

General Relativity (I)

homework for week 3

due: week 5

1. [metric tensor in flat and curved spacetime] 30%

In geometrized unit, the metric tensor in Minkowski space in Cartesian coordinate is $\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$:

(a) From $\eta^{\alpha\mu}\eta_{\mu\beta} \equiv \delta_{\beta}^{\alpha}$, where δ_{β}^{α} is the Kronecker delta, construct $\eta^{\alpha\beta}$.

(b) The determinant of the metric tensor is $\eta \equiv \det(\eta)$, where (η) is the matrix of $\eta^{\alpha\beta}$. Show that $\eta = -1$.

(c) The metric tensor in Minkowski space can also be written in terms of spherical coordinate:

$$\eta'_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix},$$

compute $\eta'^{\alpha\beta}$ and $\eta' = \det(\eta')$.

(c) The *Schwarzschild metric* describes the spacetime outside a spherical, non-rotating star with mass M :

$$g_{\alpha\beta} = \begin{pmatrix} -(1 - \frac{2M}{r}) & 0 & 0 & 0 \\ 0 & (1 - \frac{2M}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix},$$

compute $g^{\alpha\beta}$ and g .

2. [summation of indices and review of definitions] 10%

Simplify the following expressions:

(a) $\delta_{\beta}^{\alpha} u^{\beta} u^{\alpha}$

(b) $A_i g^{ij} g_{jk} B^k$

(c) $A^{\alpha} B_{\alpha} - A_{\mu} B^{\mu}$

(d) $\delta_j^i R^j - g^{ji} g_{kj} R^k$

(e) $g_{\mu\nu} u^{\mu} u^{\nu} - g^{\mu\nu} u_{\mu} u_{\nu} + k^{\alpha} k_{\beta} - \vec{k} \cdot \vec{k}$

2. [four-velocity, four-acceleration, definition of dot product] 30%

A time-like worldline S can be parameterized by $\mathcal{S}(\tau)$, where τ is the proper time. Show that four-velocity \vec{U} defined by

$$U^{\alpha} \equiv \frac{dx^{\alpha}}{d\tau},$$

and the four-acceleration \vec{A} defined by

$$A^\alpha \equiv \frac{u^\alpha}{d\tau} ,$$

are orthogonal:

$$\vec{U} \cdot \vec{A} = 0.$$

2. [Christoffel Symbol] 30%

The (natural) basis vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ for spherical coordinates in the 3D Euclidean space has been derived in the problem set 3 of the week2 homework. The Christoffel symbol $\Gamma_{\alpha\beta}^\mu$ is defined by

$$\frac{\partial \mathbf{e}_\alpha}{\partial x^\beta} = \Gamma_{\alpha\beta}^\mu \mathbf{e}_\mu .$$

That is, $\Gamma_{\alpha\beta}^\mu$ represents the μ th component of $\frac{\partial \mathbf{e}_\alpha}{\partial x^\beta}$ component. Write down all the non-vanishing Christoffel symbols for a spherical coordinate and verify

$$\Gamma_{\alpha\beta}^\mu = \Gamma_{\beta\alpha}^\mu .$$