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On involutions of type O(q, k) over a field of characteristic two ☆

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A B S T R A C T

In this article we study the involutions of $\mathrm{O}(V, \mathrm{q})$, an orthogonal group for a vector space V with quadratic form q over a field of characteristic two. The classification proceeds by discussing conjugacy classes of involutions arising as a product of transvections, involutions with respect to a hyperbolic space, and involutions acting nontrivially in the radical of V. We achieve a complete classification of the conjugacy classes of involutions when the quadratic space (V, q) is non-defective, and conclude with a discussion of the defective case.

1. Introduction

In this article we study the involutions of orthogonal groups over fields of characteristic 2. Throughout the paper k denotes a field. An understanding of these involutions is beneficial to furthering the study of symmetric k-varieties, a generalization of symmetric

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spaces, to fields of characteristic 2. Symmetric spaces were first studied by Gantmacher in [11] in order to classify simple real Lie groups. In [5] Berger provides a complete classification of symmetric spaces for simple real Lie algebras. The primary motivation is to extend Helminck's [14] study of k-involutions and symmetric k-varieties to include fields of characteristic 2. This has been studied for groups of type G_2 and A_n in [19,22] and over fields of characteristic not 2 in [6,3,2,4,16–18]. We also extend the results of Aschbacher and Seitz [1] who studied similar structures for finite fields of characteristic 2.

The study of involutions gives us a way to describe generalized symmetric spaces or symmetric k-varieties of the form $G(k)/H(k)$ where $G(k)$ is an algebraic group over k and $H(k)$ is the fixed point group of some automorphism of order 2 on $G(k)$. The notation for the theory of algebraic groups is standard and introduced as needed. We use Hoffman and Laghribi's [15] almost exclusively for notation concerning quadratic forms over fields of characteristic 2.

There have been many studies of orthogonal groups over fields of characteristic 2. In [13] Cheng Hao discusses automorphisms of the orthogonal group over perfect fields of characteristic 2 when the quadratic form is nondefective. Pollak discusses orthogonal groups over global fields of characteristic 2 in the case the quadratic form is nondefective in [21] and Connors writes about automorphism groups of orthogonal groups over fields of characteristic 2 in [7–10] for a nondegenerate quadratic form. We extend these results by including discussions of defective and degenerate quadratic forms.

We also extend the results of Wiitala from [24]. The following result appears as Theorem A in [24], where

$$\tau_u(w) = w + \frac{B(u,w)}{\mathrm{q}(u)}u.$$

Theorem 1.1. *Let* q *be a quadratic form on a vector space* V *over a field* k *of characteristic* 2 *such that* $\mathrm{rad}(V)$ *is empty with respect to* q*. If* $\tau \in \mathrm{O}(\mathrm{q},k)$*, then* τ *is an involution if and only if* $\tau = \tau_l \cdots \tau_2\tau_1$ *and*

1. $\tau_i = \tau_{u_i}$ *is a transvection with respect to* u_i *for all* i*, or*
2. *each* τ_i *is an involution with respect to a hyperbolic space.*

The author goes on to note that all such involutions of the same type and length are $\mathrm{GL}(V)$-conjugate. These results are extended in this article to a vector space with nontrivial radical and the study of conjugacy classes under $\mathrm{O}(\mathrm{q},k)$.

Our main results appear in section 3 and concern the characterization of conjugacy classes of involutions in a maximal nonsingular subspace and a characterization of what we call radical involutions. We go on to discuss the general case and some special cases within. We prove a characterization of $\mathrm{O}(\mathrm{q},k)$-conjugacy for three types of involutions. First in Theorem 3.14 we show that two diagonal involutions $\tau_{u_l}\cdots\tau_{u_2}\tau_{u_1}$

and $\tau_{x_l}\cdots\tau_{x_2}\tau_{x_1}$ are $\mathrm{O}(\mathrm{q},k)$-conjugate if and only if a bilinear form induced by the norms of $u_1, u_2, \ldots$ and u_l is equivalent to the bilinear form induced by the norms of $x_1, x_2, \ldots$ and x_l. See Section 3 for a more precise statement. Proposition 3.18 deals with involutions with respect to a hyperbolic space, which are also known as null involutions. We show that two null involutions are $\mathrm{O}(\mathrm{q},k)$-conjugate if and only if they have the same number of reduced factors. Finally, radical involutions are described in Corollary 3.23, which establishes that all radical involutions satisfying a certain norm condition are conjugate. The paper concludes with a discussion of the involutions in the case that V is singular, but not totally singular.

2. Preliminaries

The following definitions can be found in [15]. Let k be a field of characteristic 2 and V a vector space defined over k. We call $\mathrm{q} : V \to k$ a *quadratic form* if it satisfies $\mathrm{q}(av) = a^2\,\mathrm{q}(v)$ for all $a \in k$, $v \in V$and there exists a symmetric bilinear form $B : V \times V \to k$ such that $\mathrm{q}(w + w') = \mathrm{q}(w) + \mathrm{q}(w') + B(w, w')$ for all $w, w' \in V$. Over fields of characteristic 2 nonsingular symmetric bilinear forms are also symplectic.

The pair (V, q) is called a *quadratic space*. Given a quadratic form, there exists a basis of V, consisting of e_i, f_i, g_j, where $i \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, s\}$ and field elements $a_i, b_i, c_j \in k$ such that

$$\mathrm{q}(w) = \sum_{i=1}^{r}(a_i x_i^2 + x_i y_i + b_i y_i^2) + \sum_{j=1}^{s} c_j z_j^2$$

when $w = \sum_{i=1}^{r}(x_i e_i + y_i f_i) + \sum_{j=1}^{s} z_j g_j$. We denote this quadratic form by

$$\mathrm{q} = [a_1, b_1] \perp [a_2, b_2] \perp \cdots \perp [a_r, b_r] \perp \langle c_1, c_2, \ldots, c_s\rangle$$

where $\mathrm{rad}(V) = \mathrm{span}\{g_1, g_2, \ldots, g_s\}$ is the *radical* of V. We say that such a quadratic form is of type (r, s). A nonzero vector $v \in V$ is an *isotropic vector* if $\mathrm{q}(v) = 0$, V is an *isotropic vector space* if it contains isotropic elements and *anisotropic* otherwise. The vector space V is called *nonsingular* if $s = 0$, and is called *nondefective* if $s = 0$ or $\mathrm{rad}(V)$ is anisotropic. A *hyperbolic plane* has a quadratic form isometric to the form $[0, 0]$ and will be denoted by $\mathbb{H}$. We will call q' a *subform* of q if there exists a form p such that $\mathrm{q} \cong \mathrm{q}' \perp \mathrm{p}$.

Suppose $\mathcal{P}$ is a totally singular subspace of V with basis $\{p_1, p_2, \ldots, p_l\}$, then for $w = \sum_{i=1}^{l} w_i p_i$, $w' = \sum_{i=1}^{l} w'_i p_i$, and field elements $a_i \in k$, we will denote the diagonal bilinear form

$$B(w, w') = a_1 w_1 w'_1 + a_2 w_2 w'_2 + \cdots + a_l w_l w'_l,$$

by $\langle a_1, a_2, \cdots, a_l\rangle_B$, following [15].

We will denote $\mathbb{H} \perp \mathbb{H} \perp \cdots \perp \mathbb{H}$, where there are i copies of $\mathbb{H}$ in the decomposition, by $i \times \mathbb{H}$. Similarly, $\langle 0, 0, \ldots, 0\rangle$, where the 0 is repeated j times, will be denoted $j \times \langle 0\rangle$. The following is Proposition 2.4 from [15].

Proposition 2.1. *Let* q *be a quadratic form over* k*. There are integers* i *and* j *such that*

$$\mathrm{q} \cong i \times \mathbb{H} \perp \widetilde{\mathrm{q}_r} \perp \widetilde{\mathrm{q}_s} \perp j \times \langle 0\rangle,$$

with $\widetilde{q_r}$ *nonsingular,* $\widetilde{\mathrm{q}_s}$ *totally singular and* $\widetilde{\mathrm{q}_r} \perp \widetilde{q_s}$ *anisotropic. The form* $\widetilde{\mathrm{q}_r} \perp \widetilde{\mathrm{q}_s}$ *is uniquely determined up to isometry. In particular* i *and* j *are uniquely determined.*

We call i the *Witt index* and j the *defect* of q. If

$$\mathrm{q} \cong i \times \mathbb{H} \perp \widetilde{\mathrm{q}_r} \perp j \times \langle 0\rangle \perp \widetilde{\mathrm{q}_s},$$

with respect to the basis

$$\{e_1, f_1, \ldots e_i, f_i, \ldots, e_r, f_r, g_1, \ldots g_j, g_{j+1}, \ldots, g_s\},$$

we will call

$$\operatorname{def}(V) = \operatorname{span}\{g_1, \ldots g_j\},$$

the *defect* of V.

If $\mathcal{W}$ is a basis for a subspace W of V, we will refer to the restriction of q to W by $\mathrm{q}\,|_W$ or sometimes q_W.

If G is an algebraic group, then an automorphism $\theta : G \to G$ is an *involution* if $\theta^2 = \mathrm{id}$, $\theta \neq \mathrm{id}$. In addition, θ is a *k-involution* if $\theta(G(k)) = G(k)$, where $G(k)$ denotes the k-rational points of G. We define the *fixed point group* of θ in $G(k)$ by

$$G(k)^\theta = \{\gamma \in G(k) \mid \theta\gamma\theta^{-1} = \gamma\}.$$

This is often denoted $H(k)$ or H_k in the literature when there is no ambiguity with respect to θ. Notice that since θ has order 2, this group is also the centralizer of θ in $G(k)$. We will use k^* to denote the nonzero elements of k and k^2 to denote the subfield of k that consists of the squares of k. When k is a perfect field we have $k = k^2$. An l-tuple of elements of the set S will be denoted by $S^{\times l}$.

We often consider groups that leave a bilinear form or a quadratic form invariant. If B is a bilinear form on a nonsingular vector space V we will denote the *symplectic group* of (V, q) by

$$\mathrm{Sp}(B, k) = \{\varphi \in \mathrm{GL}(V) \mid B(\varphi(w), \varphi(w')) = B(w, w') \text{ for } w, w' \in V\}.$$

The classification of involutions for $\mathrm{Sp}(B,k)$ for a field k such that $\mathrm{char}(k) \neq 2$ has been studied in [3]. For any quadratic space V we will denote the *orthogonal group* of (V, q) by

$$\mathrm{O}(\mathrm{q},k) = \{\varphi \in \mathrm{GL}(V) \mid \mathrm{q}(\varphi(w)) = \mathrm{q}(w) \text{ for } w \in V\}.$$

When V is nonsingular we have $\mathrm{O}(\mathrm{q},k) \subset \mathrm{Sp}(B,k)$ if B is the bilinear form that is associated with q,

$$B(w,w') = \mathrm{q}(w+w') + \mathrm{q}(w) + \mathrm{q}(w').$$

We define $\mathrm{BL}(B,k) = \{\varphi \in \mathrm{GL}(V) \mid B(\varphi(w),\varphi(w')) = B(w,w')\}$. Notice that when V is nonsingular $\mathrm{BL}(B,k) \cong \mathrm{Sp}(B,k)$, and in general $\mathrm{BL}(B,k) \supset \mathrm{O}(\mathrm{q},k)$. We have the isomorphism

$$\mathrm{BL}(B,k) \cong (\mathrm{Sp}(B_{V_{\mathcal{B}}},k) \times \mathrm{GL}(\mathrm{rad}(V))) \ltimes \mathrm{Mat}_{2r,s}(k),$$

where $\dim_k(V_{\mathcal{B}}) = 2r$ and $V = V_{\mathcal{B}} \perp \mathrm{rad}(V)$.

We will need to make use of some simple facts about quadratic spaces stated in the following lemmas. The first outlines some standard isometries for quadratic forms over a field of characteristic 2, and the second allows us to express q using a different completion of the nonsingular space. These and more like them appear in [15].

Lemma 2.2. *Let* q *be a quadratic form on a vector space* V*, and suppose* $\alpha \in k$*. Then the following are equivalent representations of* q *on* V*:*

1. $[a,b] = [a, a+b+1] = [b,a] = [\alpha^2 a, \alpha^{-2} b]$
2. $[a,b] \perp [c,d] = [a+c, b] \perp [c, b+d] = [c,d] \perp [a,b]$

Lemma 2.3. *Let* $c_i, c_i', d_i \in k$ *for* $1 \leq i \leq n$*, and denote the subfield of squares in* k *by* k^2*. Suppose* $\{c_1, ..., c_n\}$ *and* $\{c_1', ..., c_n'\}$ *span the same vector space over* k^2 *and* $\mathrm{q} = [c_1, d_1] \perp \ldots \perp [c_n, d_n]$*. Then there exist* $d_i' \in k$*,* $1 \leq i \leq n$*, such that* $\mathrm{q} = [c_1', d_1'] \perp \ldots \perp [c_n', d_n']$*.*

3. Nonsingular involutions

Now we study the isomorphism classes of involutions of $\mathrm{O}(\mathrm{q},k)$ when (V,q) is nonsingular. Recall that in general $\mathrm{Sp}(B,k) \supset \mathrm{O}(\mathrm{q},k)$ when B is induced by q on V and V is nonsingular. A *symplectic transvection* with respect to $u \in V$ and $a \in k$ is a map of the form

$$\tau_{u,a}(w) = w + aB(u,w)u,$$

and such a map is an *orthogonal transvection* if $\mathrm{q}(u) \neq 0$ and $a = \mathrm{q}(u)^{-1}$. Notice that for a symplectic transvection a is allowed to be zero, but $\tau_{u,0} = \mathrm{id}$. The symplectic group is generated by symplectic transvections and the orthogonal group is generated by orthogonal transvections as long as V is not of the form $V = \mathbb{H} \perp \mathbb{H}$ over $\mathbb{F}_2$ as pointed out in Theorem 14.16 in [12]. A *symplectic involution* is a map of order 2 in $\mathrm{Sp}(B,k)$.

An involution $\sigma \in \mathrm{Sp}(B,k)$ is called *hyperbolic* if $B(v,\sigma(v)) = 0$ for all $v \in V$, and *diagonal* otherwise. Observe that all nontrivial hyperbolic elements of $\mathrm{Sp}(B,k)$ are involutions.

If $\sigma \in \mathrm{Sp}(B,k)$, then we call $R_\sigma = (\sigma - \mathrm{id}_V)V$ the residual space of σ and define $\mathrm{res}(\sigma) = \dim R_\sigma$. Then the following comes from [20]:

Theorem 3.1. *Let* $\sigma \in \mathrm{Sp}(B,k),\ \sigma^2 = \mathrm{id}_V,\ \sigma \neq \mathrm{id}_V$. *Then:*

1. *If σ is hyperbolic, then σ is a product of* $\mathrm{res}(\sigma)+1$, *but not of* $\mathrm{res}(\sigma)$, *symplectic transvections.*
2. *If σ is diagonal, then σ is a product of* $\mathrm{res}(\sigma)$, *but not of* $\mathrm{res}(\sigma)-1$, *symplectic transvections.*
3. *In either case, the vectors inducing transvections whose composition is σ are mutually orthogonal.*

Consider the symplectic involution of the form

$$\tau_{u_l,a_l} \cdots \tau_{u_2,a_2}\tau_{u_1,a_1}.$$

If $a = [a_i] \in k^{\times l}$ and $\mathcal{U} = \{u_1, u_2, \dots, u_l\}$, then we use $\tau_{\mathcal{U},a}$ to denote this map. We may assume $\mathcal{U}$ consists of mutually orthogonal vectors in V, thus $\mathrm{span}\,\mathcal{U}$ is a singular subspace of V with dimension less than or equal to l. A factorization of a transvection involution is called *reduced* if it is written using the least number of factors, and the number of factors in a reduced expression is called the *length* of the involution.

Lemma 3.2. *If $\sigma \in \mathrm{Sp}(B,k)$ is diagonal and we let $r = \mathrm{res}(\sigma)$, then there exists a set $\mathcal{U} = \{u_1, u_2, \dots, u_r\}$, where $B(u_i,u_j) = 0$ for all $\{i,j\} \subset [l]$, and $a = [a_i] \in (k^*)^{\times r}$ such that $\mathcal{U}$ is a basis for R_σ and $\sigma = \tau_{\mathcal{U},a}$.*

Proof. By 3.1, we know σ is a product of r transvections whose inducing vectors are mutually orthogonal. R_σ is the span of these vectors, and $r = \dim(R_\sigma)$, therefore these vectors must be linearly independent. □

We want to know when two diagonal involutions of the same length are equal, and to that end we define the following relationship. Consider a pairing consisting of a list of l orthogonal vectors contained in a nonsingular vector space over a field of characteristic 2 along with a vector in $(k^*)^{\times l}$, where k^* denotes the nonzero elements of k. This vector is our ordered list of a_i's and we take the components in k^*, since we can assume we have a reduced diagonal involution of length l. Let $\mathcal{U}$ be as above and let

$$\mathcal{X} = \{x_1, x_2, \ldots, x_l\},$$

$a = (a_1, a_2, \ldots, a_l)$ and $b = (b_1, b_2, \ldots, b_l)$. The pairing $(\mathcal{U}, a)$ and $(\mathcal{X}, b)$ is called *involution compatible* if $\mathcal{U}$ and $\mathcal{X}$ span the same l-dimensional singular subspace of V such that $u_i = \sum \alpha_{ji} x_j$ and the following hold

$$b_j = \sum a_i \alpha_{ji}^2 \text{ and} \tag{1}$$

$$0 = \sum a_i \alpha_{ji} \alpha_{ki} \text{ for all } \{j, k\} \subset [l]. \tag{2}$$

Notice that this is equivalent to

$$[\alpha_{ij}]^T_{1 \leq i,j \leq l} \operatorname{Diag}(a_1, \ldots, a_l)[\alpha_{ij}]_{1 \leq i,j \leq l} = \operatorname{Diag}(b_1, \ldots, b_l).$$

We can simplify the statement by setting $A = [\alpha_{ij}]_{1 \leq i,j \leq l}$ and $\operatorname{Diag}(a_1, \ldots, a_l) = [a_i]$

$$A^T [a_i] A = [b_i], \tag{3}$$

and we can see this is equivalent to

$$\langle a_1, a_2, \ldots, a_l \rangle_B \cong \langle b_1, b_2, \ldots, b_l \rangle_B,$$

an equivalence of bilinear forms.

Theorem 3.3. *Let $\tau_{\mathcal{U},a}$ and $\tau_{\mathcal{X},b}$ be diagonal involutions. Then $\tau_{\mathcal{U},a} = \tau_{\mathcal{X},b}$ if and only if $(\mathcal{U}, a)$ and $(\mathcal{X}, b)$ are involution compatible.*

Proof. Suppose $\tau_{\mathcal{U},a} = \tau_{\mathcal{X},b}$. Then for $w \in V$,

$$a_1 B(u_1, w)u_1 + \cdots + a_l B(u_l, w)u_l = b_1 B(x_1, w)x_1 + \cdots b_l B(x_l, w)x_l. \tag{4}$$

For each u_i there exists a v_i such that the set of v_i provide a nonsingular completion of dimension $2l$. Choosing $w = v_i$ we see that

$$a_i u_i = b_1 B(x_1, v_i)x_1 + \cdots b_l B(x_l, v_i)x_l.$$

This shows that $\mathcal{U}$ and $\mathcal{X}$ span the same nonsingular space. We choose coefficients for u_i in terms of $\mathcal{X}$ as

$$u_i = \sum_{j=1}^{l} \alpha_{ji} x_j.$$

Now substituting our new expression into Equation (4) and replacing w with y_j such that $B(x_k, y_k) = 1$ and $B(x_j, y_k) = 0$ when $j \neq k$ we have

$$a_1 B\left(\sum_{j=1}^{l}\alpha_{j1}x_j, y_k\right)\sum_{j=1}^{l}\alpha_{j1}x_j + \cdots + a_l B\left(\sum_{j=1}^{l}\alpha_{jl}x_j, y_k\right)\sum_{j=1}^{l}\alpha_{jl}x_j = b_k x_k. \tag{5}$$

Now simplifying the bilinear forms we arrive at Equation (3).

If we assume that $(\mathcal{U}, a)$ and $(\mathcal{X}, b)$ are involution compatible we can reconstruct Equation (4) from basis vectors and we have $\tau_{\mathcal{U},a} = \tau_{\mathcal{X},b}$. □

Corollary 3.4. *Two diagonal involutions $\tau_{\mathcal{U},a}$ and $\tau_{\mathcal{X},b}$ are $\mathrm{Sp}(B,k)$-conjugate if and only if there exists $\mathcal{X}'$ such that $(\mathcal{X}', a)$ is involution compatible with $(\mathcal{X}, b)$.*

In 2.1.8 of [20] the following Theorem is stated.

Theorem 3.5. *Let $\sigma \in \mathrm{Sp}(B,k)$ be hyperbolic with residual space R_σ. Let τ be any transvection such that $R_\tau \subset R_\sigma$. Then $R_{\tau\sigma} = R_\sigma$, but $\tau\sigma$ is not hyperbolic.*

The next result describes how hyperbolic maps relate to equivalent diagonal maps.

Lemma 3.6. *Let $\sigma,\ \theta \in \mathrm{Sp}(B,k)$ be hyperbolic. Then $\sigma = \theta$ if and only if there exists a symplectic transvection $\tau_{u,a} \in \mathrm{Sp}(B,k)$ where $u \in R_\sigma$ and $a \in k^*$, such that $\tau_{u,a}\sigma = \tau_{u,a}\theta$.*

Proof. If $\sigma = \theta$, then one may choose any $u \in R_\sigma = R_\theta$, $a \in k^*$. Now if such a $\tau_{u,a}$ exists, then $\sigma = \theta$ since $\tau_{u,a}^2 = \mathrm{id}_V$. □

Let $\tau_{u,a}$ be a symplectic involution and notice that $\tau_{u,a} \in \mathrm{O}(q,k)$ only if

$$\begin{aligned}
\mathrm{q}\left(\tau_{u,a}(w)\right) &= \mathrm{q}(w + aB(w,u)u)\\
&= \mathrm{q}(w) + \mathrm{q}\left(aB(w,u)u\right) + B\left(w, aB(w,u)u\right)\\
&= \mathrm{q}(w) + a^2 B(w,u)^2\,\mathrm{q}(u) + aB(w,u)^2\\
&= \mathrm{q}(w),
\end{aligned}$$

so $B(w,u)^2 a(a\,\mathrm{q}(u)+1) = 0$. So either $B(w,u) = 0$ for all w, $a = 0$ or $\mathrm{q}(u) = 1/a$. Therefore we will refer to $\tau_{u,\frac{1}{\mathrm{q}(u)}}$ by τ_u.

Proposition 3.7. *Two orthogonal transvections τ_u and τ_x are equal if and only if $x = \alpha u$ for some $\alpha \in k$.*

Proof. First assuming $x = \alpha u$, we have

$$\begin{aligned}
\tau_{\alpha u}(w) &= w + \frac{B(\alpha u, w)}{\mathrm{q}(\alpha u)}\alpha u\\
&= w + \frac{\alpha B(u,w)}{\alpha^2\,\mathrm{q}(u)}\alpha u
\end{aligned}$$

$$= \tau_u(w).$$

Therefore $\tau_u = \tau_x$.

Now consider $\tau_u = \tau_x$. Then

$$\begin{aligned} w + \frac{B(u,w)}{\mathrm{q}(u)}u &= w + \frac{B(x,w)}{\mathrm{q}(x)}x \\ \frac{B(u,w)}{\mathrm{q}(u)}u &= \frac{B(x,w)}{\mathrm{q}(x)}x \\ u &= \frac{B(x,w)\,\mathrm{q}(u)}{B(u,w)\,\mathrm{q}(x)}x. \end{aligned}$$

Therefore, setting $\alpha = \frac{B(x,w)\,\mathrm{q}(u)}{B(u,w)\,\mathrm{q}(x)}$, we have $u = \alpha x$. $\square$

Proposition 3.8. *Let $\phi \in \mathrm{O}(\mathrm{q}, k)$. Then for a product of transvections*

$$\tau_{u_1}\tau_{u_2}\cdots\tau_{u_l} \in \mathrm{O}(\mathrm{q}, k),$$

we have the conjugation relation

$$\phi\tau_{u_1}\tau_{u_2}\cdots\tau_{u_l}\phi^{-1} = \tau_{\phi(u_1)}\tau_{\phi(u_2)}\cdots\tau_{\phi(u_l)}.$$

Proof. First notice that

$$\phi\tau_u\phi^{-1}(w) = w + \frac{B(u,\phi^{-1}(w))}{\mathrm{q}(u)}\phi(u) = w + \frac{B(\phi(u),w)}{\mathrm{q}(\phi(u))}\phi(u) = \tau_{\phi(u)}(w).$$

Now we see that

$$\begin{aligned} \phi\tau_{u_1}\tau_{u_2}\cdots\tau_{u_l}\phi^{-1} &= \phi\tau_{u_1}\phi^{-1}\phi\tau_{u_2}\phi^{-1}\cdots\phi\tau_{u_l}\phi^{-1} \\ &= \tau_{\phi(u_1)}\tau_{\phi(u_2)}\cdots\tau_{\phi(u_l)}, \end{aligned}$$

as required. $\square$

Consider the reduced diagonal involution

$$\tau = \tau_{u_1}\tau_{u_2}\cdots\tau_{u_l},$$

where as before $\mathcal{U} = \{u_1, u_2, \ldots, u_l\}$ are mutually orthogonal vectors. If we consider the subspace $\operatorname{span}\mathcal{U} \subset V$, then we have

$$\mathrm{q}\,|_{\operatorname{span}\mathcal{U}} \sim \langle \mathrm{q}(u_1), \mathrm{q}(u_2), \ldots, \mathrm{q}(u_l)\rangle.$$

Proposition 3.9. *If* $\mathrm{q}(u_i) \neq 0$ *for* $1 \leq i \leq l$ *then*

$$\left\langle \frac{1}{\mathrm{q}(u_1)}, \frac{1}{\mathrm{q}(u_2)}, \ldots, \frac{1}{\mathrm{q}(u_l)} \right\rangle_B \cong \left\langle \frac{1}{\mathrm{q}(x_1)}, \frac{1}{\mathrm{q}(x_2)}, \ldots, \frac{1}{\mathrm{q}(x_l)} \right\rangle_B$$

if and only if

$$\langle \mathrm{q}(u_1), \mathrm{q}(u_2), \ldots, \mathrm{q}(u_l) \rangle_B \cong \langle \mathrm{q}(x_1), \mathrm{q}(x_2), \ldots, \mathrm{q}(x_l) \rangle_B \,.$$

Proof. If

$$\left\langle \frac{1}{\mathrm{q}(u_1)}, \frac{1}{\mathrm{q}(u_2)}, \ldots, \frac{1}{\mathrm{q}(u_l)} \right\rangle_B \cong \left\langle \frac{1}{\mathrm{q}(x_1)}, \frac{1}{\mathrm{q}(x_2)}, \ldots, \frac{1}{\mathrm{q}(x_l)} \right\rangle_B ,$$

then there exists some A such that

$$A^T \left[\frac{1}{\mathrm{q}(u_i)} \right] A = \left[\frac{1}{\mathrm{q}(x_i)} \right] .$$

Notice that

$$[\mathrm{q}(u_i)][\mathrm{q}(x_i)]A^T \left[\frac{1}{\mathrm{q}(u_i)} \right] A[\mathrm{q}(x_i)][\mathrm{q}(u_i)] = \left[\mathrm{q}(u_i)^2 \, \mathrm{q}(x_i) \right]$$

and letting $A' = [\mathrm{q}(u_i)]^{-1}[\mathrm{q}(x_i)]^{-1}A[\mathrm{q}(x_i)]\,\mathrm{q}(u_i)]$ then

$$([\mathrm{q}(x_i)]A'[\mathrm{q}(u_i)]^{-1})^T[\mathrm{q}(u_i)]([\mathrm{q}(x_i)]A'[\mathrm{q}(u_i)]^{-1}) = [\mathrm{q}(x_i)] \,.$$

This gives us

$$\left\langle \frac{1}{\mathrm{q}(u_1)}, \frac{1}{\mathrm{q}(u_2)}, \ldots, \frac{1}{\mathrm{q}(u_l)} \right\rangle_B \cong \left\langle \frac{1}{\mathrm{q}(x_1)}, \frac{1}{\mathrm{q}(x_2)}, \ldots, \frac{1}{\mathrm{q}(x_l)} \right\rangle_B$$

implies

$$\langle \mathrm{q}(u_1), \mathrm{q}(u_2), \ldots, \mathrm{q}(u_l) \rangle_B \cong \langle \mathrm{q}(x_1), \mathrm{q}(x_2), \ldots, \mathrm{q}(x_l) \rangle_B \,,$$

and the argument is reversible for the converse. □

Corollary 3.10. *If*

$$\langle \mathrm{q}(u_1), \mathrm{q}(u_2), \ldots, \mathrm{q}(u_l) \rangle_B \cong \langle \mathrm{q}(x_1), \mathrm{q}(x_2), \ldots, \mathrm{q}(x_l) \rangle_B \,,$$

then

$$\langle \mathrm{q}(u_1), \mathrm{q}(u_2), \ldots, \mathrm{q}(u_l) \rangle \cong \langle \mathrm{q}(x_1), \mathrm{q}(x_2), \ldots, \mathrm{q}(x_l) \rangle \,.$$

In general the converse of Corollary 3.10 is not true. In particular consider two diagonal involutions of length 2

$$\tau_{u_2}\tau_{u_1}, \tau_{x_2}\tau_{x_1} \in \mathrm{O}(\mathrm{q}, k)$$

over $k = \mathbb{F}_2(t_1, t_2)$ such that

$$\mathrm{q}(x_1) = \mathrm{q}(u_1) + t_1^2\,\mathrm{q}(u_2) \text{ and}$$
$$\mathrm{q}(x_2) = \mathrm{q}(u_1) + \mathrm{q}(u_2).$$

Let $\mathrm{q}(u_1) = 1$ and $\mathrm{q}(u_2) = t_2$. Notice that $\mathrm{q}(x_1), \mathrm{q}(x_2) \in k^2[\mathrm{q}(u_1, \mathrm{q}(u_2)]$, which gives us that $\langle \mathrm{q}(u_1), \mathrm{q}(u_2)\rangle \cong \langle \mathrm{q}(x_1), \mathrm{q}(x_2)\rangle$. In this case $\mathrm{q}(u_1)$ and $\mathrm{q}(u_2)$ form a basis for a k^2-vector space of dimension 2 and so do $\mathrm{q}(x_1)$ and $\mathrm{q}(x_2)$. Therefore any matrix A such that $A^T[\mathrm{q}(u_i)]A = [\mathrm{q}(x_i)]$ and $A = [\alpha_{kj}]$ must have $\alpha_{11} = \alpha_{12} = \alpha_{22} = 1$ and $\alpha_{21} = t_1$ and so

$$\begin{bmatrix} 1 & 1 \\ t_1 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & t_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ t_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 + t_1^2 t_2 & 1 + t_1 t_2 \\ 1 + t_1 t_2 & 1 + t_2 \end{bmatrix}.$$

Since that off diagonal entries, $1 + t_1 t_2$, are not zero the conditions for $\langle \mathrm{q}(u_1), \mathrm{q}(u_2)\rangle_B \cong \langle \mathrm{q}(x_1), \mathrm{q}(x_2)\rangle_B$ are not satisfied and we have a counter example.

Lemma 3.11. *Two orthogonal involutions given by reduced products of orthogonal transvections are equal,* $\tau_{u_l} \cdots \tau_{u_2}\tau_{u_1} = \tau_{x_l} \cdots \tau_{x_2}\tau_{x_1}$, *if and only if*

$$\mathrm{span}\{u_1, u_2, \ldots, u_l\} = \mathrm{span}\{x_1, x_2, \ldots, x_l\},$$

and

$$\langle \mathrm{q}(u_1), \mathrm{q}(u_2), \ldots, \mathrm{q}(u_l)\rangle_B \cong \langle \mathrm{q}(x_1), \mathrm{q}(x_2), \ldots, \mathrm{q}(x_l)\rangle_B\,.$$

Proof. Let $\{u_1, u_2, \ldots, u_l\}$ and $\{x_1, x_2, \ldots, x_l\}$ be sets of linearly independent mutually orthogonal vectors, none of which are in $\mathrm{rad}(V)$ and all of which have nonzero norm. Now assume $\tau_{u_l} \cdots \tau_{u_2}\tau_{u_1} = \tau_{x_l} \cdots \tau_{x_2}\tau_{x_1}$. Then for each set of linearly independent vectors there exists a completion of the symplectic basis. In particular there exists a set $\{v_1, v_2, \ldots, v_l\}$ of linearly independent vectors in V such that $B(u_i, v_j) = 1$ when $i = j$ and zero otherwise. Notice that we can define τ_{u_i} by

$$\tau_{u_i}(v_i) = v_i + \frac{B(u_i, v_i)}{\mathrm{q}(u_i)}u_i = v_i + \frac{1}{\mathrm{q}(u_i)}u_i,$$

and this transvection acts as the identity on every other basis vector. Setting

$$\tau_{u_l} \cdots \tau_{u_2}\tau_{u_1}(v_i) = \tau_{x_l} \cdots \tau_{x_2}\tau_{x_1}(v_i),$$

we arrive at the equation

$$\frac{1}{\mathrm{q}(u_i)}u_i = \frac{B(x_1, v_i)}{\mathrm{q}(x_1)}x_1 + \frac{B(x_2, v_i)}{\mathrm{q}(x_2)}x_2 + \cdots + \frac{B(x_l, v_i)}{\mathrm{q}(x_l)}x_l, \tag{6}$$

which tells us in particular that we can write each u_i as a linear combination of $\{x_1, x_2, \ldots, x_l\}$ and the two sets must span the same space. Notice that since we can write each x_j as a linear combination of $\{u_1, u_2, \ldots, u_l\}$ that the constants $B(x_j, v_i) = \alpha_{ij}$ are just the i-th component of x_j written in the u-basis. In other words we can write

$$x_j = \alpha_{1j}u_1 + \alpha_{2j}u_2 + \cdots + \alpha_{lj}u_l.$$

Now let us assume that we can write each u_i in the x-basis and set

$$u_i = \beta_{i1}x_1 + \beta_{i2}x_2 + \cdots + \beta_{il}x_l.$$

Solving for β_{ij} in terms of α's if $A = [\alpha_{ij}]_{1\le i,j\le l}$ we arrive at the condition

$$A^T \left[\frac{1}{\mathrm{q}(u_i)}\right] A = \left[\frac{1}{\mathrm{q}(x_i)}\right].$$

Then by Proposition 3.9 we have the result. □

We will use the following result, which is Lemma 2.6 from [15].

Lemma 3.12. *Let* q *and* q′ *be nondefective quadratic forms of the same dimension. If*

$$\mathrm{q} \perp j \times \langle 0\rangle \cong \mathrm{q}' \perp j \times \langle 0\rangle,$$

then $\mathrm{q} \cong \mathrm{q}'$.

The following is a Gram-Schmidt type theorem for characteristic 2.

Lemma 3.13. *Let V be a symplectic space of dimension $2r$. Given $\{e_1, e_2, ..., e_r\} \subset V$, a linearly independent set of vectors such that $e_i \perp e_j$, there exists $\{e_1', f_1, e_2', f_2, ..., e_r', f_r\} \subset V$ such that $B(e_i', f_j) = \delta_{ij}$, and $B(f_i, f_j) = B(e_i', e_j') = 0$.*

Proof. Choose $f_1 \in V$ such that $B(e_1, f_1) = \alpha \neq 0$. Define $e_1' = \frac{1}{\alpha}e_1$ and $e_i' = e_i + \frac{B(e_i,f_1)}{\alpha}e_1$ for $i \in \{2, 3, ..., r\}$, so that $B(f_1, e_j') = \delta_{1j}$. Then $V = \langle e_1', f_1\rangle \perp V'$, where $\dim(V') < \dim(V)$, and induction establishes the result. □

Theorem 3.14. *Let $\tau_{u_l} \cdots \tau_{u_2}\tau_{u_1}$ and $\tau_{x_l} \cdots \tau_{x_2}\tau_{x_1}$ be orthogonal diagonal involutions on V such that $\phi \in \mathrm{O}(\mathrm{q}, k)$. Then*

$$\phi\tau_{u_l} \cdots \tau_{u_2}\tau_{u_1}\phi^{-1} = \tau_{x_l} \cdots \tau_{x_2}\tau_{x_1}$$

if and only if

$$\langle \mathrm{q}(u_1), \mathrm{q}(u_2), \ldots, \mathrm{q}(u_l)\rangle_B \cong \langle \mathrm{q}(x_1), \mathrm{q}(x_2), \ldots, \mathrm{q}(x_l)\rangle_B .$$

Proof. First notice that the above condition is stronger than the two spaces having isometric norms. Recall from Proposition 3.8 that we have

$$\phi\tau_{u_l}\cdots\tau_{u_2}\tau_{u_1}\phi^{-1} = \tau_{\phi(u_l)}\cdots\tau_{\phi(u_2)}\tau_{\phi(u_1)}.$$

If we assume that the two involutions are $\mathrm{O}(\mathrm{q}, k)$-conjugate we have

$$\tau_{\phi(u_l)}\cdots\tau_{\phi(u_2)}\tau_{\phi(u_1)} = \tau_{x_l}\cdots\tau_{x_2}\tau_{x_1},$$

and so

$$\langle \mathrm{q}(\phi(u_1)), \mathrm{q}(\phi(u_2)), \ldots, \mathrm{q}(\phi(u_l))\rangle_B \cong \langle \mathrm{q}(x_1), \mathrm{q}(x_2), \ldots, \mathrm{q}(x_l)\rangle_B$$

and

$$\langle \mathrm{q}(u_1), \mathrm{q}(u_2), \ldots, \mathrm{q}(u_l)\rangle_B = \langle \mathrm{q}(\phi(u_1)), \mathrm{q}(\phi(u_2)), \ldots, \mathrm{q}(\phi(u_l))\rangle_B .$$

Now let us assume $\langle \mathrm{q}(u_1), \mathrm{q}(u_2), \ldots, \mathrm{q}(u_l)\rangle_B \cong \langle \mathrm{q}(x_1), \mathrm{q}(x_2), \ldots, \mathrm{q}(x_l)\rangle_B$. Then $\langle \mathrm{q}(u_1), \mathrm{q}(u_2), \ldots, \mathrm{q}(u_l)\rangle \cong \langle \mathrm{q}(x_1), \mathrm{q}(x_2), \ldots, \mathrm{q}(x_l)\rangle$ and there exists a map $\phi \in \mathrm{O}(\mathrm{q}, k)$ such that $\phi(\operatorname{span}\mathcal{U}) = \operatorname{span}\mathcal{X}$, where $\mathcal{U} = \{u_1, u_2, \ldots, u_l\}$ and $\mathcal{X} = \{x_1, x_2, \ldots, x_l\}$. We already know that the equivalent bilinear form condition is met so by Lemma 3.11 we have that

$$\tau_{\phi(u_l)}\cdots\tau_{\phi(u_2)}\tau_{\phi(u_1)} = \tau_{x_l}\cdots\tau_{x_2}\tau_{x_1},$$

and so the two involutions are conjugate. □

3.15. Null involutions

In this section we discuss involutions of the second type in Theorem 1.1. This definition can also be found in [23]. We note that basic null involutions are hyperbolic, in the sense of [20].

Definition 3.16. A plane $P = \operatorname{span}\{e, f\}$ is hyperbolic (or Artinian) if both of the following are satisfied:

1. $\mathrm{q}(e) = \mathrm{q}(f) = 0$
2. $B(e, f) \neq 0$.

If e, f span a hyperbolic plane, we can rescale to assume $B(e, f) = 1$. Proposition 188.2 of [23] guarantees that every nonsingular nonzero isotropic vector is contained in a hyperbolic plane.

Definition 3.17. Let η be an involution of $\mathrm{O}(\mathrm{q}, k)$ where (V, q) is a quadratic space, and let $\mathbb{P}$ be the orthogonal sum of two hyperbolic planes. Then η is called a basic null involution in $\mathbb{P}$ if all of the following are satisfied:

1. η leaves $\mathbb{P}$ invariant
2. η fixes a 2-dimensional subspace of $\mathbb{P}$ where every vector has norm zero
3. $\eta|_{\mathbb{P}^C} = \mathrm{id}_{\mathbb{P}^C}$, where $\mathbb{P}^C$ is the complement of $\mathbb{P}$ in V.

An automorphism η is a basic null involution in $\mathbb{P}$ if and only if there exists a basis with respect to which the matrix of η is a pair of 2×2 Jordan blocks, all eigenvalues are 1 and acts as the identity elsewhere. This happens precisely when there is a basis $\{e_1, f_2, e_2, f_1\}$ for A such that $B(e_i, f_i) = 1$, $\eta(e_i) = e_i$, $\eta(f_1) = e_2 + f_1$, and $\eta(f_2) = e_1 + f_2$.

Proposition 3.18. *Two null involutions are* $\mathrm{O}(\mathrm{q}, k)$*-conjugate if and only if they have the same length.*

Proof. Let $\eta_k \cdots \eta_2\eta_1$ be a null involution on V where η_i, $1 \le i \le k$ are basic null involutions. Then each η_i corresponds to a four dimensional space made up of two perpendicular hyberbolic planes. In other words for each η_i there exists a subspace N_i such that $\mathrm{q}|_{N_i} \sim [0, 0] \perp [0, 0]$. Similarly, let $\mu_k \cdots \mu_2\mu_1$ be a null involution whose basic null involutions μ_i have corresponding four dimensional hyperbolic subspaces M_i such that $\mathrm{q}|_{M_i} \sim [0, 0] \perp [0, 0]$. If we choose $\phi \in \mathrm{O}(\mathrm{q}, k)$ such that $\phi : N_i \to M_i$, then

$$\phi\mu_k \cdots \mu_2\mu_1\phi^{-1} = \eta_k \cdots \eta_2\eta_1.$$

If two null involutions do not have the same length they are not $\mathrm{GL}(V)$-conjugate. □

We recall from [24] that if a map is a product of an orthogonal transvection and a basic null involution, then it is also the product of three orthogonal transvections.

3.19. Radical involutions

In this section we characterize the involutions acting in the radical of V. Recall the bilinear form is identically zero here. First let us consider the following result.

Proposition 3.20. $\mathrm{O}(\mathrm{q}|_{\mathrm{rad}(V)}, k) \cong \mathrm{GL}_j(k) \ltimes \mathrm{Mat}_{j,s-j}(k)$ *where j is the defect of* $\mathrm{rad}(V)$.

Proof. By Proposition 2.1 every norm on $\mathrm{rad}(V)$ is isometric to

$$\langle 0, \cdots, 0, c_{j+1}, \cdots, c_s \rangle,$$

where j is the defect of q and $\dim(\mathrm{rad}(V)) = s$. Now the subform

$$\langle c_{j+1}, \cdots, c_s \rangle,$$

is anisotropic. We can choose a basis

$$\mathcal{R} = \{g_1, g_2, \ldots, g_j, g_{j+1}, g_{j+2}, \ldots, g_s\}$$

of $\mathrm{rad}(V)$ such that

$$\mathrm{q}(g_i) = 0 \text{ for } 1 \leq i \leq j$$

and

$$\mathrm{q}(g_i) = c_i \text{ for } j + 1 \leq i \leq s.$$

Let us denote the vector space spanned by the basis vectors of $\mathcal{R}$ with nonzero norms by

$$\mathrm{span}\{g_{j+1}, g_{j+2}, \ldots, g_s\} = \mathrm{def}(V)'_{\mathcal{R}}.$$

If $\phi \in \mathrm{O}(\mathrm{q}\,|_{\mathrm{rad}(V)}, k)$ then the image of ϕ is defined by the four linear maps $\chi : \mathrm{def}(V) \to \mathrm{def}(V)$, $M : \mathrm{def}(V)'_{\mathcal{R}} \to \mathrm{def}(V)$, $N : \mathrm{def}(V) \to \mathrm{def}(V)'_{\mathcal{R}}$ and $\psi : \mathrm{def}(V)'_{\mathcal{R}} \to \mathrm{def}(V)'_{\mathcal{R}}$. Let $x \in \mathrm{def}(V)$ then $\phi(x) = \chi(x) + Nx$ where $Nx \in \mathrm{def}(V)'_{\mathcal{R}}$. Also $\mathrm{q}(\phi(x)) = \mathrm{q}(x) = 0$, so

$$\mathrm{q}(\chi(x) + Nx) = \mathrm{q}(\chi(x)) + \mathrm{q}(Nx) = 0.$$

Now $\chi(x) \in \mathrm{def}(V)$ so $\mathrm{q}(\chi(x)) = 0$. Leaving $\mathrm{q}(Nx) = 0$. There are no nontrivial vectors in $\mathrm{def}(V)'_{\mathcal{R}}$ such that $\mathrm{q}(Nx) = 0$ therefore $N = 0$. In general we require $\chi \in \mathrm{GL}_j(k)$ such that $\mathrm{q}(\chi(x)) = 0$, but q is identically zero, so χ can be any element of $\mathrm{GL}_j(k)$.

Now consider $y \in \mathrm{def}(V)'_{\mathcal{R}}$. If $\phi(y) = My + \psi(y)$, then

$$\begin{aligned} \mathrm{q}(\phi(y)) &= \mathrm{q}(My + \psi(y)) \\ &= \mathrm{q}(My) + \mathrm{q}(\psi(y)) \end{aligned}$$

The vector $My \in \mathrm{def}(V)$ so $\mathrm{q}(My) = 0$ and we have $\mathrm{q}(y) = \mathrm{q}(\psi(y))$ for $\psi(y) \in \mathrm{def}(V)'_{\mathcal{R}}$, but this means $\psi(y) = y$ for all $y \in \mathrm{def}(V)'_{\mathcal{R}}$. In other words $\psi = \mathrm{id}$. We end up with $\phi(y) = y + My$ and since $\mathrm{q}(My) = 0$ for any M, the map M can be any element of $\mathrm{Mat}_{j,s-j}(k)$.

Consider two elements $\phi_1, \phi_2 \in \mathrm{O}(\mathrm{q}\,|_{\mathrm{rad}(V)}, k)$ defined by maps χ_1, M_1 and χ_2, M_2 respectively. Any element in $\mathrm{rad}(V)$ can be written as $x+y \in \mathrm{rad}(V) = \mathrm{def}(V) \oplus \mathrm{def}(V)'_{\mathcal{R}}$ where $x \in \mathrm{def}(V)$ and $y \in \mathrm{def}(V)'_{\mathcal{R}}$. We see that

$$\phi_1\phi_2(x+y) = \chi_1\chi_2(x) + y + (M_1 + \chi_1 M_2)y.$$

This is equivalent to the action of the block matrices acting on $\mathrm{rad}(V)$ so we have defined an isomorphism

$$\Psi : \mathrm{O}(\mathrm{q}\,|_{\mathrm{rad}(V)}, k) \to \begin{bmatrix} \mathrm{GL}_j(k) & \mathrm{Mat}_{j,s-j}(k) \\ 0 & \mathrm{id} \end{bmatrix},$$

such that

$$\Psi(\phi) = \Psi(\chi, M) = \begin{bmatrix} \chi & M \\ 0 & \mathrm{id} \end{bmatrix}.$$

Further, we can verify that the subgroup of the form

$$\left\{ \begin{bmatrix} \mathrm{id} & M \\ 0 & \mathrm{id} \end{bmatrix} \;\middle|\; M \in \mathrm{Mat}_{j,s-j}(k) \right\}$$

is normal in $\left[\begin{smallmatrix} \mathrm{GL}_j(k) & \mathrm{Mat}_{j,s-j}(k) \\ 0 & \mathrm{id} \end{smallmatrix}\right]$. $\square$

If $\theta \in \mathrm{O}(\mathrm{q}, V)$ such that $\dim(\mathrm{rad}(V)) > 1$, then to preserve the bilinear form we must have $\theta(\mathrm{rad}(V)) = \mathrm{rad}(V)$.

We define a *radical involution* to be an element $\rho \in \mathrm{O}(\mathrm{q}, k)$ that acts trivially outside of the $\mathrm{rad}(V)$ and is of order 2. Each nontrivial orthogonal transformation on $\mathrm{rad}(V)$ detects a defective vector in V. For example if $\rho(g) = g'$ then $\mathrm{q}(g+g') = \mathrm{q}(g)+\mathrm{q}(g') = 0$. A *basic radical involution* is a map $\rho_i \in \mathrm{O}(\mathrm{q}, k)$ such that $\rho_i(g_i) = g_i'$ where g_i, g_i' are linearly independent vectors in $\mathrm{rad}(V)$ with $\mathrm{q}(g_i) = \mathrm{q}(g_i')$.

Proposition 3.21. *Every radical involution can be written as a finite product of basic radical involutions.*

Proof. Let ρ be a radical involution on V. There is a vector $g_1 \in \mathrm{rad}(V)$ such that ρ acts nontrivially on g_1. Then there must be a vector $g_1' \in \mathrm{rad}(V)$ that is linearly independent form g_1, or else order or ρ is not 2, such that $\mathrm{q}(g_1') = \mathrm{q}(g_1)$ and $\rho(g_1) = g_1'$. Now $\{g_1, g_1'\}$ forms a basis for a two dimensional subspace $\mathrm{rad}(V)_1 \subset \mathrm{rad}(V)$ with defect ≥ 1. If g_1 and g_1' are the only vectors where ρ acts nontrivially then we are done and $\rho = \rho_1$ is a basic radical involution. If not there exists an element $g_2 \in \mathrm{rad}(V)$ such that $g_2 \notin \mathrm{rad}(V)_1$ and $\rho(g_2) = g_2'$ defines a nontrivial action. Otherwise $g_2 \in \mathrm{rad}(V)_1$ and ρ is already defined on $\mathrm{rad}(V)_1$. So g_2, g_2' are linearly independent from one another and from $\mathrm{rad}(V)_1$. We define $\mathrm{rad}(V)_2$ to be the span of $\{g_1, g_1', g_2, g_2'\}$. If ρ acts trivially outside $\mathrm{rad}(V)_2$ we are done and $\rho = \rho_2\rho_1$. For any $\mathrm{rad}(V)_i$ either ρ acts trivially outside of $\mathrm{rad}(V)_i$ and $\rho = \rho_i \cdots \rho_2\rho_1$ or there exists a new vector g_{i+1} that is linearly independent. By induction we have that there exists a basis

$$\{g_1, g_1', g_2, g_2', \dots, g_l, g_l', h_{l+1}, \dots, h_s\},$$

of $\mathrm{rad}(V)$ such that $\dim(\mathrm{rad}(V)) = s$ and $\rho(g_i) = g_i'$ for all $1 \le i \le l$ and $\rho(h_j) = h_j$ for all $l+1 \le j \le s$. Each ρ_i acts nontrivially on the subspace spanned by $\{g_i, g_i'\}$ and trivially on remaining basis vectors. So $\rho = \rho_l \cdots \rho_2\rho_1$ is a product of basis radical involutions. $\square$

Proposition 3.22. *Two basic radical involutions ρ_1, ρ_2 are $\mathrm{O}(\mathrm{q}, k)$-conjugate if and only if ρ_1 and ρ_2 act non-trivially on isometric vectors.*

Proof. Let $\rho_1(g_1) = g_1'$ and $\rho_2(g_2) = g_2'$. Then

$$\delta\rho_1\delta^{-1}(g_2) = \rho_2(g_2),$$

if and only if $\delta^{-1}(g_2) = g_1$. $\square$

Each radical involution maps an element $g_i \mapsto g_i'$ with $\mathrm{q}(g_i) = \mathrm{q}(g_i')$. We chose a basis of $\mathrm{rad}(V)$ with respect to ρ of length m to be

$$\{g_1 + g_1', g_1, g_2 + g_2', g_2, \dots, g_m + g_m', g_m, h_{2m+1}, \dots, h_s\},$$

where ρ acts nontrivially on $g_i + g_i'$ and h_j. We define the *quadratic signature* of the radical involution to be

$$\langle \mathrm{q}(g_1), \mathrm{q}(g_2), \dots, \mathrm{q}(g_m)\rangle.$$

Corollary 3.23. *All radical involutions of length m with same quadratic signature*

$$\langle \mathrm{q}(g_1), \mathrm{q}(g_2), \dots, \mathrm{q}(g_m)\rangle,$$

are conjugate.

4. Involutions of a general vector space

Elements in $\mathrm{O}(\mathrm{q}, k)$ where (V, q) is a quadratic space and $\dim(\mathrm{rad}(V)) \ge 0$, can be thought of in terms of block matrices. Consider a matrix of the form

$$(\tau, Y, \rho) = \begin{bmatrix} \tau & 0 \\ Y & \rho \end{bmatrix}, \tag{7}$$

where $\tau \in \mathrm{Sp}(B_{V_\mathcal{B}}, k)$ and where $\mathcal{B}$ is a basis of some maximal nonsingular space in V with $\dim(V_\mathcal{B}) = 2r$, $\dim(\mathrm{rad}(V)) = s$ and $\dim(V) = 2r + s$. Now we know that $\mathrm{rad}(V)$ must be left invariant by such a map so $\rho \in \mathrm{O}(\mathrm{q}_{\mathrm{rad}(V)}, k)$ and $(\tau, Y) \in \mathcal{M}(\mathrm{q}, V_\mathcal{B})$, where

$$\mathcal{M}(\mathrm{q}, V_{\mathcal{B}}) = \{(\phi, X) \in \mathrm{Sp}(B_{V_{\mathcal{B}}}, k) \ltimes \mathrm{Mat}_{2r,s} \mid \mathrm{q}(\phi(w)) = \mathrm{q}(w + Xw)\}.$$

Let q be a quadratic form of type (r, s) on a vector space V over a field k of characteristic 2 with $\dim(V) = 2r + s$. Let us define

$$\mathcal{B} = \{u_1, v_1, u_2, v_2, \dots, u_r, v_r\},$$

to be some basis of a maximal nonsingular subspace of V of dimension $2r$. Then

$$V = V_{\mathcal{B}} \perp \mathrm{rad}(V),$$

where $V_{\mathcal{B}} = \mathrm{span}\,\mathcal{B}$. We are interested in the case when $\dim(\mathrm{rad}(V)) = s > 1$ as all elements of $\mathrm{O}(\mathrm{q}, k)$ leave $\mathrm{rad}(V)$ invariant.

Proposition 4.1. $(\tau, Y, \rho)^2 = \mathrm{id}$ *if and only if* $\tau^2 = \mathrm{id}$, $\rho^2 = \mathrm{id}$ *and* $Y = \rho Y \tau$.

Proof. Thinking of (τ, Y, ρ) as a block matrix we have

$$\begin{bmatrix} \tau & 0 \\ Y & \rho \end{bmatrix}^2 = \begin{bmatrix} \tau^2 & 0 \\ Y\tau + \rho Y & \rho^2 \end{bmatrix}.$$

This matrix is order 2 if and only if $\tau^2 = \mathrm{id}$, $\rho^2 = \mathrm{id}$ and $Y = \rho Y \tau$. □

There are two main types of maps of order 2 of this form to consider. First we notice that if the above map has order 2 it is necessary that $Y\tau = \rho Y$.

Proposition 4.2. *If* $\tau, \rho \in \mathrm{O}(\mathrm{q}, k)$ *such that* τ *is a diagonal involution and* ρ *is a radical involution, then there exists a map* $Y : V_{\mathcal{B}_\tau} \to \mathrm{rad}(V)$ *such that* $\widetilde{Y} = \left[\begin{smallmatrix} \mathrm{id} & 0 \\ Y & \mathrm{id} \end{smallmatrix}\right] \in \mathrm{O}(\mathrm{q}, k)$ *and* (τ, Y, ρ) *is an involution on* V.

Proof. Let V be a vector space over a field of characteristic 2 with a quadratic form q of type (r, s),

$$\tau = \tau_{u_l} \cdots \tau_{u_2} \tau_{u_1},$$

and define

$$\mathcal{B}_\tau = \{u_1, u_1', u_2, u_2', \dots, u_l, u_l', w_1, w_1', \dots, w_{2(r-l)}, w_{2(r-l)}'\},$$

so that we have the decomposition $V = V_{\mathcal{B}_\tau} \perp \mathrm{rad}(V)$ and W is the subspace of $V_{\mathcal{B}_\tau}$ such that $\tau|_W = \mathrm{id}_W$. We can define

$$\widetilde{Y}(u_i) = u_i + h_i + \rho(h_i)$$
$$\widetilde{Y}(u_i') = u_i' + \frac{1}{\mathrm{q}(u_i)}(h_i + \rho(h_i))$$
$$\widetilde{Y}(w_j) = w_j.$$

Notice that $h_i + \rho(h_i)$ is a vector in $\mathrm{rad}(V)$ such that $\mathrm{q}(h_i + \rho(h_i)) = 0$. A direct computation shows that the properties in Proposition 4.1 are met and (τ, Y, ρ) is an involution in $\mathrm{O}(\mathrm{q}, k)$. □

Moreover, the above (τ, Y, ρ) is such that $u_i \mapsto u_i + (h_i + \rho(h_i))$ and so $\mathcal{B}_\tau$ is shifted by $h_i + \rho(h_i)$ and $\tau_{u_i} \mapsto \tau_{u_i+(h_i+\rho(h_i))}$.

Proposition 4.3. *A map of the form*

$$(\phi, X, \delta) = \begin{bmatrix} \phi & 0 \\ X & \delta \end{bmatrix},$$

is an element of $\mathrm{O}(\mathrm{q}, k)$ *if and only if* $\phi \in \mathrm{Sp}(B_{V_\mathcal{B}}, k)$, $\delta \in \mathrm{O}(\mathrm{q}_{\mathrm{rad}(V)}, k)$ *and* $\mathrm{q}(\widetilde{X}(w)) = \mathrm{q}(\phi(w))$ *for all* $w \in V_\mathcal{B}$ *and* $\widetilde{X} = \mathrm{id}_V + X$.

Proof. Let $w \in V_\mathcal{B}$ and $g \in \mathrm{rad}(V)$ and assume $(\phi, X, \delta) \in \mathrm{O}(\mathrm{q}, k)$. Then we have

$$\begin{aligned} \mathrm{q}(w+g) &= \mathrm{q}((\phi, X, \delta)(w+g)) \\ &= \mathrm{q}(\phi(w) + Xw + \delta(g)) \\ &= \mathrm{q}(\phi(w)) + \mathrm{q}(Xw) + \mathrm{q}(\delta(g)). \end{aligned}$$

Recall that $\mathrm{q}(w+g) = \mathrm{q}(w) + \mathrm{q}(g)$ and $\mathrm{q}(\delta(g)) = \mathrm{q}(g)$ to establish

$$\mathrm{q}(w) + \mathrm{q}(g) = \mathrm{q}(\phi(w)) + \mathrm{q}(Xw) + \mathrm{q}(g).$$

This is true since $\delta \in \mathrm{O}(\mathrm{q}_{\mathrm{rad}(V)}, k)$ and (ϕ, X, δ) must leave $\mathrm{rad}(V)$ invariant. So setting

$$\widetilde{X} = \begin{bmatrix} \mathrm{id} & 0 \\ X & \mathrm{id} \end{bmatrix},$$

we have

$$\mathrm{q}(w) + \mathrm{q}(Xw) = \mathrm{q}(\phi(w)) \Rightarrow \mathrm{q}(\widetilde{X}(w)) = \mathrm{q}(\phi(w)).$$

Now assuming that $\phi \in \mathrm{Sp}(B_{V_\mathcal{B}}, k)$, $\delta \in \mathrm{O}(\mathrm{q}_{\mathrm{rad}(V)}, k)$ and $\mathrm{q}(\widetilde{X}(w)) = \mathrm{q}(\phi(w))$ we can reverse the argument. □

The property in Proposition 4.3 is preserved under composition as we now note. We can consider the product

$$(\phi, X, \delta)(\phi', X', \delta') = (\phi\phi', X\phi' + \delta X', \delta\delta').$$

We may also compute

$$\begin{aligned}
\mathrm{q}((X\phi' + \delta X')(w)) &= \mathrm{q}(X\phi'(w)) + \mathrm{q}(\delta(X'w)) \\
&= \mathrm{q}(\phi\phi'(w)) + \mathrm{q}(\phi'(w)) + \mathrm{q}(X'w) \\
&= \mathrm{q}(\phi\phi'(w)) + \mathrm{q}(\phi'(w)) + \mathrm{q}(\phi'(w)) + \mathrm{q}(w) \\
&= \mathrm{q}(\phi\phi'(w)) + \mathrm{q}(w),
\end{aligned}$$

which is equivalent.

The purpose of the next result is to establish that any map of the form $(\tau_{\mathcal{U},a}, Y, \rho) \in \mathrm{O}(\mathrm{q}, k)$, where $\tau_{\mathcal{U},a}$ is a symplectic involution, can be written with an orthogonal involution in the first component.

Proposition 4.4. *Every involution of the form $(\tau_{\mathcal{U},a}, Y, \rho)$ can be written as*

$$(\tau_{\mathcal{U}'}, Y', \rho) = (\tau_{\mathcal{U}'}, 0, \mathrm{id})(\mathrm{id}, Y', \mathrm{id})(\mathrm{id}, 0, \rho),$$

where each of the three maps in the decomposition is in $\mathrm{O}(\mathrm{q}, k)$.

Proof. Assume that $a_i \in k^*$ for all i otherwise the corresponding factor would be trivial. We can choose a basis such that $\mathrm{q}(Yw) = 0$ for all $w \in V_{\mathcal{B}_{\tau_{\mathcal{U}'}}}$ by replacing u_i with

$$u_i' = u_i + \frac{1}{a_i} Y v_i.$$

To see that this works we first observe that

$$(\tau_{\mathcal{U}}, Y, \rho)(u_i) = u_i + Yu_i,$$

where $Yu_i \in \mathrm{rad}(V)$. Then computing the norm of $u_i \in \mathcal{B}_{\tau_{\mathcal{U}}}$ we have

$$\begin{aligned}
\mathrm{q}\left((\tau_{\mathcal{U}}, Y, \rho)(u_i)\right) &= \mathrm{q}(u_i + Yu_i) \\
\mathrm{q}(u_i) &= \mathrm{q}(u_i) + \mathrm{q}(Yu_i).
\end{aligned}$$

Simplifying, we see that $\mathrm{q}(Yu_i) = 0$.

There is a set of vectors in the nonsingular completion of $\mathcal{U}$, which we will label v_i such that $B(u_i, v_i) = 1$. These vectors are not fixed by $\tau_{\mathcal{U}}$. Computing the image of v_i we have

$$\begin{aligned}(\tau_{\mathcal{U}}, Y, \rho)(v_i) &= v_i + a_i B(u_i, v_i) u_i + Y v_i \\ &= v_i + a_i u_i + Y v_i.\end{aligned}$$

We take the norm of the image of v_i

$$\begin{aligned}\mathrm{q}\left((\tau_{\mathcal{U}}, Y, \rho)(v_i)\right) &= \mathrm{q}(v_i + a_i u_i + Y v_i) \\ \mathrm{q}(v_i) &= \mathrm{q}(v_i + a_i u_i) + \mathrm{q}(Y v_i) \\ &= \mathrm{q}(v_i) + a_i^2\,\mathrm{q}(u_i) + B(v_i, a_i u_i) + \mathrm{q}(Y v_i) \\ &= \mathrm{q}(v_i) + a_i^2\,\mathrm{q}(u_i) + a_i + \mathrm{q}(Y v_i).\end{aligned}$$

We can solve for $\mathrm{q}(Y v_i)$ and see that

$$\mathrm{q}(Y v_i) = a_i^2\,\mathrm{q}(u_i) + a_i.$$

Notice here that $\mathrm{q}(Y v_i) = 0$ only if $a_i = 0$ or $\mathrm{q}(u_i) = 1/a_i$. We have assumed $a_i \neq 0$ and if $\mathrm{q}(u_i) = 1/a_i$, τ_{u_i, a_i} is already an orthogonal transvection. Let us compute the norm of $u_i' = u_i + \dfrac{1}{a_i} Y v_i$,

$$\begin{aligned}\mathrm{q}\left(u_i + \frac{1}{a_i} Y v_i\right) &= \mathrm{q}(u_i) + \frac{1}{a_i^2}\,\mathrm{q}(Y v_i) \\ &= \mathrm{q}(u_i) + \frac{1}{a_i^2}\left(a_i^2\,\mathrm{q}(u_i) + a_i\right) \\ &= \mathrm{q}(u_i) + \mathrm{q}(u_i) + \frac{1}{a_i} \\ &= \frac{1}{a_i}.\end{aligned}$$

Now we can verify that $\tau_{u_i, a_i} = \tau_{u_i'}$ for all i, which is enough to say that $\tau_{\mathcal{U}, a} = \tau_{\mathcal{U}'}$. First notice that

$$B(u_i, u_i') = B\left(u_i, u_i + \frac{1}{a_i} Y v_i\right) = 0,$$

which tells us that $\tau_{\mathcal{U}'}$ fixes $\mathcal{U}$. Next we compute the image of v_i for all i and see that

$$\begin{aligned}\tau_{u_i'}(v_i) &= v_i + \frac{B\left(u_i + \frac{1}{a_i} Y v_i, v_i\right)}{\mathrm{q}\left(u_i + \frac{1}{a_i} Y v_i\right)}(u_i + \frac{1}{a_i} Y v_i) \\ &= v_i + a_i\left(u_i + \frac{1}{a_i} Y v_i\right) \\ &= v_i + a_i u_i + Y v_i.\end{aligned}$$

The map Y' acts on V by adding defective vectors to the u_i and acting as the zero map on the v_i. So we have that $\mathrm{q}(Yw) = 0$ for all $w \in V$. In the end we have that $(\tau_{\mathcal{U}'}, 0, \mathrm{id}) \in \mathrm{O}(\mathrm{q}, k)$ since $\tau_{\mathcal{U}'}$ is an orthogonal transvection involution. The map $(\mathrm{id}, Y', \mathrm{id}) \in \mathrm{O}(\mathrm{q}, k)$, since Y' can only add defective vectors to any element and so must preserve q. Finally $(\mathrm{id}, 0, \rho) \in \mathrm{O}(\mathrm{q}, k)$ since it acts isometrically on the radical and trivially elsewhere. □

Now we can prove the following theorem.

Theorem 4.5. *Two involutions* $(\tau_{\mathcal{U},a}, Y, \rho), (\tau_{\mathcal{X},b}, Z, \gamma) \in \mathrm{O}(\mathrm{q}, k)$ *are* $\mathrm{O}(\mathrm{q}, k)$*-conjugate if and only if there exists* $(\varphi, X, \delta) \in \mathrm{O}(\mathrm{q}, k)$ *such that*

1. $\varphi \tau_{\mathcal{U},a} \varphi^{-1} = \tau_{\mathcal{X},b}$
2. $\delta \rho \delta^{-1} = \gamma$
3. $X \tau_{\mathcal{U},a} + \gamma X = Z\varphi + \delta Y$.

Proof. We can consider the elements of $\mathrm{O}(\mathrm{q}, k)$ as block diagonal matrices and compute

$$\begin{bmatrix} \varphi & 0 \\ X & \delta \end{bmatrix} \begin{bmatrix} \tau_{\mathcal{U},a} & 0 \\ Y & \rho \end{bmatrix} \begin{bmatrix} \varphi & 0 \\ X & \delta \end{bmatrix}^{-1} = \begin{bmatrix} \varphi & 0 \\ X & \delta \end{bmatrix} \begin{bmatrix} \tau_{\mathcal{U},a} & 0 \\ Y & \rho \end{bmatrix} \begin{bmatrix} \varphi^{-1} & 0 \\ \delta^{-1} X \varphi^{-1} & \delta^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} \varphi \tau_{\mathcal{U},a} \varphi^{-1} & 0 \\ (X\tau_{\mathcal{U},a} + \delta Y)\varphi^{-1} + \delta\rho\delta^{-1} X \varphi^{-1} & \delta \rho \delta^{-1} \end{bmatrix}.$$

The first two equations from the statement of the Proposition can be identified by setting the upper left and lower right diagonal equal to the corresponding block in $(\tau_{\mathcal{X},b}, Z, \gamma)$. To get the final equation notice that the lower left block off the diagonal in the computation contains $\delta\rho\delta^{-1}$ which must be γ by equation 2. We then have the following equation

$$(X\tau_{\mathcal{U},a} + \delta Y)\varphi^{-1} + \gamma X \varphi^{-1} = Z.$$

Multiplying φ and then adding δY to both sides of the equation we arrive at

$$X\tau_{\mathcal{U},a} + \gamma X = Z\varphi + \delta Y. \quad \square$$

Notice that in Theorem 4.5 property 1 is equivalent to $(\mathcal{U}, a)$ and $(\mathcal{X}, b)$ being involution compatible, and property 2 is equivalent to ρ and γ having equivalent quadratic signatures.

In general the existence of a triple (φ, X, δ) depends greatly on q and k. We can consider the case when q is anisotropic when restricted to $\mathrm{rad}(V)$. In this case if $(\tau_{\mathcal{U}}, Y, \rho)$ is an orthogonal involution then $\rho = \mathrm{id}$ and $Y = 0$, since for any basis of $\mathrm{rad}(V)$ each basis vector will have a unique nonzero norm. The other extreme would be if $\mathrm{rad}(V)$ is totally isotropic, so that every vector in $\mathrm{rad}(V)$ has norm zero. In this case $\rho \in \mathrm{GL}_s(k)$ where $s = \dim(\mathrm{rad}(V))$ and $Y \in \mathrm{Mat}_{r,s}(k)$, since there are no constraints contributed by q on rad(V) and adding vectors from the radical leaves q invariant on the image of any nonsingular subspace of V.

Declaration of competing interest

There is no conflict of interest.

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