VIETNAM NATIONAL UNIVERSITY HO CHI MINH CITY
HO CHI MINH CITY UNIVERSITY OF TECHNOLOGY

PROJECT REPORT

***Subject*: Calculus 2**

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***Class*: CC14 – GROUP B**

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School year 2022 – 2023

ASSIGNMENT FOR CALCULUS II
Group: B

Question 1. Show that the function $u(x,t) = \dfrac{1}{2a\sqrt{\pi t}}\,\mathrm{e}^{-\frac{(x-b)^2}{4a^2t}}$ (a and b are constants) satisfies the differential equation $\dfrac{\partial u}{\partial t} = a^2\dfrac{\partial^2 u}{\partial x^2}$.

Question 2. Find y' and y'' of the function $y = y(x)$ given implicitly by $y = 2x\arctan\dfrac{y}{x}$.

Question 3. Find the local maximum and minimum values and saddle point(s) of the function $z = (x^2+y^2)\,\mathrm{e}^{-(x^2+y^2)}$

Question 4. Evaluate $I = \displaystyle\iint_D \left|\frac{x+y}{\sqrt{2}} - x^2 - y^2\right| dxdy$ where $D = \{x^2+y^2 \leqslant 1\}$.

Question 5. Find the volume of the solid bounded by the surfaces: $z = x^2+y^2,\ x^2+y^2 = x,\ x^2+y^2 = 2x,\ z = 0$

Question 6. Evaluate $I = \displaystyle\iiint_V \frac{dV}{(1+x+y+z)^3}$ where $V = \{x+y+z=1,\ x=0,\ y=0,\ z=0\}$.

Question 7. Evaluate the line integral

$$\int_{AmO} (\mathrm{e}^x\sin y - my)\,dx + (\mathrm{e}^x\cos y - m)\,dy$$

where AmO is the upper half of the semicircle $x^2+y^2 = ax$ from the point $A(a,0)$ to the point $O(0,0)$, (m is a constant and $a > 0$).

Question 8. Evaluate the surface integral

$$I = \iint_S (y-z)dydz + (z-x)dxdz + (x-y)dxdy$$

where S is the part of the cone $x^2+y^2 = z^2,\ 0 \leqslant z \leqslant h$ with the positive outward orientation.

Question 9. Find the area of the region bounded by the curves $xy = a^2,\ xy = 2a^2,\ y = x,\ y = 2x \quad (x>0,\ y>0)$.

Question 10. Find the area of the part of the sphere $x^2+y^2+z^2 = a^2$ that lies inside the cylinder $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, \quad (0 < b \leqslant a)$.

QUESTION 1

① Show that the function $u(x,t) = \dfrac{1}{2a\sqrt{\pi t}}\, e^{-\frac{(x-b)^2}{4a^2t}}$ (a, b are const) satisfies the differential equation $\dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}$

$$\bullet\ U_t = -\frac{1}{2}\cdot\frac{1}{2a\sqrt{\pi}\ t\sqrt{t}}\cdot e^{-\frac{(x-b)^2}{4a^2t}} + \frac{(x-b)^2}{4a^2t^2}\cdot\frac{1}{2a\sqrt{at}}\cdot e^{-\frac{(x-b)^2}{4a^2t}}$$

$$= e^{-\frac{(x-b)^2}{4a^2t}}\cdot\frac{1}{8a^3t^2\sqrt{t}}\left(-2a^2t + (x-b)^2\right)$$

$$\bullet\ U_x = -\frac{2(x-b)}{2a\sqrt{at}}\cdot\frac{e^{-\frac{(x-b)^2}{4a^2t}}}{4a^2t} = -\frac{(x-b)}{2a\sqrt{at}}\cdot\frac{e^{-\frac{(x-b)^2}{4a^2t}}}{4a^2t}$$

$$\bullet\ U_{xx} = \frac{-e^{-\frac{(x-b)^2}{4a^2t}}}{4a^2t\cdot a\sqrt{at}} + \frac{2(x-b)^2}{a\sqrt{\pi t}}\cdot\frac{1}{(4a^2t)^2}\cdot e^{-\frac{(x-b)^2}{4a^2t}} = \frac{e^{-\frac{(x-b)^2}{4a^2t}}}{8a^3t^2\sqrt{t}}\left(-2t + \frac{(x-b)^2}{a^2}\right)$$

$$\Rightarrow U_t = a^2U_{xx} \qquad \Rightarrow \frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$$

QUESTION 2

② Find y' and y'' of the function $y = y(x)$ given implicitly by $y = 2x \arctan \frac{y}{x}$.

Consider $y = 2x \arctan \frac{y}{x}$

Consider: $y = 2x \arctan \frac{y}{x}$

$$\Leftrightarrow \frac{y}{x} = 2 \arctan \frac{y}{x}$$

$$\Leftrightarrow \left(\frac{y}{x}\right)' = \left(2 \arctan \frac{y}{x}\right)'$$

$$\Leftrightarrow \frac{y'x - y}{x^2} = 2 \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{y}{x}\right)'$$

$$\Leftrightarrow \frac{y'x - y}{x^2} = 2 \cdot \frac{\frac{y'x-y}{x^2}}{1 + \frac{y^2}{x^2}}$$

$$\Leftrightarrow \frac{y'x - y}{x} - 2\,\frac{y'x - y}{x^2 + y^2} = 0$$

$$\Leftrightarrow (y'x - y)\left(\frac{1}{x^2} - \frac{2}{x^2+y^2}\right) = 0$$

$$\Leftrightarrow (y'x - y)\left(\frac{1}{x^2} - \frac{2}{x^2+y^2}\right) = 0$$

$$\Leftrightarrow y'\left(\frac{1}{x} - \frac{2x}{x^2+y^2}\right) = \frac{y}{x^2} - \frac{2y}{x^2+y^2}$$

$$\Leftrightarrow y' \cdot \frac{y^2 - x^2}{x(x^2+y^2)} = \frac{y^3 - yx^2}{x^2(x^2+y^2)}$$

$$\Leftrightarrow y' = \frac{y(y^2 - x^2) \cdot x \cdot (x^2+y^2)}{(y^2 - x^2) \cdot x^2 (x^2+y^2)} = \frac{y}{x}$$

Also: $y'' = \left(\frac{y}{x}\right)' = \frac{y'x - y}{x^2} = \frac{\frac{y}{x} \cdot x - y}{x^2} = 0$

QUESTION 3

③ Find the local maximum and minimum values and saddle point(s) of the function

$f = z = (x^2+y^2)\, e^{-(x^2+y^2)}$ $\quad$ * Critical point: $f_x = 2x.e^{-(x^2+y^2)} - 2x(x^2+y^2)e^{-(x^2+y^2)} = 0$

$f_y = 2y.e^{-(x^2+y^2)} - 2y(x^2+y^2)e^{-(x^2+y^2)} = 0$

$\Rightarrow \left[\begin{array}{l} x=0 \\ x^2+y^2=1 \end{array}\right. \wedge \left[\begin{array}{l} y=0 \\ x^2+y^2=1 \end{array}\right. \Rightarrow A(0,0) \quad B(x_1, y_1)\ \{x^2+y^2=1\}$ are critical points

* Second derivative: $f_{xx} = 2e^{-(x^2+y^2)} - 4x^2e^{-(x^2+y^2)} - 2(3x^2+y^2)e^{-(x^2+y^2)} + 4x^2(x^2+y^2)e^{-(x^2+y^2)}$

$f_{xy} = -4xye^{-(x^2+y^2)} - 2x(2y)e^{-(x^2+y^2)} + 4xy(x^2+y^2)e^{-(x^2+y^2)}$

$f_{yy} = 2e^{-(x^2+y^2)} - 4y^2e^{-(x^2+y^2)} - 2(x^2+3y^2)e^{-(x^2+y^2)} + 4y^2(x^2+y^2)e^{-(x^2+y^2)}$

$\Delta = f_{xx}f_{yy} - f_{xy} = e^{-2(x^2+y^2)}[2-4x^2-6x^2-2y^2+4x^2(x^2+y^2)][2-4y^2-6y^2-2x^2+4y^2(x^2+y^2)]$

$-[-8xy+4xy(x^2+y^2)]^2e^{-2(x^2+y^2)}$

At $A(0;0) \Rightarrow \Delta = 1.2.2-0 = 4>0 \Rightarrow A(0;0)$ is local minimum

$B(x_1,y_1)\ \{x^2+y^2=1\} \Rightarrow \Delta = e^{-2}(2-10x^2-2y^2+4x^2)(2-10y^2-2x^2+4y^2) - (-8xy+4xy)^2.e^{-2}$

$= e^{-2}(2-6x^2-2y^2)(2-6y^2-2x^2) - 16x^2y^2e^{-2} = e^{-2}(-4x^2)(-4y^2) - 16x^2y^2e^{-2} = 0$

$\Rightarrow$ Cannot conclude point $B(x_1,y_1)$ is local maximum, minimum or saddle point

We evaluate f around $x^2+y^2=1$ $\quad t = x^2+y^2 \Rightarrow f = t^2e^{-t^2} \Rightarrow f_t = 2te^{-t^2} - 2t^3e^{-t^2} = 0$

$\Rightarrow t=0 \vee t=-1 \vee t=1$

x	$-\infty$		-1		0		1		$+\infty$
f_t		$+$	0	$-$	0	$+$	0	$-$	
f	0	$\nearrow$	e^{-1}	$\searrow$	0	$\nearrow$	e^{-1}	$\searrow$	0

We see that $\begin{cases} \text{when } t<1 \to f \geq e^{-1} \\ t>1 \to f \geq e^{-1} \end{cases}$

So we conclude that every point $B(x_1, y_1)\ \{x^2+y^2=1\}$ is local maximum

So f has $\begin{cases} A(0;0) \text{ is local minimum point} \\ B(x_1,y_1)\ \{x^2+y^2=1\} \text{ is local maximum point} \\ \text{No saddle point} \end{cases}$

QUESTION 4

$$= -e^{a}\sin a + ma + ma^2\frac{\pi}{4}$$

④ Evaluate $I = \iint_D \left| \frac{x+y}{\sqrt{2}} - x^2 - y^2 \right| dx\,dy$ where $D = \{x^2+y^2 \le 1\}$ (C_1)

(C_2): $\frac{x+y}{\sqrt{2}} - x^2 - y^2 = 0 \Leftrightarrow \left(x - \frac{1}{2\sqrt{2}}\right)^2 + \left(y - \frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{4}$

$(C_1)\cap(C_2)$: $\left(x - \frac{1}{2\sqrt{2}}\right)^2 + \left(y - \frac{1}{2\sqrt{2}}\right)^2 - \frac{1}{4} = x^2+y^2 = 1 \Leftrightarrow \frac{x+y}{\sqrt{2}} = 1 \Leftrightarrow x = \sqrt{2} - y$ and $x^2+y^2=1$

$\Rightarrow 2y^2 - 2\sqrt{2}y + 1 = 0 \Rightarrow (\sqrt{2}y - 1)^2 = 0 \Rightarrow y = \frac{1}{\sqrt{2}} \Rightarrow x = \frac{1}{\sqrt{2}}$

We take a point $I\left(\frac{1}{2\sqrt{2}}; \frac{1}{2\sqrt{2}}\right)$ inside $C_2 \Rightarrow \frac{x+y}{\sqrt{2}} - x^2 - y^2 = \frac{1}{4} > 0$

So every point lies inside C_2 will make $\left(\frac{x+y}{\sqrt{2}}-x^2-y^2\right)$ positive and every point lies outside C_2 will make $\left(\frac{x+y}{\sqrt{2}}-x^2-y^2\right)$ negative

$(C_1): x^2+y^2\leq 1$ Polar coordinate $\begin{cases} x=r\cos\varphi & y=r\sin\varphi \\ 0\leq r\leq 1 & 0\leq\varphi\leq 2\pi\end{cases}$

$(C_1)\cap(C_2)$: using polar coordinate $\frac{x+y}{\sqrt{2}}-x^2-y^2=0 \Rightarrow \frac{r(\cos\varphi+\sin\varphi)}{\sqrt{2}}-r^2=0$

$$\Rightarrow \begin{array}{l} r=0 \\ r=\frac{\cos\varphi+\sin\varphi}{\sqrt{2}}=\frac{\sqrt{2}\sin\left(\varphi+\frac{\pi}{4}\right)}{\sqrt{2}}=\sin\left(\varphi+\frac{\pi}{4}\right)\end{array}$$

$I=\iint_D\left|\frac{x+y}{\sqrt{2}}-x^2-y^2\right|dx\,dy$. We devide I into 2 areas : 1st inside (C_2); 2nd outside (C_2), inside C_1

$$I=\int_0^{2\pi}\int_0^{\sin(\varphi+\frac{\pi}{4})}\left(\frac{r(\cos\varphi+\sin\varphi)}{\sqrt{2}}-r^2\right)r\,dr\,d\varphi-\int_0^{2\pi}\int_{\sin(\varphi+\frac{\pi}{4})}^{1}\left(\frac{r(\cos\varphi+\sin\varphi)}{\sqrt{2}}-r^2\right)r\,dr\,d\varphi$$

$$=\int_0^{2\pi}\int_0^{\sin(\varphi+\frac{\pi}{4})}\left(r^2\sin\left(\varphi+\frac{\pi}{4}\right)-r^3\right)dr\,d\varphi-\int_0^{2\pi}\int_{\sin(\varphi+\frac{\pi}{4})}^{1}\left[r^2\sin\left(\varphi+\frac{\pi}{4}\right)-r^3\right]dr\,d\varphi$$

$$=\int_0^{2\pi}\sin\left(\varphi+\frac{\pi}{4}\right)\cdot\frac{r^3}{3}-\frac{r^4}{4}\Bigg)\Bigg|_0^{\sin(\varphi+\frac{\pi}{4})}-\int_0^{2\pi}\left[\sin\left(\varphi+\frac{\pi}{4}\right)\frac{r^3}{3}-\frac{r^4}{4}\right]\Bigg|_{\sin(\varphi+\frac{\pi}{4})}^{1}d\varphi$$

$$=\int_0^{2\pi}\left[\sin\left(\varphi+\frac{\pi}{4}\right)\right]^4\frac{1}{12}d\varphi-\int_0^{2\pi}\frac{\sin(\varphi+\frac{\pi}{4})}{3}\left(1-\sin\left(\varphi+\frac{\pi}{4}\right)^3\right)-\frac{1}{4}\left(1-\sin\left(\varphi+\frac{\pi}{4}\right)^4\right)d\varphi$$

$$=\frac{1}{12}\int_0^{2\pi}\left[\sin\left(\varphi+\frac{\pi}{4}\right)\right]^4d\varphi-\int_0^{2\pi}\left[-\frac{1}{4}+\frac{\sin(\varphi+\frac{\pi}{4})}{3}-\frac{1}{12}\sin\left(\varphi+\frac{\pi}{4}\right)^4\right]d\varphi$$

$$=\frac{1}{6}\int_0^{2\pi}\left(\sin\left(\varphi+\frac{\pi}{4}\right)\right)^3d\varphi-\int_0^{2\pi}\left(-\frac{1}{4}+\frac{\sin\left(\varphi+\frac{\pi}{4}\right)}{3}\right)d\varphi$$

$$A=\int_0^{2\pi}\left(\sin\left(\varphi+\frac{\pi}{4}\right)\right)^3d\varphi \qquad \left(\sin\left(\varphi+\frac{\pi}{4}\right)^2\right)^2=\left(\frac{1-\cos\left(2\varphi+\frac{\pi}{2}\right)}{2}\right)^2$$

$$=\frac{1}{4}-\frac{1}{2}\cos\left(2\varphi+\frac{\pi}{2}\right)+\frac{1}{4}\cos\left(2\varphi+\frac{\pi}{2}\right)^2=\frac{1}{4}-\frac{1}{2}\cos\left(2\varphi+\frac{\pi}{2}\right)+\frac{1}{4}\left(\frac{1+\cos(4\varphi+\pi)}{2}\right)$$

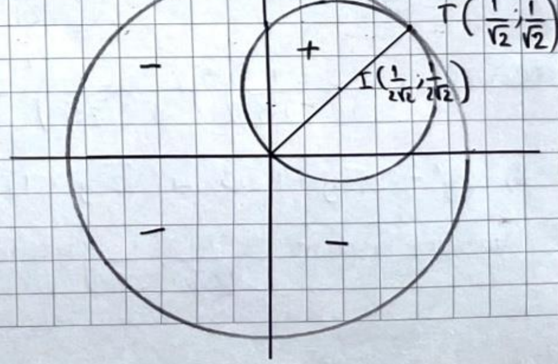
$$=\frac{3}{8}-\frac{1}{2}\cos\left(2\varphi+\frac{\pi}{2}\right)+\frac{1}{8}\cos(4\varphi+\pi)$$

$$A=\int_0^{2\pi}\left[\frac{3}{8}-\frac{1}{2}\cos\left(2\varphi+\frac{\pi}{2}\right)+\frac{1}{8}\cos(4\varphi+\pi)\right]d\varphi=\frac{3}{8}\varphi-\frac{1}{2}\cdot\frac{1}{2}\sin\left(2\varphi+\frac{\pi}{2}\right)+\frac{1}{8}\cdot\frac{1}{4}\sin(4\varphi+\pi)\Bigg|_0^{2\pi}$$

$$=\frac{3}{8}(2\pi-0)-\frac{1}{4}\sin\left(4\pi+\frac{\pi}{2}\right)-\sin\frac{\pi}{2}+\frac{1}{32}(\sin(8\pi+\pi)-\sin\pi)=\frac{3}{4}\pi-0+0=\frac{3}{4}\pi$$

$$\Rightarrow I=\frac{1}{6}\cdot\frac{3}{4}\pi-\left(-\frac{1}{4}\varphi-\frac{\cos\left(\varphi+\frac{\pi}{4}\right)}{3}\right)\Bigg|_0^{2\pi}=\frac{1}{8}\pi-\left(-\frac{1}{4}(2\pi-0)-\frac{1}{3}\left(\cos\left(\frac{\pi}{4}+2\pi\right)-\cos\left(\frac{\pi}{4}\right)\right)\right)$$

$$=\frac{1}{8}\pi+\frac{1}{2}\pi+0=\frac{5}{8}\pi$$



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QUESTION 5

⑤ Find the volume of the solid bounded by the surface : $z = x^2 + y^2$, $x^2+y^2 = x$, $x^2+y^2 = 2x$, $z = 0$

- We have : $\cos^4 x = (1 - \sin^2 x)^2$

$$= 1 - 2\sin^2 x + \sin^4 x$$

So : $\cos^4 x + \cos^4 x = (1 - 2\sin^2 x) + (\sin^4 x + \cos^4 x)$

$$= \cos(2x) + (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x . \cos^2 x$$

$$= \cos(2x) + 1 - 2\sin^2 x . \cos^2 x$$

$$(=) \quad 2\cos^4 x = \cos(2x) + 1 - \frac{1}{2} . (2\sin x . \cos x)^2$$

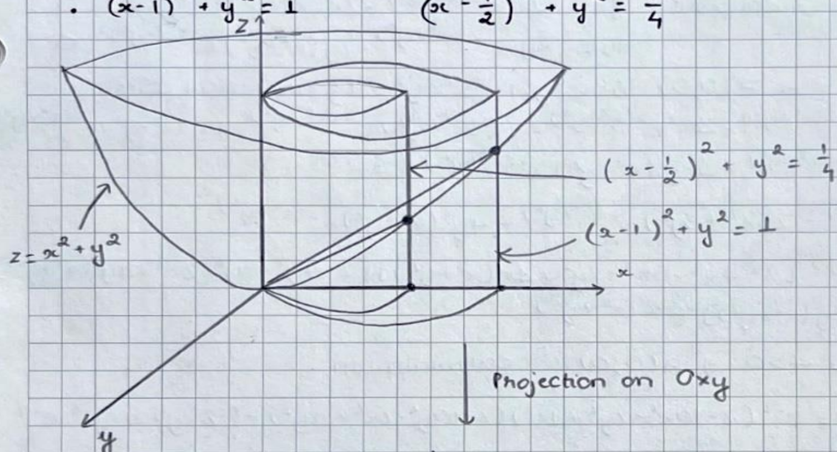
$$= \cos(2x) + 1 - \frac{1}{2}\sin^2(2x)$$

$$= \cos(2x) + 1 - \frac{1}{4}(1 - \cos(4x))$$

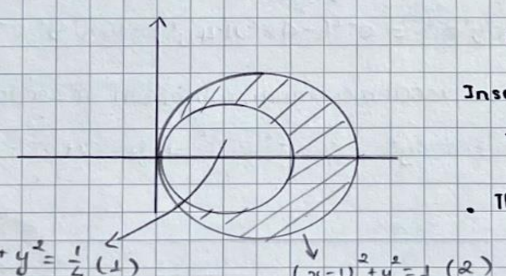
$$= \cos(2x) + \frac{\cos(4x)}{4} + \frac{3}{4}$$

$$\Rightarrow \cos^4 x = \frac{\cos(2x)}{2} + \frac{\cos(4x)}{8} + \frac{3}{8}$$

- $(x-1)^2 + y^2 = 1$ $\qquad (x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$



Projection on Oxy



Let $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$

Insert x, y into (1) & (2)

$$\rightarrow \begin{cases} r_1 = \cos\varphi \\ r_2 = 2\cos\varphi \end{cases}$$

- The backout has : $x = r\cos\varphi$

$y = r\sin\varphi$

$r \in [\cos\varphi, 2\cos\varphi]$

$\varphi \in [\frac{-\pi}{2}, \frac{\pi}{2}]$

- $V = \iint_D \int_0^{x^2+y^2} 1 \, dV = \iint_D x^2 + y^2 dA$

$$= \int_{-\pi/2}^{\pi/2} \int_{\cos\varphi}^{2\cos\varphi} r^2 . r \, dr \, d\varphi = \int_{-\pi/2}^{\pi/2} \int_{\cos\varphi}^{2\cos\varphi} r^3 dr \, d\varphi = \int_{-\pi/2}^{\pi/2} \left(\frac{r^4}{4}\right)\Bigg|_{\cos\varphi}^{2\cos\varphi} d\varphi$$

$$= \int_{-\pi/2}^{\pi/2} \frac{16(\cos\varphi)^4}{4} - \frac{(\cos\varphi)^4}{4} d\varphi = \int_{-\pi/2}^{\pi/2} \frac{15}{4}\cos^4\varphi \, d\varphi = \int_{-\pi/2}^{\pi/2} \left[\frac{15}{8}\cos(2x) + \frac{15}{32}\cos(4x) + \frac{45}{32}\right] d\varphi$$

$$= \left(\frac{15}{16}\sin(2x) + \frac{15}{128}\sin(4x) + \frac{45}{32}x\right)\Bigg|_{-\pi/2}^{\pi/2} = \frac{45}{32}\pi \approx 4.4179$$

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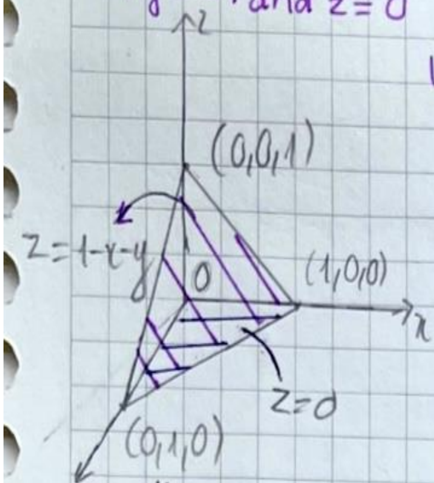
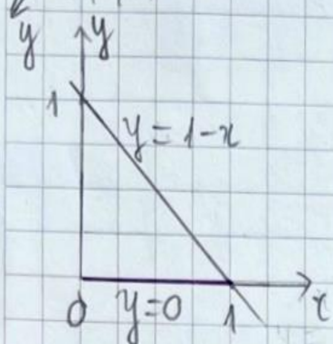
QUESTION 6

(6) Evaluate $I=\iiint_V \frac{1}{(x+y+z+1)^3}\,dV$, where V is a solid bounded by planes $x+y+z=1$, $x=0$, $y=0$, and $z=0$





We have: $E=\{x,y,z \mid 0\le x\le 1;\ 0\le y\le 1-x;\ 0\le z\le 1-x-y\}$

·Evaluate the Integral

$$I=\iiint_V \frac{1}{(1+x+y+z)^3}\,dV=\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(1+x+y+z)^3}\,dz\,dy\,dx$$

$$=\int_0^1\int_0^{1-x} -\frac{1}{(1+x+y+z)^2\cdot 2}\Big|_0^{1-x-y}\,dy\,dx=\int_0^1\int_0^{1-x} -\frac{1}{2}\cdot\left[\frac{1}{4}-\frac{1}{(1+x+y)^2}\right]dy\,dx$$

$$=-\frac{1}{2}\int_0^1\left[\frac{1}{4}y+\frac{1}{1+x+y}\right]\Big|_0^{1-x}dx=-\frac{1}{2}\cdot\int_0^1\left[\frac{1}{4}(1-x)+\frac{1}{2}-\frac{1}{1+x}\right]dx$$

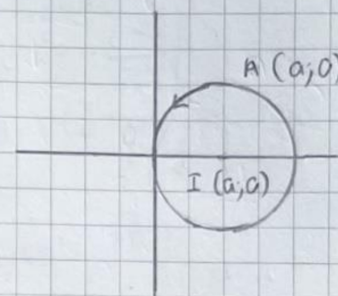
$$=-\frac{1}{2}\cdot\int_0^1 \frac{3}{4}-\frac{x}{4}-\frac{1}{1+x}\,dx=-\frac{1}{2}\cdot\left[\frac{3x}{4}-\frac{x^2}{8}-\ln(1+x)\right]\Big|_0^1$$

$$=-\frac{1}{2}\cdot\left(\frac{3}{4}-\frac{1}{8}-\ln 2\right)=-\frac{5}{16}+\frac{1}{2}\ln 2$$

(8) Evaluate the

QUESTION 7

③ Evaluate the line integral $\int_{AmO} (e^x \sin y - my)\,dx + (e^x \cos y - m)\,dy$ where AmO is the upper half of the semicircle $x^2+y^2=ax$ from point $A(a,0)$ to $O(0,0)$ (m is const and $a>0$)



(C_1): $x^2+y^2=ax \Leftrightarrow (x-a)^2+y^2=a^2$

$y=+\sqrt{a^2-(x-a)^2}$ $\quad dy=\dfrac{-(x-a)}{\sqrt{a^2-(x-a)^2}}\,dx$ (x goes from $a\to 0$)

$$I=\int_a^0 \left[e^x \sin\sqrt{a^2-(x-a)^2} - m(\sqrt{a^2-(x-a)^2} + (e^x\cos(\sqrt{a^2-(x-a)^2} - m)\,\frac{-(x-a)}{\sqrt{a^2-(x-a)^2}}\right]dx$$

$$A=\int_a^0 (e^x \sin\sqrt{a^2-(x-a)^2})\,dx$$

$u=\sin(\sqrt{a^2-(x-a)^2}) \to du=\dfrac{-(x-a)}{\sqrt{a^2-(x-a)^2}}\cos(\sqrt{a^2-(x-a)^2})$

$dv=e^x \to v=e^x$

$$A=e^x \sin(\sqrt{a^2-(x-a)^2})\Big|_a^0 - \int_a^0 \frac{-(x-a)}{\sqrt{a^2-(x-a)^2}}\cos(\sqrt{a^2-(x-a)^2})$$

$$=1\sin 0 - e^a\sin a + \int_a^0 \frac{x-a}{\sqrt{a^2-(x-a)^2}}\cos(\sqrt{a^2-(x-a)^2})\,dx$$

$$I=-e^a\sin a+\int_a^0\left[\frac{x-a}{\sqrt{a^2-(x-a)^2}}\cos\sqrt{a^2-(x-a)^2} - m\sqrt{a^2-(x-a)^2} - \frac{(x-a)}{\sqrt{a^2-(x-a)^2}}\cos(\sqrt{a^2-(x-a)^2}) + m\frac{(x-a)}{\sqrt{a^2-(x-a)^2}}\right.dx$$

$$I=-e^a\sin a + m\int_a^0\left(\frac{x-a}{\sqrt{a^2-(x-a)^2}} - \sqrt{a^2-(x-a)^2}\right)dx$$

$A=\displaystyle\int_a^0 \frac{x-a}{\sqrt{a^2-(x-a)^2}}\,dx$ $\qquad t=\sqrt{a^2-(x-a)^2}$

$dt=\dfrac{-(x-a)}{\sqrt{a^2-(x-a)^2}}$

x	0	a
t	0	a

$A=-\int_a^0 dt = -t\Big|_a^0 = a$

$B=-\displaystyle\int_a^0 \sqrt{a^2-(x-a)^2}\,dx$ $\qquad x-a=a\sin t$

x	0	a
t	$\frac{-\pi}{2}$	0

$$\Leftrightarrow \begin{cases} dx = a\cos t\,dt \\ \sqrt{a^2-(x-a)^2}=a\cos t \end{cases}$$

$$B=-\int_0^{\pi/2} a\cos t\; a\cos t\;dt = a^2\int_{-\pi/2}^0 \cos^2 t\,dt = a^2\int_{-\pi/2}^0\left(\frac{1}{2}+\cos 2t\right)dt$$

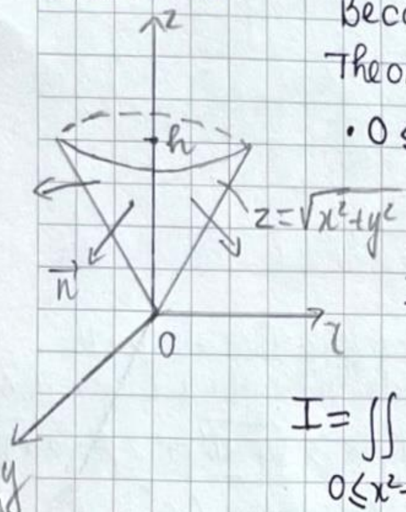
$$=a^2\left(\frac{1}{2}\left(0-\left(\frac{-\pi}{2}\right)\right)\right)+\frac{1}{2}(\sin(0)-\sin(-\pi)) = a^2.\pi/4 \Rightarrow I=-e^a\sin a + m\left(a+\frac{a^2\pi}{4}\right)$$

$$=-e^a\sin a + ma + ma^2\frac{\pi}{4}$$

④ Evaluate $I=\iint \left|\frac{x+y}{\sqrt{\ }} - x^2-y^2\right| dxdy$ where $D=\{x^2+y^2\leq 1\ldots$

QUESTION 8

(8) Evaluate the surface Integral $I = \iint_S (y-z)\,dy\,dz + (z-x)\,dx\,dz + (x-y)\,dx\,dy$ where S is the part of the cone $x^2+y^2 = z^2$, $0 \le z \le h$ with the positive outward orientation



Because S is not a closed surface, so we cannot use Divergence Theorem to evaluate the surface Integral I.

- $0 \le z \le h \Rightarrow z = \sqrt{x^2+y^2} \Rightarrow$ Domain D: $0 \le x^2+y^2 \le h^2$

Because the normal vectors are oriented downward,

$$I = -\iint_{0 \le x^2+y^2 \le h^2} -(y-z).z'_x + -(z-x).z'_y + (x-y)\, dA$$

$$I = \iint_{0 \le x^2+y^2 \le h^2} (y-\sqrt{x^2+y^2}).\frac{x}{\sqrt{x^2+y^2}} + (\sqrt{x^2+y^2}-x).\frac{y}{\sqrt{x^2+y^2}} + y - x \; dA$$

We use polar coordinates: $0 \le r \le h$; $0 \le \varphi \le 2\pi$

$$\Rightarrow I = \int_0^{2\pi}\int_0^h r.\left[(r\sin\varphi - r).\frac{r\cos\varphi}{r} + (r - r\cos\varphi).\frac{r\sin\varphi}{r} + r\sin\varphi - r\cos\varphi\right] dr\, d\varphi$$

$$I = \int_0^{2\pi}\int_0^h r\left[r\sin\varphi.\cos\varphi - r\cos\varphi + r\sin\varphi - r\sin\varphi.\cos\varphi + r\sin\varphi - r\cos\varphi\right] dr\, d\varphi$$

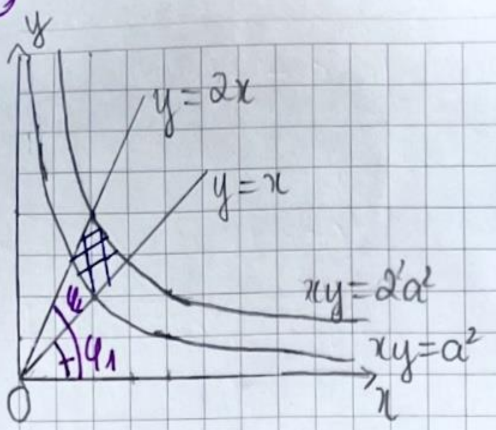
$$I = \int_0^{2\pi}\int_0^h \cancel{0\, dr\, d\varphi = 0}\; 2r(\sin\varphi - \cos\varphi)\, dr\, d\varphi = \int_0^h 2r\, dr . \int_0^{2\pi} \sqrt{2}.\sin\left(\varphi - \frac{\pi}{4}\right) d\varphi$$

$$I = r^2\Big|_0^h . \sqrt{2}. \left[-\cos\left(\varphi - \frac{\pi}{4}\right)\right]\Big|_0^{2\pi} = h^2.\sqrt{2}.\left(\frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) = 0$$

QUESTION 9

(9) Find the area of the region bounded by the curves $xy = a^2$, $xy = 2a^2$, $y = x$; $y = 2x$ $(x > 0\ ,\ y > 0)$

(9.)



We convert the Cartesian coordinates into the Polar coordinates.

$$\tan\varphi_1 = \frac{y}{x} = 1 \Rightarrow \varphi_1 = \frac{\pi}{4}$$

$$\tan\varphi_2 = \frac{y}{x} = \frac{2x}{x} = 2 \Rightarrow \varphi_2 = \arctan 2.$$

We have: $xy = 2a^2 \Leftrightarrow r_1^2 \cdot \cos\varphi \cdot \sin\varphi = 2a^2$

$$\Leftrightarrow r_1^2 = \frac{4a^2}{\sin 2\varphi} \Rightarrow r_1 = \frac{2a}{\sqrt{\sin 2\varphi}}$$

$$xy = a^2 \Leftrightarrow r_2^2 \cdot \cos\varphi \cdot \sin\varphi = a^2 \Leftrightarrow r_2^2 = \frac{2a^2}{\sin 2\varphi} \Rightarrow r_2 = \frac{a\sqrt{2}}{\sqrt{\sin 2\varphi}}$$

The area of the region bounded by these curves

$$A = \iint_D 1\, dA \text{ , in which } D = \left\{(r, \varphi) \;\middle|\; \frac{a\sqrt{2}}{\sqrt{\sin 2\varphi}} \le r \le \frac{2a}{\sqrt{\sin 2\varphi}} ; \frac{\pi}{4} \le \varphi \le \arctan 2\right\}$$

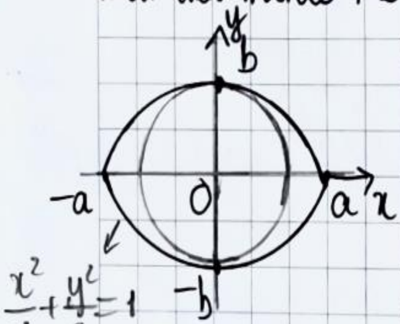
$$= \int_{\frac{\pi}{4}}^{\arctan 2} \int_{\frac{a\sqrt{2}}{\sqrt{\sin 2\varphi}}}^{\frac{2a}{\sqrt{\sin 2\varphi}}} r \cdot dr\, d\varphi = \int_{\frac{\pi}{4}}^{\arctan 2} \frac{r^2}{2} \Bigg|_{\frac{a\sqrt{2}}{\sqrt{\sin 2\varphi}}}^{\frac{2a}{\sqrt{\sin 2\varphi}}}$$

$$= \int_{\frac{\pi}{4}}^{\arctan 2} \frac{2a^2}{2\sin 2\varphi}\, d\varphi = \frac{a^2}{2} \cdot \ln(\tan\varphi) \Bigg|_{\frac{\pi}{4}}^{\arctan 2} = \frac{a^2}{2} \cdot \ln(2)$$

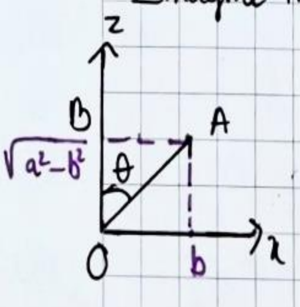
QUESTION 10

(10) Find the area of the part of the sphere $x^2+y^2+z^2=a^2$ that lies inside the cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(0<b\le a)$

The projection of the part of sphere that lies inside the cylinder onto Oxy plane



Imagine that



In Conclusion, we can find that the parts of sphere lying inside the cylinder satisfies $x^2+y^2\le b^2$ and $z\ge\sqrt{a^2-b^2}$ or $z\le-\sqrt{a^2-b^2}$.

We use spherical coordinates (ρ,φ,θ).

Because we calculate the area of these parts, every points has $\rho=a$ (Radius of the sphere)

Imagine that we consider A as the point on these surfaces.

$\Rightarrow OA=\rho=a \Rightarrow OB=\sqrt{a^2-b^2}$

$\cancel{\tan\theta=}$ $\theta=\arcsin\frac{b}{a} \Rightarrow$ We have $\rho=a$, $0\le\varphi\le 2\pi$, $0\le\theta\le\arcsin\frac{b}{a}$

The limited surface consists of 2 halves of symmetry.

Hence, the area is defined by

$$S=2.\int_0^{\arcsin\frac{b}{a}}\int_0^{2\pi}a^2.\sin\theta\, d\varphi\, d\theta=2.2\pi.a^2.(-\cos\theta)\Big|_0^{\arcsin\frac{b}{a}}$$

$$=4\pi.a^2.\left(-\frac{\sqrt{a^2-b^2}}{a}+1\right)=4\pi.a\left(-\sqrt{a^2-b^2}+a\right)$$

THE END