

Minimal parabolic k -subgroups acting on symmetric k -varieties corresponding to k -split groups

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Symmetric k -varieties are a natural generalization of symmetric spaces to general fields k . We study the action of minimal parabolic k -subgroups on symmetric k -varieties and define a map that embeds these orbits within the orbits corresponding to algebraically closed fields. We develop a condition for the surjectivity of this map in the case of k -split groups that depends only on the dimension of a maximal k -split torus contained within the fixed point group of the involution defining the symmetric k -variety.

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1. Introduction

Real reductive symmetric spaces are the homogeneous spaces $X := G/H$, where G is a reductive Lie group and H is an open subgroup of the fixed point group of an involution of G . The study of the representations of these spaces culminated in the work of Delorme [5], opening the door for the study of generalizations of the real reductive spaces.

Let k be a field of characteristic not 2. The symmetric k -variety is the homogeneous space $X_k := G_k/H_k$, where G is a reductive algebraic group defined over k , H is a k -open subgroup of the set of fixed points of a k -involution θ , and G_k (respectively, H_k) denote the k -rational points of G (respectively, H). The symmetric k -varieties have applications in diverse fields of mathematics including representation theory [22], geometry [13], the study of character sheaves [11], and in the cohomology of arithmetic subgroups [21]. Beginning in the 1980s, Helminck and Wang [9] commenced a study of the rationality properties of the symmetric k -varieties for

arbitrary fields. These have continued to be studied because of their applications, particularly in representation theory. For this reason, it is natural to consider how the parabolic orbits on a symmetric variety (i.e. where k is algebraically closed) are related to the orbits of a corresponding minimal parabolic k -subgroup acting on a symmetric k -variety. Our approach is to embed the orbits over the symmetric k -variety within the orbits corresponding the algebraic closure of k via a map which generalizes the complexification of orbits in the case $k = \mathbb{R}$:

$$\varphi : P_k \backslash G_k / H_k \hookrightarrow P \backslash G / H.$$

After providing the relevant background and notation, we recall the necessary results to discuss the orbits of Borel and parabolic subgroups acting on symmetric varieties. We then explain the relevant differences between the algebraically closed case and the general case. Finally, we restrict our attention to the case G is a k -split group and develop a condition for the surjectivity of the generalized complexification map.

2. Background

We follow the notation established in [3, 19, 10]. Throughout the paper, G will denote a connected reductive algebraic group, θ a group involution of G that leaves the k -rational points invariant and H a k -open subgroup of $G^\theta = \{g \in G \mid \theta(g) = g\}$. The variety G/H is called the symmetric variety and G_k/H_k is called the symmetric k -variety.

Define a map

$$\tau : G \rightarrow G, \quad \tau(x) = x\theta(x)^{-1}.$$

Denote the image of τ by Q , then τ induces an isomorphism between G/H and Q as well as an isomorphism between G_k/H_k and Q_k . It is sometimes more convenient in calculation to let H act from the left, in this case $\tau(x) = x^{-1}\theta(x)$.

Let $\text{Aut}(G)$ denote the set of group automorphisms $G \rightarrow G$, and for a subgroup $K \subset G$ we will use $\text{Aut}(G, K)$ to denote the set of automorphisms of G which leave K invariant. In particular, we are concerned with the order 2 elements of $\text{Aut}(G, G_k)$. For $g \in G$, we will use $\text{Int}(g)$ to denote the inner automorphism corresponding to g . We denote the algebraic closure of k by \bar{k} .

Recall that a torus T of G is a connected semisimple abelian subgroup. Let T be a torus, then $N_G(T)$ will denote the normalizer of T in G , $Z_G(T)$ will denote the centralizer of T in G , and $W_G(T) = N_G(T)/Z_G(T)$ will denote the Weyl group. The classification of orbits of minimal parabolic subgroups acting on symmetric k -varieties relies on a quotient of the Weyl group by elements having representatives in the fixed point group H , and we denote this set by $W_H(T) = N_H(T)/Z_H(T)$. These groups all have analogues for a group defined over k , replacing the group in the definition of in $N_G(T)$, $Z_G(T)$, $W_G(T)$, and $W_H(T)$ with its associated k -rational points, we obtain definitions for $N_{G_k}(T)$, $Z_{G_k}(T)$, $W_{G_k}(T)$, and $W_{H_k}(T)$.

Let $\varphi \in \text{Aut}(G, T)$, then T can be decomposed via its φ (Lie algebra) eigenspaces, i.e. $T = T_\varphi^+ T_\varphi^-$ where $T_\varphi^+ = \{t \in T \mid \theta(t) = t\}^\circ$ and $T_\varphi^- = \{t \in T \mid \theta(t) = t^{-1}\}^\circ$, where K° denotes the identity component of subgroup $K \subset G$. The product map

$$\mu : T_\varphi^+ \times T_\varphi^- \rightarrow T, \quad \mu(t_1, t_2) = t_1 t_2$$

is a separable isogeny. In fact, $T_\varphi^+ \cap T_\varphi^-$ is an elementary abelian 2-group. Of particular interest will be the case when $\varphi = \theta$, in this case we will use T^+ for T_θ^+ and T^- for T_θ^- .

Maximal tori play a fundamental role in the description of the structure of symmetric k -varieties. A torus is maximal if it is properly contained in no other torus. It is a fact that all such tori are conjugate under G .

We denote the root system of a torus T by $\Phi(T)$, its positive roots by $\Phi^+(T)$, and a basis by Δ . A torus T is called θ -split if $\theta(t) = t^{-1}$ for all $t \in T$. From [14], we know that if A is a maximal θ -split torus, then $\Phi(A)$, the set of roots of A in G , is a root system with Weyl group $W(A) = N_G(A)/Z_G(A)$. Recall that a k -torus is k -split if it can be diagonalized over the base field k . We will call a k -torus (θ, k) -split if it is θ -split and k -split. These tori yield a natural root system for the symmetric k -variety G_k/H_k since a maximal (θ, k) -split torus A of G has a root system $\Phi(A)$ [9] with Weyl group $N_{G_k}(A)/Z_{G_k}(A)$. Additionally this root system can be obtained by restricting the roots of a maximal torus of G containing A .

The rank of a group G is the dimension of a maximal torus $T \subset G$ and the k -rank of a group is the dimension of a maximal k -split torus. A group is called k -split if the k -rank is equal to the rank.

Given a Borel subgroup B , fix a maximal torus $T \subset B$. Recall that there is a one-to-one correspondence between subsets of the simple roots and parabolic subgroups containing B . A parabolic subgroup defined over k is called a parabolic k -subgroup. A parabolic k -subgroup is minimal if it properly contains no other parabolic k -subgroups, thus over algebraically closed fields the minimal parabolic subgroups are Borel subgroups. For non-algebraically closed fields it may be the case that nontrivial minimal parabolic k -subgroups do not exist. Parabolic subgroups are also self-normalizing, and can be decomposed via $P = LU$, where U is the unipotent radical $R_u(P)$ and L is the Levi factor of P . In the case that $P = B$ is a Borel subgroup, this decomposition simplifies to $B = TU$, where $T \subset B$ is a maximal torus and $U \subset B$ is unipotent.

To obtain a characterization of the orbits $B \backslash G/H$, one can separately consider B -orbits on G/H , H -orbits on G/B , or (B, H) -orbits on G . We outline these results in this section. Let $T \subset G$ be a maximal θ -stable torus, which exist by [20]. We will first consider the H -orbits. Let \mathcal{B} denote the variety of all Borel subgroups of G , then we can identify G/B with \mathcal{B} since all Borel subgroups are conjugate over G , and let \mathcal{C} denote the set of pairs (B', T') where T' is a θ -stable maximal torus contained in the Borel subgroup B' . H acts on both \mathcal{B} and \mathcal{C} by conjugation, denote the sets of H -orbits by \mathcal{B}/H and \mathcal{C}/H , respectively. The H -orbits on \mathcal{C} can

be analyzed in two steps. First, let I be a set which indexes the H -conjugacy classes of θ -stable maximal tori of G . This allows us to choose representatives $\{T_i\}_{i \in I}$ for the H -conjugacy classes of θ -stable maximal tori of G . Next, we consider the Borel subgroups containing each T_j that are not H -conjugate, which are determined by $W_G(T_j)/W_H(T_j)$. Thus, the H -orbits on \mathcal{C} are in correspondence with $\cup_{i \in I} W_G(T_i)/W_H(T_i)$.

G acts on $G/H \cong Q$ (from the left) via the θ -twisted action, i.e. $g*q = gq\theta(g)^{-1}$. Thus, the B -orbits on G/H can be viewed as B -cosets in Q , which we denote $B \backslash Q$.

The (H, B) -orbits on G are the same as the $B \times H$ -orbits on G and the action is given by $(b, h)*g := bgh^{-1}$. From [9], we know that every U -orbit on G/H , where U is the unipotent component of B , meets $N_G(T)$. Let $\mathcal{V} = \{g \in G \mid \tau(g) \in N_G(T)\}$, then \mathcal{V} is stable under left multiplication by $N_G(T)$ and right multiplication by H . We denote by V the $T \times H$ -orbits on \mathcal{V} , which in fact parameterize the $(B \times H)$ orbits on G . In fact, these characterizations are isomorphic.

Theorem 2.1 ([18, §4]). *Let B be a Borel subgroup of G and $\{T_i\}_{i \in I}$ a set of representatives of the H -conjugacy classes of θ -stable maximal tori in G . Then*

$$B \backslash G/H \cong \mathcal{B}/H \cong \bigcup_{i \in I} W_G(T_i)/W_H(T_i) \cong \mathcal{C}/H \cong B \backslash Q \cong V.$$

One can endow the set of double cosets $B \backslash G/H$ with a partial order that generalizes the usual Bruhat order on a connected reductive algebraic group, this was studied in [15, 16]. Let $\mathcal{O}_1 = Bg_1H$ and $\mathcal{O}_2 = Bg_2H$, then $\mathcal{O}_1 \leq \mathcal{O}_2$ if $\mathcal{O}_1 \subset \text{cl}(\mathcal{O}_2)$, where $\text{cl}(\mathcal{O}_2)$ denotes the Zariski closure of \mathcal{O}_2 .

The Bruhat order is a refinement of the order on the I -poset, which we now introduce. Let I be an index set of H -conjugacy classes of the θ -stable maximal tori in G , and denote these conjugacy classes by $\{[T_i]\}_{i \in I}$. We define $[T_i] \leq [T_j]$ if $T_i^- \subset T_j^-$ for some representatives T_i of $[T_i]$ and T_j of $[T_j]$. Given a set of representatives $\{T_i\}_{i \in I}$ for $\{[T_i]\}_{i \in I}$, we write $T_i \leq T_j$ if $[T_i] \leq [T_j]$. Furthermore, we can associate poset diagrams to the orbit decompositions. Since the Bruhat poset is a refinement of the I -poset, we will call the Bruhat diagram obtained from an I -diagram the expansion of the I -diagram.

3. Parabolic Subgroups acting on Symmetric k -Varieties

In this section, we collect the necessary results and set the notation needed to discuss the generalized complexification map in Sec. 5. Following [8], we outline the action of parabolic subgroups on the symmetric k -varieties in a number of cases. While the approach is similar in each case, there are important differences which effect the analysis of the generalized complexification map.

3.1. $k = \bar{k}$, P a parabolic subgroup acting G/H

This section summarizes the results of Brion and Helminck [4] for the generalized Bruhat decomposition. The important difference in this case is that for a fixed

parabolic P we are not assured that the set of G -conjugates of P includes all parabolic subgroups of G . In fact, if we identify parabolic subgroups P_1 and P_2 with their associated subsets of a base for the root system $\Gamma_1, \Gamma_2 \subset \Delta$, we see that P_1 and P_2 are G -conjugate if and only if $\Gamma_1 = \Gamma_2$. Let \mathcal{P} denote the variety of all parabolic subgroups of G and let \mathcal{D} denote the set of triples (P, B, T) , where B is a Borel subgroup of P such that $(P \cap H)B$ is open in P and T is a θ -stable maximal torus of B . Fix a parabolic subgroup P and let \mathcal{P}^P denote the set of G conjugates of P and let \mathcal{D}^P denote the set of triples $(P', B', T') \in \mathcal{D}$ such that $P' \in \mathcal{P}^P$. G acts on \mathcal{P} , \mathcal{P}^P , \mathcal{D} , and \mathcal{D}^P via conjugation; denote the H -orbits on these sets by \mathcal{P}/H , \mathcal{P}^P/H , \mathcal{D}/H , and \mathcal{D}^P/H , respectively.

Theorem 3.1. *There is a bijective map $\mathcal{P}/H \rightarrow \mathcal{D}/H$.*

Every maximal k -split torus of G is conjugate under G_k , the k -rational points of G . Furthermore, every minimal parabolic k -subgroup of G contains a maximal k -split torus. Let A be a maximal k -split torus. We wish to characterize all minimal parabolic k -subgroups which contain A . As a generalization of the algebraically closed case, we have the following proposition.

Proposition 3.1. *Suppose A_1 and A_2 are two maximal k -split tori, and P_1 and P_2 are two minimal parabolic k -subgroups containing A_1 and A_2 , respectively. Then the element of G_k that conjugates A_1 to A_2 also conjugates P_1 to P_2 .*

We now fix a parabolic subgroup P , and let T be the θ -stable maximal torus occurring in the image of P under the bijection in Theorem 3.1. Let $x_1, x_2, \dots, x_r \in G$ such that $T_1 = x_1 T x_1^{-1}, \dots, T_r = x_r T x_r^{-1}$ are representatives for the H -conjugacy classes of the θ -stable maximal tori occurring in \mathcal{D} . Let $P_1 = x_1 P x_1^{-1}, \dots, P_r = x_r P x_r^{-1}$ and denote by $W_{P_i}(T_i)$ the Weyl group of P_i . One then sees that for each T_i the H -conjugacy classes of $(P', B', T_i) \in \mathcal{D}^P$ are in bijection with $W_{P_i}(T_i) \backslash W_G(T_i) / W_H(T_i)$. As before let $\mathcal{V} = \{g \in G \mid \tau(g) \in N_G(T)\}$ and let $\mathcal{V}^P = \{g \in \mathcal{V} \mid BgH \text{ is open in } PgH\}$. Then the actions of B, H on \mathcal{V} extend to actions of P, H on \mathcal{V}^P , and every $P \times H$ -orbit on G meets \mathcal{V}^P in a unique (T, H) double coset. Let V^P denote the (T, H) -double cosets in \mathcal{V}^P . We can now generalize Springer's theorem characterizing $B \backslash G / H$ double cosets to the case of a general parabolic subgroup.

Theorem 3.2. *There is a bijective map from the set of H -orbits in \mathcal{P} onto the set of H -conjugacy classes of triples $(P, B, T) \in \mathcal{D}$. Moreover for a fixed parabolic subgroup P , we have*

$$P \backslash G / H \cong \mathcal{P}(G)^P / H \cong \mathcal{D}^P / H \cong \bigcup_{i=1}^r W_{P_i}(T_i) \backslash W_G(T_i) / W_H(T_i) \cong V^P.$$

3.2. Orbits over non-algebraically closed fields

We now turn to the double cosets which we study for the remainder. If we restrict our attention to parabolic subgroups defined over k that are minimal we can obtain

a characterization very similar to the case of a Borel subgroup acting on G/H . In this case, we are assured of the existence of θ -stable Levi factor.

Theorem 3.3 ([9]). *Let P be a minimal parabolic k -subgroup with unipotent radical U , then P contains a θ -stable maximal k -split torus, unique up to an element of $(H \cap U)_k$.*

Let P be a minimal parabolic k -subgroup and let G and H be defined as before. With a few adjustments, one can construct a characterization of the $P_k \backslash G_k / H_k$ orbits in several equivalent ways by considering P_k orbits on Q_k , H_k orbits on $P_k \backslash G_k$, or $P_k \times H_k$ orbits on G_k .

We start with the H_k -orbits on $P_k \backslash G_k$. Let \mathcal{P}_k denote the variety of all minimal parabolic k -subgroups of G , then we have that $P_k \backslash G_k$ is isomorphic to \mathcal{P}_k . G_k acts on \mathcal{P}_k by conjugation, so we can identify the double cosets with the H_k -orbits on \mathcal{P}_k , denote these orbits by \mathcal{P}_k / H_k .

Let \mathcal{C}_k denote the set of all pairs (P'_k, A'_k) , where P'_k is a minimal parabolic k -subgroup and A'_k is a θ -stable maximal k -split torus contained in P_k . G_k acts on \mathcal{C}_k by conjugation in both coordinates, i.e. $g * (P'_k, A'_k) = (gP'_k g^{-1}, gA'_k g^{-1})$. We can analyze the H_k orbits on \mathcal{C}_k (denoted \mathcal{C}_k / H_k) in two steps; first we consider the H_k -conjugacy classes of θ -stable maximal k -split tori and choose a set of representatives for these conjugacy classes, and second for each representative of an H_k -conjugacy class we consider the set of minimal parabolic k -subgroups that contain the representative but are not conjugate via H_k . This allows one to identify \mathcal{C}_k / H_k with $\cup_{i \in I} W_{G_k}(A_i) / W_{H_k}(A_i)$, where $\{A_i\}_{i \in I}$ is a set of representatives for the H_k conjugacy classes of θ -stable maximal k -split tori.

P_k acts on G_k / H_k via the θ -twisted action. Let $A \subset P$ be a maximal k -split torus. Then as in the case of a Borel subgroup acting on the symmetric space over an algebraically closed field, we have that the orbit of the unipotent radical of P_k meets $N_{G_k}(A)$. Let $\mathcal{V}_k = \{x \in G_k \mid \tau(x) \in N_{G_k}(A)\}$, then we can identify the P_k -orbits on G_k / H_k with the $Z_{G_k}(A) \times H_k$ -orbits on \mathcal{V}_k . We denote these orbits by V_k .

Theorem 3.4. *For a minimal parabolic k -subgroup P and $\{A_i\}_{i \in I}$ a set of representatives for the H_k -conjugacy classes of θ -stable maximal k -split tori, we have*

$$P_k \backslash G_k / H_k \cong \mathcal{P}_k / H_k \cong \bigcup_{i \in I} W_{G_k}(A_i) / W_{H_k}(A_i) \cong P_k \backslash Q_k \cong V_k.$$

We observe that this characterization is in direct analogy with the case of a Borel subgroup acting on the symmetric space and simpler than the case of a general parabolic subgroup acting on the symmetric space. This is because a minimal parabolic k -subgroup contains a θ -stable maximal k -split torus in light of Theorem 3.3.

Example 3.1. Let $G = \mathrm{SL}(2, \mathbb{C})$ with real form $G_{\mathbb{R}} = \mathrm{SL}(2, \mathbb{R})$. Let T denote the group of diagonal matrices and P the set of upper triangular matrices.

- (1) Let $\sigma = \text{Int} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $H_{\mathbb{R}} = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a^2 - b^2 = 1, a, b \in \mathbb{R} \right\}^{\circ}$ is connected and abelian and is diagonalizable by an orthogonal matrix. We compute the Weyl group elements

$$W_{G_{\mathbb{R}}}(T) = W_{G_{\mathbb{R}}}(H) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

The nonidentity element has a representative in H , given by $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, but this representative is not in $H_{\mathbb{R}}$. Thus, we conclude $|W_{G_{\mathbb{R}}}(T)/W_{H_{\mathbb{R}}}(T)| = |W_{G_{\mathbb{R}}}(H)/W_{H_{\mathbb{R}}}(H)| = 2$, and there are 4 orbits in $P_{\mathbb{R}} \backslash G_{\mathbb{R}}/H_{\mathbb{R}}$.

- (2) Let $\theta = \text{Int} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $H_{\mathbb{R}} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1, a, b \in \mathbb{R} \right\}$. While $H_{\mathbb{R}}$ is connected and abelian, its eigenvalues are complex and thus it is not an \mathbb{R} -split torus. Therefore there is only $H_{\mathbb{R}}$ -conjugacy class of θ -stable maximal k -split tori and only one orbit in $P_{\mathbb{R}} \backslash G_{\mathbb{R}}/H_{\mathbb{R}}$.

As with the Bruhat decomposition, we can reformulate Theorem 3.4 to obtain

$$G_k = \bigcup_{i \in I} \bigcup_{w \in W_{G_k}(A_i)} P_k \dot{w} H_k, \text{ where } \dot{w} \text{ is a representative of } w \in W_{G_k}(A_i).$$

Applying this observation to Example 3.1(b) and the fact that $\text{id} \in N_{G_{\mathbb{R}}}(T)$, we have that $G_{\mathbb{R}} = H_{\mathbb{R}} P_{\mathbb{R}}$, which is the well-known Iwasawa decomposition of a real reductive group with Cartan Involution θ .

While the characterization is similar, many of the properties from the case of an algebraically closed field do not hold. For instance, the orbits in Theorem 3.4 are always finite over an algebraically closed field, but over general fields this condition is frequently not satisfied.

In several cases, however, the number of orbits is finite. For algebraically closed fields this was proved by Springer [18], for $k = \mathbb{R}$ it was shown by Matsuki [12], Rossman [17] and Wolf [23], and for general local fields the result is due to Helminck and Wang [9].

Similar to the algebraically closed case, we can place an order on the I -poset. Let I be an index set of H_k -conjugacy classes of the θ -stable maximal k -split tori in G , and denote these conjugacy classes by $\{[A_i]\}_{i \in I}$. We define $[A_i] \leq [A_j]$ if $A_i^- \subset A_j^-$ for some representatives A_i of $[A_i]$ and A_j of $[A_j]$. Given a set of representatives $\{A_i\}_{i \in I}$ for $\{[A_i]\}_{i \in I}$, we write $A_i \leq A_j$ if $[A_i] \leq [A_j]$. As with maximal tori in the algebraically closed case, the order extends to all maximal k -split tori. Thus each representative of an H_k -conjugacy class of θ -stable maximal k -split tori with the same dimension of its split component correspond to nodes in the same level in the diagram associated to the order on the I -poset.

Example 3.2. Consider $\text{SL}(2, \mathbb{Q}_p)$, $p \equiv 1 \pmod{4}$. Let $\theta = \text{Int} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then by [1] we know that there are four H_k -conjugacy classes of θ -stable maximal (θ, \mathbb{Q}_p) -split

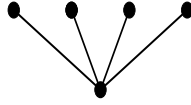


Fig. 1. I -poset diagram for $\mathrm{SL}(2, \mathbb{Q}_p)$, $p \equiv 1 \pmod{4}$.

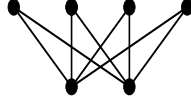


Fig. 2. Orbit diagram for $\mathrm{SL}(2, \mathbb{Q}_p)$, $p \equiv 1 \pmod{4}$.

tori and one H_k -conjugacy classes of θ -stable, θ -fixed maximal \mathbb{Q}_p -split tori. Thus the diagram for the I -poset is.

This diagram can be expanded to the order on the orbits given by closure in the Zariski topology. Each node in the I -poset diagram is expanded to the number of orbits corresponding to the torus A_i which is given by the order of $W_{G_k}(A_i)/W_{H_k}(A_i)$.

Example 3.3. Consider the setting of the previous example. Each (θ, \mathbb{Q}_p) -split torus corresponds to one orbit, while the θ -fixed torus corresponds to two. Thus, the orbit diagram is given in Fig. 2.

For a maximal torus T , the roots $\Phi(T)$ can be classified in analogy with the case $k = \mathbb{R}$. θ acts on the Weyl group, and therefore on the reflections corresponding to the roots. Thus θ acts on Φ . If $\theta(\alpha) = -\alpha$ then α is called a real root, if $\theta(\alpha) = \alpha$ then α is called imaginary, and α is called complex if $\theta(\alpha) \neq \pm\alpha$.

Definition 3.1. Two roots $\alpha, \beta \in \Phi(T)$ are called strongly orthogonal if $(\alpha, \beta) = 0$ and $\alpha \pm \beta \notin \Phi(T)$.

4. Standard k -Split Tori

We review some of the results of [7], for our purposes they will be used to determine the domain of the restriction of the generalized complexification map to the I -poset. Let \mathcal{A} denote the set of θ -stable maximal k -split tori.

Definition 4.1. Let $A_1, A_2 \in \mathcal{A}$. Then (A_1, A_2) is called a standard pair if $A_1^- \subset A_2^-$ and $A_2^+ \subset A_1^+$. A_1 is said to be standard with respect to A_2 .

The θ -stable maximal k -split tori can be arranged into chains of standard pairs as in the algebraically closed case (see [6]).

Theorem 4.1. Let $A \in \mathcal{A}$ such that A^- is maximal (θ, k) -split. Then there exists a $S \in \mathcal{A}$ standard with respect to A such that S^+ is a maximal k -split torus of H .

4.1. (θ, k) -singularity

The one-dimensional subgroups containing both a θ -fixed k -split torus and a (θ, k) -split torus depend heavily on the k -structure of the group but can still be parameterized by tori. An involution defined over k of a connected reductive group M is called split if there exists a maximal k -torus which is (θ, k) -split.

Definition 4.2. Let $A \in \mathcal{A}$ and for each $\alpha \in \Phi(A)$ let $\ker(\alpha) = \{a \in A \mid s_\alpha(a) = a\}$, where s_α is the reflection corresponding to α . Set $G_\alpha = Z_G(\ker(\alpha))$. Then α is called (θ, k) -singular if

- (1) $\theta|_{[G_\alpha, G_\alpha]}$ is split,
- (2) $k - \text{rank}([G_\alpha, G_\alpha]) = k - \text{rank}([G_\alpha, G_\alpha] \cap H)$.

Theorem 4.2. Let A be a θ -stable maximal k -split torus of G and $\Psi = \{\alpha_1, \dots, \alpha_r\} \subset \Phi(A)$ a set of strongly orthogonal roots. Let $G_\Psi = G_{\alpha_1} \cdots G_{\alpha_r}$. Then

$$[G_\Psi, G_\Psi] = \prod_{i=1}^r [G_{\alpha_i}, G_{\alpha_i}].$$

Moreover, if $\alpha_1, \dots, \alpha_r$ are (θ, k) -singular, then $\theta|_{[G_\Psi, G_\Psi]}$ is k -split and $k - \text{rank}([G_\Psi, G_\Psi]) = k - \text{rank}([G_\Psi, G_\Psi] \cap H)$.

5. Generalized Complexification

Consider the orbits $B \backslash G/H$ over an algebraically closed field. These are the orbits of a minimal parabolic subgroup acting on a symmetric variety, which can be related to the orbits $P_k \backslash G_k/H_k$ of a minimal parabolic k -subgroup acting on a symmetric k -variety. A description of how the algebraically closed orbits break up over a subfield is a fundamental question related to the representation theory of the symmetric k -varieties. In this section, we approach this problem from the reverse direction, namely by embedding the orbits over a subfield k into the orbits over its algebraic closure \bar{k} . When $k = \mathbb{R}$ this process is the complexification of the real orbits, thus we call the map yielded by the embedding generalized complexification.

Let $P \subset G$ be a minimal parabolic k -subgroup and $A \subset P$ a maximal k -split torus. We define the generalized complexification map:

$$\begin{aligned} \varphi : P_k \backslash G_k/H_k &\rightarrow P \backslash G/H, \\ P_k g H_k &\mapsto P g H. \end{aligned}$$

Recall from Theorem 3.4 that there are several equivalent characterizations of the double cosets $P_k \backslash G_k/H_k$. The generalized complexification map φ induces maps across all of these equivalent formulations. Given $v \in V_k$, let $x(v)$ be representative in $N_{G_k}(A)$ such that $v = Z_{G_k}(A)x(v)H_k$. Then, we have an induced map:

$$\begin{aligned} \varphi_V : V_k &\rightarrow V, \\ Z_{G_k}(A)x(v)H_k &\mapsto Z_G(A)x(v)H. \end{aligned}$$

Let G be a k -split group. Then minimal parabolic k -subgroups are Borel k -subgroups. In this case, we have a simpler generalized complexification map:

$$\begin{aligned}\varphi : B_k \backslash G_k / H_k &\rightarrow B \backslash G / H, \\ B_k g H_k &\mapsto B g H.\end{aligned}$$

The corresponding induced maps are also simpler in this case since the maximal k -split tori are in fact maximal tori. The generalized complexification of orbits corresponding to the $B_k \times H_k$ action on G_k becomes

$$\begin{aligned}\varphi : V_k &\rightarrow V, \\ A_k x(v) H_k &\mapsto A x(v) H.\end{aligned}$$

The greatest simplification, however, occurs in the induced map among the union of quotients of Weyl groups. Let $\{A_i\}_{i \in I}$ be a set of representatives of the H_k -conjugacy classes of θ -stable maximal k -split tori. Then $\{A_i\}_{i \in I}$ corresponds to $\{B_i\}_{i \in I'}$, a set of representatives for the H -conjugacy classes of θ -stable maximal tori. This is done in the following manner. Among the $\{A_i\}$ that correspond to the same H -conjugacy class of θ -stable maximal tori, a representative is chosen. This set is then extended with arbitrary representatives of the H -conjugacy classes of θ -stable maximal tori not obtained from the $\{A_i\}$. Therefore the generalized complexification map acts as the identity:

$$\begin{aligned}\varphi : \bigcup_{i \in I} W_{G_k}(A_i) / W_{H_k}(A_i) &\rightarrow \bigcup_{i \in I'} W_G(A_i) / W_H(A_i), \\ g W_{H_k}(A_i) &\mapsto g W_H(A_i).\end{aligned}$$

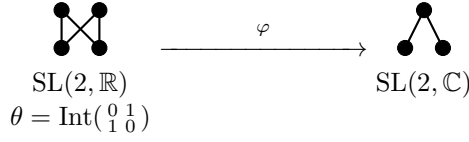
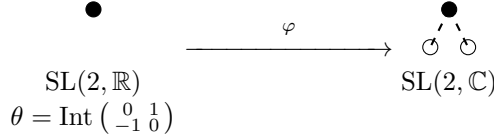
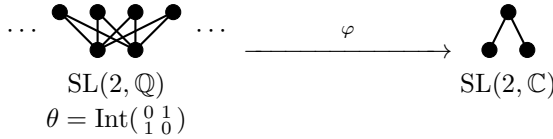
Remark 5.1. For groups that are not k -split, φ still induces a map on the union of Weyl group quotients. This map is more complicated and involves the introduction of another quotient.

5.1. Some examples

In general, the surjectivity of the generalized complexification map depends on both the choice of involution θ and the field of definition k . The first example of this section illustrates the dependence of the surjectivity of the generalized complexification map on the choice of involution.

Example 5.1. Let $G = \mathrm{SL}(2, \mathbb{C})$ with real form $G_{\mathbb{R}} = \mathrm{SL}(2, \mathbb{R})$. Note that in this case G is \mathbb{R} -split. Let T denote the set of diagonal matrices and $P = B$ the set of upper triangular matrices.

- (1) Let $\sigma = \mathrm{Int} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then from Example 3.1, we have that $|B_{\mathbb{R}} \backslash G_{\mathbb{R}} / H_{\mathbb{R}}| = 4$. Two orbits correspond to each representative of the $H_{\mathbb{R}}$ -conjugacy class of σ -stable maximal \mathbb{R} -split tori, denote the orbits corresponding to T by \mathcal{O}_1 and \mathcal{O}_2 . There


 Fig. 3. Generalized complexification of $\mathrm{SL}(2, \mathbb{R})$, $\theta = \mathrm{Int} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

 Fig. 4. Generalized complexification of $\mathrm{SL}(2, \mathbb{R})$, $\theta = \mathrm{Int} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

 Fig. 5. Generalized complexification of $\mathrm{SL}(2, \mathbb{Q})$, $\theta = \mathrm{Int} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

is only one orbit corresponding to T over \mathbb{C} , therefore the complexification maps \mathcal{O}_1 and \mathcal{O}_2 to the same orbit over \mathbb{C} . Consider $\varphi : B_{\mathbb{R}} \backslash G_{\mathbb{R}} / H_{\mathbb{R}} \rightarrow B \backslash G / H$. We can represent the action of φ diagrammatically, as in Fig. 3.

- (2) Let $\theta = \mathrm{Int} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then from Example 3.1, we have that $|B_{\mathbb{R}} \backslash G_{\mathbb{R}} / H_{\mathbb{R}}| = 1$. The complexification map has a cokernel, indicated by the empty nodes in Fig. 4.

Recall that there infinite number of orbits $B_{\mathbb{Q}} \backslash G_{\mathbb{Q}} / H_{\mathbb{Q}}$ for $G_{\mathbb{Q}} = \mathrm{SL}(2, \mathbb{Q})$. The complexification of these orbits is quite similar to the complexification of the real orbits, as shown in Figs. 5 and 6.

To illustrate the dependence on the base field k , consider $k = \mathbb{Q}(i)$. In this case, we obtain surjectivity for $\theta = \mathrm{Int} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ since in this case $H = G^{\theta}$ is a torus which splits over k .

Given a set $S \subset \mathrm{Aut}(G)$, we say two involutions $\theta, \sigma \in \mathrm{Aut}(G)$ are S -isomorphic if there exists $\gamma \in S$ such that $\theta = \gamma \sigma \gamma^{-1}$. In this case, we say that θ and σ are S -isomorphic by γ . The double coset decomposition is dependent only on the isomorphy class of involution.

Proposition 5.1. *Let θ, σ be involutions that are $\mathrm{Aut}(G)$ -isomorphic by γ , and denote their fixed point groups $H_1 = G^{\theta}$ and $H_2 = G^{\sigma}$. Then $H_2 = \gamma^{-1}(H_1)$.*

Proof. Let $h \in H_1$. Then $\sigma(\gamma^{-1}(h)) = \gamma^{-1}\theta\gamma(\gamma^{-1}(h)) = \gamma^{-1}\theta(h) = \gamma^{-1}(h)$.

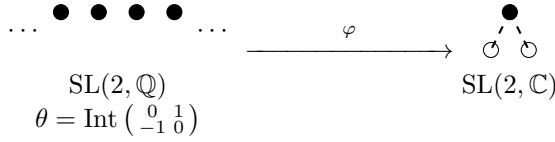


Fig. 6. Generalized complexification of $\mathrm{SL}(2, \mathbb{Q})$, $\theta = \mathrm{Int} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Therefore $\gamma^{-1}(H_1) \subset H_2$. Since γ is one-to-one, we have that $H_2 = \gamma^{-1}(H_1)$. \square

Corollary 5.1. *Assume the hypotheses of Proposition 5.1. If θ, γ are $\mathrm{Int}(G)$ -isomorphic, then H_1, H_2 are conjugate.*

Proof. Apply Proposition 5.1 to the case $\gamma = \mathrm{Int}(g)$ for some $g \in G$. \square

Theorem 5.1. *Let $\theta, \sigma \in \mathrm{Aut}(G)$ be $\mathrm{Int}(G)$ -isomorphic involutions. If $H_1 = G^\theta$ and $H_2 = G^\sigma$, then there exist Borel subgroups B_1 and B_2 such that $B_1 \backslash G / H_1 \cong B_2 \backslash G / H_2$. Furthermore, B_1 and B_2 are G -conjugate.*

Proof. Suppose $\theta = \mathrm{Int}(g)\sigma\mathrm{Int}(g)^{-1}$. From Corollary 5.1, we have that $H_2 = g^{-1}H_1g$. Let $B_2 = g^{-1}B_1g$. Given a double coset $B_1xH_1 \in B_1 \backslash G / H_1$, we compute $\mathrm{Int}(g^{-1})(B_1xH_1) = \mathrm{Int}(g^{-1})(B_1)\mathrm{Int}(g^{-1})(x)\mathrm{Int}(g^{-1})(H_1) = B_2g^{-1}xgH_2 \in B_2 \backslash G / H_2$. The inverse map is given by $\mathrm{Int}(g)$, so we have $B_1 \backslash G / H_1 \cong B_2 \backslash G / H_2$. \square

The isomorphism of the double coset decompositions extends to k -isomorphism, using the same proofs.

Theorem 5.2. *Let $\theta, \sigma \in \mathrm{Aut}(G, G_k)$ be $\mathrm{Int}(G, G_k)$ -isomorphic involutions with fixed point groups H_1 and H_2 , respectively. Then there exist Borel subgroups B_1 and B_2 such that $(B_1)_k \backslash G_k / (H_1)_k \cong (B_2)_k \backslash G_k / (H_2)_k$.*

In light of Theorem 3.4, the double cosets $B_k \backslash G_k / H_k$ are parameterized by the H_k -conjugacy of θ -stable maximal k -split tori. The H_k -conjugacy classes have not been fully classified except in a number of specific cases, notably $\mathrm{SL}(2, k)$. However, for algebraically closed fields a complete classification was achieved in [6]. Conveniently, the characterization of surjectivity of the generalized complexification map depends only on the H -isomorphism classes of θ -stable maximal tori.

If we fix θ -stable maximal k -split torus A , we can restrict the generalized complexification map

$$\varphi : \bigcup_{i \in I} W_{G_k}(A_i) / W_{H_k}(A_i) \rightarrow \bigcup_{i \in I'} W_G(A_i) / W_H(A_i)$$

to the Weyl group quotient corresponding to A :

$$\varphi_A : W_{G_k}(A) / W_{H_k}(A) \rightarrow W_G(A) / W_H(A). \quad (5.1)$$

Then, we can consider the surjectivity of φ_A . The following lemma shows that this map is in fact surjective in all cases.

Lemma 5.1. *The map φ_A of Eq. (5.1) is surjective.*

Proof. It is clear that $W_{H_k}(A) \subset W_H(A)$. Given $gW_H(A) \in W_G(A)/W_H(A)$, the fiber is nonempty since g has a representative in $W_{G_k}(A)$. \square

Thus surjectivity of the map between indexing sets of θ -stable maximal tori is sufficient to ensure surjectivity of the generalized complexification map.

Here, we recall the general Cayley transform, whose construction is quite similar to the real case.

Fix a maximal (θ, k) -split torus A of G . Then A has a root system $\Phi(A)$ and for each $\alpha \in \Phi(A)$ we can define the root group $G_\alpha = Z_G(\ker(\alpha))$. Then the commutator $[G_\alpha, G_\alpha]$ is a semisimple group of rank 1, isomorphic to $\mathrm{SL}(2, k)$. Therefore $\theta|_{[G_\alpha, G_\alpha]}$ is inner. If $[G_\alpha, G_\alpha]$ contains a nontrivial θ -fixed torus S , we define a map $\eta = \mathrm{Int}\left(\begin{pmatrix} 1 & \frac{-1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}\right)$ that acts on S . As in the real case, η maps S to a θ -split torus of $[G_\alpha, G_\alpha]$.

Lemma 5.2. *S is k -split if and only if $\theta|_{[G_\alpha, G_\alpha]} \cong \mathrm{Int}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$.*

Proof. Since $[G_\alpha, G_\alpha] \cong \mathrm{SL}(2, k)$, this follows directly from [2]. \square

Furthermore, S is k -split implies $\eta(S)$ is also k -split. We extend the action of η to the rest of the torus trivially.

In order to discuss the surjectivity of the generalized complexification map we should have a description of the I -posets in the image of the generalized complexification map, namely the I -posets for algebraically closed fields. This was carried out in [6]. In particular, we know when H contains a maximal torus.

Theorem 5.3. *$\theta \in \mathrm{Int}(G)$ if and only if $\mathrm{rank}(G) = \mathrm{rank}(H)$.*

Lemma 5.3. *Suppose H contains a nontrivial k -split torus S . Then $\Phi(S)$ consists of (θ, k) -singular roots.*

Proof. Consider $\alpha \in \Phi(S)$, then $\theta(\alpha) = \alpha$. Construct the corresponding root group $G_\alpha = Z_G(\ker(\alpha))^\circ$. Then by Lemma 5.2, we have that $\theta|_{[G_\alpha, G_\alpha]} \cong \mathrm{Int}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$. Therefore $[G_\alpha, G_\alpha]$ contains a θ -fixed k -split torus then can be flipped to a (θ, k) -split torus in $[G_\alpha, G_\alpha]$ via the Cayley transform η . Therefore α is (θ, k) -singular. \square

Corollary 5.2. *Let $S \subset H$ be a k -split torus. There exists a maximal orthogonal subset of roots of torus lying in H such that each root is (θ, k) -singular.*

We can iterate this process, performing successive Cayley transforms in the root groups of a set of strongly orthogonal roots.

Lemma 5.4. *Assume $\text{rank}(G) = n$ and let $S \subset H$ be a k -split torus of H and suppose $\Psi(S) = \{\alpha_1, \dots, \alpha_r\} \subset \Phi(S)$ is a maximal set of strongly orthogonal roots. Then G contains a (θ, k) -split torus of dimension $n - r$.*

Proof. We use induction on r . We may assume the $\Psi(S)$ consists of (θ, k) -singular roots. The case $r = 1$ is carried out explicitly in the proof of Lemma 5.3. Now, assume $r > 1$ and let ${}^{\alpha_1}S \subset H$ be the subtorus lying in $[G_{\alpha_1}, G_{\alpha_1}]$. Then $S = ({}^{\alpha_1}S)\tilde{S}$, where $\tilde{S} \subset S$ denotes the factor of S such that $[G_{\alpha_1}, G_{\alpha_1}] \cap \tilde{S} = \pm \text{id}$. Then \tilde{S} is a k -split torus in H so Lemma 5.3 applies and $|\Psi(S)| = r - 1$. \square

Lemma 5.5. *Let S be a maximal k -split torus of H . Then $Z_G(S)$ contains a maximal k -split torus of G .*

Proof. Let $S \subset H$ be a maximal k -split torus of H . Consider

$$A_1 = \left(\bigcap_{\substack{\alpha \in \Phi(T) \\ \alpha \perp \Phi(S)}} \ker(\alpha) \right)^\circ$$

Since the -1 -eigenspace of θ is orthogonal to H , $\pi(A_1)$ (where π is the usual projection) contains a maximal torus of G/H , let A_1^- be the inverse image of this torus. Then there exists a subtorus of A_1^- that is maximal k -split in G/H . Therefore $S \cdot A_1^-$ is a maximal k -split torus of G . \square

Lemma 5.6. *Given a torus S , $\dim \varphi(S)^+ = \dim S^+$ and $\dim \varphi(S)^- = \dim S^-$.*

Theorem 5.4. *Let G be a k -split group. Then the generalized complexification map φ is surjective if and only if $k - \text{rank}(H) = \text{rank}(H)$.*

Proof. First suppose $k - \text{rank}(H) \neq \text{rank}(H)$. Let $\{A_i\}_{i \in I}$ be a set of representatives for the H_k -conjugacy classes of θ -stable maximal k -split tori and let $\{B_l\}_{l \in L}$ be a set of representatives for the H -conjugacy classes of θ -stable maximal tori. Choose $A \in \{A_i\}_{i \in I}$ such that A^+ is maximal and $B \in \{B_l\}_{l \in L}$ such that B^+ is maximal. Then by assumption $\dim B^+ > \dim A^+$. In light of 5.6, B can have no preimage under the induced complexification map on the torus poset. Thus none of the orbits counted by $W_G(B)/W_H(B)$ are in the image of φ , so φ is not surjective.

Next, assume that $k - \text{rank}(H) = \text{rank}(H)$. Then let S be a maximal k -split torus of H , and let $T \subset Z_G(S)$ be a maximal k -split torus of G . Then the $\Phi(T)$ and the restricted root system $\Phi_0(T)$ consist of the same roots, thus they have the same maximal orthogonal set of roots and all such roots are (θ, k) -singular. Thus, we have surjectivity in the I -poset and thus φ is surjective. \square

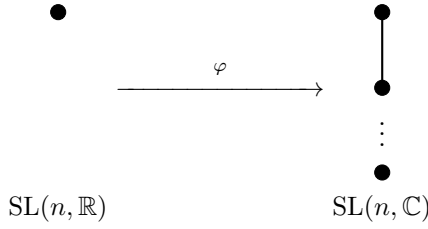


Fig. 7. Generalized complexification of $\mathrm{SL}(n, \mathbb{R})$, $\theta(g) = (g^T)^{-1}$.

Corollary 5.3. *Let G be a k -split group, suppose the generalized complexification map $\varphi : B_k \backslash G_k / H_k \rightarrow B \backslash G / H$ is surjective, and let $A \subset G$ be a k -split torus. Define $G_1 = Z_G(A)$, $B_1 \subset G_1$ a Borel subgroup, and $H_1 = H \cap G_1$. Then the restriction $\varphi|_{G_1}$ is surjective.*

Proof. G_1 is connected and reductive since A is a k -split torus. Moreover, B_1 is contained in a Borel subgroup of G , i.e. $B_1 = B \cap G_1$ for B a Borel subgroup, and therefore $(B_1)_k = (B \cap G_1)_k$. Thus the orbits $(B_1)g(H_1) \in (B_1) \backslash (G_1) / (H_1)$ can be embedded in the orbits $B_k \backslash G_k / H_k$. Therefore there is a preimage of $B_1 g H_1$ in $B_k \backslash G_k / H_k$. Then $\varphi^{-1}(B_1 g H_1) \cap (G_1)_k$ is nonempty, so surjectivity is achieved. \square

The surjectivity of the generalized complexification map implies that there exists a decomposition of G_k that is as far from the Iwasawa decomposition as possible. We conclude with some examples.

Example 5.2. Let $G = \mathrm{SL}(n, \mathbb{C})$, $G_{\mathbb{R}} = \mathrm{SL}(n, \mathbb{R})$, $\theta(g) = (g^T)^{-1}$ for all $g \in G$. Then $H_{\mathbb{R}} = \mathrm{SO}(2, \mathbb{R})$, which is compact. Therefore \mathbb{R} -rank(H) = 0, so φ is not surjective. In fact, from the Iwasawa decomposition we can deduce the generalized complexification diagram on the I -poset found in Fig. 7.

Example 5.3. Let $G_k = \mathrm{SL}(n, k)$ and $\theta = \mathrm{Int}(I_{n-i,i})$, where

$$I_{n-i,i} = \begin{bmatrix} J & 0 \\ 0 & I_{n-2i} \end{bmatrix},$$

with I_{n-2i} denoting the $n - 2i \times n - 2i$ identity matrix and

$$J = \begin{bmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The fixed point group H of θ consists of matrices of the form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A has dimensions $2i \times 2i$ and satisfies $JAJ = A$, B has dimensions $2i \times n - 2i$ and satisfies $JB = B$, C has dimensions $n - 2i \times 2i$ and satisfies $CJ = C$ and $D \in \text{GL}(n - 2i)$. Then one checks that

$$P = \begin{bmatrix} a_1 & 0 & \cdots & \cdots & 0 & b_1 \\ 0 & \ddots & & & \ddots & 0 \\ \vdots & & a_i & b_i & & \vdots \\ \vdots & & b_i & a_i & & \vdots \\ 0 & \ddots & & & \ddots & 0 \\ b_1 & 0 & \cdots & \cdots & 0 & a_1 \end{bmatrix}$$

satisfies $JPJ = P$, and has eigenvalues $a_j \pm b_j$ for $j \in \{1, 2, \dots, i\}$. Then elements of the form

$$\begin{bmatrix} P & 0 & \cdots & 0 \\ 0 & x_1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & x_{n-2i} \end{bmatrix},$$

with P as above and the x_j elements of the field, form a torus. This torus is k -split, and is maximal because its dimension is $n - 1$. Therefore $k - \text{rank}(H) = \text{rank}(H)$, and thus the generalized complexification map is surjective for all choices of n , i , and k .

Example 5.4. Let $G_k = \text{SL}(n, k)$ such that $n = 2m$ is even. Furthermore, let $\theta = \text{Int}(L_{2m, x})$, where $x \not\equiv 1 \pmod{(k^*)^2}$, $L_{2m, x} = \text{diag}(L_1, L_2, \dots, L_m)$, and

$$L_i = \begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix}.$$

The fixed point group of θ is given by $[A_{ij}]$, $i, j \in \{1, 2, \dots, m\}$, where

$$A_{ij} = \begin{bmatrix} a_{ij} & b_{ij} \\ xb_{ij} & a_{ij} \end{bmatrix}.$$

A maximal k -split torus of H is given by

$$A = \{\text{diag}\{a_1, a_1, \dots, a_m, a_m\} | a_1^2 \cdots a_m^2 = 1\}.$$

Therefore, we do not have surjectivity in these cases. Consider the centralizer in

H_k of A :

$$Z_{H_k}(A) = \begin{pmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K_{\frac{n}{2}} \end{pmatrix},$$

where each K_i is a 2×2 matrix:

$$\begin{pmatrix} a_i & b_i \\ xb_i & a_i \end{pmatrix}.$$

Then $Z_{H_k}(A)$ is diagonalizable over $\tilde{k} = k(\sqrt{x})$ since its eigenvalues are $a_i \pm b_i\sqrt{x}$, $i = 1, 2, \dots, \frac{n}{2}$. Thus, we have surjectivity of the generalized complexification over the quadratic extension field \tilde{k} .

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