General Relativity (I)

solution for week 1-7

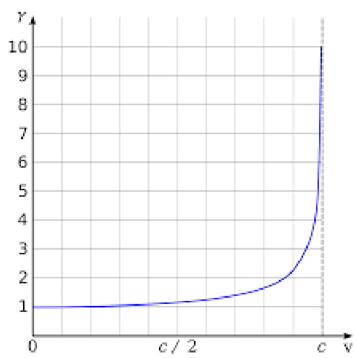
update:Nov 2020

Week 1:
1(a): key points: no acceleration, Newton's 2nd law

1(b): can be proofed by applying Lorentz transformation

1(c): you can do this

1(d):



2(a): refer to week4 homework 2(a)

2(b): refer to week4 homework 2(b)

2(c): Bob. One way to show why is to use the proper time of Bob and Alice.

3(a): you can do this!

3(b): you can do this!

3(c): you can do this!

Week2:
1(a): key points: "seeing" a object is receiving photon emitted at different historical time; this is different from "measurement", which requires the idea of "simultaneous". The subtle difference results in "invisible Lorentz contraction".

1(b): moving left to right results in a opposite rotation, moving near to far results in a opposite distortion.

2. From the idea "speed of light is finite", you can get the formula between the *transverse velocity* v_t and

the objects' velocity v

$$v_t = \frac{v \sin\theta}{1 - \frac{v}{c}\cos\theta},$$

where θ is the angle between the motion of the object and the line of sight.

3(a): you can do this

3(b): $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{\phi\phi} = r^2 \sin^2\theta$

3(c): $g^{rr} = 1/g_{rr}$, $g^{\theta\theta} = 1/g^{\theta\theta}$, $g^{\phi\phi} = 1/g^{\phi\phi}$

4(a):$A^1 = A_1 = 0$, $B^0 = -B_0 = 6$

4(b): $A^\alpha A_\alpha = 21$: time-like; $B^\alpha B_\alpha = -11$: space-like

4(c): 11

4(d): Applying the Lorentz transformation

$$\begin{aligned} t' &= \gamma(t - vx/c^2) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned} \tag{1}$$

with $\gamma = 1/\sqrt{1 - 0.8^2}$. $\vec{A}' = (-\frac{8}{3}, \frac{10}{3}, -4, 1)$

4(e): $\vec{B}' = (\frac{14}{3}, -\frac{4}{3}, 0, 3)$

4(f): 9

Week3:

1(a): $\eta^{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta_{\alpha\beta}$

1(b): you can do this

1(c): as the metric is diagonal, we have $\eta'^{\alpha\beta} = 1/\eta'_{\alpha\beta}$; $\eta' = r^4 \sin^2\theta$

1(d): $g^{\alpha\beta} = 1/g_{\alpha\beta}$;$g = r^4 \sin^2\theta$

2(a): $u^\alpha U^\alpha$ or $u^\beta U^\beta$

2(b): $A_i A^i$

2(c): 0

2(d): 0

2(e): 0 (there is a typo in the original question: k^α_β should be k^α_α)

3: $0 = \dfrac{d(\vec{U} \cdot \vec{U})}{d\tau} = 2\vec{U} \cdot \dfrac{d\vec{U}}{d\tau} = 2\vec{U} \cdot \vec{A}$

4: $\Gamma^r_{\theta\theta} = -r$

$\Gamma^r_{\phi\phi} = -r\sin^2\theta$
$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r$
$\Gamma^\theta_{\phi\phi} = -\sin^2\theta\cos\theta$
$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = 1/r$
$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta$

Week4:
1(a): you can do this

1(b): you can do this

1(c): $\Gamma^\alpha_{\mu\alpha} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\alpha} + g_{\beta\alpha,\mu} - g_{\mu\alpha,\beta})$. The first term and the third term cancel out because $alpha$ and β is *symmetric* for $g^{\alpha\beta}$ but *asymmetric* for $(g_{\beta\mu,\alpha} - g_{\mu\alpha,\beta})$

1(d): you can do this

1(e): you can do this

1(f): you can do this

1(g): you can do this

2(a): one can quickly get $dt = \gamma dt'$ by applying the Lorentz transformation from $\mathcal{O}'$ to $\mathcal{O}$

$$\begin{aligned} dt &= \gamma(dt' + v(dx')/c^2) \\ dx &= \gamma(dx' + v(dt')) \\ dy &= y \\ dz &= z \end{aligned} \tag{2}$$

when $dx' = 0$.

2(b): one can quickly get $\gamma dx = dx'$ by applying the Lorentz transformation from $\mathcal{O}$ to $\mathcal{O}'$

$$\begin{aligned} dt' &= \gamma(dt - v(dx)/c^2) \\ dx' &= \gamma(dx - v(dt)) \\ dy' &= dy \\ dz' &= dz \end{aligned} \tag{3}$$

when $dt = 0$.

Week5:
1(d):[hint]
$\Gamma^r_{\theta\theta} = \sin\theta\cos\theta$
$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \cot\theta$

1(f):$R = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi}$

2(a):[hint] start with $(g_{\alpha\beta}u^\alpha u^\beta)_{;\mu} = 0$

2(b):[hint] For dust, $T^{\mu\nu}_{;\mu} = (\rho u^\mu)_{;\mu}u^\nu + \rho u^\mu u^\nu_{;\mu} = 0$. Show the first term is zero, then it implies that $u^\mu u^\nu_{;\nu} = 0$ which is the geodesic equation.

Week6:
2(c): [hint]$g^{\mu\nu} g_{\mu\nu} = \delta^{\nu}_{\ \nu} = 4$

Week7:
you can do this