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Homework #6

$\boxed{1}$ W.N.L. Oscillator $\ddot{x} + x + \varepsilon(x^2 - 1)\dot{x}^3 = 0$

a) Using the given expressions for $x, \dot{x}$ and $\ddot{x}$ we see

$$0 = \frac{\partial^2 x_0}{\partial \tau^2} + 2\varepsilon \frac{\partial^2 x_0}{\partial \tau \partial T} + \varepsilon \frac{\partial^2 x_1}{\partial \tau^2} + x_0 + \varepsilon x_1 + O(\varepsilon^2)$$
$$+ \varepsilon\left(\left[x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots\right]^2 - 1\right)\left(\frac{\partial x_0}{\partial \tau} + \varepsilon \frac{\partial x_0}{\partial T} + \varepsilon \frac{\partial x_1}{\partial \tau} + O(\varepsilon^2)\right)^3.$$

Collecting ε^0 terms (noticing expanding the expression squared and the other cubed easily leaves us with simple ε^0 and ε^1 terms and we can ignore the rest in the $O(\varepsilon^2)$ terms since $\varepsilon \ll 1$.) Thus we have for ε^0:

$$\frac{\partial^2 x_0}{\partial \tau^2} + x_0 = 0 \quad \Rightarrow \quad x_0 = R(T)\cos(\tau + \phi(T)) \quad \square$$

b) Now collecting ε^1 terms we have

$$2\frac{\partial^2 x_0}{\partial \tau \partial T} + \frac{\partial^2 x_1}{\partial \tau^2} + x_1 + (x_0^2 - 1)\left(\frac{\partial x_0}{\partial \tau}\right)^3 = 0$$

$$\frac{\partial^2 x_1}{\partial \tau^2} + x_1 = -2\frac{\partial^2 x_0}{\partial \tau \partial T} - (x_0^2 - 1)\left(\frac{\partial x_0}{\partial \tau}\right)^3$$ Now we need to plugin

and take the appropriate derivatives of x_0. (Next page)

$$\frac{\partial^2 x_1}{\partial \tau^2} + x_1 = -2\frac{\partial^2}{\partial\tau\partial T}\left[R(T)\cos(\tau+\phi(T))\right]$$

$$-\left(R^2(T)\cos^2(\tau+\phi(T))-1\right)\left(\frac{\partial}{\partial\tau}\left[R(T)\cos(\tau+\phi(T))\right]\right)^3$$

$$= 2\left[R'(T)\sin(\tau+\phi(T)) + R(T)\phi'(T)\cos(\tau+\phi(T))\right]$$

$$+ R(T)^3\sin^3(\tau+\phi(T))\left[R(T)^2\cos^2(\tau+\phi(T))-1\right].$$

Now we employ the provided hints (trig identity & $y=\tau+\phi(T)$).
Then we have (simplifying $R(T)$ to just R... we know it depends on T.)

$$\frac{\partial^2 x_1}{\partial\tau^2}+x_1 = 2\left[R'\sin(y)+R\phi'\cos(y)\right]+R^3\sin^3(y)\left[R^2\cos^2(y)-1\right]$$

$$= 2\left[R'\sin y + R\phi'\cos y\right] - R^3\sin^3 y\left[1-R^2\cos^2 y\right]$$

$$= 2\left[R'\sin y + R\phi'\cos y\right] - R^3\left[\frac{1}{16}\left((12-2R^2)\sin y-(4+R^2)\sin(3y)+R^2\sin(5y)\right)\right]$$

$$= 2\left[R'\sin y + R\phi'\cos y\right] - \frac{R^3}{16}\left[(12-2R^2)\sin y-(4+R^2)\sin(3y)+R^2\sin(5y)\right]$$

$$= 2R'\sin y + 2R\phi'\cos y - \frac{R^3(12-2R^2)\sin y}{16} + \frac{R^3(4+R^2)}{16}\sin(3y) - \frac{R^5}{16}\sin(5y)$$

$$= \left(2R' - \frac{R^3(12-2R^2)}{16}\right)\sin y + 2R\phi'\cos y + \frac{R^3(4+R^2)}{16}\sin(3y) - \frac{R^5}{16}\sin(5y).$$

To avoid secular terms we need

$$2R' - \frac{R^3(12-2R^2)}{16} = 0 \quad \& \quad 2R\phi' = 0.$$

$\boxed{\text{F.P. for } R'=0}$

$$R' = \frac{R^3(12-2R^2)}{32} = 0$$

$R = 0, \pm\sqrt{6}$

Therefore there is a limit cycle since there is a real solution.

$\square$

problem 2

consider D.S. $\dot{x} = x(y-1)$ $\quad a,b>0$

$\dot{y} = -y(x+a)+b$

Let's first find f.p: $x(y-1) = 0 \Rightarrow x=0$ or $y=1$

$-y(x+a)+b=0$

Let $x=0$ then $(y-1)=0$ $\quad$ Let $y=1$ $\quad x(y-1)=0$

$-y(0+a)+b=0$ $\quad\quad -(x+a)+b=0$

$-x-a+b=0$

$-y^a+b=0$ $\quad\quad x=a-b$

$y=\frac{b}{a}$

Thus fixed points occur at $(x^*,y^*) \in \{(0,\frac{b}{a}),(a-b,1)\}$.

let's compute the Jacobian of the system $\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$ we have

$J = \begin{pmatrix} y-1 & x \\ -y & -(x+a) \end{pmatrix}$, Now evaluated at the two fixed points we have @ $(x,y)=(0,\frac{b}{a})$

$J\big|_{(0,\frac{b}{a})} = \begin{pmatrix} \frac{b}{a}-1 & 0 \\ -\frac{b}{a} & -a \end{pmatrix}$ and $J\big|_{(a-b,1)} = \begin{pmatrix} 0 & a-b \\ -1 & -(a-b+a) \end{pmatrix}$

$= \begin{pmatrix} 0 & a-b \\ -1 & b+2a \end{pmatrix}$

let's analyze these fixed points at two situations. First, lets assume $a<b$. Then @ $(x^*,y^*) = (0,\frac{b}{a})$

$\begin{pmatrix} \frac{b}{a}-1 & 0 \\ -\frac{b}{a} & -a \end{pmatrix} \Rightarrow \left(\frac{b}{a}-1-\lambda\right)(-a-\lambda)=0 \Rightarrow \lambda = \underbrace{\frac{b}{a}-1,\ -a}$

Thus this is a saddle. But when $a>b$ they are both negative and we have a sink.

At the other fixed point $(a-b, 1)$ we have

$$\begin{pmatrix} 0 & a-b \\ -1 & b-2a \end{pmatrix} \qquad -\lambda(b-2a-\lambda) + (a-b) = 0$$

$$+\lambda^2 - \lambda(b-2a) + a - b = 0$$

$$\lambda = \frac{+(b-2a) \pm \sqrt{(b-2a)^2 - 4(a-b)}}{2}$$

Assuming $a<b$ we get two cases depending on how much smaller a is than b. If $\frac{1}{2}b < a < b$ then the eigenvalues are negative and we a sink but if $a < \frac{1}{2}b < b$ then the eigenvalue is at least one positive and one of them is negative so it would be a saddle. Thus we see the stability switching and we have a transcritical bifurcation.

$\square$

$\boxed{\text{problem 3}}$ system of ODE's

$$\dot{x} = -ax + y + x(x^2+y^2) - a\frac{x^3}{\sqrt{x^2+y^2}}$$

$$\dot{y} = -x - ay + y(x^2+y^2) - a\frac{x^2 y}{\sqrt{x^2+y^2}}$$

(a) lets compute the Jacobian and evaluate at (0,0)

$$\begin{pmatrix} -a + 3x^2 + y^2 - a\left(\frac{3x^2\sqrt{x^2+y^2} - x^3\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}2x}{x^2+y^2}\right) & 1 + 2xy + \frac{ax^3 2y}{2(x^2+y^2)^{3/2}} \\ -1 + 2xy - a\left(\frac{2xy\sqrt{x^2+y^2} - x^2y\frac{1}{2}(x^2+y^2)^{-1/2}2x}{x^2+y^2}\right) & -a + x^2 + 3y^2 - a\left(\frac{x^2\sqrt{x^2+y^2} - x^2y\frac{1}{2}(\cdots)2y}{x^2+y^2}\right) \end{pmatrix}$$

evaluated @ (0,0)

$$\Longrightarrow \begin{pmatrix} -a & 1 \\ -1 & -a \end{pmatrix}$$

Then the characteristic eq is

$$(-a-\lambda)(-a-\lambda) + 1 = 0$$

$$a^2 + 2a\lambda + \lambda^2 + 1 = 0$$

$$\lambda^2 + 2a\lambda + a^2 + 1 = 0$$

$$\lambda = \frac{-2a \pm \sqrt{4a^2 - 4(a^2+1)}}{2} = \frac{-2a \pm \sqrt{\cancel{4a^2 - 4a^2} - 4}}{2}$$

$$\lambda = -a \pm i$$

Now as far as the stability the eigenvalues of $\lambda = -a \pm i$ gives us a stable spiral if $a > 0$ and an unstable spiral if $a < 0$.

(b) Furthermore $a_c = 0$, is a critical point for the bifurcation where the fixed point switches stability.

(c)