General Relativity (I)

homework for week 7

due: week 9

1. [Schwarzshild metric and gravitational effects] 100%
In geometrized unit, the *Schwarzschild spacetime* has the line element:

$$\boxed{ds^2 = -(1-\tfrac{2m}{r})dt^2 + (1-\tfrac{2m}{r})^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2} \; .$$

where the coordinate r is circumference radius.

(a) Taking a slice of t = constant, the infinitesimal radial physical distances dl given by θ and ϕ constant is

$$dl = (1-2m/r)^{-1/2} dr \; . \tag{1}$$

To get some insights, let us consider a *short* stick with length $dR = 1$ meter. Use the above relation to estimate the corresponding coordinate distance dr when $2M/r = 10^{-5}, 10^{-2}, 10,$ and 5. From the result, can you see that *the space is stretched along the radial distance*?

For a *long* stick has its ends at r coordinate $r1$ and $r2$, the length l of the stick is

$$l = \int_{r1}^{r2} (1-2m/r)^{-1/2} dr \; .$$

(b) Similarly, the infinitesimal proper time intervals $d\tau$ at a fixed spatial location ($r = \theta = \phi$=) constant is related to the coordinate time t by

$$d\tau = (1-2m/r)^{1/2} dt \; . \tag{2}$$

To get some insights, let us consider a *short* time elapse with length $d\tau = 1$ minute. Use the above relation to estimate the corresponding coordinate distance dr when $2M/r = 10^{-5}, 10^{-2}, 10,$ and 5. From the result, can you see effect of *gravitational time dilation*?

(c) Suppose a signal is sent from an emitter at (r_e, θ_e, ϕ_e) and received at (r_r, θ_r, ϕ_r) after the signal travels along a null geodesic. If the emitter emits n pulses during the proper time interval $\triangle\tau_e$, the measured frequency for the emitter and receiver is

$$\frac{\nu_r}{\nu_e} = \frac{n/(\triangle\tau_r)}{n/(\triangle\tau_e)} \; .$$

Show that

$$\frac{\nu_r}{\nu_e} = \left[\frac{1-2m/r_e}{1-2m/r_r}\right]^{1/2} \; .$$

Can you explain the effect of *gravitational redshift* according to this formula.

(d) In the week field limit ($r_e \gg 2m$ and $r_r \gg 2m$), show that

$$\frac{\triangle\nu}{\nu_e} \equiv \frac{\nu_r - \nu_e}{\nu_e} \approx (\frac{m}{r_r} - \frac{m}{r_e}) \; .$$

This above relation is experimentally confirmed in 1960 by Pound and Rebka with a instrument of 22.5 m vertical separation at Harvard. To interpret this relation, one can also link the gain/loss of photon energy with the gain/loss of gravitational potential energy, in a way somehow related to the equivalence principle.

2. [embedding diagram] bouns: additional 20%
The embedding diagram is a common way to visualize the curve space of a stationary spacetime. By taking $dt = 0$ and $\theta = \pi/2$ (and therefore $d\theta = 0$), we can embed the 2D Schwarzschild curved space

$$ds^2 = (1 - \frac{2m}{r})^{-1} dr^2 \ , + r^2 d\phi^2$$

in a Euclidean 3-dimensional cylindrical space

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2 \ .$$

Show that $z = \sqrt{8M(r - 2M)}$, and plot the resulting embedding diagram.

It should be clear then why the coordinate r does NOT represent the radial distance to the center ($r = 0$), but interpreted as the circumference radius (= circumference by fixed coordinate r divided by 2π).