Differential Geometry Notes

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Contents

1 Curves in Planes and Space

1.1 Parameterized Curves

Definition 1.1. A *parameterized curve* in $\mathbb{R}^n$ is a differentiable map

$$\gamma\colon\ (\alpha,\beta)\to\mathbb{R}^n$$

of an open interval $(\alpha,\beta)=\{t\in\mathbb{R}\mid\alpha<t<\beta\}$ onto $\mathbb{R}^n$ where $-\infty\leq\alpha<\beta\leq\infty$

The above definition means that γ is a correspondence from each $t\in(\alpha,\beta)$ to a point $\gamma(t)=((\gamma_1(t),\gamma_2(t),\ldots,\gamma_n(t))\in\mathbb{R}^n$ such that each function $\gamma_1(t),\gamma_2(t),\ldots,\gamma_n(t)$ is differentiable.

Definition 1.2. A *level curve* is a parametrized curve in $\mathbb{R}^3$ such that it can be defined by the pair of equations

$$f_1(x,y,z)=c_1,\text{and}f_2(x,y,z)=c_2$$

In other words, a level curve is the set of all points in the domain of f that reach a certain value c.

Example 1.1. Consider the function $y=x^2$, we seek to find a parametrization $\gamma(t)$ of the function. That means the parametrization must satisfy $\gamma_2(t)=\gamma_1(t)^2$ for all $t\in(\alpha,\beta)$ where γ is defined. An obvious solution to the equation $\gamma_2(t)=\gamma_1(t)^2$ is $\gamma_1(t)=t,\gamma_2(t)=t^2$. Since the x coordinate of $\gamma(t)$ is just t, and x can take on any value in a parabola, we know that the domain of the parametrized curve is all real numbers. Using this observation and our solution to $\gamma_2(t)=\gamma_1(t)^2$, we obtain

$$\gamma\colon\ (-\infty,\infty)\to\mathbb{R}^2,\ \ \gamma(t)=(t,t^2)$$

Remark. The equation $\gamma(t)=(2t,4t^2)$ is also a valid parametrization, as are infinitely many other pairs of coordinates that satisfy the equation.

Example 1.2. Now consider the circle $x^2+y^2=1$. An obvious solution to the equation

$$\gamma_1(t)^2 + \gamma_2(t)^2 = 1$$

is

$$\gamma_1(t) = \cos^2(t), \gamma_2 = \sin^2(t)$$

since

$$\gamma_1(t)^2 + \gamma_2(t)^2 = 1$$

, for all values of t, the domain of γ will be all real numbers. Thus, we get the curve

$$\gamma\colon (-\infty, \infty) \to \mathbb{R}^2, \ \ \gamma(t) = (\cos^2(t), \sin^2(t))$$

Remark. Taking $x = t$ instead of $x = \cos^2(t)$ would have yielded the parametrization of $\gamma(t) = (t, \sqrt{1-t^2})$ after solving for y. Although this works for the positive solutions to $x^2 + y^2 = 1$, this does not cover the negative part of the circle.

Example 1.3. Consider the parametrized curve

$$\gamma(t) = (\cos^3(t), \sin^3(t)), \ \ t \in \mathbb{R}$$

Since for all values of t,

$$\cos^2(t) + \sin^2(t) = 1$$

, the coordinates $x = \cos^3(t), y = \sin^3(t)$ of $\gamma(t)$ satisfy

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$$

(since $x^{\frac{2}{3}} + y^{\frac{2}{3}} = \cos^3(t)^{\frac{2}{3}} + \sin^3(t)^{\frac{2}{3}} = \cos^2(t) + \sin^2(t) = 1$. Thus, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ is a level curve that coincides with the image of γ

Definition 1.3. Let f be a function, f is called a *smooth function* up to order n if the derivatives of f up to order n are continuous. For the purposes of this material, all functions will be assumed to be smooth for all n

Definition 1.4. Let γ be a parametrized curve. The derivative of order 1, $\gamma'(t)$, is called the *tangent vector* of γ

Theorem 1.1. *Let γ be a parametrized curve. If the tangent vector of γ is constant, then the image of the curve is a straight line.*

Proof. Let γ be a parametrized curve with a constant tangent vector. That is, $\forall t, \gamma'(t) = \vec{a}$ where $\vec{a}$ is a constant vector. Then by component-wise integration, we have

$$\gamma(t) = \int \gamma'(t)\,\mathrm{d}t = \int \frac{\mathrm{d}\gamma}{\mathrm{d}t}\,\mathrm{d}t = \vec{a}\,\mathrm{d}t = t\vec{a} + \vec{b}$$

where $\vec{b}$ is a constant vector. This corresponds to the parametric equation for a straight line passing through point $\vec{b}$.

□

Problem 1.1. Is $\gamma(t) = (t^2, t^4)$ a parametrization of $y = x^2$?

Solution 1.1. $\gamma(t) = (t^2, t^4)$ is only a valid parametrization of the positive x side of $y = x^2$ since

$$\gamma(t) = (t^2, t^4) \implies x = t^2, y = t^4$$

and

$$y = t^4 = (t^2)^2 = x^2$$

which satisfies the equation.

Problem 1.2. Find a parametrization of the level curve $y^2 - x^2 = 1$.

Solution 1.2. We need to find a

$$\gamma\colon\ (\alpha, \beta) \to \mathbb{R}^2$$

such that $\gamma(t)$ satisfies the equation for every $t \in (\alpha, \beta)$. First try $x = t$, we get

$$y^2 - x^2 = 1 \implies x = \pm\sqrt{-1 + y^2}$$

which is inconvenient for parametrizing. Now observe that

$$\sec^2(t) - \tan^2(t) = 1$$

. So let

$$\gamma(t) = (\sec(t), \tan(t))$$

. The possible values of t is then restricted to

$$t \in (\frac{-\pi}{2}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2})$$

.

Problem 1.3. Find the Cartesian equation of the parametrization

$$\gamma(t) = (\cos^2(t), \sin^2(t))$$

We can rewrite the function $\gamma(t)$ as

$$\sin^2(t) = a\cos^2(t) + \phi(t)$$

$$= y = ax + \phi(t)$$

Observe that $\sin^2 + \cos^2 = 1 \implies \sin^2 = 1 - \cos^2 = -\cos^2 + 1$. Thus $a = -1, \phi(t) = 1$ and we get the equation $y = -x + 1$

1.2 Arc-length

Recall that for a vector v in $\mathbb{R}^n$, its magnitude or length is given by

$$\|v\| = \sqrt{v_1^2 + \ldots + v_n^2}$$

If u is another vector in $\mathbb{R}^n$, the length of the line joining u and v is given by $\|u - v\|$. So for a parametrized curve γ, if we take δt to be very small, we get a (nearly) straight line between the points $\gamma(t + \delta t)$ and $\gamma(t)$. In other words, for very small δ,

$$\gamma(t + \delta t) - \gamma(t) = \left\|\gamma(t + \delta t) - \gamma(t)\right\|$$

Furthermore, taking δ to be very small, we get

$$\frac{\gamma(t + \delta t) - \gamma(t)}{\delta t} = \gamma'(t)$$

so the length between the two points is

$$\begin{aligned}\gamma(t + \delta t) - \gamma(t) &= \left\|\gamma(t + \delta t) - \gamma(t)\right\| \\ &= \left\|\gamma'(t)\right\| \delta t\end{aligned}$$

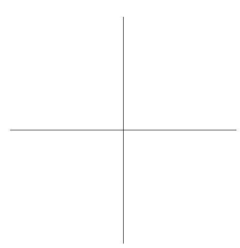
Integrating over this yields the length of the curve (arclength) for the integration bounds. This idea is defined formally below.

Definition 1.5. The *arc-length* of a curve γ starting at the point $\gamma(t_0)$ is given by the function $s(t)$ defined by

$$s(t) = \int_{t_0}^{t} \left\|g'(u)\right\| \mathrm{d}u$$

Example 1.4. Consider the logarithmic spiral given by Where k is a non-

$$\gamma(t) = (e^{kt}\cos(t), e^{kt}\sin(t))$$



zero constant. We have

$$\gamma' = (e^{kt}(k\cos(t) - \sin(t)), e^{kt}(k\sin(t) + \cos(t)))$$

and from this we get

$$\|\gamma\|^2 = e^{2kt}(k\cos(t) - \sin(t))^2 + e^{2kt}(k\sin(t)\cos(t))^2 = (k^2+1)e^{2kt}$$

We square the magnitude for simplicity of integration. Taking the starting point

$$\gamma(0) = (e^0\cos(0), e^0\sin(0)) = (1, 0)$$

for example, we get the arc-length of γ from 0 to t given by

$$\int_0^t \sqrt{k^2+1}e^{ku}\,\mathrm{d}u = \frac{\sqrt{k^2+1}}{k}(e^{kt} - 1)$$

Definition 1.6. If $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ is a parametrized curve, the *speed* at the point $\gamma(t)$ is given by $\left\|\gamma'(t)\right\|$. Additionally, γ is said to be a *unit-speed curve* if $\gamma'(t)$ is a unit vector for all $t \in (\alpha, \beta)$

Recall the dot product between two vectors

$$\vec{a} = (a_1, a_2, \ldots, a_n)\text{and}\vec{b} = (b_1, b_2, \ldots, b_n)$$

where $a, b \in \mathbb{R}^n$ is given by

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

Now, if a, b are smooth and functions of t, we can apply the 'product rule' from calculus to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{a} \cdot \vec{b}) = \frac{\mathrm{d}\vec{a}}{\mathrm{d}t} \cdot \vec{b} + \vec{a} \cdot \frac{\mathrm{d}\vec{b}}{\mathrm{d}t}$$

Proposition 1.7. *Let $\vec{n}(t)$ be a unit vector that is a smooth function of* *parameter t. Then the following dot product satisfies*

$$\vec{n}'(t) \cdot \vec{n}(t) = 0$$

for all t. In other words, $\vec{n}'(t)$ is perpendicular to $\vec{n}(t)$ for all t.

Proof. By the definition of unit vector,

$$\vec{n} \cdot \vec{n} = 1$$

Differentiating both sides with respect to t gives

$$\vec{n}' \cdot \vec{n} + \vec{n} \cdot \vec{n}' = 0$$

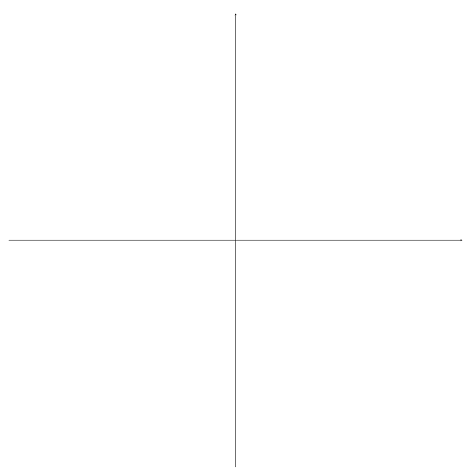
so

$$2\vec{n}' \cdot \vec{n} = 0$$

Take $\vec{n} = \gamma'$ to satisfy the desired equality. $\square$

Problem 1.4. Calculate the arc-length of the catenary

$$\gamma(t) = t, \cosh(t)$$



Solution 1.3. We begin by finding the first derivative and its length

$$\gamma'(t) = (1, \sinh(t))$$

and the length

$$\|\gamma'(t)\| = \sqrt{1 + \sinh^2(t)} = \sqrt{\cosh^2(t) = \cosh(t)}$$

The arc-length starting at the point $\gamma(0) = (0, \cosh(0)) = (0, 1)$ is then given by

$$s = \int_0^t \cosh(u)\,\mathrm{d}u = \sinh(u)\Big|_0^t = \sinh(t)$$

Problem 1.5. Show that the following curve is unit-speed

$$\gamma(t) = (\frac{1}{3}(1+t)^{\frac{3}{2}}, \frac{1}{3}(1-t)^{\frac{3}{2}}, \frac{t}{\sqrt{2}})$$

Solution 1.4. A unit vector is a vector whose magnitude is 1. We must show that $\gamma'(t)$ is a unit vector for all $t \in (-\infty, \infty)$ (since the domain of t is all real numbers). First we find the first derivative of γ with respect to t

$$\gamma'(t) = (\frac{\sqrt{t+1}}{2}, -\frac{\sqrt{t-1}}{2}, \frac{1}{2})$$

The magnitude of this vector (that is a smooth function of t) is given by

$$\|\gamma'(t)\|^2 = \frac{\sqrt{t+1}}{2}^2 + \frac{\sqrt{t-1}}{2}^2 + \frac{1}{2}^2 = \frac{t+1}{4} + \frac{-t+1}{4} + \frac{1}{2} = 1$$

Thus $\gamma(t)$ is a unit vector

1.3 Reparametrization

This section seeks to expand on the relationship between different possible parametrizations of a level curve.

Definition 1.8. A parametrized curve $\tilde{\gamma}\colon \left(\tilde{\alpha}, \tilde{\beta}\right) \to \mathbb{R}^n$ is a *reparametrization* of a parametrized curve $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$ if there is a smooth bijective map

$$\phi : \left(\tilde{\alpha}, \tilde{\beta}\right) \to (\alpha, \beta)$$

known as the *reparametrization map* such that the inverse map

$$\phi^{-1} \colon \ (\alpha, \beta) \to \left(\tilde{\alpha}, \tilde{\beta}\right)$$

is also a smooth function and

$$\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}) \ \forall \tilde{t} \in \left(\tilde{\alpha}, \tilde{\beta}\right)$$

Example 1.5. We previously found that the circle $x^2 + y^2 = 1$ has a parametrization

$$\gamma(t) = (\cos(t), \sin(t))$$

Another valid parametrization of the circle is

$$\tilde{\gamma}(t) = (\sin(t), \cos(t))$$

In order to show that $\tilde{\gamma}$ is a valid reparametrizaiton of γ, we must find a reparemtrization map such that

$$(\cos(\phi(t)), \sin(\phi(t))) = (\sin(t), \cos(t))$$

Let $\phi(t) = \frac{\pi}{2} - t$. Then we get the following

$$(\cos(\phi(t)) = \cos(\frac{\pi}{2} - t) = \sin(t), \ \sin(\phi(t)) = \sin(\frac{\pi}{2} - t) = \cos(t)$$

Definition 1.9. A point $\gamma(t)$ of a parametrized curve γ is called a *regular point* if

$$\gamma'(t) \neq \vec{0}$$

Otherwise (if $\gamma(t)$ is not a regular point), we call $\gamma(t)$ a *singular point* of γ.

A curve is regular if all of the points on the curve are regular.

Proposition 1.10. *Any reparametrization of a regular curve is regular*

Proof. Suppose γ is a regular curve and $\tilde{\gamma}$ is a reparametrization of γ. Let

$$t = \phi(\tilde{t}) \text{and} \psi = \phi^{-1}$$

Now, differentiate both sides of $\phi(\psi(t)) = t$ results in the following

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tilde{t}} \frac{\mathrm{d}\psi}{\mathrm{d}t} = 1$$

Now, since $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t})$, we can differentiate this to obtain

$$\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}\tilde{t}} = \frac{\mathrm{d}\gamma}{\mathrm{d}t} \frac{\mathrm{d}\phi}{\mathrm{d}\tilde{t}}$$

From previous two differentiated equations, we can conclude that

$$\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}\tilde{t}} \neq 0, \ \frac{\mathrm{d}\phi}{\mathrm{d}\tilde{t}} \neq 0$$

This satisfies the definition of a regular curve.

□

Proposition 1.11. *If $\gamma(t)$ is a regular curve, its arc-length s starting at any point of γ is a smooth function of t.*

Proof. Recall that by previous result, s is a differentiable function of t and

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \|\gamma'(t)\|$$

For simplicity, we will assume that γ is a plane curve of the form

$$\gamma(t) = (u(t), v(t))$$

Where u and v are smooth functions of t. Then we have

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{u'^2 + v^2}$$

Now, observe that $f(x) = \sqrt{x}$ is a smooth function on $(0, \infty)$. Furthermore, by induction,

$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n} = (-1)^{n-1}\frac{(1)3(5)...(2n-1)}{2^n}x^{\frac{-(2n+1)}{2}}$$

Furthermore, since u and v are smooth on $(0, \infty)$ so are u' and v' and an elementary composition of smooth functions is smooth so the function

$$\sqrt{u'^2 + v'^2}$$

is a smooth function. Then, we have enough information to conclude that

$$\frac{\mathrm{d}s}{\mathrm{d}t} = f(u'^2 + v'^2)$$

is a smooth function of t so s is smooth.

□

Proposition 1.12. *A parametrized curve has a unit-speed reparametrization if and only if it is a regular curve.*

Corollary 1.12.1. *Let γ be a regular curve and $\tilde{\gamma}$ a unit-speed reparametrization of γ*

$$\tilde{\gamma}(u(t)) = \gamma(t)$$

where u is a smooth function of t. Then, if s is the arc-length of γ, we have

$$u = \pm s + c$$

where c is a constant

Example 1.6. This example shows that although a unit-speed reparametrization is always possible, it is not always convenient. Consider the logarithmic spiral

$$\gamma(t) = (e^{kt}\cos(t), e^{kt}\sin(t))$$

Seeking a unit-speed paremtrization, we have

$$\gamma'(t) = (-e^{kt}\sin(t) + ke^{kt}\cos(t), e^{kt}\cos(t) + ke^{kt}\sin(t))$$

and

$$\left\|\gamma'(t)\right\|^2 = e^{2kt}\sin^2(t)+k^2e^{2kt}\cos^2(t)+e^{2kt}\cos^2(t)+k^2e^{2kt}\sin^2(t) = (k^2+1)e^{2kt}$$

so the arc-length starting from 0 is given by

$$s = \int_0^t \sqrt{k^2+1}e^{ku}\,\mathrm{d}u = \frac{1}{k}\sqrt{k^2+1}e^{kt} - 1$$

then

$$t(s) = \frac{1}{k}\ln(\frac{ks}{\sqrt{k^2+1}} + 1)$$

which gives us the ugly unit-speed reparemtrization of

$$\tilde{\gamma(s)} = ((\frac{ks}{\sqrt{k^2+1}}+1)\cos(\frac{1}{k}\ln(\frac{ks}{\sqrt{k^2+1}}+1)), (\frac{ks}{\sqrt{k^2+1}}+1)\sin(\frac{1}{k}\ln(\frac{ks}{\sqrt{k^2+1}}+1)))$$

2 Curvature and Torsion

2.1 Curvature

Definition 2.1. Let γ be a unit-speed curve with parameter t, the *curvature* of γ at point $\gamma(t)$ is given by

$$\kappa(t) = \left\|\gamma''(t)\right\|$$

Example 2.1. Consdier the circle with center (x_0, y_0) and radius r. A unit-speed parametrization is given by

$$\gamma(t) = (x_0 + r\cos\frac{t}{r}, y_0 + r\sin\frac{t}{r})$$

and γ is unit speed since

$$\left\|\gamma'(t)\right\| = \sqrt{(-\sin^2\frac{t}{r})^2 + (\cos\frac{t}{r})^2} = 1$$

Then the curvature of a circle is given by

$$\kappa(t) = \left\|\gamma''(t)\right\| = \sqrt{(-\frac{1}{r}\cos\frac{t}{r})^2 + (-\frac{1}{r}\sin\frac{t}{r})^2} = \frac{1}{r}$$

Intuitively, this makes sense since the curvature is the reciprocal of the radius, this means small circles should have large curvature and large circles should have small curvature as expected.

As dicsovered previously, every curve has a unit-speed reparemtrization but it is not always convenient or possible to express it in terms of elementary functions. The following proposition gives a way to express curvature in terms of γ itself instead of a unit-speed reparametrization of it.

Proposition 2.2. *Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$. Then its curvature is given by*

$$\kappa = \frac{\left\|\gamma'' \times \gamma'\right\|}{\left\|\gamma'\right\|^3}$$

where $\times$ denotes the cross product.

Proof. The proof of this proposition is computationally intensive so is omitted. The basic idea is to consider a unit-speed parameter s of γ. Then apply the chain rule to γ and plug it into the unit-speed curvature formula and perform some more differentiating and vector products to obtain κ in terms of just γ and its derivatives. $\square$

2.2 Space Curves

Curves in $\mathbb{R}^3$ or space curves are the main focus of study in this look at differential geometry. We will show that the previously introduced idea of curvature, together with a new concept, torsion, can determine any space curve up to a direct isometry.

Recall that a tangent vector can be expressed as the first derivative of a paremtrized curve. Then a unit-tangent vector is simply the tangent vector whose magnitude is 1. Using this idea, we have a new definition that will be helpful in introducing the concept of torsion.

Definition 2.3. Let $\gamma(s)$ be a unit-speed curve in $\mathbb{R}^3$, and let $\vec{t} = \gamma'$ be the unit-tangent vector of γ. If the curvature is non-zero, we define the *principal normal* of γ at the point $\gamma(s)$ to be the vector

$$\vec{n}(s) = \frac{1}{\kappa(s)}\vec{t}'(s)$$

Observe that since $\left\|\vec{t}'\right\| = \kappa$, $\vec{n}$ is a unit vector. Furthermore, $\vec{t} \cdot \vec{t}' = 0$ so $\vec{n}$ and $\vec{t}$ are perpendicular. This leads to our next definition.

Definition 2.4. Let $\vec{t}$ be a unit-tangent vector of a paremtrized curve γ and let $\vec{n}$ be its principal norm. Then the vector

$$\vec{b}(s) = \vec{t}(s) \times \vec{n}(s)$$

is called the *binormal vector* of γ at the point $\gamma(s)$.

The set of the unit-tangent vector, the principal norm and the binormal vector $\{\vec{t}, \vec{n}, \vec{b}\}$ form an orthonormal basis of $\mathbb{R}^3$. If we apply the vector product rule to the binormal norm, we obtain the equation

$$\vec{b}' = \vec{t}' \times \vec{n} + \vec{t} \times \vec{n}' = \vec{t} \times \vec{n}'$$

Furthermore, by definition,

$$\vec{n}(s) = \frac{1}{\kappa(s)}\vec{t}'(s) \implies \vec{t}' \times \vec{n} = \kappa\vec{n} \times \vec{n} = 0$$

so $\vec{b}'$ is perpendicular to both $\vec{b}$ and $\vec{t}$. Thus it is parallel to n. This leads us to our next definition

Definition 2.5. Let $\vec{b}$ be a binormal vector of a parametrized curve γ. The *torsion* of γ at a point $\gamma(s)$ is given by τ where

$$\vec{b}' = -\tau\vec{n}$$

Similar to curvature, it is possible to express the torsion of a curve in terms of γ itself rather than a unit-speed reparametrization.

Proposition 2.6. *Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$ with non-zero curvature. Then the torsion of γ is given by*

$$\tau = \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{\|\gamma' \times \gamma''\|^2}$$

Proof. Similar to the proof for curvature, this proof is omitted due to lengthy computations. $\square$

Proposition 2.7. *Let γ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Then the image of γ is contained in a plane if and only if τ is zero at every point of the curve*

Proof. Let γ be unit-speed (reparametrizing does not change torsion). Let s be the parameter of γ. Next, suppose the image of γ is contained by the plane $\vec{n} \cdot \vec{N} = d$ where $\vec{N}$ is a constant unit vector and $\vec{v} \in \mathbb{R}^3$. Differentiating $\gamma \cdot \vec{N} = d$ with respect to s, we have

$$\vec{t} \cdot \vec{N} = 0 \implies \vec{t}' \cdot \vec{N} = 0 \implies \kappa \vec{n} \cdot \vec{N} = 0$$

Since $\vec{t}$ and $\vec{n}$ are perpendicular to $\vec{N}$ so $\vec{b}$ is parallel to $\vec{N}$ and is a constant vector. So $\vec{b}' = 0$ so $\tau = 0$. The converse is straightforward. $\square$

Theorem 2.1. *Let γ be a unit-speed curve in $\mathbb{R}^3$ with non-zero curvature. Then the following relations hold*

$$\vec{t}' = \kappa \vec{n}$$

$$\vec{n}' = -\kappa \vec{t} + \tau \vec{b}$$

$$\vec{b} = -\tau \vec{n}$$

These equations are called the Frenet-Serret equations

The following result is an immediate consequence of the Frenet-Serret equations

Proposition 2.8. *Let γ be a unit-speed curve in $\mathbb{R}^3$ with constant curvature and zero torsion. Then γ is a parametrizaiton of a circle.*

Proof. This immediately follows from the previous proposition which states that any curve with zero torsion contains the image of γ (since a circle has a constant curvature). However, a more instructive proof follows. We previously showed that the binormal vector of a zero torsion curve is constant and γ is contained in a plane perpendicular to $\vec{b}$. Now

$$\frac{\mathrm{d}}{\mathrm{d}s}(\gamma + \frac{1}{\kappa}\vec{n} = \vec{t} + \frac{1}{\kappa}\vec{n}' = 0)$$

From the Frenet-Serret equation, and the fact that curvature is constant, we have

$$\vec{n}' = -\kappa\vec{t} + \tau\vec{b} = -\kappa\vec{t}$$

Then $\gamma + \frac{1}{\kappa}\vec{n}$ is a constant vector and

$$\left\|\gamma - \gamma + \frac{1}{\kappa}\vec{n}\right\| = \frac{1}{k}$$

So γ lies on a sphere S with center $\gamma + \frac{1}{\kappa}\vec{n}$ and radius $\frac{1}{k}$. $\square$

Theorem 2.2. *Let $\gamma(s)$ and $\tilde{\gamma}(s)$ be two unit-speed curves in $\mathbb{R}^3$ with the same $\kappa(s) > 0$ and τ for all s. Then there is a direct isometry M of $\mathbb{R}^3$ such that*

$$\tilde{\gamma}(s) = M(\gamma(s))$$

3 Global Properties of Curves

3.1 Simple Closed Curves

Definition 3.1. Let $\gamma\colon (\alpha, \beta) \to \mathbb{R}^2$ be a curve. We say γ is a *closed curve* if $\gamma(\alpha) = \gamma(\beta)$, i.e., the start and endpoints are the same, creating an enclosed area.

Definition 3.2. Let $\gamma(t)$ be a parametrized curve described by $x(t), y(t)$. The curve γ has a *self intersection* if there is a point x, y such that $x(w) = x(t) = x$ and $y(w) = y(t) = y$ where $w \neq t$

Definition 3.3. A *simple closed curve* in $\mathbb{R}^2$ is a closed curve that has no self-intersections

The Jordan Curve Theorem gives the result that any simple closed curve has an interior and an exterior that describe the complement of the image of γ by

1. $\text{int}(\gamma)$ is bounded
2. $\text{ext}(\gamma)$ is unbounded
3. The regions $\text{int}(\gamma)$ and $\text{ext}(\gamma)$ are connected, i.e., any two points in one of the respective regions can be joined by a curve that is contained by the region. However, a curve joining a point from the two regions must cross the curve γ.

Example 3.1. The curve $\gamma(t) = (p\cos(t), q\sin(t))$ is a simple closed curve. The interior of γ is given by

$$\text{int}(\gamma) = \{(x, y) \in \mathbb{R}^2 | \frac{x^2}{p^2} + \frac{y^2}{q^2} < 1\}$$

and the exterior is given by

$$\text{ext}(\gamma) = \{(x, y) \in \mathbb{R}^2 | \frac{x^2}{p^2} + \frac{y^2}{q^2} > 1\}$$

For a simple closed curve, we can use the interior and exterior to distinguish two orientations of γ. We say a curve γ is *positively oriented* if the signed unit-normal vector points towards the interior of the curve.

Theorem 3.1. *The total signed curvature of a simple closed curve in* $\mathbb{R}^2$ *is* $\pm 2\pi$.

Proof. The proof is omitted, it is beyond the scope of this course □

3.2 Isoperimetric Inequality

We begin this section with recalling a theorem from multivariable calculus, Green's Theorem.

Theorem 3.2. *Let $f(x,y)$ and $g(x,y)$ be smooth functions and let γ be a positively oriented simple closed curve, then*

$$\int_{int(\gamma)} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \mathrm{d}x \, \mathrm{d}y = \int_\gamma f(x,y) \, \mathrm{d}x + g(x,y) \, \mathrm{d}y$$

Proposition 3.4. *Let $\gamma(t) = (x(t), y(t))$ be a positively oriented simple closed curve with period T. Then the area contained by γ is*

$$A(\gamma) = \frac{1}{2} \int_0^T (xy' - yx') \, \mathrm{d}t$$

Proof. let $f = -\frac{1}{2}y, g = \frac{1}{2}x$. Applying Green's Theorem, we get

$$A(\gamma) = \frac{1}{2} \int_\gamma x \, \mathrm{d}y - y \, \mathrm{d}x$$

$\square$

Theorem 3.3. *Let γ be a simple clsoed curve, let $l(\gamma)$ be its length and $A(\gamma)$ the area contained by it. Then the following inequality holds*

$$A(\gamma) \leq \frac{1}{4\pi} l(\gamma)^2$$

4 Surfaces in $\mathbb{R}^3$

4.1 Surface Parametrization

Recall from analysis the concept of an open set. A subset U of $\mathbb{R}^n$ is called open if

$$a \in U, \|u - a\| < \epsilon \implies u \in U$$

Furthermore, a function f is said to be continuous if

$$u \in X, \|u - a\| < \delta \implies \big\|f(u) - f(a)\big\| < \epsilon$$

Using the definition of an open set to express a continuous function, we have that f is continuous if and only if for every open set $V \in \mathbb{R}^n$, there is an open set $U \in \mathbb{R}^n$ such that $U \cap X = \{x \in X | f(x) \in V\}$.

Definition 4.1. Let $f\colon X \to Y$ be a continuous, bijective function. The function f is called a *homeomorphism* if the inverse map $f^{-1}\colon Y \to X$ is also continuous. Furthermore, X and Y are said to be *homeomorphic*

Using the above definitions, we now have the information needed to construct a definition for a surface in $\mathbb{R}^3$ and a way to describe the concept of a parametrization in $\mathbb{R}^3$.

Definition 4.2. Let $S \subseteq \mathbb{R}^3$. If every point $p \in S$ has open sets $U \in \mathbb{R}^2$ and $W \in \mathbb{R}^3$ containing p such that $S \cap W$ is homemorphic to U, then S is a *surface*. A homeomorphism $\sigma\colon U \to S \cap W$ is called a *surface patch* or *surface parametrization* of the open subset $S \cap W \subseteq S$. A set of surface patches whose images cover S is called a *atlas* of S.

Example 4.1. Consider the unit cylinder given by

$$S = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = 1\}$$

which has a parametrization given by

$$\sigma(u, v) = (\cos u, \sin u, v)$$

Here, σ is continuous but not injective (so not bijective) since $\sigma(u, v) = (u + 2\pi, v)$ for all (u, v). Restricting the interval of u such that $0 \le u \le 2\pi$ yields an injective map. However, even when we restrict σ to this subset of $\mathbb{R}^2$

$$V = \{(u, v) \in \mathbb{R}^2 | 0 \le u < 2\pi\}$$

we do not have a homeomorphism because V is not an *open* subset of $\mathbb{R}^2$

Example 4.2. Consider the unit sphere denoted by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$$

A common way to parametrize the sphere is with a latitude θ and a longitude

ϕ where

$$\sigma(\theta, \phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$$

This is known as the *latitude-longitude parametrization* of S^2. Like the cylinder, σ is not an injective map since $\sigma(\theta, \phi) = \sigma(\theta, \phi + 2\pi)$.

5 Homework

5.1 homework

Problem 5.1. Consider the circular helix

$$\gamma(t) = (at, b\sin t, b\cos t)$$

where a, b are non-zero constants

Solution 5.1.

(a.) To find the arc length, we first differentiate γ with respect to t.

$$\gamma'(t) = (a, b\cos t, -b\sin t)$$

Next, we must find the magnitude of $\gamma'(t)$.

$$\big\|(a, b\cos t, -b\sin t)\big\| = \sqrt{t^2 + b^2\cos^2 t + b^2\sin^2 t} = \sqrt{a^2 + b^2}$$

Finally, we integrate the magnitude from 0 to t to obtain our arc length.

$$s = \int_0^t \sqrt{a^2 + b^2}\,\mathrm{d}u = \sqrt{b^2 + a^2}t$$

(b.) For all t, $\gamma'(t) = (a, b\cos t, -b\sin t) \neq \vec{0}$, so γ is a regular curve. Thus γ has a unit-speed reparametrization. By taking the inverse of $s(t)$,

$$s(t) = \sqrt{a^2 + b^2}t \implies t(s) = t(s) = \frac{s}{\sqrt{a^2 + b^2}}$$

the unit-speed paremtrization is then given by

$$c(s) = \gamma(t(s)) = (\frac{as}{a^2+b^2}, b\sin(\frac{s}{a^2+b^2}, b\cos(\frac{s}{a^2+b^2})))$$

(c.) The following calculations are needed

$$\gamma'(t(s)) = (a, b\cos\frac{s}{a^2+b^2}, -b\sin\frac{s}{a^2+b^2})$$

$$\gamma''(t(s)) = (0, -b\sin\frac{s}{a^2+b^2}, -b\cos\frac{s}{a^2+b^2})$$

$$\gamma'''(t(s)) = (0, -b\cos\frac{s}{a^2+b^2}, b\sin\frac{s}{a^2+b^2})$$

$$\gamma' \times \gamma'' = (ab\sin t, -ab\cos t, b^2)$$

$$\|\gamma' \times \gamma''\| = \sqrt{a^2b^2(\sin^2\frac{s}{a^2+b^2} + \cos^2\frac{s}{a^2+b^2}) + b^4} = b\sqrt{a^2+b^2}$$

$$(\gamma' \times \gamma'')\dot{\gamma}''' = ab^2\sin^2\frac{s}{a^2+b^2} + ab^2\cos^2\frac{s}{a^2+b^2} = ab^2$$

$$\vec{T}(s) = (\frac{a}{\sqrt{a^2+b^2}}, \frac{b\cos(\frac{s}{a^2+b^2})}{\sqrt{a^2+b^2}}, \frac{-b\sin(\frac{s}{a^2+b^2})}{\sqrt{a^2+b^2}})$$

$$\vec{B}(s) = \frac{(a, b\cos\frac{s}{a^2+b^2}, -b\sin\frac{s}{a^2+b^2}) \times (0, -b\sin(\frac{s}{a^2+b^2}), -b\cos(\frac{s}{a^2+b^2}))}{\left\|(a, b\cos\frac{s}{a^2+b^2}, -b\sin\frac{s}{a^2+b^2}) \times (0, -b\sin(\frac{s}{a^2+b^2}), -b\cos(\frac{s}{a^2+b^2}))\right\|}$$

$$\kappa = \frac{\|(\gamma' \times \gamma'')\|}{\|\gamma\|^3} = \frac{b}{b^2+a^2}$$

$$\tau = \frac{(\gamma' \times \gamma'').\gamma'''}{\|\gamma' \times \gamma''\|^2} = \frac{ab^2}{b^2(a^2+b^2)}$$

(d.) Let $a = \frac{3}{3^2+(-4)^2}, b = \frac{-4}{3^2+4^2}$ then every curve of constant curvature 3 and torsion -4 can be obtained by isometry to the helix given by the unit speed curve of

$$\gamma(t) = (at, b\sin t, b\cos t)$$

This is true because any curves with the same torsion and curvature have a direct isometry by theorem 2.3.6.

5.2 Homework 2

Problem 5.2. Let D be the region in $\mathbb{R}^2$ enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Denote the ellipse by ∂D and consider the paremtrization of ∂D given by

$$c(t) = (a\cos(2t), b\sin(2t))$$

Solution 5.2. The period of γ is π

Solution 5.3. Using the given parametrization c, we can apply Green's Theorem to find the area of D given by

$$A(D) = \frac{1}{2}\int_0^\pi a\cos(2t)[b\sin(2t)]' - b\sin(2t)[a\cos(2t)]' \, \mathrm{d}t = \frac{1}{2}\int_0^\pi 2ab\cos^2(2t) + 2ab\sin^2(2t) \, \mathrm{d}t$$

$$= \int_0^\pi ab \, \mathrm{d}t = \pi ab$$

Solution 5.4. Consider the parametrization of the ellipse given by $c(t) = (a\cos(2t), b\sin(2t))$, we previously found the area of this curve is given by πab. Now, observe that the length l of c is given by

$$l = \int_0^\pi \|c'\| = \int_0^\pi \sqrt{a^2\sin^2(2t) + b^2\cos^2(2t)} \, \mathrm{d}t$$

By the isoperemtric inequality, the length l satisfies $l \geq \pi\sqrt{ab}$. Furthermore, the theorem states that equality holds if and only if c is a circle which happens only if $a = b$.

Problem 5.3. Let S^2 denote the unit sphere $x^2 + y^2 + z^2 = 1$ and define the parametrization $\sigma\colon U \to \mathbb{R}^3$ by

$$\sigma(\theta, \phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\phi)$$

where

$$U = \{(\theta, \phi) \in \mathbb{R}^2 | -\frac{\pi}{2} < \theta < \frac{\pi}{2}, 0 < \phi < 2\pi\} \text{and} C = \{(x, 0, z) \in S^2 | x \geq 0\}$$

Solution 5.5. Let $(\theta_0, \phi_0) \in U$. For the curve $\gamma_1(\theta) = \sigma(\theta, \phi_0)$, the curvature is given by

$$\kappa_1 = \left\|\gamma_1(\theta)''\right\| = \left\|\sigma(\theta, \phi_0)''\right\| = \sqrt{\cos^2\theta\cos^2\phi_0 + \cos^2\theta\sin^2\phi_0 + \sin^2\phi}$$

$$= \sqrt{\cos^2(\theta_0) + \sin^2(\phi)} = 1$$

and the curvature of the curve $\gamma_2(\theta) = \sigma(\theta_0, \phi)$ is given by

$$\kappa_2 = \left\|\gamma_2(\theta)''\right\| = \left\|\sigma(\theta_0, \phi)''\right\| = \sqrt{-\cos^2\theta_0\cos^2\phi + -\cos^2\theta_0\sin^2} = \cos\theta_0$$

Definitions