General Relativity (I)

homework for week 11

due: week 13

1. [time-like geodesic in the Schwarzschild spacetime] 40%
In geometrized unit, the *Schwarzschild spacetime* has the line element:

$$\boxed{\mathrm{ds}^2 = -(1-\tfrac{2M}{r})dt^2 + (1-\tfrac{2M}{r})^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2} \,,$$

where M is the mass of the black hole.

(a)The absence of t and ϕ in $g_{\mu\nu}$ implies that the covariant component of the four momentum p_t and p_ϕ are conserved along the trajectories of a freely moving particle. Explain why?

(b) Making use of the result of (a), for a test particle with mass m, we can define

$$P_t/m \equiv -E \,, \tag{1}$$

and

$$P_\phi/m \equiv L \,. \tag{2}$$

Show that the time-like geodesic has the form:

$$(\frac{dr}{d\tau})^2 = E^2 - V_{\mathrm{eff}}^2 \,, \tag{3}$$

with the effective potential

$$V_{\mathrm{eff}}^2 = (1-\frac{2M}{r})(1+\frac{L^2}{r^2}) \,. \tag{4}$$

See **Fig 1.** for the example profiles of $V_{\mathrm{eff}}^2(L^2)$.

(c) Show that for a stable circular orbit at radius r, the specific angular momentum L follows

$$L_{GR}^2 = \frac{Mr}{1-\frac{3M}{r}} \,, \tag{5}$$

and the corresponding specific energy E follows

$$E_{GR}^2 = \frac{(1-\frac{2M}{r})^2}{1-\frac{3M}{r}} \,, \tag{6}$$

(hint: apply $dV_{\mathrm{eff}}^2/dr = 0$, and $E^2 = V_{\mathrm{eff}}^2$ for circular orbits)

From **Fig 2.**, it is clear that particles must be *sub-Keplerian* when they enter the event horizon, as can be understood by the requirement of causality. In comparison, the specific angular momentum in the Newtonian case is $L_N^2 = Mr$: a stable circular orbit can be found at any radius.

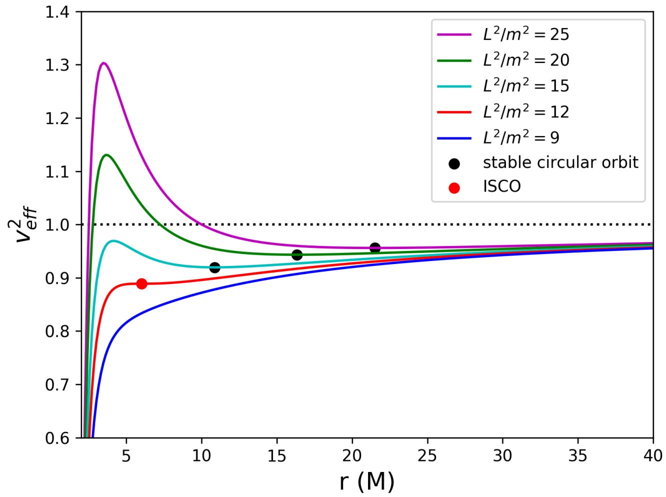

Figure 1: V^2_{eff} as function of different L^2, and the corresponding location of stable circular orbits for $L^2 = (25, 20, 15, 12)$.

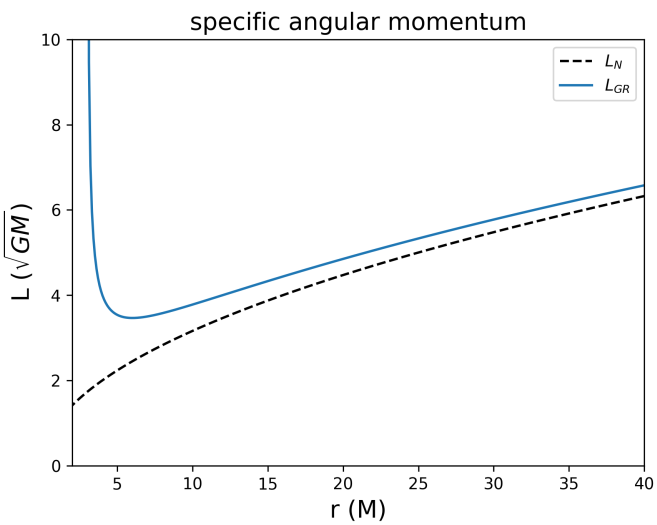

Figure 2: The angular momentum for a stable circular orbit rapidly increase within the ISCO.

(d) Show that there is no stable circular orbit when $L < 12M^2$ and there exists a *innermost stable circular orbit* **(ISCO)** at $r = 6M$. See also **Fig 1.** for reference. What is the corresponding E_{GR} at ISCO? (hint: solve for r by using eqn [3])

2. [observer with radial motion] 30%
(a) From eqn [3], argue that $E = 1$ for a free-fall particle ($L = 0$) falling from rest at infinity, and that $E = 0$ for a free-fall particle falling from rest at $r = 2M$.

(b) For a free-fall observer falls from rest at infinity, show that the proper time it takes to fall from $r = 2m$ to $r = 0$ is 4M/3.

(c) What mass of black hole would allow an observer in (b) to survive for a year inside the the horizon before she/he reaches the singularity?

3. [hovering observer] 30%
An astronaut use rocket-pack to stay hovering ($dr = d\theta = d\phi = 0$) at $R > 2M$.

(a) What is the four-velocity u^α of the astronaut?

(b) Show that $u^\alpha u_\alpha = -1$.

(c) Argue that it is impossible to stay hovering inside the horizon ($R < 2M$), since $g_{tt} > 0$ inside the horizon. [hint: can $u^\alpha u_\alpha = -1$ be satisfied in that case?]

4. [circular orbiting observer] bonus: 20%
An astronaut is circulating the black hole ($dr = d\theta = 0$) at $R > 6M$, following the geodesic.

(a) What is the four-velocity u^α of the astronaut?
(hint: We can easily construct the four velocity of the circular observer, by the relation

$$u^t = \frac{dt}{d\tau} = g^{tt} E \tag{7}$$

$$u^\phi = \frac{d\phi}{d\tau} = g^{\phi\phi} L \tag{8}$$

and the helps of eqns [5] and [6].)

(b) For the orbiting observer, what is the proper time $\triangle\tau_{\text{orb}}$ for one orbit?

(c) With the result of (a), show that

$$\frac{d\phi}{dt} = \sqrt{\frac{M}{r^3}} \tag{9}$$

and therefore the coordinate time elapses for one orbit is $\triangle t = 2\pi\sqrt{\frac{M}{r^3}}$.

(d) Based on the problem set 3, for the hovering observer, what is the corresponding proper time $\triangle\tau_{\text{hov}}$ for the time span $\triangle t$? One should get the relation:

$$\frac{\triangle\tau_{\text{hov}}}{\triangle\tau_{\text{orb}}} = \left(\frac{r-2M}{r-3M}\right)^{1/2} . \tag{10}$$

Note : unlike the twin paradox in special relativity, the observer follows the geodesic (the orbiting observer) find himself younger *than his sister who is hovering at the same radius! Such result become obvious by seeing that $ds^2 = g_{tt}\,dt^2$ for a hovering observer but $ds^2 = g_{tt}\,dt^2 + g_{\phi\phi}\,d\phi^2$ for a orbiting observer.*