

Hunter Lybbert  
Student ID: 2426454  
04-09-25  
AMATH 503

## HOMework 1

Exercises come from *Introduction to Partial Differential Equations by Peter J. Olver* as well as supplemented by instructor provided exercises.

---

### 1: Olver 1.1

*Solution:*

- (a)  $\frac{du}{dx} + xu = 1$  : Ordinary equilibrium differential equation of the first order.
- (b)  $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = x$  : Partial dynamic differential equation of the first order.
- (c)  $u_{tt} = 9u_{xx}$  : Partial dynamic differential equation of the second order.
- (d)  $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$  : Partial dynamic differential equation of the second order.
- (e)  $-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = x^2 + y^2$  : Partial equilibrium differential equation of the second order.
- (f)  $\frac{\partial^2 u}{\partial t^2} + 3u = \sin t$  : Ordinary dynamic differential equation of the second order.
- (g)  $u_{xx} + u_{yy} + u_{zz} + (x^2 + y^2 + z^2)u = 0$  : Partial equilibrium differential equation of the second order.
- (h)  $u_{xx} = x + u^2$  : Ordinary equilibrium differential equation of the second order.
- (i)  $\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u \frac{\partial u}{\partial x} = 0$  : Partial dynamic differential equation of the third order.
- (j)  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y \partial z} = u$  : Partial equilibrium differential equation of the second order.
- (k)  $u_{tt} = u_{xxxx} + 2u_{xxyy} + u_{yyyy}$  : Partial dynamic differential equation of the fourth order.

□

### 2: Olver 1.17

*Solution:*

- (a)  $u_t = x^2 u_{xx} + 2xu_x$  : homogeneous linear
- (b)  $-u_{xx} = u_{yy} = \sin u$  : nonlinear
- (c)  $u_{xx} + 2yu_{yy} = 3$  : inhomogeneous linear
- (d)  $u_t + uu_x = 3u$  : nonlinear
- (e)  $e^y u_x = e^x u_y$  : homogeneous linear
- (f)  $u_t = 5u_{xxx} + x^2 u + x$  : inhomogeneous linear

□

**3:** Olver 1.22

- (a) Prove that the Laplacian  $\Delta = \partial_x^2 + \partial_y^2$  defines a linear differential operator.

*Solution:* We need to show that for some appropriate functions  $u, v$  and two scalars  $a, b \in \mathbb{R}$

$$\Delta[au + bv] = a\Delta[u] + b\Delta[v].$$

We will do this directly,

$$\begin{aligned}\Delta[au + bv] &= (\partial_x^2 + \partial_y^2)(au + bv) = (\partial_x^2 + \partial_y^2)au + (\partial_x^2 + \partial_y^2)bv \\ &= \partial_x^2 au + \partial_y^2 au + \partial_x^2 bv + \partial_y^2 bv \\ &= a\partial_x^2 u + a\partial_y^2 u + b\partial_x^2 v + b\partial_y^2 v \\ &= au_{xx} + au_{yy} + bv_{xx} + bv_{yy} \\ &= a(u_{xx} + u_{yy}) + b(v_{xx} + v_{yy}) \\ &= a(\partial_x^2 u + \partial_y^2 u) + b(\partial_x^2 v + \partial_y^2 v) \\ &= a(\partial_x^2 + \partial_y^2)u + b(\partial_x^2 + \partial_y^2)v \\ &= a\Delta[u] + b\Delta[v].\end{aligned}$$

□

- (b) Write out the Laplace equation  $\Delta[u] = 0$  and the Poisson equation  $-\Delta[u] = f$ .

*Solution:* The Laplace equation is

$$\Delta[u] = (\partial_x^2 + \partial_y^2)u = u_{xx} + u_{yy} = 0$$

and the Poisson equation is

$$-\Delta[u] = -(\partial_x^2 + \partial_y^2)u = -u_{xx} - u_{yy} = f.$$

□

- 4: We derive the advection-diffusion equation from the microscopic view. Define  $u(x, t)$  as the density of the particles at location  $x$  and time  $t$ . Define the probability of jumping from the left as  $p(x - \Delta x \rightarrow x, t) \approx \frac{1}{2} + \Delta x$  when  $\Delta x$  is small, and the probability of jumping from the right as  $q(x + \Delta x \rightarrow x, t) \approx \frac{1}{2} - \Delta x$  with small  $\Delta x$ . Assume

$D := \lim_{\Delta x, \Delta t \rightarrow 0} \frac{(\Delta x)^2}{\Delta t}$ . Establish the equation of  $u(x, t)$  in the continuum limit.

*Solution:*

We begin by Taylor expanding  $u(x, t + \Delta t)$ ,  $u(x - \Delta x, t)$ , and  $u(x + \Delta x, t)$

$$\begin{aligned} u(x, t + \Delta t) &= u(x, t) + u_t \Delta t + \mathcal{O}((\Delta t)^2) \\ u(x - \Delta x, t) &= u(x, t) - u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}((\Delta x)^3) \\ u(x + \Delta x, t) &= u(x, t) + u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}((\Delta x)^3). \end{aligned}$$

Additionally, we have the following relationship for the evolution of the system in one time step

$$\begin{aligned} u(x, t + \Delta t) &= p(x - \Delta x \rightarrow x, t) u(x + \Delta x, t) + q(x + \Delta x \rightarrow x, t) u(x - \Delta x, t) \\ u(x, t + \Delta t) &\approx \left( \frac{1}{2} + \Delta x \right) u(x + \Delta x, t) + \left( \frac{1}{2} - \Delta x \right) u(x - \Delta x, t). \end{aligned}$$

Combining this with the Taylor expansions from earlier we have

$$\begin{aligned} u(x, t) + u_t \Delta t &\approx \left( \frac{1}{2} + \Delta x \right) \left( u(x, t) + u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}((\Delta x)^3) \right) \\ &\quad + \left( \frac{1}{2} - \Delta x \right) \left( u(x, t) - u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}((\Delta x)^3) \right) \\ u(x, t) + u_t \Delta t &\approx \frac{1}{2} \left( u(x, t) + u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}((\Delta x)^3) \right) + \Delta x \left( u(x, t) + u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}((\Delta x)^3) \right) \\ &\quad + \frac{1}{2} \left( u(x, t) - u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}((\Delta x)^3) \right) - \Delta x \left( u(x, t) - u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}((\Delta x)^3) \right) \\ u_t \Delta t &\approx \frac{1}{2} u_{xx} (\Delta x)^2 + \Delta x \left( u(x, t) + u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}((\Delta x)^3) \right) \\ &\quad - \Delta x \left( u(x, t) - u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}((\Delta x)^3) \right) \\ u_t \Delta t &\approx \frac{1}{2} u_{xx} (\Delta x)^2 + 2u_x (\Delta x)^2 \\ u_t &\approx \left( \frac{1}{2} u_{xx} + 2u_x \right) \frac{(\Delta x)^2}{\Delta t} \\ u_t &= D \left( \frac{1}{2} u_{xx} + 2u_x \right). \end{aligned}$$

This is the differential equation for the equation  $u(x, t)$  in the continuum limit.

□

5: (a) Consider the following boundary value problem (BVP).

$$\begin{cases} X''(x) + \lambda X = 0, & x \in (0, L) \\ X(0) = X(L) = 0, \end{cases}$$

where  $L > 0$  is a constant. Solve the eigenpair:

$$(X_k, \lambda_k) = \left\{ \sin\left(\frac{k\pi x}{L}\right), \left(\frac{k\pi}{L}\right)^2 \right\}_{k=1}^{\infty}$$

*Solution:*

I begin by rewriting the second order ODE as a 2D first order ODE. Let  $X_1 = X$  and  $X_2 = X'$  then we have

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}' = \begin{bmatrix} X_2 \\ -\lambda X_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\lambda & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

Using this system we can solve for the eigenvalues of this system, that is we want to solve for  $\gamma$  (since  $\lambda$  is already in use in this function we choose a stand in variable) in the following equation

$$(-\gamma)^2 - (-\lambda \cdot 1) = \gamma^2 + \lambda = 0$$

Therefore the eigenvalues are  $\gamma = \pm i\sqrt{\lambda}$ . Since the eigenvalues are both imaginary we know that the solution will be of the form

$$X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x).$$

Now using the boundary values we can determine

$$\begin{aligned} X(0) &= C_1 \cos(\sqrt{\lambda}0) + C_2 \sin(\sqrt{\lambda}0) \\ 0 &= C_1. \end{aligned}$$

Furthermore, we need

$$\begin{aligned} X(L) &= C_2 \sin(\sqrt{\lambda}L) \\ 0 &= C_2 \sin(\sqrt{\lambda}L) \\ 0 &= \sin(\sqrt{\lambda}L) \end{aligned}$$

assuming  $C_2 \neq 0$  to avoid arriving at the uninteresting trivial solution. We know  $\sin(x) = 0$  where  $x = \pi k$  for  $k \in \mathbb{Z}$ . Thus we need  $\sqrt{\lambda}L = \pi k$  which gives us  $\lambda = \left(\frac{\pi k}{L}\right)^2$ . Finally, our general solution currently is the following

$$X_k(x) = \sin\left(\frac{\pi k x}{L}\right)$$

where  $\lambda_k = \left(\frac{\pi k}{L}\right)^2$ .

**TODO: Do the other cases.**

□

(b) Consider the following boundary value problem (BVP).

$$\begin{cases} X''(x) + \lambda X = 0, & x \in (0, L) \\ X'(0) = X'(L) = 0, \end{cases}$$

where  $L > 0$  is a constant. Solve the eigenpair:

$$(X_k, \lambda_k) = \left\{ \cos\left(\frac{k\pi x}{L}\right), \left(\frac{k\pi}{L}\right)^2 \right\}_{k=0}^{\infty}$$

*Solution:*

We reuse much of the work for  $X(x)$  from part (a), however the final steps using the boundary values will vary slightly. Beginning from the solution following the form

$$X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x),$$

we now apply boundary conditions. We first need this time to calculate

$$X'(x) = -C_1 \sqrt{\lambda} \sin(\sqrt{\lambda}x) + C_2 \sqrt{\lambda} \cos(\sqrt{\lambda}x)$$

Plugging in our boundary conditions we have

$$\begin{aligned} X'(0) &= -C_1 \sqrt{\lambda} \sin(\sqrt{\lambda}0) + C_2 \sqrt{\lambda} \cos(\sqrt{\lambda}0) \\ 0 &= C_2. \end{aligned}$$

Once more, we have

$$\begin{aligned} X'(L) &= -C_1 \sqrt{\lambda} \sin(\sqrt{\lambda}L) \\ 0 &= -C_1 \sqrt{\lambda} \sin(\sqrt{\lambda}L) \end{aligned}$$

which only holds when  $\sqrt{\lambda} = \frac{\pi k}{L}$  for  $k \in \mathbb{Z}$ . This in total gives the solution

$$X(x) = C_1 \cos\left(\frac{\pi k x}{L}\right)$$

where  $\lambda = \left(\frac{\pi k}{L}\right)^2$ .

□

**6:** Consider the following IBVP in a rectangle:

$$\begin{cases} u_t = \Delta u, & (x, y) \in (0, L_1) \times (0, L_2), t > 0 \\ \partial_{\mathbf{n}} u(x, y, t) = 0, & (x, y) \in \partial((0, L_1) \times (0, L_2)), t > 0 \\ u(x, y, 0) = u_0(x, y) \geq 0, \neq 0 & (x, y) \in (0, L_1) \times (0, L_2) \end{cases}$$

where  $\mathbf{n}$  denotes the unit outer normal derivative and  $L_1, L_2 > 0$  are given constants. Solve to get the general solution. Recall that  $\Delta = \partial_{xx} + \partial_{yy}$ .

*Solution:*

We have

$$u_t = u_{xx} + u_{yy},$$

assuming we have the ability to use separation of variables we let  $u(x, y, t)$  take the form  $u(x, y, t) = X(x)Y(y)T(t)$ . Then plugging this in we have:

$$\begin{aligned} X(x)Y(y)T'(t) &= X''(x)Y(y)T(t) + X(x)Y''(y)T(t) \\ \frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\lambda. \end{aligned}$$

Where the  $-\lambda$  came from following the example in class.

**TODO: use 5 b) to skip the cases**