General Relativity (I)

homework for week 13

due: week 15

1. [linearized gravity and GW] 20%
The **metric perturbation** to the flat spacetime $\eta_{\alpha\beta}$:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \ , \tag{1}$$

where $|h_{\alpha\beta}| \ll 1$, can be used to describe the gravitational wave propagation in flat spacetime.

With the help of proper **gauge conditions**, and keeping terms linear in $h_{\alpha\beta}$, the solution to the **linearized EFE** (in vacuum) $G_{\alpha\beta} = 0$ reduces to a wave equation:

$$(-\frac{\partial^2}{\partial t^2} + \nabla^2) h^{\alpha\beta} = 0 \tag{2}$$

and the following example solution represents a transverse gravitational wave (GW) propagating along the z-direction:

$$h_{\alpha\beta}(t, z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_x & 0 \\ 0 & h_x & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} f(t - z) \ , \tag{3}$$

where $f(t - z)$ is a solution to the save equation.

The non-zero sub-matrix in parenthesis in eqn. [3] can be written as $h_+ \mathbf{e_1} + h_x \mathbf{e_2}$, where $\mathbf{e_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{e_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are basis of the "plus" and "cross" polarization mode, respectively.

(1) Show that, to the linear order, $g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$

(2) To obtain the result of eqn. [3], what gauge condition(s) has been applied?

2. [power of GW, equal mass binary] 20%
Let us consider the following system to estimate the power of gravitational wave (in **geometrized unit**): two stars of equal masses m with total separation R, orbiting in a circular orbit with angular frequency Ω. .

Keeping only the relevant physical quantities, the amplitude for the perturbation $h_{\mu\nu}$ roughly follows

$$h \sim mR^2\Omega^2/r \ , \tag{4}$$

which corresponds to the **mass quadrupole** radiation.

(1) In terms of dimension, eqn. [4] can be written as $h \simeq v^2 m/r$, where v is the orbital velocity. Assuming $v^2 = O(0.1)$, estimate the perturbation amplitude h for a binary neutron star system with $M = M_\odot$ each, and located at a galaxy with $r \approx 10$ Mpc.

(2)It is expected that the energy density of the wave $\epsilon_{\rm gw}$ is proportional to the square of the amplitude, therefore $\epsilon_{\rm gw} \propto h^2$. From the dimension analysis, argue that

$$\epsilon_{\rm gw} \propto \Omega^2 h^2 \ ,$$

(3) From the dimension analysis, argue that the energy flux F has the same dimension of the energy density ϵ.

(4) What's the relation between the luminosity(power) L and the flux F? According to the relation, show that

$$L_{\rm gw} \propto \Omega^2 h^2 \propto m^2 R^4 \Omega^6 \ .$$

Note: both h and L are dimensionless in geomertirized unit. The relation for L between geometrized unit to cgs unit follows $L[\text{cgs unit}] = c^5/G \times L[\text{geometrized unit}]$.

3. [chirp mass, unequal mass binary, orbital evolution due to GW] 60%
With the insights gained from problem set 2, let us now consider the following system with further details: two stars of unequal masses $m1$ and $m2$ with total separation R, orbiting in a circular orbit with angular frequency Ω.

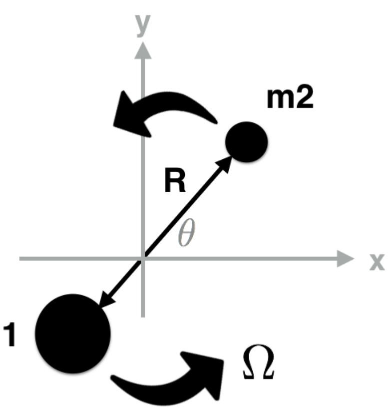

Figure 1: A unequal mass binary orbiting on the x-y plane in a circular orbit.

(1) Assuming the system is orbiting in the x-y plane, and putting the origin of coordinates at the center of mass of the system (as shown in **Fig 1**), verify that the xx-component of the mass quadrupole moment $M_{xx} = \mu R^2 \cos^2(\Omega t)$, where $\mu = m1m2/(m1+m2)$ is the reduced mass.

(2) Since the gravitational radiation only depends on the time varying part of the quadrupole moment, show that we can rewrite M_{xx} as $M_{xx} = \frac{1}{2}\mu R^2 \cos(\Omega_{\rm gw} t)$, with $\Omega_{\rm gw} = 2\Omega$. [hint: apply $2\cos^2 x = 1 + \cos(2x)$]

Similarly, work out that $M_{yy} = -\frac{1}{2}\mu R^2 \cos(\Omega_{\rm gw} t)$ and $M_{xy} = \frac{1}{2}\mu R^2 \sin(\Omega_{\rm gw} t)$.

(3) Replacing m by μ, the power can be now rewrite as $L_{\rm gw} \propto \mu^2 R^4 \Omega^6$. The separation R can be further eliminated by

$$R^3 = \frac{m1+m2}{\Omega^2} \ .$$

Show that $L_{\rm gw} \propto \mu^2 (m1+m2)^{4/3} \Omega^{10/3}$.

(4) Define the **chirp mass** as

$$\mathcal{M} = \mu^{3/5}(m1 + m2)^{2/5} = \frac{(m1m2)^{3/5}}{(m1 + m2)^{1/5}} \,. \tag{5}$$

Verify that

$$\boxed{\mathrm{h} \propto \frac{\mathcal{M}^{5/3}\Omega_{\mathrm{gw}}^{2/3}}{r}} \,. \tag{6}$$

and

$$\boxed{\mathrm{L_{gw}} \propto (\mathcal{M}\Omega_{\mathrm{gw}})^{10/3}} \,. \tag{7}$$

It turns out that the GW property is solely related to the chirp mass (rather than other combinations of individual masses)!

(5) The GW power is supplied by the orbital energy $E_{\mathrm{orb}} = -m1m2/R$, that is, $-\frac{dE_{\mathrm{orb}}}{dt} = L_{\mathrm{gw}}$. From the relation, derive the frequency evolution of the system:

$$\dot{\Omega}_{\mathrm{gw}} \propto \mathcal{M}^{5/3}\Omega_{\mathrm{gw}}^{11/3} \,.$$

Remarkably, Chirp mass is again involved! How the system would "chirp" (increase of frequency) simply depends on the chirp mass. As a result, a chirping binary with a circular orbit can serve as a **standard candle**: the distance of the system can be inferred once h, Ω_{gw}, $\dot{\Omega}_{\mathrm{gw}}$ are known.

(6) Similarly, show the period evolution of the system follows

$$\dot{P} \propto \mathcal{M}^{5/3}P^{-5/3} \,.$$

Hello again, chirp mass!