訊號與系統
SIGNAL AND SYSTEM

Lecture 6
FOURIER TRANSFORM

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5.2 FOURIER TRANSFORM

$$x(t) \Leftrightarrow X(\omega)$$

$$\boxed{X(\omega) = \mathcal{F}\left[x(t)\right] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt}$$

$$\boxed{x(t) = \mathcal{F}\left[X(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega}$$

- If a signal is real-valued, its magnitude spectrum is even, that is, $|X(\omega)| = |X(-\omega)|$ and its phase spectrum is odd, that is, $\angle X(\omega) = -\angle X(-\omega)$.

Recall Laplace Transform

$$\mathcal{L}[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

Example 5.1

Find the Fourier transform of the following functions: (a)$\delta(t)$, (b) $e^{j\omega_0 t}$, (c) $\sin\omega_0 t$, (d) $e^{-at}u(t)$.

Solution

(a)

$$X(\omega) = \mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = e^{-j\omega t}\Big|_{t=0} = 1$$

$$\mathcal{F}[\delta(t)] = 1$$

(b)

$$\delta(t) = \mathcal{F}^{-1}[1] = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1e^{j\omega t}d\omega$$

$$\int_{-\infty}^{\infty} e^{j\omega t}dt = 2\pi\delta(\omega)$$

$$\boxed{\mathcal{F}[e^{j\omega_o t}] =} \int_{-\infty}^{\infty} e^{j\omega_o t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{j(\omega_o-\omega)t}dt = 2\pi\delta(\omega_0 - \omega)$$

$$\mathcal{F}[e^{j\omega_0 t}] = 2\pi\delta(\omega - \omega_0)$$

$$\mathcal{F}[1] = 2\pi\delta(\omega)$$

Example 5.1

Find the Fourier transform of the following functions: (a)$\delta(t)$, (b) $e^{j\omega_0 t}$, (c) $\sin\omega_0 t$, (d) $e^{-at}u(t)$.

Solution

(c)

$$\mathcal{F}[\sin\omega_0 t] = \mathcal{F}\left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right]$$

$$= \frac{1}{2j}\mathcal{F}\left[e^{j\omega_0 t}\right] - \frac{1}{2j}\mathcal{F}\left[e^{-j\omega_0 t}\right]$$

$$= j\pi\left[\delta\left(\omega + \omega_0\right) - \delta\left(\omega - \omega_0\right)\right]$$

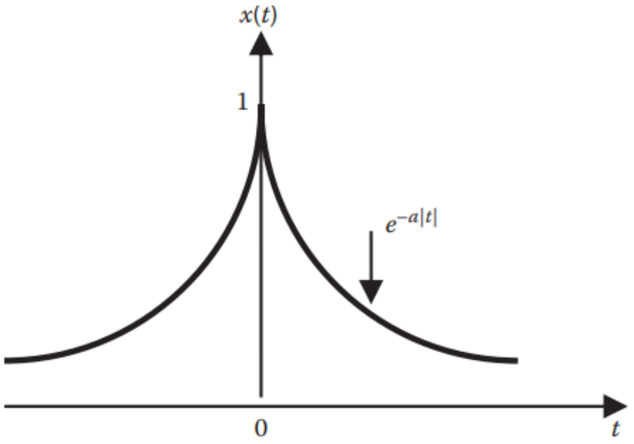
(d)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$\mathcal{F}\left[e^{-at}u(t)\right] = X(\omega) = \frac{-1}{a+j\omega}e^{-(a+j\omega)t}\Big|_{0}^{\infty} = \frac{1}{a+j\omega}$$

Example 5.3

Find the Fourier transform of the two-sided exponential pulse shown in Figure 5.2. Sketch the transform.



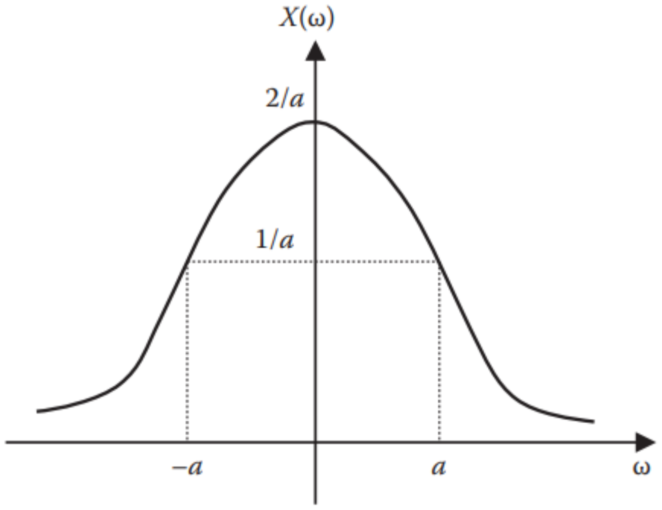
Solution

Let $x(t) = e^{-a|t|} = \begin{cases} e^{at}, & t < 0 \\ e^{-at}, & t > 0 \end{cases}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega}$$

$$= \frac{2a}{a^2 + \omega^2}$$



- Duality

$$\boxed{\mathcal{F}\left[x(t)\right]=X(\omega)\Rightarrow\mathcal{F}\left[X(t)\right]=2\pi x(-\omega)}$$

Proof

$$x(t)=\mathcal{F}^{-1}\left[X(\omega)\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}\,d\omega$$

$$2\pi x(t)=\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}\,d\omega$$

$$2\pi x(-t)=\int_{-\infty}^{\infty}X(\omega)e^{-j\omega t}\,d\omega$$

$$2\pi x(-\omega)=\int_{-\infty}^{\infty}X(t)e^{-j\omega t}\,dt=\mathcal{F}\left[X(t)\right]$$

Example 5.4

A signal $x(t)$ has a Fourier transform given by

$$X(\omega) = \frac{5(1+j\omega)}{8-\omega^2+6j\omega}$$

Without finding $x(t)$, find the Fourier transform of

(a) $x(t-3)$
(b) $x(4t)$
(c) $e^{-j2t}x(t)$
(d) $x(-2t)$

Solution

(a) $\mathcal{F}[x(t-3)] = e^{-j\omega 3}X(\omega) = \dfrac{5(1+j\omega)e^{-j\omega 3}}{8-\omega^2+j6\omega}$

(b) $\mathcal{F}[x(4t)] = \dfrac{1}{4}X\left(\dfrac{\omega}{4}\right) = \dfrac{\frac{5}{4}(1+j\omega/4)}{8-\omega^2/16+j6\omega/4} = \dfrac{5(4+j\omega)}{128-\omega^2+j24\omega}$

(c) $\mathcal{F}[e^{-j2t}x(t)] = X(\omega+2) = \dfrac{5[1+j(\omega+2)]}{8-(\omega+2)^2+6j(\omega+2)} = \dfrac{5(1+j\omega+j2)}{4-\omega^2-4\omega+6j\omega+j12}$

(d) $\mathcal{F}[x(-2t)] = \dfrac{1}{2}X\left(\dfrac{\omega}{-2}\right) = \dfrac{\frac{5}{2}(1-j\omega/2)}{8-\frac{\omega^2}{4}-\frac{6j\omega}{2}} = \dfrac{5(2-j\omega)}{32-\omega^2-12j\omega}$

TABLE 5.1
Properties of the Fourier Transform

No.	Property	$x(t)$	$X(\omega)$
1.	Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
2.	Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
3.	Time shift	$x(t-a)$	$e^{-j\omega a}X(\omega)$
4.	Frequency shift	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
5.	Modulation	$\cos(\omega_0 t)x(t)$	$\frac{1}{2}[X(\omega+\omega_0)+X(\omega-\omega_0)]$
		$\sin(\omega_0 t)x(t)$	$\frac{j}{2}[X(\omega+\omega_0)-X(\omega-\omega_0)]$
6.	Time differentiation	$\frac{dx}{dt}$	$j\omega X(\omega)$
		$\frac{d^n x}{dt^n}$	$(j\omega)^n X(\omega)$
7.	Frequency differentiation	$(-jt)^n x(t)$	$\frac{d^n}{d\omega^n}X(\omega)$
8.	Time integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{X(\omega)}{j\omega}+\pi X(0)\delta(\omega)$
9.	Time reversal	$x(-t)$	$X(-\omega)$ or $X^*(\omega)$
10.	Duality	$X(t)$	$2\pi x(-\omega)$
11.	Time convolution	$x_1(t)*x_2(t)$	$X_1(\omega)X_2(\omega)$
12.	Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega)*X_2(\omega)$
13.	Parseval's relation	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2 dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\omega)\rvert^2 d\omega$

- Time Differentiation

$$\boxed{\mathcal{F}\left[x'(t)\right] = j\omega X(\omega)}$$

Proof

$$x(t) = \mathcal{F}^{-1}\left[X(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

$$\frac{dx(t)}{dt} = \frac{j\omega}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega = j\omega\ x(t)$$

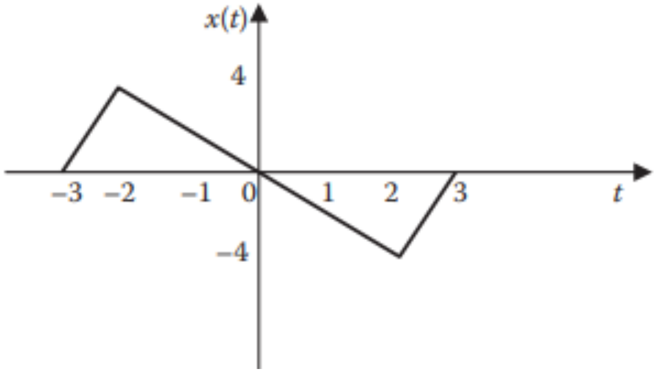
$$\mathcal{F}\frac{dx(t)}{dt} = \mathcal{F}\ j\omega\ x(t) = j\omega\ \left[X(\omega)\right]$$

$$\mathcal{F}\left[x'(t)\right] = j\omega X(\omega)$$

$$\mathcal{F}\left[x^{(n)}(t)\right] = (j\omega)^n X(\omega)$$

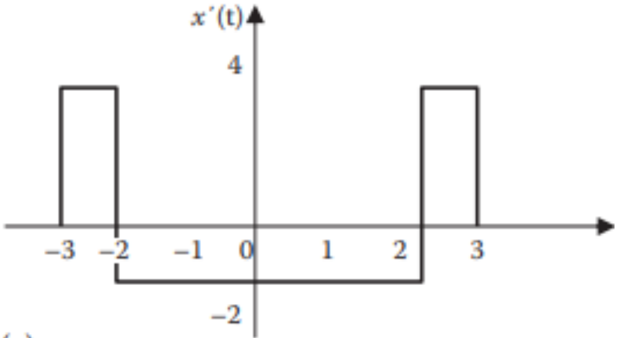
Example 5.5

Determine the Fourier transform of the signal in Figure 5.6.



Solution

$$x''(t) = 4\delta(t+3) - 6\delta(t+2) + 6\delta(t-2) - 4\delta(t-3)$$



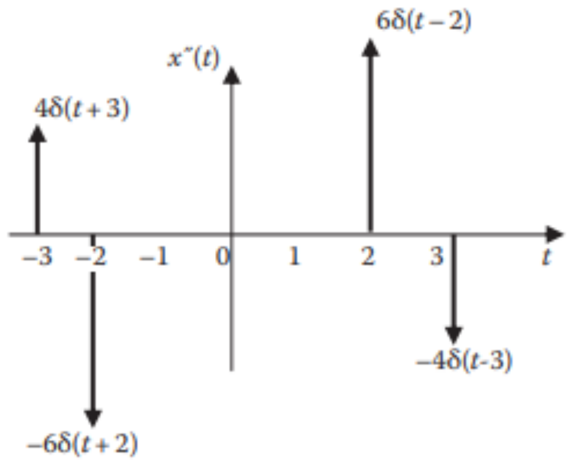
(a)

$$(j\omega)^2 X(\omega) = 4e^{j3\omega} - 6e^{j2\omega} + 6e^{-j2\omega} - 4e^{-j3\omega}$$

$$-\omega^2 X(\omega) = 4\left(e^{j3\omega} - e^{-j3\omega}\right) + 6\left(e^{j2\omega} - e^{-j2\omega}\right)$$

$$= j8\sin 3\omega - j12\sin 2\omega$$

$$X(\omega) = \frac{j}{\omega^2}\left(12\sin 2\omega - 8\sin 3\omega\right)$$



(b)

Example 5.6

Find the inverse Fourier transform of

(a) $G(\omega) = \dfrac{10j\omega}{(-j\omega+2)(j\omega+3)}$

(b) $Y(\omega) = \dfrac{\delta(\omega)}{(j\omega+1)(j\omega+2)}$

Solution

(a)

$$G(s) = \frac{10s}{(2-s)(3+s)} = \frac{-10s}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}, \quad s = j\omega$$

$$A = (s-2)G(s)\Big|_{s=2} = \frac{-10(2)}{2+3} = -4$$

$$B = (s+3)G(s)\Big|_{s=-3} = \frac{-10(-3)}{-3-2} = -6$$

$$G(\omega) = \frac{-4}{j\omega-2} - \frac{6}{j\omega+3}$$

Taking the inverse Fourier transform of each term,

$$g(t) = -4e^{2t}u(-t) - 6e^{-3t}u(t)$$

Example 5.6

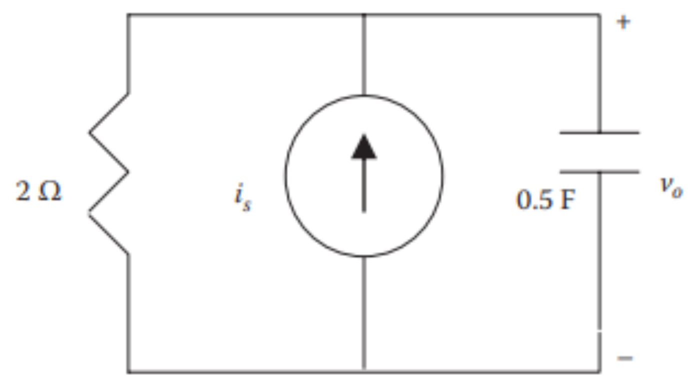
$$\text{(b)} \quad Y(\omega) = \frac{\delta(\omega)}{(j\omega+1)(j\omega+2)}$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\delta(\omega)e^{j\omega t}d\omega}{(2+j\omega)(j\omega+1)} = \frac{1}{2\pi}\frac{e^{j\omega t}}{(2+j\omega)(j\omega+1)}\bigg|_{\omega=0} = \frac{1}{2\pi}\frac{1}{2} = \frac{1}{4\pi}$$

5.5 APPLICATIONS

Example 5.7

Determine $v_o(t)$ in the circuit of Figure 5.9, where $i_s = 10e^{-2t}u(t)A$.



Solution

$$0.5\text{F} \rightarrow \frac{1}{j\omega C} = \frac{1}{j\omega 0.5} = \frac{2}{j\omega},$$

$$i_s = 10e^{-2t} \rightarrow I_s = \frac{10}{2 + j\omega}$$

$$I_o = \frac{2}{2 + \frac{2}{j\omega}} I_s = \frac{j\omega}{1 + j\omega} \cdot \frac{10}{2 + j\omega}$$

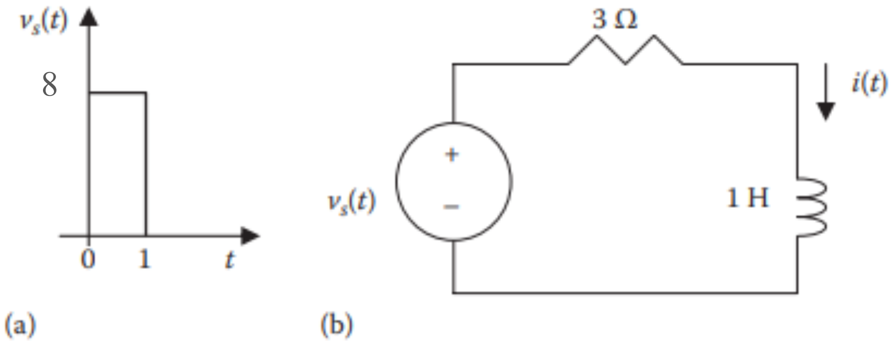
$$V_o = I_o \frac{2}{j\omega} = \frac{20}{(1 + j\omega)(2 + j\omega)} = \frac{20}{(s+1)(s+2)}, s = j\omega$$

$$V_o = \frac{A}{s+1} + \frac{B}{s+2} = 20\left[\frac{1}{s+1} - \frac{1}{s+2}\right]$$

$$v_o(t) = 20(e^{-t} - e^{-2t})u(t)\ V$$

Example 5.8

Given the circuit in Figure 5.11, with its excitation, determine the Fourier transform of $i(t)$.



Solution

$$v_s'(t) = 8\delta(t) - 8\delta(t-1)$$

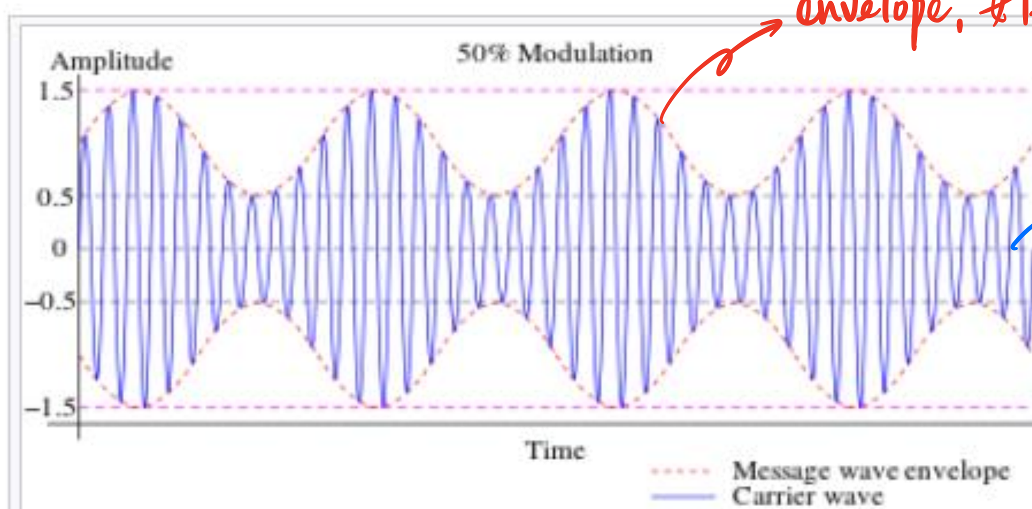
$$j\omega V_s = 8(1 - e^{-j\omega}) \rightarrow V_s = \frac{8}{j\omega}\left(1 - e^{-j\omega}\right)$$

$$I(\omega) = \frac{V_s}{3 + j\omega 1} = \frac{8(1 - e^{-j\omega})}{j\omega(3 + j\omega)}$$

Amplitude modulation (AM) 頻率不同的訊號疊加

- One way of transmitting low-frequency audio information (50 Hz to 20 kHz) is to transmit it along with a high-frequency signal, called a carrier.
- Any of the three characteristics (amplitude, frequency, or phase) of a carrier can be controlled, to be able to carry the intelligent signal, called the modulating signal.
- AM is a process whereby we let the modulating signal control the amplitude of the carrier.

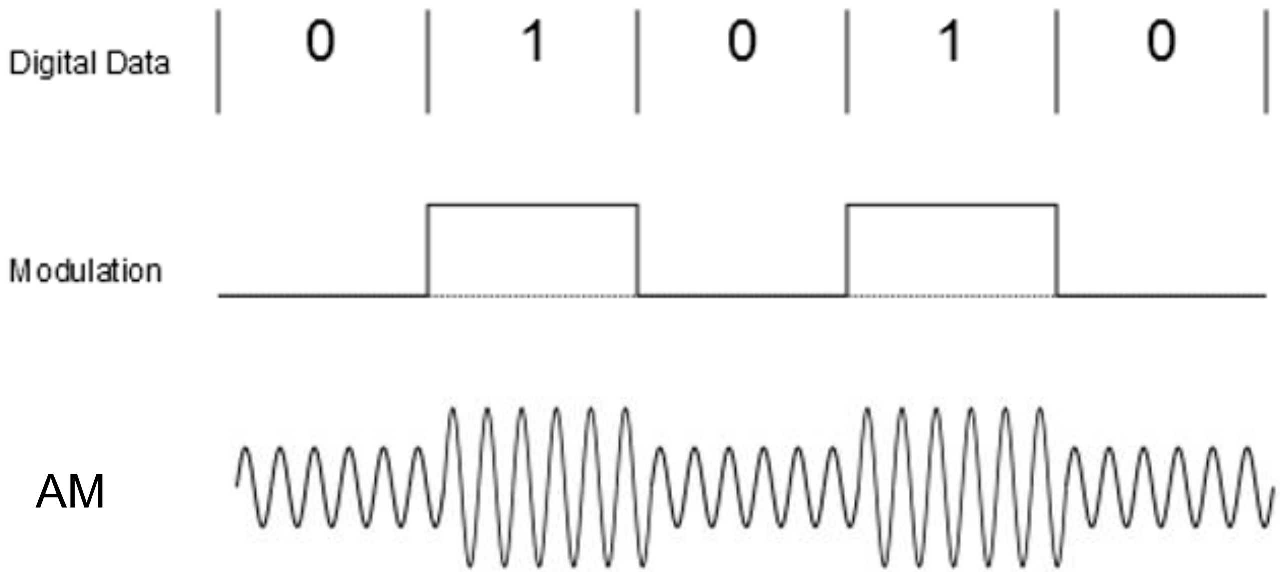
envelope, 要傳送的資訊(低頻)



載波
(高頻)

https://en.wikipedia.org/wiki/Amplitude_modulation

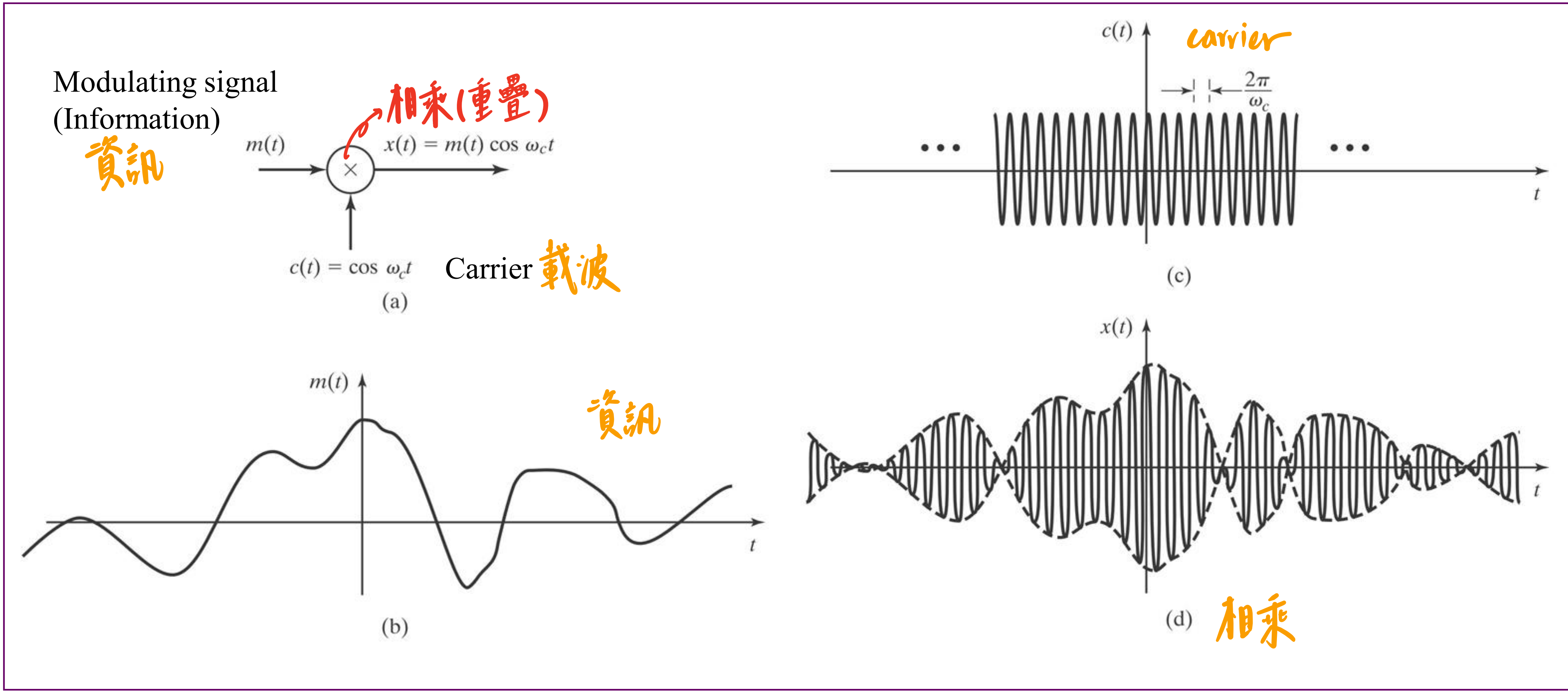
Digital Modulation

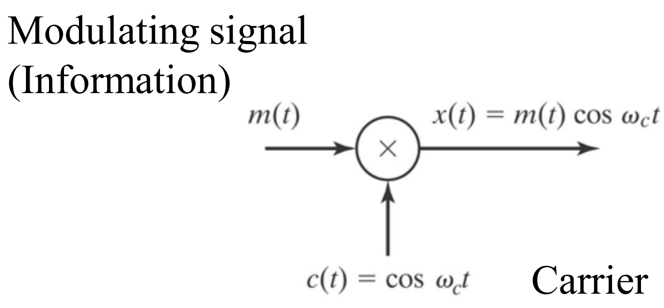


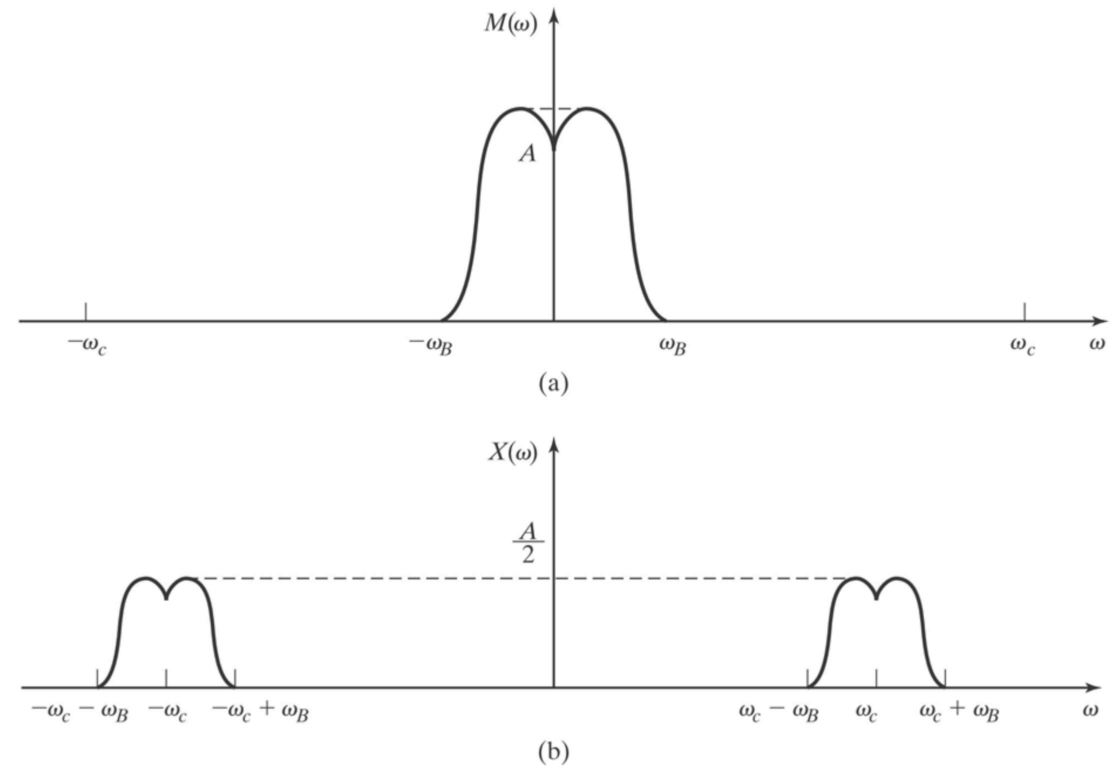
↳ 用數位
⇒ 抗雜訊

https://www.5gtechnologyworld.com/digital-modulation-basics-part-1/

Amplitude modulation (AM)







$$x(t) = m(t)c(t) = m(t)\cos(\omega_c t).$$

$$x(t) = m(t) \times \tfrac{1}{2}\left[e^{j\omega_c t} + e^{-j\omega_c t}\right]$$

F

$$= \tfrac{1}{2} m(t) e^{j\omega_c t} + \tfrac{1}{2} m(t) e^{-j\omega_c t}.$$

$$X(\omega) = \int_{-\infty}^{\infty} m(t)\left(\frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}\right) e^{-j\omega t}\, dt$$

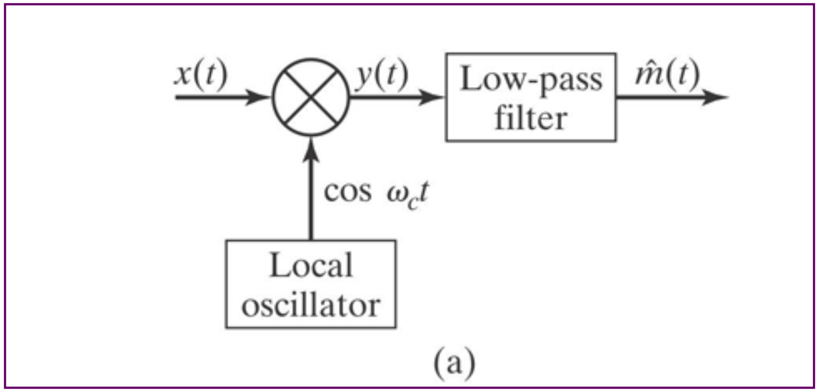
$$X(\omega) = \tfrac{1}{2}\left[M(\omega - \omega_c) + M(\omega + \omega_c)\right].$$

Recall

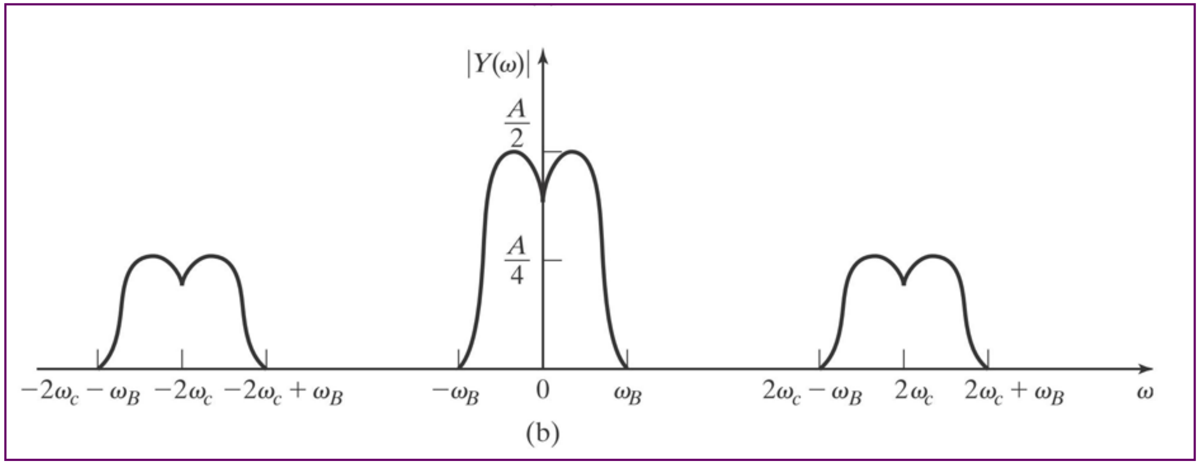
- Frequency Shifting $\boxed{\mathcal{L}\left[e^{-at}x(t)u(t)\right] = X(s+a)}$

$$\mathcal{L}\left[e^{-at}x(t)u(t)\right] = \int_0^{\infty} e^{-at}x(t)e^{-st}\,dt = \int_0^{\infty} x(t)e^{-(s+a)t}\,dt = X(s+a)$$

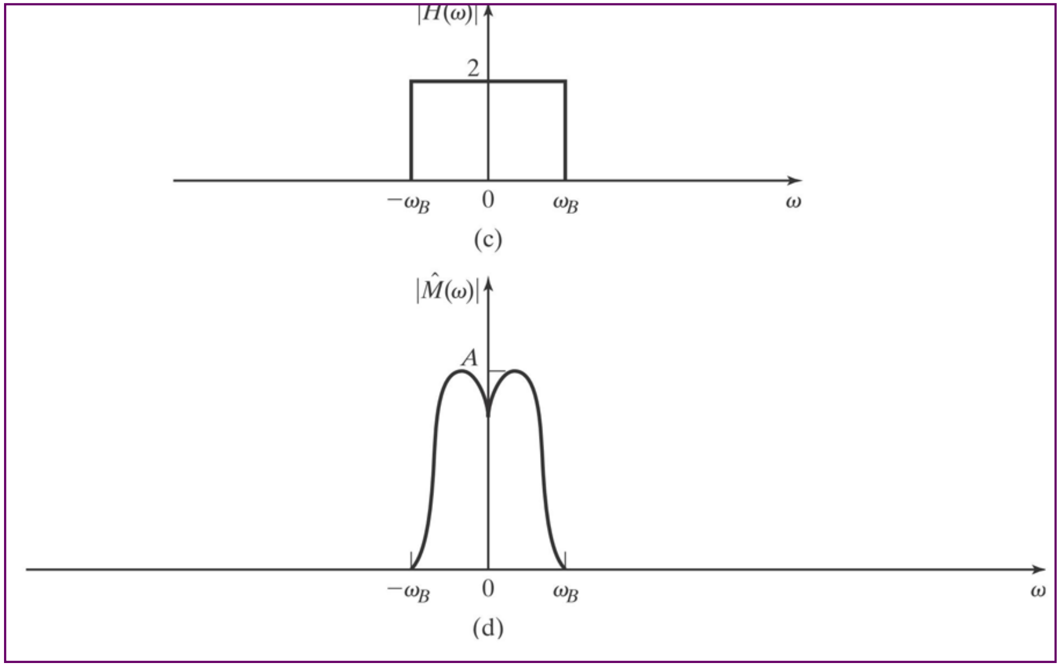
Demodulation



(a)



(b)



(d)

$$y(t) = x(t)\cos(\omega_c t).$$

$$Y(\omega) = \tfrac{1}{2}[X(\omega-\omega_c) + X(\omega+\omega_c)].$$

$$X(\omega) = \tfrac{1}{2}[M(\omega-\omega_c) + M(\omega+\omega_c)].$$

$$Y(\omega) = \tfrac{1}{2}M(\omega) + \tfrac{1}{4}M(\omega-2\omega_c) + \tfrac{1}{4}M(\omega+2\omega_c),$$

Example 5.9

Recall

- One way of transmitting low-frequency audio information (50 Hz to 20 kHz) is to transmit it along with a high-frequency signal, called a carrier.

An AM signal is given by

$$x(t) = \cos 200\pi t \boxed{\cos 10^4 \pi t}$$

Identify the upper sideband and the lower sideband.

Solution

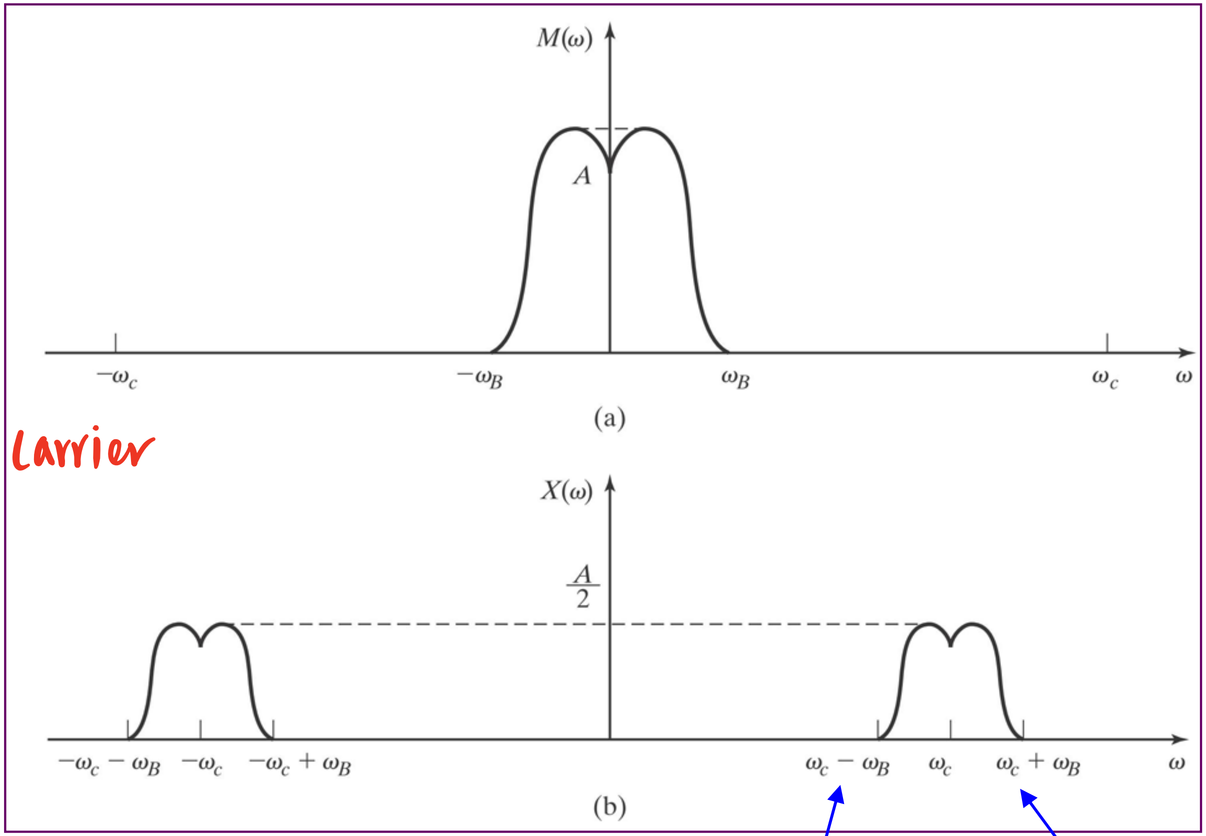
$w = 2\pi f$

$$\omega_o = 200\pi \rightarrow f_o = \frac{\omega_o}{2\pi} = 100 \text{ 低}$$

$$\omega_c = 10^4\pi \rightarrow f_c = \frac{\omega_c}{2\pi} = 5000 \text{ 高} \Rightarrow \text{carrier}$$

$$\text{USB} = f_c + f_o = 5100 = 5.1 \text{ kHz}$$

$$\text{LSB} = f_c - f_o = 4900 = 4.9 \text{ kHz}$$



upper sideband

lower sideband

Practice Problem 5.9

A music signal whose frequencies range from 80 Hz 低 to 12 kHz is used to modulate a 2-MHz 高 carrier. Find the range of frequencies for the lower and upper sidebands.

Solution

$$\mathrm{LSB} = f_c - f_o$$

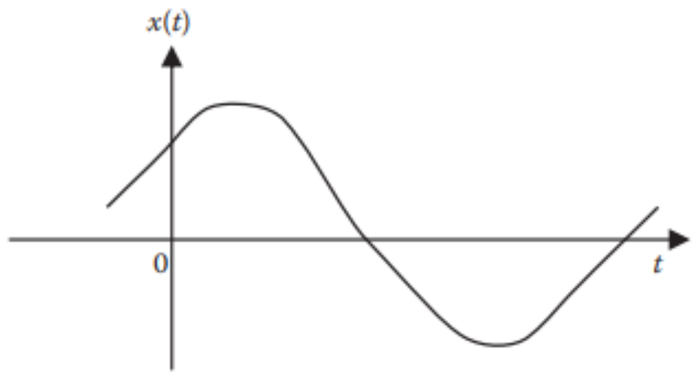
2,000,000-[80 12000]= [1,999,200 1,988,000] Hz

$$\mathrm{USB} = f_c + f_o$$

2,000,000+[80 12000]= [2,000,080 2,012,000] Hz

Sampling

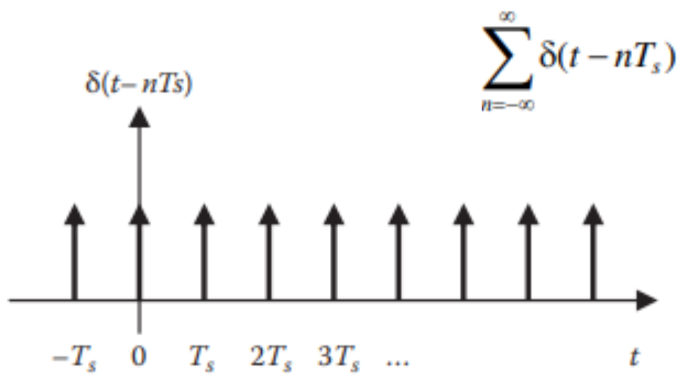
- Sampling is an important operation in signal processing. It may be regarded as a way of reducing analog signals to discrete signals.
- We multiply $x(t)$ by a train of impulses $\delta(t-nTs)$, where Ts is the sampling interval and $fs = 1/Ts$ is the sampling frequency (or rate). The value of $x(t)$ at point nT.



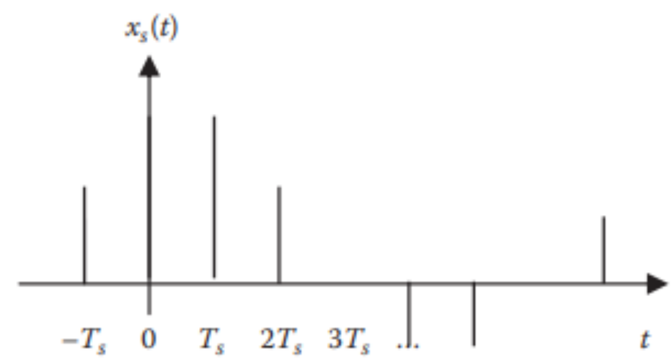
(a)

$$x_s(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$$

t=nTs有值



(b)



(c)

$$x_s(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$$

Refer to Fourier Series

$$\sum_{n=-\infty}^{\infty}\delta(t-nT_0) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t},$$

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\left[\sum_{m=-\infty}^{\infty}\delta(t-mT_0)\right]e^{-jn\omega_0 t}dt. = \frac{1}{T_0}.$$

$$= \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

$$\mathcal{F}[e^{j\omega_0 t}] = 2\pi\delta(\omega-\omega_0)$$

$$\mathcal{F}\left[\sum_{n=-\infty}^{\infty}\delta(t-nT_0)\right] = \frac{2\pi}{T_0}\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0) = \omega_s\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_s)$$

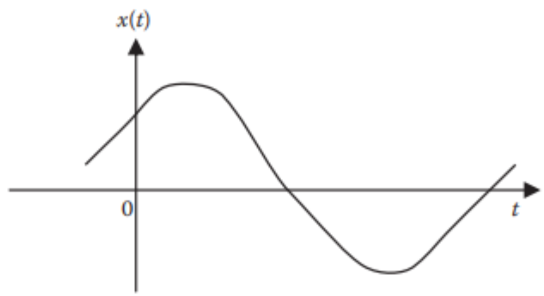
We take the Fourier transform of this and apply the frequency convolution property.

$$X_s(\omega) = \frac{1}{2\pi}\left[X(\omega)\right]*\omega_s\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_s)$$

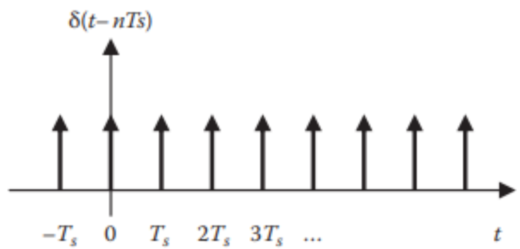
Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega)*X_2(\omega)$

$$= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega)*\delta(\omega-n\omega_s)$$

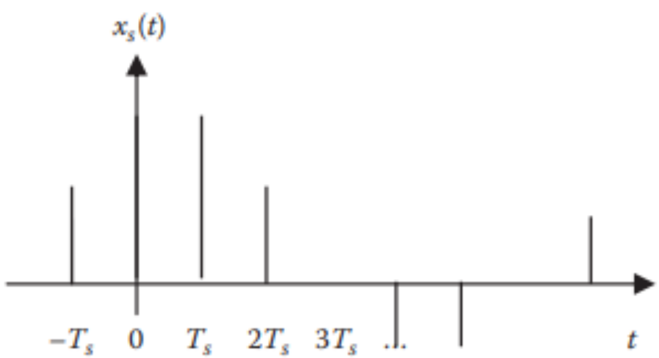
$$= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega-n\omega_s)$$



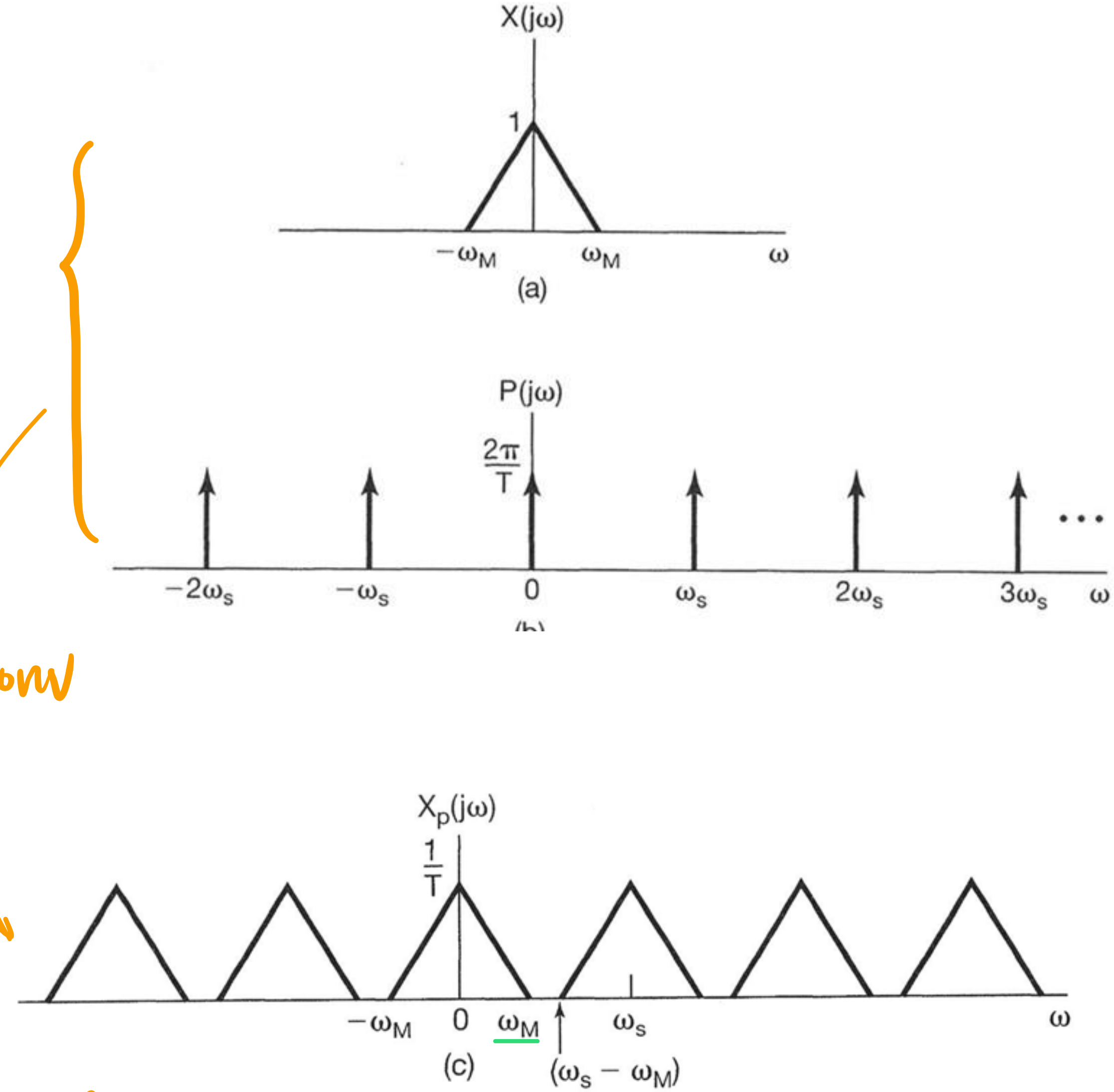
(a)



(b)



(c)



lower band upper band

$\omega_s - \omega_M > \omega_M$

$\omega_s > 2\,\omega_M$

to avoid overlapping

↳ 避免重疊相互干擾

Band-limited Signal

- A band-limited signal, with bandwidth W hertz, may be completely recovered from its samples if taken at a frequency at least twice as high as $2W$ samples.

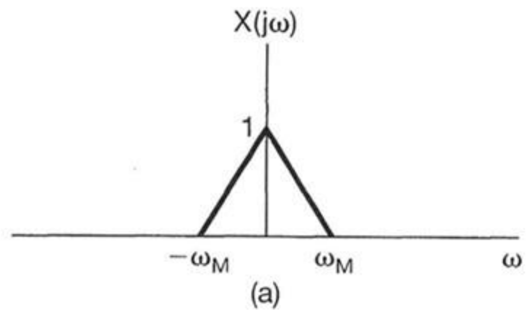
$$\frac{1}{T_s} = f_s \geq 2W$$

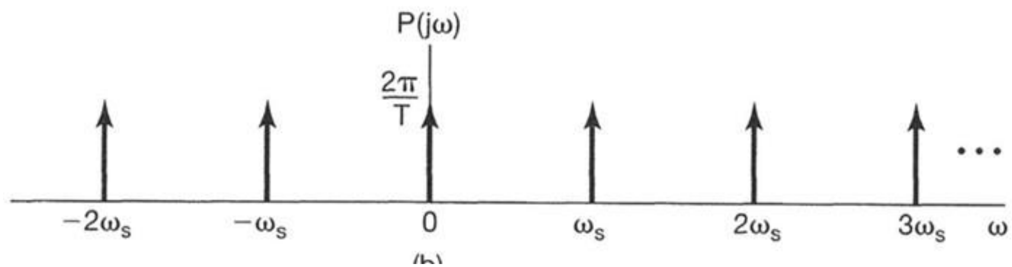
取樣頻率 ≥ 2倍頻寬

Nyquist frequency

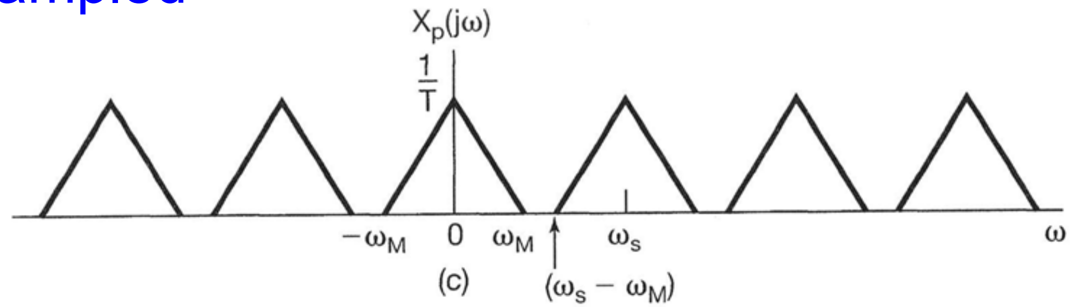
- The minimum sampling frequency $fs = 2W$ is known as the Nyquist frequency or rate, and the maximum spacing $1/fs$ is the Nyquist interval.
- A band-limited signal, with bandwidth W hertz, may be completely recovered from its samples if taken at a frequency at least twice as high as $2W$ samples.
- A signal is said to be **oversampled** if it is sampled at a rate **greater** than its Nyquist rate.
- It is **undersampled** if it is sampled at **less** than the Nyquist rate.

↳ 會overlapping

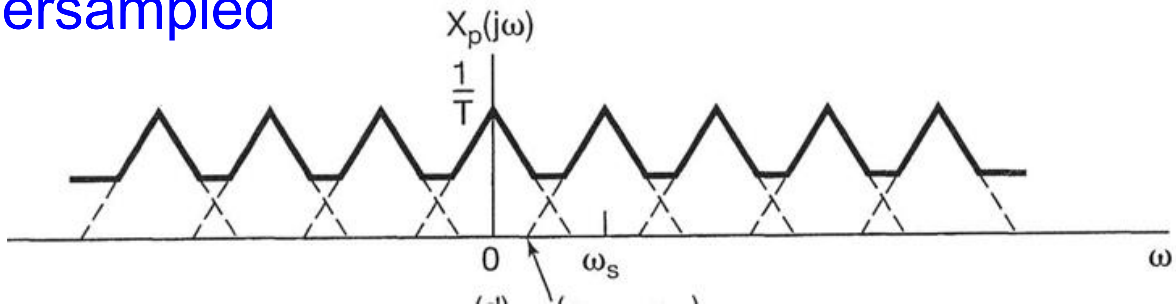




oversampled

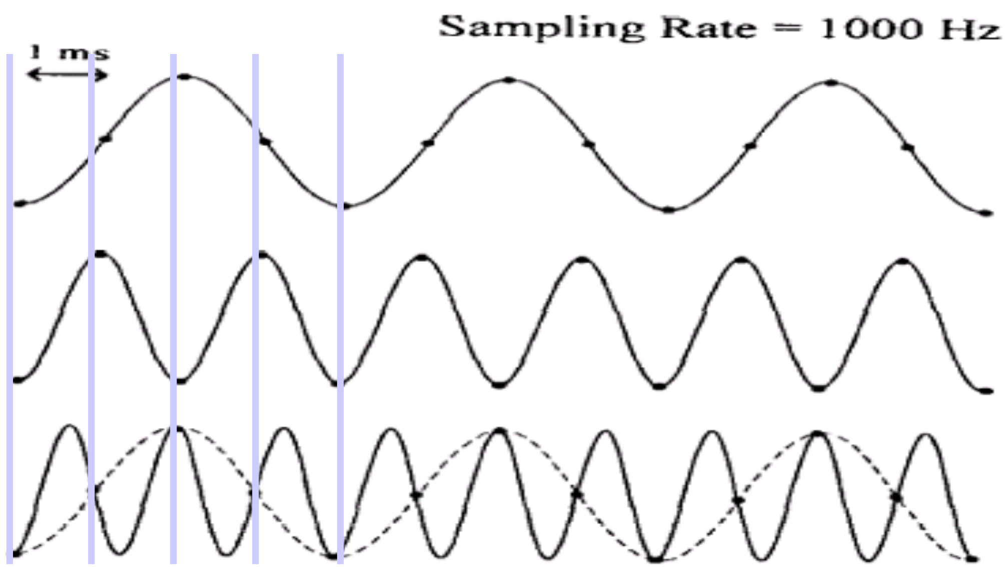


undersampled



→ 互相干擾

Aliasing



250Hz

500Hz

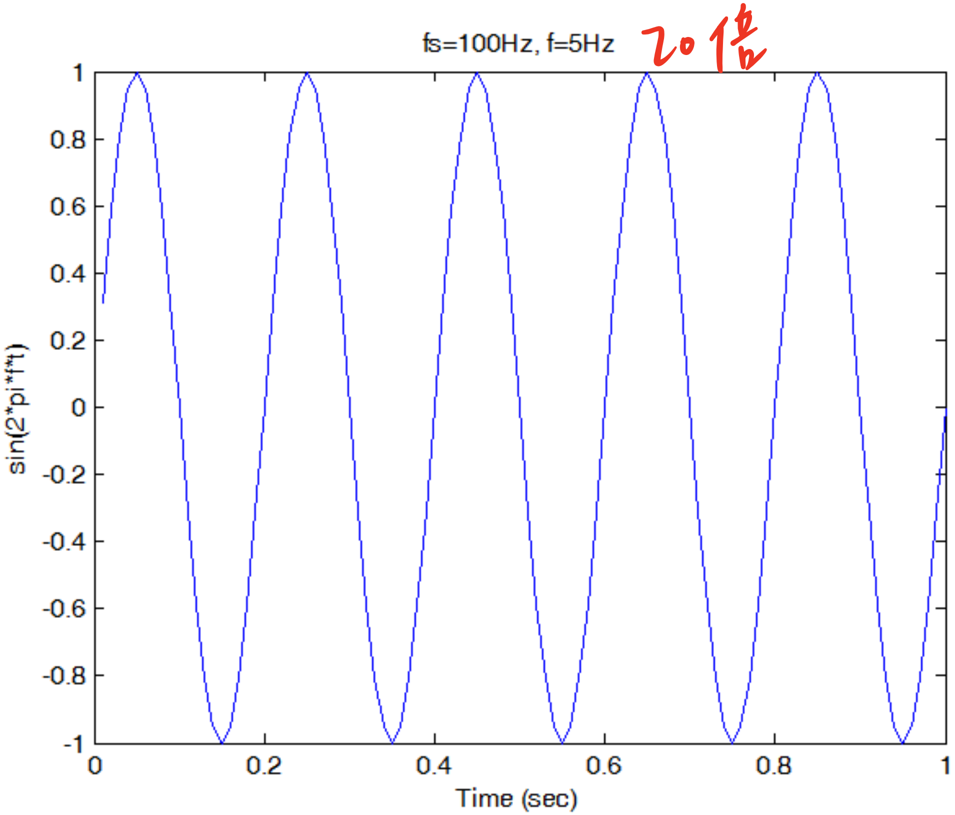
750Hz ⇒ 取樣f至少要1500Hz

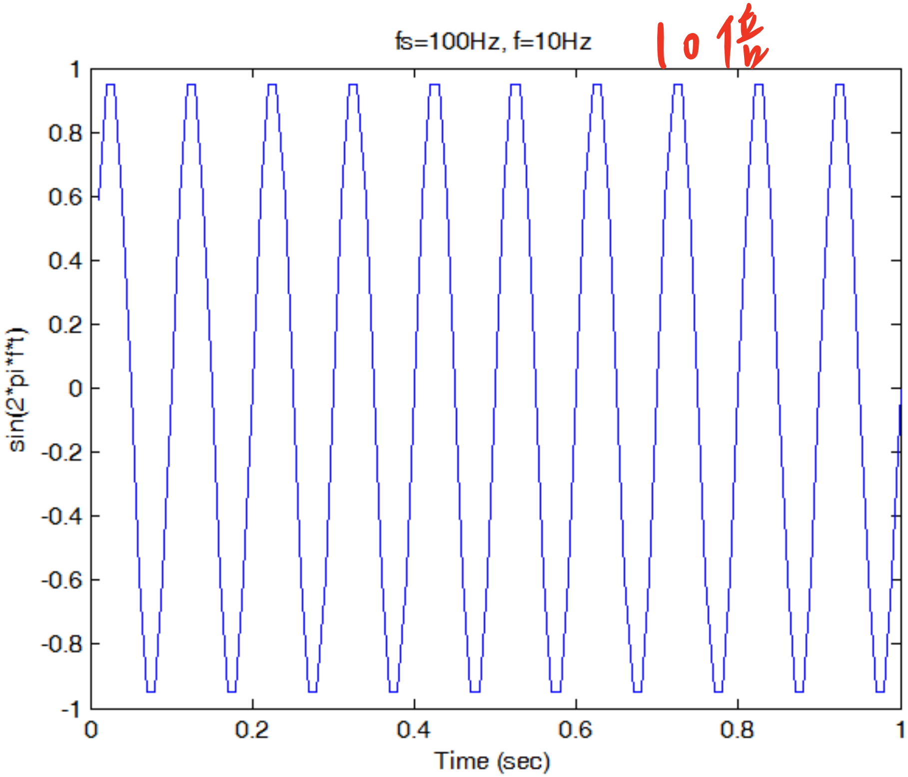
In this example, A-D conversion is 1000 Hz and the Nyquist frequency is 500 Hz. Signals of 250 and 500 Hz can be adequately portrayed in digital form. However, a frequency of 750 Hz will appear as a frequency of 250 Hz. Frequency above the Nyquist frequency are thus aliased into the recording.

Aliasing

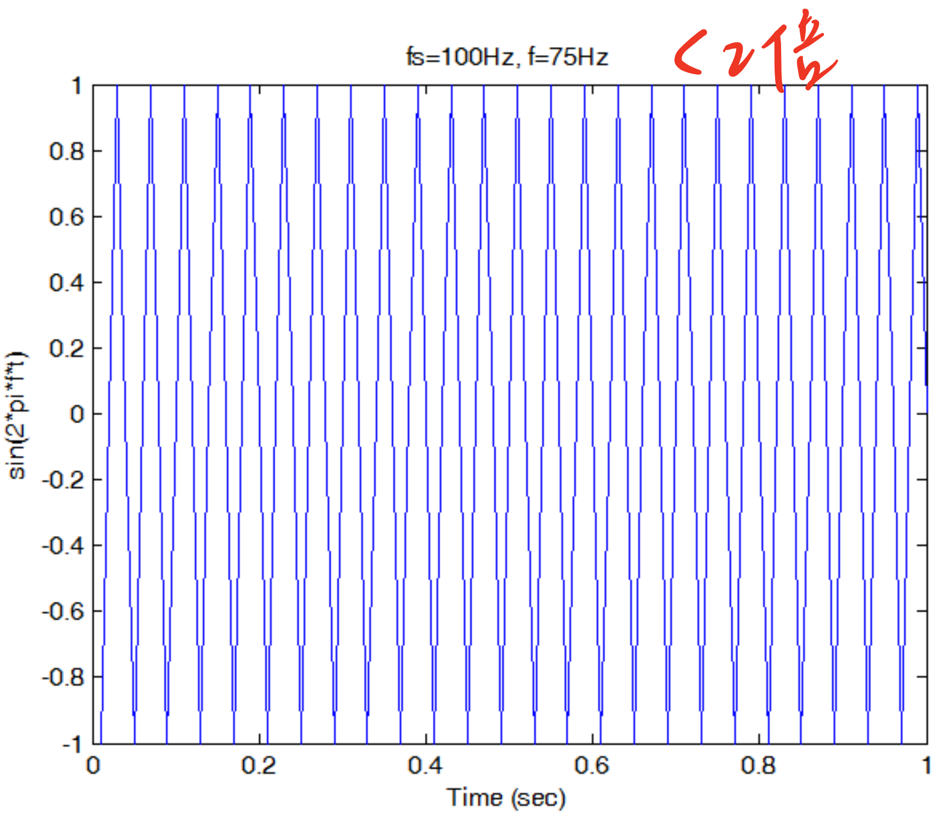
- If there is a 250 Hz component in a digitized data with 1000 Hz sampling rate, the component may come from
  - A real 250 Hz component
  - An aliasing of a 750 Hz component
- So we may need analog lowpass filter for anti-aliasing before sampling (if the 750 Hz component is not needed).

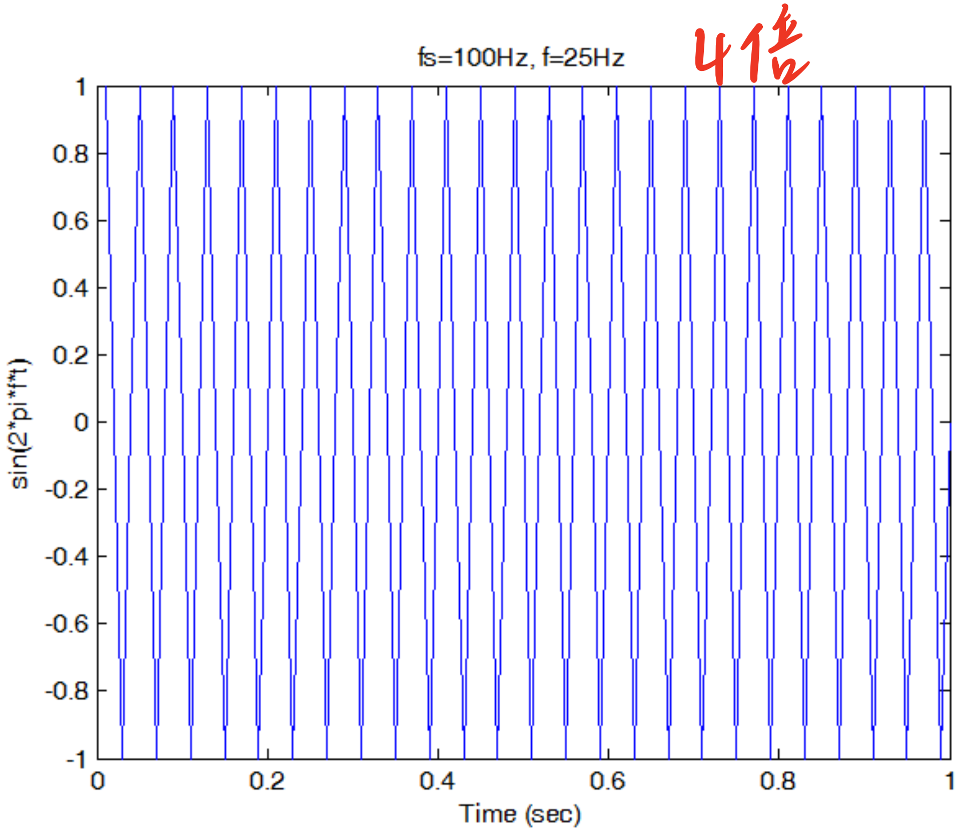
Aliasing


fs=100Hz, f=5Hz
20倍
sin(2*pi*f*t)
Time (sec)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
1

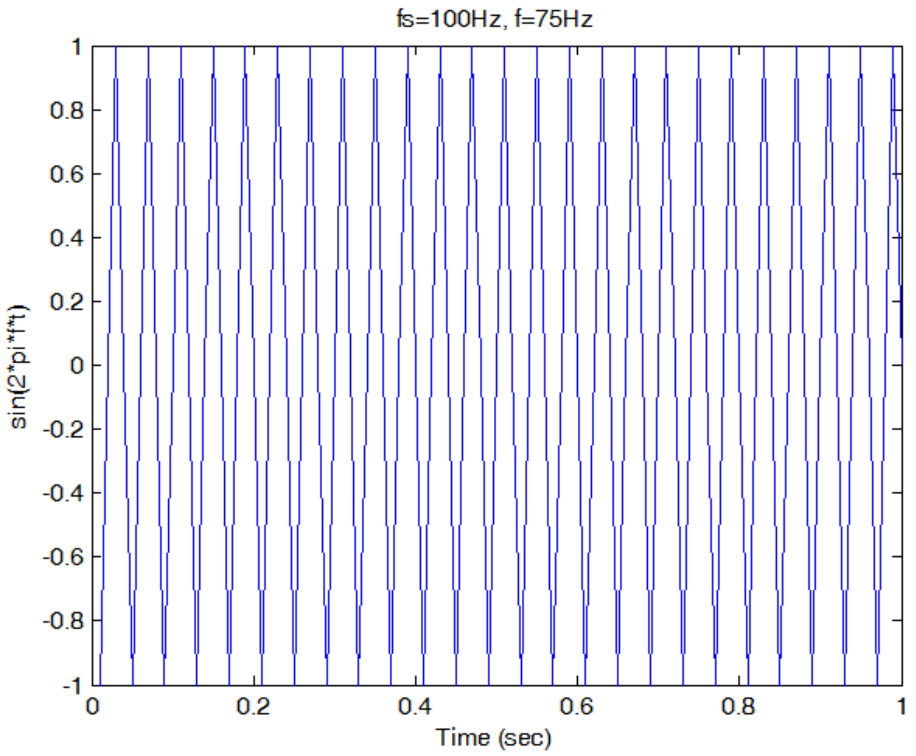

fs=100Hz, f=10Hz
10倍
sin(2*pi*f*t)
Time (sec)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
1

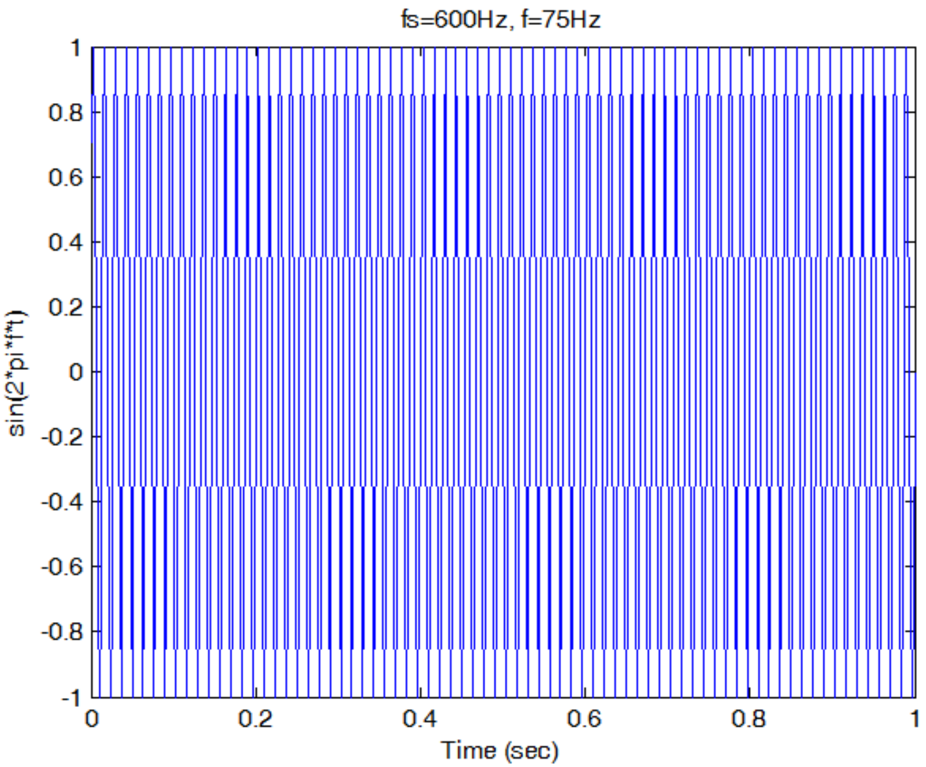
Aliasing


fs=100Hz, f=75Hz
<2倍
sin(2*pi*f*t)
Time (sec)


fs=100Hz, f=25Hz
4倍
sin(2*pi*f*t)
Time (sec)

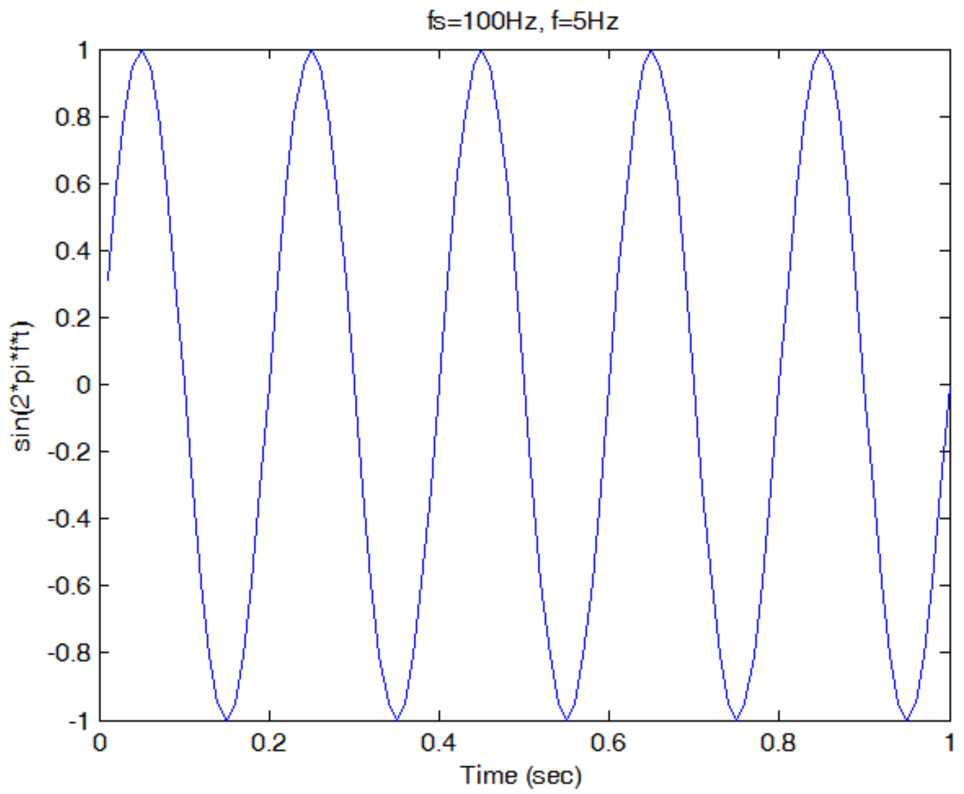
Aliasing


fs=100Hz, f=75Hz
sin(2*pi*f*t)
Time (sec)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
1


fs=600Hz, f=75Hz
sin(2*pi*f*t)
Time (sec)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
1

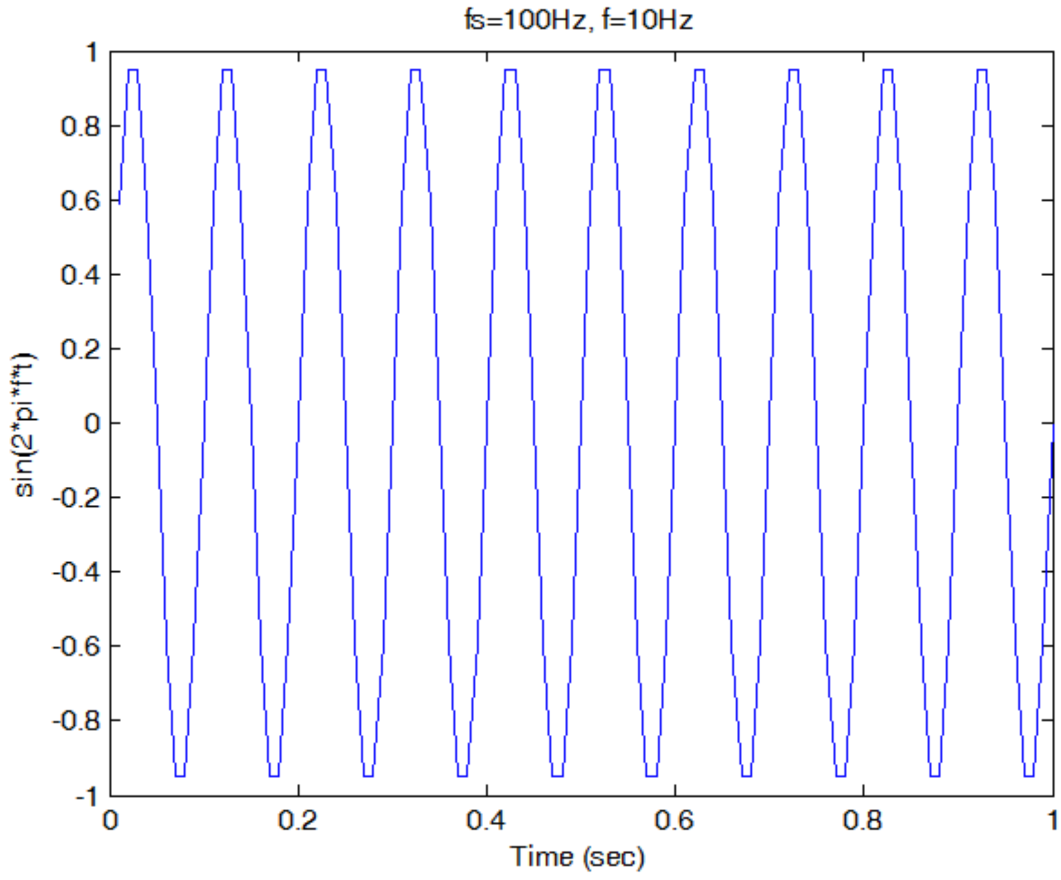
Simulation-Aliasing

```
%%fs=100 Hz
N = 100;

fs = 100;
t = (1:N)/fs;

% case1:f=5Hz
f5=5;
n5=2*pi*f5*t;
x5 = sin(n5);

figure(1)
plot(t,x5)
xlabel('Time (sec)');
ylabel('sin(2*pi*f*t)');
title('fs=100Hz, f=5Hz')
```



Simulation-Aliasing

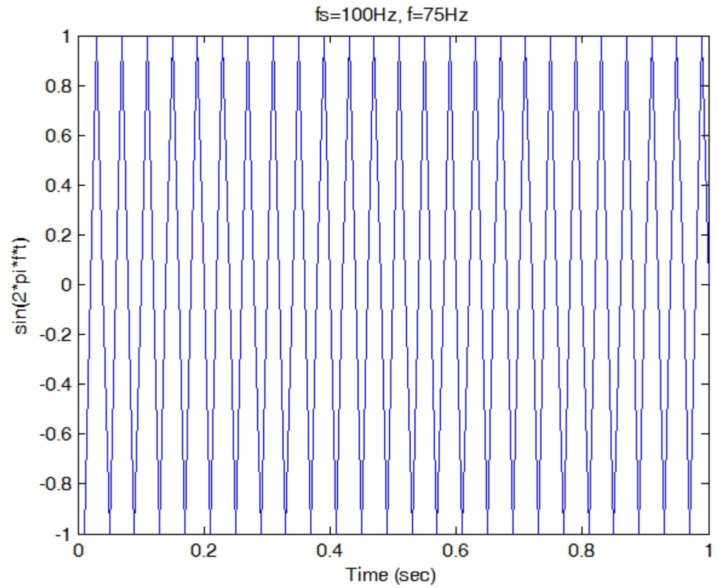
```
%case2:f=10 Hz
f10=10;
n10=2*pi*f10*t;
x10 = sin(n10);

figure(2)
plot(t,x10)
xlabel('Time (sec)');
ylabel('sin(2*pi*f*t
)');
title('fs=100Hz,
f=10Hz')
```

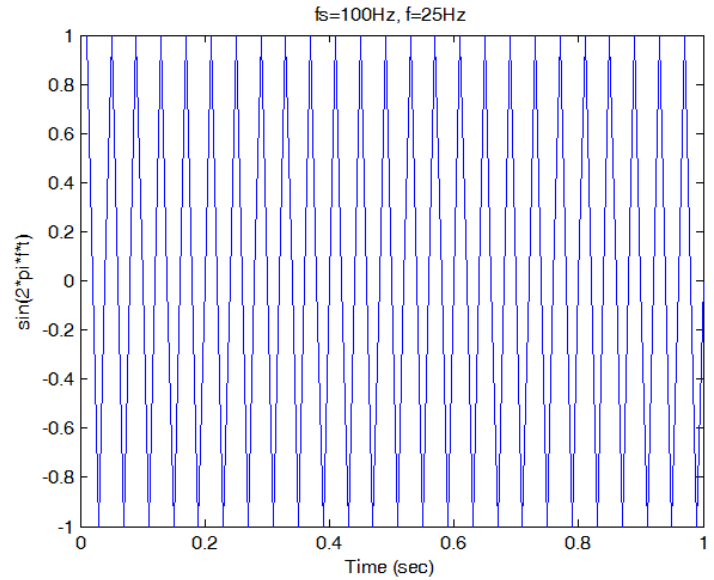


Simulation-Aliasing

```
%case3:f=75 Hz
f75=75;
n75=2*pi*f75*t;
x75 = sin(n75);

figure(3)
plot(t,x75)
xlabel('Time (sec)');
ylabel('sin(2*pi*f*t)');
title('fs=100Hz, f=75Hz')
```

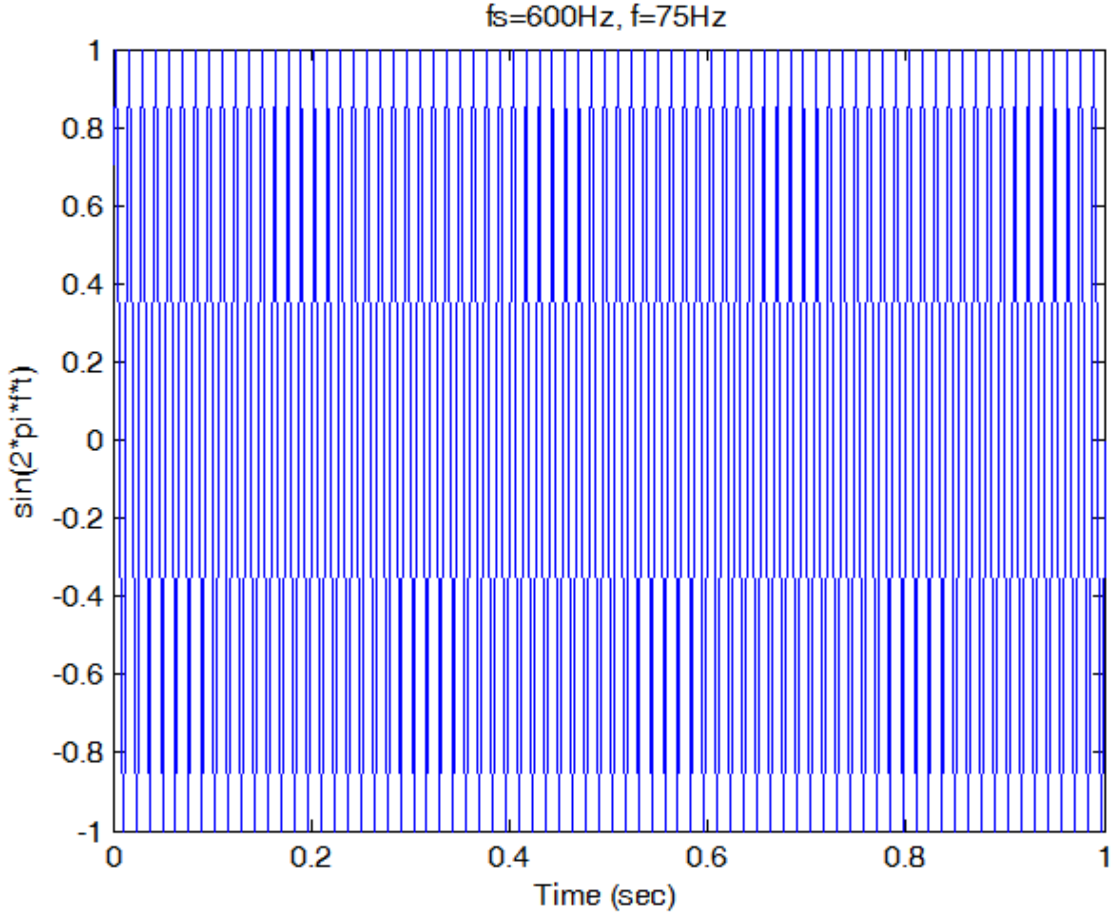
```
%case4:f=25 Hz
f25=25;
n25=2*pi*f25*t;
x25 = sin(n25);

figure(4)
plot(t,x25)
xlabel('Time (sec)');
ylabel('sin(2*pi*f*t)');
title('fs=100Hz, f=25Hz')
```





Simulation-Aliasing

```
%%fs600=600
%case5:f=75Hz,
fs=600Hz
N600 = 600;
fs600 = 600;
t600 = (1:N600)/fs600;
f75=75;
t600 = (1:N600)/fs600;
n75_600=2*pi*f75*t600;
% Generate data
x75_600 = sin(n75_600);

figure(5)
plot(t600,x75_600)
xlabel('Time (sec)');
ylabel('sin(2*pi*f*t)');
title('fs=600Hz, f=75Hz')
```



Example 5.10 ☆☆

For frequencies above 4 kHz, the spectrum of a signal is zero. To sample the signal, find the maximum time spacing between samples.

頻寬 = 4kHz

Solution

$$W = 4 \text{ kHz}$$

The Nyquist rate is

$$f_s = 2W = 8 \text{ kHz} \Rightarrow \text{最小取樣} f$$

||

The Nyquist interval is

$$T_s = \frac{1}{f_s} = \frac{1}{8 \times 10^3} = 125 \,\mu\text{s} \quad \text{最大取樣} T$$

Example 5.12

Use MATLAB to find the Fourier transform of

$$x(t) = u(t-2) - e^{-2t}u(t)$$

Solution

```
syms  x t
 x=heaviside(t-2)-exp(-2*t)*Heaviside(t) %define the signal
 X = fourier(x)
pretty(X) % simplify

 x =
 heaviside(t - 2) - heaviside(t)/exp(2*t)
  X =
 - (1/exp(2*w*i))*(- pi*dirac(-w) + i/w) - 1/(2 + w*i)

          1      /                      i \      1
  - ---------- | - pi dirac(-w) + - | - -------
    exp(2 w i) \                      w /   2 + w i
```

Refer to
Symbolic Math Toolbox

F
From this, we obtain

$$X(\omega) = e^{-j2\omega}\left[-\pi\delta(-\omega) + \frac{j}{\omega}\right] - \frac{1}{2+j\omega}$$

Example 5.14

Consider a system with transfer function

$$H(s) = \frac{100s^2}{s^4 + 25s^3 + 50s^2 + 400s + 6000}, \qquad s = j\omega$$

Use MATLAB to obtain the plot of $H(\omega)$.

Solution

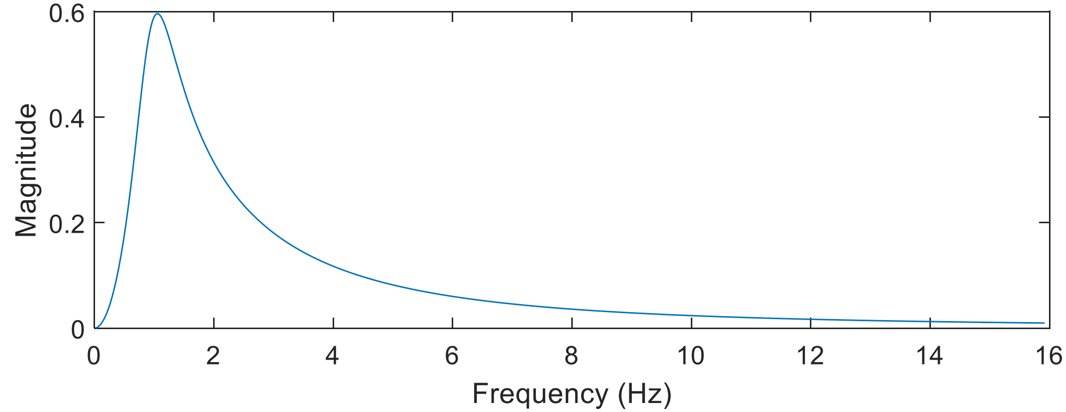
```
freqs Laplace-transform (s-domain) frequency response.
   H = freqs(B,A,W) returns the complex frequency response vector H
   of the filter B/A:
                        nb-1          nb-2
            B(s)   b(1)s      +  b(2)s      + ... +  b(nb)
     H(s) = ---- = ------------------------------------
                        na-1          na-2
            A(s)   a(1)s      +  a(2)s      + ... +  a(na)

   given the numerator and denominator coefficients in vectors B and A.
   The frequency response is evaluated at the points specified in
   vector W (in rad/s).  The magnitude and phase can be graphed by
   calling freqs(B,A,W) with no output arguments.
```

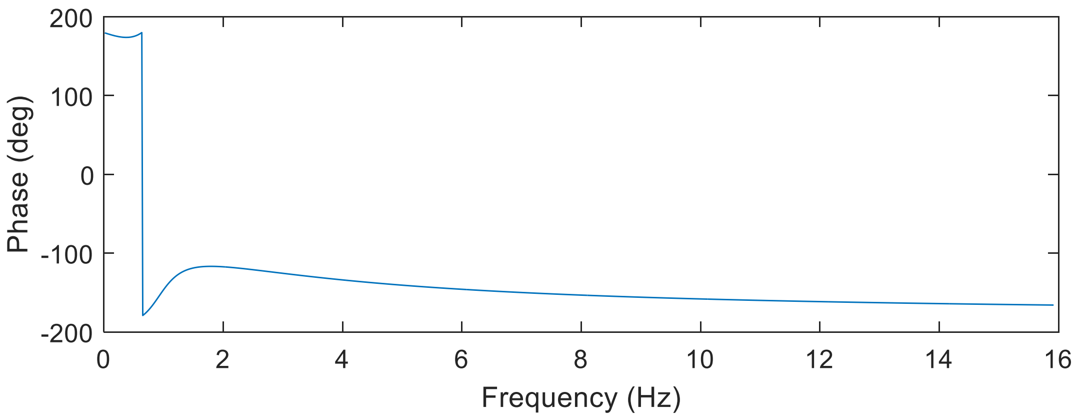
$$w = 2\pi f$$

$$f = w/2\pi$$

$$H(s)=\frac{100s^2}{s^4+25s^3+50s^2+400s+6000}, \qquad s=j\omega$$

```
num = [100 0 0];
den = [ 1  25  50  400  6000];
w = 0.1:0.1:100
H = freqs(num, den,w);
mag =abs(H);
phase = angle(H)*180/pi   % converts phase to degrees
subplot(2,1,1)
 plot(w/(2*pi), mag)
  xlabel('Frequency (Hz)')
  ylabel('Magnitude')
subplot(2,1,2)
 plot(w/(2*pi), phase)
  xlabel('Frequency (Hz)')
ylabel('Phase (deg)')
```





Appendix

Frequency Convolution

Frequency convolution $x_1(t)x_2(t) \xrightarrow{FT} \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$

$$F[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$x(t) = F^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

相乘的傅立葉

$$F[x_1(t) \,.\, x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) \,.\, x_2(t)]\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \underbrace{\left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(p) e^{jpt}\, dp \right]}_{\text{inverse } X_1(w)} x_2(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(p) \left[\int_{-\infty}^{\infty} x_2(t)\ e^{-j\omega t}\, e^{jpt}\, dt \right] dp$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(p) \left[\int_{-\infty}^{\infty} x_2(t)\ e^{-j(\omega - p)t}\, dt \right] dp$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(p)\, X_2(\omega - p)\ dp$$

$\frac{1}{2\pi} \times$ (兩個傅立葉後的 conv)

https://www.tutorialspoint.com/frequency-convolution-theorem

FOURIER SERIES

- Fourier series, a premier tool for analyzing periodic signals.
- Fourier analysis leads to the frequency spectrum of a continuous-time signal.
- The frequency spectrum displays the various sinusoidal components that make up the signal.
- The Fourier series can be represented in three ways, the sine–cosine, amplitude–phase, and complex exponential.

A periodic signal is one that repeats itself every T s.

$$x(t) = x(t + nT)$$

$$= a_0 + a_1 \cos \omega_0 t + b_1 \sin \omega_0 t + a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t$$

$$+ a_3 \cos 3\omega_0 t + b_3 \sin 3\omega_0 t + \cdots$$

$$= \underbrace{\frac{a_0}{\quad}}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}}$$

$$\omega_0 = \frac{2\pi}{T}$$

Dirichlet Conditions

- A periodic function x(t) can be expanded as a Fourier series only if

1. $x(t)$ should be integrable over any period; that is,

$$\int_{t_0}^{t_o+T} |x(t)|\,dt \underline{< \infty}$$

一週期內積分要<∞

2. $x(t)$ has only a finite number of maxima and minima over any period
3. $x(t)$ has only a finite number of discontinuities over any period

Fourier Analysis

- Fourier Analysis is the process of determining the Fourier coefficients a_0, a_n, and b_n.

$$x(t) = a_0 + a_1\cos\omega_0 t + b_1\sin\omega_0 t + a_2\cos 2\omega_0 t + b_2\sin 2\omega_0 t$$
$$+ a_3\cos 3\omega_0 t + b_3\sin 3\omega_0 t + \cdots \quad = \underset{\text{dc}}{a_0} + \underbrace{\sum_{n=1}^{\infty}(a_n\cos n\omega_0 t + b_n\sin n\omega_0 t)}_{\text{ac}}$$

$$\boxed{a_0 = \frac{1}{T}\int_0^T x(t)dt} \qquad \boxed{a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\, dt} \qquad \boxed{b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t\, dt}$$

- The sine and cosine functions are orthogonal over a period T leads to the following trigonometric integrals:

$$\int_0^T \sin n\omega_0 t\, dt = 0 = \int_0^T \cos n\omega_0 t\, dt \qquad \int_0^T \sin n\omega_0 t\cos n\omega_0 t\, dt = 0$$

$$\int_0^T \sin n\omega_0 t\sin m\omega_0 t\, dt = 0 = \int_0^T \cos n\omega_0 t\cos m\omega_0 t\, dt, \quad m \neq n \qquad \int_0^T \sin^2 n\omega_0 t\, dt = \frac{T}{2} = \int_0^T \cos^2 n\omega_0 t\, dt$$

Complex Exponential Form

- The exponential Fourier series of a periodic signal $x(t)$ is a representation that is the sum of the complex exponentials at positive and negative harmonic frequencies.

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \qquad c_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt$$

$$\cos n\omega_0 t = \frac{1}{2}\left[e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right] \qquad \sin n\omega_0 t = \frac{1}{2j}\left[e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right] = -\frac{j}{2}\left[e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right]$$

$$x(t) = a_0 + \frac{1}{2}\sum_{n=1}^{\infty} a_n\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right) - jb_n\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

$$= a_0 + \frac{1}{2}\sum_{n=1}^{\infty}\left[(a_n - jb_n)e^{jn\omega_0 t} + (a_n + jb_n)e^{-jn\omega_0 t}\right]$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \quad = c_0 + \sum_{n=1}^{\infty}\left[c_n e^{jn\omega_0 t} + c_{-n}e^{-jn\omega_0 t}\right] \quad = \sum_{n=-\infty}^{\infty} |c_n| e^{j(n\omega_0 t + \theta_n)}$$

$$c_0 = a_0, \qquad c_n = \frac{(a_n - jb_n)}{2} \qquad c_{-n} = c_n^* = \frac{(a_n + jb_n)}{2}$$

Complex Exponential Form

To prove

$$\sum_{n=-\infty}^{\infty} \delta(t-nT_0) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

Proof

$$c_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt \quad = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\left[\sum_{m=-\infty}^{\infty}\delta(t-mT_0)\right]e^{-jn\omega_0 t}dt. = \frac{1}{T_0}.$$

$\Longrightarrow$

$$\sum_{n=-\infty}^{\infty} \delta(t-nT_0) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$