

General Relativity (I)

homework for week 6

due: week 8

1. [equation of motion from stress-energy tensor] 40%

The stress-energy tensor for a perfect fluid reads

$$T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta + P g^{\alpha\beta},$$

where ρ and P are *rest-frame* energy density and pressure.

By perfect fluid we mean there is no viscosity (and therefore $T^{ij} = 0$) and no heat conduction (and therefore $T^{0i} = T^{i0} = 0$) in the MCRF (*Momentarily Comoving Reference Frame*). The tensor is symmetric and the relation

$$T^{\mu\nu}_{;\nu} = 0 \tag{1}$$

gives the equation of motion.

In the Newtonian limit, we have $P \ll \rho$, $u^0 \simeq 1$, $u^i \simeq v^i$, and $g^{\alpha\beta} = \eta^{\alpha\beta}$.

(a) Under Newtonian limit, show that the zeroth component of eqn. (4) reduces to the classical continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0.$$

With the relation between partial time derivative and total time derivatives,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla),$$

the above equation can also be rewritten as

$$\frac{1}{\rho} \frac{d\rho}{dt} = -\nabla \cdot \mathbf{v}$$

(b) Under Newtonian limit, show that the spatial component of eqn. (4) reduces to the classical Euler equation

$$\frac{d\mathbf{v}}{dt} = -\frac{\nabla p}{\rho}.$$

2. [Einstein's field equation] 60 %

The Einstein's field equation,

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = \kappa T^{\mu\nu},$$

describes the relation between the spacetime geometry and the matter and energy, the latter is described by the stress-energy tensor $T^{\mu\nu}$. The κ is some constant.

(a) A few months before Einstein reached the field equation, he also published the equation

$$R^{\mu\nu} = \kappa T^{\mu\nu} \text{ (this is wrong).}$$

Later on, he realized that $R^{\mu\nu}_{;\mu} \neq 0$ and therefore not consistent with that $T^{\mu\nu}_{;\nu} = 0$. By using the *Bianchi identity*,

$$R^{\alpha}_{\beta\mu\nu;\lambda} + R^{\alpha}_{\beta\lambda\mu;\nu} + R^{\alpha}_{\beta\nu\lambda;\mu} = 0,$$

show that $R^{\mu\nu}_{;\mu} \neq 0$.

Note that, although $g^{\mu\nu}_{;\nu} = 0$, the guess

$$g^{\mu\nu} = \kappa T^{\mu\nu} \text{ (this is wrong),}$$

is also not viable: $g_{\mu\nu}$ has the dimension of that of the gravitational potentials, this indicates a symmetric tensor involving the second derivatives of $g_{\mu\nu}$ is needed.

(b) Proof the Bianchi identity.

(Hint: Remember at any point \mathcal{P} in a curved spacetime we can construct a coordinate system $\Gamma^{\alpha}_{\beta\gamma}|_{\mathcal{P}} = 0$. Apply this to eqn. (1) of week 5 homework for all the terms in the LHS of the Bianchi identity, then validate the RHS is zero. As the choice of \mathcal{P} is arbitrary, the result is true everywhere. **Why this approach works?** The technique here is to adopt a local coordinate system in which $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and therefore $\Gamma^{\alpha}_{\beta\gamma} = 0$. Although such coordinate is specially selected, however, once we can construct a *tensor equation* in such coordinate, then the same tensor equation applies to other coordinates and to other points in spacetime. This is the **power of tensor equations!** a coordinate independent way to describe physics!)

(c) Show the field equation can also be written as

$$\boxed{R^{\mu\nu} = \kappa \left(T^{\mu\nu} - \frac{1}{2} T g^{\mu\nu} \right)},$$

where $T = T^{\mu}_{\mu}$. (Hint: it may be helpful to first show $R = -\kappa T$.)