General Relativity (I)

homework for week 6

due: week 8

1. [equation of motion from stress-energy tensor] 40%
The stress-energy tensor for a perfect fluid reads

$$\boxed{\mathrm{T}^{\alpha\beta} = (\rho + P)u^{\alpha}u^{\beta} + Pg^{\alpha\beta}} \, ,$$

where ρ and P are *rest-frame* energy density and pressure.

By perfect fluid we mean there is no viscosity (and therefore $T^{ij} = 0$) and no heat conduction (and therefore $T^{0i} = T^{i0} = 0$) in the MCRF (*Momentarily Comoving Reference Frame*). The tensor is symmetric and the relation

$$\boxed{\mathrm{T}^{\mu\nu}_{\ \ ;\nu} = 0} \tag{1}$$

gives the equation of motion.

In the Newtonian limit, we have $P \ll \rho$, $u^0 \simeq 1$, $u^j \simeq v^j$, and $g^{\alpha\beta} = \eta^{\alpha\beta}$.

(a) Under Newtonian limit, show that the zeroth component of eqn. (4) reduces to the classical continuity equation

$$\frac{\partial\rho}{\partial t} + \bigtriangledown \cdot (\rho\mathbf{v}) = 0 \, .$$

With the relation between partial time derivative and total time derivatives,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \bigtriangledown) \, ,$$

the above equation can also be rewritten as

$$\frac{1}{\rho}\frac{d\rho}{dt} = - \bigtriangledown \cdot \mathbf{v}$$

(b) Under Newtonian limit, show that the spatial component of Eqn. (1) reduces to the classical Euler equation

$$\frac{d\mathbf{v}}{dt} = -\frac{\bigtriangledown p}{\rho} \, .$$

2. [Einstein's field equation] 60 %
The Einstein's field equation,

$$\boxed{\mathrm{R}^{\mu\nu} - \tfrac{1}{2}Rg^{\mu\nu} = \kappa T^{\mu\nu}} \, ,$$

describes the relation between the spacetime geometry and the matter and energy, the latter is described by the stress-energy tensor $T^{\mu\nu}$. The κ is some constant.

(a) A few months before Einstein reached the field equation, he also publish the equation

$$R^{\mu\nu} = \kappa T^{\mu\nu} \ (this\ is\ wrong).$$

Later on, he realized that $R^{\mu\nu}_{\ \ ;\mu} \neq 0$ and therefore not consistent with that $T^{\mu\nu}_{\ \ ;\nu} = 0$. By using the *Bianchi identity*,

$$R^{\alpha}_{\beta\mu\nu;\lambda} + R^{\alpha}_{\beta\lambda\mu;\nu} + R^{\alpha}_{\beta\nu\lambda;\mu} = 0 \ ,$$

show that $R^{\mu\nu}_{\ \ ;\mu} \neq 0$.

Note that, although $g^{\mu\nu}_{\ \ ;\nu} = 0$, the guess

$$g^{\mu\nu} = \kappa T^{\mu\nu} \ (this\ is\ wrong),$$

is also not viable: $g_{\mu\nu}$ has the dimension of that of the gravitational potentials, this indicates a symmetric tensor involving the second derivatives of $g_{\mu\nu}$ is needed.

(b) Proof the Bianchi identity.
(Hint: Remember at any point $\mathcal{P}$ in a curved spacetime we can construct a coordinate system $\Gamma^{\alpha}_{\beta\gamma}|_{\mathcal{P}} = 0$. Apply this to eqn. (1) of week 5 homework for all the terms in the LHS of the Bianchi identity, then validate the RHS is zero. As the choose of $\mathcal{P}$ is arbitrary, the result is true everywhere. **Why this approach works?** The technique here is to adopt a local coordinate system in which $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and therefore $\Gamma^{\alpha}_{\beta\gamma} = 0$. Although such coordinate is specially selected, however, once we can construct a *tensor equation* in such coordinate, then the same tensor equation applies to other coordinates and to other points in spacetime. This is the **power of tensor equations!** a coordinate independent way to describe physics!)

(c)Show the field equation can also be written as

$$\boxed{\mathrm{R}^{\mu\nu} = \kappa(T^{\mu\nu} - \tfrac{1}{2}Tg^{\mu\nu})} \ ,$$

where $T = T^{\mu}_{\mu}$. (Hint: it may be helpful to first show $R = -\kappa T$.)