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AMATH 502:
Dynamical Systems and Chaos

HOMEWORK 1

Exercises come from *Nonlinear Dynamics and Chaos by Steven H. Strogatz*

1: 2.2.3
Solution: Looking closely at the DS $\dot{x} = x - x^3$, we can recognize there are three fixed points at $x^* = -1, 0, 1$. Now we can look at the plots in Figure 1, to identify that $x^* = -1, 1$ are stable fixed points and $x^* = 0$ is unstable.

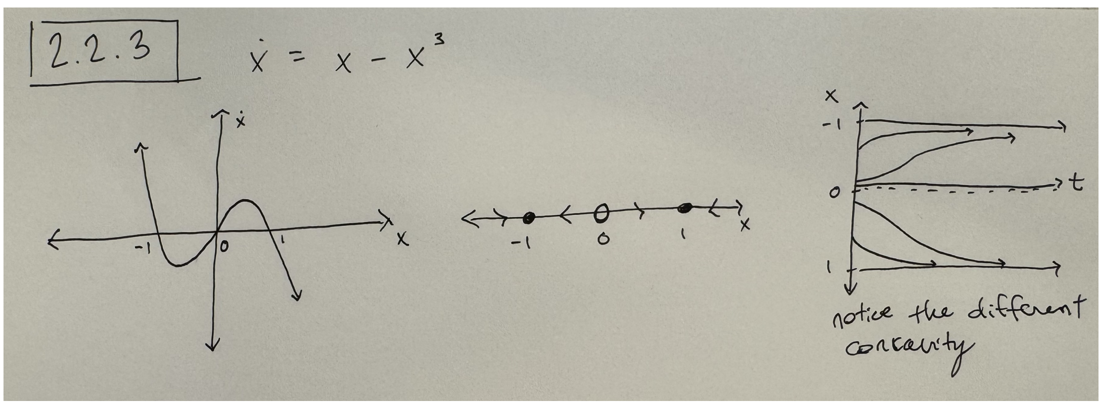

FIGURE 1. We have sketched the vector field on the real line, identified all three fixed points, classified their stability and sketched the $x(t)$ for different initial conditions.

Attempting an analytic solution now by separation of variables

$$\begin{aligned}
\dot{x} &= x - x^3 \\
\frac{dx}{dt} &= x - x^3 \\
\int \frac{dx}{x - x^3} &= \int dt \\
\int -\frac{1}{x^3}\frac{dx}{1 - \frac{1}{x^2}} &= t + C.
\end{aligned}$$

Let's now use a substitution $u = 1 - \frac{1}{x^2}$ and $du = \frac{2}{x^3}dx$ on the left hand side, this gives us. Note, I am lazy with my notation for the constants which show up and just let C

continue to absorb them. Furthermore,

$$\begin{aligned}
\int -\frac{1}{2}\frac{du}{u} &= t + C \\
-\frac{1}{2}\log u &= t + C \\
-\frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) &= t + C.
\end{aligned}$$

Further solving for x in terms of t we have

$$\begin{aligned}
\log\left(1 - \frac{1}{x^2}\right) &= -2t + C \\
\frac{1}{x^2} &= -\,\mathrm{e}^{-2t}\,C + 1 \\
x^2 &= \frac{1}{1 - \mathrm{e}^{-2t}\,C} \\
x(t) &= \frac{1}{\pm\sqrt{1 - \mathrm{e}^{-2t}\,C}} \\
x(t) &= \pm\frac{1}{\sqrt{1 - \mathrm{e}^{-2t}\,C}}.
\end{aligned}$$

Which aligns with our analysis at least in terms of as $t \to \infty$ we see $x(t)$ converges to either 1 or -1.

$\square$

2: 2.2.7
Solution: Looking at the DS $\dot{x} = e^x - \cos x$ is tricky so let's plot in Figure 2 each of e^x and $\cos x$ on the same plot to determine where the fixed points are and what their stability is. Notice, the two functions intersect at $x = 0$ and infinitely more points as $x \to -\infty$. Since e^x is decaying to 0 as $x \to -\infty$ the points at which the two functions intersect approach odd multiples of $\pi/2$ such that the fixed points can be written as $x^* = 0$ and $x^* \approx -\frac{2k+1}{2}$ for the integer k which is sufficiently less than 0, denoted as $k << 0$.

Furthermore, note that to the right of the fixed point $x^* = 0$ we have that $e^x > \cos x$, therefore $e^x - \cos x < 0$ and the flow is to the right in this interval. Finally, note as depicted in the graph in Figure 2, that the oscillating nature of the cos function we end up with alternating flow directions and thus with alternating stability for the fixed points. So $x^* = 0$ is an unstable fixed point, and each fixed point alternates stability as $x \to -\infty$.

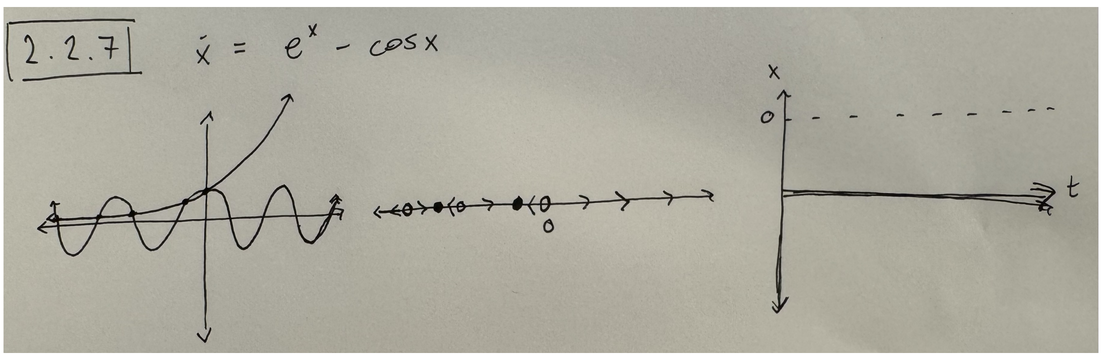

FIGURE 2. We have sketched the vector field on the real line, identified the approximate location of all the fixed points, and classified their stability (which is alternating stable and unstable). I am not actually going to come back and fill in the sketch of $x(t)$ for different initial conditions since there are an unending number of fixed points and each has two concavities due to the alternating stability and oscillating nature of the cos function.

□

3: 2.2.8

Solution:

Based on the given flow diagram $\dot{x} = x(x+1)^2(x-2)$. Now to verify that this satisfies the criteria we are given I will plug in values in the ranges given and verify they match the expected flow direction. We also have depicted this in Figure 3

$$\dot{x} = x(x+1)^2(x-2)\big|_{x=-2}$$
$$\dot{x} = -2(-2+1)^2(-2-2)$$
$$\dot{x} = -2(-1)^2(-4)$$
$$\dot{x} = 8 > 0$$

$$\dot{x} = x(x+1)^2(x-2)\big|_{x=-.5}$$
$$\dot{x} = -.5(-.5+1)^2(-.5-2)$$
$$\dot{x} = -.5(.5)^2(-2.5) > 0$$

$$\dot{x} = x(x+1)^2(x-2)\big|_{x=1}$$
$$\dot{x} = 1(1+1)^2(1-2)$$
$$\dot{x} = 1(1+1)^2(-1) < 0$$

$$\dot{x} = x(x+1)^2(x-2)\big|_{x=3}$$
$$\dot{x} = 3(3+1)^2(3-2)$$
$$\dot{x} = 3(3+1)^2(1) > 0$$

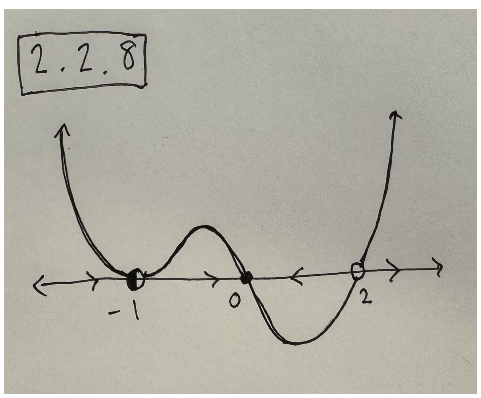

FIGURE 3. On the x-axis I have written the provided phase diagram and plotted over it one possible polynomial which satisfies the required flows.

□

4: 2.2.10

Find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not.

(a) Every real number is a fixed point.
Solution: Let $\dot{x} = f(x) = 0$ be a constant function.

(b) Every integer is a fixed point and there are no others.
Solution: Let $\dot{x} = f(x) = \sin(\pi x)$.

(c) Precisely three fixed points, and all three are stable.
Solution: This is not possible since a fixed point has to have a negative slope at that point and you can't cross the x-axis from the upper left to the lower right 3 times without having crossed the x-axis once in between to return above the x-axis. This would force there to be an additional fixed point or for one of the 3 fixed points to be unstable (namely the one in between).

(d) There are no fixed points.
Solution: Let $\dot{x} = f(x) = \mathrm{e}^x$.

(e) There are precisely 100 fixed points.
Solution: Let $x_i \in \mathbb{R}$ for $i = 1, 2, ..., 100$ and $x_i \neq x_j$ for $i \neq j$. Therefore we can say

$$\dot{x} = f(x) = \prod_{i=1}^{100} (x - x_i).$$

$\square$

5: 2.2.13 (a,b,c,d)

(a) $m\dot{v} = mg - kv^2$ with initial condition $v(0) = 0$

Solution:
Let's begin by dividing by m and factoring out a g term.

$$\begin{aligned} m\dot{v} &= mg - kv^2 \\ \dot{v} &= g - \frac{kv^2}{m} \\ \dot{v} &= g\left(1 - \frac{kv^2}{gm}\right). \end{aligned}$$

Now rewriting $\dot{v}$ as the derivative of v with respect to t we have

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}t} &= g\left(1 - \frac{kv^2}{gm}\right) \\ \frac{1}{\left(1 - \frac{kv^2}{gm}\right)}\mathrm{d}v &= g\mathrm{d}t \\ \int \frac{1}{\left(1 - \sqrt{\frac{k}{gm}}v\right)\left(1 + \sqrt{\frac{k}{gm}}v\right)}\mathrm{d}v &= \int g\mathrm{d}t. \end{aligned}$$

Now, we can do partial fractions on the left and integrate both sides

$$\begin{aligned} \int \frac{1}{\left(1 - \sqrt{\frac{k}{gm}}v\right)\left(1 + \sqrt{\frac{k}{gm}}v\right)}\mathrm{d}v &= \int g\mathrm{d}t \\ \int \frac{1/2}{\left(1 - \sqrt{\frac{k}{gm}}v\right)}\mathrm{d}v + \int \frac{1/2}{\left(1 + \sqrt{\frac{k}{gm}}v\right)}\mathrm{d}v &= gt + C \\ -\frac{1}{2}\sqrt{\frac{gm}{k}}\log\left(1 - \sqrt{\frac{k}{gm}}v\right) + \frac{1}{2}\sqrt{\frac{gm}{k}}\log\left(1 + \sqrt{\frac{k}{gm}}v\right) &= gt + C \\ \frac{1}{2}\sqrt{\frac{gm}{k}}\left(\log\left(1 + \sqrt{k/(gm)}\,v\right) - \log\left(1 - \sqrt{k/(gm)}\,v\right)\right) &= gt + C \end{aligned}$$

Interesting ... now I need to input the initial condition

$$\begin{aligned} \frac{1}{2}\sqrt{\frac{gm}{k}}\left(\log\left(1 + \sqrt{k/(gm)}\,0\right) - \log\left(1 - \sqrt{k/(gm)}\,0\right)\right) &= g0 + C \\ \frac{1}{2}\sqrt{\frac{gm}{k}}\left(\log(1) - \log(1)\right) &= C \\ 0 &= C. \end{aligned}$$

Plugging this in and simplifying we have

$$\begin{aligned}\frac{1}{2}\sqrt{\frac{gm}{k}}\left(\log\left(1+\sqrt{k/(gm)}\,v\right)-\log\left(1-\sqrt{k/(gm)}\,v\right)\right) &= gt\\ \log\left(\frac{1+\sqrt{k/(gm)}\,v}{1-\sqrt{k/(gm)}\,v}\right) &= 2gt\sqrt{k/(gm)}.\end{aligned}$$

Now we can exponentiate and solve for v

$$\begin{aligned}\frac{1+\sqrt{k/(gm)}\,v}{1-\sqrt{k/(gm)}\,v} &= \mathrm{e}^{2gt\sqrt{k/(gm)}}\\ 1+\sqrt{k/(gm)}\,v &= \mathrm{e}^{2gt\sqrt{k/(gm)}}(1-\sqrt{k/(gm)}\,v)\\ 1+\sqrt{k/(gm)}\,v &= \mathrm{e}^{2gt\sqrt{k/(gm)}}-\sqrt{k/(gm)}\,v\,\mathrm{e}^{2gt\sqrt{k/(gm)}}\\ \sqrt{k/(gm)}\,v+\sqrt{k/(gm)}\,v\,\mathrm{e}^{2gt\sqrt{k/(gm)}} &= \mathrm{e}^{2gt\sqrt{k/(gm)}}-1\\ v &= \frac{\mathrm{e}^{2gt\sqrt{k/(gm)}}-1}{\sqrt{k/(gm)}+\sqrt{k/(gm)}\,\mathrm{e}^{2gt\sqrt{k/(gm)}}}.\end{aligned}$$

Some final simplifications gives us

$$\begin{aligned}v &= \frac{1}{\sqrt{k/(gm)}}\frac{\mathrm{e}^{2gt\sqrt{k/(gm)}}-1}{1+\mathrm{e}^{2gt\sqrt{k/(gm)}}}\\ v &= \sqrt{\frac{gm}{k}}\left(\frac{\mathrm{e}^{2gt\sqrt{k/(gm)}}-1}{\mathrm{e}^{2gt\sqrt{k/(gm)}}+1}\right).\end{aligned}$$

Therefore our final analytical solution is

$$v=\sqrt{\frac{gm}{k}}\left(\frac{\mathrm{e}^{2gt\sqrt{k/(gm)}}-1}{\mathrm{e}^{2gt\sqrt{k/(gm)}}+1}\right).$$

$\square$

(b) Determine the limit of $v(t)$ as $t\to\infty$. We will need to utilize L'Hôpital's rule since the numerator and the denominator go to infinity.

$$\begin{aligned}\lim_{t\to\infty}\sqrt{\frac{gm}{k}}\left(\frac{\mathrm{e}^{2gt\sqrt{k/(gm)}}-1}{\mathrm{e}^{2gt\sqrt{k/(gm)}}+1}\right) &= \lim_{t\to\infty}\sqrt{\frac{gm}{k}}\left(\frac{2gt\sqrt{k/(gm)}\,\mathrm{e}^{2gt\sqrt{k/(gm)}}}{2gt\sqrt{k/(gm)}\,\mathrm{e}^{2gt\sqrt{k/(gm)}}}\right)\\ &= \lim_{t\to\infty}\sqrt{\frac{gm}{k}}\\ &= \sqrt{\frac{gm}{k}}.\end{aligned}$$

Therefore, the terminal velocity is $\sqrt{\frac{gm}{k}}$.

$\square$

(c) Graphical argument.

Solution: Beginning with $m\dot{v} = mg - kv^2$ we have

$$\dot{v} = g - \frac{k}{m}v^2.$$

By inspection we can see that $\dot{v} = g - \frac{k}{m}v^2 = (\sqrt{\frac{gm}{k}} - v)(\sqrt{\frac{gm}{k}} + v) = 0$. Therefore, the fixed points are $v = \pm\sqrt{\frac{gm}{k}}$.

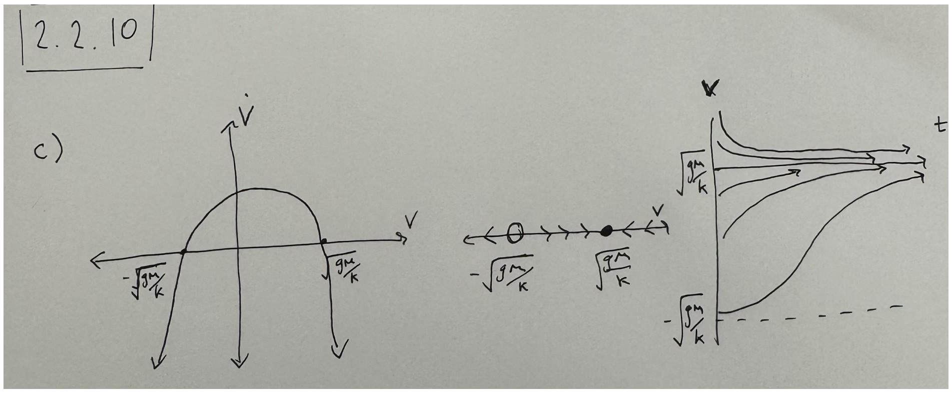

FIGURE 4. We have sketched the vector field on the real line, identified the location of all the fixed points, classified their stability, and sketched the $x(t)$ for different initial conditions (ignore the typo in the photograph this is for 2.2.13 c) not 2.2.10 c) as it is written)

$\square$

(d) Calculate the average velocity.

Solution:

$$\frac{31,400 - 2,100\text{ft}}{116\text{s}} = \frac{29,300\text{ft}}{116\text{s}} \approx 253\text{ft/s} = 172\text{mph}.$$

$\square$

6: 2.3.2 Analyze the DS governed by $\dot{x} = k_1 a x - k_{-1} x^2$

(a) Find all the fixed points and classify their stability
Solution:

Notice $\dot{x}$ has fixed points at $x^* = 0, \frac{k_1 a}{k_{-1}}$, since

$$\dot{x} = k_1 a x - k_{-1} x^2$$
$$\dot{x} = x(k_1 a - k_{-1} x).$$

Then $\dot{x} = x(k_1 a - k_{-1} x) = 0$ implies the above fixed points. We will analyze this graphically. Looking at Figure 5 makes it obvious that $x^* = 0$ is unstable and $x^* = \frac{k_1 a}{k_{-1}}$ is stable.

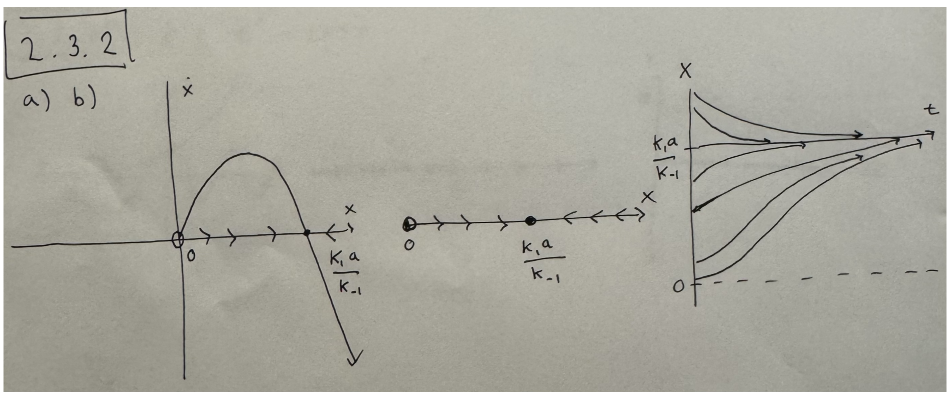

FIGURE 5. We have sketched the vector field on the real line, identified the location of all the fixed points, classified their stability, and sketched the $x(t)$ for different initial conditions.

□

(b) Sketch the graph of $x(t)$ for various initial values of x_0.

Solution: See Figure 5

□

7: 2.3.6 We are looking at the model governed by $\dot{x} = s(1-x)x^a - (1-s)x(1-x)^a$

(a) Show that this equation for $\dot{x}$ has three fixed points.

Solution:
We fist begin by factoring the expression on the rhs first to get

$$\dot{x} = x(1-x)\big(sx^{a-1} - (1-s)(1-x)^{a-1}\big).$$

Now we know the fixed points occur at $x^* = 0, 1$ and when the third factor is equal to 0. Let's algebraically solve for this third fixed point, by setting it equal to zero and solving for x.

$$\begin{aligned}
sx^{a-1} - (1-s)(1-x)^{a-1} &= 0 \\
sx^{a-1} - (1-x)^{a-1} + s(1-x)^{a-1} &= 0 \\
s\big(x^{a-1} + (1-x)^{a-1}\big) &= (1-x)^{a-1} \\
s\frac{\big(x^{a-1} + (1-x)^{a-1}\big)}{x^{a-1}} &= \frac{(1-x)^{a-1}}{x^{a-1}} \\
s\left(1 + \left(\frac{1-x}{x}\right)^{a-1}\right) &= \left(\frac{1-x}{x}\right)^{a-1} \\
s &= \left(\frac{1-x}{x}\right)^{a-1} - s\left(\frac{1-x}{x}\right)^{a-1} \\
s &= (1-s)\left(\frac{1-x}{x}\right)^{a-1} \\
\left(\frac{s}{1-s}\right)^{\frac{1}{a-1}} &= \frac{1-x}{x} \\
x + x\left(\frac{s}{1-s}\right)^{\frac{1}{a-1}} &= 1 \\
x\left(1 + \left(\frac{s}{1-s}\right)^{\frac{1}{a-1}}\right) &= 1 \\
x &= \frac{1}{1 + \left(\frac{s}{1-s}\right)^{\frac{1}{a-1}}}.
\end{aligned}$$

Therefore we have there are 3 distinct fixed points.

$\square$

(b) Show that for all $a > 1$, the fixed points at $x = 0$ and $x = 1$ are both stable.

Solution:
Let's do so by LSA. So we first need to take the derivative of f with respect to x from

$$\dot{x} = f(x) = s(1-x)x^a - (1-s)x(1-x)^a.$$

Then,

$$\begin{aligned}
f'(x) &= \frac{df}{dx}\big(s(1-x)x^a - (1-s)x(1-x)^a\big) \\
&= \frac{df}{dx}s(1-x)x^a - \frac{df}{dx}(1-s)x(1-x)^a \\
&= -sx^a + sa(1-x)x^{a-1} - (1-s)(1-x)^a + (1-s)x(1-x)^{a-1}.
\end{aligned}$$

Now we can evaluate this at $x^* = 0$ and $x^* = 1$ as follows,

$$\begin{aligned}
f'(x^*=0) &= -sx^a + sa(1-x)x^{a-1} - (1-s)(1-x)^a + (1-s)x(1-x)^{a-1}\big|_0 \\
&= -s0^a + sa(1-0)0^{a-1} - (1-s)(1-0)^a + (1-s)0(1-0)^{a-1} \\
&= -(1-s)1^a \\
&= s-1 \leq 0.
\end{aligned}$$

Therefore, $f'(x^*=0) \leq 0$ since s could be equal to 1. Assuming $s < 1$ we have that the fixed point $x^* = 0$ is stable by LSA.

$$\begin{aligned}
f'(x^*=1) &= -sx^a + sa(1-x)x^{a-1} - (1-s)(1-x)^a + (1-s)x(1-x)^{a-1}\big|_0 \\
&= -s1^a + sa(1-1)1^{a-1} - (1-s)(1-1)^a + (1-s)1(1-1)^{a-1} \\
&= -s \leq 0.
\end{aligned}$$

Therefore, $f'(x^*=1) \leq 0$ since s could be equal to 0. Assuming $s > 0$ we have that the fixed point $x^* = 1$ is stable by LSA.

□

(c) Show that the third fixed point, $0 < x^* < 1$, is unstable.

Solution: We know there are three fixed points by part a), by 2.2.10 we established that a DS with precisely three fixed points can't all be stable, finally by part b) we know that $x^* = 0$ and 1 are stable then the fixed point which lies within the interval $0 < x^* < 1$ must be unstable.

□

8: 2.4.7 Use LSA to classify the fixed points of the system, $\dot{x} = ax - x^3$. If linear stability analysis fails because $f'(x^*) = 0$, use a graphical argument to decide the stability.

(a) Where $a > 0$

Solution: The fixed points are going to be given by $x^* = 0, \pm\sqrt{a}$ which are all real valued fixed points since $a > 0$. Now to use LSA we need to calculate the derivative of f with respect to x, therefore we have

$$f'(x) = a - 3x^2.$$

Then

$$f'(x^* = 0) = a > 0$$

thus $x^* = 0$ is unstable. Lastly

$$f'(x^* = \pm\sqrt{a}) = a - 3a = -2a < 0$$

thus both $\pm\sqrt{a}$ are stable.

□

(b) Where $a = 0$

Solution: In this case the system reduces to $\dot{x} = -x^3$ which only has one fixed point at $x^* = 0$. The derivative becomes $f'(x) = -3x^2$ which evidently gives us $f'(x^* = 0) = 0$ therefore LSA is inconclusive and we will analyze graphically. By the analysis in Figure 6 we can see that $x^* = 0$ is a stable fixed point.

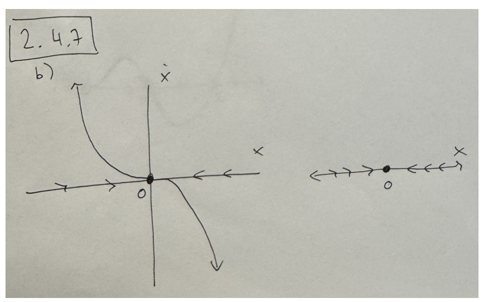

FIGURE 6. We have sketched the vector field on the real line, identified the location of all the fixed points, and classified their stability.

(c) Where $a < 0$

Solution: Again the fixed points are at $x^* = 0, \pm\sqrt{a}$. However, since $a < 0$ and we are not currently handling non-real fixed points, we conclude there is just one fixed point at $x^* = 0$.

Furthermore, performing LSA using the $f'(x) = a - 3x^2$, gives us

$$f'(x^* = 0) = a < 0,$$

thus $x^* = 0$ is a stable fixed point in this scenario.

□