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04-24-25
AMATH 503

HOMEWORK 3

Exercises come from *Introduction to Partial Differential Equations by Peter J. Olver* as well as supplemented by instructor provided exercises.

1: Olver: 3.2.6 (a,c,e)

Solution:
TODO:

2: Olver: 3.3.2 and 3.3.3

- 3.3.2 Find the Fourier series for the function $f(x) = x^3$. If you differentiate your series, do you recover the Fourier series for $f'(x) = 3x^2$? If not, explain why not.

  Solution:
  We begin by calculating the coefficients a_k and b_k.

  $$\begin{aligned} a_k = \langle x^3, \cos kx \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^3 \cos kx dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^3 \cos kx dx \end{aligned}$$

- 3.3.3 Repeat exercise 3.3.2 but starting with $f(x) = x^4$.
  Solution:
  TODO:

Solution:
TODO:

3: Olver: 3.2.55

(a) Find the complex Fourier series for $x\,\mathrm{e}^{\mathrm{i}x}$

Solution:
First of all we define the complex Fourier series for a piecewise continuous real or complex function f is the doubly infinite series

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k\,\mathrm{e}^{\mathrm{i}kx}$$

where the c_k are given by

$$c_k = \langle f, \mathrm{e}^{\mathrm{i}kx} \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,dx.$$

Therefore, the bulk of our work here is to establish what the coefficients c_k need to be. In other words we need to calculate

$$\begin{aligned} c_k = \langle x\,\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{\mathrm{i}kx} \rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,\mathrm{e}^{\mathrm{i}x}\,\mathrm{e}^{-\mathrm{i}kx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,\mathrm{e}^{\mathrm{i}x(1-k)}\,dx. \end{aligned}$$

I believe integration by parts would be useful. Let $u = x$ and let $dv = \mathrm{e}^{\mathrm{i}x(1-k)}\,dx$ these then also give rise to $du = dx$ and $v = \frac{1}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}x(1-k)}$, respectively. Then we have

$$\int u dv = uv - \int v du$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x\,\mathrm{e}^{\mathrm{i}x(1-k)}\,dx = \frac{1}{2\pi}\left[\frac{x}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}x(1-k)}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{1}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}x(1-k)}\,dx\right]$$

Let's take the right hand piece by piece in order to keep the calculations clean. First with the uv term

$$\begin{aligned} uv &= \frac{x}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}x(1-k)}\Big|_{-\pi}^{\pi} \\ &= \frac{\pi}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}\pi(1-k)} - \frac{(-\pi)}{\mathrm{i}(1-k)}\,\mathrm{e}^{-\mathrm{i}\pi(1-k)} \\ &= \frac{\pi}{\mathrm{i}(1-k)}\left(\mathrm{e}^{\mathrm{i}\pi(1-k)} + \mathrm{e}^{-\mathrm{i}\pi(1-k)}\right) \\ &= \frac{2\pi\cos(\pi(1-k))}{\mathrm{i}(1-k)} \\ &= -\frac{2\pi\mathrm{i}\cos(\pi(1-k))}{1-k}. \end{aligned}$$

Now we proceed with the integral on the right hand side

$$
\begin{aligned}
\int v du &= \int_{-\pi}^{\pi} \frac{1}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}x(1-k)}\,dx \\
&= \frac{1}{(\mathrm{i}(1-k))^2}\,\mathrm{e}^{\mathrm{i}x(1-k)}\Big|_{-\pi}^{\pi} \\
&= \frac{1}{(\mathrm{i}(1-k))^2}\,\mathrm{e}^{\mathrm{i}\pi(1-k)} - \frac{1}{(\mathrm{i}(1-k))^2}\,\mathrm{e}^{-\mathrm{i}\pi(1-k)} \\
&= \frac{1}{\mathrm{i}^2(1-k)^2}\left(\mathrm{e}^{\mathrm{i}\pi(1-k)} - \mathrm{e}^{-\mathrm{i}\pi(1-k)}\right) \\
&= -\frac{1}{(1-k)^2}\,(2\mathrm{i}\sin(\pi(1-k))) \\
&= 0
\end{aligned}
$$

where the final equality holds due to the fact that $\sin(\pi(1-k))$ is always 0 since $1-k$ is an integer. Using these in our initial IBP step we have

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} x\,\mathrm{e}^{\mathrm{i}x(1-k)}\,dx &= \frac{1}{2\pi}\left[\frac{x}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}x(1-k)}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{1}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}x(1-k)}\,dx\right] \\
&= \frac{1}{2\pi}\left[-\frac{2\pi\mathrm{i}\cos(\pi(1-k))}{1-k}\right] \\
&= -\frac{\mathrm{i}\cos(\pi(1-k))}{1-k}.
\end{aligned}
$$

Notice, since $\cos(\ell\pi) = \begin{cases} 1, & \text{if } \ell \text{ is even} \\ -1, & \text{if } \ell \text{ is odd} \end{cases}$, therefore, when k is odd $1-k$ is even but if k is even then $1-k$ is odd. Hence

$$
\begin{aligned}
-\frac{\mathrm{i}\cos(\pi(1-k))}{1-k} &= -\frac{\mathrm{i}(-1)^{(1-k)}}{1-k} \\
&= \frac{\mathrm{i}(-1)^{(2-k)}}{1-k} \\
&= \frac{\mathrm{i}(-1)^{k}}{1-k}
\end{aligned}
$$

Thus we have calculated the c_k to be

$$c_k = \frac{\mathrm{i}(-1)^k}{1-k}$$

and thus our Fourier series of the function $x\,\mathrm{e}^{\mathrm{i}x}$ is given by

$$f(x) \sim \sum_{k=-\infty}^{\infty} \frac{\mathrm{i}(-1)^k}{1-k}\,\mathrm{e}^{\mathrm{i}kx}$$

□

(b) Use your result to write down the real Fourier series for $x\cos x$ and $x\sin s$
Solution:

Notice we can rewrite the previous Fourier series as

$$f(x) \sim \sum_{k=-\infty}^{\infty} \frac{\mathrm{i}(-1)^k}{1-k}\, \mathrm{e}^{\mathrm{i}kx}$$

$$x\,\mathrm{e}^{\mathrm{i}x} \sim \sum_{k=-\infty}^{\infty} \frac{\mathrm{i}(-1)^k}{1-k}\, \mathrm{e}^{\mathrm{i}kx}$$

$$x\cos x + \mathrm{i}x\sin x \sim \sum_{k=-\infty}^{\infty} \left[\frac{\mathrm{i}(-1)^k}{1-k}\cos kx - \frac{(-1)^k}{1-k}\sin kx\right].$$

Pairing up the real and imaginary parts of this we get

$$x\cos x \sim \sum_{k=-\infty}^{\infty} \frac{(-1)^{k+1}}{1-k}\sin kx$$

and

$$x\sin x \sim \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{1-k}\cos kx.$$

However, for the real Fourier series we want the indices to start at $k = 1$ rather than $-\infty$. Thus we have

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} \frac{(-1)^{k+1}}{1-k}\sin kx &= \sum_{k=1}^{\infty}\left[\frac{(-1)^{k+1}}{1-k}\sin kx + \frac{(-1)^{-k+1}}{1+k}\sin(-kx)\right] \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\Big(\sin kx(1+k) - \sin kx(1-k)\Big)}{1-k^2} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}2k\sin kx}{1-k^2}
\end{aligned}$$

However, we actually want to exclude $k = 1$ to avoid dividing by zero. Also it is helpful to note that we have already removed $k = 0$ since $\sin 0 = 0$ Thus

$$x\cos x \sim \sum_{k=2}^{\infty} \frac{(-1)^{k+1}2k\sin kx}{1-k^2}.$$

Finally for $x\sin x$ we have

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{1-k}\cos kx &= 1 + \sum_{k=1}^{\infty}\left(\frac{(-1)^k}{1-k}\cos kx + \frac{(-1)^{-k}}{1+k}\cos(-kx)\right) \\
&= 1 + \sum_{k=1}^{\infty}(-1)^k\cos kx\left(\frac{1}{1-k} + \frac{1}{1+k}\right) \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k\cos kx}{1-k^2}.
\end{aligned}$$

And thus

$$x\sin x \sim 1 + \sum_{k=1}^{\infty} \frac{(-1)^k\cos kx}{1-k^2}.$$

TODO: revisit this and verify how to handle the $k = 1$ case.

4: Olver: 3.4.6 Write down formulas for the Fourier series of both even and odd functions on $[-\ell, \ell]$.

Solution:
TODO:

5: Olver: 3.5.29 Let $f(x) \in L^2[a, b]$ be square integrable. Which constant function $g(x) \equiv c$ best approximates f in the least squares sense?

Solution:
TODO:

6: Olver: 3.5.43 For each $n = 1, 2, ...,$ define the function

$$f_n(x) = \begin{cases} 1, & \frac{k}{m} \le x \le \frac{k+1}{m}, \\ 0, & \text{otherwise} \end{cases},$$

where $n = \frac{1}{2}m(m+1)+k$ and $0 \le k \le m$. Show first that m, k are uniquely determined by n. Then prove that, on the interval $[0, 1]$, the sequence $f_n(x)$ converges in norm to 0 but does not converge pointwise *anywhere*!

Solution:
TODO:

7: We consider the complex orthonormal basis

$$\varphi_n = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{inx}$$

where $n = 0, 1, -1, 2, -2, \ldots$. Consider the function $f_a(x) = \mathrm{e}^{ax}$ with real number $a \neq 0$ and compute the Fourier coefficient

$$\langle f_a, \varphi_n \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f_a(x)\,\mathrm{e}^{-inx}\,dx.$$

Then prove the formula

$$\sum_{n=1}^{\infty} \frac{1}{a^2 + n^2} = \frac{\pi}{2a}\coth(\pi a) - \frac{1}{2a^2}$$

(Hint: Plancherel's formula: the relation between L^2 norm of coefficients and $\langle f_a, f_a \rangle$.)

Solution:
We begin by calculating the Fourier coefficient as requested

$$\begin{aligned}
\langle f_a, \varphi_n \rangle &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f_a(x)\,\mathrm{e}^{-inx}\,dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \mathrm{e}^{ax}\,\mathrm{e}^{-inx}\,dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \mathrm{e}^{(a-\mathrm{i}n)x}\,dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{(a-\mathrm{i}n)}\,\mathrm{e}^{(a-\mathrm{i}n)x}\Big|_{-\pi}^{\pi} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{(a-\mathrm{i}n)}\,\mathrm{e}^{(a-\mathrm{i}n)\pi} - \frac{1}{(a-\mathrm{i}n)}\,\mathrm{e}^{-(a-\mathrm{i}n)\pi} \right) \\
&= \frac{1}{\sqrt{2\pi}(a-\mathrm{i}n)} \left(\mathrm{e}^{(a-\mathrm{i}n)\pi} - \mathrm{e}^{-(a-\mathrm{i}n)\pi} \right) \\
&= \frac{1}{\sqrt{2\pi}(a-\mathrm{i}n)} \left(\mathrm{e}^{a}\,\mathrm{e}^{-\mathrm{i}n\pi} - \mathrm{e}^{-a}\,\mathrm{e}^{\mathrm{i}n\pi} \right) \\
&= \frac{1}{\sqrt{2\pi}(a-\mathrm{i}n)} \left(\frac{\mathrm{e}^{2a}\,\mathrm{e}^{-\mathrm{i}n\pi}}{\mathrm{e}^{a}} - \frac{\mathrm{e}^{\mathrm{i}n\pi}}{\mathrm{e}^{a}} \right) \\
&= \frac{1}{\sqrt{2\pi}(a-\mathrm{i}n)\,\mathrm{e}^{a}} \left(\mathrm{e}^{2a} \left(\cos n\pi - \cancelto{0}{\mathrm{i}\sin n\pi} \right) - \left(\cos n\pi + \cancelto{0}{\mathrm{i}\sin n\pi} \right) \right) \\
&= \frac{(\mathrm{e}^{2a} - 1)\cos n\pi}{\sqrt{2\pi}(a-\mathrm{i}n)\,\mathrm{e}^{a}} \\
&= \frac{(\mathrm{e}^{2a} - 1)(-1)^n}{\sqrt{2\pi}(a-\mathrm{i}n)\,\mathrm{e}^{a}}
\end{aligned}$$

Next, let's prove the formula provided. **TODO:**