

General Relativity (I)

homework for week 4

due: week 6

1. [Christoffel Symbol from the metric] 60%

In the problem set 3 of the week 3 homework, we computed the Christoffel symbol $\Gamma_{\alpha\beta}^{\mu}$ for the spherical coordinates in the 3D Euclidean space according to the definition:

$$\frac{\partial \mathbf{e}_{\alpha}}{\partial x^{\beta}} = \Gamma_{\alpha\beta}^{\mu} \mathbf{e}_{\mu} .$$

(a) We can also compute the Cristoffel symbol directly from the metric. The flat spacetime described by a spherical coordinates (r, θ, ϕ) reads:

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} .$$

Compute the Cristoffel symbols via

$$\Gamma_{\alpha\beta}^{\mu} = \frac{1}{2} g^{\mu\nu} (g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}) . \quad (1)$$

From the relation, can you see

$$\Gamma_{\alpha\beta}^{\mu} = \Gamma_{\beta\alpha}^{\mu} .$$

(b) The relation shown in equation (1) is obtained by that the covariant derivative of the metric is zero:

$$\nabla_{\gamma} g_{\mu\nu} \equiv g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} - \Gamma_{\beta\gamma}^{\mu} g_{\alpha\mu} - \Gamma_{\alpha\gamma}^{\mu} g_{\mu\beta} = 0 . \quad (2)$$

Check $g_{\theta\theta;r} = g_{\phi\phi;r} = g_{\phi\phi;\theta} = 0$.

(c) With equation (1), show that

$$\Gamma_{\mu\alpha}^{\alpha} = \frac{1}{2} g^{\alpha\beta} g_{\alpha\beta,\mu} . \quad (3)$$

(d) Verify that

$$g_{,\mu} = g g^{\alpha\beta} g_{\alpha\beta,\mu} \quad (4)$$

where g is the determinant of the metric tensor as mentioned in the problem set 1 of the week 3 homework. Note that g is NOT a scalar.

(e) From equations (3) and (4), derive

$$\Gamma_{\mu\alpha}^{\alpha} = \frac{(\sqrt{-g})_{,\mu}}{\sqrt{-g}} . \quad (5)$$

As a result, the covariant derivative for an arbitrary field V^α ,

$$V^\alpha_{;\beta} = V^\alpha_{,\beta} + \Gamma^\alpha_{\mu\beta} V^\mu,$$

reduces to

$$V^\alpha_{;\alpha} = \frac{(\sqrt{-g} V^\alpha)_{,\alpha}}{\sqrt{-g}}.$$

Note that $\Gamma^\alpha_{\mu\beta}$ are not components of a single tensor, instead, the combination $V^\alpha_{,\beta} + \Gamma^\alpha_{\mu\beta} V^\mu$ are components of a single tensor, $V^\alpha_{;\beta}$.

(f) In fact, equation (2) is valid in *both* flat and curved spacetime. This implies that a *local* Cartesian inertial frame ($g_{\alpha\beta,\gamma}|_{\mathcal{P}} = \eta_{\alpha\beta,\gamma}|_{\mathcal{P}} = 0$ and $\Gamma^\alpha_{\beta\gamma}|_{\mathcal{P}} = 0$) can always be found at any point \mathcal{P} of a spacetime manifold, and the curved spacetime at the point \mathcal{P} can be *approximated* by

$$g_{\mu\nu} \approx \eta_{\mu\nu} + \frac{1}{2}(g_{\mu\nu,\alpha\beta}|_{\mathcal{P}})x^\alpha x^\beta.$$

It turns out the nonvanishing second order derivatives of the metric $g_{\mu\nu,\alpha\beta}$, and hence the first order derivatives of the Christoffel symbol are our probe for the spacetime curvature. The **Riemann curvature tensor** is defined by:

$$R^\alpha_{\beta\mu\nu} \equiv \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\sigma\mu} \Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu} \Gamma^\sigma_{\beta\mu},$$

and a spacetime is flat *if and only if* when Riemann curvature tensor is zero. That is,

$$R^\alpha_{\beta\mu\nu} = 0 \iff \text{flat spacetime.}$$

Verify, for example, $R^t_{rtr} = R^\theta_{r\theta r} = 0$, by using the Christoffel symbols obtained in (a).

(g)[bonus: additional 20%]

Verify also $R^\theta_{\phi\theta\phi} = 0$.

2. [length contraction and time dilation revisited] 40%

As there are typos in the problem set 2 in the week 1 homework, here we revisit the idea of length contraction and time dilation with a more detailed description.

For a inertial frame $\mathcal{O}' : (t', x', y', z')$ moving uniformly with respect to another inertial frame $\mathcal{O} : (t, x, y, z)$ along the x -axis with speed v , the connection between inertial frames are given by the *Lorentz transform*:

$$\begin{aligned} t' &= \gamma(t - vx/c^2) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

where c is the speed of light, and $\gamma = (1 - v^2/c^2)^{-1/2} \geq 1$.

(a) For a time interval dt' measured by clocks comoving with \mathcal{O}' (we call dt' as the **proper time**), show that the corresponding coordinate time interval dt measured by stationary clocks at \mathcal{O} follows

$$\boxed{dt = \gamma dt'} > dt'.$$

This is usually described by the slogan "moving clocks run slow."

(b) For a rod with its length dx' comoving with \mathcal{O}' (we call dx' as the **proper length**), show that the length of the rod measured at \mathcal{O} follows

$$\boxed{dx = dx'/\gamma} < dx'.$$

That is, a moving object's length is measured to be shorter than its proper length.