General Relativity (I)

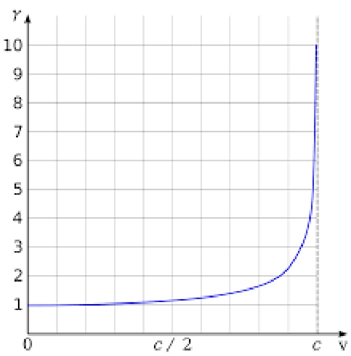
solutions for week 1-7

update:Nov 2020

Week 1:
1(a): key points: no acceleration, Newton's 2nd law

1(b): simply verify that $ds^2 = -cdt^2 + dx^2 + dy^2 + dz^2 = -cdt'^2 + dx'^2 + dy'^2 + dz'^2$

1(d):



2(a): refer to week4 homework 2(a)

2(b): refer to week4 homework 2(b)

2(c): Bob is older than Alice when they meet again. One way to show why is to use the proper time of Bob and Alice.

Week2:
1(a): key points: "seeing" a object is receiving photon emitted at different historical time; this is different from "measurement", which requires the idea of "simultaneous". The subtle difference results in "invisible Lorentz contraction".

1(b): moving left to right results in a opposite rotation, moving near to far results in a opposite distortion.

2. From the idea "speed of light is finite", you can get the formula between the *transverse velocity* v_t and the objects' velocity v

$$v_t = \frac{v\sin\theta}{1 - \frac{v}{c}\cos\theta} ,$$

where θ is the angle between the motion of the object and the line of sight.

3(b): $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{\phi\phi} = r^2\sin^2\theta$

3(c): $g^{rr} = 1/g_{rr}$, $g^{\theta\theta} = 1/g^{\theta\theta}$, $g^{\phi\phi} = 1/g^{\phi\phi}$

4(a):$A^1 = A_1 = 0$, $B^0 = -B_0 = 6$

4(b): $A^\alpha A_\alpha = 21$: time-like; $B^\alpha B_\alpha = -11$: space-like

4(c): 11

4(d): Applying the Lorentz transformation

$$\begin{aligned} t' &= \gamma(t - vx/c^2) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned} \tag{1}$$

with $\gamma = 1/\sqrt{1-0.8^2}$. $\vec{A}' = (-\frac{8}{3}, \frac{10}{3}, -4, 1)$

4(e): $\vec{B}' = (\frac{14}{3}, -\frac{4}{3}, 0, 3)$

4(f): 11 (the same as the answer to 4(c))

Week3:

1(a): $\eta^{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta_{\alpha\beta}$

1(c): as the metric is diagonal, we have $\eta'^{\alpha\beta} = 1/\eta'_{\alpha\beta}$; $\eta' = r^4 \sin^2\theta$

1(d): $g^{\alpha\beta} = 1/g_{\alpha\beta}$;$g = r^4 \sin^2\theta$

2(a): $u^\alpha U^\alpha$ or $u^\beta U^\beta$

2(b): $A_i A^i (= \vec{A}\cdot\vec{A} = \vec{A}^2)$

2(c): 0

2(d): 0

2(e): 0 (there is a typo in the original question: k^α_β should be k^α_α)

3: $0 = \dfrac{d(\vec{U}\cdot\vec{U})}{d\tau} = 2\vec{U}\cdot\dfrac{d\vec{U}}{d\tau} = 2\vec{U}\cdot\vec{A}$

4: $\Gamma^r_{\theta\theta} = -r$
$\Gamma^r_{\phi\phi} = -r\sin^2\theta$
$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r$
$\Gamma^\theta_{\phi\phi} = -\sin^2\theta\cos\theta$
$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = 1/r$
$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta$

Week4:
1(a): Note that, comparing with Week 3 homework 4, we additionally consider $g_{tt} = -1$. However, the non-vanishing Christoffel symbol is just the same as in Week 3 homework 4.

1(b): Note that we can actually show $g_{\beta;\gamma} = 0$ by inserting the definition of $\Gamma^\alpha_{\beta\gamma}$ back into equation (2).

This means the Christoffel symbol is actually *chosen* to satisfy that the covariant derivative of the metric tensor is zero. In other words, $g_{\alpha\beta;\gamma} = 0$ is a condition for us to choose a specific connection $\Gamma^{\alpha}_{\beta\gamma}$.

1(c): $\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\alpha} + g_{\beta\alpha,\mu} - g_{\mu\alpha,\beta})$. The first term and the third term cancel out because α and β is *symmetric* for $g^{\alpha\beta}$ but *asymmetric* for $(g_{\beta\mu,\alpha} - g_{\mu\alpha,\beta})$

2(a): one can quickly get $dt = \gamma dt'$ by applying the Lorentz transformation from $\mathcal{O}'$ to $\mathcal{O}$

$$\begin{aligned} dt &= \gamma(dt' + v(dx')/c^2) \\ dx &= \gamma(dx' + v(dt')) \\ dy &= y \\ dz &= z \end{aligned} \tag{2}$$

when $dx' = 0$.

2(b): one can quickly get $\gamma dx = dx'$ by applying the Lorentz transformation from $\mathcal{O}$ to $\mathcal{O}'$

$$\begin{aligned} dt' &= \gamma(dt - v(dx)/c^2) \\ dx' &= \gamma(dx - v(dt)) \\ dy' &= dy \\ dz' &= dz \end{aligned} \tag{3}$$

when $dt = 0$.

Week5:
1(a): (thanks to Biu Hong-Nhung)

1) The curvature tensor 2D Riemann manifold

a) Show that $(\nabla_\beta \nabla_\gamma + \nabla_\gamma \nabla_\beta) P_\alpha = R^{\mu}_{\alpha\beta\gamma} P_\mu$

Using definitions of covariant derivatives:

$\nabla_\beta P_\alpha = \partial_\beta P_\alpha - \Gamma^{\mu}_{\alpha\beta} P_\mu$ ~ rank-2 tensor

$\nabla_\gamma T_{\beta\alpha} = \partial_\gamma T_{\beta\alpha} - \Gamma^{\sigma}_{\gamma\beta} T_{\sigma\alpha} - \Gamma^{\sigma}_{\gamma\alpha} T_{\sigma\beta}$

$$\begin{aligned}\nabla_\gamma \nabla_\beta P_\alpha &= \partial_\gamma \nabla_\beta P_\alpha - \Gamma^{\sigma}_{\gamma\beta} \nabla_\sigma P_\alpha - \Gamma^{\sigma}_{\gamma\alpha} \nabla_\sigma P_\beta \\ &= \partial_\gamma \partial_\beta P_\alpha - \Gamma^{\mu}_{\alpha\beta} \partial_\gamma P_\mu - \Gamma^{\sigma}_{\gamma\beta} (\partial_\sigma P_\alpha - \Gamma^{\mu}_{\sigma\alpha} P_\mu) + \\ &\quad - (\partial_\lambda \Gamma^{\mu}_{\alpha\beta}) P_\mu - \Gamma^{\sigma}_{\gamma\alpha} (\partial_\sigma P_\beta - \Gamma^{\mu}_{\sigma\beta} P_\mu)\end{aligned}$$

$$\begin{aligned}\nabla_\beta \nabla_\gamma P_\alpha &= \partial_\beta \nabla_\gamma P_\alpha - \Gamma^{\sigma}_{\beta\gamma} \nabla_\sigma P_\alpha - \Gamma^{\sigma}_{\beta\alpha} \nabla_\sigma P_\gamma \\ &= \partial_\beta \partial_\gamma P_\alpha - \Gamma^{\mu}_{\alpha\gamma} \partial_\beta P_\mu - \Gamma^{\sigma}_{\beta\gamma} (\partial_\sigma P_\alpha - \Gamma^{\mu}_{\sigma\alpha} P_\mu) + \\ &\quad - (\partial_\beta \Gamma^{\mu}_{\alpha\gamma}) P_\mu - \Gamma^{\sigma}_{\beta\alpha} (\partial_\sigma P_\gamma - \Gamma^{\mu}_{\sigma\gamma} P_\mu)\end{aligned}$$

Note that $\Gamma^{\sigma}_{\gamma\beta} = \Gamma^{\sigma}_{\beta\gamma}$

$$(\nabla_\beta \nabla_\gamma - \nabla_\gamma \nabla_\beta) P_\alpha = \left(\Gamma^{\mu}_{\alpha\beta}\,\partial_\gamma - \Gamma^{\mu}_{\alpha\gamma}\,\partial_\beta + \Gamma^{\sigma}_{\beta\alpha} \Gamma^{\mu}_{\sigma\gamma} - \Gamma^{\sigma}_{\gamma\alpha} \Gamma^{\mu}_{\sigma\beta}\right) P_\mu$$

Compared to $R^{\mu}_{\alpha\beta\gamma} = \Gamma^{\mu}_{\alpha\gamma,\beta} - \Gamma^{\mu}_{\alpha\beta,\gamma} + \Gamma^{\sigma}_{\alpha\gamma} \Gamma^{\mu}_{\sigma\beta} - \Gamma^{\sigma}_{\alpha\beta} \Gamma^{\mu}_{\sigma\gamma}$ 1 Riemann curvature tensor

So $[\nabla_\gamma, \nabla_\beta] P_\alpha = R^{\mu}_{\alpha\beta\gamma} P_\mu$ □.

1(d): Comparing with Week 3 homework 4, all the $r-$related Chrstoffel symbols are gone. From
$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$
$\Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \cot\theta$
$\rightarrow R_{\theta\theta} = R^{\alpha}_{\ \theta\alpha\theta} = 1$ and $R_{\phi\phi} = R^{\alpha}_{\ \phi\alpha\phi} = \sin^2\theta$

1(e): Apply $\mu = \alpha$ into $R^{\alpha}_{\beta\mu\nu} + R^{\alpha}_{\nu\beta\mu} + R^{\alpha}_{\mu\nu\beta} = 0$, we get $R_{\beta\nu} - R_{\nu\beta} + 0 = 0$. As a result, $R_{\beta\nu} = R_{\nu\beta}$.

1(f): $R = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = 2/a^2$

1(g): As $G^{\mu\nu}$ is symmetric, we have $G^{\mu\nu}_{;\mu} = G^{\nu\mu}_{;\mu}$. The RHS can be write as $G^{\mu\nu}_{;\nu}$ by rename the indices.

2(a): Start with $(g_{\alpha\beta}u^\alpha u^\beta)_{;\mu} = g_{\alpha\beta}(u^\alpha u^\beta)_{;\mu} = 0$

2(b): For dust, $T^{\mu\nu}_{;\mu} = (\rho u^\mu)_{;\mu}u^\nu + \rho u^\mu u^\nu_{;\mu} = 0$. The first term is zero, as can be seen by applying $T^{\mu\nu}_{;\mu}u_\nu = 0$ (the same trick we used during the class). As a result, the second term $u^\mu u^\nu_{;\mu} = 0$ which is the geodesic equation.

Week6:
2(a): applying $\mu = \alpha$ and $\lambda = \beta$, then put $R_{\beta\nu;\beta}$ at the LHS. One can see that, in general, the RHS would not be zero.

2(b): In the local frame inertial frame $g_{\mu\nu}|_p \approx \eta_{\mu\nu}$, the last two terms in the equation $R^{\alpha}_{\beta\mu\nu} \equiv \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} +$

$\Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}$ vanishes. You can then verify a *tensor equation* (for our case, the Bianchi identity) in this frame.

2(c): use $g^{\mu\nu}g_{\mu\nu} = \delta^{\nu}_{\nu} = 4$

Week7:
1(a): you can see that $dl \to dr$ at large distance (asymptotically flat) and $dl \gg dr$ near 2m: space is stretched along the radial distance.

1(b): you can see $d\tau \to dt$ at large distance (asymptotically flat) and $d\tau \gg dt$ near 2m: the effect that time runs slower near the massive object is gravitational dilation

1(c): If $r_r < r_e$, the received frequency ν_r would be smaller the the emitted frequency: gravitational red-shift.

2: The top plot below shows $z(r)$. Rotate the curve with respective to r=0 with 2π, you can recover the $phi-$ coordinates and get the bottom plot below. Note that the curve stops at r=2M (location of the event horizon): this is because $g_{rr} < 0$ in the region $r < 2M$.