Appendix: maths revision

Many people find maths challenging and are very happy to drop it after school. This appendix contains simple descriptions of the main mathematical concepts that will help you to understand MRI. These are:

- vectors;
- sine and cosine waves;
- exponentials;
- complex numbers;
- simple Fourier analysis.

A.1 Vectors

A vector quantity is one that has both magnitude and direction. For example, velocity describes the rate of movement in a particular direction (in comparison, speed is a scalar quantity which just measures the rate of travel). Vectors are commonly depicted using arrows with the length denoting the magnitude. They can be added together by putting the arrows end-to-end and then joining the start and end points, creating the resultant vector. Alternatively a vector may be divided into components along the x, y and z axes in any frame of reference (Figure A.1).

In equations vectors are either shown with little arrows over the top (like this $\vec{\mathrm{M}}$) or in bold typeface (**M**). In this book we have used the bold notation, but only in equations where the vector directions are important. Components of vectors can be denoted by their magnitude and unit vectors **i**, **j** and **k** along the three directions. Vectors can be multiplied together using either the dot product (which has a scalar result) or the cross product (which has a vector result).

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta = A_xB_x + A_yB_y + A_zB_z.$$

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} = \left(A_yB_z - A_zB_y\right)\mathbf{i} + \left(A_zB_x - A_xB_z\right)\mathbf{j} + \left(A_xB_y - A_yB_x\right)\mathbf{k} = AB\sin\theta\mathbf{n}$$

where θ is the angle between the two vectors, and **n** is a unit vector perpendicular to both A and B.

A.2 Sine and Cosine Waves

The simplest waves are pure sine or cosine waves, which have the same shape but are shifted with respect to each other (Figure A.2a). They have three basic properties: the amplitude is the peak height (how large it is), the frequency, measured in hertz (Hz), describes how rapidly in time the magnitude of the wave is changing, and the phase describes where we are within the cyclic variation.

A unit vector rotating around a circle produces sine and cosine components along the x and y axes (Figure A.2b), and this gives us an easy way of thinking of phase, which is just the angle between the unit vector and the

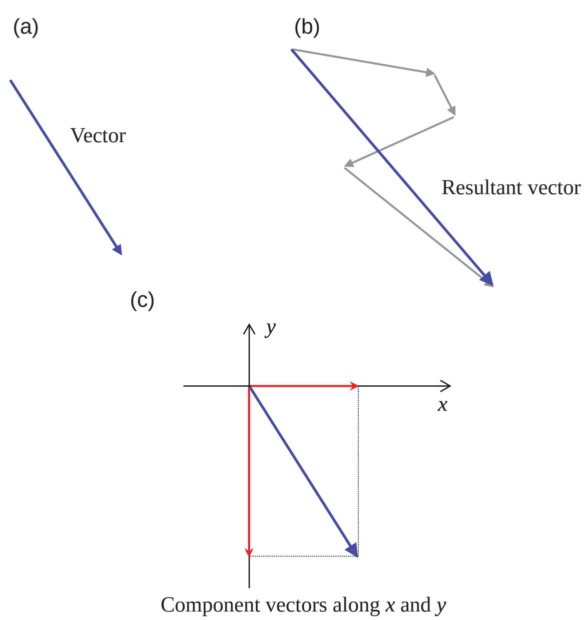

Figure A.1 (**a**) A vector is shown as an arrow. (**b**) Several vectors can be joined together end-to-end to create the resultant vector. (**c**) A vector can be broken down into components along principal axes.

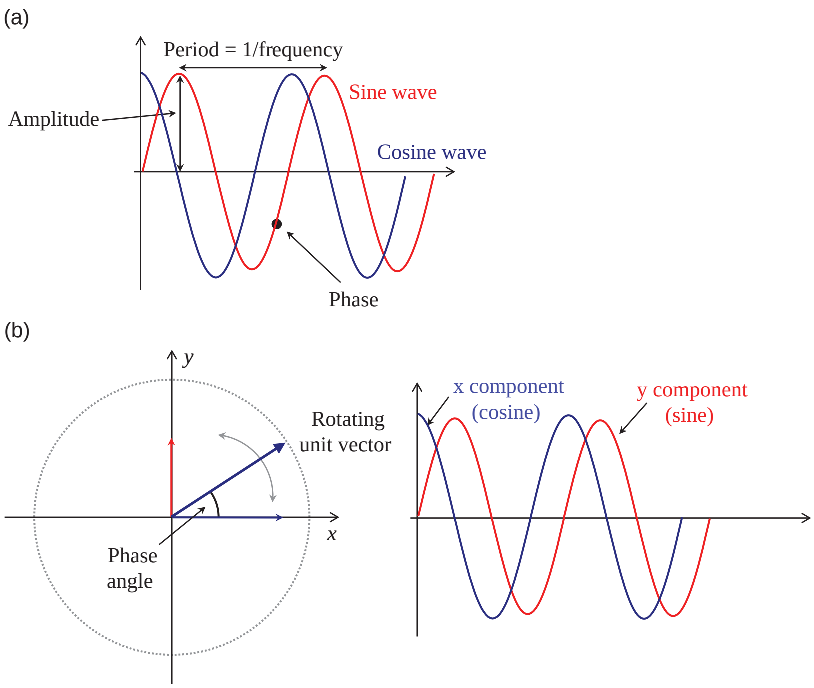

Figure A.2 (**a**) A wave has amplitude, frequency and phase. A cosine wave differs from a sine wave by a phase angle of 90°. (**b**) A rotating unit vector creates two waves along the *x* and *y* axes.

reference axis. Phase angles can vary from 0° to 360°, and angles larger than 360° just overlay themselves.

Angles around the circle can also be measured in radians, with the full 360° being equal to 2π rads. π (pronounced 'pie') is the ratio of the circumference of a circle to its diameter, and is approximately equal to 3.14. There are some important angles to know: $\pi/2$ is 90°, π is 180°, and in general $2n\pi$ is in line with 0°.

A.3 Exponentials

Exponentials are rather difficult to explain but you really only need to know some important properties of these useful numbers. They are based on the number e which is approximately equal to 2.718. In equations exponentials may be denoted either as e^x or $\exp(x)$ – we have used the latter in this book – and x is known as the exponent. Exponentials are the inverse of natural logarithms (denoted ln), and in particular if $y = \exp(x)$ then $\ln(y) = x$.

Let's start with some general results for exponentials:

$$\exp(-x) = \frac{1}{\exp(x)}$$
$$\exp(x)\exp(y) = \exp(x + y)$$
$$\exp(0) = 1$$
$$\exp(\infty) = \infty$$
$$\exp(-\infty) = 0$$

In MRI we are particularly interested in exponential decays, $\exp(-x)$, which reduce down to zero (see Figure A.3). This not only describes radioactive decay and the free induction decay in MRI, it also shows how the temperature of a cup of coffee drops – fast at first but getting slower and slower until it is stone cold. In contrast, exponential growth, when the exponent is positive, just keeps getting bigger and bigger – like the world's population!

A.4 Complex Numbers

Complex numbers have a real part and an imaginary part. The imaginary part is multiplied by i, the square root of −1, so a complex number can be written as

$$A = \mathrm{R} + i\mathrm{I}$$

The complex conjugate of A is defined by

$$A^* = \mathrm{R} - i\mathrm{I}$$

so when A is multiplied by its complex conjugate the result is a pure real number

$$A \cdot A^* = \mathrm{R}^2 - \mathrm{I}^2$$

You may see j used instead of i, particularly in engineering textbooks. Of course in real life it is impossible to find the square root of a negative number, so the imaginary parts don't really exist. However, complex numbers are useful for describing the rotational movement in MRI. We can define any point on the circle by a complex number

$$A = x + iy = \cos\theta + i\sin\theta$$

We can use a complex exponent to define the rotational operator $\exp(i\theta)$, so that

$$\exp(i\theta) = \cos\theta + i\sin\theta$$

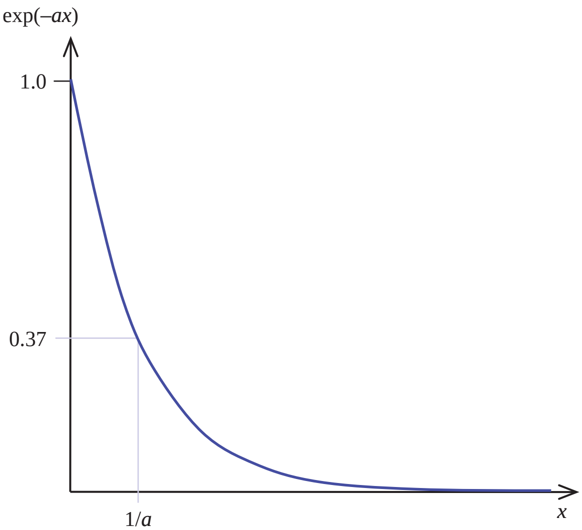

Figure A.3 Exponential decay describes spin–spin relaxation in MRI as well as radioactive decay.

and if we use $\theta = \omega t/2\pi$, we can denote a rotating vector as $\exp(i\omega t/2\pi)$, which allows us to combine any number of rotational movements simply by adding their exponents.

A.5 Simple Fourier Analysis

Fourier's theorem states that any complex waveform can be created from a sum of sine and cosine waves with appropriate frequencies and amplitudes. We write this as

$$s(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=-\infty}^{\infty} a_n \exp(inx)$$

where a and b are the amplitudes and $i = \sqrt{-1}$. For MRI the most important feature of Fourier analysis is the Fourier transform, which is defined by the equations

$$S(k) = \int_{-\infty}^{\infty} s(x)\exp(-i2\pi kx)\,\mathrm{d}x$$

$$s(x) = \int_{-\infty}^{\infty} S(k)\exp(i2\pi kx)\,\mathrm{d}k$$

Although these integrals look nasty, you don't have to work them out, you just need to recognize them. $S(k)$ and $s(x)$ are functions of k and x respectively, called a Fourier transform pair, and k and x have an inverse relationship. Some important Fourier transform pairs are shown in Figure A.4. In MRI, x is real space and k

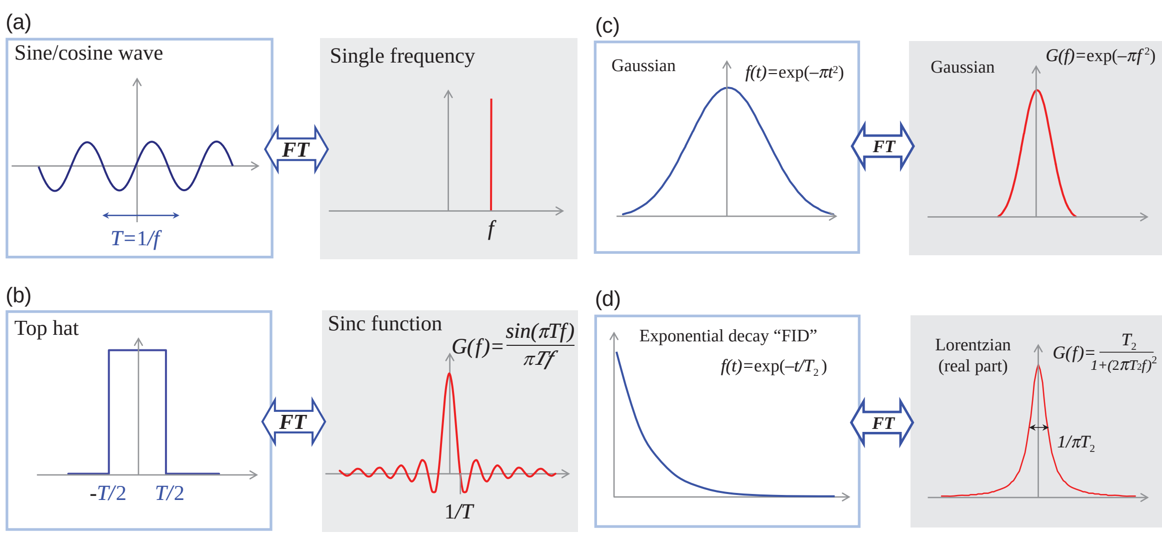

Figure A.4 Some important Fourier transform pairs. (**a**) A simple sine wave with frequency f has a period T, and its Fourier transform is a spike function at frequency f. (**b**) The FT of a top-hat function with width $2T$ is a sinc function, with the first zero crossing points at $1/T$. (**c**) A Gaussian function transforms to another Gaussian, and finally (**d**) an exponential decay transforms to a Lorentzian function.

is spatial frequency. Another useful Fourier transform pair is time t and frequency f. A two-dimensional Fourier transformation involves integration over two directions and 3D FT is over all three:

$$S(k_x, k_y) = \iint_{x,y} s(x,y)\exp(-i2\pi k_x x)\exp(-i2\pi k_y y)\,\mathrm{d}x\,\mathrm{d}y$$

$$S(k_x, k_y, k_z) = \iiint_{x,y,z} s(x,y,z)\exp(-i2\pi k_x x)\exp(-i2\pi k_y y)\exp(-i2\pi k_z z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

A.6 Some Useful Constants

The Boltzmann constant	k_B	1.38×10^{-23} J K^{-1}
Planck's constant	h	6.63×10^{-34} J s
	$\hbar = \frac{h}{2\pi}$	1.05×10^{-34} J s
Proton gyromagnetic ratio	γ	2.68×10^{8} rad s^{-1} T^{-1}
	$ƛ = \frac{\gamma}{2\pi}$	42.57 MHz T^{-1}