

# General Relativity (I)

homework for week 13

due: week 15

1. [linearized gravity and GW] 20%

The **metric perturbation** to the flat spacetime  $\eta_{\alpha\beta}$ :

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} , \quad (1)$$

where  $|h_{\alpha\beta}| \ll 1$ , can be used to describe the gravitational wave propagation in flat spacetime.

With the help of proper **gauge conditions**, and keeping terms linear in  $h_{\alpha\beta}$ , the solution to the **linearized EFE** (in vacuum)  $G_{\alpha\beta} = 0$  reduces to a wave equation:

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)h^{\alpha\beta} = 0 \quad (2)$$

and the following example solution represents a transverse gravitational wave (GW) propagating along the z-direction:

$$h_{\alpha\beta}(t, z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_x & 0 \\ 0 & h_x & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} f(t - z) , \quad (3)$$

where  $f(t - z)$  is a solution to the save equation.

The non-zero sub-matrix in parenthesis in eqn. [3] can be written as  $h_+ \mathbf{e}_1 + h_x \mathbf{e}_2$ , where  $\mathbf{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  and  $\mathbf{e}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  are basis of the "plus" and "cross" polarizaton mode, respectively.

(1) Show that, to the linear order,  $g^{ab} = \eta_{\alpha\beta} - h_{\alpha\beta}$

(2) To obtain the result of eqn. [3], what gauge condition(s) has been applied?

2. [power of GW, equal mass binary] 20%

Let us consider the following system to estimate the power of gravitational wave (in geometrized unit): two stars of equal masses  $m_1$  and  $m_2$  with total separation  $R$ , orbiting in a circular orbit with angular velocity  $\Omega$ .

Keeping only the relevant physical quantities, the amplitude for the perturbation  $h_{\mu\nu}$  roughly follows

$$h \sim mR^2\Omega^2/r , \quad (4)$$

which corresponds to the **mass quadrupole** radiation.

(1) In terms of dimension, eqn. [4] can be written as  $h \simeq v^2 M/r$ , where  $v$  is the orbital velocity. Assuming  $v^2 = O(0.1)$ , estimate the perturbation amplitude  $h$  for a binary neutron star system with  $M = M_\odot$  each, and located at a galaxy with  $r \approx 10$  Mpc.

(2) It is expected that the energy density of the wave  $\epsilon_{\text{gw}}$  is proportional to the square of the amplitude, therefore  $\epsilon_{\text{gw}} \propto h^2$ . From the dimension analysis, argue that

$$\epsilon_{\text{gw}} \propto \Omega_{\text{gw}}^2 h^2 ,$$

(3) From the dimension analysis, argue that the energy flux  $F$  has the same dimension of the energy density  $\epsilon$ .

(4) What's the relation between the luminosity(power)  $L$  and the flux  $F$ ? According to the relation, show that

$$L_{\text{gw}} \propto \Omega^2 h^2 \propto m^2 R^4 \Omega^6 .$$

**Note:** both  $h$  and  $L$  are dimensionless in geometrized unit. The relation for  $L$  between geometrized unit to cgs unit follows  $L[\text{cgs unit}] = c^5/G \times L[\text{geometrized unit}]$ .

3. [chirp mass, unequal mass binary, orbital evolution due to GW] 60%

With the insights gained from problem set 2, let us now consider the following system with further details: two stars of unequal masses  $m_1$  and  $m_2$  with total separation  $R$ , orbiting in a circular orbit with angular velocity  $\Omega$ .

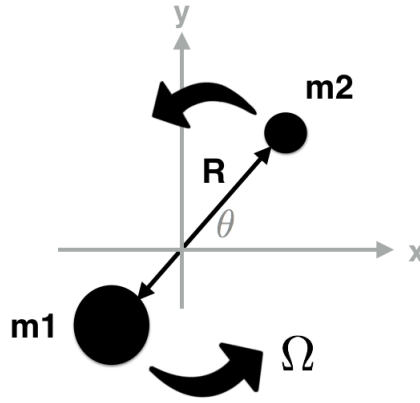


Figure 1: A unequal mass binary orbiting on the x-y plane in a circular orbit.

(1) Assuming the system is orbiting in the x-y plane, and putting the origin of coordinates at the center of mass of the system (as shown in **Fig 1**), verify that the xx-component of the mass quadrupole moment  $M_{xx} = \mu R \cos^2(\Omega t)$ , where  $\mu = m_1 m_2 / (m_1 + m_2)$  is the reduced mass.

(2) Since the gravitational radiation only depends on the time varying part of the quadrupole moment, show that we can rewrite  $M_{xx}$  as  $M_{xx} = \frac{1}{2} \mu R \cos^2(\Omega_{\text{gw}} t)$ , with  $\Omega_{\text{gw}} = 2\Omega$ . Similarly, work out that  $M_{yy} = -\frac{1}{2} \mu R \cos^2(\Omega_{\text{gw}} t)$

(3) Replacing  $m$  by  $\mu$ , the power can be now rewrite as  $L_{\text{gw}} \propto \mu^2 R^4 \Omega^6$ . The separation  $R$  can be further eliminated by

$$R^3 = \frac{m_1 + m_2}{\Omega^2} .$$

Show that  $L_{\text{gw}} \propto \mu^2 (m_1 + m_2)^{4/3} \Omega^{10/3}$ .

(4) Define the **chirp mass** as

$$\mathcal{M} = \mu^{3/5} (m_1 + m_2)^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} . \quad (5)$$

Verify that

$$\boxed{h \propto \frac{\mathcal{M}^{3/5} \Omega_{\text{gw}}^{2/3}}{r}} . \quad (6)$$

and

$$\boxed{L_{\text{gw}} \propto (\mathcal{M} \Omega_{\text{gw}})^{10/3}} . \quad (7)$$

**It turns out that the GW property is solely related to the chirp mass (rather than other combinations of individual masses)!**

(5) The GW power is supplied by the orbital energy  $E_{\text{orb}} = -m_1 m_2 / R$ , that is,  $-\frac{dE_{\text{orb}}}{dt} = L_{\text{gw}}$ . From the relation, derive the frequency evolution of the system:

$$\dot{\Omega}_{\text{gw}} \propto \mathcal{M}^{5/3} \Omega_{\text{gw}}^{11/3} .$$

(6) Similarly, show the period evolution of the system follows

$$\dot{P} \propto \mathcal{M}^{5/3} P^{-5/3} .$$