Linear Algebra Notes

Adapted From Freidberg's Linear Algebra

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1 Vector Spaces

1.1 Vector Spaces

Definition 1.1.1. A *vector space* V over a field F consists of a set on which two operations are defined so that for each $x, y \in V$ there exists $x + y \in V$ and $ax \in V \ \forall a \in F$ such that the following hold:

$$\forall x, y \in V,\ x + y = y + x$$

$$\forall x, y, z \in V,\ (x + y) + z = x + (y + z)$$

$$\exists 0 \in V | x + 0 = x,\ \forall x \in V$$

$$\forall x \in V,\ \exists y \in V | x + y = 0$$

$$\forall x \in V,\ 1x = x$$

$$\forall a, b \in F, x \in V,\ (ab)x = a(bx)$$

$$\forall a \in F \text{ and } x, y \in V,\ a(x + y) = ax + ay$$

$$\forall a, b \in F, x \in V,\ (a + b)x = ax + bx$$

Example 1.1.1. The set of n-tuples from a field F is denoted F^n. This set is a vector space over F with the operations of vector addition and scalar multiplication. That is, if $u = (a_1, a_2, ..., a_n), v = (b_1, b_2, ..., b_n)$ and $c \in F$ then $u + v = (a_1 + b_1, a_2 + b_2, ..., a_n + b_n)$ and $cu = (ca_1, ca_2, ..., ca_n)$.

Example 1.1.2. The set of $m \times n$ matrices with entries from a field F is a vector space, denoted $M_{m\times n}(F)$, with the operations of matrix addition and scalar multiplication. Let $A, B \in M_{m\times n}(F)$ and let $c \in F$. Then

$$(A + B)_{ij} = A_{ij} + B_{ij}, c(A_{ij}) = cA_{ij}$$

Example 1.1.3. Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 \text{ and } g(x) = b_n x^n + b_{n-1} x^{n-1} + ... + b_1 x + b_0$$

be polynomials with coefficients from a field F. Define

$$f(x) + g(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + ... + (a_1 + b_1)x + (a_0 + b_0)$$

and

$$cf(x) = ca_n x^n + ca_{n-1} x^{n-1} + ... + ca_1 x + ca_0$$

Under this construction, the set of polynomials with coefficients from a field F is a vector space and is denoted $P(F)$

Example 1.1.4. Let $S = \{(a_1, a_2) | a_1, a_2\} \in \mathbb{R}$ and define the following two operations:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2)$$
$$c(a_1, a_2) = (ca_1, ca_2)$$

Notice that conditions 1,2 and 8 of a vector space fail to hold so S is not a vector space.

Theorem 1.1.1 (Cancellation Law of Vector Addition). *if $x, y, z \in V$ and $x + z = y + z$, then $x = y$*

Proof. By condition 4 of a vector space, there is a vector v such that $z+v = 0$. Now,

$$x = x+0 = x+(z+v) = (x+z)+v = (y+z)+v = y+(z+v) = y+0 = y$$

by conditions 2 and 3 of a vector space □

Theorem 1.1.2 (Zero Vector Properties). *In any vector space V, the following hold:*

$$forall x \in V,\ 0x = 0$$
$$forall a \in F, x \in V,\ (-a)x = -a(x) = a(-x)$$
$$forall a \in F,\ a0 = 0$$

Problems

Problem 1.1.1. Let $S = \{0, 1\}$ and $F = R$. Show that $f = g$ and $f+g = h$ where $f(x) = st + 1$, $g(x) = 1 + 4t + 2t^2$, $h(x) = 5^t + 1$.

Solution 1.1.1. Since the set S consists of only 0 and 1, the following is sufficient to show that $f = g$ and $f + g = h$:

$$f(0) = 2(0) + 1 = 1 = 1 + 4(0) + 2(0)^2 = g(0)$$
$$f(1) = 2(1) + 1 = 3 = 1 + 4(1) + 2(1)^2 = g(1)$$
$$f(0) + g(0) = 1 + 1 = 5^0 + 1 = h(0)$$
$$f(1) + g(1) = 3 + 3 = 6 = 5^1 + 1 = h(1)$$

Problem 1.1.2. Show that

$$(a + b)(x + y) = ax + ay + bx + by)$$

for any $x, y \in V$ and $a, b \in F$

Solution 1.1.2. *Proof.* Let V be a vector space with $x, y \in V$. Then by condition 7 and 8 of a vector space,

$$(a+b)(x+y) = a(x+y) + b(x+y) = ax + ay + bx + by$$

$\square$

1.2 Subspaces

Definition 1.2.1. A subset W of a vector space V over a field F is called a *subspace* of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V.

Theorem 1.2.1 (Subspace of a Vector Space)**.** *Let V be a vector space and $W \subseteq V$. Then W is a subspace of V if and only if the following conditions hold:*

$$0 \in V$$

$$\forall x, y \in W,\ x + y \in W$$

$$\forall c \in F, x \in W,\ cx \in W$$

Example 1.2.1. The set W of all symmetric matrices (matrices such that $A^t = A$) is a subspace of $M_{m\times n}(F)$. Since the zero matrix has all zero entries, clearly the zero matrix is in W. If $A, B \in W$, then $(A+B)^t = A^t + B^t = A + B$ so W is closed under addition. If $A \in W$ and $c \in F$, then $cA^t = c(A^t) = c(A) = cA$ so W is closed under scalar multiplication.

Example 1.2.2. The set W of diagonal matrices is a subspace of $M_{n\times n}(F)$. Clearly, the zero matrix is diagonal so it is in W. Further, if $A, B \in W$ then $(A+B)_{ij} = A_{ij} + B_{ij}$ and when $i \neq j$, $A_{ij} = B_{ij} = 0$. So $A_{ij} + B_{ij}$ is diagonal and in W. The same argument applies to closure under scalar multiplication.

Theorem 1.2.2 (Intersection of Subspaces)**.** *Any intersection of subspaces of a vector space V is a subspace.*

Proof. Since every subspace contains the zero vector, the intersection of subspaces contains the zero vector. The closure of addition and multiplication follow from the definition of closure. $\square$

Problems

Problem 1.2.1. Prove that $(A^t)^t = A$ for all $A \in M_{m\times n}(F)$

Solution 1.2.1. Let $A \in M_{m\times n}(F)$. Then by the definition of transpose,

$$(A^t)^t = (A^t)^t_{ij} = (A_{ji})^t = A_{ij} = A$$

Problem 1.2.2. Prove that diagonal matrices are symmetric matrices

Solution 1.2.2. Let A be a diagonal matrix. Then where $i \neq j$, $A_{ij} = 0$. Additionally, for all $i = j$, we have $A_{ij} = A_{ji}$. Then

$$A^t = (A^t)_{ij} = A_{ji} = A_{ij} = A$$

and A is symmetric by definition.

Problem 1.2.3. Prove that the set $W_1 = \{(a_1, a_2, ..., a_n \in F^n | a_1 + a_2 + ... + a_n = 0\}$ is a subspace and the space $W_2 = \{(a_1, a_2, ..., a_n \in F^n | a_1 + a_2 + ... + a_n = 1\}$ is not.

Solution 1.2.3. Let W_1 be defined as above. Since $a_1 + a_2 + ... + a_n = 0$, the zero vector is in W_1. Let $\{a_n\}, \{b_n\} \in W_1$. Then $\{a_n\} + \{b_n\} = (a_1 + a_2 + ... + a_n) + (b_1 + b_2 + ... + b_n) = 0 + 0 = 0 \in W_1$. For $c \in F$, $c\{a_n\} = c(a_1 + a_2 + ... + a_n) = c(0) = 0 \in W_1$. So W_1 is a subspace. Define W_2 as above, the zero vector is not in W_2 so it is not a subspace.

1.3 Linear Combinations and Systems of Equations

Definition 1.3.1. Let V be a vector space and S a nonempty subset of V. A vector $v \in V$ is called a *linear combination* of vectors of S if there exist a finite number of vectors $u_1, u_2, ..., u_n \in S$ and scalars $a_1, a_2, ..., a_n \in F$ such that $v = a_1u_1 + a_2u_2 + ... + a_nu_n$.

Example 1.3.1. The claim is that

$$2x^3 - 2x^2 + 12x - 6$$

is a linear combination of

$$x^3 - 2x^2 - 5x - 3 \text{ and } 3x^3 - 5x^2 - 4x - 9$$

in $P_3(R)$, but

$$3x^3 - 2x^2 + 7x + 8$$

is not. For case 1, we need to find scalars a, b such that

$$2x^3 - 2x^2 + 12x - 6 = a(x^3 - 2x^2 - 5x - 3) + b(3x^3 - 5x^2 - 4x - 9)$$

$$= (a + 3b)x^3 + (-2a - 5b)x^2 + (-5a - 4b)x + (-3a - 9b)$$

Which results in the following

$$a + 3b = 2$$

$$-2a - 5b = -2$$

$$-5a - 4b = 12$$

$$-3a - 9b = -6$$

Now, we take a multiple of the first equation, and add it to the remaining equations to eliminate a (2, 5 and 3 respectively). This results in the following equations

$$a + 3b = 2$$

$$2(a + 3b) + (-2a - 5b) = b = 2(2) - 2 = 2$$

$$5(a + 3b) + (-5a - 4b) = 11b = 5(2) + 12$$

$$3(a + 3b) + (-3a - 9b) = 0b = 3(2) - 6 = 0$$

Solving this yields $a = -4, b = 2$ hence

$$2x^3 - 2x^2 + 12x - 6 = -4(x^3 - 2x^2 - 5x - 3) + 2(3x^3 - 5x^2 - 4x - 9)$$

For the case involving

$$3x^3 - 2x^2 + 7x + 8$$

,we need to show that there are no scalars a, b such that

$$3x^3 - 2x^2 + 7x + 8 = a(x^3 - 2x^2 - 5x - 3) + b(3x^3 - 5x^2 - 4x - 9)$$

$$= (a + 3b)x^3 + (-2a - 5b)x^2 + (-5a - 4b)x + (-3a - 9b)$$

Using the same technique as previous, we get the system of equations

$$a + 3b = 3$$

$$-2a - 5b = -2$$

$$-5a - 4b = 7$$

$$-3a - 9b = 8$$

then

$$a + 3b = 3$$
$$2(a + 3b) + (-2a - 5b) = b = 2(3) - 2 = 4$$
$$5(a + 3b) + (-5a - 4b) = 11b = 5(3) + 7 = 22$$
$$3(a + 3b) + (-3a - 9b) = 0b = 0 = 3(3) + 8 = 17$$

The inconsistency $0 = 17$ means that his equation has no solutions. So

$$3x^3 - 2x^2 + 7x + 8$$

is not a linear combination of the two equations.

Definition 1.3.2. Let S be a nonempty subset of a vector space V. The *span* of S denoted by $\text{span}(S)$, is the set consisting of all linear combinations of the vectors in S.

Observe that in $\mathbb{R}^3$, the spanning set of $\{(1, 0, 0), (0, 1, 0)\}$ consists of all vectors in $\mathbb{R}^3$ of the form $a(1, 0, 0) + b(0, 1, 0)$ (by the definition of linear combinations) which is equivalent to $(a, b, 0)$ or a point on the xy plane. Thus, this spanning set contains all points of the xy plane and is a subset of $\mathbb{R}$.

Theorem 1.3.1 (Span is a Subspace)**.** *The span of any subset S of a vector space V is a subspace of V that contains S. Moreover, any subspace of V that contains S must also contain the span of S.*

Incomplete Proof. If $S = \emptyset$, the proof is immediate since the $\text{span}(\emptyset) = \{0\}$ and $\{0\}$ is a subspace of any subspace V (and in this case, $\{0\}$ contains S since any set contains the empty set). Let S be a nonempty subset of V. Then S contains a vector, call it z. Then $0z = 0$ so by the definition of span, $0 \in \text{span}(S)$. Let $x, y \in \text{span}(S)$. Then there exists vectors $u_1, u_2, ..., u_m$, $v_1, v_2, ...v_n \in S$ and scalars $a_1, a_2, ..., a_m$, $b_1, b_2, ...b_n \in V$ such that

$$x = a_1u_1 + a_2u_2 + ... + a_mu_m \text{ and } y = b_1v_1 + b_2v_2 + ... + b_nv_n$$

Then

$$x + y = a_1u_1 + a_2u_2 + ... + a_mu_m + b_1v_1 + b_2v_2 + ... + b_nv_n$$

is a linear combination of vectors in S so $x + y \in \text{span}(S)$. Similarly,

$$cx = ca_1u_1 + ca_2u_2 + ... + ca_mu_m$$

which is also in $\text{span}(S)$. Thus $\text{span}(S)$ contains the zero vector, is closed under addition and scalar multiplication, so it is a subspace of V. □

Definition 1.3.3. A subset S of a vector space V *generates* or *spans* V if $\text{span}(S) = V$.

Example 1.3.2. The vectors

$$(1, 1, 0), (1, 0, 1) \text{ and } (0, 1, 1)$$

span $\mathbb{R}^3$ since any arbitrary vector in $\mathbb{R}^3$ is a linear combination of these 3 vectors. It is possible to compute the scalars r, s, t which solve the equation

$$r(1, 1, 0) + s(1, 0, 1) + t(0, 1, 1) = (a_1, a_2, a_3)$$

by

$$r = \frac{1}{2}(a_1 + a_2 - a_3), s = \frac{1}{2}(a_1 - a_2 + a_3), t = \frac{1}{2}(-a_1 + a_2 + a_3)$$

Example 1.3.3. The polynomials

$$x^2 + 3x - 2, 2x^2 + 5x - 3 \text{ and } - x^2 - 4x + 4$$

span or generate $P_2(R)$ since each polynomial in $P_2(R)$ is a linear combination of these 3 polynomials, and they are all in $P_2(R)$.

Definitions