General Relativity (I)

homework for week 1

due: Sep. 28th, 2020

1. [Lorentz Transform] 40% For a inertial frame $\mathcal{O}' : (t', x', y', z')$ moving uniformly with respect to another inertial frame $\mathcal{O} : (t, x, y, z)$ along the x-axis with speec v, the connection between inertial frames are given by the *Lorentz transform*:

$$\begin{aligned} t' &= \gamma(t - vx/c^2) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

where c is the speed of light, and $\gamma = (1 - v^2/c^2)^{-1/2} \geq 1$.

(a) explain what is a inertial frame.

(b) show that $ds^2 = -c^2 dt'^2 + dx'^2 + dy'^2 + dz'^2$ is an invariant under Lorentz transformation

(c) show that when $v/c \ll 1$, the Lorentz transformation reduce to the *Galilean transformation*:

$$\begin{aligned} t' &= t \\ x' &= (x - vt) \\ y' &= y \\ z' &= z \end{aligned} \tag{1}$$

(d) plot the value of γ as a function of v/c.

2. [length contraction, time dilation, and twin paradox] 30%
With the same coordinate setup described in the problem 1, consider the following cases.

(a) For a time interval dt' measured by clocks comoving with $\mathcal{O}'$ (we call dt' as the **proper time**), show that the corresponding coordinate time interval dt measured by stationary clocks at $\mathcal{O}$ follows

$$\boxed{\text{dt=}\gamma dt'} > dt' \; .$$

This is usually described by the slogan "moving clocks run slow."

(b) For a rod with its length dx' comoving with $\mathcal{O}'$ (we call dx' as the **proper length**), show that the length of the rod measured at $\mathcal{O}$ follows

$$\boxed{\text{dx= dx}'/\gamma} < dx' \; .$$

That is, a moving object's length is measured to be shorter than its proper length.

(c)Two twins, Alice and Bob, start from rest at (t_1, x) in an inertial frame. Bob remains at rest at x. Alice moves away from x and return back to x and meet Bob at time t_2. Who ages more, Alice or Bob? Why?

3. [unit] 30%
In *geometrized unit*, the speed of light c and the gravitational constant G are set to be unity ($c = G = 1$):

(a) verify that the radius of the Earth ($R_\oplus$) can be represented by $R_\oplus \sim 10^9 M_\oplus$, and $M_\oplus \approx 0.5$ cm

(b) verify the radius of the Sun ($R_\odot$) can be represented by $R_\odot \sim 10^6 M_\odot$ and $M_\odot \approx 1.5$ km. In comparison, the *event horizon*, R_{BH}, of a non-rotating black hole with mass M_{BH} has the size $R_{\mathrm{BH}} = 2M_{\mathrm{BH}}$.

(c) show that the angular momentum has the dimension [length^2] and the force has the dimension [1] (i.e., dimensionless).