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AMATH 502

HOMEWORK 8

Exercises come from the assignment sheet provided by the professor on canvas.

1: See assignment document for background on the *Sierpinski Carpet.*
   (a) Show that the (Lebesgue) measure of the resulting fractal is 0. Justify your work.

   Solution:
   Notice the following pattern in the measure (or area in this case) of each step denoted as S_n:

   $$S_0 = 1 = (8/9)^0$$

   $$S_1 = 1 \cdot \frac{8}{9} = \left(\frac{8}{9}\right)^1$$

   $$S_2 = \sum_{i=1}^{8} \frac{1}{9} \cdot \frac{8}{9} = \left(\frac{8}{9}\right)^2$$

   $$\vdots$$

   $$S_n = \left(\frac{8}{9}\right)^n.$$

   Finally, the limiting set forming the fractal of interest S_∞ has measure 0 since

   $$\lim_{n \to \infty} \left(\frac{8}{9}\right)^n = 0.$$

   $\square$

   (b) Find the similarity dimension of the limiting fractal. Show and explain your work.

   Solution:
   Following the examples in Strogatz we define the similarity dimension of the limiting fractal to be

(1)
   $$d = \frac{\ln m}{\ln r}$$

   where m is the number of copies and r is the scale factor of the fractal. In our particular fractal we have $m = 8$ and $r = 3$. Therefore, our similarity dimension for this carpet fractal is

   $$d = \ln 8 / \ln 3$$

   $\square$

(c) Show that the box-counting dimension of this fractal is the same as the similarity dimension. (You may not be able to complete this until box-counting dimension is defined in class).

Solution:
Notice, that by definition of the box counting dimension is

$$d = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)}.$$
(2)

Now for this carpet fractal, each S_n consists of 8^n squares with side length $(1/3)^n$. So if we pick $\epsilon = (1/3)^n$ we need all 8^n of these squares to cover this carpet fractal. Hence $N = 8^n$ when $\epsilon = (1/3)^n$. Since $\epsilon \to 0$ as $n \to \infty$, we have the boxing dimension of

$$d = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)} = \frac{\ln(8^n)}{\ln(3^n)} = \frac{n \ln 8}{n \ln 3} = \frac{\ln 8}{\ln 3}.$$

Which is exactly what we got for the similarity dimension.

□

(d) Show that there are uncountably many points on the interior of the limiting fractal (i.e. infinitely many points without x $= 0$, x $= 1$, y $= 0$, or y $= 1$).
Hint: Try showing this in just one dimension, e.g., show that there are uncountably many points (x, y) with a fixed value of y, perhaps y $= 0.5$ is a good choice.

Solution:
Notice, if we fix a value of $y = 0.5$, then the x values in this cross section of the carpet fractal follow the pattern and sequence of the middle 1/3 cantor set. Citing example 11.2.3 from Strogatz, we know that the middle 1/3 cantor set is uncountable, therefore the set of points in the fractal such that x can vary and $y = 0.5$ are uncountable. Since this is a subset of the limiting carpet fractal, then the whole set of points in the limiting fractal are uncountable.

□

2: In this problem we construct what is called a middle-halves Cantor set. Consider the following Cantor set construction. Start with the interval $[0, 1]$, then remove the middle half. Continue this process for each sub-interval.

(a) Draw S_1 and S_2.
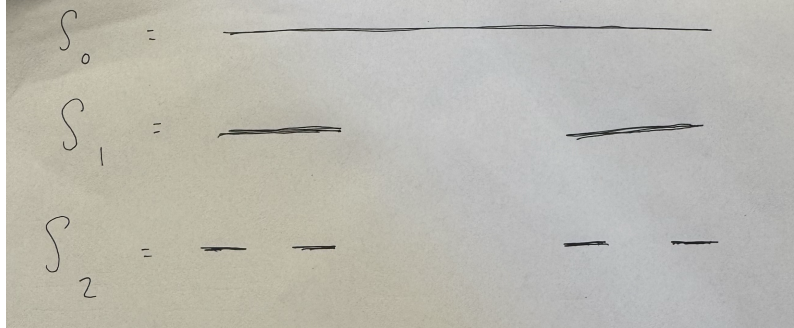
Solution:
See Figure (1).

FIGURE 1. We have sketched the first 3 steps of the middle halves Cantor set.

☐

(b) Find the similarity dimension of the set.

Solution:
We have the number of copies $m = 2$ and the scale factor $r = 4$ therefore, using equation (1) the similarity dimension is
$$d = \frac{\ln 2}{\ln 4}.$$

☐

(c) Find the measure of the set.

Solution:
Similar to problem 1 a), we can calculate the measure of the set as $S_n = (1/2)^n$ which leads to
$$\lim_{n \to \infty} (1/2)^n = 0.$$

☐

3: See assignment sheet for details of the *fat fractal*.

    (a) Find the (Lebesgue) measure of this cantor set. Show your work.

        Solution:
        Let's do this by noting the measure of the unremoved line segment is 1, therefore
        the measure of the fat fractal will be $1 - R$, where R is the total measure which
        is removed. Notice, to go from the n step to the $(n+1)$th step we are removing a
        piece the size of $(2/7)^{n+1}$ and we are removing 2^n of them. Written in summation
        form this is

$$R = \sum_{n=0}^{\infty} 2^n (2/7)^{n+1} = (2/7) \sum_{n=0}^{\infty} (4/7)^n = (2/7)\frac{1}{1-4/7} = \frac{2}{7}\frac{7}{3} = 2/3$$

        where we used the geometric series with $r = 4/7$. Therefore, the measure of the
        fat fractal is $1 - R = 1 - (2/3) = 1/3$.

                                            □

    (b) Is the fractal self similar? Justify your answer.
        Hint: Can you find the similarity dimension of this set? What happens when you
        try?)
        Note: You may find this part to be difficult. If you are struggling with it, you
        may want to skip it for now and come back to it later.

        Solution:
        If we were to try and calculate the similarity dimension we need to calculate m,
        the number of additional copies we get each iteration and r the scale factor for
        these new copies. We can determine that m is 2 in our case, however, the scale
        factor is not constant with so there is no one r that works for all steps. Therefore,
        the fat fractal is not self similar.

                                            □

4: The tent map on the interval $[0, 1]$ is defined by $x_{n+1} = f(x_n)$, where

$$f(x) = \begin{cases} rx, & 0 \le x \le \frac{1}{2} \\ r(1-x), & \frac{1}{2} < x \le 1 \end{cases}.$$

Assume that $r > 2$. Then some points get mapped outside of the interval $[0, 1]$. If $f(x_0) > 1$ then we say that x_0 has "escaped" after one iteration. Similarly, if $f^n(x_0) > 1$ for some finite n and n is the smallest integer for which this is true, then we say x_0 has escaped after n iterations.

Hint: You should plot the function $f(x)$ and think about orbits of different initial conditions when you are solving this problem, it will make your work much easier. Use these plots to guide your analysis, the plots alone are not enough.

(a) Find the set of initial conditions x_0 that escape after one iteration.

Solution:
The set of initial conditions for which x_0 will escape in one iteration is all

$$x_0 \in \left\{ \frac{1}{r} < x \le 1 - \frac{1}{r} \right\}.$$

Now we calculated this by determining that any x_0 from in that range for example, the minimum $\frac{1}{r}$ we have $f(1/r) = r(1/r) = 1$ which is the lower bound for the output of the points in the set we have given. It is easy and left to the reader to verify the upper bound as well.

□

(b) Find the set of initial conditions x_0 that escape after two iterations.

Solution:
Another way to state which x_0's we are looking for is to say the set of x_0 which map to the set from part a) in one step. And thus in the next iteration will escape in one (but two in total). Essentially we are trying to find the pre-image of $\left\{ \frac{1}{r} < x \le 1 - \frac{1}{r} \right\}$ with respect to $f(x)$. Using the example of $r = 3$, we see we are looking for the pre-image of $\left\{ \frac{1}{3} < x \le \frac{2}{3} \right\}$. Notice, if $f(x) = 1/3$ we get $x = 1/9$ or $8/9$ and for $f(x) = 2/3$ we get $x = 2/7$ or $7/9$. Hence,

$$x_0 \in \left\{ \frac{1}{9} < x \le \frac{2}{9} \right\} \cup \left\{ \frac{7}{9} < x \le \frac{8}{9} \right\} \implies f(x_0) \in \left\{ \frac{1}{3} < x \le \frac{2}{3} \right\}.$$

And thus $f(f(x_0)) \notin [0, 1]$ and hence x_0 escaped in two steps. Therefore, in general r the set of initial conditions x_0 which escape in two steps exactly is

$$x_0 \in \left\{ \frac{1}{r} < x \le \frac{1}{r} - \frac{1}{r^2} \right\} \cup \left\{ 1 - \frac{1}{r} + \frac{1}{r^2} < x \le 1 - \frac{1}{r^2} \right\}.$$

□

(c) Describe the set of x_0 that never escape. This is called the invariant set.
Hint: First look at what happens for $r = 3$. Does this look like a set you recognize?

Solution:
After inspecting this for $r = 3$ this is basically just the middle third Cantor set. I could redraw that here, but that would be redundant at this point. This set is

quite interesting though, seeing as it has come up so many different ways in this homework assignment.

□

(d) Find the box dimension of the invariant set (for general r, not $r = 3$).

Solution:
The box dimension is going to be easy to calculate. Using the setup from problem 1 c) and Equation ((2)) we can determine that in this case each step in the iteration consists of 2^n intervals which remain in the interval $[0, 1]$. Furthermore, if we pick $\epsilon = (1/r)^n$ we need all 2^n of these line segments to cover the ultimate fractal structure. Hence, $N(\epsilon) = 2^n$ and $\epsilon = (1/r)^n$. Since $\epsilon \to 0$ as $n \to \infty$, we have the boxing dimension of

$$d = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)} = \frac{\ln(2^n)}{\ln(r^n)} = \frac{n \ln 2}{n \ln r} = \frac{\ln 2}{\ln r}.$$

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