

# 訊號與系統

# SIGNAL AND SYSTEM

## Lecture 3

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# Laplace Transform

- Laplace transform is a **frequency-domain** representation that makes analysis and design of **linear systems** simpler.
- Laplace transform is powerful for providing us in one single operation the complete response, that is, the steady-state plus transient.
- It allows us to convert **ordinary differential** equations into **algebraic** equations, which are easier to manipulate and solve.
- It converts convolution into a simple multiplication.
- We can apply Laplace transform to generate the **transfer function** representation of a continuous-time LTI system.

# TRANSFER FUNCTION

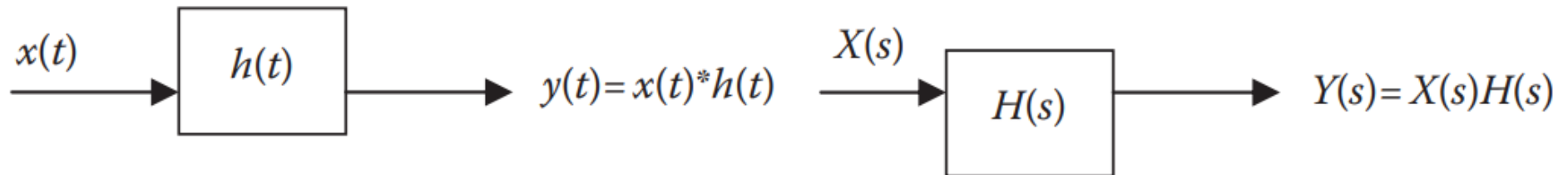
The transfer function  $H(s)$  is defined as the ratio of the output response  $Y(s)$  to the input excitation  $X(s)$ , assuming all initial conditions are zero.

$$H(s) = \frac{Y(s)}{X(s)}$$

$$y(t) = x(t) * h(t)$$
$$\underline{Y(s) = X(s)H(s)}$$

Laplace

方便計算




# APPLICATIONS

- Integro-Differential Equations

Use the Laplace transform to solve the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{-t}u(t) \quad y(0) = 1, \quad \frac{dy(0)}{dt} = 0.$$

Solution

$$\left[ s^2Y(s) - sy(0) - y'(0) \right] + 3[sY(s) - y(0)] - 4Y(s) = \frac{1}{s+1}$$


$$(s^2 + 3s - 4)Y(s) = s + 3 + \frac{1}{s+1} = \frac{s^2 + 4s + 4}{s+1}$$

$$Y(s) = \frac{s^2 + 4s + 4}{(s^2 + 3s - 4)(s+1)} = \frac{s^2 + 4s + 4}{(s-1)(s+1)(s+4)}$$

- Circuit Analysis

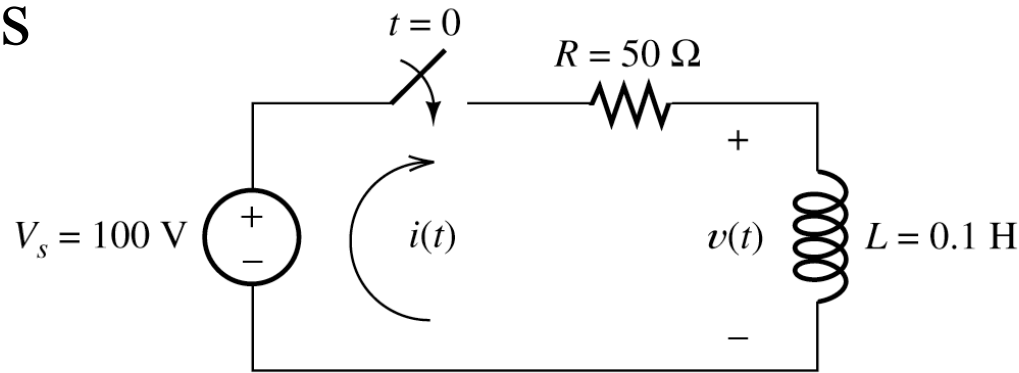


Figure 4.7 The circuit analyzed in Example 4.2.

$$Ri(t) + L \frac{di}{dt} = V_s$$

$$RI(s) + L[sI(s) - i(0)] = V_s(s) = \frac{V_s}{s}$$

一次微分

$$I(s) = \frac{V_s}{s(R + Ls)} = \frac{V_s}{L} \frac{1}{s(s + R/L)}$$

$$= \frac{V_s}{L} \left( \frac{A}{s} + \frac{B}{s + R/L} \right)$$

$$As + A(R/L) + Bs = 1$$

$$B = -A \quad A = L/R$$

$$I(s) = \frac{V_s}{L} \left( \frac{L/R}{s} + \frac{-L/R}{s + R/L} \right)$$

$$= \frac{V_s}{R} \left( \frac{1}{s} - \frac{1}{s + R/L} \right)$$

$$i(t) = 2 - 2e^{-(R/L)t}$$

## 3.2 DEFINITION OF LAPLACE TRANSFORM

The Laplace transform of a signal  $x(t)$  is the integration of the product of  $x(t)$  and  $e^{-st}$  over the interval from 0 to  $+\infty$  (commonly)

Bilateral

$$\mathcal{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$$

*time domain*      *複數平面*

Unilateral (more commonly)

$$\mathcal{L}[x(t)] = X(s) = \int_{0^-}^{\infty} x(t)e^{-st} dt$$

*t=0 開始算*

$s$  is the complex frequency given by  $s = \sigma + j\omega$   
*Laplace 變*

Inverse Laplace transform

$$\mathcal{L}^{-1}[X(s)] = x(t) = \frac{1}{2\pi j} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} X(s)e^{st} ds$$

*反函數*      *s → t (不常用)*

where the integration is performed along a straight line ( $\sigma_1 + j\omega$ ,  $-\infty < \omega < \infty$ )

- Laplace transformable  
The integral must converge.

A signal  $x(t)$  is Laplace transformable if the integral exists

$$\left| \int_{0^-}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt \right| < \infty$$

↪ 存在

$$\left| \int_{0^-}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt \right| \leq \int_{0^-}^{\infty} |x(t) e^{-(\sigma + j\omega)t}| dt \leq \int_{0^-}^{\infty} \underbrace{|x(t)|}_{=0} |e^{-(\sigma + j\omega)t}| dt < \infty$$

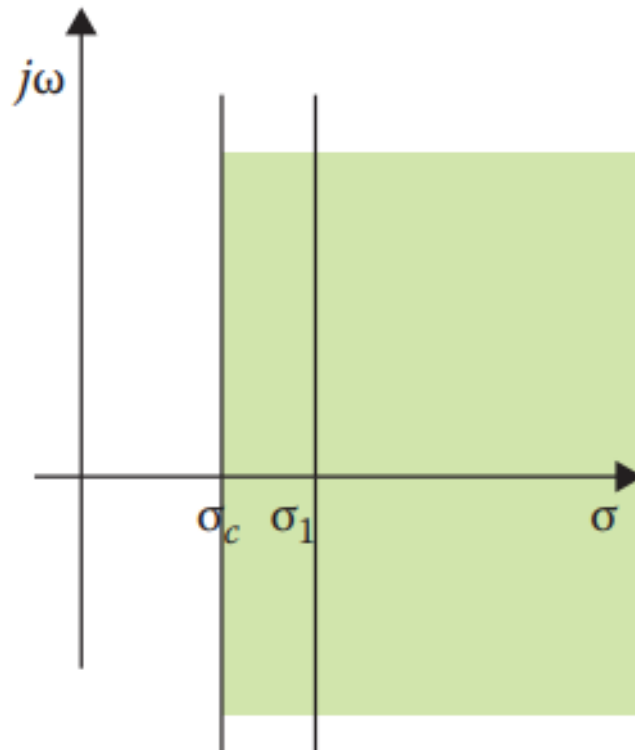
Since  $|e^{j\omega t}| = 1$  for any value of  $t$ ,

$$\int_{0^-}^{\infty} |x(t)| e^{-\sigma t} dt < \infty \quad \text{for some real value of } \sigma = \sigma_c.$$

- Region of Convergence (ROC)

The range of  $s$  for which the Laplace transform converges.

The region of convergence (ROC) for Laplace transform is  $\text{Re}(s) = \sigma > \sigma_c$



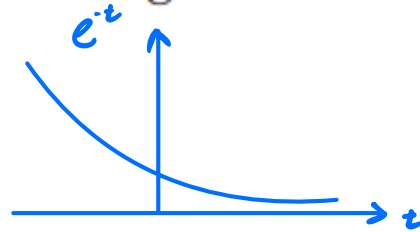


# Example 3.1

Find the Laplace transform of the following functions and establish the ROC for each case.

(a)  $e^{-5t}u(t)$

(b)  $\delta(t)$



$$\mathcal{L}[e^{-at}u(t)] = \frac{1}{s+a}$$

Solution

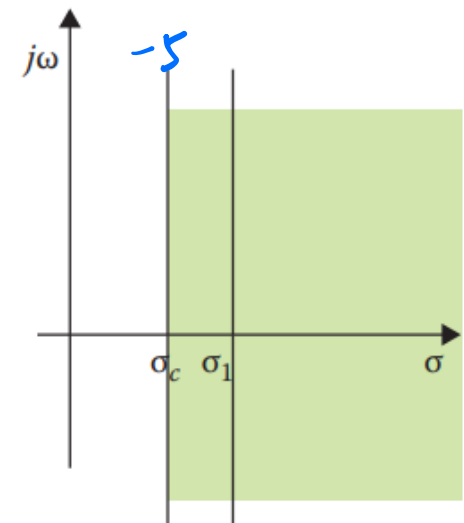
(a)

$$\mathcal{L}[e^{-5t}u(t)] = \int_{0^-}^{\infty} \underbrace{e^{-5t}}_{\substack{e^{-(s+5)t} \\ \parallel}} \underbrace{e^{-st}}_{\substack{0 - (-\frac{1}{s+5}) \\ \parallel}} dt = -\frac{1}{s+5} e^{-(s+5)t} \bigg|_{0^-}^{\infty} = \frac{1}{s+5}$$

The ROC is obtained from

$$|e^{-(s+5)t}| = |e^{-(\sigma+5)t}| \underbrace{|e^{-j\omega t}|}_{\substack{\text{wavy line} \\ \parallel \\ 1}} = |e^{-(\sigma+5)t}| < \infty$$

which is valid when  $\sigma + 5 > 0$  or  $\sigma > -5 = \sigma_c$ .



## Example 3.1

### Solution

(b)

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = e^{-0} = 1 \quad \text{for all } s$$

*Handwritten notes: // 0 - (-0)*

The transform does not depend on  $s$  and hence the region of convergence is the entire  $s$  plane.

## Example 3.2

Find the Laplace transform of  $x(t) = \sin \omega t u(t)$ .

Solution

$$\begin{aligned} e^{i\theta} &= \cos \theta + i \sin \theta \\ e^{-i\theta} &= \cos \theta - i \sin \theta \end{aligned}$$

$$\begin{aligned} X(s) &= \mathcal{L}[\sin \omega t] = \int_0^{\infty} (\sin \omega t) e^{-st} dt = \int_0^{\infty} \left( \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right) e^{-st} dt \\ &= \frac{1}{2j} \int_0^{\infty} \left( e^{-(s-j\omega)t} - e^{-(s+j\omega)t} \right) dt \\ &= \frac{1}{2j} \left( \frac{1}{s-j\omega} - \frac{1}{s+j\omega} \right) = \frac{\omega}{s^2 + \omega^2} \end{aligned}$$

$\mathcal{L}[e^{-at} u(t)] = \frac{1}{s+a}$
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## 3.3 PROPERTIES OF THE LAPLACE TRANSFORM

- 線性  
Linearity

$$\boxed{\mathcal{L}[a_1x_1(t) + a_2x_2(t)] = a_1X_1(s) + a_2X_2(s)}$$

$$\begin{aligned}\mathcal{L}[a_1x_1(t) + a_2x_2(t)] &= \int_0^{\infty} [a_1x_1(t) + a_2x_2(t)] e^{-st} dt \\ &= a_1 \int_0^{\infty} x_1(t) e^{-st} dt + a_2 \int_0^{\infty} x_2(t) e^{-st} dt \\ &= a_1X_1(s) + a_2X_2(s)\end{aligned}$$

伸縮

- Scaling

$$\mathcal{L}[x(at)] = \frac{1}{a} X\left(\frac{s}{a}\right), \quad a > 0$$

Let  $\lambda = at$ ,  $d\lambda = a dt$ ,  $\tau = \frac{\lambda}{a}$   
變數變換

$$\begin{aligned}\mathcal{L}[x(at)] &= \int_0^{\infty} x(at) e^{-st} dt = \frac{1}{a} \int_0^{\infty} x(\lambda) e^{-\lambda \left(\frac{s}{a}\right)} d\lambda \\ &= \frac{1}{a} X\left(\frac{s}{a}\right), \quad a > 0\end{aligned}$$

平移時間

- Time Shifting

$$\mathcal{L}[x(t-a)u(t-a)] = e^{-as}X(s)$$

↳ 頻率不變

$$\begin{aligned}\mathcal{L}[x(t-a)u(t-a)] &= \int_0^{\infty} x(t-a)u(t-a)e^{-st}dt, \quad a \geq 0 \\ &= \int_a^{\infty} x(t-a)(1)e^{-st}dt\end{aligned}$$

substituting  $\lambda = t - a$ ,  $d\lambda = dt$ , and  $t = \lambda + a$ . As  $t \rightarrow a$ ,  $\lambda \rightarrow 0$ , and as  $t \rightarrow \infty$ ,  $\lambda \rightarrow \infty$

$$\mathcal{L}[x(t-a)u(t-a)] = \int_0^{\infty} x(\lambda)e^{-s(\lambda+a)}d\lambda$$

$$= e^{-as} \int_0^{\infty} x(\lambda)e^{-s\lambda}d\lambda$$

$$= e^{-as}X(s)$$

• 平移頻率  
Frequency Shifting

$$\mathcal{L}[e^{-at}x(t)u(t)] = X(s+a)$$

↳ domain shift a

$$\mathcal{L}[e^{-at}x(t)u(t)] = \int_0^{\infty} e^{-at}x(t)e^{-st}dt = \int_0^{\infty} x(t)e^{-(s+a)t}dt = X(s+a)$$

• 微分  
Time Differentiation

$$\mathcal{L}[x'(t)] = sX(s) - x(0^-)$$

初值

$$\frac{d}{dt}x(t)e^{-st} = \frac{dx}{dt}e^{-st} + x(t)(-s e^{-st})$$

$$\frac{dx}{dt}e^{-st} = \frac{d}{dt}x(t)e^{-st} - x(t)(-s e^{-st})$$

左右積分  
⇒

$$\int_0^{\infty} \frac{dx}{dt}e^{-st}dt = \int_0^{\infty} \left( \frac{d}{dt}x(t)e^{-st} \right)dt - \int_0^{\infty} x(t)(-s e^{-st})dt$$

$$= \underbrace{x(t)e^{-st}}_{0^-} \Big|_0^{\infty} - \int_0^{\infty} x(t)[-se^{-st}]dt = 0 - x(0^-) + s \int_0^{\infty} x(t)e^{-st}dt = sX(s) - x(0^-)$$

- Time Convolution  $\xrightarrow{\text{conv} \rightarrow \text{相乘}}$   $\mathcal{L}[x(t) * h(t)] = X(s)H(s)$

$$\mathcal{L}[x(t) * h(t)] = \int_0^{\infty} \left[ \int_0^{\infty} h(\tau) x(t - \tau) d\tau \right] e^{-st} dt$$

$$\mathcal{L}[x(t) * h(t)] = \int_0^{\infty} h(\tau) \left[ \int_0^{\infty} x(t - \tau) e^{-st} dt \right] d\tau$$

$$= \int_0^{\infty} h(\tau) \left[ \int_0^{\infty} x(\lambda) e^{-s(\tau + \lambda)} d\lambda \right] d\tau$$

$$= \int_0^{\infty} h(\tau) e^{-s\tau} d\tau \int_0^{\infty} x(\lambda) e^{-s\lambda} d\lambda = H(s)X(s)$$



+

➤ Laplace transform of  $u(t)$ ?

$$\mathcal{L}[e^{-at}u(t)] = \frac{1}{s+a}$$

Let  $a = 0$ , we get  $L[u(t)] = 1/s$

➤ Laplace transform of  $r(t)$ ?

$$r(t) = u(t) * u(t), \quad t \geq 0$$



$$L[t] = \frac{1}{s} \frac{1}{s} = \frac{1}{s^2}, \quad t \geq 0$$

$\gamma(t)$  or  $t \cdot u(t)$

↳ 確保 0 以下為 0

積分

- Time Integration

$$\mathcal{L}\left[\int_0^t x(t) dt\right] = \frac{1}{s} X(s)$$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

$$\mathcal{L}\left[\int_0^t x(\lambda) d\lambda\right] = \mathcal{L}\left[\int_0^\infty x(\lambda) u(t-\lambda) d\lambda\right] = \mathcal{L}[u(t) * x(t)] = \frac{1}{s} X(s)$$

$$\mathcal{L}\left[\int_0^t x(t) dt\right] = \frac{1}{s} X(s)$$

## + • Frequency Differentiation

Let  $X(s)$  be the Laplace transform of the signal  $x(t)$ . Then Laplace transform of the frequency differentiation is given as

$$\mathcal{L}[tx(t)] = -\frac{dX(s)}{ds}$$

↪ 對 s 微分

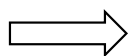
$$X(s) = \int_0^{\infty} x(t) e^{-st} dt$$

Differentiating both sides, we get

$$\frac{dX(s)}{ds} = \int_0^{\infty} x(t)(-te^{-st})dt = \int_0^{\infty} (-tx(t))e^{-st}dt = \mathcal{L}[-tx(t)]$$

The frequency differentiation property is given as

$$\mathcal{L}[tx(t)] = -\frac{dX(s)}{ds}$$



推廣  $\mathcal{L}[t^n x(t)] = (-1)^n \frac{d^n X(s)}{ds^n}$

# 3.4 THE INVERSE LAPLACE TRANSFORM

Inverse Laplace transform

$$\mathcal{L}^{-1}[X(s)] = x(t) = \frac{1}{2\pi j} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} X(s)e^{st} ds \quad \text{不用}$$

$$X(s) = \frac{N(s)}{D(s)} \quad \begin{array}{l} N(s) \text{ is the numerator polynomial} \\ D(s) \text{ is the denominator polynomial} \end{array}$$

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + b_{m-2} s^{m-2} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0}$$

$$X(s) = \frac{N(s)}{D(s)} = k \frac{(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)} \quad \begin{array}{l} \text{roots of } N(s) = 0 \text{ are called the } \textit{zeros} \text{ of } X(s) \\ \text{roots of } D(s) = 0 \text{ are the } \textit{poles} \text{ of } X(s), \end{array}$$

where  $k = b_m/a_n$ .

- Simple Poles

$$X(s) = \frac{N(s)}{(s + p_1)(s + p_2) \cdots (s + p_n)}$$

$$* \mathcal{L}\{e^{-at} \cdot u(t)\} = \frac{1}{s+a}$$

$$X(s) = \frac{k_1}{s + p_1} + \frac{k_2}{s + p_2} + \cdots + \frac{k_n}{s + p_n}$$

$$(s + p_1)X(s) = k_1 + \frac{(s + p_1)k_2}{s + p_2} + \cdots + \frac{(s + p_1)k_n}{s + p_n} \quad \text{同乘}(s+p_1)$$

$$(s + p_1)X(s) \Big|_{s=-p_1} = k_1$$

$$\boxed{k_i = (s + p_i)X(s) \Big|_{s=-p_i}} \quad \text{用此法算出各項係數}$$

$$\mathcal{L}^{-1}[k / (s + a)] = ke^{-at}u(t)$$

$$\boxed{x(t) = \left( k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \cdots + k_n e^{-p_n t} \right) u(t)}$$

- Repeated Poles

$$X(s) = \frac{k_n}{(s+p)^n} + \frac{k_{n-1}}{(s+p)^{n-1}} + \cdots + \frac{k_2}{(s+p)^2} + \frac{k_1}{s+p} + X_1(s) \quad \text{同乘 } (s+p)^n$$

$$k_n = (s+p)^n X(s) \Big|_{s=-p}$$

$$\star \mathcal{L}^{-1} \left[ \frac{1}{(s+a)^n} \right] = \frac{t^{n-1} e^{-at}}{(n-1)!}$$

$$k_{n-1} = \frac{d}{ds} \left[ (s+p)^n X(s) \right] \Big|_{s=-p}$$

$$x(t) = \left( k_1 e^{-pt} + k_2 t e^{-pt} + \frac{k_3}{2!} t^2 e^{-pt} + \cdots + \frac{k_n}{(n-1)!} t^{n-1} e^{-pt} \right) u(t) + x_1(t)$$

$$k_{n-2} = \frac{1}{2!} \frac{d^2}{ds^2} \left[ (s+p)^n X(s) \right] \Big|_{s=-p}$$

$$k_{n-m} = \frac{1}{m!} \frac{d^m}{ds^m} \left[ (s+p)^n X(s) \right] \Big|_{s=-p}$$

- Complex Poles

$$X(s) = \frac{A_1 s + A_2}{s^2 + as + b} + X_1(s)$$

$$s^2 + as + b = s^2 + 2\alpha s + \alpha^2 + \beta^2 = (s + \alpha)^2 + \beta^2$$

$$A_1 s + A_2 = A_1(s + \alpha) + B_1 \beta$$

$$X(s) = \frac{A_1(s + \alpha)}{(s + \alpha)^2 + \beta^2} + \frac{B_1 \beta}{(s + \alpha)^2 + \beta^2} + X_1(s)$$

$$x(t) = \left( A_1 e^{-\alpha t} \cos \beta t + B_1 e^{-\alpha t} \sin \beta t \right) u(t) + x_1(t)$$

$$\mathcal{L} \left[ e^{-at} x(t) u(t) \right] = X(s + a)$$

$$u(t) \cos(\omega_o t) \Leftrightarrow \frac{s}{s^2 + \omega_o^2}$$

$$u(t) \sin(\omega_o t) \Leftrightarrow \frac{\omega_o}{s^2 + \omega_o^2}$$

The sine and cosine terms can be combined if desired.

$$A \cos \theta + B \sin \theta = R \cos(\theta - \phi)$$

$$R = \sqrt{A^2 + B^2}$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\phi = \tan^{-1}(B/A)$$

## Example 3.8

Find the inverse Laplace transform of

$$X(s) = 1 + \frac{2}{s} + \frac{4}{s-1} - \frac{3s}{s^2+9}$$

背公式反向化簡

Solution

$$\begin{aligned} x(t) &= \overset{\delta(t)}{\mathcal{L}^{-1}(1)} + \overset{2u(t)}{\mathcal{L}^{-1}\left(\frac{2}{s}\right)} + \overset{4e^{t}u(t)}{\mathcal{L}^{-1}\left(\frac{4}{s-1}\right)} - \overset{-3\sin(3t)u(t)}{3\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right)} \\ &= \delta(t) + (2 + 4e^t - 3\cos 3t)u(t) \end{aligned}$$



## Example 3.9

Obtain  $h(t)$  given that

$$H(s) = \frac{4}{(s+1)(s+3)}$$

Solution

 Residue method

拆成部分分式

$$H(s) = \frac{4}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$$\underline{A = (s+1)H(s)|_{s=-1}} = \frac{4}{(s+3)} \Big|_{s=-1} = \frac{4}{(2)} = 2$$

$$\underline{B = (s+3)H(s)|_{s=-3}} = \frac{4}{(s+1)} \Big|_{s=-3} = \frac{4}{(-2)} = -2$$

$$H(s) = \frac{4}{(s+1)(s+3)} = \frac{2}{s+1} - \frac{2}{s+3}$$

$$h(t) = (2e^{-t} - 2e^{-3t})u(t)$$

Algebraic method

$$4 = A(s+3) + B(s+1)$$

# Example 3.10

Determine  $x(t)$  given that

$$X(s) = \frac{2s^2 + 4s + 1}{(s+1)(s+2)^3}$$

Solution

拆成部分分式

$$X(s) = \frac{2s^2 + 4s + 1}{(s+1)(s+2)^3} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3} = -e^{-t} + e^{-2t} + 3 \cdot \frac{t^1}{1!} e^{-2t} - \frac{t^2}{2!} e^{-2t}$$

$$A = F(s)(s+1) \Big|_{s=-1} = \frac{2-4+1}{-1+2} = -1$$

$$D = F(s)(s+2)^3 \Big|_{s=-2} = \frac{8-8+1}{-2+1} = -1$$

首、尾項才能看出來  $\Rightarrow$  中間項用通分較快

$$\begin{aligned} C &= \frac{d}{ds} \left[ (s+2)^3 X(s) \right] \Big|_{s=-2} = \frac{d}{ds} \left[ \frac{2s^2 + 4s + 1}{(s+1)} \right] \Big|_{s=-2} \\ &= \frac{(s+1)(4s+4) - (2s^2 + 4s + 1) \times 1}{(s+1)^2} \Big|_{s=-2} = \frac{2s^2 + 4s + 3}{(s+1)^2} \Big|_{s=-2} \\ &= \frac{8-8+3}{1} = 3 \end{aligned}$$

$$\begin{aligned} B &= \frac{1}{2} \frac{d^2}{ds^2} \left[ (s+2)^3 X(s) \right] \Big|_{s=-2} = \frac{1}{2} \frac{d}{ds} \left[ \frac{2s^2 + 4s + 3}{(s+1)^2} \right] \Big|_{s=-2} \\ &= \frac{(s+1)^2(4s+4) - (2s^2 + 4s + 3) \times 2(s+1)}{2(s+1)^4} \Big|_{s=-2} = \frac{-2}{2(s+1)^3} \Big|_{s=-2} \\ &= \frac{-2}{-2} = 1 \end{aligned}$$

# Example 3.11

Find the inverse transform of

$$G(s) = \frac{s+1}{(s+2)(s^2+2s+5)}$$

## Solution

$G(s)$  has a pair of complex poles at  $s^2 + 2s + 5 = 0$  or  $s = -1 \pm j2$ . We let

拆成部分分式

$$G(s) = \frac{s+1}{(s+2)(s^2+2s+5)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+2s+5}$$

$$A = G(s)(s+2)|_{s=-2} = \frac{-1}{5}$$

$$s+1 = A(s^2+2s+5) + B(s^2+2s) + C(s+2)$$

$$s^2 : 0 = A + B \rightarrow B = -A = \frac{1}{5} \quad \Rightarrow$$

$$s^1 : 1 = 2A + 2B + C = 0 + C \rightarrow C = 1$$

$$s^0 : 1 = 5A + 2C = -1 + 2 = 1$$

硬凑

$$G(s) = \frac{-1/5}{s+2} + \frac{1/5 \cdot s+1}{(s+1)^2+2^2} = \frac{-1/5}{s+2} + \frac{1/5(s+1)}{(s+1)^2+2^2} + \frac{4/5}{(s+1)^2+2^2}$$

||  
 $\frac{2 * 0.4}{(s+1)^2+2^2}$

$$g(t) = (-0.2e^{-2t} + 0.2e^{-t} \cos(2t) + 0.4e^{-t} \sin(2t))u(t)$$

## Example 3.12

Consider the rectangular pulse or gate function  $x(t) = u(t) - u(t - 2)$ . Obtain  $y(t) = x(t) * x(t)$ , that is, the convolution of the rectangular pulse with itself.

### Solution

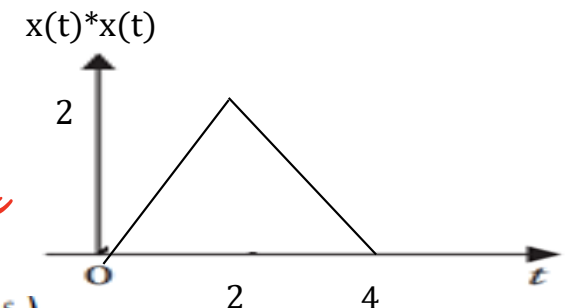
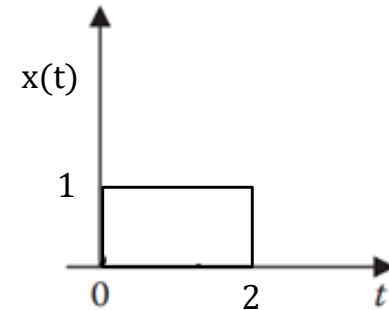
The Laplace transform of  $x(t)$  is

$$X(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

Using the convolution property,

$$Y(s) = X(s)X(s) = X^2(s) = \frac{1}{s^2} \left( 1 - 2e^{-2s} + e^{-4s} \right)$$

*t shift*



Taking the inverse Laplace transform of each term,

$$\begin{aligned} y(t) &= x(t) * x(t) = tu(t) - 2(t-2) + (t-4)u(t-4) \\ &= r(t) - 2r(t-2) + r(t-4) \end{aligned}$$


*$\mathcal{L}\{t \cdot u(t)\} = \frac{1}{s^2}$*

# Example 3.13

Determine the inverse Laplace transform of

$$X(s) = \frac{se^{-2s} + e^{-3s}}{s(s^2 + 5s + 4)}$$

## Solution

Let  $X(s) = X_1(s)e^{-2s} + X_2(s)e^{-3s}$  

$$X_1(s) = \frac{s}{s(s^2 + 5s + 4)} = \frac{1}{(s+1)(s+4)} = \frac{A}{s+1} + \frac{B}{s+4}$$

$$A = \frac{1}{s+4} \Big|_{s=-1} = \frac{1}{-1+4} = \frac{1}{3} \quad B = \frac{1}{s+1} \Big|_{s=-4} = \frac{1}{-4+1} = -\frac{1}{3}$$

$$X_2(s) = \frac{1}{s(s^2 + 5s + 4)} = \frac{1}{s(s+1)(s+4)} = \frac{C}{s} + \frac{D}{s+1} + \frac{E}{s+4}$$

$$C = \frac{1}{(s+1)(s+4)} \Big|_{s=0} = \frac{1}{(1)(4)} = \frac{1}{4} \quad D = \frac{1}{s(s+4)} \Big|_{s=-1} = \frac{1}{(-1)(3)} = -\frac{1}{3}$$

$$E = \frac{1}{s(s+1)} \Big|_{s=-4} = \frac{1}{(-4)(-3)} = \frac{1}{12}$$

$$X_1(s) = \frac{1}{3} \left[ \frac{1}{s+1} - \frac{1}{s+4} \right]$$

$$x_1(t) = \frac{1}{3} (e^{-t} - e^{-4t}) u(t)$$

$$X_2(s) = \frac{1/4}{s} - \frac{1/3}{s+1} + \frac{1/12}{s+4}$$

$$x_2(t) = \left[ \frac{1}{4} - \frac{1}{3} e^{-t} + \frac{1}{12} e^{-4t} \right] u(t)$$

$$x(t) = x_1(t-2)u(t-2) + x_2(t-3)u(t-3)$$

$$= \frac{1}{3} (e^{-(t-2)} - e^{-4(t-2)}) u(t-2) + \left[ \frac{1}{4} - \frac{1}{3} e^{-(t-3)} + \frac{1}{12} e^{-4(t-3)} \right] u(t-3)$$

# Example 3.14

Obtain  $g(t)$  given that  $G(s) = \frac{s^3 + 5s^2 + 10}{s^2 + 3s + 2}$

## Solution

$$\begin{array}{r} s+2 \\ s^2+3s+2 \overline{) s^3+5s^2+0s+10} \\ \underline{s^3+3s^2+2s} \phantom{+10} \\ 2s^2-2s+10 \\ \underline{2s^2+6s+4} \\ -8s+6 \end{array} \quad \left. \vphantom{\begin{array}{r} s+2 \\ s^2+3s+2 \overline{) s^3+5s^2+0s+10} \\ \underline{s^3+3s^2+2s} \phantom{+10} \\ 2s^2-2s+10 \\ \underline{2s^2+6s+4} \\ -8s+6 \end{array}} \right\} \text{長除法}$$



$$G(s) = s + 2 + \frac{-8s + 6}{s^2 + 3s + 2}$$

$$G(s) = \underline{s + 2} + \frac{14}{s+1} - \frac{22}{s+2}$$

$$\text{Let } Y(s) = \frac{-8s+6}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$\underline{\underline{g(t) = \frac{d}{dt} \delta(t) + 2\delta(t) + (14e^{-t} - 22e^{-2t})u(t)}}$$

$$A = \left. \frac{-8s+6}{s+2} \right|_{s=-1} = \frac{8+6}{-1+2} = 14$$

$$B = \left. \frac{-8s+6}{s+1} \right|_{s=-2} = \frac{16+6}{-2+1} = -22$$

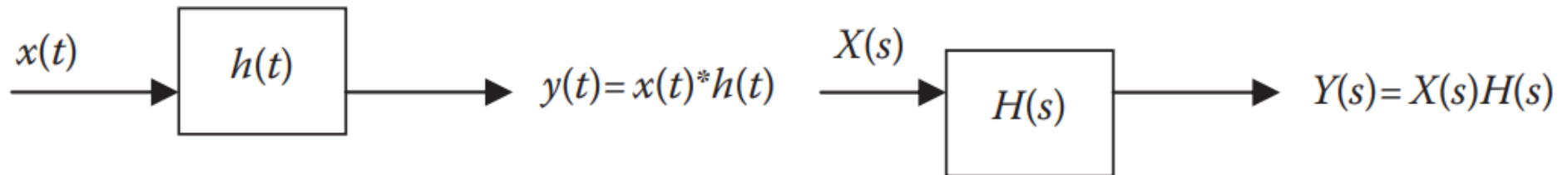
## 3.5 TRANSFER FUNCTION

The transfer function  $H(s)$  is defined as the ratio of the output response  $Y(s)$  to the input excitation  $X(s)$ , assuming all initial conditions are zero.

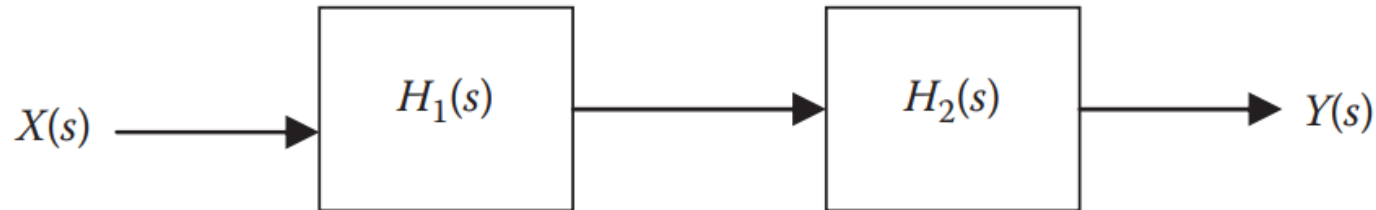
$$H(s) = \frac{Y(s)}{X(s)}$$

$$y(t) = x(t) * h(t)$$

$$Y(s) = X(s)H(s)$$



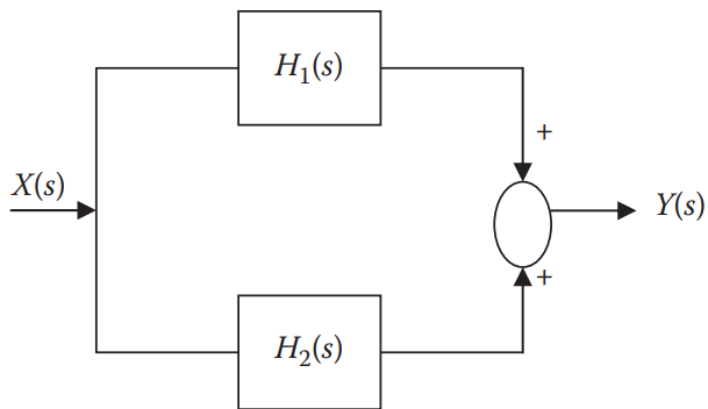
- Cascade connection



$$Y(s) = \underline{H_1(s)H_2(s)}X(s) \qquad H(s) = \frac{Y(s)}{X(s)} = H_1(s)H_2(s)$$

串聯  $\Rightarrow$  conv

- Parallel interconnection



$$Y(s) = [H_1(s) + H_2(s)]X(s)$$

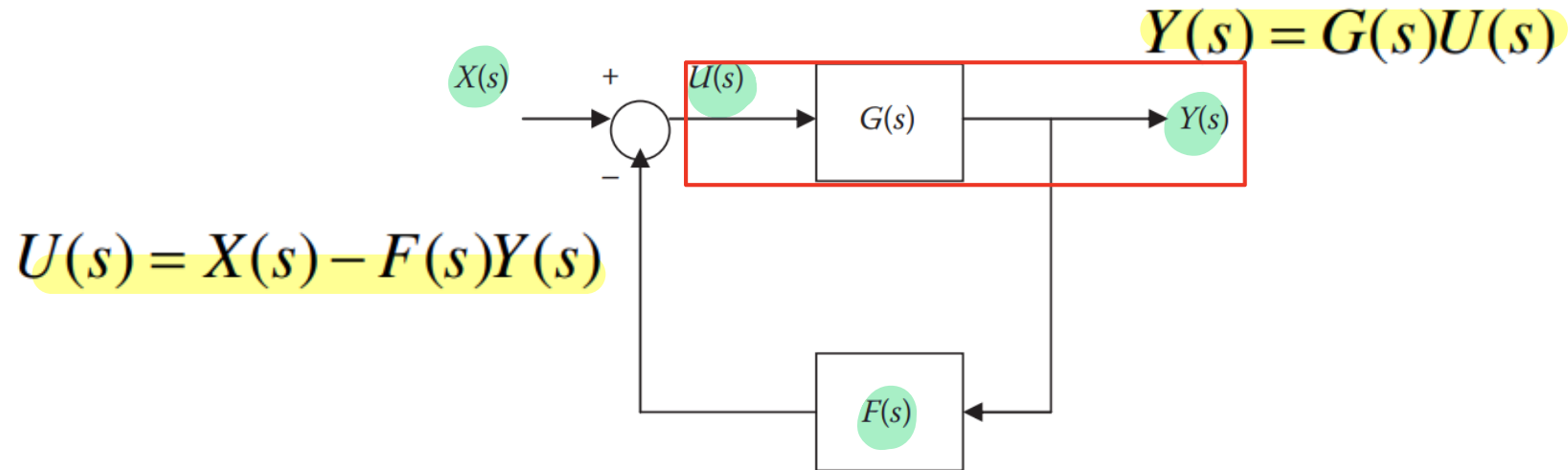
$$H(s) = \frac{Y(s)}{X(s)} = H_1(s) + H_2(s)$$

---

並聯  $\Rightarrow$  相加



- Feedback interconnection



$$Y(s) = G(s) \underbrace{(X(s) - F(s)Y(s))}_{U(s)}$$

$$Y(s) + G(s)F(s)Y(s) = G(s)X(s)$$

目標

$$H(s) = \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + F(s)G(s)}$$

## Example 3.15

The output of a linear system is  $y(t) = 10e^{-t}\cos 4t u(t)$ , when the input is  $x(t) = e^{-t}u(t)$  ;  
Find the transfer function of the system.

### Solution

$$X(s) = \frac{1}{s+1} \quad \text{and} \quad Y(s) = \frac{10(s+1)}{(s+1)^2 + 4^2}$$



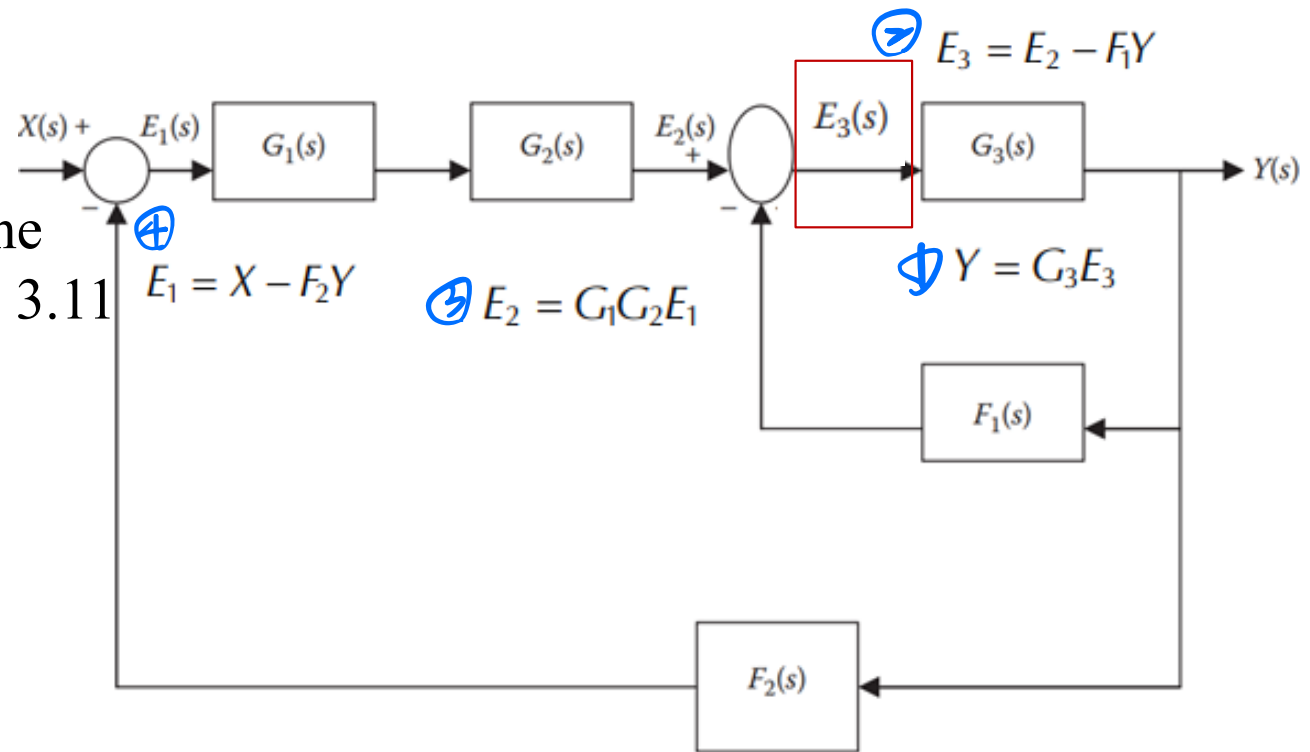
$$H(s) = \frac{Y(s)}{X(s)} = \frac{10(s+1)^2}{(s+1)^2 + 16} = \frac{10(s^2 + 2s + 1)}{s^2 + 2s + 17} \quad \text{長除法}$$

$$= 10 - 40 \frac{4}{s^2 + 2s + 17} = \frac{4}{(s+1)^2 + 4^2}$$

$$h(t) = 10\delta(t) - 40e^{-t} \sin 4t u(t)$$

## Example 3.16

Find the transfer function for the feedback system shown in Fig. 3.11



Solution

$$\frac{Y}{G_3} = E_3 - F_1Y = G_1G_2E_1 - F_1Y = G_1G_2[X - F_2Y] - F_1Y$$

$$Y = G_1G_2G_3X - G_1G_2G_3F_2Y - F_1G_3Y$$

$$H = \frac{Y}{X} = \frac{G_1G_2G_3}{1 + F_1G_3 + F_2G_1G_2G_3}$$

## 3.6 APPLICATIONS

- Integro-Differential Equations

### Example 3.17

Use the Laplace transform to solve the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{-t}u(t) \quad y(0) = 1, \quad \overset{y'(0)}{\frac{dy(0)}{dt}} = 0.$$

### Solution

$$\left[ s^2 Y(s) - \overset{\text{常數}}{s y(0)} - \overset{\text{常數}}{y'(0)} \right] + 3 \left[ s Y(s) - y(0) \right] - 4 Y(s) = \frac{1}{s+1}$$

$$(s^2 + 3s - 4)Y(s) = s + 3 + \frac{1}{s+1} = \frac{s^2 + 4s + 4}{s+1}$$

$$Y(s) = \frac{s^2 + 4s + 4}{(s^2 + 3s - 4)(s+1)} = \frac{s^2 + 4s + 4}{(s-1)(s+1)(s+4)}$$

$$Y(s) = \frac{s^2 + 4s + 4}{(s^2 + 3s - 4)(s + 1)} = \frac{s^2 + 4s + 4}{(s - 1)(s + 1)(s + 4)} = \frac{A}{(s - 1)} + \frac{B}{(s + 1)} + \frac{C}{(s + 4)}$$

$$A = (s - 1)Y(s)\big|_{s=1} = \frac{1 + 4 + 4}{2(5)} = \frac{9}{10}$$

$$B = (s + 1)Y(s)\big|_{s=-1} = \frac{1 - 4 + 4}{(-2)(3)} = \frac{1}{-6}$$

$$C = (s + 4)Y(s)\big|_{s=-4} = \frac{16 - 16 + 4}{(-5)(-3)} = \frac{4}{15}$$

$$Y(s) = \frac{9/10}{(s - 1)} - \frac{1/6}{(s + 1)} + \frac{4/15}{(s + 4)}$$

$$y(t) = \left( \frac{9}{10} e^t - \frac{1}{6} e^{-t} + \frac{4}{15} e^{-4t} \right) u(t)$$

## Example 3.18

Solve the integro-differential equation

$$\frac{dv}{dt} + 2v + 5 \int_0^t v(\lambda) d\lambda = 4u(t)$$

with  $v(0) = -1$  and determine  $v(t)$  for  $t > 0$ .

Solution

$$[sV(s) - \overset{\text{常數}}{v(0)}] + 2V(s) + \frac{5}{s}V(s) = \frac{4}{s}$$

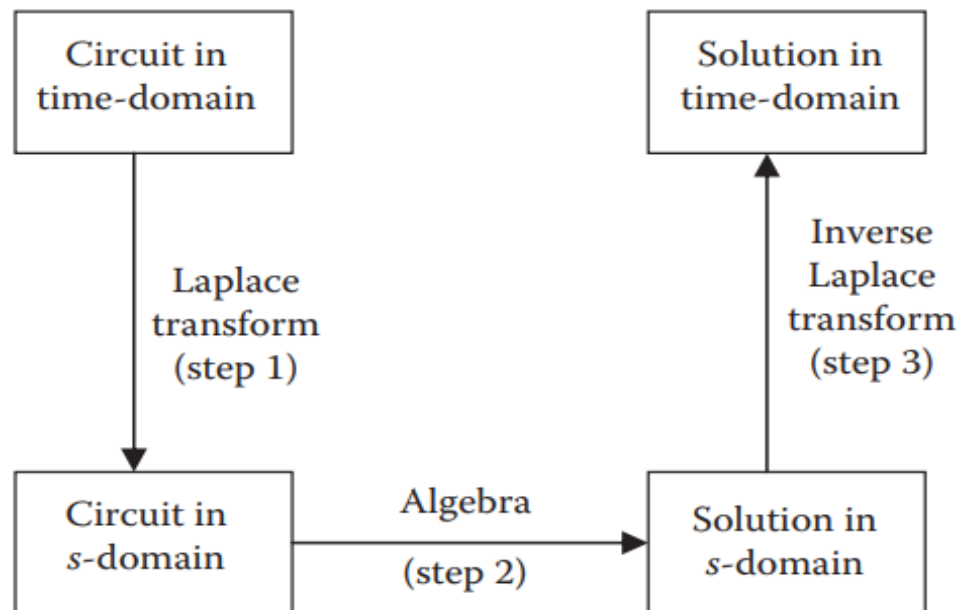
$$[sV(s) + 1] + 2V(s) + \frac{5}{s}V(s) = \frac{4}{s} \rightarrow V(s) = \frac{4-s}{s^2 + 2s + 5}$$

$$V(s) = \frac{-(s+1) + 5}{(s+1)^2 + 2^2} = \frac{-(s+1)}{(s+1)^2 + 2^2} + \frac{5}{2} \frac{2}{(s+1)^2 + 2^2}$$

$$v(t) = (-e^{-t} \cos 2t + 2.5e^{-t} \sin 2t)u(t)$$

# • Circuit Analysis 不考!

1. Laplace transform the circuit from the time-domain to the frequency domain (or  $s$ -domain).
2. The circuit in  $s$ -domain is solved using circuit analysis techniques (such as voltage division, current division, nodal analysis, mesh analysis, source transformation, and superposition) and we obtain the desired quantity  $X(s)$ .
3. Obtain the inverse Laplace transform of  $X(s)$  to get the desired solution  $x(t)$  in the time domain.



- Resistor

$$v(t) = Ri(t)$$

$$V(s) = RI(s)$$

- Inductor

$$v(t) = L \frac{di}{dt}$$

$$V(s) = L \left[ sI(s) - i(0^-) \right] = sLI(s) - Li(0^-)$$

- Capacitor

$$i(t) = C \frac{dv(t)}{dt}$$

$$I(s) = C \left[ sV(s) - v(0^-) \right] = sCV(s) - Cv(0^-)$$

$$V(s) = \frac{1}{sC} I(s) + \frac{v(0^-)}{s}$$



## Example 3.19

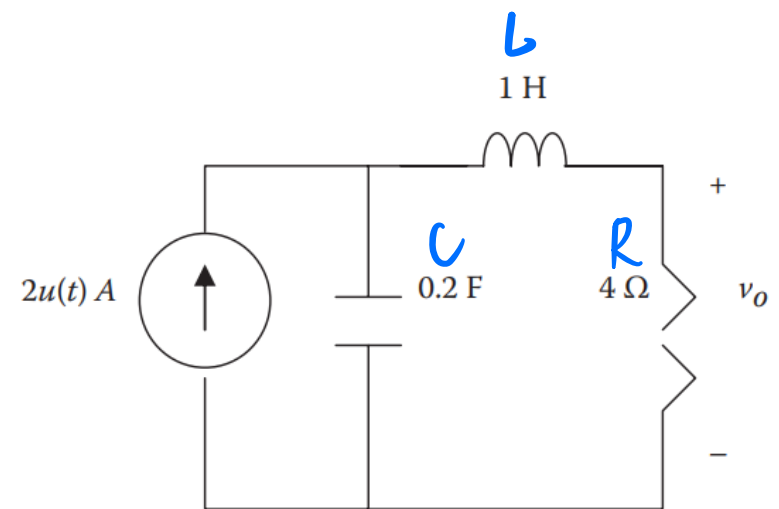
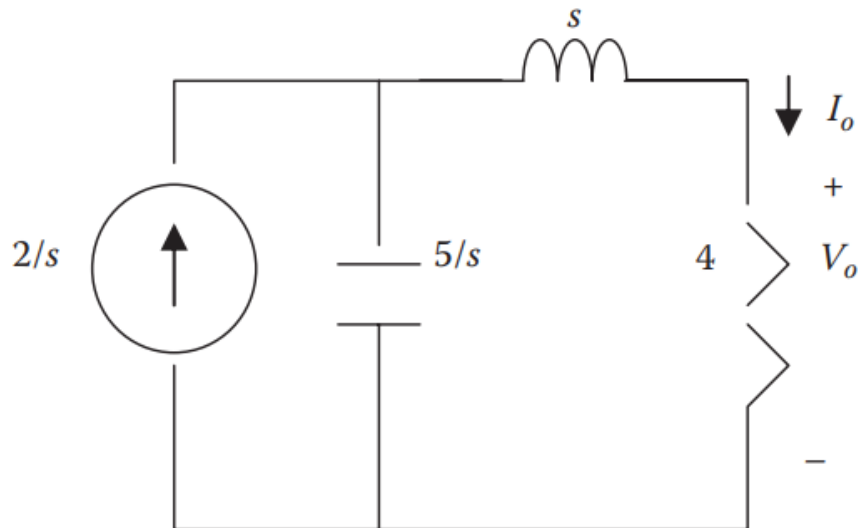
Find  $v_o(t)$  in the circuit in Figure 3.14.

### Solution

$$2u(t) \Rightarrow \frac{2}{s}$$

$$1\text{ H} \Rightarrow sL = s$$

$$0.2\text{ F} \Rightarrow \frac{1}{sC} = \frac{1}{0.2s} = \frac{5}{s}$$



$$I_o = \frac{5/s}{\frac{5/s + s + 4}{\text{分流}}} \times \frac{2}{s} = \frac{5}{s^2 + 4s + 5} \times \frac{2}{s}$$

$$V_o = 4I_o = \frac{40}{s(s^2 + 4s + 5)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 5}$$

$$40 = A(s^2 + 4s + 5) + Bs^2 + Cs$$

$$A = 8 \quad C = -4A = -32 \quad B = -A = -8$$

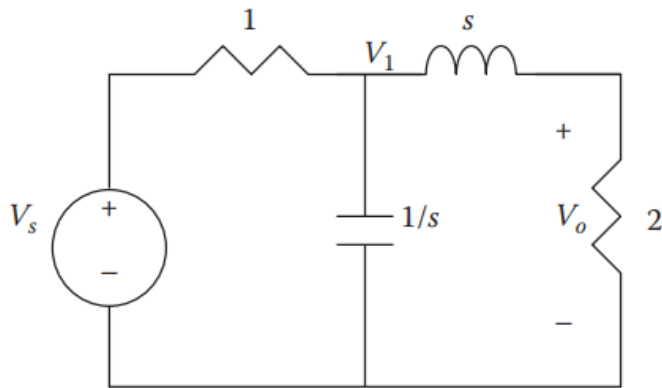
$$V_o = \frac{8}{s} - \frac{8s + 32}{(s+2)^2 + 1} = \frac{8}{s} - \frac{8(s+2)}{(s+2)^2 + 1} - \frac{16}{(s+2)^2 + 1}$$

$$v_o(t) = (8 - 8e^{-2t} \cos t - 16e^{-2t} \sin t)u(t)$$

## Example 3.20

For the circuit in Figure 3.17, find  $H(s) = V_o(s)/V_s(s)$ . Assume zero initial conditions.

Solution

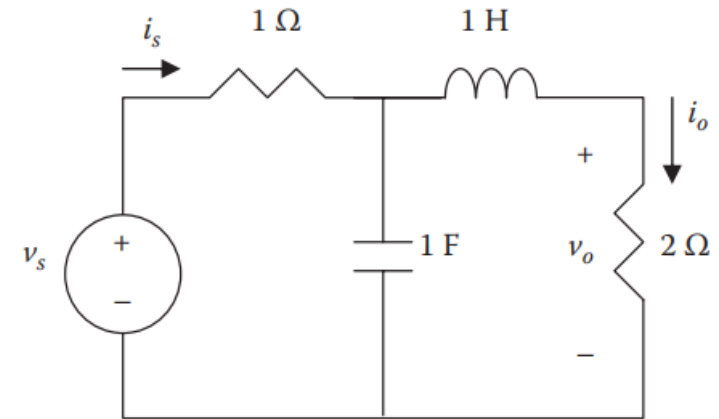


$$Z = \frac{1}{s} \parallel (s+2) = \frac{1/s(s+2)}{1/s + s+2} = \frac{(s+2)}{(s^2 + 2s + 1)}$$

$$V_1 = \frac{Z}{Z+1} V_s$$

$$V_o = \frac{2}{s+2} V_1 = \frac{2}{s+2} \frac{Z}{Z+1} V_s$$

分壓



$$H(s) = \frac{V_o}{V_s} = \frac{2}{s+2} \times \frac{\frac{s+2}{s^2 + 2s + 1}}{\frac{s+2}{s^2 + 2s + 1} + 1}$$

$$= \frac{\boxed{2}}{s^2 + 3s + 3}$$

# Recall 數位電路導論

## Example 4.2 RL Transient Analysis

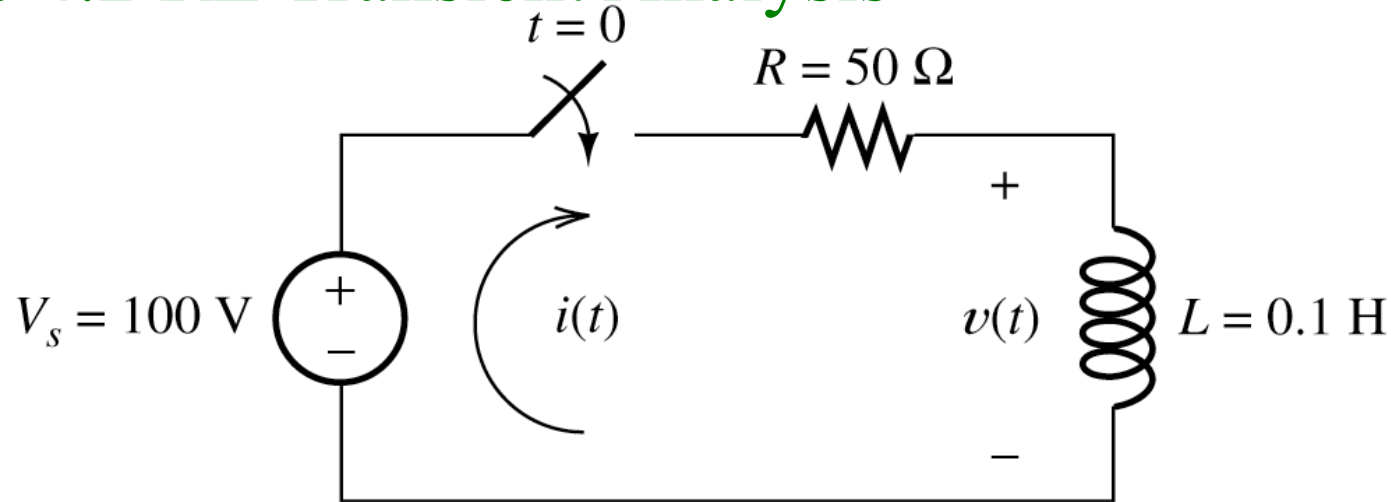


Figure 4.7 The circuit analyzed in Example 4.2.

1.  $t < 0$ ,  $i(t) = 0$ .

2.  $t > 0$

KVL

$$Ri(t) + L \frac{di}{dt} = V_s \quad \left( v(t) = L \frac{di}{dt} \right)$$

## Example 4.2 RL Transient Analysis

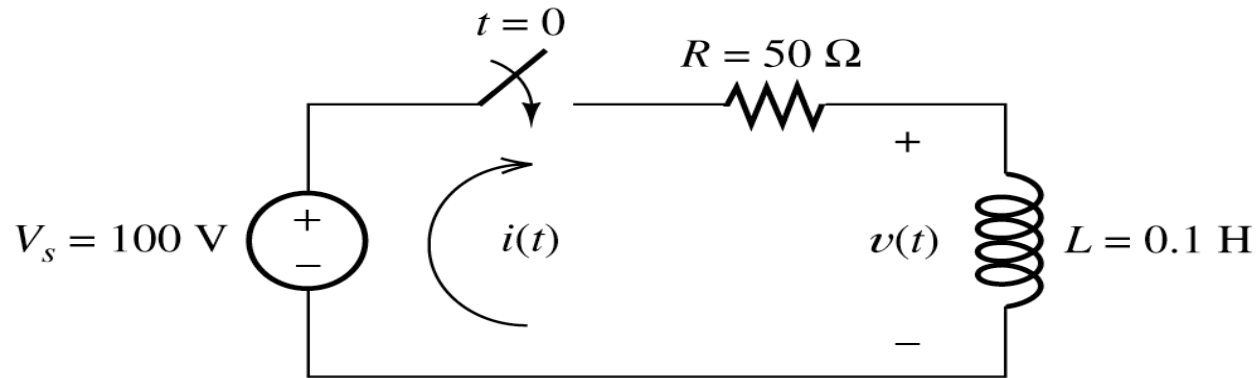


Figure 4.7 The circuit analyzed in Example 4.2.

Assume 猜

$$i(t) = K_1 + K_2 e^{st}$$



$$RK_1 + (RK_2 + sLK_2)e^{st} = V_s$$



$$K_1 = \frac{V_s}{R} = 2 \qquad s = \frac{-R}{L}$$

## Example 4.2 RL Transient Analysis

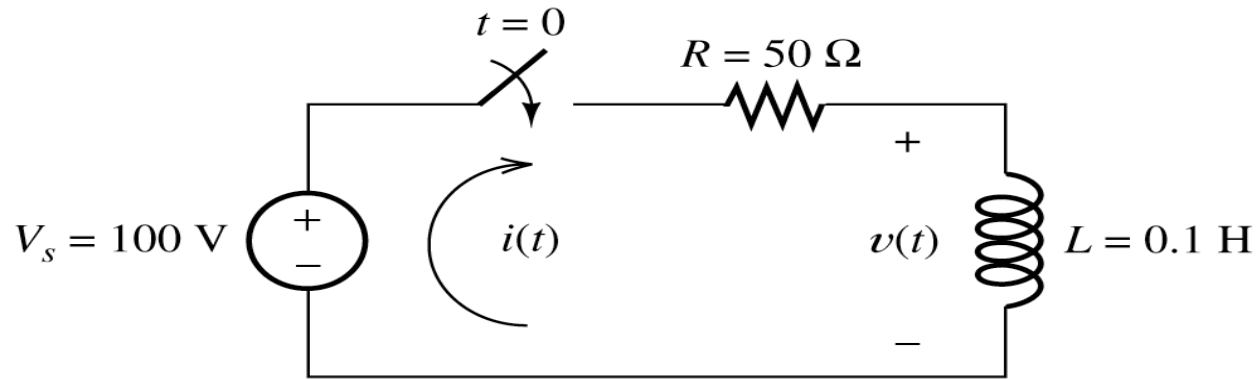


Figure 4.7 The circuit analyzed in Example 4.2.

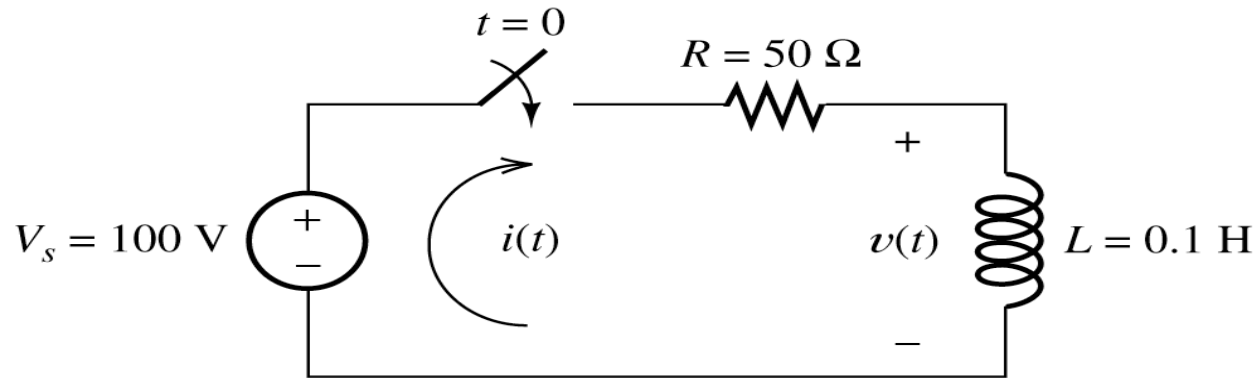


$$i(t) = 2 + K_2 e^{-tR/L}$$

$$\therefore i(0_+) = 0 = 2 + K_2 e^0 = 2 + K_2 \quad \longrightarrow \quad K_2 = -2$$

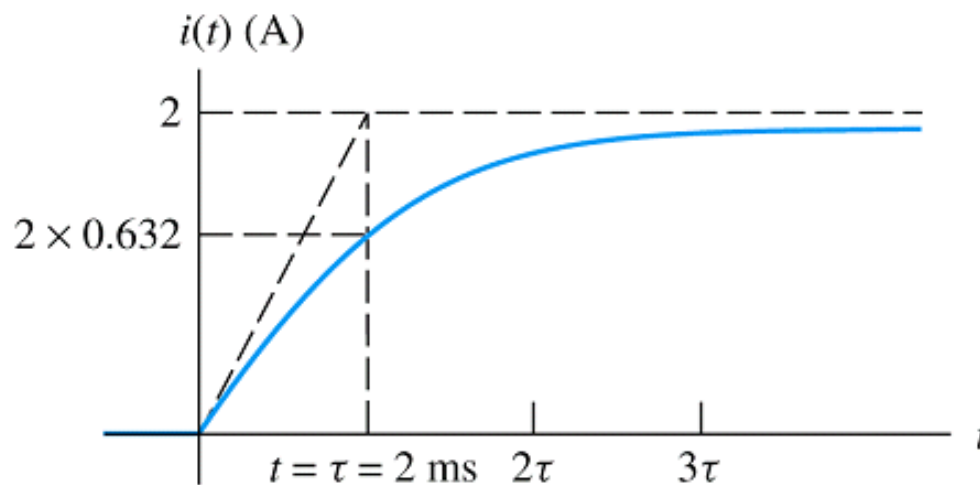
$$i(t) = 2 - 2e^{-t/\tau} \quad \text{for } t > 0 \quad \tau = \frac{L}{R}$$

## Example 4.2 RL Transient Analysis



**Figure 4.7** The circuit analyzed in Example 4.2.

$$i(t) = 2 - 2e^{-t/\tau} \quad \text{for } t > 0 \quad \tau = \frac{0.1}{50} = 2 \times 10^{-3} \text{ (sec)}$$



## Example 4.2 *RL* Transient Analysis

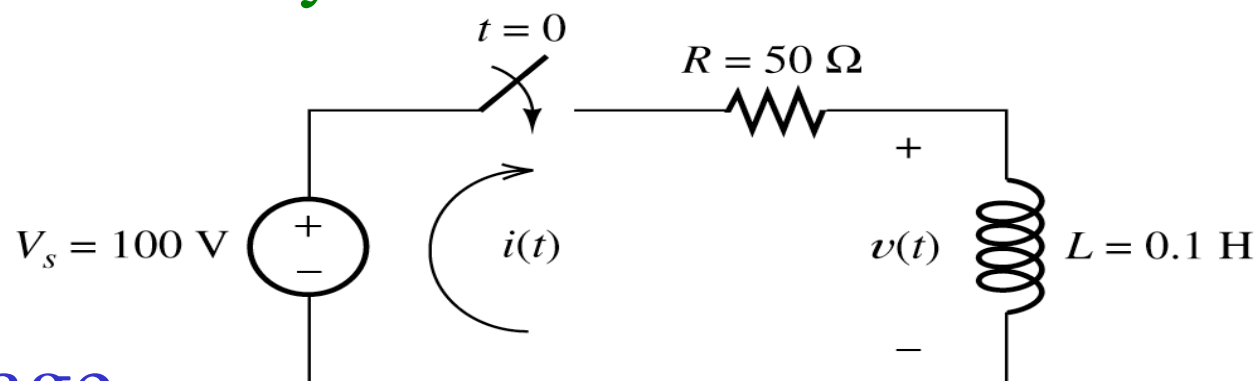


Figure 4.7 The circuit analyzed in Example 4.2.

Consider the voltage

1.  $t < 0$ ,  $v(t) = 0$ .

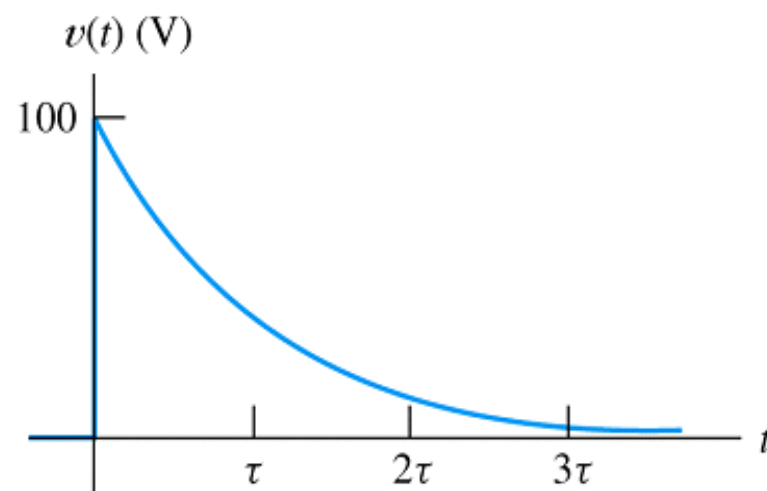
2.  $t > 0$

$$v(t) = 100 - 50i(t) \quad \text{for } t > 0 \quad \left( i(t) = 2 - 2e^{-t/\tau} \right)$$

→  $v(t) = 100e^{-t/\tau}$

or

$$v(t) = L \frac{di}{dt} = 0.1 \cdot (2/\tau) e^{-t/\tau}$$
$$= 0.1 \cdot 1000 \cdot e^{-t/\tau} = 100e^{-t/\tau}$$



# Laplace Transform

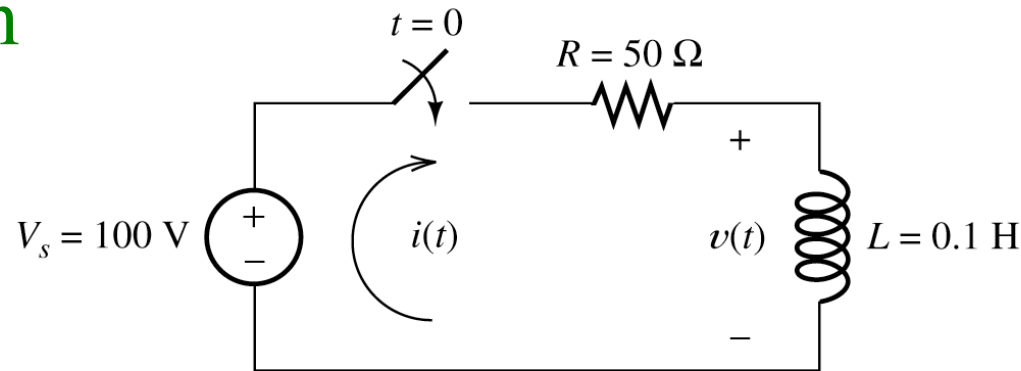


Figure 4.7 The circuit analyzed in Example 4.2.

$$Ri(t) + L \frac{di}{dt} = V_s u(t)$$

$$RI(s) + L[sI(s) - \cancel{i(0)}] = V_s(s) = \frac{V_s}{s}$$

$$I(s) = \frac{V_s}{s(R + Ls)} = \frac{V_s}{L} \frac{1}{s(s + R/L)}$$

$$= \frac{V_s}{L} \left( \frac{A}{s} + \frac{B}{s + R/L} \right)$$

$$As + A(R/L) + Bs = 1$$

$$B = -A \quad A = L/R$$

$$I(s) = \frac{V_s}{L} \left( \frac{L/R}{s} + \frac{-L/R}{s + R/L} \right)$$

$$= \frac{V_s}{R} \left( \frac{1}{s} - \frac{1}{s + R/L} \right)$$

$$i(t) = 2 - 2e^{-(R/L)t}$$



## Example 3.22

Use MATLAB to find the Laplace transform of

$$x(t) = 2\delta(t) + e^{-3t}$$

### Solution

We recall that the commands **dirac**( $t$ ) and **heaviside**( $t$ ) are used to represent the unit impulse  $\delta(t)$  and unit step  $u(t)$ , respectively. The MATLAB commands are:

```
syms x t
x = 2*dirac(t) + exp(-3*t);
X = laplace(x)
```

This produces the following result:

```
X =
1/(s+3) + 2
```

### Example 3.23

Use MATLAB to find the inverse Laplace transform of

$$V(s) = \frac{10s^2 + 4}{s(s+1)(s+2)^2}$$

#### Solution

The MATLAB code is as follows:

```
syms s v V
V = (10*s^2 + 4) / ( s*(s+1)*(s+2)^2 );
v = ilaplace(V)
```

This produces the following result:

```
v =
1-14*exp(-t) + (13+22*t)*exp(-2*t)
```

so that

$$v(t) = 1 - 14e^{-t} + 13e^{-2t} + 22te^{-2t}, \quad t > 0$$

## Example 3.24

Use the **residue** command to find the Laplace inverse of

$$X(s) = \frac{4s^5 + 20s^4 + 16s^3 + 10s^2 - 12}{s^4 + 5s^3 + 8s^2 + 4s}$$

### Solution

This is an indirect way of finding the inverse Laplace transform. We specify the numerator (num) and the denominator (den) of the transfer function  $X(s)$ . We find the residues of  $X(s)$  using the following code.

```
num = [4 20 16 10 0 -12]; % numerator coefficients in descending
powers of s
den = [1 5 8 4 0]; % denominator coefficients in descending
powers of s
[r,p,k] = residue(num,den); % call residue
```

```
r =
-15.0000
```

This produces a vector  $r$  that has the residues and a vector  $p$  that has the corresponding poles.

```
46.0000
```

```
2.0000
```

```
-3.0000
```

Notice that pole  $-2$  is repeated. Also, since the order of the numerator of  $X(s)$  is one greater than the order of the denominator of  $X(s)$ ,  $k$  contains two values. From  $r$ ,  $p$ , and  $k$ , we can write  $X(s)$  as

```
p =
```

```
-2.0000
```

```
-2.0000
```

```
-1.0000
```

```
0
```

```
k =
```

```
4 0
```

$$\begin{aligned} X(s) &= 4s + 0s^0 + \frac{-15}{s - (-2)} + \frac{46}{[s - (-2)]^2} + \frac{2}{s - (-1)} + \frac{-3}{s - 0} \\ &= 4s - \frac{15}{s + 2} + \frac{46}{(s + 2)^2} + \frac{2}{s + 1} - \frac{3}{s} \end{aligned}$$

Using Table 3.2, we obtain the inverse  $v(t)$  as

$$x(t) = 4\delta'(t) - 15e^{-2t} + 46te^{-2t} + 2e^{-t} - 3u(t), \quad t \geq 0$$

### Example 3.25

Use MATLAB to find the zeros and poles of

$$H(s) = \frac{s^2 + 3s + 1}{s^3 + 4s^2 + 3s}$$

### Solution

We use the command **roots** to find the roots of the numerator to get the zeros, and denominator to get the poles.

```
num = [1 3 1];  
den = [1 4 3 0];  
z=roots(num);  
p=roots(den);  
This result is:  
z =  
-2.6180  
-0.3820  
p =  
0  
-3  
-1
```

### Example 3.26

Use MATLAB to plot the step response of a system whose transfer function is

$$H(s) = \frac{s+1}{(s+2)(s+3)}$$

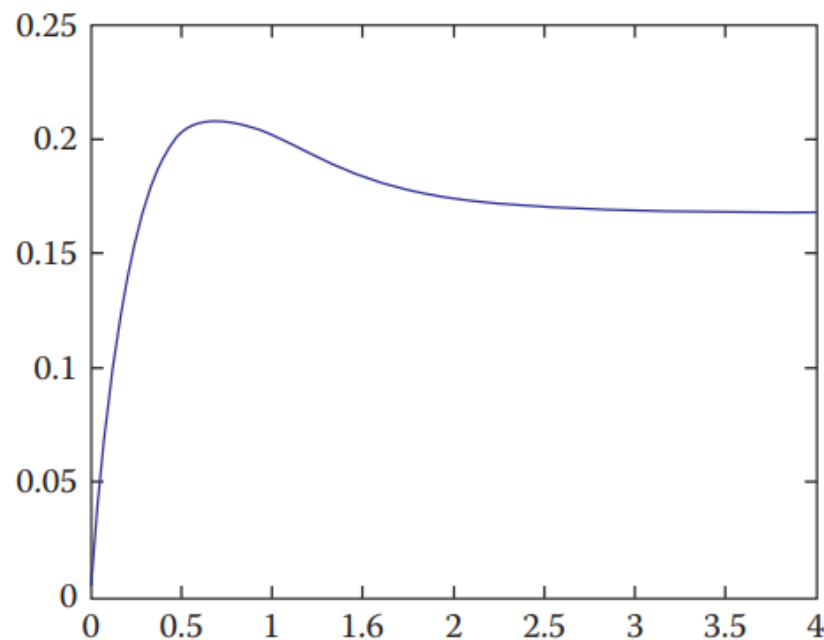
### Solution

We first need to expand the denominator:

$$H(s) = \frac{s+1}{s^2 + 5s + 6}$$

By definition, the step response is the response when the input to the system is the unit step  $u(t)$ . Using the MATLAB script below, we obtain the response as plotted in Figure 3.19.

```
num = [1 1];  
den = [1 5 6];  
t = 0: 0.1: 4;  
y=step(num,den,t);  
plot(t,y)
```



### Example 3.27

Use MATLAB to obtain the Bode plots for the transfer function

$$H(s) = \frac{100s}{s^2 + 12s + 20}$$

### Solution

The MATLAB script for the transfer is shown below, while the Bode plots are in Figure 3.21.

```
num = [100 0];  
den = [1 12 20];  
bode(num,den); %determines and draws Bode plots
```

