General Relativity (I)

solutions for week 11-13

Week 11:
1(a): one way to see why there are conserved quantities is to check the Euler-Lagrage equation.

1(b): We already have $u^t = -g^{tt}E$ and $u^\phi = g^{\phi\phi}L$. Without loss of generality, we can consider the motion on the equatorial plane and therefore $d\phi = 0$. By $g_{\alpha\beta}u^\alpha u_\beta = 1$ we can represent $(u^r)^2 = (\dfrac{dr}{d\tau})^2$ as function of g_{tt}, E, and L.

1(c): At a stable circular orbit, $\dfrac{dV_{\text{eff}}^2}{dr} = 0$ leads to $r = \dfrac{L^2 + L\sqrt{L^2 - 12M^2}}{2M}$. Invert the relation, then we can express L as function of r. By $E^2 = V_{\text{eff}}^2$ we can have E^2 as a function of r.

1(d): from 1(c), when the square root equals zero, we have $L^2 = 12M^2$, and therefore $r = 6M$. The corresponding $E^2 = \sqrt{8/9}$.

2(a): With $L = 0$ and $dr/d\tau = 0$, $E^2 = V_{\text{eff}}^2 = 1 - \frac{2M}{r}$. At rest at infinity ($r \to \infty$) implies$E^2 = 1$. At rest at $r = 2M$ implies $E^2 = 0$.

2(b): For a radial in falling observer at rest at infinity $E = 1$, as shown in 2(a). We then have $d\tau = \frac{2M}{r}dr$. Finally, $\int_{2M}^{0} \frac{2M}{r}dr = \frac{4M}{3}$.

2(c): The cgs unit expression for the answer of 2(b) is $t = \frac{4M}{3}\frac{G}{c^3}$. For t=1 year, $M = 5 \times 10^{12}$ solar mass.

3(a): $d\tau^2 = -ds^2 = -g_{tt}dt^2$ implies $u^t = dt/d\tau = (-g_{tt})^{-\frac{1}{2}} = (-g^{tt})^{\frac{1}{2}} = 1/\sqrt{1 - \frac{2M}{r}}$ $(u^r = u^\theta = u^\phi = 0)$

3(b): $g_{\alpha\beta}u^\alpha u^\beta = g_{tt}(u^t)^2 = g_{tt}(-g^{tt}) = -1$

3(c): If hovering, $u^r = u^\theta = u^\phi = 0$. When $g_{tt} > 0$, $u^\alpha u_\alpha = g_{tt}(u^t)^2 > 0$ which conflicts with the requirement $u^\alpha u_\alpha < 0$.

4(a): $u^\alpha = (1 - \frac{3M}{r})^{-1/2}(1, 0, 0, \sqrt{\frac{M}{r^3}})$.

4(b): From 4(a), we have $d\tau = (1 - \frac{3M}{r})^{1/2}dt$.
From 4(a) we also have $\frac{d\phi}{dt} = \frac{u^\phi}{u^t} = \sqrt{\frac{M}{r^3}}$. For one orbit, $d\phi = 2\pi = \sqrt{\frac{M}{r^3}}dt$.
Finally, we get $d\tau = 2\pi\left[\frac{r^3}{M}(1 - \frac{3M}{r})\right]^{1/2} dt$.

4(c): $\frac{d\phi}{dt} = \frac{u^\phi}{u^t} = \sqrt{\frac{M}{r^3}}$. **Note that there is a typo in the homework, the correct form should be $\triangle t = 2\pi\sqrt{\frac{r^3}{M}}$**

4(d): The relation can be obtained from of results 3(a) and 4(b).

Week 12:
1(a): See 2(c) of week 11

1(b): The relation can be helpful:

$$\begin{pmatrix} g^{tt} & g^{t\phi} \\ g^{t\phi} & g^{\phi\phi} \end{pmatrix} = \frac{1}{D} \begin{pmatrix} g_{\phi\phi} & -g_{t\phi} \\ -g_{t\phi} & g_{tt} \end{pmatrix}$$

where $D = g_{tt}g_{\phi\phi} - g_{t\phi}^2$

1(c): $(u^t)^2 = \dfrac{-1}{g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}^2\omega^2} = \dfrac{-1}{g_{tt} - \omega^2 g_{\phi\phi}} = -\dfrac{g_{\phi\phi}}{g_{tt}g_{\phi\phi} - g_{t\phi}^2} = \dfrac{-g_{\phi\phi}}{\Delta \sin^2\theta}$

1(d): As can be inffered from 1(b)

2(a): Even for fixed M, $A|_{a>0} < A|_{a=0}$.

2(b): The energy release is related to the mass difference $dM = M_{\text{after}} - M_{\text{before}}$ before and after the collision, where $M_{\text{before}} = 2 \times M$. We would like to maximize dM, while the area theorem asks $dA \geq 0$ should be satisfied. The maximum dM take place if (1)$dA = 0$ and (2) M_{final} is as small as possible. From (1), we have $2 \times 16\pi M^2 = 8\pi M_{\text{after}}[M_{\text{after}} + \sqrt{M_{\text{after}}^2 - a_{\text{after}}^2}\,]$. From (2) we have $a_{\text{after}}^2 = 0$. As a result, $M_{\text{after}} = \sqrt{2}M$ and the energy release is $dM = (2 - \sqrt{2})M$.

Week13:

1(a): just verify that $g_{\alpha\beta}g^{\alpha\gamma} \approx \delta_\beta^\gamma$ is satisfied up to $\mathcal{O}(h)$

1(b): *Lorentz gauge* and *transverse-traceless gauge*

2(a): Note we are using geometrized unit and $r \approx 10\text{Mpc} \approx 10^{21} M_\odot$. Therefore, $h \approx \mathcal{O}(0.1 \frac{M_\odot}{10^{21} M_\odot}) = \mathcal{O}(10^{-22})$

2(b): In geometrized unit, the energy density has dimension $[M^{-2}]$. While h is dimensionless, one should expect $\epsilon \propto h^2\Omega^2$ or $\epsilon \propto h^2/R^2$ or $\epsilon \propto h^2\Omega/R$, which one should be pick? From the analogue of a wave described by $y \propto \cos(\omega t + kx)$, its kinetic energy $dy^2/d^2t \propto \omega^2$, indicating we should take $\epsilon \propto h^2\Omega^2$.

2(c): F(energy per unit time per area) $= \epsilon$(energy per volumn) $\times c$, the speed of light $c = 1$ is dimensionless in geometrized unit.

2(d): L(energy per unit time) $= \int F\ r^2 \sin^2\theta d\theta d\phi$. From the dimension point of view, one should expect $L \propto Fr^2$

3(a): $M_{xx} = m1\left(\dfrac{m2R\cos(\Omega t)}{m1 + m2}\right)^2 + m2\left(\dfrac{m1R\cos(\Omega t)}{m1 + m2}\right)^2 = \mu R^2\cos^2(\Omega t)$

3(b): $M_{xx} = \mu R^2\cos^2(\Omega t) = \mu R^2 \dfrac{1 + \cos(2\Omega t)}{2}$, and we can drop the time-independent term since gravitational wave is only related to the time varying term of the quadrupole ($h \sim \ddot{M}/r$)

3(c)-3(f): should be straightforward