

General Relativity (I)

homework for week 7

due: week 9

1. [Schwarzschild metric and gravitational effects] 100%

In geometrized unit, the *Schwarzschild spacetime* has the line element:

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 .$$

where the coordinate r is circumference radius.

(a) Taking a slice of $t = \text{constant}$, the infinitesimal radial physical distances dl given by θ and ϕ constant is

$$dl = \left(1 - 2m/r\right)^{-1/2}dr . \quad (1)$$

To get some insights, let us consider a *short* stick with length $dR = 1$ meter. Use the above relation to estimate the corresponding coordinate distance dr when $2M/r = 10^{-5}, 10^{-2}, 10$, and 5 . From the result, can you see that *the space is stretched along the radial distance*?

For a *long* stick has its ends at r coordinate r_1 and r_2 , the length l of the stick is

$$l = \int_{r_1}^{r_2} \left(1 - 2m/r\right)^{-1/2}dr .$$

(b) Similarly, the infinitesimal proper time intervals $d\tau$ at a fixed spatial location ($r = \theta = \phi = \text{constant}$) is related to the coordinate time t by

$$d\tau = \left(1 - 2m/r\right)^{1/2}dt . \quad (2)$$

To get some insights, let us consider a *short* time elapse with length $d\tau = 1$ minute. Use the above relation to estimate the corresponding coordinate distance dr when $2M/r = 10^{-5}, 10^{-2}, 10$, and 5 . From the result, can you see effect of *gravitational time dilation*?

(c) Suppose a signal is sent from an emitter at (r_e, θ_e, ϕ_e) and received at (r_r, θ_r, ϕ_r) after the signal travels along a null geodesic. If the emitter emits n pulses during the proper time interval $\Delta\tau_e$, the measured frequency for the emitter and receiver is

$$\frac{\nu_r}{\nu_e} = \frac{n/(\Delta\tau_r)}{n/(\Delta\tau_e)} .$$

Show that

$$\frac{\nu_r}{\nu_e} = \left[\frac{1 - 2m/r_e}{1 - 2m/r_r} \right]^{1/2} .$$

Can you explain the effect of *gravitational redshift* according to this formula.

(d) In the weak field limit ($r_e \gg 2m$ and $r_r \gg 2m$), show that

$$\frac{\Delta\nu}{\nu_e} \equiv \frac{\nu_r - \nu_e}{\nu_e} \approx \left(\frac{m}{r_r} - \frac{m}{r_e} \right) .$$

This above relation is experimentally confirmed in 1960 by Pound and Rebka with a instrument of 22.5 m vertical separation at Harvard. To interpret this relation, one can also link the gain/loss of photon energy with the gain/loss of gravitational potential energy, in a way somehow related to the equivalence principle.

2. [embedding diagram] bouns: additional 20%

The embedding diagram is a common way to visualize the curve space of a stationary spacetime. By taking $dt = 0$ and $\theta = \pi/2$ (and therefore $d\theta = 0$), we can embed the 2D Schwarzschild curved space

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\phi^2$$

in a Euclidean 3-dimensional cylindrical space

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2 .$$

Show that $z = \sqrt{8M(r - 2M)}$, and plot the resulting embedding diagram.

It should be clear then why the coordinate r does NOT represent the radial distance to the center ($r = 0$), but interpreted as the circumference radius (= circumference, of fixed coordinate r , divided by 2π).