Hunter Lybbert
Student ID: 2426454
04-24-25
AMATH 503

HOMEWORK 3

Exercises come from *Introduction to Partial Differential Equations by Peter J. Olver* as well as supplemented by instructor provided exercises.

1: Olver: 3.2.6 (a,c,e)

Solution:

□

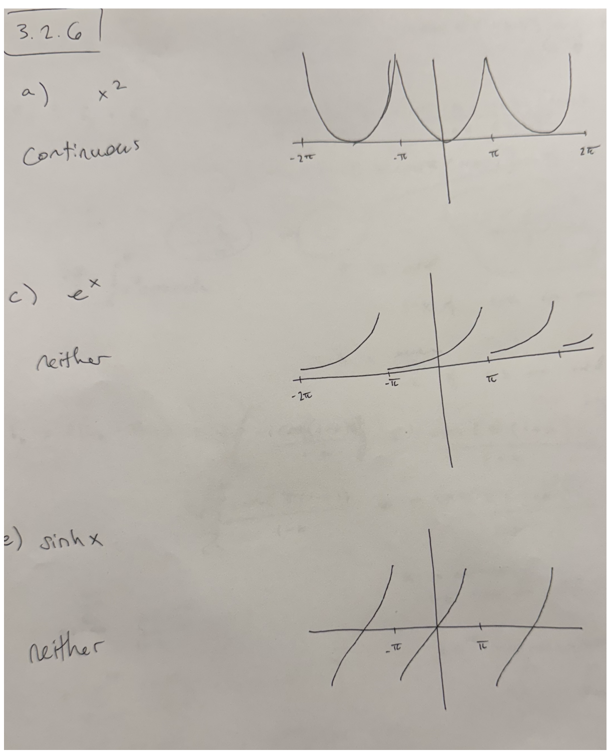

FIGURE 1. A sketch of the various 2π periodic extensions of the requested functions.

2: Olver: 3.3.2 and 3.3.3

- 3.3.2 Find the Fourier series for the function $f(x) = x^3$. If you differentiate your series, do you recover the Fourier series for $f'(x) = 3x^2$? If not, explain why not.

  Solution:
  We begin by calculating the coefficients a_k and b_k We first have,

$$a_k = \langle x^3, \cos kx \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \cos kx dx$$

  which leads us to use integration by parts where $u = x^3$, $dv = \cos kx dx$. Notice we are going to need to do this iteratively where u is always set to the polynomial part of the integrand and dv is always the trigonometric part. Therefore,

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \cos kx dx &= \frac{1}{\pi}\left[x^3 \frac{\sin kx}{k}\bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 3x^2 \frac{\sin kx}{k} dx \right] \\
&= \frac{1}{\pi}\left[x^3 \frac{\sin kx}{k}\bigg|_{-\pi}^{\pi} + \left(3x^2 \frac{\cos kx}{k^2}\bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 6x \frac{\cos kx}{k^2} dx \right)\right] \\
&= \frac{1}{\pi}\left[x^3 \frac{\sin kx}{k}\bigg|_{-\pi}^{\pi} + \left(3x^2 \frac{\cos kx}{k^2}\bigg|_{-\pi}^{\pi} - \left(6x \frac{\sin kx}{k^3}\bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 6 \frac{\sin kx}{k^3} dx \right)\right)\right] \\
&= \frac{1}{\pi}\left[x^3 \frac{\sin kx}{k}\bigg|_{-\pi}^{\pi} + \left(3x^2 \frac{\cos kx}{k^2}\bigg|_{-\pi}^{\pi} - \left(6x \frac{\sin kx}{k^3}\bigg|_{-\pi}^{\pi} + 6 \frac{\cos kx}{k^4}\bigg|_{-\pi}^{\pi} \right)\right)\right] \\
&= \frac{1}{\pi}\left[x^3 \frac{\sin kx}{k} + 3x^2 \frac{\cos kx}{k^2} - 6x \frac{\sin kx}{k^3} - 6 \frac{\cos kx}{k^4}\bigg|_{-\pi}^{\pi} \right] \\
&= \frac{1}{\pi}\left[\left((\pi)^3 \frac{\sin k\pi}{k} + 3(\pi)^2 \frac{\cos k\pi}{k^2} - 6\pi \frac{\sin k\pi}{k^3} - 6 \frac{\cos k\pi}{k^4} \right) \right. \\
&\quad \left. - \left((-\pi)^3 \frac{\sin(-k\pi)}{k} + 3(-\pi)^2 \frac{\cos(-k\pi)}{k^2} + 6\pi \frac{\sin(-k\pi)}{k^3} - 6 \frac{\cos(-k\pi)}{k^4} \right) \right].
\end{aligned}$$

  Each of the sin terms are 0 and the cos terms of different signs end up canceling one another out. Thus we have

$$= \frac{1}{\pi}\left[3(\pi)^2 \frac{\cos k\pi}{k^2} - 6 \frac{\cos k\pi}{k^4} - 3(\pi)^2 \frac{\cos(k\pi)}{k^2} + 6 \frac{\cos(k\pi)}{k^4} \right] = 0$$

  Which checks out since x^3 is an odd function, therefore it's Fourier series is going to be solely comprised of the b_k and sin terms. Calculating b_k using repeated integration by parts we have

$$
\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin kx dx &= \frac{1}{\pi}\left[- x^3\frac{\cos kx}{k}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} 3x^2\frac{\cos kx}{k}dx\right] \\
&= \frac{1}{\pi}\left[- x^3\frac{\cos kx}{k}\Big|_{-\pi}^{\pi} + \left(3x^2\frac{\sin kx}{k^2}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 6x\frac{\sin kx}{k^2}dx\right)\right] \\
&= \frac{1}{\pi}\left[- x^3\frac{\cos kx}{k}\Big|_{-\pi}^{\pi} + \left(3x^2\frac{\sin kx}{k^2}\Big|_{-\pi}^{\pi} + \Big(6x\frac{\cos kx}{k^3}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 6\frac{\cos kx}{k^3}dx\Big)\right)\right] \\
&= \frac{1}{\pi}\left[- x^3\frac{\cos kx}{k}\Big|_{-\pi}^{\pi} + \left(3x^2\frac{\sin kx}{k^2}\Big|_{-\pi}^{\pi} + \Big(6x\frac{\cos kx}{k^3}\Big|_{-\pi}^{\pi} - 6\frac{\sin kx}{k^4}\Big|_{-\pi}^{\pi}\Big)\right)\right] \\
&= \frac{1}{\pi}\left[-x^3\frac{\cos kx}{k} + 3x^2\frac{\sin kx}{k^2} + 6x\frac{\cos kx}{k^3} - 6\frac{\sin kx}{k^4}\Big|_{-\pi}^{\pi}\right] \\
&= \frac{1}{\pi}\left[\left(-(\pi)^3\frac{\cos k\pi}{k} + 3(\pi)^2\frac{\sin k\pi}{k^2} + 6\pi\frac{\cos k\pi}{k^3} - 6\frac{\sin k\pi}{k^4}\right)\right. \\
&\qquad \left. - \left(-(-\pi)^3\frac{\cos(-k\pi)}{k} + 3(-\pi)^2\frac{\sin(-k\pi)}{k^2} - 6\pi\frac{\cos(-k\pi)}{k^3} - 6\frac{\sin(-k\pi)}{k^4}\right)\right].
\end{aligned}
$$

Once more, we utilize the fact that the sin terms are all 0, and thus we have

$$
\begin{aligned}
&= \frac{1}{\pi}\left[-\pi^3\frac{\cos k\pi}{k} + 6\pi\frac{\cos k\pi}{k^3} - \pi^3\frac{\cos(k\pi)}{k} + 6\pi\frac{\cos(-k\pi)}{k^3}\right] \\
&= \frac{1}{\pi}\left[-2\pi^3\frac{\cos k\pi}{k} + 12\pi\frac{\cos k\pi}{k^3}\right] \\
&= -2\pi^2\frac{\cos k\pi}{k} + 12\frac{\cos k\pi}{k^3} \\
&= \cos k\pi\left(\frac{12}{k^3} - \frac{2\pi^2}{k}\right) \\
&= (-1)^k\left(\frac{12}{k^3} - \frac{2\pi^2}{k}\right).
\end{aligned}
$$

Therefore, the Fourier series of x^3 is

$$x^3 \sim \sum_{k=1}^{\infty}(-1)^k\left(\frac{12}{k^3} - \frac{2\pi^2}{k}\right)\sin kx.$$

Now we are interested in seeing if the derivative of this series is the same as the Fourier series for the derivative of x^3.

$$
\begin{aligned}
\frac{d}{dx}\left[\sum_{k=1}^{\infty}(-1)^k\left(\frac{12}{k^3} - \frac{2\pi^2}{k}\right)\sin kx\right] &= \sum_{k=1}^{\infty}\frac{d}{dx}\left[(-1)^k\left(\frac{12}{k^3} - \frac{2\pi^2}{k}\right)\sin kx\right] \\
&= \sum_{k=1}^{\infty}(-1)^k\left(\frac{12}{k^3} - \frac{2\pi^2}{k}\right)\cos kx
\end{aligned}
$$

This does not match the Fourier coefficient for $2x^3$ we computed manually.

$\square$

- 3.3.3 Repeat exercise 3.3.2 but starting with $f(x) = x^4$.
  Solution:
  Notice, x^4 is an even function therefore we will forgo calculating the b_k which will all be 0. Instead we calculate the a_k coeffs

$$
\begin{aligned}
a_k = \langle x^4, \cos kx\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^4 \cos kx dx \\
&= \frac{1}{\pi}\left[x^4\frac{\sin kx}{k}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 4x^3\frac{\sin kx}{k}dx\right] \\
&= \frac{1}{\pi}\left[x^4\frac{\sin kx}{k}\Big|_{-\pi}^{\pi} + \left(4x^3\frac{\cos kx}{k^2}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 12x^2\frac{\cos kx}{k^2}dx\right)\right] \\
&= \frac{1}{\pi}\left[x^4\frac{\sin kx}{k}\Big|_{-\pi}^{\pi} + \left(4x^3\frac{\cos kx}{k^2}\Big|_{-\pi}^{\pi} - \left(12x^2\frac{\sin kx}{k^3}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 24x\frac{\sin kx}{k^3}dx\right)\right)\right] \\
&= \frac{1}{\pi}\left[x^4\frac{\sin kx}{k}\Big|_{-\pi}^{\pi} + \left(4x^3\frac{\cos kx}{k^2}\Big|_{-\pi}^{\pi} - \left(12x^2\frac{\sin kx}{k^3}\Big|_{-\pi}^{\pi} + \left(24x\frac{\cos kx}{k^4}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 24\frac{\cos kx}{k^4}dx\right)\right)\right)\right] \\
&= \frac{1}{\pi}\left[x^4\frac{\sin kx}{k} + 4x^3\frac{\cos kx}{k^2} - 12x^2\frac{\sin kx}{k^3} - 24x\frac{\cos kx}{k^4} + 24\frac{\sin kx}{k^5}\Big|_{-\pi}^{\pi}\right] \\
&= \frac{1}{\pi}\Bigg[\left((\pi)^4\frac{\sin k\pi}{k} + 4(\pi)^3\frac{\cos k\pi}{k^2} - 12\pi^2\frac{\sin k\pi}{k^3} - 24\pi\frac{\cos k\pi}{k^4} + 24\frac{\sin k\pi}{k^5}\right) \\
&\quad - \left((-\pi)^4\frac{\sin(-k\pi)}{k} + 4(-\pi)^3\frac{\cos(-k\pi)}{k^2} - 12(-\pi)^2\frac{\sin(-k\pi)}{k^3} + 24\pi\frac{\cos(-k\pi)}{k^4} + 24\frac{\sin(-k\pi)}{k^5}\right)\Bigg]
\end{aligned}
$$

Once, again the sin terms are 0 but the cos terms won't cancel since they switch signs with odd power coefficients this time

$$
\begin{aligned}
&= \frac{1}{\pi}\left[4(\pi)^3\frac{\cos k\pi}{k^2} - 24\pi\frac{\cos k\pi}{k^4} - \left(4(-\pi)^3\frac{\cos(-k\pi)}{k^2} + 24\pi\frac{\cos(-k\pi)}{k^4}\right)\right] \\
&= \frac{1}{\pi}\left[4(\pi)^3\frac{\cos k\pi}{k^2} - 24\pi\frac{\cos k\pi}{k^4} + 4(\pi)^3\frac{\cos(k\pi)}{k^2} - 24\pi\frac{\cos(k\pi)}{k^4}\right] \\
&= \cos k\pi\left(\frac{8\pi^2}{k^2} - \frac{48}{k^4}\right) \\
&= (-1)^k\left(\frac{8\pi^2}{k^2} - \frac{48}{k^4}\right).
\end{aligned}
$$

Therefore, the Fourier series of x^4 is

$$
x^4 \sim \sum_{k=1}^{\infty}(-1)^k\left(\frac{8\pi^2}{k^2} - \frac{48}{k^4}\right)\cos kx.
$$

Similar to part 1, we have

$$\begin{aligned}
\frac{d}{dx}\left[\sum_{k=1}^{\infty}(-1)^k\left(\frac{8\pi^2}{k^2}-\frac{48}{k^4}\right)\cos kx\right] &= \sum_{k=1}^{\infty}\frac{d}{dx}\left[(-1)^k\left(\frac{8\pi^2}{k^2}-\frac{48}{k^4}\right)\cos kx\right] \\
&= \sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{8\pi^2}{k^2}-\frac{48}{k^4}\right)\sin kx.
\end{aligned}$$

Which once again does not have the right coefficients for $4x^3$.

$\square$

3: Olver: 3.2.55

(a) Find the complex Fourier series for $x\,\mathrm{e}^{\mathrm{i}x}$

Solution:
First of all we define the complex Fourier series for a piecewise continuous real or complex function f is the doubly infinite series

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k\,\mathrm{e}^{\mathrm{i}kx}$$

where the c_k are given by

$$c_k = \langle f, \mathrm{e}^{\mathrm{i}kx} \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,dx.$$

Therefore, the bulk of our work here is to establish what the coefficients c_k need to be. In other words we need to calculate

$$\begin{aligned} c_k = \langle x\,\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{\mathrm{i}kx} \rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,\mathrm{e}^{\mathrm{i}x}\,\mathrm{e}^{-\mathrm{i}kx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,\mathrm{e}^{\mathrm{i}x(1-k)}\,dx. \end{aligned}$$

I believe integration by parts would be useful. Let $u = x$ and let $dv = \mathrm{e}^{\mathrm{i}x(1-k)}\,dx$ these then also give rise to $du = dx$ and $v = \frac{1}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}x(1-k)}$, respectively. Then we have

$$\int u dv = uv - \int v du$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x\,\mathrm{e}^{\mathrm{i}x(1-k)}\,dx = \frac{1}{2\pi}\left[\frac{x}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}x(1-k)}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{1}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}x(1-k)}\,dx\right]$$

Let's take the right hand piece by piece in order to keep the calculations clean. First with the uv term

$$\begin{aligned} uv &= \frac{x}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}x(1-k)}\Big|_{-\pi}^{\pi} \\ &= \frac{\pi}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}\pi(1-k)} - \frac{(-\pi)}{\mathrm{i}(1-k)}\,\mathrm{e}^{-\mathrm{i}\pi(1-k)} \\ &= \frac{\pi}{\mathrm{i}(1-k)}\left(\mathrm{e}^{\mathrm{i}\pi(1-k)} + \mathrm{e}^{-\mathrm{i}\pi(1-k)}\right) \\ &= \frac{2\pi\cos(\pi(1-k))}{\mathrm{i}(1-k)} \\ &= -\frac{2\pi\mathrm{i}\cos(\pi(1-k))}{1-k}. \end{aligned}$$

Now we proceed with the integral on the right hand side

$$
\begin{aligned}
\int v du &= \int_{-\pi}^{\pi} \frac{1}{\mathrm{i}(1-k)} \,\mathrm{e}^{\mathrm{i}x(1-k)}\, dx \\
&= \frac{1}{(\mathrm{i}(1-k))^2} \,\mathrm{e}^{\mathrm{i}x(1-k)} \Big|_{-\pi}^{\pi} \\
&= \frac{1}{(\mathrm{i}(1-k))^2} \,\mathrm{e}^{\mathrm{i}\pi(1-k)} - \frac{1}{(\mathrm{i}(1-k))^2} \,\mathrm{e}^{-\mathrm{i}\pi(1-k)} \\
&= \frac{1}{\mathrm{i}^2(1-k)^2} \left(\mathrm{e}^{\mathrm{i}\pi(1-k)} - \mathrm{e}^{-\mathrm{i}\pi(1-k)}\right) \\
&= -\frac{1}{(1-k)^2} \left(2\mathrm{i}\sin(\pi(1-k))\right) \\
&= 0
\end{aligned}
$$

where the final equality holds due to the fact that $\sin(\pi(1-k))$ is always 0 since $1-k$ is an integer. Using these in our initial IBP step we have

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} x\,\mathrm{e}^{\mathrm{i}x(1-k)}\, dx &= \frac{1}{2\pi}\left[\frac{x}{\mathrm{i}(1-k)} \,\mathrm{e}^{\mathrm{i}x(1-k)}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{1}{\mathrm{i}(1-k)} \,\mathrm{e}^{\mathrm{i}x(1-k)}\, dx\right] \\
&= \frac{1}{2\pi}\left[-\frac{2\pi\mathrm{i}\cos(\pi(1-k))}{1-k}\right] \\
&= -\frac{\mathrm{i}\cos(\pi(1-k))}{1-k}.
\end{aligned}
$$

Notice, since $\cos(\ell\pi) = \begin{cases} 1, & \text{if } \ell \text{ is even} \\ -1, & \text{if } \ell \text{ is odd} \end{cases}$, therefore, when k is odd $1-k$ is even but if k is even then $1-k$ is odd. Hence

$$
\begin{aligned}
-\frac{\mathrm{i}\cos(\pi(1-k))}{1-k} &= -\frac{\mathrm{i}(-1)^{(1-k)}}{1-k} \\
&= \frac{\mathrm{i}(-1)^{(2-k)}}{1-k} \\
&= \frac{\mathrm{i}(-1)^{k}}{1-k}
\end{aligned}
$$

Thus we have calculated the c_k to be

$$c_k = \frac{\mathrm{i}(-1)^k}{1-k}$$

and thus our Fourier series of the function $x\,\mathrm{e}^{\mathrm{i}x}$ is given by

$$f(x) \sim \sum_{k=-\infty}^{\infty} \frac{\mathrm{i}(-1)^k}{1-k} \,\mathrm{e}^{\mathrm{i}kx}$$

□

(b) Use your result to write down the real Fourier series for $x\cos x$ and $x\sin s$
Solution:

Notice we can rewrite the previous Fourier series as

$$
\begin{aligned}
f(x) &\sim \sum_{k=-\infty}^{\infty} \frac{\mathrm{i}(-1)^k}{1-k}\,\mathrm{e}^{\mathrm{i}kx} \\
x\,\mathrm{e}^{\mathrm{i}x} &\sim \sum_{k=-\infty}^{\infty} \frac{\mathrm{i}(-1)^k}{1-k}\,\mathrm{e}^{\mathrm{i}kx} \\
x\cos x + \mathrm{i}x\sin x &\sim \sum_{k=-\infty}^{\infty} \left[\frac{\mathrm{i}(-1)^k}{1-k}\cos kx - \frac{(-1)^k}{1-k}\sin kx\right].
\end{aligned}
$$

Pairing up the real and imaginary parts of this we get

$$
x\cos x \sim \sum_{k=-\infty}^{\infty} \frac{(-1)^{k+1}}{1-k}\sin kx
$$

and

$$
x\sin x \sim \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{1-k}\cos kx.
$$

However, for the real Fourier series we want the indices to start at $k=1$ rather than $-\infty$. Thus we have

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} \frac{(-1)^{k+1}}{1-k}\sin kx &= \sum_{k=1}^{\infty}\left[\frac{(-1)^{k+1}}{1-k}\sin kx + \frac{(-1)^{-k+1}}{1+k}\sin(-kx)\right] \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\Big(\sin kx(1+k) - \sin kx(1-k)\Big)}{1-k^2} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}2k\sin kx}{1-k^2}
\end{aligned}
$$

However, we actually want to exclude $k=1$ to avoid dividing by zero. Also it is helpful to note that we have already removed $k=0$ since $\sin 0 = 0$ Thus

$$
x\cos x \sim \sum_{k=2}^{\infty} \frac{(-1)^{k+1}2k\sin kx}{1-k^2}.
$$

Finally for $x\sin x$ we have

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{1-k}\cos kx &= 1 + \sum_{k=1}^{\infty}\left(\frac{(-1)^k}{1-k}\cos kx + \frac{(-1)^{-k}}{1+k}\cos(-kx)\right) \\
&= 1 + \sum_{k=1}^{\infty}(-1)^k\cos kx\left(\frac{1}{1-k} + \frac{1}{1+k}\right) \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k\cos kx}{1-k^2}.
\end{aligned}
$$

And thus

$$
x\sin x \sim 1 + \sum_{k=1}^{\infty} \frac{(-1)^k\cos kx}{1-k^2}.
$$

TODO: revisit this and verify how to handle the $k=1$ case.

4: Olver: 3.4.6 Write down formulas for the Fourier series of both even and odd functions on $[-\ell, \ell]$.

Solution:
When f is even on $[-\ell, \ell]$ we have

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{\ell}\right).$$

Additionally, when f is odd on $[-\ell, \ell]$ we have

$$f(x) \sim + \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{\ell}\right).$$

□

5: Olver: 3.5.29 Let $f(x) \in L^2[a,b]$ be square integrable. Which constant function $g(x) \equiv c$ best approximates f in the least squares sense?

Solution:
In the least squares sense, we want to find c which solves

$$\min ||f(x) - c||^2$$

Given our norm for L^2 we have

$$\begin{aligned}
\frac{d}{dc}||f(x) - c||^2 &= \frac{d}{dc}\left[\left(\sqrt{\langle f(x) - c, f(x) - c\rangle}\right)^2\right] \\
&= \frac{d}{dc}\left[\int_a^b (f(x) - c)^2 dx\right] \\
&= \frac{d}{dc}\left[\int_a^b (f(x))^2 - 2cf(x) + c^2 dx\right] \\
&= \frac{d}{dc}\left[\int_a^b (f(x))^2 dx - \int_a^b 2cf(x)dx + \int_a^b c^2 dx\right] \\
&= -\int_a^b 2f(x)dx + \int_a^b 2c dx.
\end{aligned}$$

Setting this equal to zero and solving for c we have

$$\begin{aligned}
-\int_a^b 2f(x)dx + \int_a^b 2cdx = 0 \\
c &= \frac{\int_a^b 2f(x)dx}{\int_a^b 2dx} \\
c &= \frac{\int_a^b f(x)dx}{\int_a^b dx} \\
c &= \frac{1}{b-a}\int_a^b 2f(x)dx.
\end{aligned}$$

Therefore, this value of c for the constant function $g(x) \equiv c$ best approximates f in teh least squares sense.

□

6: Olver: 3.5.43 For each $n = 1, 2, ...$, define the function

$$f_n(x) = \begin{cases} 1, & \frac{k}{m} \le x \le \frac{k+1}{m}, \\ 0, & \text{otherwise} \end{cases},$$

where $n = \frac{1}{2}m(m+1) + k$ and $0 \le k \le m$. Show first that m, k are uniquely determined by n. Then prove that, on the interval $[0, 1]$, the sequence $f_n(x)$ converges in norm to 0 but does not converge pointwise *anywhere*!

Solution:
We begin by showing that n uniquely determines k and m. First, notice

$$n = \frac{1}{2}m(m+1) + k \implies 0 = \frac{1}{2}m^2 + \frac{1}{2}m + (k - n).$$

Hence,

$$m = \frac{-1/2 \pm \sqrt{\frac{1}{4} - 2(k-n)}}{2(1/2)} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2(k-n)}$$

TODO: Revisit...

Now we want to show that the sequence converges in norm to 0. Thus we need to show

$$\lim_{n\to\infty} ||f_n(x) - 0|| = 0.$$

Without further ado,

$$\begin{aligned}
\lim_{n\to\infty} ||f_n(x) - 0|| &= \lim_{n\to\infty} ||f_n(x)|| \\
&= \lim_{n\to\infty} \sqrt{\langle f_n(x), f_n(x) \rangle} \\
&= \lim_{n\to\infty} \sqrt{\int_0^1 f_n(x)^2 dx} \\
&= \lim_{n\to\infty} \sqrt{\int_0^{\frac{k}{m}} f_n(x)^2 dx + \int_{\frac{k}{m}}^{\frac{k+1}{m}} f_n(x)^2 dx + \int_{\frac{k+1}{m}}^1 f_n(x)^2 dx} \\
&= \lim_{n\to\infty} \sqrt{\int_{\frac{k}{m}}^{\frac{k+1}{m}} dx} \\
&= \lim_{n\to\infty} \sqrt{\frac{k+1}{m} - \frac{k}{m}} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{m}} \to 0
\end{aligned}$$

Where we know m is going to infinity as n goes to infinity since n depends on n and k. Therefore, f_n converges to 0 in norm. However, we claim that it does not converge point wise *anywhere*! The definition of point wise convergence is states as follows for all $\epsilon > 0$ and every $x \in I$ there exists $N \in \mathbb{N}$ depending on ϵ and x such that

$$|v_n(x) - v_*(x)| < \epsilon$$

for all $n \ge N$. I think I should show this by way of contradiction.

$\square$

7: We consider the complex orthonormal basis

$$\varphi_n = \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{inx}$$

where $n = 0, 1, -1, 2, -2, \ldots$. Consider the function $f_a(x) = \mathrm{e}^{ax}$ with real number $a \neq 0$ and compute the Fourier coefficient

$$\langle f_a, \varphi_n \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f_a(x)\, \mathrm{e}^{-inx}\, dx.$$

Then prove the formula

$$\sum_{n=1}^{\infty} \frac{1}{a^2 + n^2} = \frac{\pi}{2a} \coth(\pi a) - \frac{1}{2a^2}$$

(Hint: Plancherel's formula: the relation between L^2 norm of coefficients and $\langle f_a, f_a \rangle$.)

Solution:
We begin by calculating the Fourier coefficient as requested

$$\begin{aligned}
\langle f_a, \varphi_n \rangle &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f_a(x)\, \mathrm{e}^{-inx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \mathrm{e}^{ax}\, \mathrm{e}^{-inx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \mathrm{e}^{(a-\mathrm{i}n)x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{(a-\mathrm{i}n)}\, \mathrm{e}^{(a-\mathrm{i}n)x} \Big|_{-\pi}^{\pi} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{(a-\mathrm{i}n)}\, \mathrm{e}^{(a-\mathrm{i}n)\pi} - \frac{1}{(a-\mathrm{i}n)}\, \mathrm{e}^{-(a-\mathrm{i}n)\pi} \right) \\
&= \frac{1}{\sqrt{2\pi}(a-\mathrm{i}n)} \left(\mathrm{e}^{(a-\mathrm{i}n)\pi} - \mathrm{e}^{-(a-\mathrm{i}n)\pi} \right) \\
&= \frac{1}{\sqrt{2\pi}(a-\mathrm{i}n)} \left(\mathrm{e}^{a\pi}\, \mathrm{e}^{-\mathrm{i}n\pi} - \mathrm{e}^{-a\pi}\, \mathrm{e}^{\mathrm{i}n\pi} \right) \\
&= \frac{1}{\sqrt{2\pi}(a-\mathrm{i}n)} \left(\mathrm{e}^{a\pi} \left(\cos n\pi - \cancelto{0}{\mathrm{i}\sin n\pi} \right) - \mathrm{e}^{-a\pi} \left(\cos n\pi + \cancelto{0}{\mathrm{i}\sin n\pi} \right) \right) \\
&= \frac{\cos n\pi}{\sqrt{2\pi}(a-\mathrm{i}n)} \left(\mathrm{e}^{a\pi} - \mathrm{e}^{-a\pi} \right) \\
&= \frac{2(-1)^n \sinh a\pi}{\sqrt{2\pi}(a-\mathrm{i}n)}.
\end{aligned}$$

This is the requested coefficient.

Next, let's prove the formula provided using **Theorem 3.43** from Olver, which states

$$||f||^2 = \sum_{k=1}^{\infty} |c_k|^2 = \sum_{k=1}^{\infty} \langle f, \varphi_n \rangle^2.$$

Notice, we have already calcualted $c_k = \langle f, \varphi_n \rangle$, thus

$$\begin{aligned}
\sum_{k=1}^{\infty} |c_k|^2 &= \sum_{k=-\infty}^{\infty} \left| \frac{2(-1)^n \sinh a\pi}{\sqrt{2\pi}(a - \mathrm{i}n)} \right|^2 \\
&= \sum_{k=-\infty}^{\infty} \frac{4 \sinh^2 a\pi}{2\pi |a - \mathrm{i}n|^2} \\
&= \sum_{k=-\infty}^{\infty} \frac{4 \sinh^2 a\pi}{2\pi |a - \mathrm{i}n|^2} \\
&= \sum_{k=-\infty}^{\infty} \frac{2 \sinh^2 a\pi}{\pi (a^2 + n^2)} \\
&= \frac{2 \sinh^2 a\pi}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{a^2 + n^2}
\end{aligned}$$

On the other hand we can calculate

$$||f_a||^2 = \langle \mathrm{e}^a, \mathrm{e}^a \rangle = \int_{-\pi}^{\pi} \mathrm{e}^{2ax} \, dx = \frac{1}{2a} \left(\mathrm{e}^{2a\pi} - \mathrm{e}^{-2a\pi} \right) = \frac{\sinh 2a\pi}{a}$$

Hence,

$$\begin{aligned}
||f_a||^2 &= \sum_{k=1}^{\infty} |c_k|^2 \\
\frac{\sinh 2a\pi}{a} &= \frac{2 \sinh^2 a\pi}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{a^2 + n^2} \\
\frac{\pi}{2 \sinh^2 a\pi} \frac{\sinh 2a\pi}{a} &= \sum_{k=-\infty}^{\infty} \frac{1}{a^2 + n^2} \\
\frac{\pi}{2a} \frac{\sinh 2a\pi}{\sinh^2 a\pi} &= \sum_{k=-\infty}^{\infty} \frac{1}{a^2 + n^2}
\end{aligned}$$

We now employ the trig identity $\sinh 2x = 2\sinh x \cosh x$, thus

$$\begin{aligned}
\frac{\pi}{2a}\frac{\sinh 2a\pi}{\sinh^2 a\pi} &= \sum_{k=-\infty}^{\infty}\frac{1}{a^2+n^2} \\
\frac{\pi}{2a}\frac{2\cancel{\sinh a\pi}\cosh a\pi}{\sinh^{\cancel{2}\,1} a\pi} &= \sum_{k=-\infty}^{\infty}\frac{1}{a^2+n^2} \\
\frac{\pi}{2a}\frac{2\cosh a\pi}{\sinh a\pi} &= \sum_{k=-\infty}^{\infty}\frac{1}{a^2+n^2} \\
\frac{\pi}{a}\coth a\pi &= \sum_{k=-\infty}^{\infty}\frac{1}{a^2+n^2} \\
\frac{\pi}{a}\coth a\pi &= \frac{1}{a^2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\frac{1}{a^2+n^2} \\
\frac{\pi}{a}\coth a\pi &= \frac{1}{a^2} + 2\sum_{k=1}^{\infty}\frac{1}{a^2+n^2} \\
\frac{\pi}{2a}\coth a\pi - \frac{1}{2a^2} &= \sum_{k=1}^{\infty}\frac{1}{a^2+n^2}
\end{aligned}$$

as requested.

$\square$