General Relativity (I)

homework for week 4

due: week 6

1. [Christoffel Symbol from the metric] 60%
In the problem set 4 of the week 3 homework, we computed the Christoffel symbol $\Gamma^{\mu}_{\alpha\beta}$ for the spherical coordinates in the 3D Euclidean space according to the definition:

$$\frac{\partial \mathbf{e}_\alpha}{\partial x^\beta} = \Gamma^{\mu}_{\alpha\beta}\mathbf{e}_\mu \ .$$

(a) We can also compute the Cristoffel symbol directly from the metric. The flat spacetime described by a spherical coordinates (r, θ, ϕ) reads:

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \ .$$

Compute the Cristoffel symbols via

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}) \ . \tag{1}$$

From the relation, can you see

$$\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha} \ .$$

(b) The relation shown in equation (1) is obtained by that the covariant derivative of the metric is zero:

$$\nabla_\gamma g_{\mu\nu} \equiv g_{\alpha\beta\,;\gamma} = g_{\alpha\beta,\gamma} - \Gamma^{\mu}_{\beta\gamma}g_{\alpha\mu} - \Gamma^{\mu}_{\alpha\gamma}g_{\mu\beta} = 0 \ . \tag{2}$$

Check $g_{\theta\theta\,;r} = g_{\phi\phi\,;r} = g_{\phi\phi\,;\theta} = 0$.

(c) With equation (1), show that

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2} g^{\alpha\beta} g_{\alpha\beta\,,\mu} \ . \tag{3}$$

(d) Verify that

$$g_{,\mu} = g g^{\alpha\beta} g_{\alpha\beta\,,\mu} \tag{4}$$

where g is the determinant of the metric tensor as mentioned in the problem set 1 of the week 3 homework. Note that g is NOT a scalar.

(e) From equations (3) and (4), derive

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{(\sqrt{-g})_{,\mu}}{\sqrt{-g}} \ . \tag{5}$$

As a result, the covariant derivative for an arbitrary field V^α,

$$V^\alpha_{;\beta} = V^\alpha_{,\beta} + \Gamma^\alpha_{\mu\beta} V^\mu \ ,$$

reduces to

$$V^\alpha_{;\alpha} = \frac{(\sqrt{-g}V^\alpha)_{,\alpha}}{\sqrt{-g}} \ .$$

Note that $\Gamma^\alpha_{\mu\beta}$ are not components of a single tensor, instead, the combination $V^\alpha_{,\beta} + \Gamma^\alpha_{\mu\beta} V^\mu$ are components of a single tensor, $V^\alpha_{;\beta}$.

(f) In fact, equation (2) is valid in *both* flat and curved spacetime. This implies that a *local* Cartesian inertial frame ($g_{\alpha\beta,\gamma}|_{\mathcal{P}} = \eta_{\alpha\beta,\gamma}|_{\mathcal{P}} = 0$ and $\Gamma^\alpha_{\beta\gamma}|_{\mathcal{P}} = 0$) can always be found at any point $\mathcal{P}$ of a spacetime manifold, and the curved spacetime at the point $\mathcal{P}$ can be *approximated* by

$$g_{\mu\nu} \approx \eta_{\mu\nu} + \frac{1}{2}(g_{\mu\nu,\alpha\beta}|_{\mathcal{P}}) x^\alpha x^\beta \ .$$

It turns out the nonvanishing second order derivatives of the metric $g_{\mu\nu,\alpha\beta}$, and hence the first order derivatives of the Christoffel symbol are our probe for the spacetime curvature. The **Riemann curvature tensor** is defined by:

$$R^\alpha_{\beta\mu\nu} \equiv \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu} \ ,$$

and a spacetime is flat *if and only if* when Riemann curvature tensor is zero. That is,

$$R^\alpha_{\beta\mu\nu} = 0 \iff \text{flat} \quad \text{spacetime.}$$

Verify, for example, $R^t_{rtr} = R^\theta_{r\theta r} = 0$, by using the Christoffel symbols obtained in (a).

(g)[bonus: additional 20%]
Verify also $R^\theta_{\phi\theta\phi} = 0$.

2. [length contraction and time dilation revisited] 40%
Here we revisit the idea of length contraction and time dilation with a more detailed description.

For a inertial frame $\mathcal{O}' : (t', x', y', z')$ moving uniformly with respect to another inertial frame $\mathcal{O} : (t, x, y, z)$ along the x-axis with speec v, the connection between inertial frames are given by the *Lorentz transform*:

$$\begin{aligned} t' &= \gamma(t - vx/c^2) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

where c is the speed of light, and $\gamma = (1 - v^2/c^2)^{-1/2} \geq 1$.

(a) For a time interval dt' measured by clocks comoving with $\mathcal{O}'$ (we call dt' as the **proper time**), show that the corresponding coordinate time interval dt measured by stationary clocks at $\mathcal{O}$ follows

$$\boxed{\text{dt=}\gamma dt'} > dt' \ .$$

This is usually described by the slogan "moving clocks run slow."

(b) For a rod with its length dx' comoving with $\mathcal{O}'$ (we call dx' as the **proper length**), show that the length of the rod measured at $\mathcal{O}$ follows

$$\boxed{\text{dx= dx}'/\gamma} < dx' \ .$$

That is, a moving object's length is measured to be shorter than its proper length.