6.1 (a) Show that $X(\Omega)$ is periodic with period 2π, that is, $X(\Omega + 2\pi) = X(\Omega)$.

(b) Specifically show that $X(\Omega) = \dfrac{1}{1 - ae^{-j\Omega}}$ is periodic.

(a)

给定 $X(\Omega) = \sum\limits_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$

则 $X(\Omega + 2\pi) = \sum\limits_{n=-\infty}^{\infty} x[n]\, e^{-j(\Omega+2\pi)n} = \sum\limits_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}\cdot \underline{e^{-jn2\pi}}$

但 $e^{-jn2\pi} = \cos(2n\pi) - j\sin(2n\pi) = 1 - j0 = 1$

$\therefore X(\Omega + 2\pi) = X(\Omega) \Rightarrow$ periodic #

(b)

$X(\Omega + 2\pi) = \dfrac{1}{1 - ae^{-j(\Omega+2\pi)}} = \dfrac{1}{1 - a^{-j\Omega}e^{-j2\pi}}$

但 $e^{-j2\pi} = \cos(2\pi) - j\sin(2\pi) = 1 - j0 = 1$

$\therefore X(\Omega + 2\pi) = \dfrac{1}{1 - a^{-j\Omega}} = X(\Omega) \Rightarrow$ periodic #

6.10 The DTFT of a signal $x[n]$ is

$$X(\Omega) = \frac{2}{3 + e^{-j\Omega}}$$

4. Frequency-shifting (modulation)	$e^{j\Omega_0 n} x[n]$	$X(\Omega - \Omega_0)$

Find the DTFT of the following signals:
(a) $y[n] = x[-n]$
(b) $z[n] = nx[n]$
(c) $w[n] = x[n] + x[n-1]$
(d) $v[n] = x[n]\cos(n\pi)$

(a) $x[-n] \Rightarrow X(-\Omega)$

$\therefore y(\Omega) = X(-\Omega) = \dfrac{2}{3 + e^{j\Omega}}$ #

(b) $nx[n] \Rightarrow j\dfrac{dX(\Omega)}{d\Omega}$

$\therefore z(\Omega) = j\dfrac{dX(\Omega)}{d\Omega} = j\dfrac{0 - 2\cdot(-j)e^{-j\Omega}}{(3 + e^{-j\Omega})^2}$

$= \dfrac{-2e^{-j\Omega}}{(3 + e^{-j\Omega})^2}$ #

(c) $x[n - n_0] \Rightarrow e^{-j\Omega n_0} X(\Omega)$

$\therefore W(\Omega) = \dfrac{2}{3 + e^{-j\Omega}} + e^{-j\Omega}\dfrac{2}{3 + e^{-j\Omega}}$

$= \dfrac{2(1 + e^{-j\Omega})}{3 + e^{-j\Omega}}$ #

(d) $x[n]\cos(n\pi) = x[n]\dfrac{1}{2}(e^{jn\pi} + je^{-jn\pi})$

$\therefore V(\Omega) = \dfrac{1}{2}X(\Omega + \pi) + \dfrac{1}{2}X(\Omega - \pi)$

$= \dfrac{1}{3 + e^{-j(\Omega+\pi)}} + \dfrac{1}{3 + e^{-j(\Omega-\pi)}}$

但 $e^{j\pi} = \cos\pi + j\sin\pi = -1$, $e^{-j\pi} = \cos(-\pi) + j\sin(-\pi) = -1$

$= \dfrac{2}{3 - e^{-j\Omega}}$ #

6.13 Determine the signal $x[n]$ corresponding to each of the following Fourier transforms:

(a) $X(\Omega) = 1 + 3e^{-j2\Omega} - 2e^{j4\Omega} + e^{j5\Omega}$

(b) $X(\Omega) = \dfrac{e^{-j\Omega} - \dfrac{1}{2}}{1 - \dfrac{1}{2}e^{-j\Omega}}$

(c) $X(\Omega) = 3\pi[\delta(\Omega-2) + \delta(\Omega + 2)]$ $\Omega_0 = 2$

$\cos(\Omega_0 n)$ $\pi \sum\limits_{k=-\infty}^{\infty} \{\delta(\Omega - \Omega_0 - 2\pi k) + \delta(\Omega + \Omega_0 - 2\pi k)\}$

$\sin(\Omega_0 n)$ $\dfrac{\pi}{j} \sum\limits_{k=-\infty}^{\infty} \{\delta(\Omega - \Omega_0 - 2\pi k) - \delta(\Omega + \Omega_0 - 2\pi k)\}$

(a)

$$x[n] = \delta[n] + 3\delta[n-2] - 2\delta[n+4] + \delta[n+5] \quad \#$$

(b)

$$X(\Omega) = \frac{e^{-j\Omega}}{1 - \frac{1}{2}e^{j\Omega}} - \frac{1}{2} \frac{1}{1 - \frac{1}{2}e^{j\Omega}}$$

$$= \left(\frac{1}{2}\right)^{n-1} u[n-1] - \left(\frac{1}{2}\right)^{n+1} u[n] \quad \#$$

$$\mathcal{F}\{e^{j2n}\} = 2\pi\delta(\Omega-2)$$
$$\mathcal{F}\{e^{-j2n}\} = 2\pi\delta(\Omega+2)$$

(c) $X(\Omega) = \dfrac{3}{2} \cdot 2\pi[\delta(\Omega-2) + \delta(\Omega+2)]$

$$\Rightarrow x[n] = \frac{3}{2}(e^{j2n} + e^{-j2n}) = 3\cos(2n) \quad \#$$

6.18 Find the convolution $y[n] = h[n]* x[n]$ of the following pairs of signals:

(a) $x[n] = \left(\dfrac{1}{4}\right)^n u[n], \quad h[n] = 1$

(b) $x[n] = \left(\dfrac{1}{3}\right)^n u[n], \quad h[n] = \delta[n] + \delta[n-1]$

(c) $x[n] = \left(\dfrac{1}{2}\right)^n u[n], \quad h[n] = \left(\dfrac{1}{3}\right)^n u[n]$

(a)

$$X(\Omega) = \frac{1}{1 - \frac{1}{4}e^{-j\Omega}} \qquad H(\Omega) = \frac{1}{1 - e^{-j\Omega}}$$

$$Y(\Omega) = X(\Omega)H(\Omega) = \frac{A}{1 - \frac{1}{4}e^{-j\Omega}} + \frac{B}{1 - e^{-j\Omega}}$$

$$A = -\frac{1}{3}, \qquad B = \frac{4}{3}$$

$$\Rightarrow y[n] = -\frac{1}{3}\left(\frac{1}{4}\right)^n u[n] + \frac{4}{3} u[n] \quad \#$$

(b)

$$X(\Omega) = \frac{1}{1 - \frac{1}{3}e^{j\Omega}} \qquad H(\Omega) = 1 + e^{-j\Omega}$$

$$Y(\Omega) = X(\Omega)H(\Omega) = \frac{1}{1 - \frac{1}{3}e^{j\Omega}} + \frac{e^{-j\Omega}}{1 - \frac{1}{3}e^{j\Omega}}$$

$$\Rightarrow y[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{3}\right)^{n-1} u[n-1]_*$$

(c)

$$X(\Omega) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} \qquad H(\Omega) = \frac{1}{1 - \frac{1}{3}e^{-j\Omega}}$$

$$Y(\Omega) = X(\Omega)H(\Omega) = \frac{A}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{B}{1 - \frac{1}{3}e^{-j\Omega}}$$

$$A = 3 \quad B = -2$$

$$\Rightarrow y[n] = 3\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[n]_{\#}$$

6.20 Prove the following DFT properties:

(a) $X[0] = \sum_{n=0}^{N-1} x[n]$

(b) $X[N/2] = \sum_{n=0}^{N-1} (-1)^n x[n]$

(a)

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi n k/N}$$

$$k = 0, \quad X[0] = \sum_{n=0}^{N-1} x[n] e^{-j0} = \sum_{n=0}^{N-1} x[n]_*$$

(b)

$$k = \frac{N}{2}, \quad X\left[\frac{N}{2}\right] = \sum_{n=0}^{N-1} x[n] e^{-jn\pi}, \quad 1 \circ e^{-jn\pi} = \cos(n\pi) - j\sin(n\pi) = (-1)^n$$

$$= \sum_{n=0}^{N-1} (-1)^n x[n]_{\#}$$

6.21 Find the DFT of the following sequences:

(a) $x[n] = \{0, 1, 2, 3\}$

(b) $y[n] = \{1, 1, -1, -1, 1, 1, -1, -1\}$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}$$

(a) $N = 4$, $k = 0, 1, 2, 3$

$$X[k] = x[0] + x[1]e^{-j\pi k/2} + x[2]e^{-j\pi k} + x[3]e^{-j\pi 3k/2}$$

$$= e^{-j\pi k/2} + 2e^{-j\pi k} + 3e^{-j\pi 3k/2}, \quad k = 0, 1, 2, 3$$

$$= [6, -2+j2, -2, -2-j2]_{\#}$$

(b) $N = 8$, $k = 0 \sim 7$

$$Y[k] = y[0] + y[1]e^{-j\pi k/4} + y[2]e^{-j\pi k/2} + y[3]e^{-j\pi 3k/4} + y[4]e^{-j\pi k} + y[5]e^{-j\pi 5k/4} + y[6]e^{-j\pi 3k/2} + y[7]e^{-j\pi 7k/4}$$

$$= 1 + e^{-j\pi k/4} - e^{-j\pi k/2} - e^{-j\pi 3k/4} + e^{-j\pi k} + e^{-j\pi 5k/4} - e^{-j\pi 3k/2} - e^{-j\pi 7k/4}, \quad k = 0 \sim 7$$

$$= [0, 0, 4-j4, 0, 0, 0, 4+j4, 0]_{\#}$$

6.23 Show that

(a) $x[n]\cos\left(\dfrac{2\pi km}{N}\right) \longleftrightarrow \dfrac{1}{2}[X(m-k) + X(m+k)]$

(b) $x[n]\sin\left(\dfrac{2\pi km}{N}\right) \longleftrightarrow \dfrac{1}{2j}[X(m-k) - X(m+k)]$

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \quad x[n] = \dfrac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}$$

(a) ① $x[n]\cos\left(\dfrac{2\pi km}{N}\right) = x[n] \cdot \dfrac{1}{2}\left(e^{j\frac{2\pi km}{N}} + e^{-j\frac{2\pi km}{N}}\right)$

$$= \dfrac{1}{2}\left(x[n]e^{j\frac{2\pi km}{N}} + x[n]e^{-j\frac{2\pi km}{N}}\right) \xrightarrow{\text{DFT}} \dfrac{1}{2}[X(k-m) + X(k+m)]$$

② $\dfrac{1}{2}[X(k-m) + X(k+m)] \xrightarrow{\text{IDFT}} \dfrac{1}{2}\left(x[n]e^{-j\frac{2\pi km}{N}} + x[n]e^{j\frac{2\pi km}{N}}\right) = x[n]\cos\dfrac{2\pi km}{N}$

由①② 得證 #

(b)

① $x[n]\sin\left(\dfrac{2\pi km}{N}\right) = x[n] \cdot \dfrac{1}{2j}\left(e^{j\frac{2\pi km}{N}} - e^{-j\frac{2\pi km}{N}}\right)$

$$= \dfrac{1}{2j}\left(x[n]e^{j\frac{2\pi km}{N}} - x[n]e^{-j\frac{2\pi km}{N}}\right) \xrightarrow{\text{DFT}} \dfrac{1}{2j}[X(k-m) - X(k+m)]$$

② $\dfrac{1}{2j}[X(k-m) - X(k+m)] \xrightarrow{\text{IDFT}} \dfrac{1}{2j}\left(x[n]e^{\frac{2\pi km}{N}} - x[n]e^{-j\frac{2\pi km}{N}}\right) = x[n]\sin\dfrac{2\pi km}{N}$

由①② 得證 #

6.25 The DFT of a signal $x[n]$ is

$X(0)=1, \quad X(1)=1+j2, \quad X(2)=1-j, \quad X(3)=1+j, \quad X(4)=1-j2$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi nk/N}$$

Compute $x[n]$.

$N = 5$

$$x[0] = \frac{1}{5}\sum_{k=0}^{4} X(k)\, e^{j0} = \frac{1}{5}\left(X(0)+\cdots+X(4)\right) = 1$$

$$x[1] = \frac{1}{5}\sum_{k=0}^{4} X(k)\, e^{j2\pi k/5} \doteq -0.5257$$

$$x[2] = \frac{1}{5}\sum_{k=0}^{4} X(k)\, e^{j4\pi k/5} \doteq -0.8507$$

6.25 $x[0] = 1$, $x[1] = -0.5257$, $x[2] = -0.8507$, $x[3] = 0.8507$, $x[4] = 0.5257$

$$x[3] = \frac{1}{5}\sum_{k=0}^{4} X(k)\, e^{j6\pi k/5} \doteq 0.8507$$

$$x[4] = \frac{1}{5}\sum_{k=0}^{4} X(k)\, e^{j8\pi k/5} \doteq 0.5257$$ #
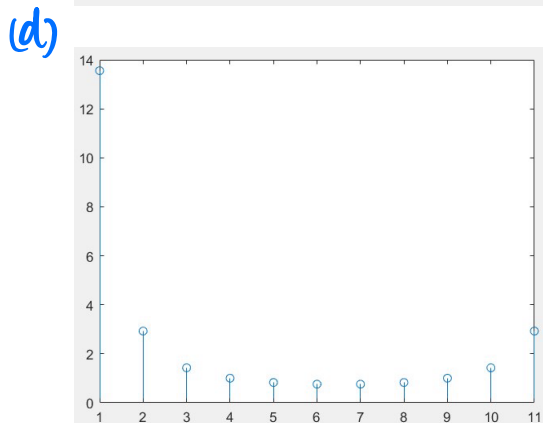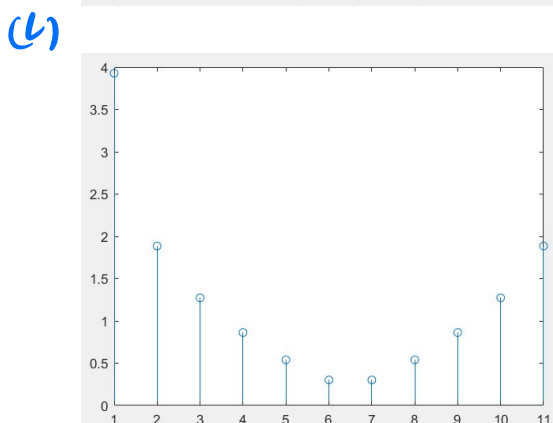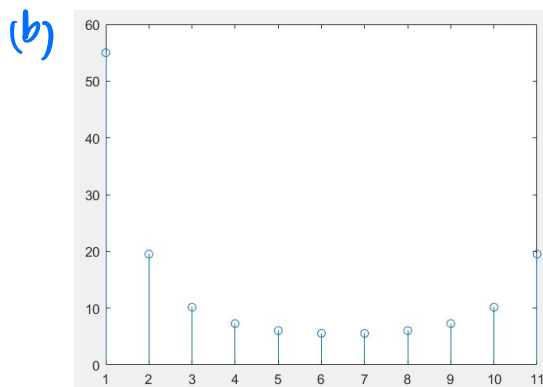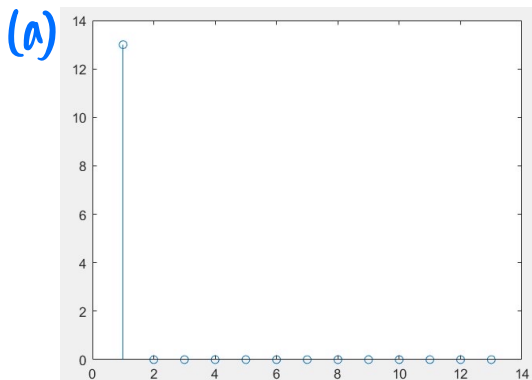
6.28 Use MATLAB to compute the FFT of the following signals. For each signal, plot $|X(k)|$.
(a) $x[n] = 1, 0 \le n \le 12$
(b) $x[n] = n, 0 \le n \le 10$

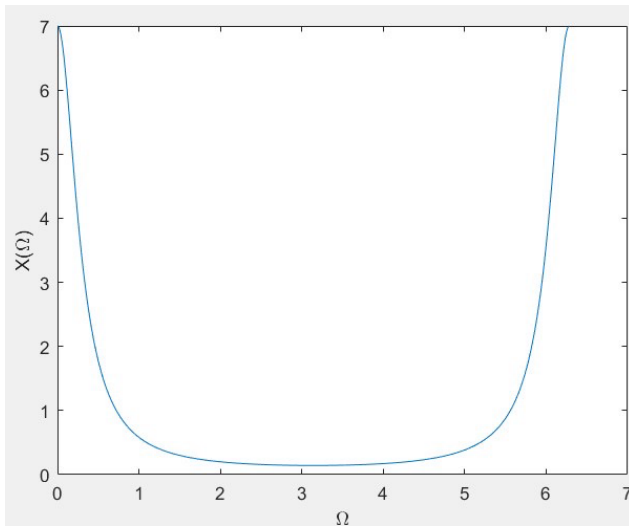(c) $x[n] = \begin{cases} 1, & n = 0 \\ 1/n, & n = 1,2,\ldots,10 \\ 0, & \text{otherwise} \end{cases}$

(d) $x[n] = n(0.8)^n, \quad 0 \le n \le 10$

(a)

(b)

(c)

(d)

6.29 In Example 6.3, the DTFT of the signal $x[n] = a^{|n|}$ is

$$X(\Omega) = \frac{1-a^2}{1-2a\cos\Omega + a^2}$$

For $a = 0.75$ and $0 < \Omega < 2\pi$, plot $|X(\Omega)|$.

6.30 Use MATLAB to find the DFT of the discrete signal

$$x[n] = \{1, 2, 0, -1, -2, 1, 5, 4\}$$

```
>> HW5_30
  Columns 1 through 3

  10.0000 + 0.0000i    7.2426 + 7.8284i   -6.0000 + 0.0000i

  Columns 4 through 6

  -1.2426 - 2.1716i   -2.0000 + 0.0000i   -1.2426 + 2.1716i

  Columns 7 through 8

  -6.0000 + 0.0000i    7.2426 - 7.8284i
```