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HMS 581: Infectious Disease Modeling

QUIZ

Solutions to the problems which are described in the entrance quiz to *HMS 581*.

Question 1 (Linear Algebra):

(a) Dominant Eigenvector and corresponding eigenvalue of

$$A = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & -0.125 \end{bmatrix}$$

Solution:
Solving using the characteristic equation gives us the following

$$\begin{aligned}
(0.5-\lambda)(-0.125-\lambda) - 0.5(0.25) &= 0 \\
\lambda^2 - 0.5\lambda + 0.125\lambda + 0.5(-0.125) - 0.5(0.25) &= 0 \\
\lambda^2 - 0.375\lambda - 0.0625 - 0.125 &= 0 \\
\lambda^2 - 0.375\lambda - 0.1875 &= 0 \\
\implies \lambda = \frac{0.375 \pm \sqrt{(-0.375)^2 - 4(-0.1875)}}{2}&.
\end{aligned}$$

I have verified this with the results from calculating it with NumPy in Python. The dominant eigenvector is the eigenvector corresponding to the eigenvalue which is largest in absolute value. In our case the larger eigenvalue in absolute value is $\lambda = 0.65936465$ with the eigenvector

$$v_1 = \begin{bmatrix} 0.84324302 \\ 0.53753252 \end{bmatrix}.$$

(b) If $x = [1 \;\; 1]^T$, let $y = Ax$. Further let $|\cdot|_2$ denote the l^2 norm or Euclidean distance. Without doing the calculation which will be larger $|x|_2$ or $|y|_2$? Why?

Solution:
I believe $|x|_2$ will be larger. If my linear algebra memory is serving me correctly the dominant eigenvalue being less than 1 would imply that the linear transformation which the matrix A represents is compressing input vectors that it is operating on. Therefore, the resultant $|y|_2$ would be smaller since A is compressing input vectors.

(c) Let $\hat{x}$ and $\hat{y}$ denote the normalized vectors. We calculated $|\hat{x} - v_1|$ and $|\hat{y} - v_1|$.
Solution:
I might have been able to guess that the second quantity is smaller since the matrix A is compressing values and they are getting closer to the dominant eigenvalue.

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Question 2 (ODE Question):

(a) Solve the following differential equation:

$$\frac{dy}{dx} = ry, \qquad \text{with } y(0) = y_0.$$

Solution:
We begin by using separation of variables

$$\begin{aligned} \frac{dy}{dx} &= ry \\ \frac{1}{y}dy &= rdx \\ \int \frac{1}{y}dy &= \int rdx \\ \log y &= rx + C \\ y(x) &= \mathrm{e}^{rx}\,\mathrm{e}^{C}\,. \end{aligned}$$

Now let's utilize our initial condition $y(0) = y_0$ to determine the constant e^C.

$$\begin{aligned} y(0) &= \mathrm{e}^{r0}\,\mathrm{e}^{C} \\ y_0 &= \mathrm{e}^{0}\,\mathrm{e}^{C} \\ y_0 &= \mathrm{e}^{C}\,. \end{aligned}$$

Therefore our solution given this initial condition is

$$y(x) = y_0\,\mathrm{e}^{rx}\,.$$

(b) Solve the following differential equation:

$$\frac{dy}{dx} = ry\left(1 - \frac{y}{K}\right) \qquad \text{with } y(0) = y_0$$

Solution:
Similarly let's use separation of variables

$$\begin{aligned} \frac{dy}{dx} &= ry\left(1 - \frac{y}{K}\right) \\ \frac{1}{y - \frac{y^2}{K}}dy &= rdx \\ \int \frac{1}{y - \frac{y^2}{K}}dy &= \int rdx \\ \int -\frac{1}{y^2}\frac{1}{\frac{1}{K} - \frac{1}{y}}dy &= \int rdx \end{aligned}$$

Consider using a u sub at this point. Perhaps let $u = \frac{1}{K} - \frac{1}{y}$ then $du = \frac{1}{y^2}$. plugging this in we now have

$$\begin{aligned}
\int -\frac{1}{u} du &= \int r dx \\
-\log u &= rx + C \\
\log\left(\frac{1}{K} - \frac{1}{y}\right) &= -rx + C \\
\frac{1}{K} - \frac{1}{y} &= \mathrm{e}^{-rx} C \\
-\frac{1}{y} &= \mathrm{e}^{-rx} C + \frac{1}{K} \\
y(x) &= \frac{1}{-\mathrm{e}^{-rx} C - \frac{1}{K}}.
\end{aligned}$$

Our initial condition gives us

$$\begin{aligned}
y(0) &= \frac{1}{-\mathrm{e}^{-r0} C - \frac{1}{K}} \\
y_0 &= \frac{1}{-C - \frac{1}{K}} \\
-\frac{1}{y_0} - \frac{1}{K} &= C.
\end{aligned}$$

Finally, we have

$$y(x) = \frac{1}{-\mathrm{e}^{-rx}\left(-\frac{1}{y_0} - \frac{1}{K}\right) - \frac{1}{K}}.$$

(c) For the second equation, assuming $r > 0$ and $K > 0$, identify all the equilibrium and assess their stability.

Solution:
Let $\dot{y} = \frac{dy}{dx}$, the equilibrium occur where $\dot{y} = 0$ that is

$$\begin{aligned}
\frac{dy}{dx} &= ry\left(1 - \frac{y}{K}\right) \\
\dot{y} &= ry\left(1 - \frac{y}{K}\right) \\
0 &= ry\left(1 - \frac{y}{K}\right)
\end{aligned}$$

Which occurs when $y^* = 0, K$. I will use Linear Stability Analysis (LSA) to determine the stability of these fixed points (or equilibrium). Let's additionally let $f(y) = ry\,(1 - y/K)$. Then using LSA we want to determine if $f'(y^*)$ is greater than or less than 0 for each fixed point, y^*. Since we have $f'(y) = r - 2ry/K$, then

$$\begin{aligned}
f'(0) = r > 0 &\implies \text{the fixed point at 0 is unstable} \\
f'(K) = -r < 0 &\implies \text{the fixed point at K is stable}
\end{aligned}$$

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Question 3 (Coding Question...with probability):

Question 4 (Epidemiology question):