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AMATH 502

HOMEWORK 1

Exercises come from *Nonlinear Dynamics and Chaos by Steven H. Strogatz*

1: 2.2.3
Solution: Looking closely at the DS $\dot{x} = x - x^3$, we can recognize there are three fixed points at $x^* = -1, 0, 1$. Now we can look at the plots in Figure 1, to identify that $x^* = -1, 1$ are stable fixed points and $x^* = 0$ is unstable.

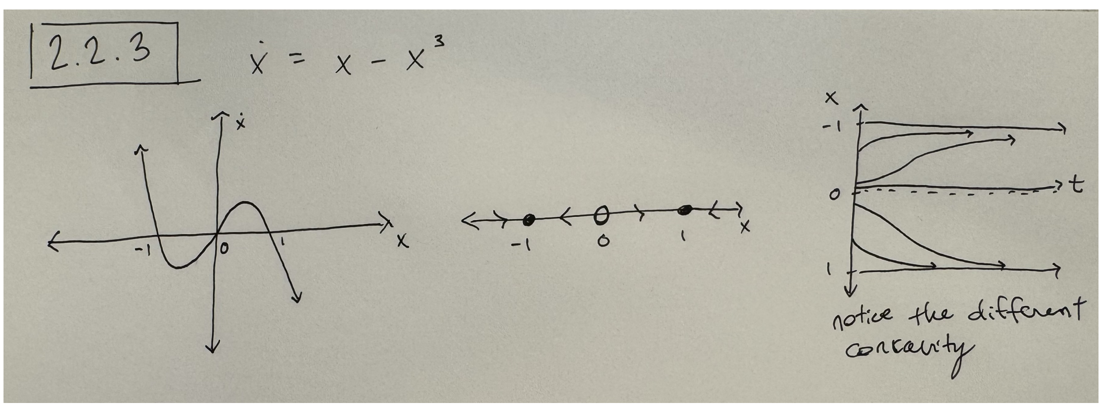

FIGURE 1. We have sketched the vector field on the real line, identified all three fixed points, classified their stability and sketched the $x(t)$ for different initial conditions.

$\square$

2: 2.2.7

Solution: Looking at the DS $\dot{x} = e^x - \cos x$ is tricky so let's plot each of e^x and $\cos x$ on the same plot to determine where the fixed points are and what their stability is.

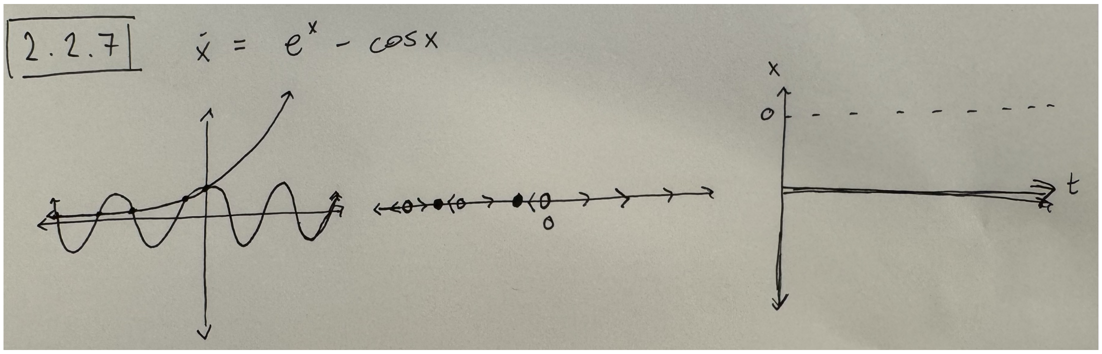

FIGURE 2. We have sketched the vector field on the real line, identified the approximate location of all the fixed points, and classified their stability (which is alternating stable and unstable) sketched the $x(t)$ for different initial conditions.

$\square$

3: 2.2.8

Solution:

TODO

4: 2.2.10

Solution:

TODO

5: 2.2.13 (a,b,c,d)

(a) $m\dot{v} = mg - kv^2$ with initial condition $v(0) = 0$

Solution:
Let's begin by dividing by m and factoring out a g term.

$$\begin{aligned} m\dot{v} &= mg - kv^2 \\ \dot{v} &= g - \frac{kv^2}{m} \\ \dot{v} &= g\left(1 - \frac{kv^2}{gm}\right). \end{aligned}$$

Now rewriting $\dot{v}$ as the derivative of v with respect to t we have

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}t} &= g\left(1 - \frac{kv^2}{gm}\right) \\ \frac{1}{\left(1 - \frac{kv^2}{gm}\right)}\mathrm{d}v &= g\mathrm{d}t \\ \int \frac{1}{\left(1 - \sqrt{\frac{k}{gm}}v\right)\left(1 + \sqrt{\frac{k}{gm}}v\right)}\mathrm{d}v &= \int g\mathrm{d}t. \end{aligned}$$

Now, we can do partial fractions on the left and integrate both sides

$$\begin{aligned} \int \frac{1}{\left(1 - \sqrt{\frac{k}{gm}}v\right)\left(1 + \sqrt{\frac{k}{gm}}v\right)}\mathrm{d}v &= \int g\mathrm{d}t \\ \int \frac{1/2}{\left(1 - \sqrt{\frac{k}{gm}}v\right)}\mathrm{d}v + \int \frac{1/2}{\left(1 + \sqrt{\frac{k}{gm}}v\right)}\mathrm{d}v &= gt + C \\ -\frac{1}{2}\sqrt{\frac{gm}{k}}\log\left(1 - \sqrt{\frac{k}{gm}}v\right) + \frac{1}{2}\sqrt{\frac{gm}{k}}\log\left(1 + \sqrt{\frac{k}{gm}}v\right) &= gt + C \\ \frac{1}{2}\sqrt{\frac{gm}{k}}\left(\log\left(1 + \sqrt{k/(gm)}\,v\right) - \log\left(1 - \sqrt{k/(gm)}\,v\right)\right) &= gt + C \end{aligned}$$

Interesting ... now I need to input the initial condition

$$\begin{aligned}
\frac{1}{2}\sqrt{\frac{gm}{k}}\left(\log\left(1+\sqrt{k/(gm)}\,0\right)-\log\left(1-\sqrt{k/(gm)}\,0\right)\right) &= g0+C\\
\frac{1}{2}\sqrt{\frac{gm}{k}}\left(\log(1)-\log(1)\right) &= C\\
0 &= C.
\end{aligned}$$

Plugging this in and simplifying we have

$$\begin{aligned}
\frac{1}{2}\sqrt{\frac{gm}{k}}\left(\log\left(1+\sqrt{k/(gm)}\,v\right)-\log\left(1-\sqrt{k/(gm)}\,v\right)\right) &= gt\\
\log\left(\frac{1+\sqrt{k/(gm)}\,v}{1-\sqrt{k/(gm)}\,v}\right) &= 2gt\sqrt{k/(gm)}.
\end{aligned}$$

Now we can exponentiate and solve for v

$$\begin{aligned}
\frac{1+\sqrt{k/(gm)}\,v}{1-\sqrt{k/(gm)}\,v} &= \mathrm{e}^{2gt\sqrt{k/(gm)}}\\
1+\sqrt{k/(gm)}\,v &= \mathrm{e}^{2gt\sqrt{k/(gm)}}(1-\sqrt{k/(gm)}\,v)\\
1+\sqrt{k/(gm)}\,v &= \mathrm{e}^{2gt\sqrt{k/(gm)}}-\sqrt{k/(gm)}\,v\,\mathrm{e}^{2gt\sqrt{k/(gm)}}\\
\sqrt{k/(gm)}\,v+\sqrt{k/(gm)}\,v\,\mathrm{e}^{2gt\sqrt{k/(gm)}} &= \mathrm{e}^{2gt\sqrt{k/(gm)}}-1\\
v &= \frac{\mathrm{e}^{2gt\sqrt{k/(gm)}}-1}{\sqrt{k/(gm)}+\sqrt{k/(gm)}\,\mathrm{e}^{2gt\sqrt{k/(gm)}}}.
\end{aligned}$$

Some final simplifications gives us

$$\begin{aligned}
v &= \frac{1}{\sqrt{k/(gm)}}\,\frac{\mathrm{e}^{2gt\sqrt{k/(gm)}}-1}{1+\mathrm{e}^{2gt\sqrt{k/(gm)}}}\\
v &= \sqrt{\frac{gm}{k}}\left(\frac{\mathrm{e}^{2gt\sqrt{k/(gm)}}-1}{\mathrm{e}^{2gt\sqrt{k/(gm)}}+1}\right).
\end{aligned}$$

Therefore our final analytical solution is

$$v = \sqrt{\frac{gm}{k}}\left(\frac{\mathrm{e}^{2gt\sqrt{k/(gm)}}-1}{\mathrm{e}^{2gt\sqrt{k/(gm)}}+1}\right).$$

$\square$

(b) Determine the limit of $v(t)$ as $t \to \infty$. We will need to utilize L'Hôpital's rule since the numerator and the denominator go to infinity.

$$\begin{aligned}
\lim_{t\to\infty} \sqrt{\frac{gm}{k}} \left(\frac{\mathrm{e}^{2gt\sqrt{k/(gm)}} - 1}{\mathrm{e}^{2gt\sqrt{k/(gm)}} + 1} \right) &= \lim_{t\to\infty} \sqrt{\frac{gm}{k}} \left(\frac{2gt\sqrt{k/(gm)}\,\mathrm{e}^{2gt\sqrt{k/(gm)}}}{2gt\sqrt{k/(gm)}\,\mathrm{e}^{2gt\sqrt{k/(gm)}}} \right) \\
&= \lim_{t\to\infty} \sqrt{\frac{gm}{k}} \\
&= \sqrt{\frac{gm}{k}}.
\end{aligned}$$

Therefore, the terminal velocity is $\sqrt{\frac{gm}{k}}$.

$\square$

(c) **TODO:** insert pictures.

(d)

6: 2.3.2

Solution:

TODO

7: 2.3.6

Solution:

TODO

8: 2.4.7

Solution:

TODO