# Synchronization and modularity in complex networks

A. Arenas[1] and A. Díaz-Guilera[2]

[1] Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili,
   43007 Tarragona, Spain
[2] Departament de Física Fonamental, Universitat de Barcelona, Martí i Franquès 1,
   08028 Barcelona, Spain

**Abstract.** We investigate the connection between the dynamics of synchronization and the modularity on complex networks. Simulating the Kuramoto's model in complex networks we determine patterns of meta-stability and calculate the modularity of the partition these patterns provide. The results indicate that the more stable the patterns are, the larger tends to be the modularity of the partition defined by them. This correlation works pretty well in homogeneous networks (all nodes have similar connectivity) but fails when networks contain hubs, mainly because the modularity is never improved where isolated nodes appear, whereas in the synchronization process the characteristic of hubs is to have a large stability when forming its own community.

## 1 Introduction

The theory of complex networks has reported major advances in the understanding of the networked substrate in which many natural, social and technological processes take place. Complex networks are representative of the intricate connections between elements in systems as diverse as the Internet and the WWW, metabolic networks, neural networks, food webs, communication networks, transport networks, and social networks [1,2]. The availability of wide databases of entities (nodes) and relations (links) as well as the advances in computation have provided scientists with the necessary tools to unravel the statistical properties of complex networks [3–5].

One of the subjects that has received more attention, in the recent years, is the detection and characterization of intermediate topological scales in their structure. In particular, the problem of detection of *community structure*, meaning the appearance of densely connected groups of vertices, with only sparser connections between groups, has been intensely attacked from the scientific community [6,7]. The most successful solutions, in terms of accuracy and computational cost required, are those based on the optimization of a magnitude called *modularity* proposed by Newman [8] that allows the comparison of different partitioning of the network. The modularity of a given partition is, up to a multiplicative constant, the number of edges falling within groups minus the expected number in an equivalent network with edges placed at random. Given a network partitioned into communities, being $c_i$ the community to which node $i$ is assigned, the mathematical definition of modularity is expressed in terms of the adjacency matrix $A_{ij}$ and the total number of links $m = \frac{1}{2}\sum_i k_i$ where $k_i$ is the degree of node $i$ as

$$Q = \frac{1}{2m} \sum_{ij} \left( A_{ij} - \frac{k_i k_j}{2m} \right) \delta(c_i, c_j). \qquad (1.1)$$

The search for the optimal (largest) modularity value is an NP-hard problem [9] that means that the space of possible partitions grows faster than any power of the system size. For this reason, a heuristic search strategy is mandatory to restrict the search space while preserving the optimization goal [10–15]. Indeed, it is possible to relate the current optimization problem for $Q$ with classical problems in statistical physics, e.g. the spin glass problem of finding the ground state energy [16], where algorithms inspired in natural optimization processes as simulated annealing and genetic algorithms have been successfully used.

In a different scenario, physicists have largely studied the dynamics of complex biological systems, and in particular the paradigmatic analysis of large populations of coupled oscillators [17–19]. The connection between the study of synchronization processes and complex networks is interesting by itself. Indeed, the original inspiration of Watts and Strogatz in the development of the Small-World network structure [1] was to understand the synchronization of cricket chirps. These synchronization phenomena as many others e.g. asian fireflies flashing at unison, pacemaker cells in the heart oscillating in harmony, etc. have been mainly described under the mean field hypothesis that assumes that all oscillators behave identically and interact with the rest of the population. Recently, the emergence of synchronization phenomena in these systems has been shown to be closely related to the underlying topology of interactions at mesoscopic scales [20].

Here we analyze the effect of the community structure in the path towards synchronization. We study the dynamics towards synchronization in several types of structured complex networks and find an evolving community structure based on the recruitment of groups of nodes towards complete synchronization. We will also provide a connection between the emergence of synchronized groups and the way nodes are grouped in some of the agglomerative methods of community detection based on the maximization of the modularity, as defined in (1.1) [7,10].

The paper is structured as follows: in section II we present the synchronization model studied. In section III we describe a method to construct synthetic networks with a well prescribed hierarchical community structure. In section IV, we expose the analysis of the route towards synchronization and their relationship with the topological structure. Finally, we conclude with a discussion about the communities revealed by synchronization processes in complex networks.

## 2 The dynamical model

The first successful attempt to understand synchronization phenomena, from a physicist's perspective, was due to Kuramoto [19], who analyzed a model of phase oscillators coupled through the sine of their phase differences. The model is rich enough to display a large variety of synchronization patterns and sufficiently flexible to be adapted to many different contexts [21]. The Kuramoto model consists of a population of $N$ coupled phase oscillators where the phase of the $i$-th unit, denoted by $\theta_i(t)$, evolves in time according to the following dynamics

$$\frac{d\theta_i}{dt} = \omega_i + \sum_j K_{ij} \sin(\theta_j - \theta_i) \quad i = 1, \ldots, N \tag{2.1}$$

where $\omega_i$ stands for its natural frequency and $K_{ij}$ describes the coupling between units. The original model studied by Kuramoto assumed mean-field interactions $K_{ij} = K, \forall i, j$. In absence of noise the long time properties of the population are determined by analyzing the only two factors which play a role in the dynamics: the strength of the coupling $K$ whose effect tends to synchronize the oscillators (same phase) versus the width of the distribution of natural frequencies, the source of disorder which drives them to stay away each other by running at different velocities. For unimodal distributions, there is a critical coupling $K_c$ above which synchronization emerges spontaneously.

### 2.1 Synchronization in complex networks

Recently, due to the realization that many networks in nature have complex topologies, synchronization studies have been extended to systems with heterogeneous connectivity patterns

[22–29]. Usually, due to the complexity of the analysis in these cases some further assumptions have been introduced. For instance, it has been a normal practice to assume that the oscillators are identical. In absence of disorder, i.e. if $(\omega_i = \omega \; \forall i)$ there is only one attractor of the dynamics: the fully synchronized regime where $\theta_i = \theta, \; \forall i$. In this context the interest concerns not the final locked state in itself but the route to the attractor. In particular, it has been shown [30,31] that high densely interconnected sets of oscillators (motifs) synchronize more easily that those with sparse connections. This scenario suggests that for a complex network with a non-trivial connectivity pattern, starting from random initial conditions, those highly interconnected units forming local clusters will synchronize first and then, in a sequential process, larger and larger spatial structures also will do it up to the final state where the whole population should have the same phase [32]. We have shown [33,34] this process to occur at different time scales if a clear community structure exists. Thus, the dynamical route towards the global attractor reveals different topological structures, indeed some of them very similar to those which represent communities in partitions with high modularity.

## 2.2 Order parameter

It is a normal practice to define, for the Kuramoto model, a global "order parameter" to characterize the level of entrainment between oscillators. The normal choice is to use the following complex-valued order-parameter

$$re^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}.$$  (2.2)

where $r(t)$, with $0 \leq r(t) \leq 1$, measures the coherence of the oscillator population, and $\psi(t)$ is the average phase. However, this definition, although suitable for mean-field models is not efficient to identify local dynamic effects. In particular it does not give information about the route to the attractor (fully synchronization) in terms of local clusters which is so important to identify functional groups or communities. For this reason, instead of considering a global observable, we define a local order parameter measuring the average of the correlation between pairs of oscillators

$$\rho_{ij}(t) = \langle cos(\theta_i(t) - \theta_j(t)) \rangle$$  (2.3)

where the brackets stand for the average over initial random phases. The main advantage of this approach is that it allows to trace the time evolution of pairs of oscillators and therefore to identify compact clusters of synchronized oscillators reminiscent of the existence of communities.

In previous works [33,34] we have analyzed the dynamics towards synchronization in different networks with community structure. From the average correlations between pairs of oscillators $(\rho_{ij})$ we define a dynamical connectivity matrix. We consider that two nodes are linked if their correlation is above some fixed threshold. In this way we start with a system of disconnected nodes. As time goes on, nodes merge into groups until they form a single synchronized component, for a time long enough.

For networks with a clear community structure, we have been able to identify the jumps of the number of connected components in time with the complete eigenvalue spectrum of the Laplacian matrix, showing a striking similarity. Nevertheless, here we will focus in the relation between a magnitude that describes the quality of the community partitioning, the modularity (1.1), and the relative stability of the dynamical structures that are formed in the merging process described above.

## 3 Networks

In the present work we analyze the same type of networks than in [33,34]. Those are structured networks with a clear community structure. Some of them are homogeneous in degree and are generalizations of the model networks proposed in [35] as a benchmark for community detection

**Table 1.** Maximum modularity and number of communities in the corresponding partition obtained with our dynamical method and with the Extremal Optimization method [14]. Labels for the networks correspond to those in Figs. 1–6, in this order.

| | Extremal | | Synchronization | |
|---|---|---|---|---|
| | $Q_{max}$ | comms | $Q_{max}$ | comms |
| 13_4 | 0.696 | 4 | 0.696 | 4 |
| 15_2 | 0.772 | 16 | 0.772 | 16 |
| 3n64 | 0.714 | 6 | 0.714 | 6 |
| inhomo | 0.609 | 6 | 0.609 | 5 |
| RB 25 | 0.551 | 5 | 0.551 | 5 |
| RB 125 | 0.642 | 11 | 0.626 | 9 |

algorithms. Other networks have special nodes that act as hubs. For a detailed description and visualization of the networks the reader is pointed to [34].

The networks we analyze are:

- *Networks with 1 level of community with in-homogeneous distribution of community sizes*: it is a kind of network that has been proposed as a better benchmark for community detection algorithms, since in real networks the community sizes are not homogeneously distributed [36].
- *Networks with two and three hierarchical levels of homogeneous communities*: This generalization was proposed [33,34] to show that the synchronization dynamics is able to find communities at different levels.
  In general, to construct such a network, one takes a set of $N$ nodes and divide it into $n_1$ groups of equal size; each of these groups is then divided into $n_2$ groups and so on up to a number of steps $k$ which defines the number of hierarchical levels of the network. Then we add links to the networks in such a way that at each node we assign at random a number of $z_1$ neighbours within its group at the first level, $z_2$ neighbours within the group at the second level and so on. There is a remaining number of links that each node has to the rest of the network, that we will call $z_{out}$. In this case it is easy to compute the modularity of the partition [35] at any level $l \leq k$

$$Q_{n_1 \cdot n_2 \cdot \ldots \cdot n_l} = \frac{z_l + \cdots + z_k}{z_{out} + z_1 + \cdots + z_k} - \frac{1}{n_1 \cdot n_2 \cdot \ldots \cdot n_l} \qquad (3.1)$$

  and its numerical value tells us how good as partition into a given community structure is. Here we will consider networks with two hierarchical levels, 256 nodes, and $n_1 = n_2 = 4$; this gives two possible partitions: one with 4 communities and the other one with 16 communities. In the case of three levels we take 64 nodes and $n_1 = n_2 = n_3 = 2$, and hence there are three possible partitions, 2, 4, and 8 equal size communities.
- *Hierarchical networks with hubs*: There is a set of self-similar deterministic networks that has been used as an example of hierarchical scale-free networks, proposed by Ravasz and Barabasi [37]. This type of networks, apart from its hierarchical structure has some nodes with a special role in terms of number of connections (hubs) in contrast to the networks discussed previously that are essentially homogeneous in degree.

## 4 Synchronization and community structure

We have simulated the Kuramoto's model (2.1) with a constant natural frequency for the whole population on the above mentioned structured complex networks. According to the dynamics described in Sect. 2 we analyze the evolution of the system in time by averaging over random initial phases distributed homogeneously in the range $[0, 2\pi]$. Two oscillators are synchronized when its correlation, given by eq. (2.3), is above some fixed threshold.
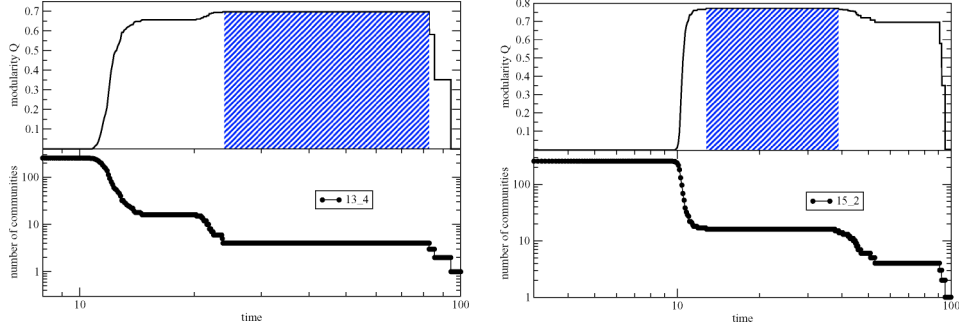
**Fig. 1.** Merging of groups of oscillators in time for a network of 256 nodes grouped in 2 hierarchical levels of communities, left with $z_1 = 13$, $z_2 = 4$, $z_{out} = 1$, and right with $z_1 = 15$, $z_2 = 2$, $z_{out} = 1$. Bottom: Evolution of the number of communities, identified as groups of synchronized oscillators. Top: modularity computed according to the partition given by the synchronized groups. The shadow area corresponds to the largest modularity. Time is in a logarithmic scale and units are arbitrary.
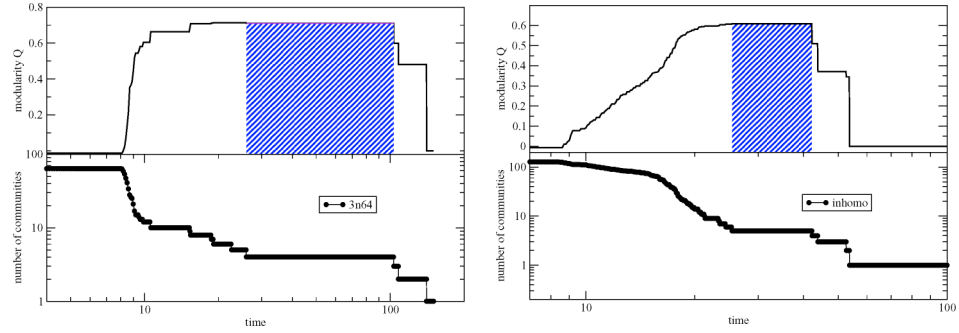


**Fig. 2.** Same as Fig. 1 for: left a network of 3 hierarchical levels of two branches each, with 64 nodes; right a network with 4 communities of 16 nodes each plus an additional one of 64 nodes.
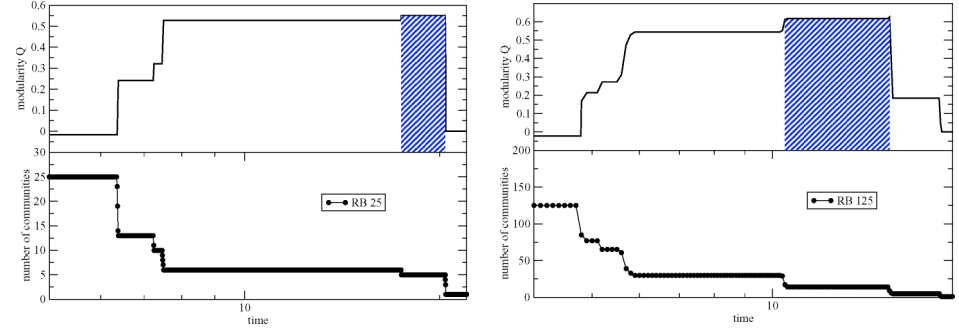


**Fig. 3.** Same as Fig. 1 for: left a hierarchical Ravasz-Barabasi network of 25 nodes; for a hierarchical Ravasz-Barabasi network of 125 nodes [37].

In Figs. 1–3 we plot the time evolution for a set of selected networks. In the bottom panels we represent the number of communities as a function of time (in a logarithmic scale). A community here is identified with a group of synchronized units. We use then this partition of the network into dynamical communities to compute the modularity in the top panel of the figures.

We observe that the partition corresponding to the most stable synchronization groups in time also have large values of modularity. This interesting effect is showing that the meta-stability of the synchronized groups is related to the specific topological structure where

the dynamics takes place. Keeping in mind this idea, it is natural to propose a method for community detection based on the dynamics towards synchronization, however this should be carefully considered. The first problem we face is that the optimal partition into communities given by maximizing the modularity $Q$ does not corresponds exactly to the most stable conformation of groups of synchronization. We have used and heuristic algorithm to optimize $Q$ based on extremal optimization [14] obtaining larger values for the modularity than those presented by the synchronization communities for one of the networks, see Table 1. Still more striking that this difference is the observation of the communities that present larger stability in the Ravasz-Barabasi type networks. These hierarchical networks are characterized by the presence of hubs, the role of hubs in the synchronization process is very different that the role played by the rest of nodes. As shown in [33] the equations involving hubs in the synchronization process are topological averages of the phases of the nodes they are connected with. In terms of meta-stable patterns of synchronization, hubs persist during long times as isolated communities. However, this fact could never be detected via optimization of the modularity, because modularity of any partition with isolated nodes can never be optimal. This last fact is proved analytically from the definition of modularity.

## 5 Conclusions

We have shown that meta-stable patterns of synchronization in the path towards complete synchronization are closely related to the partitions obtained optimizing modularity on complex networks. However, the communities at different scales from topological analysis, including the possibility of having nodes forming its own community will be more representative of the topological role of the structure in the dynamics taking place on it.

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