General Relativity (I)

homework for week 12

due: week 14

In Boyer-Lindquist coordinate. the Kerr geometry which describes the spacetime of a rotating black hole can be written as (in terms of geometrized unit):

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\rho^2}dt^2 - 2a\frac{2Mr\sin^2\theta}{\rho^2}dtd\phi + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \ , \tag{1}$$

where

$$\Delta \equiv r^2 - 2Mr + a^2 \ ,$$
$$\rho^2 \equiv r^2 + a^2\cos^2\theta \ ,$$

As the angular momentum of the black hole, J, has dimension m^2, conventionally defines $a = J/M$ (and the dimensionless spin parameter can be defined by $a* = a/M$). When $a = 0$, the metric reduces to the Schwarzschild metric.

1. [horizon, ergosphere, and zero angular momentum observer] 80%
(a) Show that it is possible for a observer remain at constant spatical coordinates within the *ergosphere* where $g_{tt} < 0$.

As a result, the surface

$$r_0 = M + \sqrt{M^2 - a^2\cos^2\theta} \tag{2}$$

where $g_{tt} = 0$ is called **static limit**. If $a > 0$, the static limit is located outside the **event horizon**

$$r_+ = M + \sqrt{M^2 - a^2} \ , \tag{3}$$

where $\Delta = 0$. The region outside the event horizon but within the static limit is the region where a rotating black hole stores its rotational energy, and therefore it has the name **ergosphere**. See **Fig. 1** for an example of the structure of a rotating black hole.

(b) Show that even for a zero angular momentum particle ($p_t = 0$), its coordinate angular velocity

$$\omega(r,\theta) \equiv \frac{d\phi}{dt} = \frac{p^\phi}{p^t} = \frac{g^{t\phi}}{g^{tt}} = -\frac{g_{t\phi}}{g_{\phi\phi}}$$

is not zero due to the non-zero $g_{t\phi}$ term. The effects resulting from the spacetime rotation is usually called the **frame dragging effect** or **dragging of inertial frame**, see also **Fig. 1** for an example. Can you see $\omega \propto a/r^3$, which implies that the particle must *co-rotates* with the same direction black hole rotates for a distant observer.

(c) It is possible to define a **zero angular momentum observer (ZAMO)** observer who carries a rocket-pack and stays at the same $r-$coordinate and $\theta-$coordinate. The 4-velocity of a ZAMO observer can be written as

$$u^\alpha = u^t(1, 0, 0, \omega) \ .$$

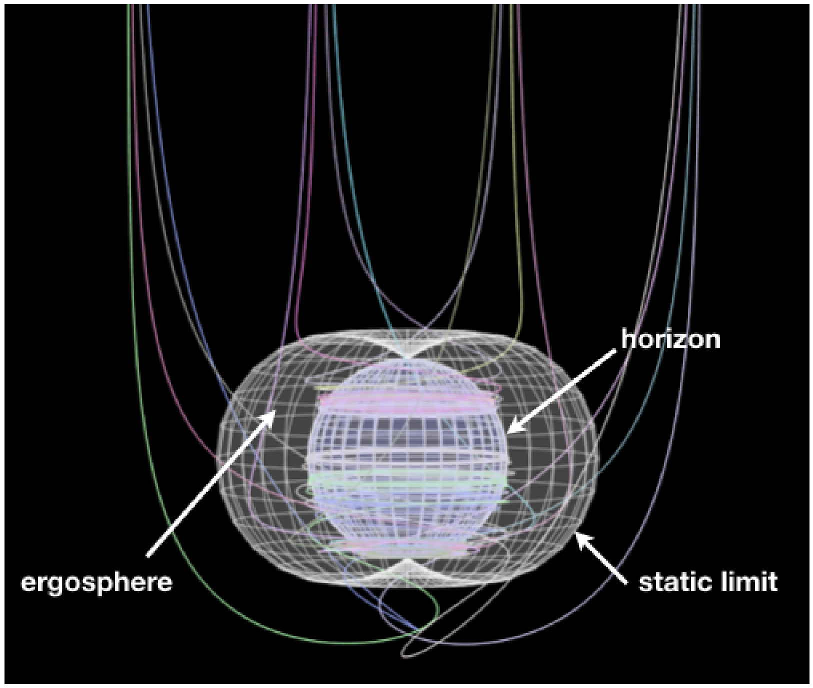

Figure 1: Example of frame dragging effect: color coded light rays (incoming along the rotational axis) captured by the rotating black hole. Students who are interested in exploring null geodesics in Kerr spacetime can refer to the free educational software: Odyssey_Edu. A instruction (in Chinese) for how to use Odyssey_Edu is available here.

Compute u^t by applying $u^\alpha u_\alpha = -1$.

Note that the 4-velocity of a ZAMO reduces to Schwarzwschild hovering observers, as in problem 3(a) of week 11 homework.

(d) Show that the proper time elapse for a ZAMO observer is

$$d\tau|_{\text{ZAMO}} = |g_{tt} - \omega^2 g_{\phi\phi}|^{1/2} dt\ .$$

2. [Hawking's area theorem] 20%
The area of the horizon is

$$A = \int_0^{2\pi} \int_0^{\pi} (\sqrt{g_{\theta\theta} g_{\phi\phi}}|_{r=r+}) d\theta d\phi = 4\pi(r_+^2 + a^2) = 8\pi M[M + \sqrt{M^2 - a^2}]\ . \tag{4}$$

Hawking's area theorem (1972) states that the horizon cannot decrease in time.

(a) Show that, for a fixed M the largest area is obtained when $a = 0$. This implies that spinning down a rotating black hole (via, e.g., **Penrose process** (1971) or **Blandford-Znajek process** (1977) discussed during the class) would not violated the are theorem.

(b) Show that the *maximum* energy released for a collision event of two Schwarzschild black holes (with equal mass M) is $(2 - \sqrt{2})M$.