訊號與系統
SIGNAL AND SYSTEM

Lecture 5
Discrete-time Fourier Transform

梁　勝　富
成功大學 資訊工程系
sfliang@ncku.edu.tw
Office:資訊系館 12F 65C06, Tel: Ext. 62549

Lab:神經運算與腦機介面實驗室
(3F 65301, Tel: 62530-2301)

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6.2 DISCRETE-TIME FOURIER TRANSFORM

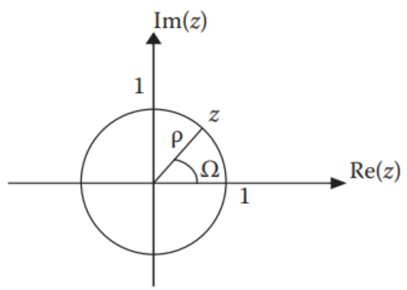
$$x[n] \xleftrightarrow{\text{DTFT}} X(\Omega)$$

$$\text{DFT: } X(\Omega) = F\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

$$\text{IDFT: } x[n] = F^{-1}\{X(\Omega)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}\,d\Omega$$

- Since $X(\Omega)$ is periodic, the integral can be evaluated over any interval of length 2π. For example,

$$x[n] = \frac{1}{2\pi}\int_{0}^{2\pi} X(\Omega)e^{j\Omega n}\,d\Omega$$



Recall $z = \rho e^{j\Omega}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\rho^{-n}e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n](\rho e^{j\Omega})^{-n} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Example 6.1

$$x[n] = \begin{cases} 1, & n = -1 \\ -2, & n = 0 \\ 0, & n = 1 \\ 3, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the discrete-time Fourier transform.

Solution

值代入

n代入

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\underline{e^{-j\Omega n}}$$

$$= e^{j\Omega} - 2e^{j0} + 0e^{j\Omega} + 3e^{-j2\Omega}$$

$$= e^{j\Omega} - 2 + 3e^{-j2\Omega}$$

Example 6.2

Find the DTFT of the discrete-time signal

(a) $x[n] = \delta[n]$ 常見

(b) $x[n] = \begin{cases} a^n, & 0 \le n \le m \\ 0, & \text{otherwise} \end{cases}$

where a is a constant.

Solution

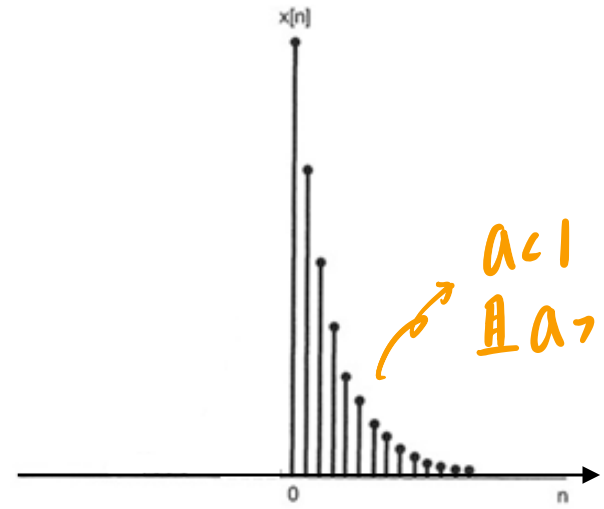
$$\text{(a)} \quad X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=0}^{m} \delta[n]e^{-j\Omega n} = e^{-j\Omega n}\Big|_{n=0} = 1$$

δ特性

Thus,

$$\delta[n] \xleftrightarrow{\text{DTFT}} 1$$

(b) $x[n] = \begin{cases} a^n, & 0 \le n \le m \\ 0, & \text{otherwise} \end{cases}$



a<1 且a>0

(a為負的話其值會正負交錯)

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} = \sum_{n=0}^{m} a^n e^{-j\Omega n} = \sum_{n=0}^{m} (\underbrace{ae^{-j\Omega}}_{q})^n$$

But the geometric series sums as

$$\sum_{n=0}^{m} q^n = \frac{1-q^{m+1}}{1-q}, \quad |q|<1$$

等比級數 $= \dfrac{首項(1-公比^{項數})}{1-公比}$

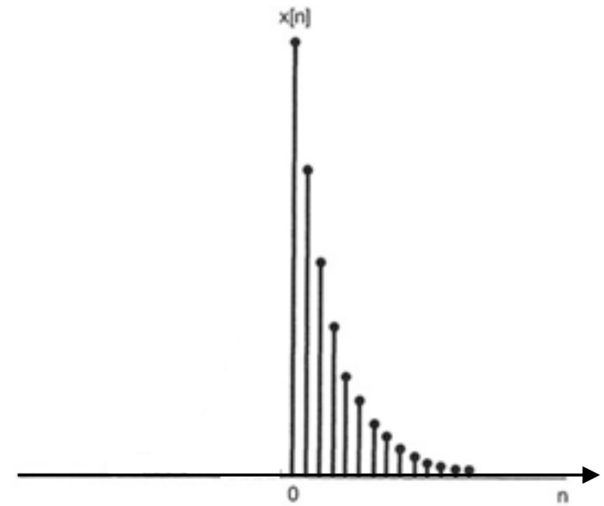
$$\Rightarrow \quad X(\Omega) = \frac{1-(ae^{-j\Omega})^{m+1}}{1-ae^{-j\Omega}}, \quad |a|<1$$

For $m \to \infty$ and $|a|<1$

$$X(\Omega) = \frac{1}{1-ae^{-j\Omega}}$$

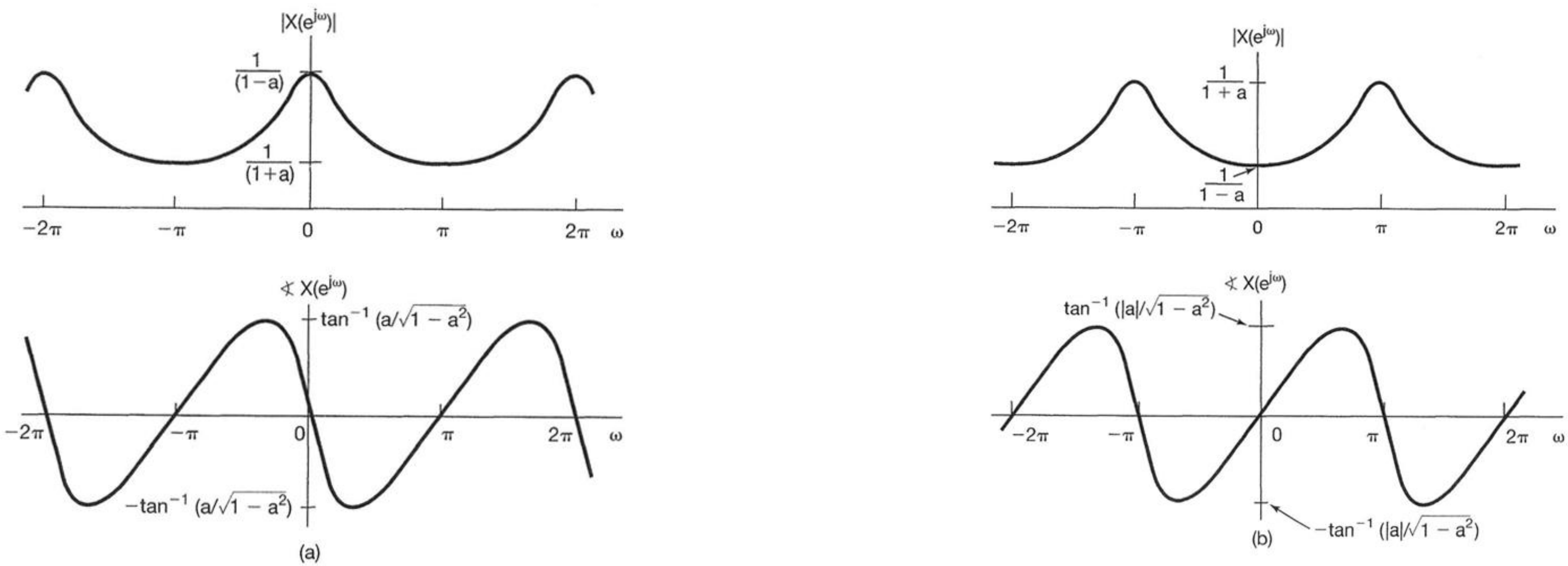
⇒ 無窮等比

For $m \to \infty$ and $|a|<1$



$$X(\Omega) = \frac{1}{1 - a\underbrace{e^{-j\Omega}}_{\cos\Omega - j\sin\Omega}}$$

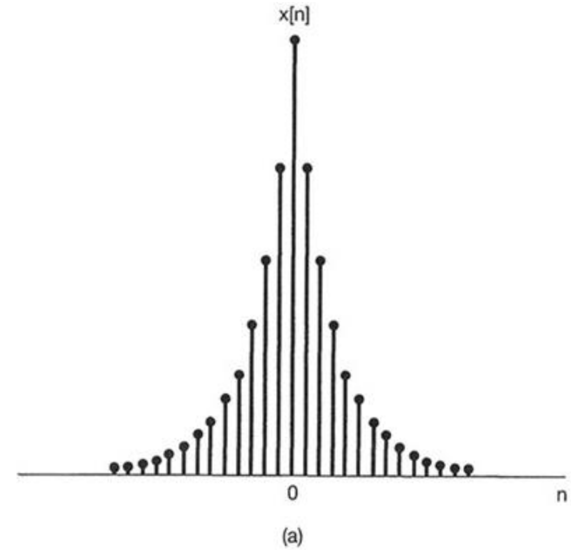
(a) $a > 0$ and (b) $a < 0$.



Oppenheim, Signals and Systems

Example 6.3

Obtain the DTFT of the signal $x[n] = a^{|n|}$, $|a| < 1$.



Solution

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} a^{|n|}e^{-j\Omega n}$$

$$= \underbrace{\sum_{n=0}^{\infty} a^{n}e^{-j\Omega n}}_{n>0} + \underbrace{\sum_{n=-\infty}^{-1} a^{-n}e^{-j\Omega n}}_{n<0} = \sum_{n=0}^{\infty} (ae^{-j\Omega})^{n} + \sum_{m=1}^{\infty} (ae^{j\Omega})^{m} \Rightarrow m=-n$$

① $$\sum_{n=0}^{\infty} q^{n} = 1 + q + q^{2} + \cdots = \frac{1}{1-q}, \quad |q|<1$$

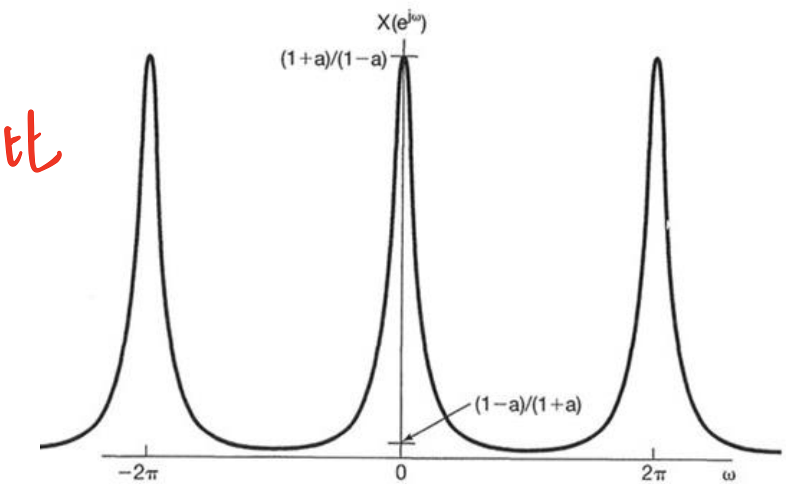
② $$\sum_{n=1}^{\infty} q^{n} = q + q^{2} + \cdots = \frac{1}{1-q} - 1 = \frac{q}{1-q}, \quad |q|<1$$

無窮等比

$$X(\Omega) = \overset{①}{\frac{1}{1-a\underline{e^{-j\Omega}}}} + \overset{②}{\frac{ae^{j\Omega}}{1-a\underline{e^{j\Omega}}}} = \frac{1-a^{2}}{1-2a\cos\Omega + a^{2}}$$

通分化簡

$\cos\Omega - \cancel{j\sin\Omega}$ $\quad$ $\cos\Omega + \cancel{j\sin\Omega}$



作圖

Oppenheim, Signals and Systems

- Time Reversal and Conjugation

If $x[n] \xleftrightarrow{\text{DTFT}} X(\Omega)$, then

$$\boxed{x[-n] \xleftrightarrow{\text{DTFT}} X(-\Omega)}$$

$$F\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n] e^{-j\Omega n}$$

If we let $-n = k$, then

$$F\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[k] e^{j\Omega k} = \sum_{n=-\infty}^{\infty} x[k] e^{-j(-\Omega)k} = X(-\Omega)$$

A related property is the conjugation of $x[n]$. That is,

$$\boxed{x^*[n] \xleftrightarrow{\text{DTFT}} X^*(-\Omega)}$$

where * denotes the complex conjugate.

- Time Scaling

$$\boxed{x_{(k)}[n] \xleftrightarrow{\text{DTFT}} X(k\Omega)}$$

$$x_{(k)}[n] = \begin{cases} x[n/k] = x[m], & \text{if } n = km, m = \text{integer} \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$

跳k個 ⇒ 拉長

⇒ 中間為0

Since $x_{(k)}[n]$ is zero unless n is a multiple of k, the DTFT of $x_{(k)}[n]$ is given by

$$X_{(k)}(\Omega) = \sum_{n=-\infty}^{\infty} x_{(k)}[n]e^{-j\Omega n} = \sum_{m=-\infty}^{\infty} x_{(k)}[mk]e^{-j\Omega mk}$$

Replacing $x_{(k)}[n]$ by $x[m]$ gives

$$X_{(k)}(\Omega) = \sum_{m=-\infty}^{\infty} x[m]e^{-j(k\Omega)m} = X(k\Omega)$$

- Time Shifting

If $x[n] \xleftrightarrow{\text{DTFT}} X(\Omega)$,

$$\boxed{x[n-n_0] \xleftrightarrow{\text{DTFT}} e^{-j\Omega n_0} X(\Omega)}$$ ⇒ delay n_0

Proof

$$F\{x[n-n_0]\} = \sum_{n=-\infty}^{\infty} x[n-n_0] e^{-j\Omega n}$$

Let $n - n_0 = k$ on the right side of this equation so that $n = k + n_0$.

$$F\{x[n-n_0]\} = \boxed{\sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega(k+n_0)}} = \boxed{e^{-j\Omega n_0} \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega k}}$$

$$= e^{-j\Omega n_0} X(\Omega)$$

Practice Problem 6.4

Obtain the DTFT of

$$x[n] = e^{j\Omega_0 n}$$

↳ $\cos\Omega_0 n + j\sin\Omega_0 n$ → 複數

Solution

$$X(\Omega) = F\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \qquad X(\Omega + 2\pi) = X(\Omega), \quad -\infty < \Omega < \infty$$

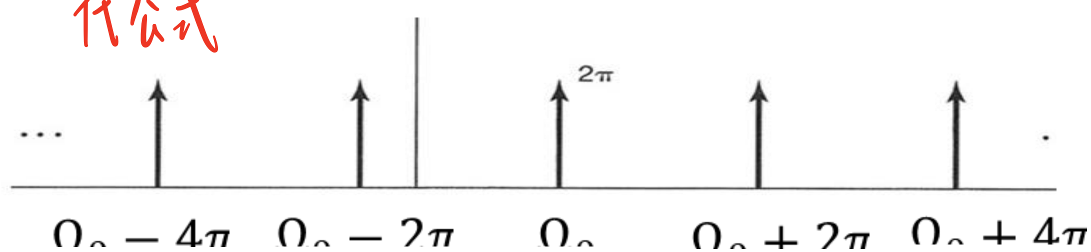
$$x[n] = F^{-1}\{X(\Omega)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}\,d\Omega$$

→ 對角度積分再除掉

$$F^{-1}\left\{\sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k)\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\Omega - \Omega_0)e^{i\Omega n}\,d\Omega = \frac{1}{2\pi}e^{i\Omega_0 n}\int_{-\pi}^{\pi} \delta(\Omega - \Omega_0)\,d\Omega$$

代公式

$$= \boxed{\frac{1}{2\pi}}e^{i\Omega_0 n}$$



2π 乘回去

$$\Rightarrow \quad F\{e^{i\Omega_0 n}\} = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k)$$

Example

Find the DTFT of the sinusoidal sequence

$$\underline{x[n] = \cos\Omega_0 n}, \quad |\Omega_0| \le \pi$$

$e^{j\Omega_0 n}$在實軸投影

Solution

Recall

$$F\{e^{i\Omega_0 n}\} = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k)$$

$$\cos\Omega_0 n = \underline{\frac{1}{2}\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right)}$$

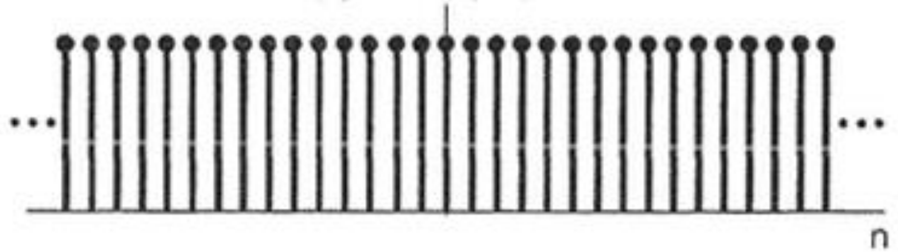
代入

$$X(\Omega) = \boxed{\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k) + \delta(\Omega + \Omega_0 - 2\pi k)}$$

Example 6.4

Obtain the DTFT of the constant Signal $x[n]$ for all n. That is,

$$x[n] = 1, \quad n = 0, \pm 1, \pm 2, \ldots$$



Solution

$$F\{e^{i\Omega_0 n}\} = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k)$$

Let

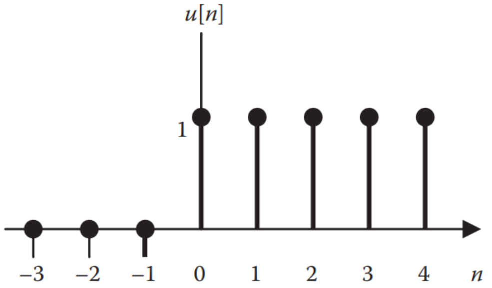
$$x[n] = 1, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$x[n] = e^{i\Omega_0 n}, \Omega_0 = 0 \quad \text{for all } n$$

$$F\{x[n]\} = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 0 - 2\pi k) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$$

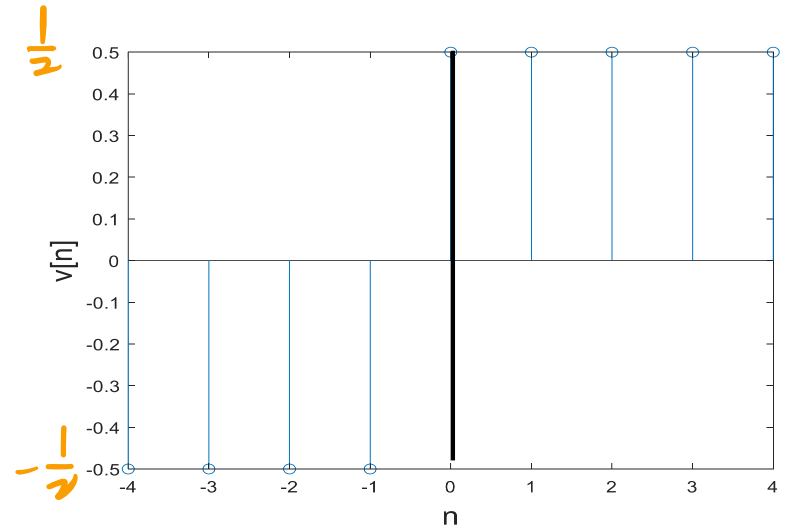
Example

Obtain the DTFT of $u[n]$



Solution

Let 技巧



$$v[n] = u[n] - 1/2$$

$$v[n] - v[n-1] = \delta[n]$$

$$F\{v[n] - v[n-1]\} = F\{\delta[n]\}$$

$$V(\Omega) - e^{-j\Omega}V(\Omega) = 1, \quad V(\Omega) = \frac{1}{1 - e^{-j\Omega}}$$

$$u[n] = v[n] + 1/2$$

$$\mathcal{U}(\Omega) = V(\Omega) + \mathrm{F}\{1/2\} = \frac{1}{1 - e^{-j\Omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$$

- Accumulation

$$\boxed{\sum_{k=-\infty}^{n} x[k] \xleftrightarrow{\text{DTFT}} \frac{1}{1-e^{-j\Omega}} X(\Omega) + \pi X(0) \sum_{n=-\infty}^{\infty} \delta(\Omega - 2n\pi)}$$

Proof

$$y[n] = \boxed{\sum_{k=-\infty}^{n} x[k]} = \underline{x[n] * u[n]}$$

$$Y(\Omega) = \underline{X(\Omega)U(\Omega)} = X(\Omega)\left[\frac{1}{1-e^{-j\Omega}} + \pi \sum_{n=-\infty}^{\infty} \underline{\delta(\Omega - 2n\pi)}\right]$$

↳ $\Omega = 0, 2\pi, 4\pi \cdots$ 才有值 ⇒ $X(0)$

$$= \pi X(0) \sum_{n=-\infty}^{\infty} \delta(\Omega - 2n\pi) + \frac{1}{1-e^{-j\Omega}} X(\Omega)$$

- Frequency Differentiation

$$\boxed{nx[n] \xleftrightarrow{\text{DTFT}} j\frac{dX(\Omega)}{d\Omega}}$$

Solution

$$\frac{dX(\Omega)}{d\Omega} = \sum_{n=-\infty}^{\infty} -jnx[n]e^{-j\Omega n}$$

移到左側

$$j\frac{dX(\Omega)}{d\Omega} = \sum_{n=-\infty}^{\infty} nx[n]e^{-j\Omega n}$$

$$j = -\frac{1}{j}$$

- # Parseval's Relation

The energy of a discrete-time signal $x[n]$ is

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$y[m] \triangleq \sum_{n=-\infty}^{\infty} \underbrace{x[n] \cdot x[n-m]}_{\text{算相關係數}}.$$

Note that $y[n] = x[n] * x[-n]$, and in particular, $y[0] = \sum_{n=-\infty}^{\infty} x^2[n]$. Applying the convolution theorem, $y[n]$'s Fourier transform of $Y(\omega)$ can be expressed in terms of $X(\omega)$ as follows,

$$\boxed{x[-n] \xleftrightarrow{\text{DTFT}} X(-\Omega)}$$

$$Y(\omega) = X(\omega) \cdot \left(\sum_{n=-\infty}^{\infty} x[-n] e^{-j\omega n} \right) = X(\omega) \cdot X^*(\omega) = |X(\omega)|^2 .$$

Calculating the inverse DTFT at time 0, then we reach that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x^2[n] = y[0] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(\omega) e^{j\omega \cdot 0} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega. \end{aligned}$$

（劉奕文數位聲訊分析與合成）

Example 6.6

Given the signal

$$x[n] = \begin{cases} 1, & |n| \le 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Sketch $x[n]$ and its DTFT $X(\Omega)$.
(b) Sketch the time-scaled signal $x_{(2)}[n]$ and its DTFT $X_{(2)}(\Omega)$.

Solution

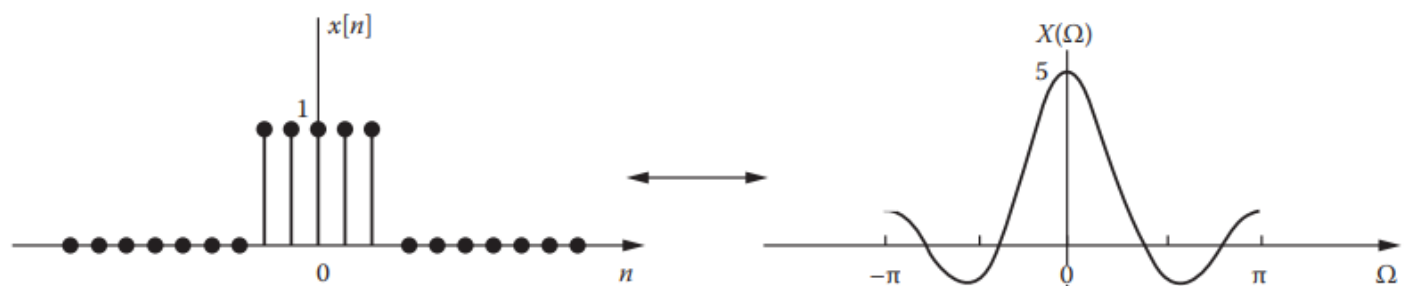
$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=-(N-1)/2}^{(N-1)/2} (1)\underbrace{(e^{-j\Omega})}_{a}{}^{n}, \quad \underline{N = 5} \quad \frac{5-1}{2} = 2$$

$$\underline{\sum_{k=M}^{N} a^k = \frac{a^{N+1} - a^M}{a - 1} \quad a \ne 1}$$ 等比 $\dfrac{a^M(a^{N-M+1} - 1)}{a-1}$

$$X(\Omega) = \frac{e^{-j[(N+1)/2]\Omega} - e^{j[(N-1)/2]\Omega}}{e^{-j\Omega} - 1} = \frac{e^{-j\Omega/2}\left(e^{-j(N/2)\Omega} - e^{j(N/2)\Omega}\right)}{e^{-j\Omega/2}(e^{-j\Omega/2} - e^{j\Omega/2})}$$

$$= \frac{\sin(N\Omega/2)}{\sin(0.5\Omega)}, \quad N = 5$$

$$= \frac{\sin(2.5\Omega)}{\sin(0.5\Omega)}$$



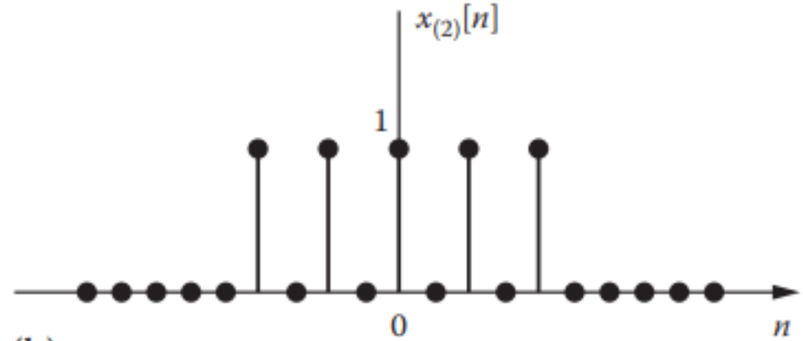
(a)

(b) Using the time-scaling property

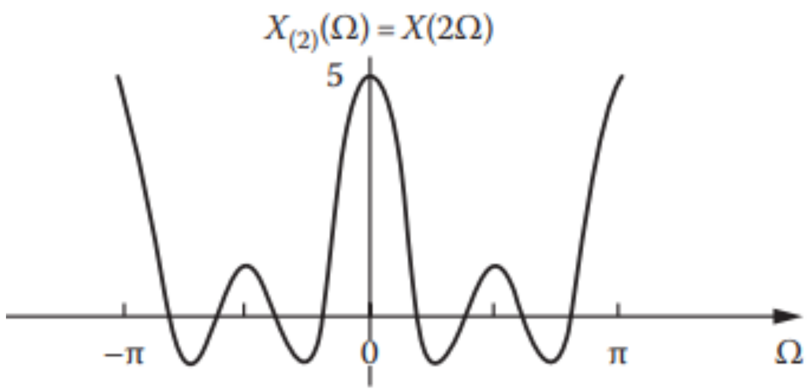
Ω用2Ω代入

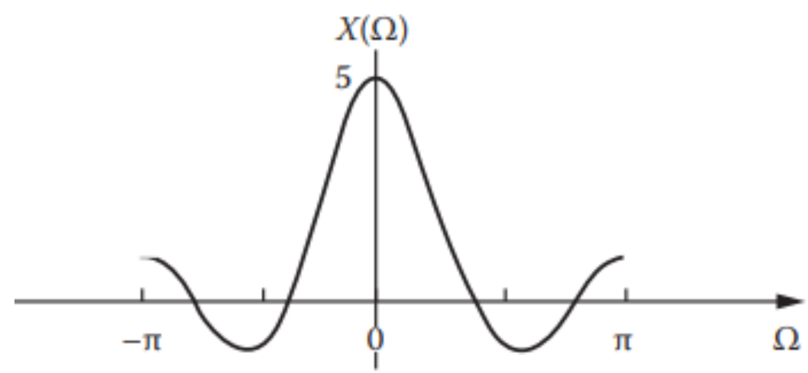
$$X_{(2)}(\Omega) = X(2\Omega) = \frac{\sin(5\Omega)}{\sin(\Omega)}$$

$k=2$



(b)





Example 6.7

Determine the Inverse of the Function

$$X(\Omega) = \frac{1}{(1 - ae^{-j\Omega})^2}, \quad |a| < 1$$

Solution

$$a^n u[n] \xleftrightarrow{\text{DTFT}} \frac{1}{1 - ae^{-j\Omega}}, \quad |a| < 1$$

The given function can be expressed as

$$X(\Omega) = \frac{1}{(1 - ae^{-j\Omega})^2} = \left(\frac{1}{1 - ae^{-j\Omega}}\right)\left(\frac{1}{1 - ae^{-j\Omega}}\right)$$

拆成 conv.

$$x[n] = a^n u[n] * a^n u[n] = \sum_{k=-\infty}^{\infty} a^k u[k] a^{n-k} u[n-k]$$

$$= a^n \sum_{k=0}^{n} 1 = (n+1)a^n u[n] \quad |a| < 1$$

6.5 DISCRETE FOURIER TRANSFORM

- The discrete Fourier transform (DFT) may be regarded as a logical extension of the discrete-time Fourier transform (DTFT).
- DFT is obtained by sampling the DTFT $X(\Omega)$ at uniformly spaced frequencies $\Omega = 2\pi k/N$, where $k = 0, 1, 2, \ldots, N-1$.
- DFT $X[k]$ is a periodic sequence with period N. 特定位置才有值

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}, \qquad k = 0,1,2,\ldots, N-1$$

拆開

$$X[k] = R[k] + jI[k]$$

$$\boxed{\text{DFT: } X[k] = F[x[n]] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}}$$

實 $$R[k] = x[0] + \sum_{n=1}^{N-1} x[n]\cos\left(\frac{2\pi kn}{N}\right)$$

$$\boxed{\text{IDFT: } x[n] = F^{-1}\big[X[k]\big] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{+j2\pi nk/N}}$$

虛 $$I[k] = -\sum_{n=1}^{N-1} x[n]\sin\left(\frac{2\pi kn}{N}\right)$$

Linear Convolution and Circular Convolution

- Linear convolution

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- Circular (periodic)convolution 有限點的 conv.

  It is often regarded as the evaluation of two signals around two concentric circles.
  - Writing the N values of $x[n]$ equally spaced around an outer circle in a counterclockwise direction,
  - while the N values of $h[n]$ are equally spaced in a clockwise direction on an inner circle.

$$y[n] = x[n] \otimes h[n] = \sum_{k=0}^{N-1} x[k]h[n-k]$$

Example 6.8

Find the DFT of the sequence $x[n] = \{1, -2, 1, 3\}$.

Solution

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N} = \sum_{n=0}^{3} x[n]e^{-j\pi nk/2}, \qquad k = 0,1,2,3$$

$$= x[0] + x[1]e^{-j\pi k/2} + x[2]e^{-j\pi k} + x[3]e^{-j\pi 3k/2}, \qquad k = 0,1,2,3$$

$$= 1 - 2e^{-j\pi k/2} + e^{-j\pi k} + 3e^{-j\pi 3k/2}, \qquad k = 0,1,2,3$$

The real part of this is

實

$$R[k] = 1 - 2\cos(-\pi k/2) + \cos(-\pi k) + 3\cos(-3\pi k/2), \quad k = 0,1,2,3$$
$$= 1 - 2\cos(\pi k/2) + \cos(\pi k) + 3\cos(3\pi k/2), \quad k = 0,1,2,3$$

The imaginary part of $X[k]$ is

虛

$$I[k] = -2\sin(-\pi k/2) + \sin(-\pi k) + 3\sin(-\pi 3k/2)$$
$$= 2\sin(\pi k/2) - \sin(\pi k) - 3\sin(3\pi k/2), \quad k = 0,1,2,3$$

$$R[k] = \begin{cases} 3, & k = 0 \\ 0, & k = 1 \\ 1, & k = 2 \\ 0, & k = 3 \end{cases} \qquad I[k] = \begin{cases} 0, & k = 0 \\ 5, & k = 1 \\ 0, & k = 2 \\ -5, & k = 3 \end{cases} \quad \Rightarrow \quad X[k] = \begin{cases} 3, & k = 0 \\ j5, & k = 1 \\ 1, & k = 2 \\ -j5, & k = 3 \end{cases}$$

代不同的k

Example 6.9

Find the periodic convolution of the following two sequences:

$$x[n] = [\,1, 0, -2, 3\,] \quad \text{and} \quad h[n] = [\,3, 1, 2, -1\,] \qquad N = 4$$

Solution

$$\frac{2\pi nk}{N}$$

$$X[k] = x[0] + x[1]e^{-\underline{j\pi k/2}} + x[2]e^{-j\pi k} + x[3]e^{-j\pi 3k/2}, \quad k = 0, 1, 2, 3$$

$$X[0] = x[0] + x[1] + x[2] + x[3] = 2$$

$$X[1] = x[0] + x[1]e^{-j\pi/2} + x[2]e^{-j\pi} + x[3]e^{-j3\pi/2} = 3 + j3$$

$$e^{jw} = \cos w + j \sin w$$

$$X[2] = x[0] + x[1]e^{-j\pi} + x[2]e^{-j2\pi} + x[3]e^{-j3\pi} = -4$$

$$X[3] = x[0] + x[1]e^{-j3\pi/2} + x[2]e^{-3j\pi} + x[3]e^{-j9\pi/2} = 3 - j3$$

$$H[0] = h[0] + h[1] + h[2] + h[3] = 5$$

$$H[1] = h[0] + h[1]e^{-j\pi/2} + h[2]e^{-j\pi} + h[3]e^{-j3\pi/2} = 1 - j2$$

$$H[2] = h[0] + h[1]e^{-j\pi} + h[2]e^{-j2\pi} + h[3]e^{-j3\pi} = 5$$

$$H[3] = h[0] + h[1]e^{-j3\pi/2} + h[2]e^{-3j\pi} + h[3]e^{-j9\pi/2} = 1 + j2$$

共軛虛根

Convolution in the time domain produces multiplication in the frequency domain,

$$y[n] = h[n] \otimes x[n] \leftrightarrow Y[k] = H(k)X(k)$$

速 $x[n] = [1, 0, -2, 3]$ and $h[n] = [3, 1, 2, -1]$

$$Y[0] = H[0]X[0] = 10$$
$$Y[1] = H[1]X[1] = 9 - j3$$
$$Y[2] = H[2]X[2] = -20$$
$$Y[3] = H[3]X[3] = 9 + j3$$

inverse

$$y[n] = F^{-1}[Y[k]] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k]e^{j2\pi nk/N}$$

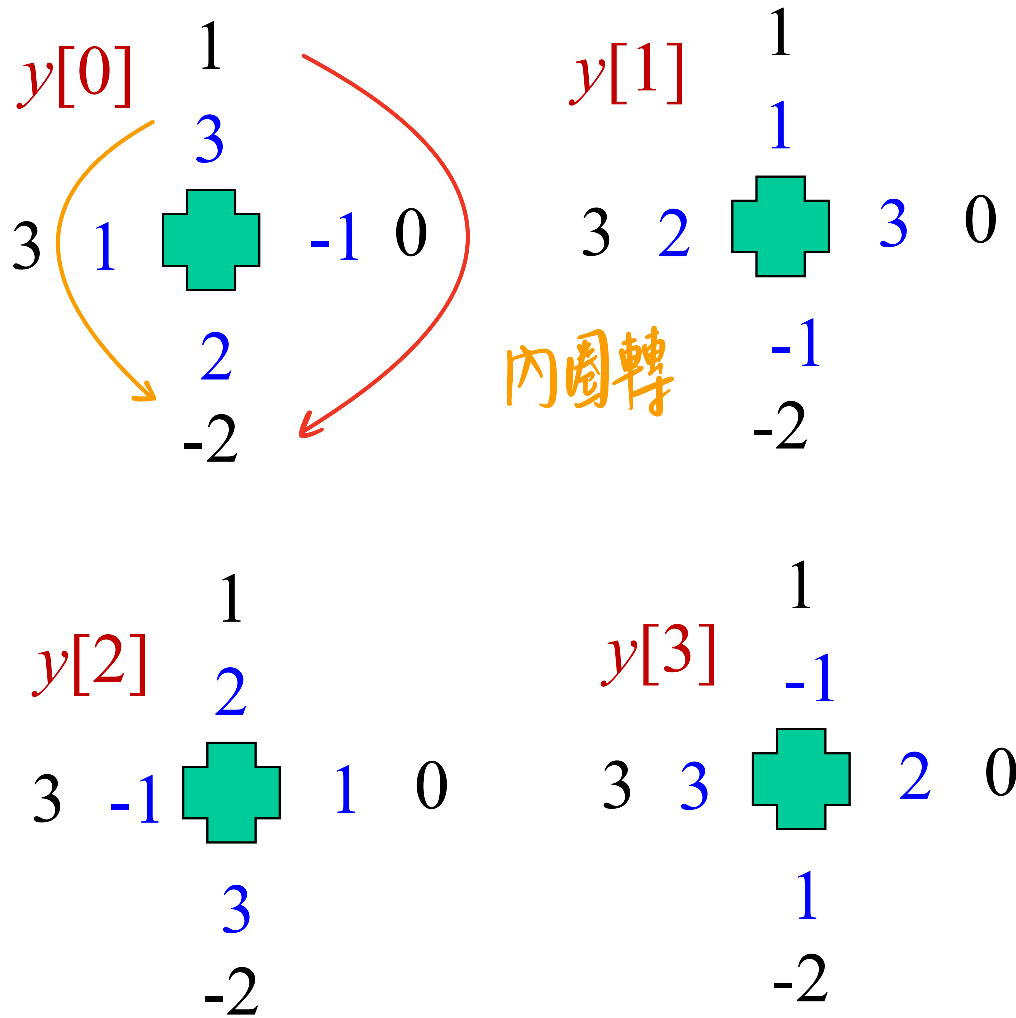
$$y[0] = \frac{1}{4}[Y[0] + Y[1] + Y[2] + Y[3]] = 2$$

$4 = N$

$$y[1] = \frac{1}{4}\left[Y[0] + Y[1]e^{j\pi/2} + Y[2]e^{j\pi} + Y[3]e^{j3\pi/2}\right] = 9$$

$$y[2] = \frac{1}{4}\left[Y[0] + Y[1]e^{j\pi} + Y[2]e^{j2\pi} + Y[3]e^{j3\pi}\right] = -7$$

$$y[3] = \frac{1}{4}\left[Y[0] + Y[1]e^{j3\pi/2} + Y[2]e^{3j\pi} + Y[3]e^{j9\pi/2}\right] = 6$$



$\Rightarrow$ $y[n] = [2, 9, -7, 6]$.

Example 6.10

$N=4$

Verify Parseval's relation using the sequence $x[n] = \{1, -2, 1, 3\}$.

Solution

energy $$E_x = \sum_{n=0}^{N-1} |x[n]|^2 = 1+4+1+9 = 15$$

DFT

$$X[k] = \begin{cases} 3, & k=0 \\ j5, & k=1 \\ 1, & k=2 \\ -j5, & k=3 \end{cases}$$

energy $$E_x = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 = \frac{1}{4}(9+25+1+25) = 15$$

6.5 FAST FOURIER TRANSFORM

- Fast Fourier Transform (FFT) dramatically reduces the number of computational operations.
- FFT is an algorithm—a fast computation of DFT.
- FFT was developed by James Cooley and John Tukey in 1965.
- For each $X[k]$, it requires N complex multiplications and $N-1$ complex additions. Because $k = 0, 1, 2, \ldots, N-1$, the entire computation of DFT requires N^2 complex multiplications and $N(N-1)$ complex additions.
- The number of complex multiplications required for FFT is

$\left(\frac{N}{2}\log_2 N\right)$ 較快

DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N} \quad k = 0, 1, \ldots, N-1$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi nk/N} \quad n = 0, 1, 2, \ldots, N-1$$

偶 奇

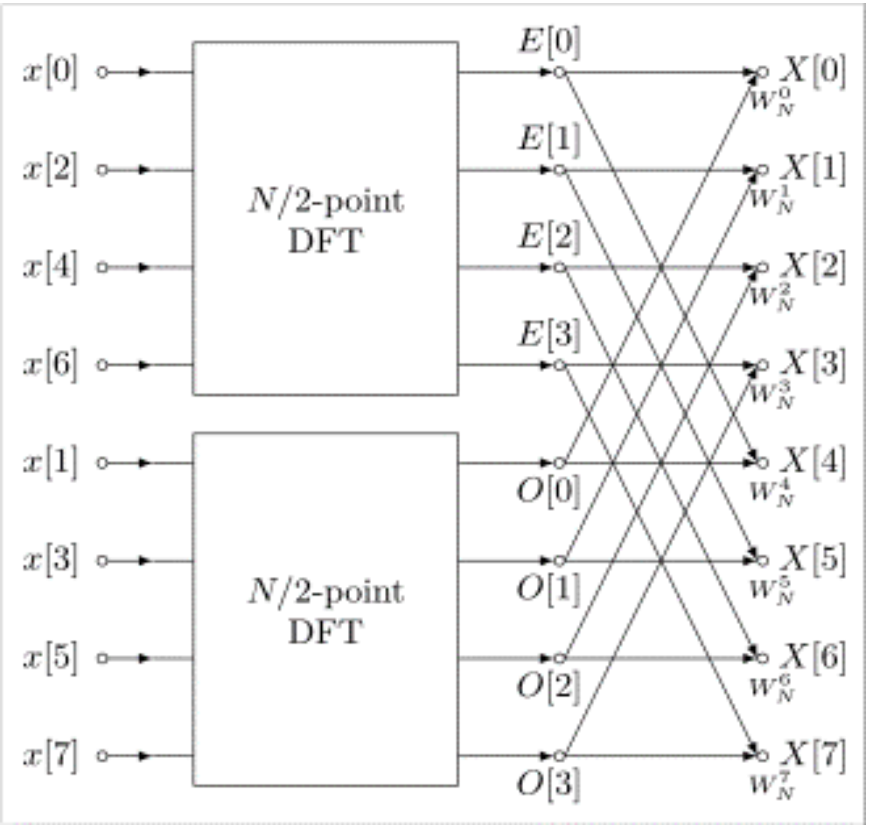
$$X[k] = \sum_{r=0}^{(N/2)-1} x[2r]w_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1]w_N^{(2r+1)k}$$

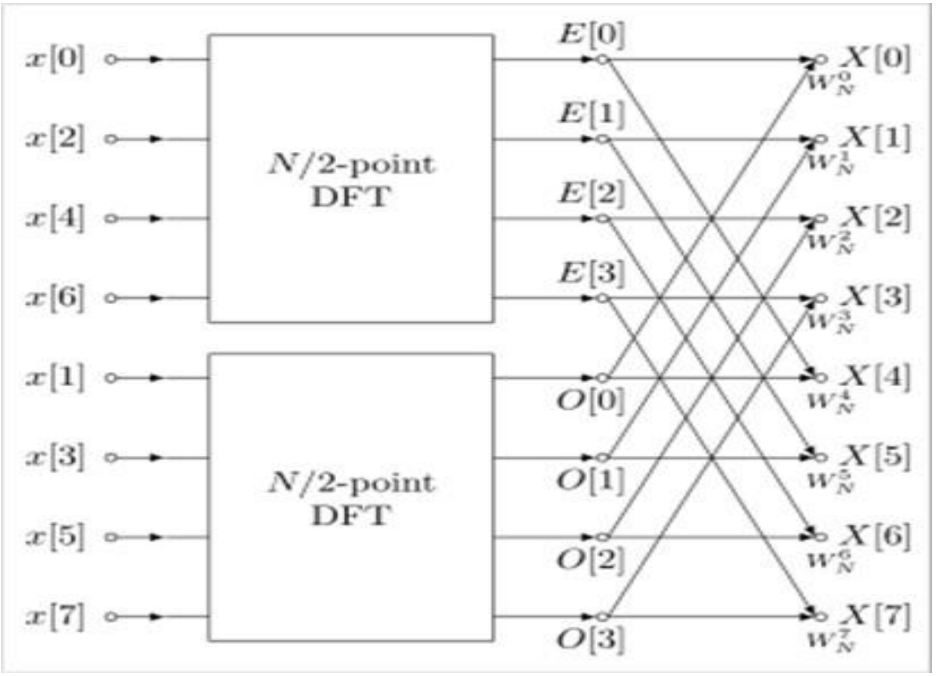
$$= \sum_{r=0}^{(N/2)-1} x[2r]w_N^{2rk} + w_N^k \sum_{r=0}^{(N/2)-1} x[2r+1]w_N^{2rk}$$

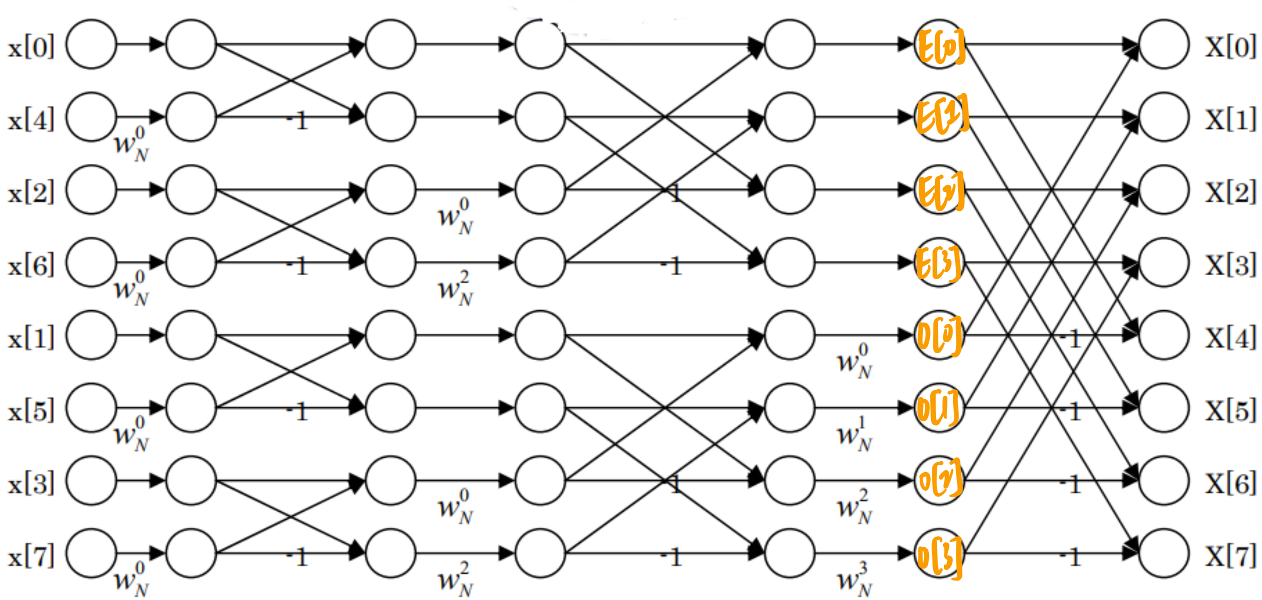
$$= \sum_{r=0}^{(N/2)-1} x[2r]w_{N/2}^{rk} + w_N^k \sum_{r=0}^{(N/2)-1} x[2r+1]w_{N/2}^{rk}$$

TABLE 6.4
Number of Multiplications Required in DFT and FFT

M	$N = 2^m$	DFT	FFT
1	2	4	1
2	4	16	4
3	8	64	12
4	16	256	32
5	32	1,024	80
6	64	4,096	192
7	128	16,384	448
8	256	65,536	1,024
9	512	261,144	2,304
10	1,024	1,048,576	5,120

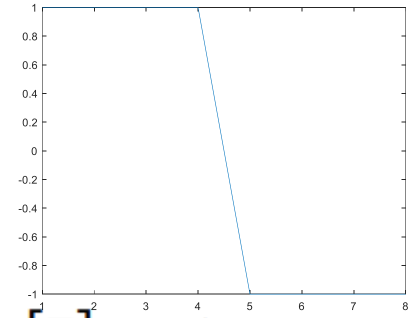






https://ir.nctu.edu.tw/bitstream/11536/70434/4/360104.pdf

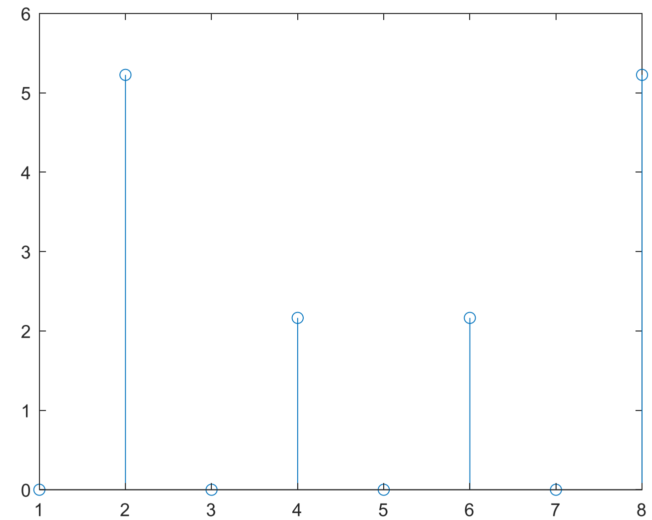
Example 6.11



A square wave is represented by the sequence $x[n]$ defined as

$$x[0]=x[1]=x[2]=x[3]=1 \text{ and } x[4]=x[5]=x[6]=x[7]=-1$$

Use MATLAB to find the FFT.



Solution

```
x = [ 1 1 1 1 -1 -1 -1 -1];
X = fft(x)
stem(abs(X))
```

↳絕對值(取大小)

This results in

X = [0 2 − j4.8284 0 2 − j0.8284 0 2 + j0.8284 0 2 + j4.8284]

The MATLAB command stem is used to plot the absolute value of the result, as shown in Figure 6.3. To ensure that the result is correct, we can find the IFFT of X.

```
x = ifft(X)
```

↳inverse

```
ans =

    1.0000    1.0000    1.0000    1.0000   -1.0000   -1.0000   -1.0000   -1.0000
```

Example 6.12

Use Matlab to find the periodic convolution of the following two sequences: (Exampe 6.9) ↳ circular conv

$$x[n]=[1,0,-2,3] \quad \text{and} \quad h[n]=[3,1,2,-1]$$

Solution

```
x = [ 1  0 -2 3];
h = [ 3  1  2 -1];
X = fft(x);
H = fft(h);
Y = X.*H;
y = ifft(Y)
```

→ circular conv

```
X =

  2.0000 + 0.0000i   3.0000 + 3.0000i  -4.0000 + 0.0000i   3.0000 - 3.0000i

>> H

H =

  5.0000 + 0.0000i   1.0000 - 2.0000i   5.0000 + 0.0000i   1.0000 + 2.0000i

y =

     2     9    -7     6
```

Example 6.13

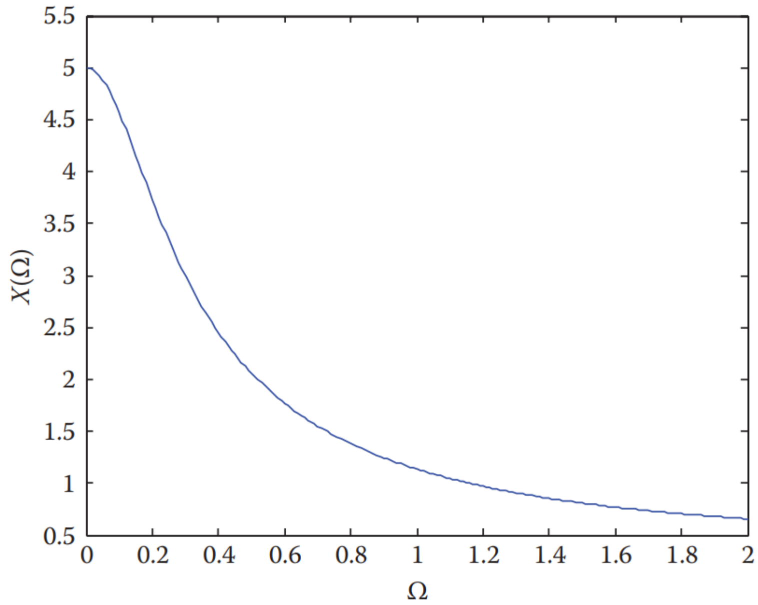
Although MATLAB does not have a function for DTFT, we can use it to plot the Fourier spectrum $X(\Omega)$. In Practice Problem 6.2, the DTFT of the discrete-time signal $x[n] = a^n u[n]$ is

$$X(\Omega) = \frac{1}{1 - ae^{-j\Omega}}$$

Plot this for $a = 0.8$,

Solution

```
a=0.8;
Omega= 0:0.01:2.0;
X =1./(1- a*exp(-j*Omega));
plot(Omega,abs(X));
xlabel('\Omega')
ylabel('X(\Omega)')
```



Example 6.14

Compute the DTFT of the discrete sequence $x[n] = 0.8^n u[n]$. Use $N = 32$ and compare the exact spectrum with the one obtained using the Hamming window.

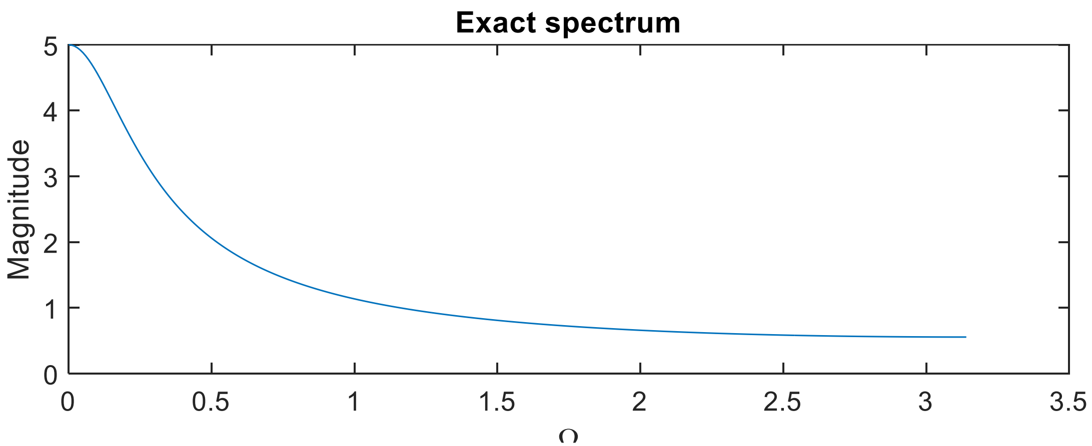
Solution

$$X(\Omega) = \frac{1}{1 - ae^{-j\Omega}} \qquad |a| \leq 1$$

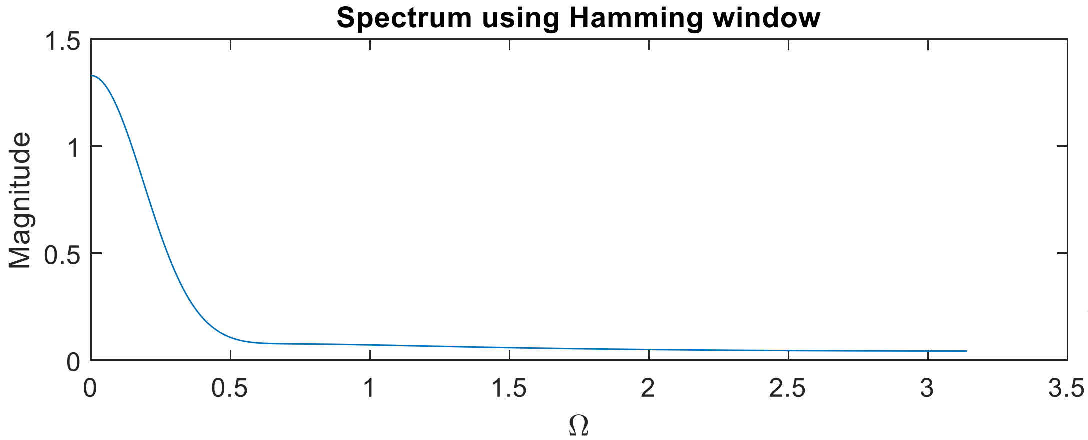
where $a = 0.8$. We use the following MATLAB code for computing $X(\Omega)$ and Hamming window.

```
a=0.8; N = 32;
n = 0:N-1
w = 0:0.01:pi; % values of Omega
fe = abs(1./(1 - a*exp(-j*w))); % exact DTFT
subplot(2,1,1)
  plot(w,fe)
  title('Exact spectrum');
  xlabel('\Omega'); ylabel('Magnitude');
   wh = 0.54 - 0.46*cos(2*pi*n'/(N-1)) %Hamming window
fh=abs((a.^n'.*wh)'*exp(-j*n'*w));
fhs = sum(fh,1); %sums columns of N x length(w) matrix fh
subplot(2,1,2)
   plot(w,fhs)
   title('Spectrum using Hamming window');
  xlabel('\Omega'); ylabel('Magnitude');
```

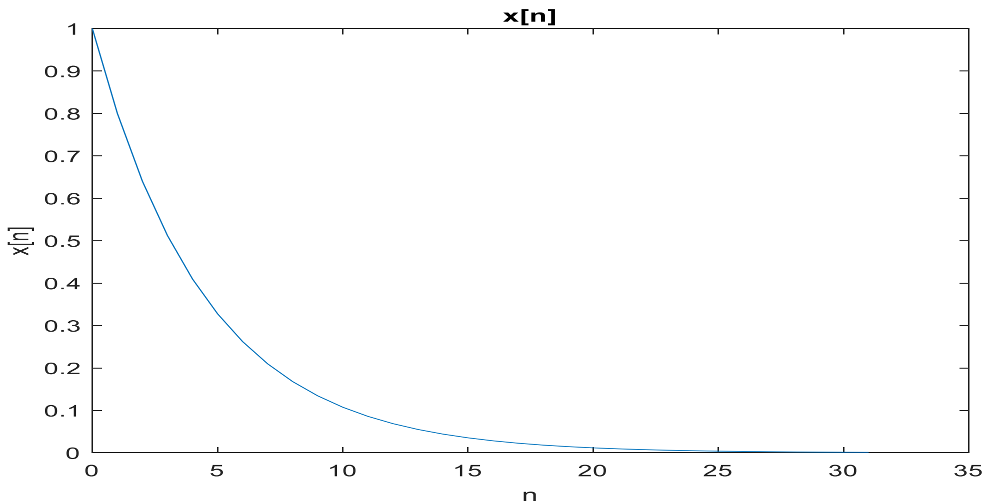
Example 6.14



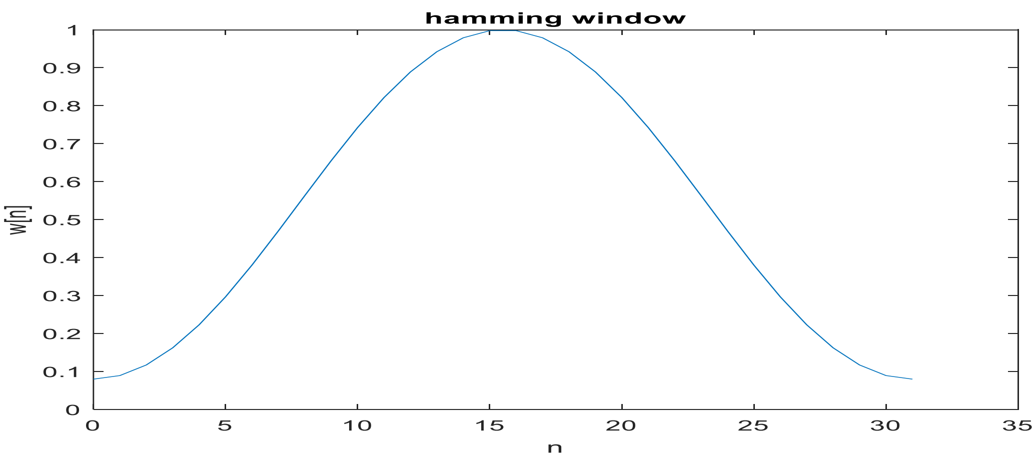

```
a=0.8; N = 32;
n = 0:N-1
w = 0:0.01:pi; % values of Omega
fe = abs(1./(1 - a*exp(-j*w))); % exact DTFT
subplot(2,1,1)
  plot(w,fe)
  title('Exact spectrum');
  xlabel('\Omega'); ylabel('Magnitude');
```



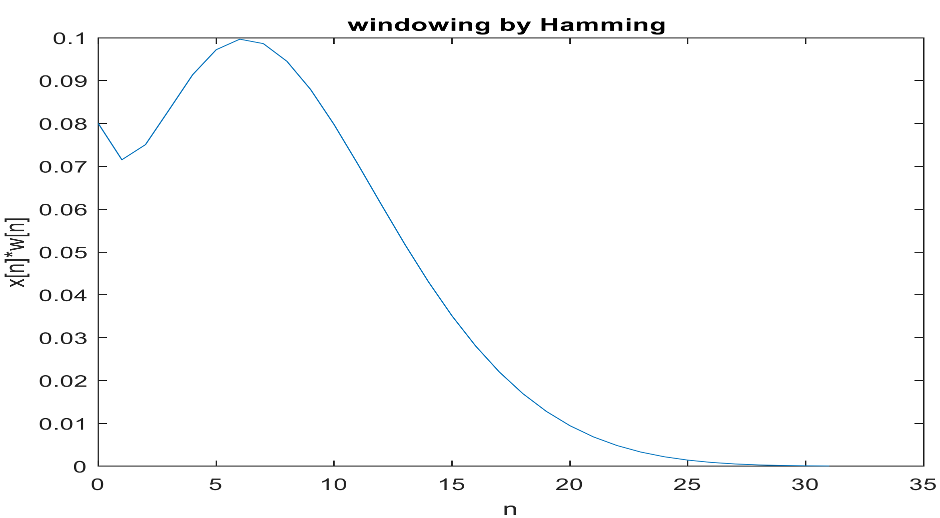

```
   wh = 0.54 - 0.46*cos(2*pi*n'/(N-1)) %Hamming window
fh=abs((a.^n'.*wh)'*exp(-j*n'*w));
fhs = sum(fh,1); %sums columns of N x length(w) matrix fh
subplot(2,1,2)
   plot(w,fhs)
   title('Spectrum using Hamming window');
  xlabel('\Omega'); ylabel('Magnitude');
```




```
figure
  plot(n,a.^n')
  title('x[n]');
  xlabel('n'); ylabel('x[n]');
```




```
figure
  plot(n,wh)
  title('hamming window');
  xlabel('n'); ylabel('w[n]');
```




```
figure
  plot(n,a.^n'.*wh)
  title('windowing by Hamming ');
  xlabel('n'); ylabel('x[n]*w[n]');
```