TTK4115 – Linear System Theory

Helicopter lab assignment

Department of Engineering Cybernetics
NTNU

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3.6, August 2006, JB
3.5, August 2005, JH
3.4, August 2005, JS
3.3, August 2003, JS
3.2, September 2002, ESI
3.1, September 2001, ESI
3.0, June 2001, TAJ

1 Preliminaries

1.1 Practical information

- The completion of this assignment is required for the completion of the course. The resulting report and presentation will be given a score that accounts for 30% of the final grade of the course. The assignment will be carried out in groups of two or three students. Each group of students will be given a common score. It is expected that each group will deliver their original work. The evaluation will consist of practical demonstrations in the lab, accompanied by oral questioning of each group based on a lab report and observations made during the demonstrations. The report is delivered the same day as the presentation. The evaluation will be done in week 43.

- The amount of work required to complete the assignment should be approximately 40 hours.

- The ten helicopters are found in "Elektrobygget", rooms B117 (Helicopters 1–3), B121 (Helicopters 4–7) and B125 (Helicopters 8–10). You need your key card to access the room[1]. You are assigned to a specific helicopter, and should use that throughout the lab to avoid conflicts with other groups and to minimize time spent on adapting your code to a new helicopter, as they vary slightly.

- Each group will have four whole days (8 hours each) reserved in the lab. This means that you will have 32 hours exclusively reserved in the lab for your group. A student assistant will be present for 3 hours each day. Note that it takes approximately 40 hours to complete the assignment. This means that you have to work on the assignment outside the lab as well. **Start working on the lab assignment as soon as possible.** You will need your time.

- Always read the bulletin board on "It's learning" if you run into problems. All practical information will be posted there.

- The lab computers are **not connected to the internet**, which implies there are no e-mail or other internet-based ways to store your data. The easiest way to store your data is to bring a memory stick to the lab. **Note that you need the put the MATLAB files that are posted on "It's learning" on your memory stick before you come to the lab.** It is also possible to use the "sambastud" server; see Section 4.4 for more details.

[1] If you are denied access, talk to Unni Johansen who works in the Department's reception (room D144 in "Elektrobygget"), or contact the teaching assistant.

1.2 Purpose of the helicopter lab assignment

The purpose of this assignment is to:

- Derive a model for a given system (the helicopter) that can be used for control purposes.
- Derive and apply PD and multivariable controllers in a real-time environment.
- Demonstrate how states can be estimated that are not directly measured.
- Give an introduction to optimal control by developing a linear quadratic regulator (LQR) for the helicopter.

1.3 Lab report and presentation

To complete the helicopter-lab assignment, a report should be written and an oral presentation should be given. The report should contain your solutions to the problems in the assignment. The report should be **brief** and to the point. For each problem in the assignment, it should include:

- Relevant analyses, deductions and calculations. All relevant mathematical expressions have to be derived. A discussion of the validity of the derivations should be given.
- Documented MATLAB code and Simulink diagrams which were used in the assignment. Do not include the whole scripts and diagrams but rather just the bits that show how things were done. The documented MATLAB scripts and Simulink diagrams need to be clear and easy to understand, and should be accompanied by sufficient explanation.
- Relevant plots of measured time series. The plots should be clear and support the point you want to make. Make sure that you label the included figures appropriately.
- Calculated and chosen parameter values. For each calculated constant and chosen controller/observer gain, the corresponding parameter values should be given. Do not forget to state the correct units.
- A short discussion of the results. Compare different results and explain possible differences.

You do not need to include a detailed chapter to introduce the helicopter system. The report has to be written in **English**. The report has to be handed in the same time as your presentation is held at the helicopter lab. Start the writing process early! It is highly recommended to write the report using LaTeX. See the ITK Labreport guide/template on Github for some useful hints[2].
At the oral presentation, you will be asked to explain and demonstrate the work done for each problem of the assignment. Prepare a separate MATLAB file for each problem of the assignment for the demonstration (including initialization, generation of the necessary plots, etc.). A pdf file of the report, as well as all MATLAB/Simulink files needed for the demonstration, should be uploaded in accordance with the specifications found on Blackboard.

2 Description of the system

2.1 The helicopter

The helicopter consists of a base on which a long arm is attached; see Fig. 1. The helicopter head, with two motors and propellers, is attached to one side of the arm. On the other side, a counterweight is placed. The helicopter has three rotational joints. The helicopter arm can rotate about the vertical axis. This will be referred to as the travel of the helicopter. The helicopter head can move up and down with respect to the base. This rotation of the helicopter arm will be referred to as the elevation of the helicopter. Moreover, the helicopter head can tilt with respect to the arm, which will referred to as the pitch of the helicopter. All three joint angles are measured using encoders. The helicopter is actuated by the two motors on the helicopter head, which are connected to the propellers. The propellers deliver the necessary thrust to lift the helicopter head up from the table.

2.2 Hardware

The helicopter lab stations use the Quanser Q8-USB data aquisition (DAQ) cards. These cards are used for both inputs and outputs. The joint angles for pitch, elevation and travel are measured using special counter circuits to read the encoders on the helicopter. The control signals (motor voltages) are set using a digital-to-analog converter (DAC). This signal is amplified by a linear amplifier inside the power supply and fed to the motors.

[2] The files are also posted on Blackboard

Fig. 1: Helicopter.

2.2.1 Joystick

A joystick is used for controlling the helicopter. The joystick displacement is measured by potentiometers in the joystick, and are transmitted to the PC via USB. In the *Joystick* Simulink block this signal is interpreted, and translated into a value in the range from -1 to 1. Within the *Joystick*-block the buttons on the joystick can be read. However, this functionality will not be used in this lab.

2.2.2 Power supply

The power supplied to the motors comes from a power supply located under each lab station's desk. It should be labeled "Quanser VoltPAQ-X2".

3 Safety

- Before you start, read the safety instructions attached to your work station.
- Keep your fingers (and limbs) away from the propellers while they is running, and also as long as the power supply is turned on! The DAQ card will output transient signals now and again, which cannot be controlled, e.g. when the computer is switched on or off.
- With respect to fire safety: remember to turn off the power supply when you leave the lab.
- The system is often in an unstable state and behaves badly. It is also expensive and a bit fragile, so try to avoid crash landings! Be prepared to either turn off the power supply, and/or to stop the execution of the controller application. It is good practice to prepare yourself to grab the counterweight

during take-off and landing; grabbing the counterweight can avoid crashes while you end the execution of the controller in case of instability.

- It should not be necessary to alter the helicopter setup in any way. If you experience hardware problems, notify the student assistants, teaching assistant or technical staff. **Do not try to fix it yourself.**

4 Getting started

4.1 Starting up

Before you get started, make sure that the power supply of the helicopter is switched off: the power switch on top of your desk should be on “Stopp”. To turn on the computer, press the power button on the computer below the helicopter platform. Once Windows has started up, click on the ”Switch User” button and subsequently on the “Other User” button. Log in with your student user name and password. Make sure you log in onto the right domain by using “win-ntnu-no“xxxx” as your user name, where “xxxx” is your student user name. Once you are logged in, double click on the MATLAB icon on the desktop or click on the MATLAB icon in the task bar to start up MATLAB.

4.2 MATLAB, Simulink, QuaRC

The controllers for the helicopter are to be made with MATLAB and Simulink. When MATLAB has loaded, you should see a window similar to the one shown in Fig. 2. From this window Simulink can be run by clicking the Simulink button in the toolbar, or entering `simulink` in the MATLAB Command Window and pressing enter.

When Simulink has been loaded, a window similar to the one shown in Fig. 3 should appear. In this window you should be able to find all the Simulink blocks needed to create your controllers.

A template Simulink model has been made, which you can use as a starting point for the assignment work. This file is described in Section 4.3. A Simulink model file can be opened from Windows, MATLAB, and Simulink. If you open such a file from Windows, a new instance of MATLAB is run, so it is usually preferable to open such files in MATLAB or Simulink directly.

When a block diagram implementing your controller has been made in Simulink, QuaRC is used to automatically generate source code for this model. QuaRC will also automatically compile the source code. When the source code has been built, Simulink is used to run the implementation in real time. With QuaRC installed, a new menu is available in Simulink. It is labelled `QuaRC`, and is shown in Fig. 4.

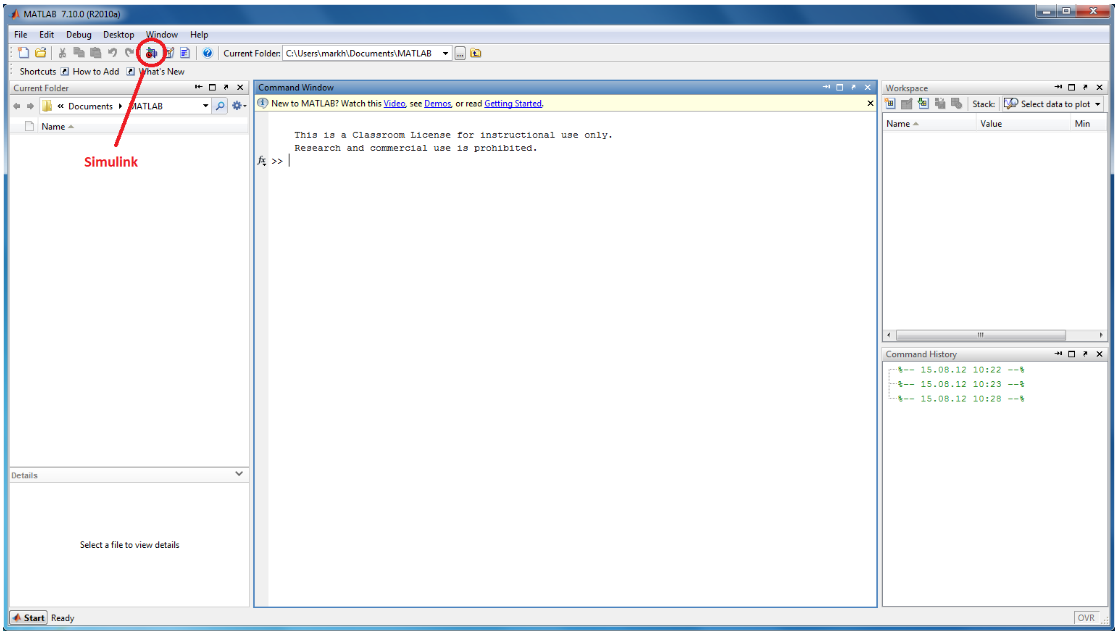

Fig. 2: MATLAB Desktop.

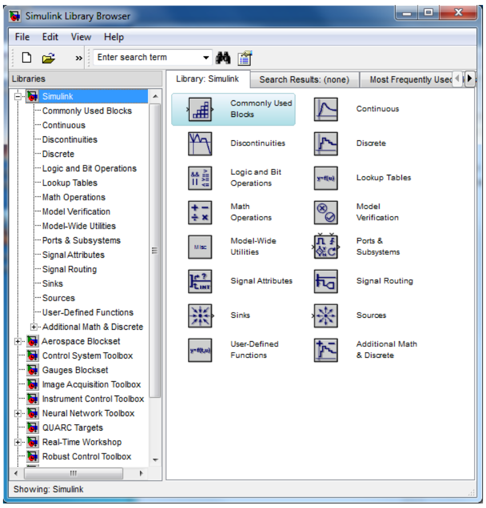

Fig. 3: Simulink Library Browser.

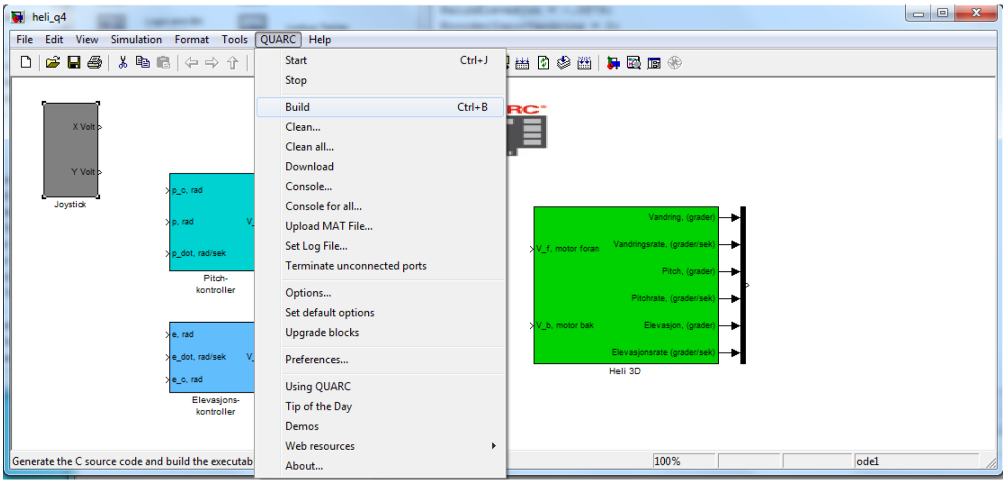

Fig. 4: QuaRC menu in Simulink.

The menu item `QuaRC` → `Build` should be the only item in this menu needed throughout this assignment. The following happens when `Build` is selected:

- QuaRC generates ANSI C code for the Simulink block diagram.
- QuaRC compiles and links the source code.
- The executable code is loaded into the QuaRC Real-Time server.

After this ordeal, your Simulink block diagram implementation should be ready to run. You can now turn on the power supply. The power supply is the big black box underneath the table, labeled "Quanser VoltPAQ-X2". The power switch on your desk turns them on. Please turn them off when you are not using the helicopter or if your helicopter becomes unstable.

When the power supply is on, your controller application can be started by subsequently clicking the "Connect" and "Play" buttons in Simulink. This is shown in Fig. 5.

When you make changes to the Simulink diagram, by e.g. adding new blocks, you will have to `Build` again before you can "Connect" and "Play". However changes in `Gain` blocks can be done while the helicopter is running, and does not require a new `Build`. Errors can occur during building. It is usually good practice to wait until QuaRC finishes building, and for the model to be loaded into the QuaRC Server application before the controller application is started.

If you get strange error messages during building, there are a couple of tricks that usually help. Firstly, try to delete all compiled code by deleting the QuaRC files from the folder containing your model. Then, try to build everything again. Secondly, if this does not help your situation, restarting the computer might.

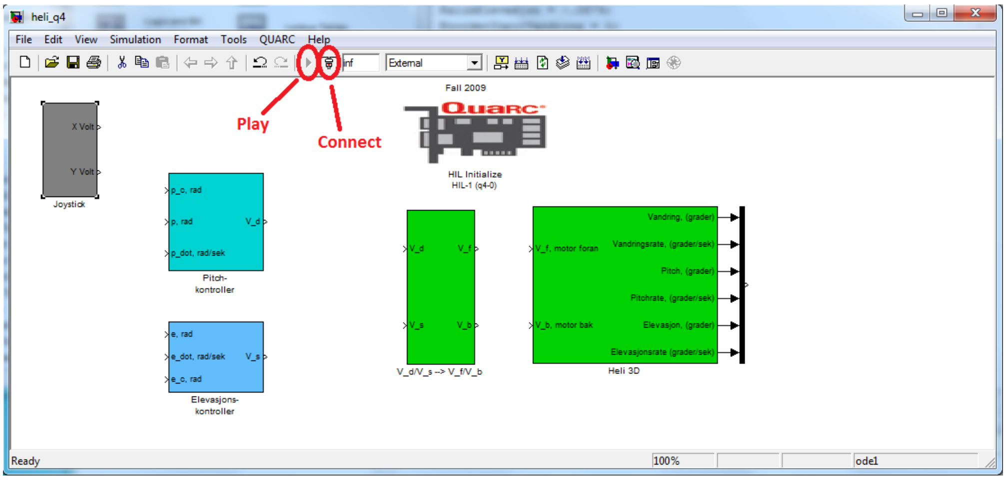

Fig. 5: Start the Simulink program. Connect before playing.

To save measured data from the system, you may use a ”To File” block. Use the save format “Array” to export your data to a file; see Fig. 6. You can load the stored data into the MATLAB Workspace by using the command `load xxxx`, where `xxxx.mat` is the name of the stored MAT file. Note that the first row of the array in which the data is stored contains the time sequence; the other rows contain the stored signals. Note that you can store multiple signals in one file by “muxing” the signals with the “Mux” block in the Simulink Library Browser. Due to a limit in the QuaRC driver, the following steps has to be made to store data sequences longer than 10 seconds: `Code` (in the Simulink menu) → `External mode control panel` → `Signals and Triggering`. Under `Trigger options` you should increase the duration to an appropriate value.

4.3 Starting point for the programming

A template Simulink model, `heli_q8.slx`, has been made which you can use as a starting point for the assignment. In addition to this file, initialization file containing the physical parameter values for the helicopter, `init_heli.m`, has been prepared. These files can be downloaded from “Blackboard”[3].

The initialization file must be run prior to the `QuaRC` → `Build` command to avoid any error messages during building. The reason for this is that in addition to the physical parameter values, calibration data for the encoders and other configuration specific data is also in this file.

It should **not** be necessary to rewire/reconnect any cables on the helicopter, the

[3]The files have been tested on all the helicopters. However, due to some differences between the helicopters, some adjustments of the parameters might improve performance.

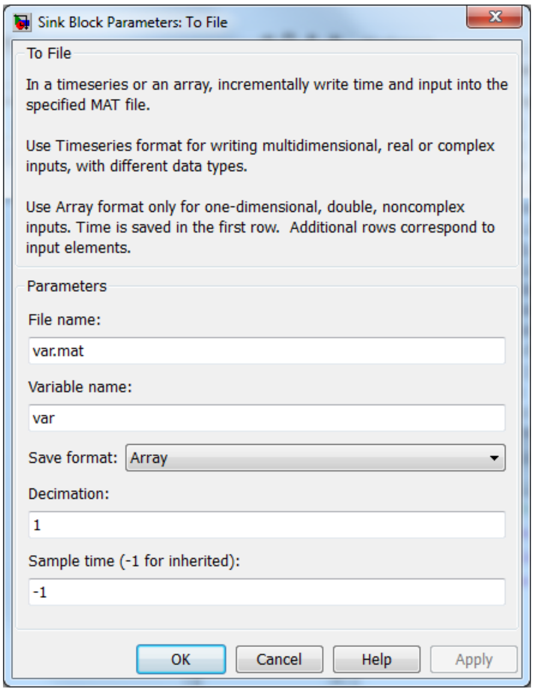

Fig. 6: Sending data to a file

power supply, the joystick, the DAQ card's terminal board, etc. It is important that your end product uses the configuration given in the initialization file.

For a faster building procedure, copy the MATLAB/Simulink files to your folder on the hard-drive of the computer:

`C:\Users\xxxx`

where `xxxx` is your username. Do not forget to copy the files to your memory stick after you are done! Note that no backups are made of the files on the lab's computers. Thus, it is your own responsibility to back up your files.

4.4 Data storage

Because there is **no internet in the lab**, it is not possible to store and transfer your data via e-mail or other internet-based methods. The easiest way to store your data is on a memory stick, so **do not forget to bring a memory stick to the lab**.

Alternatively, you can store your data on the "sambastud" server, which should be accessible from the lab's computers. Windows should automatically connect to this server when you log on, but in case it does not, you can try to following:

right-click on “My Computer”, select “Map Network Drive”, enter

```
\\sambastud.stud.ntnu.no\username
```

as the path, and your username in the “Connect As” field. Be sure the “Reconnect at Logon” checkbox is unchecked. Click “OK” and fill in your password. To connect to the “sambastud” server outside the lab, you need to make sure that you are connected to the NTNU network (possibly via a VPN connection).

5 Problems

The helicopter lab assignment is divided into four parts:

1. **Mathematical modeling:** A mathematical model of the system will be derived to form the entry point for the rest of the assignment.

2. **Mono-variable control:** Standard controllers for pitch and travel – P and PD – will be designed to control one of the system's output.

3. **Multi-variable control:** Controllers to control several system outputs simultaneously will be developed.

4. **State estimation:** The system has sensors for three states, but state feedback requires knowledge of all states. Various methods for state estimation of unknown states will be studied.

5.1 Part I – Mathematical modeling

The helicopter is modeled as three point masses: two point masses represent the two motors that are connected to the propellers, and one point mass represents the counterweight. The model of the helicopter is depicted in Fig. 7. The cubes in Fig. 7 represent the point masses. The cylinders represent the helicopter joints, where the axis of the cylinder is equal to the axis of rotation of the joint. The rotations of the joints are defined as follows: p denotes the pitch angle of the helicopter head, e denotes its elevation angle, and λ denotes the travel angle of the helicopter. By the helicopter head, the part of the helicopter on which the two motors are attached is meant. In Fig. 7, all joint angles are zero. This means that $p = 0$ if the helicopter head is horizontal, and that $e = 0$ if the arm between the elevation axis and the helicopter head is horizontal.

The propeller forces for the front and back propeller are given by F_f and F_b, respectively. It is assumed that there is a linear relation between the voltages V_f

and V_b supplied to the motors and the forces generated by the propellers:

$$F_f = K_f V_f \tag{1a}$$
$$F_b = K_f V_b \tag{1b}$$

where K_f is the motor force constant. The two propellers are placed symmetrically in relation to the pitch axis. The sum of the forces $F_f + F_b$ developed by the two propellers determines the lift of the helicopter. The difference between the forces $F_f - F_b$ will result in a rotation about the pitch axis.

The gravitational forces for the front and the back motor are denoted by $F_{g,f}$ and $F_{g,b}$, while the gravitational force of the counterweight is denoted by $F_{g,c}$. The amplitude of the gravitational forces can be computed as mass times the gravitational constant g. **Note that the gravitational forces always point in a vertical direction, while the direction of the propeller forces is dependent on the joint angles.**

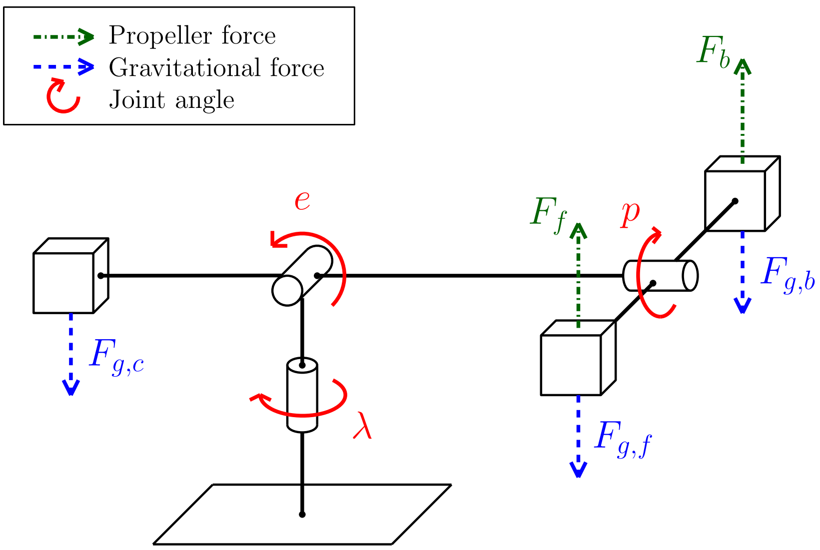

Fig. 7: Helicopter model

The masses of the front and the back motor are assumed to be equal and given by m_p. The mass of the counterweight is given by m_c. The distance from the elevation axis to the head of the helicopter is given by l_h, while the distance from the elevation axis to the counterweight is given by l_c. Because the two propellers are placed symmetrically in relation to the pitch axis, the distance from the pitch axis to both motors is the same and is given by l_p. The masses and distances are depicted in Fig. 8. **Other forces, such as friction and centripetal forces, are neglected.**

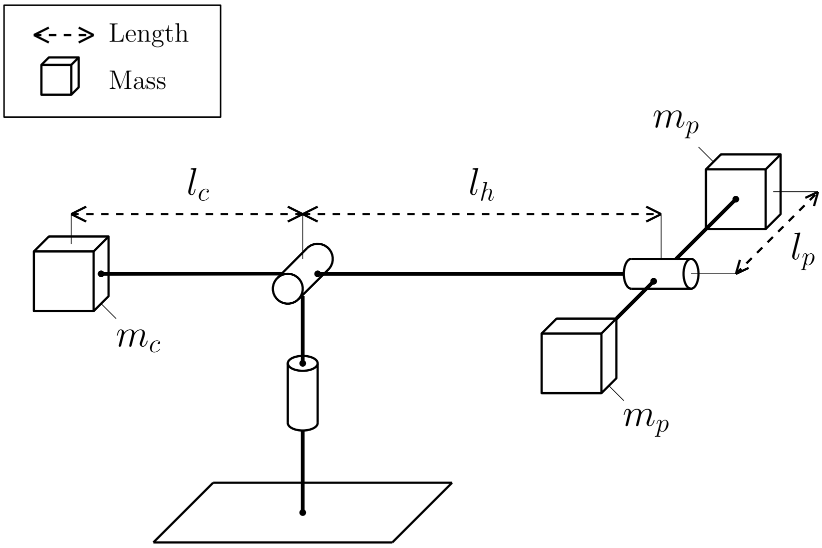

Fig. 8: Masses and distances

Note: The actual values for the masses and distances differ from helicopter to helicopter, and are therefore not given here. These values can be found in an initialization file which can be downloaded from "Blackboard". The masses of the helicopters vary, and one can expect that the heavier helicopters are more difficult to control than the lighter ones. The motor force constant K_f differs quite a bit from helicopter to helicopter. This constant will be determined in one of the problems of the assignment.

5.1.1 Problem 1

Compute the equations of motion (differential equations) for the pitch angle p, the elevation angle e, and the travel angle λ. Let the moments of inertia about the pitch, elevation and the travel axes be denoted by J_p, J_e, and J_λ, respectively. Show that the equations of motion can be stated in the form

$$J_p\ddot{p} = L_1 V_d \tag{2a}$$
$$J_e\ddot{e} = L_2\cos(e) + L_3 V_s\cos(p) \tag{2b}$$
$$J_\lambda\ddot{\lambda} = L_4 V_s\cos(e)\sin(p) \tag{2c}$$

where L_i, $i = 1, 2, 3, 4$ are constants, and where the sum of the motor voltages V_s and the difference of the motor voltages V_d are given by

$$V_s = V_f + V_b \tag{3a}$$

$$V_d = V_f - V_b \tag{3b}$$

5.1.2 Problem 2

In order to design a linear controller for the system, we want to linearize the equations of motion around the point $(p, e, \lambda)^T = (p^*, e^*, \lambda^*)^T$, with $p^* = e^* = \lambda^* = 0$. Determine the voltages V_s^* and V_d^* such that $(p^*, e^*, \lambda^*)^T$ is an equilibrium point of the system, i.e., find the values of V_s^* and V_d^* such that for all time $\dot{p} = \dot{e} = \dot{\lambda} = 0$ for $(p, e, \lambda)^T = (p^*, e^*, \lambda^*)^T$ and $(V_s, V_d)^T = (V_s^*, V_d^*)^T$. (Note that $\dot{p} = \dot{e} = \dot{\lambda} = 0$ for all time implies that $\ddot{p} = \ddot{e} = \ddot{\lambda} = 0$ for all time.)

Next, we introduce the coordinate transformation

$$\begin{bmatrix} \tilde{p} \\ \tilde{e} \\ \tilde{\lambda} \end{bmatrix} = \begin{bmatrix} p \\ e \\ \lambda \end{bmatrix} - \begin{bmatrix} p^* \\ e^* \\ \lambda^* \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \tilde{V}_s \\ \tilde{V}_d \end{bmatrix} = \begin{bmatrix} V_s \\ V_d \end{bmatrix} - \begin{bmatrix} V_s^* \\ V_d^* \end{bmatrix} \tag{4}$$

This transformation simplifies the further analysis, as the equilibrium point of this new coordinate system is located in the origin. Using (4), transfer the equations of motion in (2a)-(2c) to the new coordinate system with state vector $(\tilde{p}, \tilde{e}, \tilde{\lambda})^T$ and input vector $(\tilde{V}_s, \tilde{V}_d)^T$.

Linearize the resulting system around the point $(\tilde{p}, \tilde{e}, \tilde{\lambda})^T = (0, 0, 0)^T$ and $(\tilde{V}_s, \tilde{V}_d)^T = (0, 0)^T$. Assume that the moments of inertia are constant and given by

$$J_p = 2 m_p l_p^2 \tag{5a}$$

$$J_e = m_c l_c^2 + 2 m_p l_h^2 \tag{5b}$$

$$J_\lambda = m_c l_c^2 + 2 m_p (l_h^2 + l_p^2) \tag{5c}$$

Show that the linearized equations of motion can be written as

$$\ddot{\tilde{p}} = K_1 \tilde{V}_d \tag{6a}$$

$$\ddot{\tilde{e}} = K_2 \tilde{V}_s \tag{6b}$$

$$\ddot{\tilde{\lambda}} = K_3 \tilde{p} \tag{6c}$$

where K_i, $i = 1, 2, 3$ are constants.

5.1.3 Problem 3

The first attempt to control of the helicopter will be done using feed forward. Let the signal from the x-axis of the joystick connect directly to the voltage difference V_d, and the signal from the y-axis of the joystick connect directly to the voltage sum V_s. The motors do not need a large voltage difference to increase the pitch angle. It might therefore be a good idea to add a small gain from the output of the joystick to the voltage difference V_d. It is rather difficult to control the helicopter this way. Before you try it, guesstimate the input you think you need to control the helicopter. Be careful not to crash the helicopter! It is good practice to hold the counterweight every time you turn off the power supply or disconnect Simulink; this will prevent the helicopter from hard landings.

Compare the physical behaviour of the helicopter with the theoretical models (2a)-(2c) and (6a)-(6c). Discuss the reasons for the discrepancies.

Note: Before you use the joystick, check if the joystick outputs are zero when the joystick handle is at rest. If necessary, adjust the offset of the output signals by turning the wheels on the side of the joystick.

5.1.4 Problem 4

The encoder values are set to zero every time Simulink is connected to the helicopter. Assuming that the helicopter head rests on the table when Simulink is connected, the encoder outputs are not the same as the joint angles p, e and λ. Add constants to the encoder outputs such that the output of the encoder for the pitch is zero when the helicopter head is horizontal, and such that output for the elevation is zero when the arm between the elevation axis and the helicopter head is horizontal. Moreover, add appropriate gains to convert the encoder output values from degrees to radians.

Before we can implement a controller that is based on the linearized equations of motion in (6a)-(6c), we need to determine the motor force constant K_f. By measurement, obtain the value for the voltage V_s which makes the helicopter maintain the equilibrium value $e = e^* = 0$ (the arm between the elevation axis and the helicopter head is horizontal). Note that this values is equal to V_s^*. Use this value to calculate the motor force constant K_f. Use the calculated value of K_f in the remaining problems.

5.2 Part II – Monovariable control

In this part of the assignment you will use the elevation controller already given in the Simulink diagram available from "Blackboard". Note that the inputs of the elevation controller are $\tilde{e}$, $\dot{\tilde{e}}$ and $\tilde{e}_c$, and the output is $\tilde{V}_s$, where $\tilde{e}_c$ is the reference value for $\tilde{e}$ and should be set to zero. Use the coordinate transformation in (4) to make sure the correct values for $\tilde{e}$, $\dot{\tilde{e}}$ and $\tilde{V}_s$ are implemented.

In this part of the assignment, we will design a controller for the pitch angle $\tilde{p}$ and the travel rate $\dot{\tilde{\lambda}}$. Before feeding any reference signal for the pitch angle or the travel angle to the controller, the elevation controller should be given a few seconds to place the helicopter at an elevation angle around $\tilde{e}_c$.

After the desired elevation angle has been reached, you can start to input a reference trajectory for the pitch angle/travel rate controller by using the joystick. Only the joystick's x-axis should be used for this.

5.2.1 Problem 1

A PD controller is added to control the pitch angle p. This controller is given as

$$\tilde{V}_d = K_{pp}(\tilde{p}_c - \tilde{p}) - K_{pd}\dot{\tilde{p}} \tag{7}$$

with constants $K_{pp}, K_{pd} > 0$, where $\tilde{p}_c$ is the desired reference for the pitch angle $\tilde{p}$. Substitute (7) in the equation of motion for the pitch angle in (6a). Apply the Laplace transform to the resulting differential equation to find the transfer function $\frac{\tilde{p}(s)}{\tilde{p}_c(s)}$.

Use the obtained transfer function to find reasonable values for K_{pp} and K_{pd}. Note that the values for K_{pp} and K_{pd} influence the eigenvalues of the closed-loop system and, therefore, the response of the helicopter. Note that the linearized pitch dynamics can be regarded as a second-order linear system, i.e. a "mass-spring-damper system". Keep the eigenvalues of the closed-loop system in mind when you tune the controller gains K_{pp} and K_{pd}. The controller should control the pitch angle rapidly to its desired value. However, it should not give rise to excessive oscillations. Discuss how the tuning of the controller gains K_{pp} and K_{pd} influences the closed-loop eigenvalues and the pitch response.

Include the PD controller in the Simulink diagram. Connect the output of the x-axis of the joystick to the reference for the pitch angle $\tilde{p}_c$ of the controller. Test if controlling the helicopter is easier now than it was when only feed forward was used. The controller might need some additional tuning before it gives a good result. Moreover, you might need to scale down the output of the joystick. Do this by adding an appropriate gain to the output of the joystick.

5.2.2 Problem 2

The travel rate $\dot{\tilde{\lambda}}$ is to be controlled using a simple P controller:

$$\tilde{p}_c = K_{rp}(\dot{\tilde{\lambda}}_c - \dot{\tilde{\lambda}}) \tag{8}$$

with constant $K_{rp} < 0$. Assume that the pitch angle is controlled perfectly, that is $\tilde{p} = \tilde{p}_c$. Show that the transfer function from the reference $\dot{\tilde{\lambda}}_c$ to the travel rate $\dot{\tilde{\lambda}}$ can be written as

$$\frac{\dot{\tilde{\lambda}}(s)}{\dot{\tilde{\lambda}}_c(s)} = \frac{\rho}{s+\rho} \tag{9}$$

where ρ is a constant. Add the P controller (8) to the Simulink diagram. Choose the gain K_{rp} such that the response of the helicopter is fast and accurate. Similar to the previous problem, test the helicopter using the joystick output as reference. This time the output from the joystick should be connected to the reference for the travel rate $\dot{\tilde{\lambda}}_c$. To limit the joystick sensitivity, it might again be a good idea to scale the output of the joystick by adding an appropriate gain to the joystick output.

5.3 Part III – Multivariable control

In this part of the assignment, you will control the pitch angle $\tilde{p}$ and the elevation rate $\dot{\tilde{e}}$ with a multivariable controller, where the reference for the pitch angle and the elevation rate are provided by the output of the joystick.

5.3.1 Problem 1

Put the system of equations given by the relations for pitch and elevation in (6a)-(6b) in a state-space formulation of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{10}$$

where $\mathbf{A}$ and $\mathbf{B}$ are matrices. The state vector and the input vector should be

$$\mathbf{x} = \begin{bmatrix} \tilde{p} \\ \dot{\tilde{p}} \\ \dot{\tilde{e}} \end{bmatrix} \qquad \text{and} \qquad \mathbf{u} = \begin{bmatrix} \tilde{V}_s \\ \tilde{V}_d \end{bmatrix} \tag{11}$$

5.3.2 Problem 2

We aim to track the reference $\mathbf{r} = [\tilde{p}_c,\ \dot{\tilde{e}}_c]^T$ for the pitch angle $\tilde{p}$ and elevation rate $\dot{\tilde{e}}$, which will be given by the joystick output. The x-axis of the joystick is used to provide the reference $\tilde{p}_c$ for the pitch angle, while the y-axis is used for the reference $\dot{\tilde{e}}_c$ for the elevation rate. Firstly, examine the controllability of the system.

We are looking for a controller of the form

$$\mathbf{u} = \mathbf{P}\mathbf{r} - \mathbf{K}\mathbf{x} \tag{12}$$

The matrix $\mathbf{K}$ corresponds to the linear quadratic regulator (LQR) for which the control input $\mathbf{u} = -\mathbf{K}\mathbf{x}$ optimizes the cost function

$$J = \int_0^\infty \left(\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right) dt \tag{13}$$

For simplicity, let the weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ be diagonal. The matrices $\mathbf{Q}$ and $\mathbf{R}$ should be chosen such that the response of the helicopter is fast and accurate. The corresponding matrix $\mathbf{K}$ can be obtained using the MATLAB command `K=lqr(A,B,Q,R)`.

Choose the matrix $\mathbf{P}$ such that (in theory) $\lim_{t\to\infty} \tilde{p}(t) = \tilde{p}_c$ and $\lim_{t\to\infty} \dot{\tilde{e}}(t) = \dot{\tilde{e}}_c$ for fixed values of $\tilde{p}_c$ and $\dot{\tilde{e}}_c$. Comment on your choices for the controller design.

5.3.3 Problem 3

Modify the controller from the previous problem to include an integral effect for the elevation rate and the pitch angle (a PI controller). Note that this results in two additional states γ and ζ, for which the differential equations are given by

$$\begin{aligned} \dot{\gamma} &= \tilde{p} - \tilde{p}_c \\ \dot{\zeta} &= \dot{\tilde{e}} - \dot{\tilde{e}}_c \end{aligned} \tag{14}$$

The state vector and the input vector are now given by

$$\mathbf{x} = \begin{bmatrix} \tilde{p} \\ \dot{\tilde{p}} \\ \dot{\tilde{e}} \\ \gamma \\ \zeta \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} \tilde{V}_s \\ \tilde{V}_d \end{bmatrix} \tag{15}$$

Note that by adding an integral effect the dimensions of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{K}$ and $\mathbf{Q}$ change. Compare the behavior of the helicopter with and without integral effect.

5.4 Part IV – State estimation

The rest of the assignment will deal with state estimation. The system has sensors for measuring the pitch angle, the elevation angle and the travel angle. In the previous problems of this assignment, the angular velocities corresponding to these angles have been computed using numerical differentiation. In this part of the assignment, an observer will be developed in order to estimate these nonmeasured states instead.

5.4.1 Problem 1

Derive a state-space formulation of the system in (6a)-(6c) of the form

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x}\end{aligned} \tag{16}$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are matrices. The state vector, the input vector and the output vector should be

$$\mathbf{x} = \begin{bmatrix} \tilde{p} \\ \dot{\tilde{p}} \\ \tilde{e} \\ \dot{\tilde{e}} \\ \tilde{\lambda} \\ \dot{\tilde{\lambda}} \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} \tilde{V}_s \\ \tilde{V}_d \end{bmatrix} \qquad \text{and} \qquad \mathbf{y} = \begin{bmatrix} \tilde{p} \\ \tilde{e} \\ \tilde{\lambda} \end{bmatrix} \tag{17}$$

5.4.2 Problem 2

Examine the observability of the system and create a linear observer for the system of the form

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \tag{18}$$

where $\mathbf{L}$ is the observer gain matrix.

Try to control the helicopter (with both controllers from Part III), but now use the estimated states as the feedback to the controllers. Plot the estimated states together with the measured states and compare. Discuss how the placement of the closed-loop observer poles influences the behavior of the helicopter.

5.4.3 Problem 3

Show that the system is observable if one only measures $\tilde{e}$ and $\tilde{\lambda}$, but that it is not observable if one only measures $\tilde{p}$ and $\tilde{e}$. Create a linear observer based on the measurement vector

$$\mathbf{y} = \begin{bmatrix} \tilde{e} \\ \tilde{\lambda} \end{bmatrix} \tag{19}$$

That is, the pitch angle $\tilde{p}$ is not used. Note that the dimensions of the matrices $\mathbf{C}$ and $\mathbf{L}$ change. Plot the estimated states together with the measured states and compare. Test the observer and try to explain (using physical considerations and the model of the system) why some state estimates are poor.

Table 1: Nomenclature (1 of 2)

e	-	elevation angle [rad]
$\tilde{e}$	-	elevation angle (after coordinate transformation) [rad]
e^*	-	linearization point elevation angle [rad]
$\tilde{e}_c$	-	reference for elevation [rad]
F_f	-	force generated by front propeller [N]
F_b	-	force generated back propeller [N]
$F_{g,b}$	-	gravitational force of back motor [N]
$F_{g,c}$	-	gravitational force of counterweight [N]
$F_{g,f}$	-	gravitational force of front motor [N]
g	-	gravitational constant [m/s^2]
J	-	cost function of linear quadratic regulator
J_e	-	moment of inertia about elevation axis [kg m^2]
J_p	-	moment of inertia about pitch axis [kg m^2]
J_λ	-	moment of inertia about travel axis [kg m^2]
$\mathbf{K}$	-	gain matrix of linear quadratic regulator
K_f	-	motor force constant [N/V]
K_{pd}	-	controller gain [$-$]
K_{pp}	-	controller gain [$-$]
K_{rp}	-	controller gain [$-$]
$\mathbf{L}$	-	gain matrix of linear observer
l_c	-	distance from elevation axis to counterweight [m]
l_h	-	distance from elevation axis to helicopter head [m]
l_p	-	distance from pitch axis to motor [m]
m_p	-	motor mass [kg]
m_c	-	counterweight mass [kg]

Nomenclature

The nomenclature used throughout this assignment is given in Tables 1 and 2.

Table 2: Nomenclature (2 of 2)

p	-	pitch angle [rad]
$\mathbf{P}$	-	gain matrix
$\tilde{p}$	-	pitch angle (after coordinate transformation) [rad]
p^*	-	linearization point pitch angle [rad]
$\tilde{p}_c$	-	reference for pitch [rad]
$\mathbf{Q}$	-	weighting matrix of linear quadratic regulator
$\mathbf{r}$	-	reference vector
$\mathbf{R}$	-	weighting matrix of linear quadratic regulator
t	-	time [s]
$\mathbf{u}$	-	input vector
V_b	-	voltage back motor [V]
V_d	-	voltage difference, $V_f - V_b$ [V]
$\tilde{V}_d$	-	voltage difference (after coordinate transformation) [V]
V_d^*	-	linearization point voltage difference [V]
V_f	-	voltage front motor [V]
V_s	-	voltage sum, $V_f + V_b$ [V]
$\tilde{V}_s$	-	voltage sum (after coordinate transformation) [V]
V_s^*	-	linearization point voltage sum [V]
$\mathbf{x}$	-	state vector
$\mathbf{y}$	-	output vector
λ	-	travel angle [rad]
$\tilde{\lambda}$	-	travel angle (after coordinate transformation) [rad]
λ^*	-	linearization point travel angle [rad]
$\dot{\tilde{\lambda}}_c$	-	reference for travel rate [rad/s]