

General Relativity (I)

homework for week 13

due: week 15

1. [linearized gravity and GW] 20%

The **metric perturbation** to the flat spacetime $\eta_{\alpha\beta}$:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} , \quad (1)$$

where $|h_{\alpha\beta}| \ll 1$, can be used to describe the gravitational wave propagation in flat spacetime.

With the help of proper **gauge conditions**, and keeping terms linear in $h_{\alpha\beta}$, the solution to the **linearized EFE** (in vacuum) $G_{\alpha\beta} = 0$ reduces to a wave equation:

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)h^{\alpha\beta} = 0 \quad (2)$$

and the following example solution represents a transverse gravitational wave (GW) propagating along the z-direction:

$$h_{\alpha\beta}(t, z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_x & 0 \\ 0 & h_x & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} f(t - z) , \quad (3)$$

where $f(t - z)$ is a solution to the wave equation.

The non-zero sub-matrix in parenthesis in eqn. [3] can be written as $h_+ \mathbf{e}_1 + h_x \mathbf{e}_2$, where $\mathbf{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are basis of the "plus" and "cross" polarization mode, respectively.

(1) Show that, to the linear order, $g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$

(2) To obtain the result of eqn. [3], what gauge condition(s) has been applied?

2. [power of GW, equal mass binary] 20%

Let us consider the following system to estimate the power of gravitational wave (in **geometrized unit**): two stars of equal masses m with total separation R , orbiting in a circular orbit with angular frequency Ω .

Keeping only the relevant physical quantities, the amplitude for the perturbation $h_{\mu\nu}$ roughly follows

$$h \sim mR^2\Omega^2/r , \quad (4)$$

which corresponds to the **mass quadrupole** radiation.

(1) In terms of dimension, eqn. [4] can be written as $h \simeq v^2 m/r$, where v is the orbital velocity. Assuming $v^2 = O(0.1)$, estimate the perturbation amplitude h for a binary neutron star system with $M = M_\odot$ each, and located at a galaxy with $r \approx 10$ Mpc.

(2) It is expected that the energy density of the wave ϵ_{gw} is proportional to the square of the amplitude, therefore $\epsilon_{\text{gw}} \propto h^2$. From the dimension analysis, argue that

$$\epsilon_{\text{gw}} \propto \Omega^2 h^2,$$

(3) From the dimension analysis, argue that the energy flux F has the same dimension of the energy density ϵ .

(4) What's the relation between the luminosity(power) L and the flux F ? According to the relation, show that

$$L_{\text{gw}} \propto \Omega^2 h^2 \propto m^2 R^4 \Omega^6.$$

Note: both h and L are dimensionless in geometrized unit. The relation for L between geometrized unit to cgs unit follows $L[\text{cgs unit}] = c^5/G \times L[\text{geometrized unit}]$.

3. [chirp mass, unequal mass binary, orbital evolution due to GW] 60%

With the insights gained from problem set 2, let us now consider the following system with further details: two stars of unequal masses m_1 and m_2 with total separation R , orbiting in a circular orbit with angular frequency Ω .

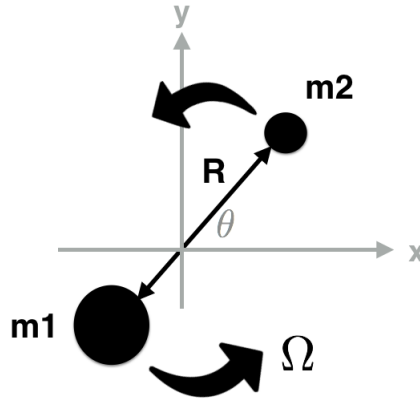


Figure 1: A unequal mass binary orbiting on the x-y plane in a circular orbit.

(1) Assuming the system is orbiting in the x-y plane, and putting the origin of coordinates at the center of mass of the system (as shown in **Fig 1**), verify that the xx-component of the mass quadrupole moment $M_{xx} = \mu R^2 \cos^2(\Omega t)$, where $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass.

(2) Since the gravitational radiation only depends on the time varying part of the quadrupole moment, show that we can rewrite M_{xx} as $M_{xx} = \frac{1}{2} \mu R^2 \cos^2(\Omega_{\text{gw}} t)$, with $\Omega_{\text{gw}} = 2\Omega$. Similarly, work out that $M_{yy} = -\frac{1}{2} \mu R^2 \cos^2(\Omega_{\text{gw}} t)$ and $M_{xy} = \frac{1}{2} \mu R^2 \sin^2(\Omega_{\text{gw}} t)$.

(3) Replacing m by μ , the power can be now rewrite as $L_{\text{gw}} \propto \mu^2 R^4 \Omega^6$. The separation R can be further eliminated by

$$R^3 = \frac{m_1 + m_2}{\Omega^2}.$$

Show that $L_{\text{gw}} \propto \mu^2 (m_1 + m_2)^{4/3} \Omega^{10/3}$.

(4) Define the **chirp mass** as

$$\mathcal{M} = \mu^{3/5} (m_1 + m_2)^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}. \quad (5)$$

Verify that

$$h \propto \frac{\mathcal{M}^{5/3} \Omega_{\text{gw}}^{2/3}}{r}. \quad (6)$$

and

$$L_{\text{gw}} \propto (\mathcal{M} \Omega_{\text{gw}})^{10/3}. \quad (7)$$

It turns out that the GW property is solely related to the chirp mass (rather than other combinations of individual masses)!

(5) The GW power is supplied by the orbital energy $E_{\text{orb}} = -m_1 m_2 / R$, that is, $-\frac{dE_{\text{orb}}}{dt} = L_{\text{gw}}$. From the relation, derive the frequency evolution of the system:

$$\dot{\Omega}_{\text{gw}} \propto \mathcal{M}^{5/3} \Omega_{\text{gw}}^{11/3}.$$

Remarkably, Chirp mass is again involved! How the system would “chirp” (increase of frequency) simply depends on the chirp mass. As a result, a chirping binary with a circular orbit can serve as a **standard candle**: the distance of the system can be inferred once h , Ω_{gw} , $\dot{\Omega}_{\text{gw}}$ are known.

(6) Similarly, show the period evolution of the system follows

$$\dot{P} \propto \mathcal{M}^{5/3} P^{-5/3}.$$

Hello again, chirp mass!