General Relativity (I)

homework for week 5

due: week 7

1. [the curvature tensor and related tensors; use 2D Riemann manifold as an example] 80%
The **Riemann curvature tensor** can be computed by:

$$\boxed{\mathrm{R}^{\alpha}_{\beta\mu\nu} \equiv \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}}\,, \tag{1}$$

and a associated $\begin{pmatrix}0\\4\end{pmatrix}$ tensor can be obtained by

$$\boxed{\mathrm{R}_{\alpha\beta\mu\nu} \equiv g_{\alpha\kappa}R^{\kappa}_{\beta\mu\nu}}\,.$$

Consider a 2D sphere with coordinate (θ, ϕ) and radius a, the metric tensor is

$$g_{\alpha\beta} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix}\,.$$

(a) In the class we have learned that the covairant derivative of a one-form p_α is

$$p_{\alpha;\beta} = p_{\alpha,\beta} - p_\mu \Gamma^{\mu}_{\alpha\beta}\,.$$

From this, show that

$$p_{\alpha;\beta\gamma} - p_{\alpha;\gamma\beta} = R^{\mu}_{\alpha\beta\gamma}p_\mu\,.$$

That is, unlike partial derivatives, the order of covariant derivatives matters (unless $R^{\mu}_{\alpha\beta\gamma} = 0$).

(b) As seen in problem set 1(f) of the week 4 homework, in a locally inertial frame at a point $\mathcal{P}$, we have find $\Gamma^{\alpha}_{\beta\mu}|_{\mathcal{P}} = 0$ and $\Gamma^{\alpha}_{\beta\mu,\nu}|_{\mathcal{P}} = 0$ (that is, second derivatives of $g_{\mu,\nu}$ cannot be zero in general). From eqn. (1) and

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu})\,,$$

we get

$$R^{\alpha}_{\beta\mu\nu} = \frac{1}{2}g^{\alpha\sigma}(g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu})\,. \tag{2}$$

Show the cyclic identity:

$$R^{\alpha}_{\beta\mu\nu} + R^{\alpha}_{\nu\beta\mu} + R^{\alpha}_{\mu\nu\beta} = 0 \tag{3}$$

or, alternatively

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$$

Note that eqn. (2) is not a valid tensor equation since it invlves partial derivative rather than covariant ones. However, eqn. (3) is a tensor equation since it is constructed by the (Riemann) tensors.

(c) According to the relation

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} \;,$$
$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu} \;,$$
$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta} \;,$$

argue that $R^{\alpha}_{\alpha\mu\nu} = 0$ and $R^{\alpha}_{\beta\mu\alpha} = -R^{\alpha}_{\beta\alpha\mu}$. Therefore, for the 2D sphere considered here, all the Riemann tensor are either zero or $\pm R_{\theta\phi\theta\phi}$.

(d)As a result of (c), the only non-zero contraction of the Riemann tensor is the **Ricci tensor**

$$\boxed{\mathrm{R}_{\alpha\beta} \equiv R^{\mu}_{\alpha\mu\beta} = R_{\beta\alpha}} \;.$$

Show that $R_{\theta\theta} = -1$ and $R_{\phi\phi} = -\sin^2\theta$.

Note that the Riemann curvature tensor in a spherical surface considered here is NOT zero, as expected. In comparison, the Riemann curvature tensor vanishes for a *spherical coordinate* in a 3D space, since it is a *flat* space. This was verified in problem set 1(g) in the week 4 homework.

(e) The Ricci tensor is symmetric ($R_{\alpha\beta} = R_{\beta\alpha}$). Show this by contacting the cyclic identity, eqn. (3).

(f) The **Ricci scalar** is defined by

$$\boxed{\mathrm{R} \equiv g^{\mu\nu} R_{\mu\nu}} \;.$$

Show that $R = -2/a^2$.

(g) We can further define an symmetric tensor, the **Einstein tensor**

$$\boxed{\mathrm{G}_{\mu\nu} \equiv R^{\mu\nu} - \tfrac{1}{2} g^{\alpha\beta} R}$$

Can you see there is only one divergence: $G^{\mu\nu}_{\;;\mu}$. That is, $G^{\mu\nu}_{\;;\mu} = G^{\mu\nu}_{\;;\nu}$.

Although you are not asked to show the covariant of the Einstein tensor is zero (the property is extremely important when constructing *Einstein's field equation*), it is good to know such property can be shown by using the Bianchi identity

$$R^{\alpha}_{\beta\mu\nu} + R^{\alpha}_{\nu\beta\mu} + R^{\alpha}_{\mu\nu\beta} = 0$$

2. [stress-energy tensor] 20%
The stress-energy tensor for a perfect fluid reads

$$\boxed{\mathrm{T}^{\alpha\beta} = (\rho + P) u^{\alpha} u^{\beta} + P g^{\alpha\beta}} \;,$$

where ρ and P are *rest-frame* energy density and pressure.

By perfect fluid we mean there is no viscosity (and therefore $T^{ij} = 0$) and no heat conduction (and therefore $T^{0i} = T^{i0} = 0$) in the MCRF (*Momentarily Comoving Reference Frame*). The tensor is symmetric and the relation

$$\boxed{\mathrm{T}^{\mu\nu}_{\;;\nu} = 0} \tag{4}$$

gives the equation of motion.

(a) *Dust* is a fluid without internal streaa or pressure. Show that $T^{\mu\nu}_{\;;\nu} = 0$ implise that the dust partticles follow geodesics. [hint: proof and use the fact that $u^{\nu}_{\;;\mu} u_{\nu} = 0$.]

(b) In the limit $P \ll \rho$, $u^0 \simeq 1$, $u^j \simeq v^j$, and $g^{\alpha\beta} = \eta^{\alpha\beta}$, show that the zeroth component of eqn. (4) reduces to the classical continuity equation

$$\frac{\partial \rho}{\partial t} + \bigtriangledown \cdot (\rho \mathbf{v}) = 0 \, .$$

With the relation between partial time derivative and total time derivatives,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \bigtriangledown) \, ,$$

the above equation can also be rewritten as

$$\frac{1}{\rho} \frac{d\rho}{dt} = - \bigtriangledown \cdot \mathbf{v}$$

(c)[bonus: additional 20%]
In the limit $P \ll \rho$, $u^0 \simeq 1$, $u^j \simeq v^j$, and $g^{\alpha\beta} = \eta^{\alpha\beta}$, show that the spatial component of eqn. (4) reduces to the classical Euler equation

$$\frac{d\mathbf{v}}{dt} = -\frac{\bigtriangledown p}{\rho} \, .$$