

General Relativity (I)

homework for week 11

due: week 13

1. [time-like geodesic in the Schwarzschild spacetime] 40%

In geometrized unit, the *Schwarzschild spacetime* has the line element:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2,$$

where M is the mass of the black hole.

(a) The absence of t and ϕ in $g_{\mu\nu}$ implies that the covariant component of the four momentum p_t and p_ϕ are conserved along the trajectories of a freely moving particle. Explain why?

(b) Making use of the result of (a), for a test particle with mass m , we can define

$$P_t/m \equiv -E, \quad (1)$$

and

$$P_\phi/m \equiv L. \quad (2)$$

Show that the time-like geodesic has the form:

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - V_{\text{eff}}^2, \quad (3)$$

with the effective potential

$$V_{\text{eff}}^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right). \quad (4)$$

See **Fig 1.** for the example profiles of $V_{\text{eff}}^2(L^2)$.

(c) Show that for a stable circular orbit at radius r , the specific angular momentum L follows

$$L_{GR}^2 = \frac{Mr}{1 - \frac{3M}{r}}, \quad (5)$$

and the corresponding specific energy E follows

$$E_{GR}^2 = \frac{\left(1 - \frac{2M}{r}\right)^2}{1 - \frac{3M}{r}}, \quad (6)$$

(hint: apply $dV_{\text{eff}}^2/dr = 0$, and $E^2 = V_{\text{eff}}^2$ for circular orbits)

From **Fig 2.**, it is clear that particles must be *sub-Keplerian* when they enter the event horizon, as can be understood by the requirement of causality. In comparison, the specific angular momentum in the Newtonian case is $L_N^2 = Mr$: a stable circular orbit can be found at any radius.

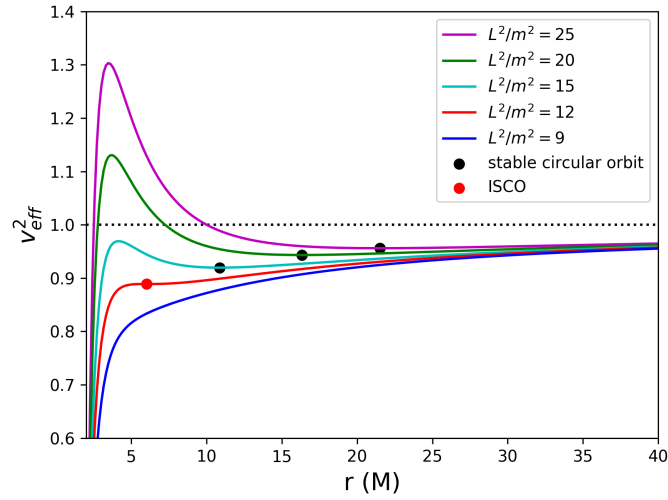


Figure 1: V_{eff}^2 as function of different L^2 , and the corresponding location of stable circular orbits for $L^2 = (25, 20, 15, 12)$.

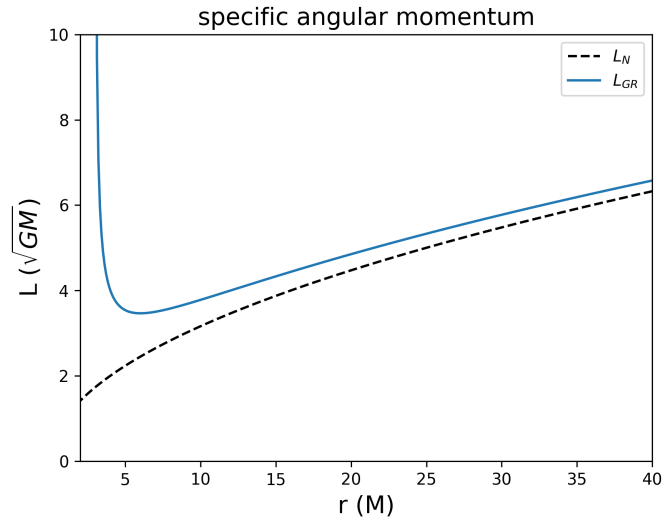


Figure 2: The angular momentum for a stable circular orbit rapidly increase within the ISCO.

(d) Show that there is no stable circular orbit when $L < 12M^2$ and there exists a *innermost stable circular orbit (ISCO)* at $r = 6M$. See also Fig 1. for reference. What is the corresponding E_{GR} at ISCO? (hint: solve for r by using eqn [3])

2. [observer with radial motion] 30%

(a) From eqn [3], argue that $E = 1$ for a free-fall particle ($L = 0$) falling from rest at infinity, and that $E = 0$ for a free-fall particle falling from rest at $r = 2M$.

(b) For a free-fall observer falls from rest at infinity, show that the proper time it takes to fall from $r = 2m$ to $r = 0$ is $4M/3$.

(c) What mass of black hole would allow an observer in (b) to survive for a year inside the the horizon before she/he reaches the singularity?

3. [hovering observer] 30%

An astronaut use rocket-pack to stay hovering ($dr = d\theta = d\phi = 0$) at $R > 2M$.

(a) What is the four-velocity u^α of the astronaut?

(b) Show that $u^\alpha u_\alpha = -1$.

(c) Argue that it is impossible to stay hovering inside the horizon ($R < 2M$), since $g_{tt} > 0$ inside the horizon. [hint: can $u^\alpha u_\alpha = -1$ be satisfied in that case?]

4. [circular orbiting observer] bonus: 20%

An astronaut is circulating the black hole ($dr = d\theta = 0$) at $R > 6M$, following the geodesic.

(a) What is the four-velocity u^α of the astronaut?

(hint: We can easily construct the four velocity of the circular observer, by the relation

$$u^t = \frac{dt}{d\tau} = g^{tt} E \quad (7)$$

$$u^\phi = \frac{d\phi}{d\tau} = g^{\phi\phi} L \quad (8)$$

and the helps of eqns [5] and [6].)

(b) For the orbiting observer, what is the proper time $\Delta\tau_{\text{orb}}$ for one orbit?

(c) With the result of (a), show that

$$\frac{d\phi}{dt} = \sqrt{\frac{M}{r^3}} \quad (9)$$

and therefore the coordinate time elapses for one orbit is $\Delta t = 2\pi\sqrt{\frac{r^3}{M}}$.

(d) Based on the problem set 3, for the hovering observer, what is the corresponding proper time $\Delta\tau_{\text{hov}}$ for the time span Δt ? One should get the relation:

$$\frac{\Delta\tau_{\text{hov}}}{\Delta\tau_{\text{orb}}} = \left(\frac{r-2M}{r-3M}\right)^{1/2}. \quad (10)$$

Note : unlike the twin paradox in special relativity, the observer follows the geodesic (the orbiting observer) find himself younger than his sister who is hovering at the same radius! Such result become obvious by seeing that $ds^2 = g_{tt} dt^2$ for a hovering observer but $ds^2 = g_{tt} dt^2 + g_{\phi\phi} d\phi^2$ for a orbiting observer.