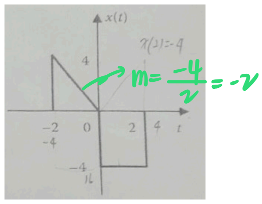
1. (15%) Given $x(t)$ in Figure 1, sketch
(a) $y(t) = -x(t-1)$
(b) $z(t) = 4x(t/2)$
(c) $h(t) = x(2-t)$

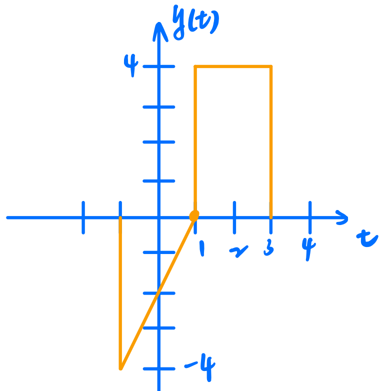


$$x(t) = \begin{cases} -2t, & -2 < t < 0 \\ -4, & 0 < t < 2 \end{cases}$$

(a)

$$y(t) = \begin{cases} -(-2(t-1)), & -2 < t-1 < 0 \\ -(-4), & 0 < t-1 < 2 \end{cases}$$

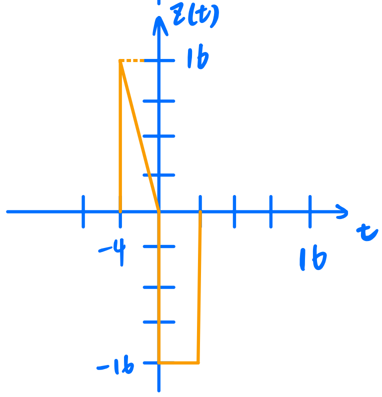
$$= \begin{cases} 2t-2, & -1 < t < 1 \\ 4, & 1 < t < 3 \end{cases}$$



(b)

$$z(t) = \begin{cases} 4\cdot\left(-2\cdot\frac{t}{2}\right), & -2 < \frac{t}{2} < 0 \\ 4(-4), & 0 < \frac{t}{2} < 2 \end{cases}$$

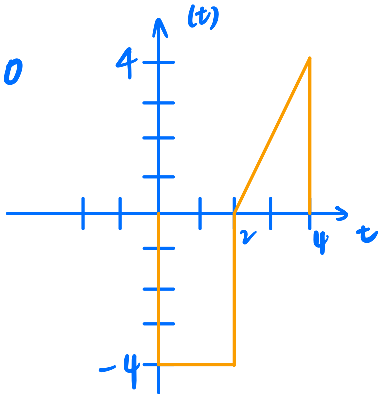
$$= \begin{cases} -4t, & -4 < t < 0 \\ -16, & 0 < t < 4 \end{cases}$$



(c)

$$h(t) = \begin{cases} -2(2-t), & -2 < (2-t) < 0 \\ -4, & 0 < (2-t) < 2 \end{cases}$$

$$= \begin{cases} 2t-4, & 2 < t < 4 \\ -4, & 0 < t < 2 \end{cases}$$



2. (12%) Determine and explain the following systems are linear or nonlinear, time varying or time invariant?
(a) $y(t) = \exp[x(t)]$
(b) $y(t) = \cos x(t)$
(c) $y(t) = t^2 x(t)$

☆ Linear : $T(k_1x_1 + k_2x_2) = k_1y_1 + k_2y_2$
☆ time invariant : $T(x\{t-\tau\}) = y(t-\tau)$

$y = Tx$

(a) $\exp[k_1x_1(t) + k_2x_2(t)] \neq k_1\underline{\exp[x_1(t)]} + k_2\underline{\exp[x_2(t)]} \Rightarrow$ nonlinear

$\exp(x\{t-\tau\}) = y(t-\tau) = \exp[x(t-\tau)] \Rightarrow$ time invariant

(b) $\cos(x_1(t) + x_2(t)) \neq \underline{\cos x_1(t)} + \underline{\cos x_2(t)} \Rightarrow$ nonlinear (k1, k2用1代)

$\cos(x\{t-\tau\}) = y(t-\tau) = \cos(x(t-\tau)) \Rightarrow$ time invariant

(c) $t^2(k_1x_1(t) + k_2x_2(t)) = k_1t^2x_1(t) + k_2t^2x_2(t) \Rightarrow$ linear

$t^2x\{t-\tau\} \neq y(t-\tau) = (t-\tau)^2x(t-\tau) \Rightarrow$ time varying

3. (18%) Two systems are described by

$$h_1[n] = (0.4)^n u[n], \qquad h_2[n] = \delta[n] + 0.5\delta[n-1]$$

Determine the response to the input $x[n] = (0.4)^n u[n]$

(a) The two systems are connected in parallel

(b) The two systems are connected in cascade

* impulse response：算 h 即可

* response：算 $h(t)\,x(t)$，即 output

(a) parallel：並聯 $h_1 + h_2$

$$h[n] = h_1[n] + h_2[n] = (0.4)^n u[n] + \delta[n] + 0.5\delta[n-1]$$

$$y[n] = h[n] * x[n] = \{\underbrace{(0.4)^n u[n]}_{①} + \underbrace{\delta[n]}_{②} + \underbrace{0.5\delta[n-1]}_{③}\} * \{(0.4)^n u[n]\}$$

$$= (n+1)(0.4)^n u[n] + (0.4)^n u[n] + 0.5(0.4)^{n-1} u[n-1] = (n+2)(0.4)^n u[n] + 0.5(0.4)^{n-1} u[n-1]$$

①
$$\sum_{k=0}^{n} (0.4)^k u[k] \cdot (0.4)^{n-k} u[n-k]$$
$$= \sum_{k=0}^{n} (0.4)^n u[k]u[n-k] = (0.4)^n \sum_{k=0}^{n} 1 = (n+1)(0.4)^n u[n]$$

自己 * 自己 = (n+1) 自己

② 自己 $* \delta[n] =$ 自己

③ 自己 $* \delta[n-1] =$ 自己$[n-1]$

(b) cascade：串聯 $h_1 * h_2$

$$h[n] = h_1[n] * h_2[n] = \{(0.4)^n u[n]\} * \{\delta[n] + 0.5\delta[n-1]\} = (0.4)^n u[n] + 0.5(0.4)^{n-1} u[n-1]$$

$$y[n] = h[n] * x[n] = \{(0.4)^n u[n] + \underline{0.5(0.4)^{n-1} u[n-1]}\} * \{\underline{(0.4)^n u[n]}\}$$

$$= (n+1)(0.4)^n u[n] + 0.5(0.4)^{n-1} n\, u[n-1]$$

$$\sum_{k=0}^{n} (0.5)(0.4)^{k-1} u[k-1] \cdot (0.4)^{n-k} u[n-k]$$

$$= (0.5)(0.4)^{n-1} \sum_{k=0}^{n} u[\underset{右1}{k-1}]\, u[n-k] = 0.5(0.4)^{n-1} \cdot n \cdot u[n-1]$$

↳ $k=0 \sim n$，但 $k=0$ 時 $u[-1]u[n]=0$

$$\Rightarrow a^n u[n] * a^{n-1} u[n-1] = n a^{n-1} u[n-1]$$

4. (15%) The input $x[n] = [3, 0, 2, 6]$ to a system produces the output $y[n] = [6, 12, 25, 20, 38, 42]$. Determine the impulse response of the system $h[n]$.

$$y[n] = h[n] * x[n] \Rightarrow h[n] = \frac{y[n]}{x[n]}$$

$$\begin{array}{r|l}
 & 2 + 4 + 7 \\
\hline
3+0+2+6 & 6 + 12 + 25 + 20 + 38 + 42 \\
 & 6 + 0 + 4 + 12 \\
\hline
 & 12 + 21 + 8 + 38 \\
 & 12 + 0 + 8 + 24 \\
\hline
 & 21 + 0 + 14 + 42 \\
 & 21 + 0 + 14 + 42 \\
\hline
 & 0
\end{array}$$

$$\Rightarrow h[n] = [2, 4, 7]$$

5. (15%) Find the Laplace $X(s)$ given that $x(t)$ is
(a) $2tu(t-4)$
(b) $5\cos t\ \delta(t-2)$
(c) $e^{-t}u(t-\tau)$

能化成公式 ⇒ 用公式

不行 ⇒ $\int_0^\infty x(t)\, e^{-st}\, dt$

(a)

$$\mathcal{L}\{2tu(t-4)\} = \mathcal{L}\{2(t-4)u(t-4) + 8u(t-4)\} = 2e^{-4s}\left(-\frac{d}{ds}s^{-1}\right) + 8e^{-4s}\cdot\frac{1}{s}$$

$$= e^{-4s}\cdot\frac{2}{s^2} + e^{-4s}\cdot\frac{8}{s}$$

(b)

$$\mathcal{L}\{5\cos t\,\delta(t-2)\} = \int_0^\infty 5\cos t\,\delta(t-2)\,e^{-st}\,dt = 5\cos t\, e^{-st}\Big|_{t=2} = 5e^{-2s}\cos 2$$

(c)

$$\mathcal{L}\{e^{-t}u(t-\tau)\} = \mathcal{L}\{e^{-(t-\tau)-\tau}\cdot u(t-\tau)\} = \mathcal{L}\{\underbrace{e^{-\tau}}_{\text{常數}}\cdot e^{-(t-\tau)}\cdot u(t-\tau)\} = e^{-\tau}\cdot e^{-\tau s}\cdot\frac{1}{s+1}$$

6. (10%) Obtain the inverse Laplace transform of the following functions:

(a) $X(s)=\frac{1}{s}+\frac{2}{s}e^{-s}$ (b) $P(s)=\frac{6s^2+36s+20}{(s+1)(s+2)(s+3)}$

$\mathcal{L}\{\frac{t^{n-1}}{(n-1)!}e^{-at}\}=\frac{1}{(s+a)^n}$, 不要忘了$u(t)$!

(a)

$$\mathcal{L}^{-1}\{X(s)\}=u(t)+2u(t-1)$$

(b)

$$P(s)=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{s+3}$$

$$A=(s+1)P(s)\Big|_{s=-1}=\frac{6-36+20}{1\cdot 2}=\frac{-10}{2}=-5$$

$$B=(s+2)P(s)\Big|_{s=-2}=\frac{24-72+20}{(-1)(1)}=28$$

$$C=(s+3)P(s)\Big|_{s=-3}=\frac{54-108+20}{(-2)(-1)}=\frac{-34}{2}=-17$$

$$\mathcal{L}^{-1}\{P(s)\}=\left(-5e^{-t}+28e^{-2t}-17e^{-3t}\right)u(t)$$

7. (15%) Solve the differential equation

$$y''(t)+7y'(t)+12y(t)=e^{-t}u(t)$$

subject to $y(0)=-1$, $y'(0)=2$.

$$[s^2Y(s)-sy(0)-y'(0)]+7[sY(s)-y(0)]+12Y(s)=\frac{1}{s+1}$$

$$\Rightarrow [s^2Y(s)+s-2]+7[sY(s)+1]+12Y(s)=\frac{1}{s+1}$$

$$\Rightarrow Y(s)(s^2+7s+12)=\frac{1}{s+1}-s-5=\frac{-s^2-6s-4}{s+1}$$

$$\Rightarrow Y(s)=\frac{-s^2-6s-4}{(s+1)(s^2+7s+12)}=\frac{A}{s+1}+\frac{B}{s+3}+\frac{C}{s+4}$$

$$A=(s+1)Y(s)\Big|_{s=-1}=\frac{1}{6}$$

$$B=(s+3)Y(s)\Big|_{s=-3}=\frac{-9+18-4}{(-2)\cdot 1}=-\frac{5}{2}$$

$$C=(s+4)Y(s)\Big|_{s=-4}=\frac{-16+24-4}{(-3)\cdot(-1)}=\frac{4}{3}$$

$$\Rightarrow y(t)=\left(\frac{1}{6}e^{-t}-\frac{5}{2}e^{-3t}+\frac{4}{3}e^{-4t}\right)\times u(t)$$