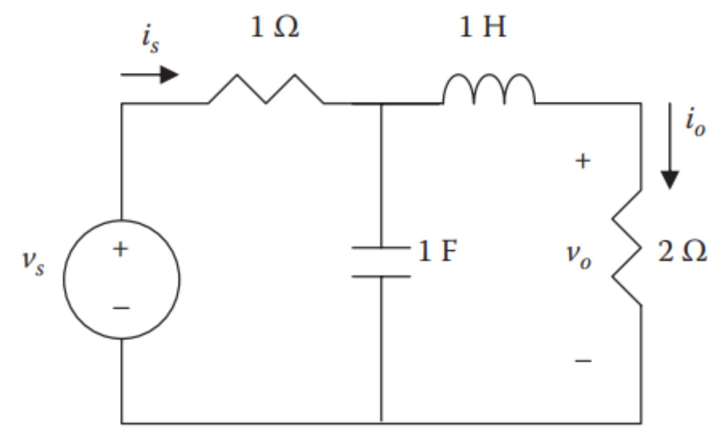
$$A = 8 \quad C = -4A = -32 \quad B = -A = -8$$

$$V_o = \frac{8}{s} - \frac{8s + 32}{(s+2)^2 + 1} = \frac{8}{s} - \frac{8(s+2)}{(s+2)^2 + 1} - \frac{16}{(s+2)^2 + 1}$$

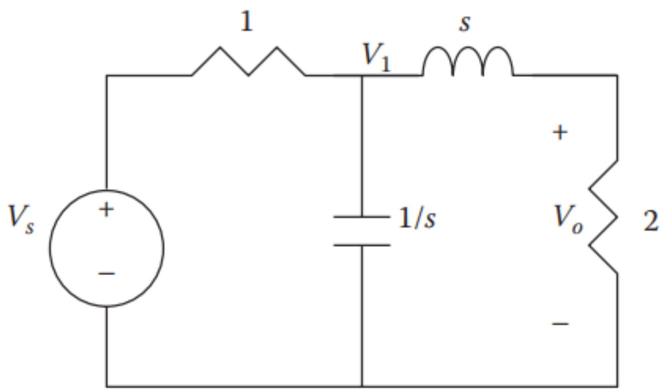
$$v_o(t) = \left(8 - 8e^{-2t}\cos t - \boxed{16}\, e^{-2t}\sin t\right)u(t)$$

Example 3.20

For the circuit in Figure 3.17, find $H(s) = V_o(s)/V_s(s)$. Assume zero initial conditions.



Solution



$$Z = \frac{1}{s} \,||\, (s+2) = \frac{1/s(s+2)}{1/s + s + 2} = \frac{(s+2)}{(s^2+2s+1)}$$

$$V_1 = \frac{Z}{Z+1} V_s$$

$$V_o = \frac{2}{s+2} V_1 = \frac{2}{s+2} \frac{Z}{Z+1} V_s$$

$$H(s) = \frac{V_o}{V_s} = \frac{2}{s+2} \times \frac{\dfrac{s+2}{s^2+2s+1}}{\dfrac{s+2}{s^2+2s+1}+1}$$

$$= \frac{\boxed{2}}{s^2+3s+3}$$

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Example 4.2 RL Transient Analysis

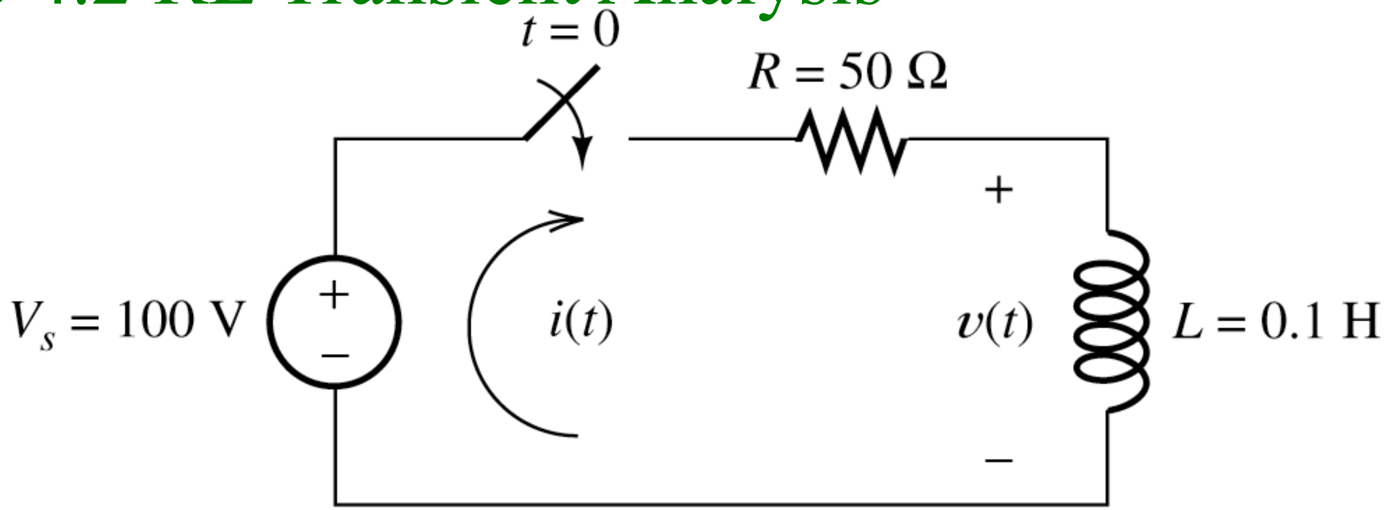


Figure 4.7 The circuit analyzed in Example 4.2.

1. $t<0$, $i(t) = 0$.

2. $t > 0$

KVL

$$Ri(t) + L\frac{di}{dt} = V_s \qquad \left(v(t) = L\frac{di}{dt} \right)$$

Example 4.2 RL Transient Analysis

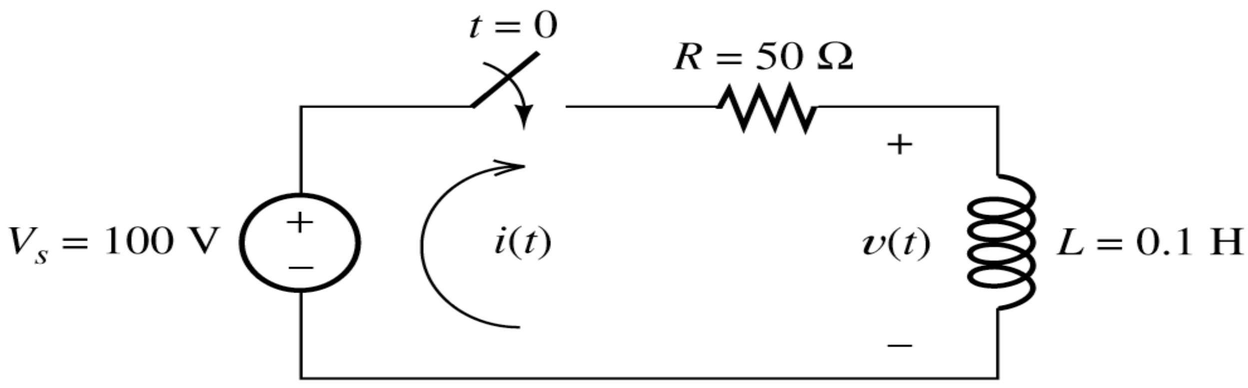


Figure 4.7 The circuit analyzed in Example 4.2.

Assume

$$i(t) = K_1 + K_2 e^{st}$$

$$RK_1 + (RK_2 + sLK_2)e^{st} = V_s$$

$$K_1 = \frac{V_s}{R} = 2 \qquad s = \frac{-R}{L}$$

Example 4.2 RL Transient Analysis

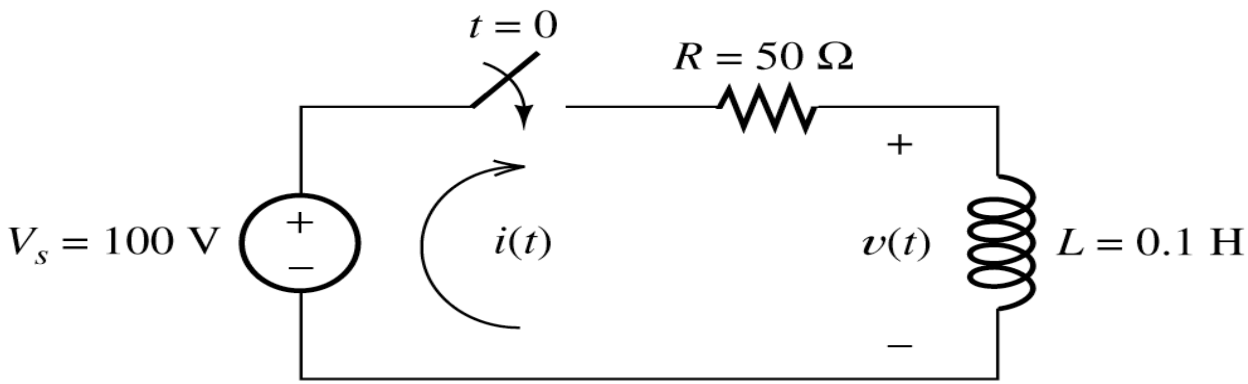


Figure 4.7 The circuit analyzed in Example 4.2.

$$i(t) = 2 + K_2 e^{-tR/L}$$

$$\because \quad i(0_+) = 0 = 2 + K_2 e^0 = 2 + K_2 \quad \Rightarrow \quad K_2 = -2$$

$$i(t) = 2 - 2e^{-t/\tau} \quad for \;\; t > 0 \qquad \tau = \frac{L}{R}$$

Example 4.2 RL Transient Analysis

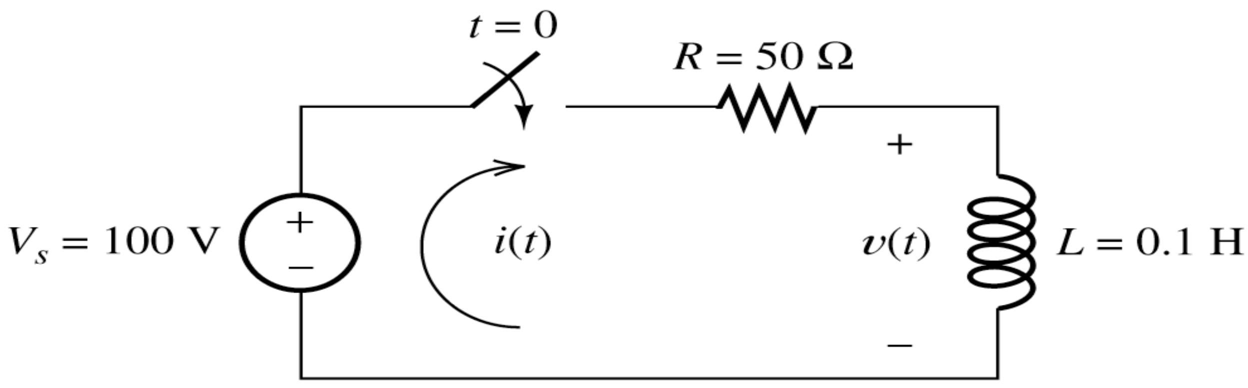
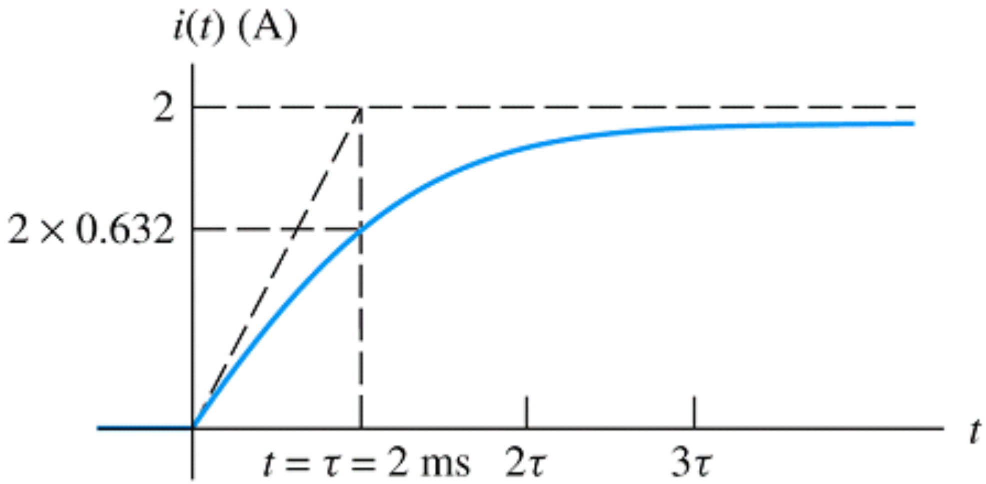


Figure 4.7 The circuit analyzed in Example 4.2.

$$i(t) = 2 - 2e^{-t/\tau} \quad for \;\; t > 0 \qquad \tau = \frac{0.1}{50} = 2 \times 10^{-3}\,(\text{sec})$$



Example 4.2 *RL* Transient Analysis

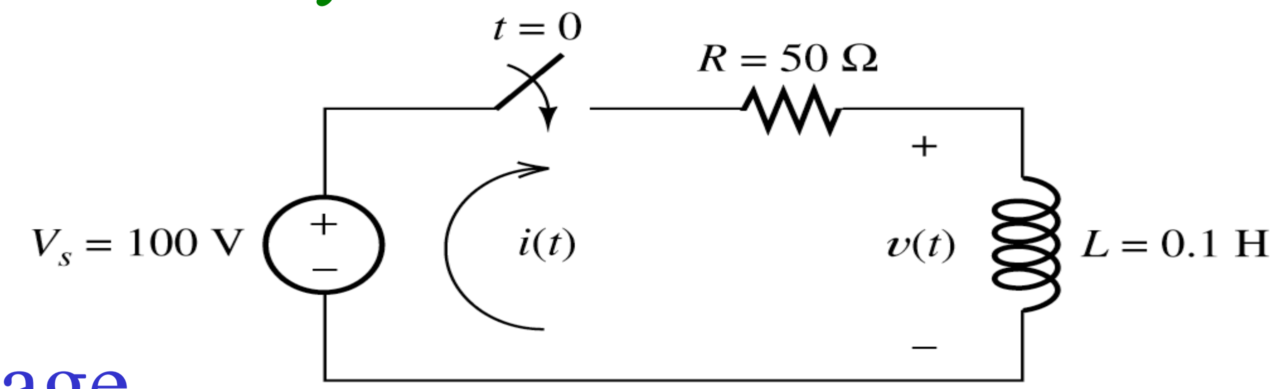


Figure 4.7 The circuit analyzed in Example 4.2.

Consider the voltage

1. $t<0$, $v(t) = 0$.

2. $t > 0$

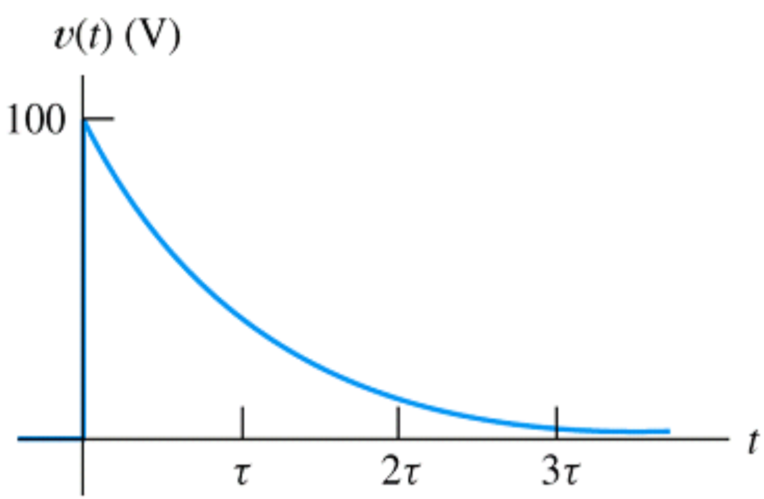
$$v(t) = 100 - 50 i(t) \quad for\ t > 0 \quad \left(i(t) = 2 - 2e^{-t/\tau} \right)$$

$$\Rightarrow \quad v(t) = 100\, e^{-t/\tau}$$

or

$$v(t) = L\frac{di}{dt} = 0.1 \cdot (2/\tau) e^{-t/\tau}$$

$$= 0.1 \cdot 1000 \cdot e^{-t/\tau} = 100 e^{-t/\tau}$$



Laplace Transform

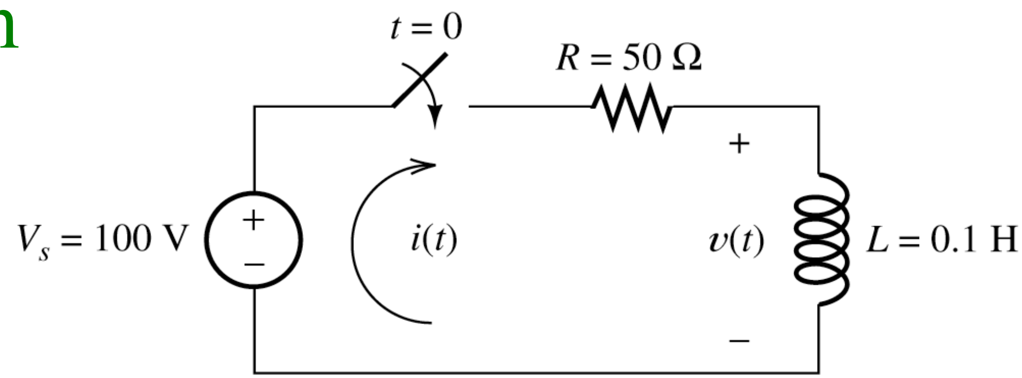


Figure 4.7 The circuit analyzed in Example 4.2.

$$Ri(t) + L\frac{di}{dt} = V_s$$

$$RI(s) + L[sI(s) - i(0)] = V_s(s) = \frac{V_s}{s}$$

$$I(s) = \frac{V_s}{s(R + Ls)} = \frac{V_s}{L}\frac{1}{s(s + R/L)}$$

$$= \frac{V_s}{L}\left(\frac{A}{s} + \frac{B}{s+R/L}\right)$$

$$As + A\left({}^{R}\!/_{L}\right) + Bs = 1$$

$$B = -A \qquad A = {}^{L}\!/_{R}$$

$$I(s) = \frac{V_s}{L}\left(\frac{{}^{L}\!/_{R}}{s} + \frac{-{}^{L}\!/_{R}}{s + {}^{R}\!/_{L}}\right)$$

$$= \frac{V_s}{R}\left(\frac{1}{s} - \frac{1}{s + {}^{R}\!/_{L}}\right)$$

$$i(t) = 2 - 2e^{-({}^{R}\!/_{L})t}$$

Example 3.22

Use MATLAB to find the Laplace transform of

$$x(t) = 2\delta(t) + e^{-3t}$$

Solution

We recall that the commands **dirac**(t) and **heaviside**(t) are used to represent the unit impulse $\delta(t)$ and unit step $u(t)$, respectively. The MATLAB commands are:

```
syms x t
x = 2*dirac(t) + exp(-3*t);
X = laplace(x)
```

This produces the following result:

```
X =
1/(s+3) + 2
```

Example 3.23

Use MATLAB to find the inverse Laplace transform of

$$V(s) = \frac{10s^2 + 4}{s(s+1)(s+2)^2}$$

Solution

The MATLAB code is as follows:

```
syms s v V
V =(10*s^2 +4)/( s*(s+1)*(s+2)^2 );
v = ilaplace(V)
```

This produces the following result:

```
v =
1-14*exp(-t)+(13+22*t)*exp(-2*t)
```

so that

$$v(t) = 1 - 14e^{-t} + 13e^{-2t} + 22te^{-2t}, \quad t > 0$$

Example 3.24

Use the **residue** command to find the Laplace inverse of

$$X(s) = \frac{4s^5 + 20s^4 + 16s^3 + 10s^2 - 12}{s^4 + 5s^3 + 8s^2 + 4s}$$

Solution

This is an indirect way of finding the inverse Laplace transform. We specify the numerator (num) and the denominator (den) of the transfer function $X(s)$. We find the residues of $X(s)$ using the following code.

```
num = [4 20 16 10 0 -12]; % numerator coefficients in descending
powers of s
den = [1 5 8 4 0]; % denominator coefficients in descending
powers of s
[r,p,k] = residue(num,den); % call residue
r =
-15.0000
46.0000
2.0000
-3.0000
p =
-2.0000
-2.0000
-1.0000
0
k =
4 0
```

This produces a vector r that has the residues and a vector p that has the corresponding poles.

Notice that pole –2 is repeated. Also, since the order of the numerator of $X(s)$ is one greater than the order of the denominator of $X(s)$, k contains two values. From r, p, and k, we can write $X(s)$ as

$$X(s) = 4s + 0s^0 + \frac{-15}{s-(-2)} + \frac{46}{[s-(-2)]^2} + \frac{2}{s-(-1)} + \frac{-3}{s-0}$$

$$= 4s - \frac{15}{s+2} + \frac{46}{(s+2)^2} + \frac{2}{s+1} - \frac{3}{s}$$

Using Table 3.2, we obtain the inverse $v(t)$ as

$$x(t) = 4\delta'(t) - 15e^{-2t} + 46te^{-2t} + 2e^{-t} - 3u(t), \quad t \geq 0$$

Example 3.25

Use MATLAB to find the zeros and poles of

$$H(s) = \frac{s^2 + 3s + 1}{s^3 + 4s^2 + 3s}$$

Solution

We use the command **roots** to find the roots of the numerator to get the zeros, and denominator to get the poles.

```
num = [1 3 1];
den = [1 4 3 0];
z=roots(num);
p=roots(den);
This result is:
z =
-2.6180
-0.3820
p =
0
-3
-1
```

Example 3.26

Use MATLAB to plot the step response of a system whose transfer function is

$$H(s) = \frac{s+1}{(s+2)(s+3)}$$

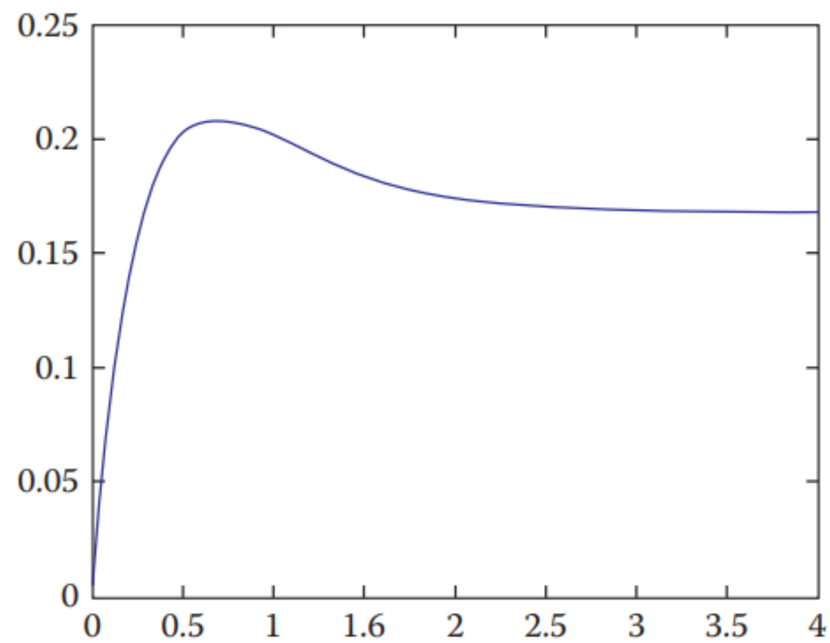
Solution

We first need to expand the denominator:

$$H(s) = \frac{s+1}{s^2+5s+6}$$

By definition, the step response is the response when the input to the system is the unit step $u(t)$. Using the MATLAB script below, we obtain the response as plotted in Figure 3.19.

```
num = [1 1];
den = [1 5 6];
t = 0: 0.1: 4;
y=step(num,den,t);
plot(t,y)
```



Example 3.27

Use MATLAB to obtain the Bode plots for the transfer function

$$H(s) = \frac{100s}{s^2 + 12s + 20}$$

Solution

The MATLAB script for the transfer is shown below, while the Bode plots are in Figure 3.21.

```
num = [100 0];
den = [1 12 20];
bode(num,den); %determines and draws Bode plots
```