General Relativity (I)

homework for week 15

due: week 18

1. [time-evoving universe] 30%
Consider the following example of an evolving Universe which is spatially **homogeneous** and **isotropic** at all cosmological times t:

$$ds^2 = -dt^2 + a(t)(dr^2 + r^2 d\Omega^2) \ , \tag{1}$$

where $a(t)$ is called the scale factor. Eqn (1) is a **comoving coordinate** as there is no g_{ti} terms and g_{tt} is independent of the spatial coordinates.

Representing $(0,1,2,3) = (t, r, \theta, \phi)$, the non-vanishing Christoffel symbols are:

$$\begin{array}{lll}
\Gamma^0_{11} = a\dot{a} & \Gamma^0_{22} = a\dot{a}r^2 & \Gamma^0_{33} = a\dot{a}r^2 \sin^2\theta \\
\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \dfrac{\dot{a}}{a} & \Gamma^1_{22} = -r & \Gamma^1_{33} = -r\sin^2\theta \\
\Gamma^2_{12} = \Gamma^3_{13} = \dfrac{1}{r} & \Gamma^2_{33} = -\sin\theta\cos\theta & \Gamma^3_{23} = \cot\theta
\end{array}$$

(a) The (ideal) fluid at rest in co moving coordinates can be described by $T^\alpha_\beta = \text{diag}(-\rho, p, p, p)$. By the 00-term of EFE, show that $R_{00} = T_{00} - \frac{1}{2} g_{00} T$ reduces to the so-called **acceleration equation** or the *second* Friedmann equation:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p)} \tag{2}$$

(b) Show that $g_{0\mu} T^{\mu\nu}_{;\nu} = 0$ reduces to the so-called **fluid equation**;

$$\boxed{\frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a}} \tag{3}$$

with the help of the equation of state:

$$p = \omega\rho \ , \tag{4}$$

where $\omega = 0$ for matter and $\omega = 1/3$ for radiation.

(c) Using Eqn (2) and (3) to obtain the (*first*) **Friedmann equation**:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho \tag{5}$$

[hint: eqn (5) can be inferred from $\dfrac{d}{dt}(\dot{a}^2) = \dfrac{d}{dt}(\dfrac{8\pi}{3}\rho a^2)$]

2. [Cosmological model: matter + radiation] 70%:
Eqn (1) is actually one possible realization of the **Robertson-Walker metric**

$$ds^2 = -dt^2 + a(t)(\frac{1}{1-\kappa r^2} dr^2 + r^2 d\Omega^2) \ , \tag{6}$$

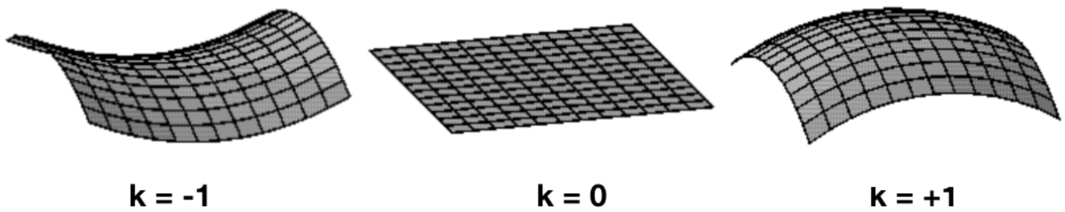

Figure 1: The geometry of spaces corresponding to different curvature. Figure adopted from here.

where the curvature parameter $\kappa = (1, 0, -1)$ (which corresponds to a [closed, open, open] spatial topology, respectively) describes different 3D geometry (see **Fig. 1**).

In turns out that, after including the κ contribution, both Eqn. (2) and (3) would remain the same, while the Friedmann equation becomes

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{\kappa}{a^2}} \,. \tag{7}$$

Technically, we need to derived the above equation from the EFE, with Christoffel symbols including the κ contributions.

(a) As Eqn (2) does not depend on κ, argue that *a Universe includes only matter and radiation must be decelerating* ($\ddot{a} < 0$).

(b) Eqn (3) implies the energy density evolution as a function of the scale factor. Show that

$$\rho(a) \propto a^{-3(1+\omega)} \,.$$

From the relation we can inferred that the Universe is dominated by the radiation (or matter) energy density when a is sufficiently small (or large).

(c) Show that the second term in Eqn (7) is negligible when a is sufficiently small.

(d) Show that a **matter dominated** epoch, Eqn (7) can be rewritten in a form

$$\dot{a}^2 = \frac{\mathcal{C}}{a} - \kappa \,, \tag{8}$$

where $\mathcal{C}$ is a constant. The requirement $\dot{a}^2 > 0$ can always be satisfied when $\kappa = -1$ and $\kappa = 0$, but not for $\kappa = 1$ when a is sufficiently large.

(e) For cases of either $k = 0$ or a is sufficiently small, show that Eqn (7) implies

$$a(t) \propto t^{2/3(1+\omega)} \,.$$

(f) Our current cosmological observations is consistent with a flat Universe ($\kappa = 0$). With $\kappa = 0$, show that the energy density evolution of radiation $\rho_\gamma(t)$ in the **radiation dominated** epoch follows $\rho_\gamma(t) \propto t^a$, and the energy density evolution of matter $\rho_m(t)$ in the **matter dominated** epoch follows $\rho_m(t) \propto t^b$, with $a = b = -2$.

(g) The radiation energy and its temperature has the relation $\rho_\gamma \propto T^4$. Show that, in the early Universe (dominated by radiation), the temperature decreases with time by $T \propto t^{-1/2}$.

Further estimation shows $t \simeq \frac{10^{20}}{T^2}$, where t is in unit of *second* and T is in unit of K. As **nucleosyntheis** is taking place at $T \simeq 10^9 K$, corresponds to $t \simeq 10^2$ sec. As the Universe further cools down to $T \simeq 3000K$

(corresponds to the typical atomic binding energy), neutral atoms form and the photon is therefore decoupled with matters. The Universe becomes transparent to photons! Such **photon-decoupling** time is about $t \simeq 10^{13}$ sec.