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AMATH 503 HW05

Problem 1: Olver 2.3.2 (a)

Solve the initial value problem (IVP)

$$u_t + 3uu_x = 0$$

$$u(0,x) = \begin{cases} 2, & x > 1 \\ 0, & x > 1 \end{cases};$$

we have $\frac{dt}{1} = \frac{dx}{3u} = \frac{du}{0} \Rightarrow \frac{dx}{dt} = 3u$; $\frac{du}{dt} = 0 \Rightarrow u$ is const.

Now using our initial conditions we have taking $x = x(t) = \xi$

$$u(0,x) = u(x(0)) = u(\xi) = \begin{cases} 2, & \xi < 1 \\ 0, & \xi > 1 \end{cases}$$

$\xi < 1$ means $u = 2 \Rightarrow x(t) = 6t + \xi$

$\xi > 1$ means $u = 0 \Rightarrow x(t) = \xi$

Shock speed is $\frac{d\sigma}{dt} = \frac{c(u^-(t)) - c(u^+(t))}{u^-(t) - u^+(t)}$, $c(u) = \int 3u\,du = \frac{3}{2}u^2$

Thus

$$\frac{d\sigma}{dt} = \frac{\frac{3}{2}(u^-(t)^2 - u^+(t)^2)}{u^-(t) - u^+(t)} = \frac{3}{2}(u^-(t) + u^+(t)) = \frac{3}{2}(2+0) = 3$$

Thus

$$u(t,x) = \begin{cases} 2, & x < 3t+1 \\ 0, & x > 3t+1 \end{cases}$$

$\square$

Problem 2: Olver 2.3.3 let $u(0,x) = \frac{1}{x^2+1}$.

Resulting solution to $u_t + u u_x = 0$ produce shock wave?

$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{0}$ Then $\frac{du}{dt} = 0 \Rightarrow u$ const.

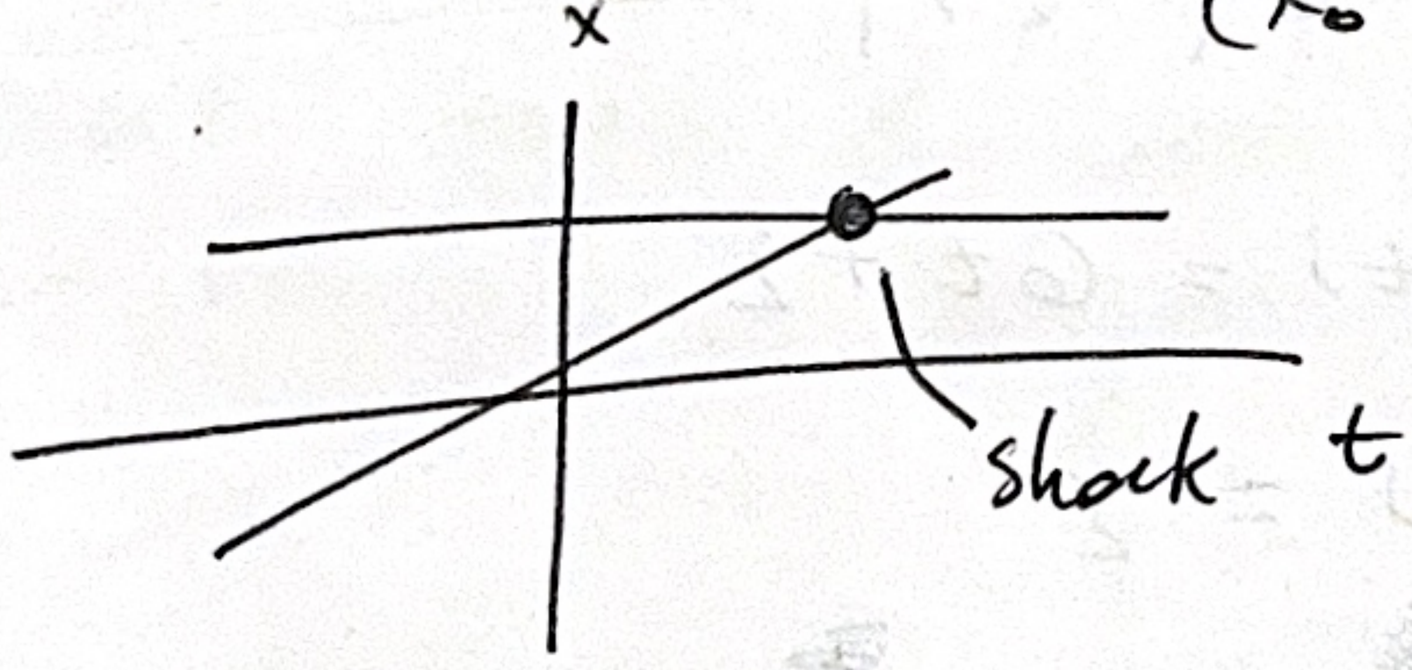
$\frac{dx}{dt} = u$ (which is const. Thus $X(t) = x_0 + ut$ thus

$x_0 = x - ut$.

We have $u(t,x) = f(x_0) = f(x - u(t,x)t)$.

For the shock we need characteristics to intersect

$$X(t) = x_0 + u_0 t = x_0 + \frac{1}{(x_0^2+1)} t$$



Find time of onset using (2.41) we have

$$t_* := \min\left\{ -\frac{1}{f'(x)} \;\middle|\; f'(x) < 0 \right\}.$$

We have $f'(x) = -\frac{2x}{(x^2+1)^2}$ thus we want to min $\frac{(x^2+1)^2}{2x}$.

Where is $f'(x) < 0$? as long as $x > 0$ we have it. Thus

$$t_* = \min\left\{ \frac{(x^2+1)^2}{2x} \;\middle|\; x > 0 \right\}.$$

$$\frac{d}{dx}\left[\frac{(x^2+1)^2}{2x}\right] = \frac{4x(x^2+1)2x - 2(x^2+1)^2}{4x^2} = \frac{\cancel{8x^4} + \cancel{8x^2} - \cancel{2x^4} - \cancel{4x^2} - \cancel{2}}{\cancel{4x^2}} =$$

$$= \frac{4x^4 + 4x^2 - x^4 - 2x^2 - 1}{2x^2} = \frac{3x^4 + 2x^2 - 1}{2x^2} = 0$$ continued...

2 cont.

This is ~~minimized when the numerator~~ is 0 if the numerator is 0 Thus

$$3x^4 + 2x^2 - 1 = 0 \implies (3x^2 - 1)(x^2 + 1) = 0$$

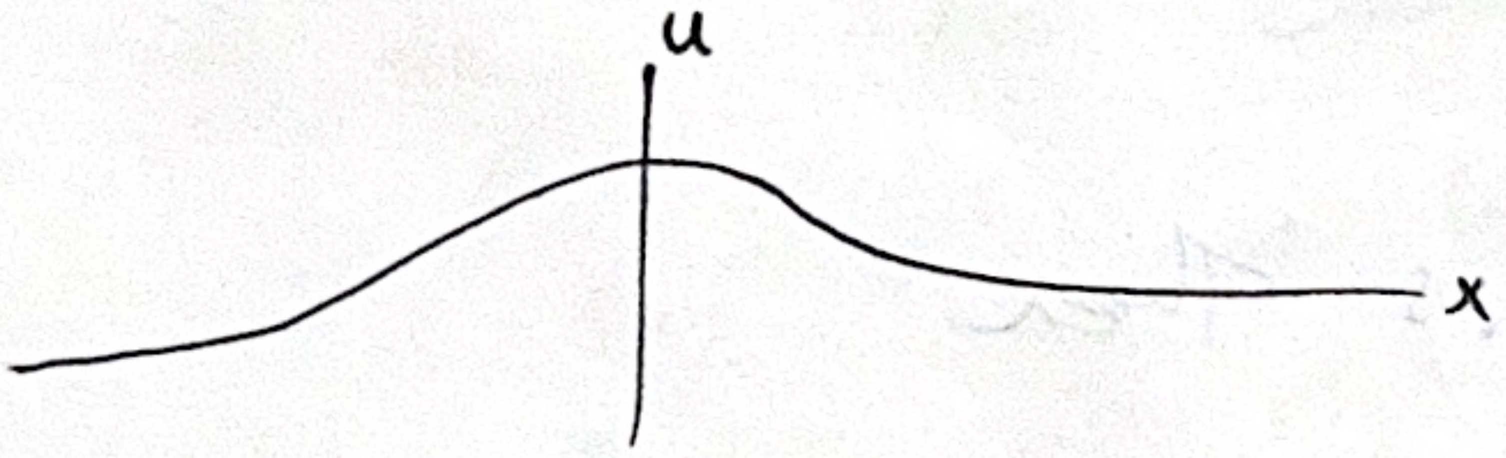
$$x_{1,2} = \pm\sqrt{\tfrac{1}{3}}$$

Since $x > 0$ $x = \sqrt{\tfrac{1}{3}}$

Therefore

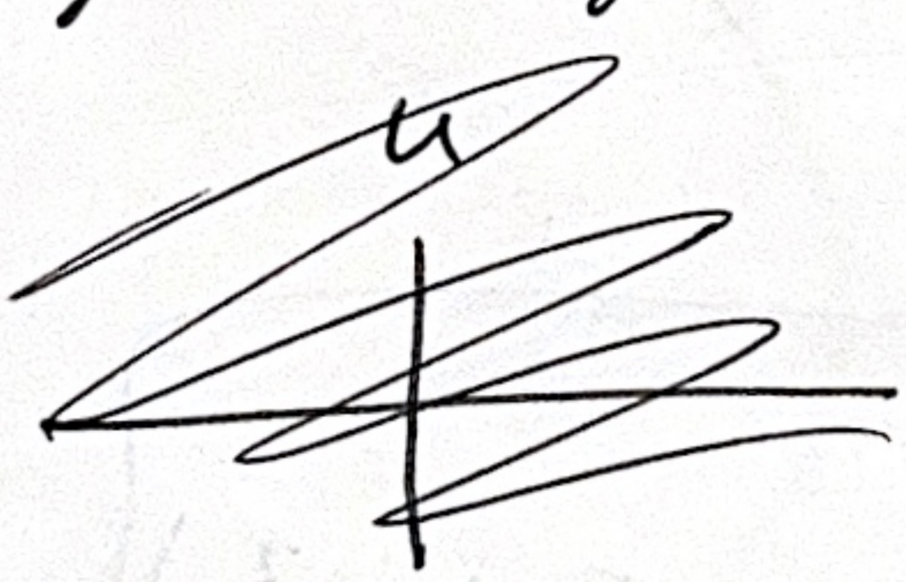
$$t_* = \left(\frac{(x^2+1)^2}{2x}\right) = \left(\frac{(\tfrac{1}{3}+1)^2}{2\sqrt{\tfrac{1}{3}}}\right) = \frac{16/9}{2\sqrt{1/3}} = \frac{16 \cdot \sqrt{3}}{9 \cdot 2} = \boxed{\frac{8\sqrt{3}}{9}}$$

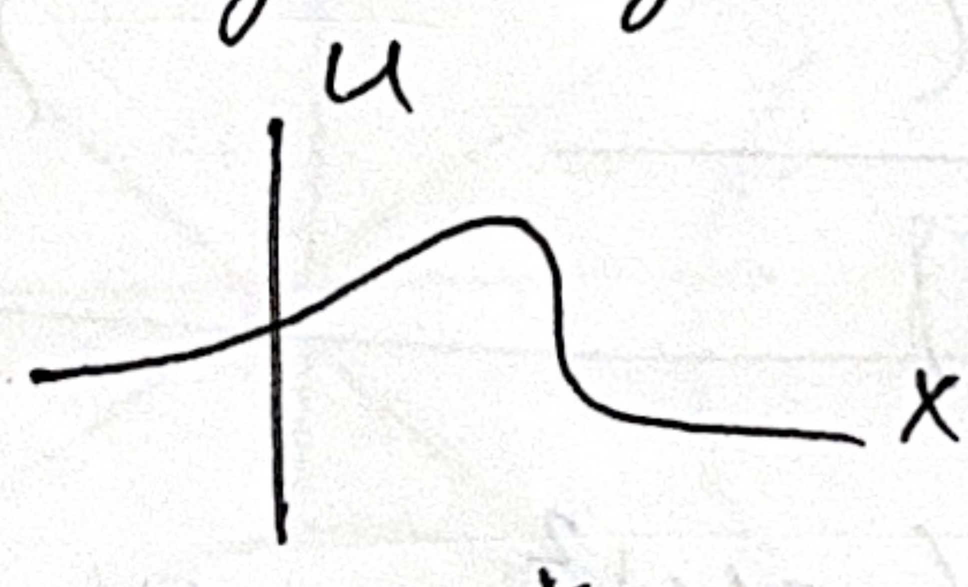
Before



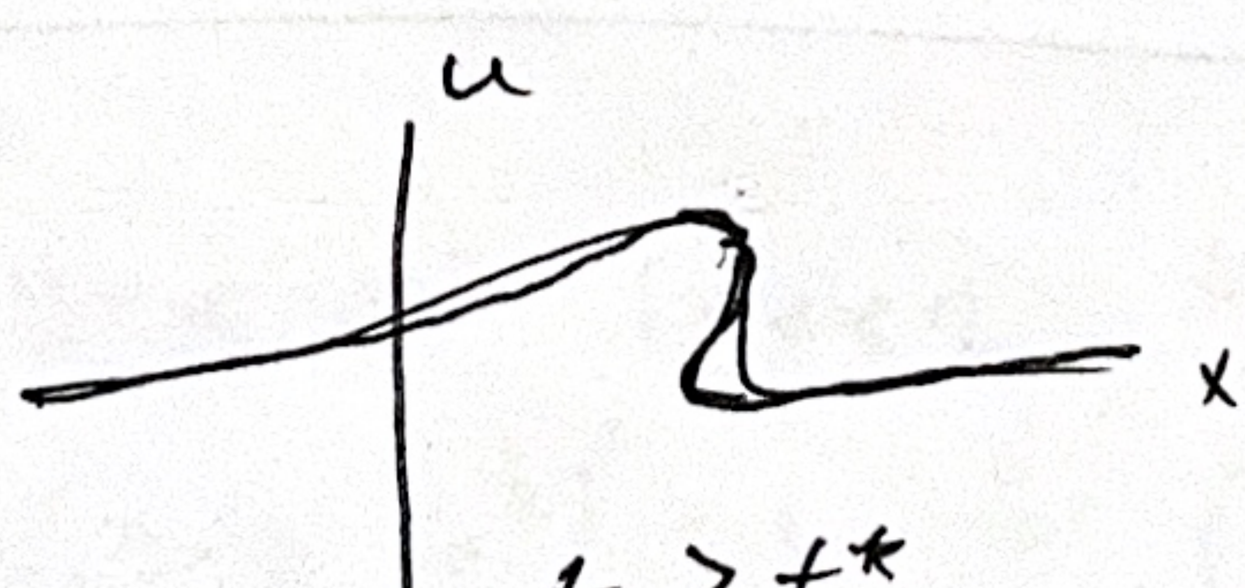
$t=0$

$x = x_0 + ut$ means the wave moves to the right and with larger velocity as u gets larger.





$0 < t < t^*$



$t > t^*$

□

Olver 2.3.17

problem 3: (a)

Determine the Rankine-Hugoniot condition, based on conservation of mass, for the speed of a shock for eq

$$u_t + u^2 u_x = 0.$$

$f(u) = u^2$ so we can $\int f(u)du = \int u^2 du = \frac{1}{3}u^3 = C(u)$ then

$$u_t + \left(\frac{u^3}{3}\right)_x = 0$$

The Rankine-Hugoniot cond is then

$$\frac{C(u^-(t)) - C(u^+(t))}{u^-(t) - u^+(t)} = \frac{\frac{1}{3}\left(u^-(t)^3 - u^+(t)^3\right)}{u^-(t) - u^+(t)}$$

$$= \frac{\frac{1}{3}\cancel{(u^-(t) - u^+(t))}\left(\quad\right)}{\cancel{(u^-(t) - u^+(t))}}$$

$$= \boxed{\frac{1}{3}\left(u^-(t)^2 + u^-(t)u^+(t) + u^+(t)^2\right)}$$

□

Problem 4

first order PDE $u(x,y)$ given by $yu_x + xu_y = xyu$.

(i) general solution & Characteristics

$$\frac{dx}{y} = \frac{dy}{x} = \frac{du}{xyu}$$

$x\,dx = y\,dy \Rightarrow \int x\,dx = \int y\,dy$

$\Rightarrow \frac{1}{2}x^2 = \frac{1}{2}y^2 + C_0$ Thus $\frac{1}{2}(x^2 - y^2) = C_0$

$x^2 - y^2 = C_0$

Also $\frac{dy}{x} = \frac{du}{xyu} \Rightarrow y\,dy = \frac{1}{u}du \Rightarrow \frac{1}{2}y^2 = \ln u$ $\quad u = C_1 e^{\frac{1}{2}y^2}$

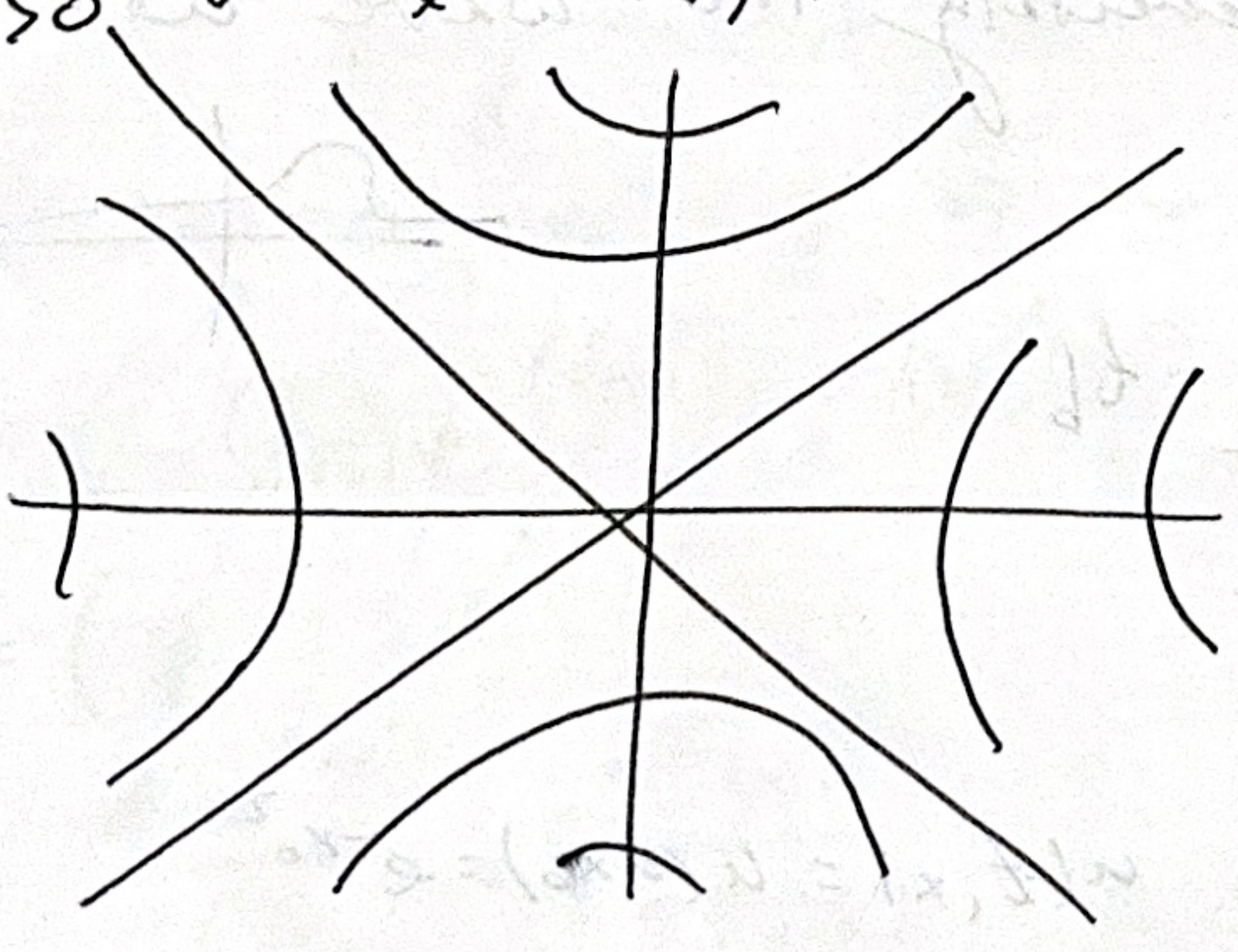
$$\boxed{u = f(x^2 - y^2)\, e^{\frac{1}{2}y^2}}$$

Plotting Char Curves

if $C_0 = 0$ we have $x^2 = y^2$ and Thus $x = \pm y$ (vice versa)

$C_0 < 0$ $\quad x^2 - y^2 = C_0 \Rightarrow y^2 = x^2 - C_0 \Rightarrow y = \pm\sqrt{x^2 - C_0}$

alternatively $C_0 > 0$ $\quad x = \pm\sqrt{y^2 + C_0}$



ii) data: $u = 1$ on $y = x$ for $1 < x < 2$, $u(x,y) = f(x^2 - y^2)e^{\frac{1}{2}y^2}$

Then we have $f(x^2 - y^2)e^{\frac{1}{2}y^2} = 1$ for $(x,y) \in (1,2)\times(1,2)$ since $y = x$.

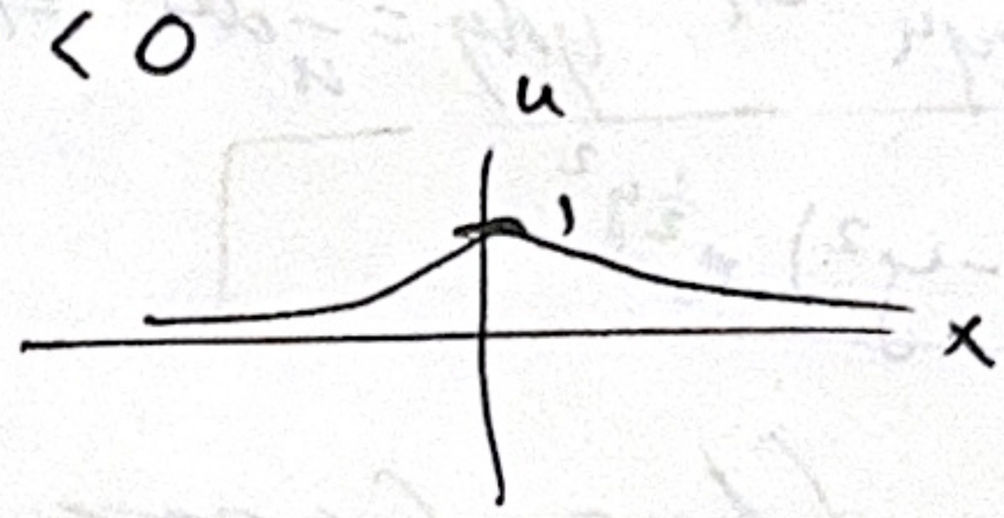
we then have $f(0)e^{\frac{1}{2}y^2} = 1 \Rightarrow f(0) = e^{-\frac{1}{2}y^2}$.

$f(0)$ should be constant but $e^{-\frac{1}{2}y^2}$ is not so no solution since $y = x$ on a characteristic curve.

problem 5 : let $c(u)$ be a smooth function satisfying both $c'(u) < 0$ and $c(u) < 0$ $\forall u > 0$. Assume Also $c(0) = 0$ quasi-linear 1st order PDE

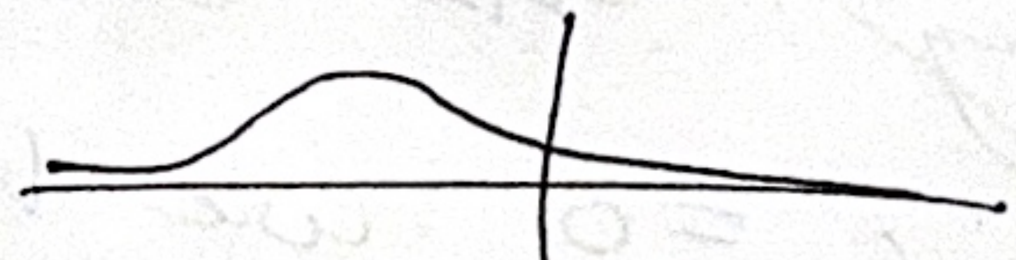
$$u_t + c(u) u_x = 0, \quad -\infty < x < \infty, \quad t \geq 0 \quad u(x, 0) = e^{-x^2}$$

(i) $\dfrac{dt}{1} = \dfrac{dx}{c(u)} = \dfrac{du}{0} \Rightarrow \dfrac{dx}{dt} = c(u) < 0$



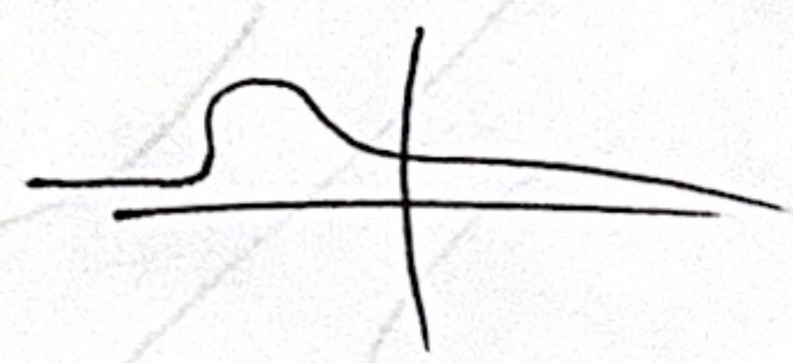
$t = 0$ $\quad u(x) = e^{-x}$

$t \geq 0$ $\quad \dfrac{dx}{dt} < 0$ so it moves left



$c'(u) < 0$ means larger acceleration as u becomes more negative.

for really large t eventually the wave will break & shock.



(ii) finding breaking time t_b

$$\frac{dt}{1} = \frac{dx}{c(u)} = \frac{du}{0}$$

$\Rightarrow u$ is a constant $u(t, x) = u(0, x_0) = e^{-x_0^2}$

$$\frac{dx}{dt} = c(u) \Rightarrow x(t) = x_0 + c(u)\, t.$$

$$\frac{dx}{dx_0} = 1 + c'(u) \frac{du}{dx_0}\, t = 1 + c'(e^{-x_0^2})(-2x_0 e^{-x_0^2})\, t = 0$$

Then $t = + \dfrac{1}{+2x_0 e^{-x_0^2} C'(e^{-x_0^2})}$ thus $t_b = \min\limits_{x_0} \left\{ \dfrac{1}{+2x_0 e^{-x_0^2} C'(e^{-x_0^2})} \middle| t \geq 0 \right\}$

(iii) if $c(u) = -u^4$ then we have

$c'(u) = -4u^3$ and thus

$$t = \left(+2x_0 e^{-x_0^2} 4e^{-3x_0^2}\right)^{-1}$$

$$= \left(+8x_0 e^{-4x_0^2}\right)^{-1} = \frac{e^{4x_0^2}}{8x_0} \Rightarrow \text{derivative} \cancel{\text{is } t'(x_0) =}$$

$$\Rightarrow t'(x_0) = \frac{(8x_0)^2 e^{4x_0^2} - 8e^{4x_0^2}}{(8x_0)^2} = 0$$

$$\frac{((8x_0)^2 - 8)e^{4x_0^2}}{(8x_0)^2} = 0$$

$$\frac{(8x_0^2 - 1)e^{4x_0^2}}{8x_0^2} = 0$$

only true if $(8x_0^2 - 1)e^{4x_0^2} = 0$

Thus $8x_0^2 - 1 = 0$ $\quad x_0^2 = \frac{1}{8} \Rightarrow x_0 = \frac{1}{2\sqrt{2}}$ since we need the pos.

Then $t_b = \cancel{\left(2\left(\frac{1}{2\sqrt{2}}\right)e^{-\left(\frac{1}{2\sqrt{2}}\right)^2} 4e^{-3\left(\frac{1}{2\sqrt{2}}\right)^2}\right)^{-1}}$

$= \cancel{(8}$

$$t_b = \frac{e^{4x_0^2}}{8x_0} = \frac{e^{4\left(\frac{1}{2\sqrt{2}}\right)^2}}{8\left(\frac{1}{2\sqrt{2}}\right)} = \frac{e^{\frac{4}{4\cdot 2}}}{8/2\sqrt{2}} = \frac{e^{1/2}}{4/\sqrt{2}} = \frac{\sqrt{e}\sqrt{2}}{4}$$

$$\boxed{= \frac{\sqrt{2e}}{4}}$$

| problem 6 |

Solve the following

$u_t + uu_x = 0$ with $u(x,0) = \begin{cases} 0, & -\infty < x < 0 \\ 1, & 0 < x < 1 \\ -1, & x > 1 \end{cases}$

$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{0}$ u constant along $u(t,x) = u(0,x_0)$

$\frac{dx}{dt} = u$ then $x = x_0 + u(0,x_0)t$

$-\infty < x < 0$ gives $x = x_0$ Thus $x_0 < 0$ as well.

$x \in (0,1)$: $x = x_0 + t \Rightarrow x_0 = x - t \Rightarrow x \in (t, t+1)$

Additionally,

$x > 1$ we have $x = x_0 - t$ which implies
$x_0 = x + t$ and thus $x > 1 - t$

Therefore

$u(x,t) = \begin{cases} 0, & x < 0 \\ 1, & t < x < t+1 \\ -1, & x > 1 - t \end{cases}$

~~Shock~~ Shock wave hitting expansion fan

$x = 0$ gives $u^-(t) = 0$, $u^+(0) = 1$

then $x = t$ for $x \in (0,t)$, $t \geq 0$

then $x = x_0 + u(x,t) \cdot t$

Thus $u(x,t) = \frac{x}{t}$ in this region of expansion fan

$x=1$ gives $u^-(0)=1$, $u^+(0)=-1$

$$u(x,t)=\begin{cases} 0, & x<0 \\ x/t, & 0<x<t \\ 1, & t<x<1 \\ -1, & t>1 \end{cases}$$

The shock hits expansion when $t=1$.