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AMATH 502

HOMEWORK 2

Exercises come from *Nonlinear Dynamics and Chaos by Steven H. Strogatz*

1: 2.6.1 Explain this paradox: a simple harmonic oscillator $m\ddot{x} = -kx$ is a system which oscillates in one dimension (along the x-axis). But the text says one-dimensional systems can't oscillate.

Solution:
Not a formal method of finding a solution but I can see that if $x = \sin\left(\frac{k}{m}t\right)$ then

$$\begin{aligned}\dot{x} &= \frac{k}{m}\cos\left(\frac{k}{m}t\right)\\ \ddot{x} &= -\frac{k^2}{m^2}\sin\left(\frac{k}{m}t\right)\\ \ddot{x} &= -\frac{k^2}{m^2}x\\ m^2\ddot{x} &= -k^2x\end{aligned}$$

Therefore to adjust the constants k and m so they agree with the original solution we need to actually have $x = \sin\left(\sqrt{k/m}\,t\right)$. Furthermore, we could similarly have arrived at a similar solution with $x = \cos\left(\sqrt{k/m}\,t\right)$. Therefore, we have

$$x = c_1\sin\left(\sqrt{k/m}\,t\right) + c_2\cos\left(\sqrt{k/m}\,t\right).$$

Now setting all this aside, we have to look more closely at the original system. Since this is a second derivative in x, we are actually implicitly depending on the position and velocity the value of x and it's first derivative. Therefore this is not actually a first order system and thus is not a contradiction to the statements in the text about first order systems. $\square$

2: 3.1.1 $\dot{x} = 1 + rx + x^2$

Sketch all the qualitatively different vector fields that occur as r is varied. Show that a saddle-node bifurcation occurs at a critical value of r, to be determined. Finally sketch the bifurcation diagram of fixed points x^* vs. r.

Solution:

Well we know that we have fixed points at

$$x = \frac{-r \pm \sqrt{r^2 - 4}}{2}$$

which implies there are no fixed points when $|r| < 2$, a single fixed point when $r = \pm 2$, and two fixed points when $|r| > 2$.

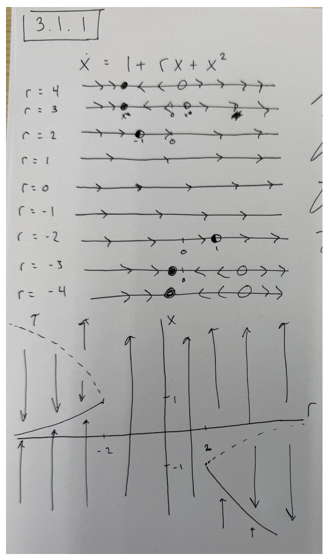

FIGURE 1. We have sketched the vector field of x for various values of r. Additionally we have sketched the bifurcation diagram.

□

3: 3.1.5 (Unusual bifurcations) In discussing the normal form of the saddle-node bifurcation, we mentioned the assumption that $a = \partial f/\partial r|_{(x^*,r_c)} \neq 0$. To see what can happen if $a = \partial f/\partial r|_{(x^*,r_c)} = 0$, sketch the vector fields for the following examples, and then plot the fixed points as a function of r.

(a) $\dot{x} = r^2 - x^2$ *Solution:*
We know that the rhs is equal to 0 when $x^* = \pm r$. So we plot a series of the vector fields for various values of r. See Figure 2.

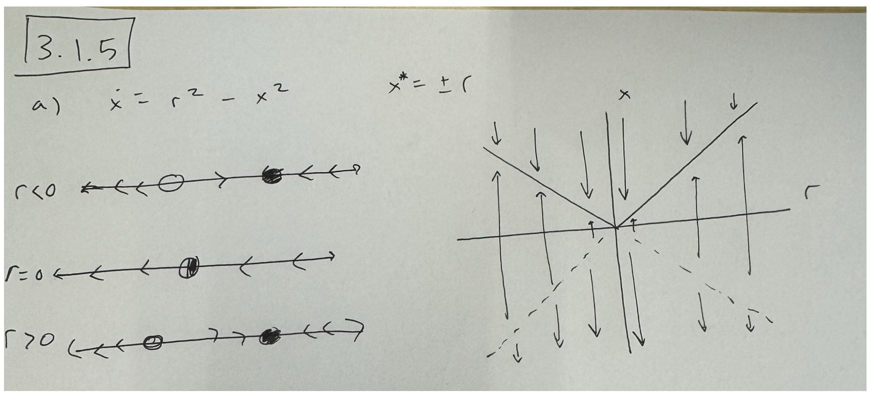

FIGURE 2. We have sketched the vector field of x for various values of r. Additionally we have sketched the bifurcation diagram.

(b) $\dot{x} = r^2 + x^2$
Solution: In this instance there is only one fixed point when $r = 0$ and no fixed points for any other value of r. See Figure 3.

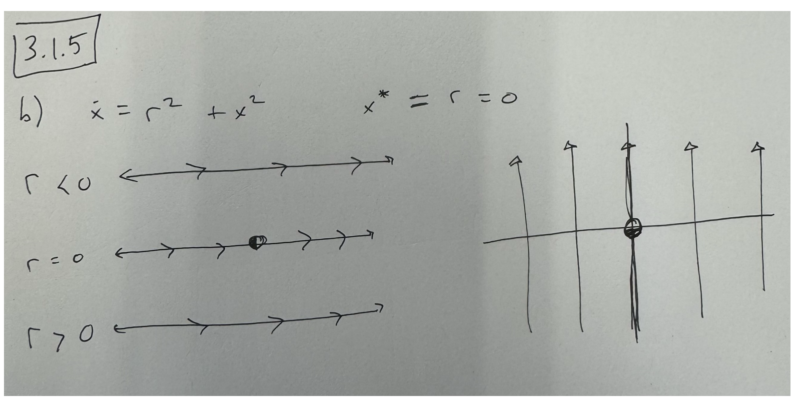

FIGURE 3. We have sketched the vector field of x for various values of r. Additionally we have sketched the bifurcation diagram.

$\square$

4: 3.2.3 $\dot{x} = x - rx(1-x)$

Sketch all the qualitatively different vector fields that occur as r is varied. Show that transcritical bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x^* vs. r.

Solution:

Notice we have

$$\dot{x} = x - rx(1-x)$$
$$\dot{x} = (1-r)x(1-x)$$

Therefore we have fixed points at $x^* = 0, 1$ independent of values of r. However, if $r = 1$, then $\dot{x} = 0$ trivially

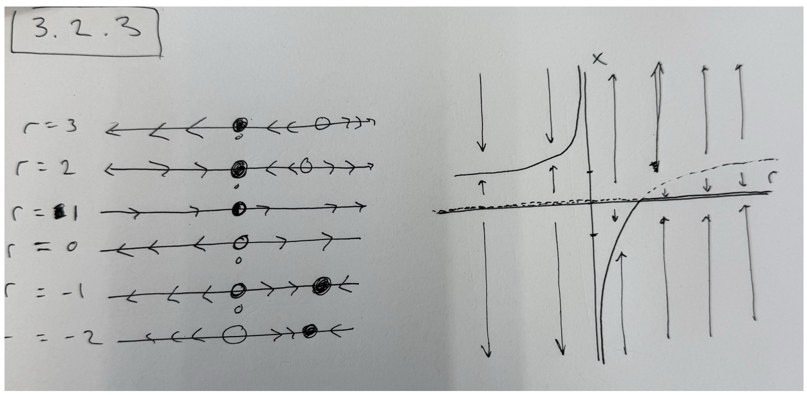

FIGURE 4. We have sketched the vector field of x for various values of r. Additionally we have sketched the bifurcation diagram.

□

5: 3.4.11 (An interesting bifurcation diagram) Consider $\dot{x} = rx - \sin x$

(a) For the case $r = 0$, find and classify all the fixed points, and sketch the vector field.

Solution:
When $r = 0$, then we have $\dot{x} = -\sin x$. Therefore we have fixed points at $x^* = k\pi$ for $k \in \mathbb{Z}$. See Figure 5 below.

$\square$

(b) Show that when $r > 1$, there is only one fixed point. What kind of fixed point is it?

Solution:
There is only one fixed point at $x^* = 0$ and it is unstable. See Figure 5 below.

$\square$

(c) As r decreases from ∞ to 0, classify *all* the bifurcations that occur.

Solution:
As r decreases from ∞ to 0 there is only one fixed point until $r < 1$ then rapidly more fixed points and bifurcations occur. Additionally when $r < 1$ the stability of the fixed point at 0 changes and becomes stable. They are added in pairs on both sides of 0 each pair alternating from stable and unstable.

$\square$

(d) For $0 < r << 1$, find an approximate formula for values of r at which bifurcations occur.

Solution:
We begin by looking at when $rx - \sin x = 0$ as well as when $r - \cos x = 0$. These in turn give us $r = \cos x$ and $r = \frac{\sin x}{x}$. Furthermore, if r is really small and nearly 0, then we basically have fixed points when $\cos x = 0$ in other words at $x^* = \frac{\pi}{2} + 2\pi k$ for $k \in \mathbb{Z}$. Combining this with the information we have from before we then have

$$r = \frac{\sin\left(\frac{\pi}{2} + 2\pi k\right)}{\frac{\pi}{2} + 2\pi k} = \frac{1}{\frac{\pi}{2} + 2\pi k}.$$

$\square$

(e) Now classify all the bifurcations that occur as r decreases from 0 to $-\infty$.

Solution:
Similar to going from ∞ to 0 we have only one stable fixed point at $x^* = 0$ as r goes from $-\infty$ to some small negative value approximately close to $r \approx -.224$ then as r get's closer and closer to 0 additional fixed points are added to the system. They are added in a similar manner as in part c) in pairs and alternating stability as you go out to negative and positive infinity.

$\square$

(f) Plot the Bifurcation diagram for $-\infty < r < \infty$, and indicate the stability of the various branches of fixed points.

Solution:
See Figure 5, note the dotted and solid lines indicating the stability of the branches of fixed points.

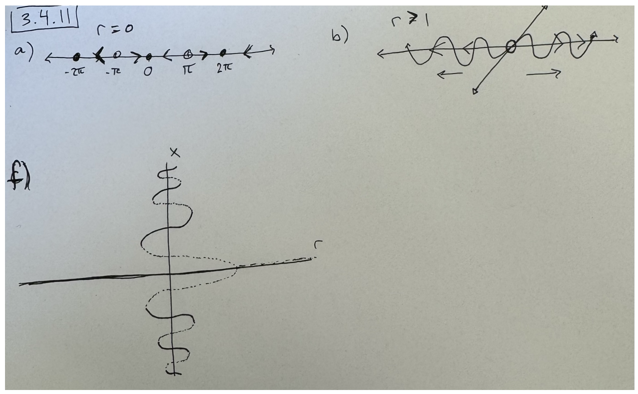

FIGURE 5. We have sketched the vector field of x for various values of r. Additionally we have sketched the bifurcation diagram.

$\square$

6: 3.5.8 (Nondimensionalizing the subcritical pitchfork) The first-order system $\dot{u} = au + bu^3 - cu^5$, where $b, c > 0$, has a subcritical pitchfork bifurcation at $a = 0$. Show that this equation can be rewritten as

$$\frac{dx}{d\tau} = rx + x^3 - x^5$$

where $x = u/U$, $\tau = t/T$, and U, T, and r are to be determined in terms of a, b, and c.

Solution:
We are using the substitutions provided so we can also note that they result in $Udx = du$ and $Td\tau = dt$. Then we have

$$\begin{aligned}
\dot{u} &= au + bu^3 - cu^5 \\
\frac{du}{dt} &= au + bu^3 - cu^5 \\
\frac{Udx}{Td\tau} &= a(Ux) + b(Ux)^3 - c(Ux)^5 \\
\frac{dx}{d\tau} &= Tax + TbU^2x^3 - TcU^4x^5. \qquad (1)
\end{aligned}$$

In order for this to give us the final equation we are looking for we need the following to all hold

$$Ta = r, \quad TbU^2 = 1, \quad TcU^4 = 1.$$

Let's solve these so we have U, T, and r in terms of a, b, and c. Beginning with the second equation we have $T = 1/bU^2$ and plugging that into the third equation we get $\frac{1}{bU^2}cU^4 = 1$ then simplifying gives us

$$\begin{aligned}
\frac{c}{b}U^2 &= 1 \\
U^2 &= \frac{b}{c} \\
U &= \sqrt{\frac{b}{c}}.
\end{aligned}$$

Thus we also have

$$T = \frac{1}{bU^2} = \frac{1}{b\frac{b}{c}} = \frac{c}{b^2}.$$

Hence

$$T = \frac{c}{b^2}, \quad U = \sqrt{\frac{b}{c}}, \quad r = \frac{ac}{b^2}.$$

Finally we can combine this with (1) to see

$$\begin{aligned}
\dot{u} &= au + bu^3 - cu^5 \\
\frac{dx}{d\tau} &= Tax + TbU^2x^3 - TcU^4x^5 \\
\frac{dx}{d\tau} &= \frac{ac}{b^2}ax + \frac{cb}{b^2}\sqrt{\frac{b}{c}}^2 x^3 - \frac{c^2}{b^2}\sqrt{\frac{b}{c}}^4 x^5 \\
\frac{dx}{d\tau} &= rx + x^3 - x^5.
\end{aligned}$$

$\square$

7: 3.7.3 (A model of a fishery) The equation

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - H$$

provides an extremely simple model of a fishery. In the absence of fishing, the population is assumed to grow logistically. The effects of fishing are modeled by the term $-H$, which says that fish are caught or "harvested" at a constant rate $H > 0$, independent of their population N. (This assumes that the fisherman aren't worried about fishing the population dry–they simply catch the same number of fish every day.)

(a) Show that the system can be rewritten in dimensionless form as

$$\frac{dx}{d\tau} = x(1 - x) - h$$

for suitably defined dimensionless quantities x, τ, and h

Solution:
Let's use the substitutions $Mx = N$ and $T\tau = t$ and their derivative forms $Mdx = dN$ and $Td\tau = dt$. Now we can use these as follows

$$\begin{aligned}
\dot{N} &= rN\left(1 - \frac{N}{K}\right) - H \\
\frac{dN}{dt} &= rN\left(1 - \frac{N}{K}\right) - H \\
\frac{Mdx}{Td\tau} &= rMx\left(1 - \frac{Mx}{K}\right) - H \\
\frac{dx}{d\tau} &= Trx\left(1 - \frac{Mx}{K}\right) - \frac{TH}{M}.
\end{aligned}$$

Then we need the following to hold

$$Tr = 1, \quad \frac{M}{K} = 1, \quad \frac{TH}{M} = h.$$

This implies we have to choose $M = K$, $T = \frac{1}{r}$, and $h = \frac{H}{Kr}$. Therefore,

$$\begin{aligned}
\frac{dx}{d\tau} &= Trx\left(1 - \frac{Mx}{K}\right) - \frac{TH}{M} \\
\frac{dx}{d\tau} &= \frac{1}{r}rx\left(1 - \frac{Kx}{K}\right) - \frac{H}{Kr} \\
\frac{dx}{d\tau} &= x\left(1 - x\right) - h
\end{aligned}$$

$\square$

(b) Plot the vector field for different values of h

Solution:
Notice in this form we have fixed points at

$$x^* = \frac{-1 \pm \sqrt{1-4h}}{-2} = \frac{1 \mp \sqrt{1-4h}}{2}.$$

In Figure 6 we can see the different vector fields for various values of h

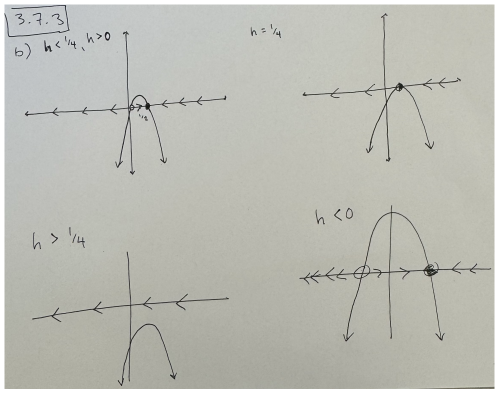

FIGURE 6. We have sketched the vector field of x for various values of h.

□

(c) Show that a bifurcation occurs at a certain value of h_c, and classify this bifurcation.

Solution: The bifurcation occurs at $h_c = \frac{1}{4}$. And it is a saddle node bifurcation. For further evidence, please refer to Figure 6. □

(d) Discuss the long-term behavior of the fish population for $h < h_c$ and $h > h_c$. Give the biological interpretation in each case.

Solution: For $h > h_c$ we would have overfishing or fishing at a rate faster than they can be born or replenished and thus would result in an extinction. Furthermore, if $h < h_c$ the fishing rate and birth rates would arrive to an equilibrium of sorts and reach a steady state solution at $x^* = \frac{1+\sqrt{1-4h}}{2}$.

□