General Relativity (I)

homework for week 3

due: week 5

1. [metric tensor in flat and curved spacetime] 30%

In geometrized unit, the metric tensor in Minkowski space in Cartesian coordinate is $\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$:

(a) From $\eta^{\alpha\mu}\eta_{\mu\beta} \equiv \delta^\alpha_\beta$, where δ^α_β is the Kronecker delta, construct $\eta^{\alpha\beta}$.

(b) The determinant of the metric tensor is $\eta \equiv \det(\eta)$=$|\eta_{\alpha\beta}|$, where (η) is the matrix of $\eta_{\alpha\beta}$. Show that $\eta = -1$.

(c) The metric tensor in Minkowski space can also be written in terms of spherical coordinate:

$$\eta'_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} ,$$

compute $\eta'^{\alpha\beta}$ and $\eta' = \det(\eta')$.

(d) The *Schwarzschild metric* describes the spacetime outside a spherical, non-rotating star with mass M:

$$g_{\alpha\beta} = \begin{pmatrix} -(1-\frac{2M}{r}) & 0 & 0 & 0 \\ 0 & (1-\frac{2M}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} ,$$

compute $g^{\alpha\beta}$ and g.

2. [summation of indces and review of definitions] 10%
Simplify the following expressions:

(a) $\delta^\alpha_\beta u^\beta u^\alpha$

(b)$A_i g^{ij} g_{jk} B^k$

(c)$A^\alpha B_\alpha - A_\mu B^u$

(d)$\delta^i_j R^j - g^{ji} g_{kj} R^k$

(e)$g_{\mu\nu} u^\mu u^\nu - g^{\mu\nu} u_\mu u_\nu + k^\alpha k_\alpha - \vec{k}\cdot\vec{k}$

3. [four-velocity, four-acceleration, definition of dot product] 30%
A time-like worldline S can be parameterized by $\mathcal{S}(\tau)$,where τ is the proper time. Show that four-velocity $\vec{U}$ defined by

$$U^\alpha \equiv \frac{dx^\alpha}{d\tau} ,$$

and the four-acceleration $\vec{A}$ defined by

$$A^{\alpha} \equiv \frac{u^{\alpha}}{d\tau} \ ,$$

are orthogonal:

$$\vec{U} \cdot \vec{A} = 0.$$

4. [Christoffel Symbol] 30%
The (natural) basis vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ for spherical coordinates in the 3D Euclidean space has been derived in the problem set 3 of the week2 homework. The Christoffel symbol $\Gamma^{\mu}_{\alpha\beta}$ is defined by

$$\frac{\partial \mathbf{e}_\alpha}{\partial x^\beta} = \Gamma^{\mu}_{\alpha\beta} \mathbf{e}_\mu \ .$$

That is, $\Gamma^{\mu}_{\alpha\beta}$ represents the μth comppnent of $\dfrac{\partial \mathbf{e}_\alpha}{\partial x^\beta}$ component. Write down all the non-vanishing Christoffel symbols for a spherical coordinate and verify

$$\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha} \ .$$