General Relativity (I)

bonus homework for week 16

1. [cosmological constant] 20%
The EFE including cosmological constant Λ reads

$$G^{\alpha\beta} + \Lambda g^{\alpha\beta} = 8\pi T^{\alpha\beta} ,$$

one can write the above equaiton as

$$G^{\alpha\beta} = T^{\alpha\beta} + T_{\Lambda}^{\alpha\beta} ,$$

with $T_{\Lambda}^{\alpha\beta} = -(\Lambda/8\pi)g^{\alpha\beta}$.

(a) Show that the energy of the cosmological constant $\rho = \Lambda/8\pi$, and associated with the pressure by $p_\Lambda = -\rho_\Lambda$.

(b) From the acceleration equation

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p)} , \tag{1}$$

argue that *a Universe includes dark matter could be accelerating* ($\ddot{a} > 0$) if Λ is positive. (cf. problem 2.a of the week 15 homework).

2. [cosmological model: matter + curvature + Λ] 50%
(a) The first relativistic cosmological model is Einstein's static Universe (1917). Show that, in a Universe includes matter and Λ, $\ddot{a} = 0$ requires the matter energy follows $\rho_m = 2\rho_\Lambda$.

Einstein's static model is actually unstable, and it is discarded later as people found that our Universe is expanding, not static.

(b) The Friemann equation including cosmological constant reads

$$\boxed{\mathrm{H}^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}(\rho_m + \rho_\Lambda) - \frac{\kappa}{a^2}} , \tag{2}$$

where H is the Hubble parameter. Show that a **big bang** (a=0) in the past of the Universe is inevitable if $\Lambda > 0$.

(c) Show that the Friedmann eqn can be written as

$$1 = \frac{\rho_m + \rho_\Lambda + \rho_k}{\rho_c} \equiv \Omega_m(t) + \Omega_\Lambda(t) + \Omega_k(t) ,$$

where $\rho_c = 3H^2/(8\pi)$ is the critical density and $\rho_\kappa = -3\kappa/(8\pi a^2)$ is the effective (energy) density due to the curvature.

From observations, current **concordance cosmological parameters** are:

$$\Omega_{\kappa,0} = 0 \qquad \Omega_{\Lambda,0} = 0.7 \qquad \Omega_{m,0} (= \Omega_{b,0} + \Omega_{d,0} = 0.04 + 0.26) = 0.3$$

where the subscript b and d corresponds to the **baryonic matter** and **dark matter**, and '0' indicates the values at present. With $H_0 \approx 71$km/s/Mpc, the critical density $\rho_{c,0} \approx 10^{-26}$ kg/m^3.

(d) Show that the Friemann eqn can be written in a form (cf. eqn 10 in the week 15 homework)

$$\dot{a}^2 = \frac{\mathcal{C}}{a} - \kappa + \frac{1}{3}a^2\Lambda \,, \tag{3}$$

where $\mathcal{C} > 0$ and Λ can be positive and negative. During the class we have discussed about the corresponding different cosmological models from this equation (see also **fig. 1**). **Note that the future of the Universe is not simply determined by the geometry once the cosmological constant is presence**.

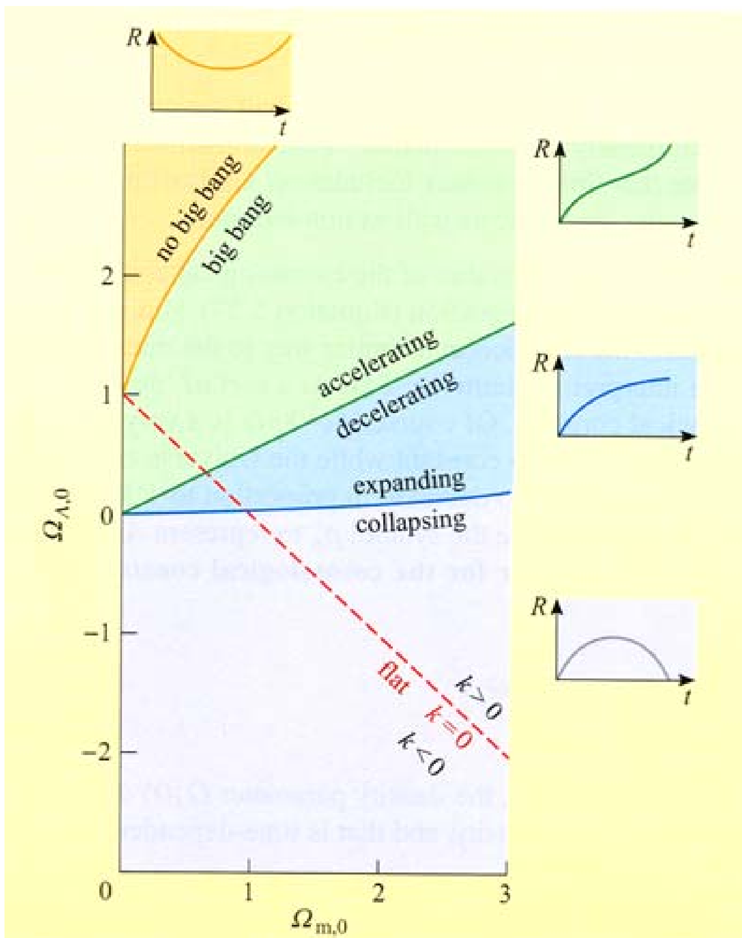

Figure 1: Fates of different cosmological models according to the content of the Universe (the parameter 'R' in the plot is 'a' in our notation). Figure from "Galaxies and Cosmology" (and here).

(e) Show that when the Universe is dominated by Λ, the scale factor has the form

$$a \propto e^{\frac{1}{\tau_\Lambda}t} \,,$$

where $\tau_\Lambda^2 = 3/(8\pi\rho_\Lambda)$.

2. [Cosmological redshift] 30%:
(a) The expansion of the space would cause the change of the wavelength emitted at t_e and received at t_0,

with the relation

$$\frac{\lambda(t_0)}{\lambda(t_e)} = \frac{a(t_0)}{a(t_e)} \; .$$

The redshift of the spectrum z is defined by

$$z \equiv \frac{\lambda(t_0) - \lambda(t_e)}{\lambda(t_e)} \; .$$

With $a(t_0) = 1$, show that

$$1 + z = \frac{1}{a(t_e)} \; . \tag{4}$$

(b) The proper distance d_{p} in the Roberson- Walker metric is

$$d_{\mathrm{p}} = -\int_d^0 \frac{dr}{\sqrt{1-\kappa r^2}} \; .$$

With the help of Eqn. (4), the **redshift-distance** relation can be established by considering

$$d_p = c\int_{t_e}^{t_0} \frac{dt}{a(t)} = c\int_{a(t_e)}^{a(t_0)=1} \frac{da}{a\dot{a}} = -c\int_z^{z(t_0)=0} \frac{dz}{H(z)} \; .$$

By replacing $\rho(a)$ with $\rho(z)$, show that

$$H(z) = H_0\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}}$$

Actually, cosmological constant (with the property $p = -\rho$) is one possible form of **dark energy** (anything with $\omega < -1/3$, where $p = \omega\rho$). A general form of H(z) with the dark energy is therefore

$$H(z) = H_0\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{de,0}(1+z)^{3(1+\omega)}}$$

where the subscript 'de' corresponds to 'dark energy'. As a result, if we can measure distances and redshifts for galaxies at different distances, a(t) and κ can be inferred.

(c) The 2011 Nobel Prize in physics was awarded to the the discovery of an **accelerating Universe**, supported by that the high redshift type Ia supernovae observations (see also **Fig. 2**). In a expanding Universe, should those supernovae found to be *brighter* or *fainter* than expected?

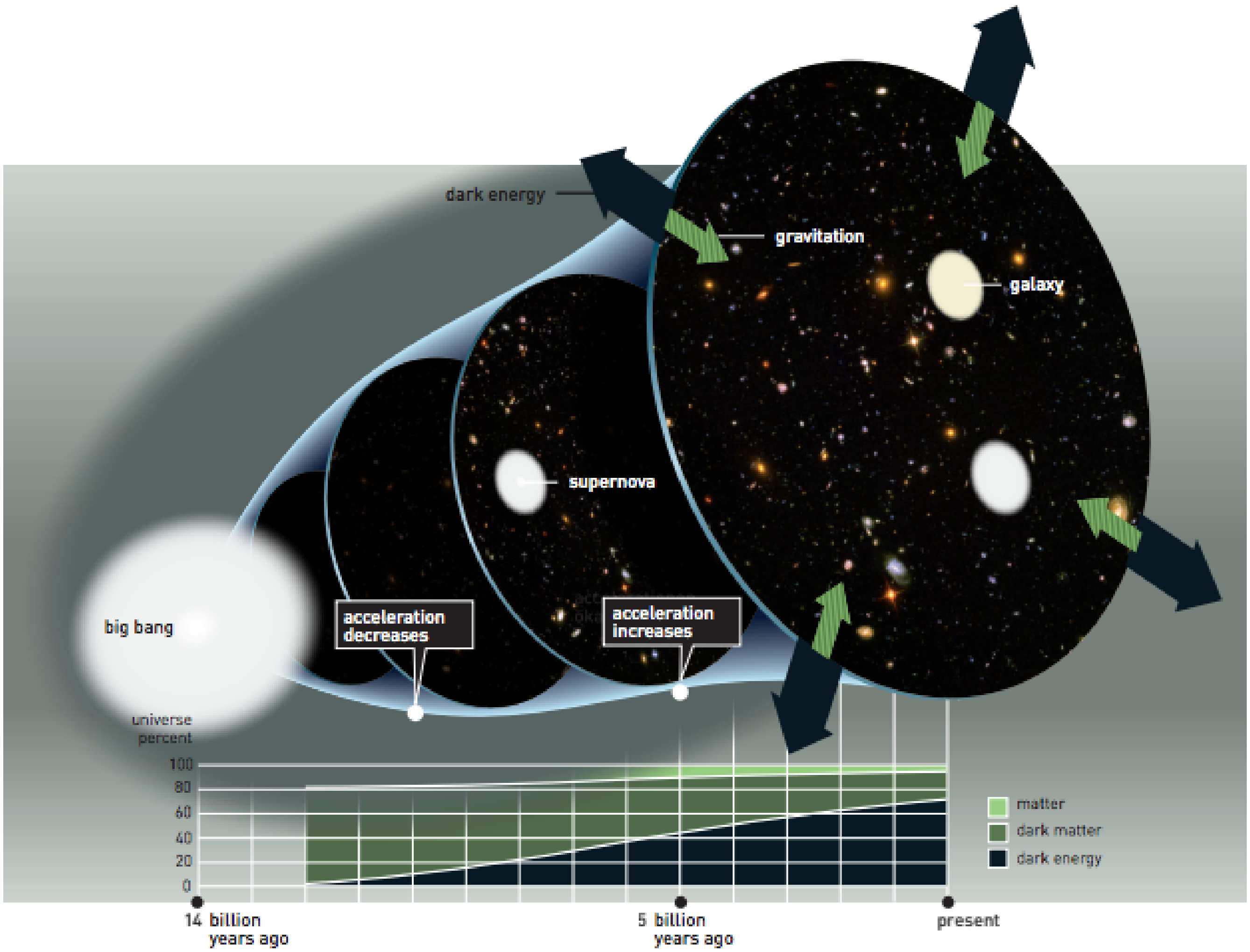

Figure 1. The world is growing. The expansion of the Universe began with the Big Bang 14 billion years ago, but slowed down during the first several billion years. Eventually it started to accelerate. The acceleration is believed to be driven by dark energy, which in the beginning constituted only a small part of the Universe. But as matter got diluted by the expansion, the dark energy became more dominant.

Figure 2: The growth of our Universe. Figure and its caption are adopted from figure 1 of this supplementary article of 2011 Nobel Prize in physics.