

Hunter Lybbert
Student ID: 2426454
01-23-25
AMATH 502

HOMEWORK 2

Exercises come from *Nonlinear Dynamics and Chaos by Steven H. Strogatz*

- 1:** 2.6.1 Explain this paradox: a simple harmonic oscillator $m\ddot{x} = -kx$ is a system which oscillates in one dimension (along the x -axis). But the text says one-dimensional systems can't oscillate.

Solution:

Not a formal method of finding a solution but I can see that if $x = \sin(\frac{k}{m}t)$ then

$$\begin{aligned}\dot{x} &= \frac{k}{m} \cos\left(\frac{k}{m}t\right) \\ \ddot{x} &= -\frac{k^2}{m^2} \sin\left(\frac{k}{m}t\right) \\ \ddot{x} &= -\frac{k^2}{m^2}x \\ m^2\ddot{x} &= -k^2x\end{aligned}$$

Therefore to adjust the constants k and m so they agree with the original solution we need to actually have $x = \sin(\sqrt{k/m}t)$. Furthermore, we could similarly have arrived at a similar solution with $x = \cos(\sqrt{k/m}t)$. Therefore, we have

$$x = c_1 \sin(\sqrt{k/m}t) + c_2 \cos(\sqrt{k/m}t).$$

Now setting all this aside, we have to look more closely at the original system. Since this is a second derivative in x , we are actually implicitly depending on the position and velocity the value of x and its first derivative. Therefore this is not actually a first order system and thus is not a contradiction to the statements in the text about first order systems. \square

2: 3.1.1 $\dot{x} = 1 + rx + x^2$

Sketch all the qualitatively different vector fields that occur as r is varied. Show that a saddle-node bifurcation occurs at a critical value of r , to be determined. Finally sketch the bifurcation diagram of fixed points x^* vs. r .

Solution:

Well we know that we have fixed points at

$$x = \frac{-r \pm \sqrt{r^2 - 4}}{2}$$

which implies there are no fixed points when $|r| < 2$, a single fixed point when $r = \pm 2$, and two fixed points when $|r| > 2$.

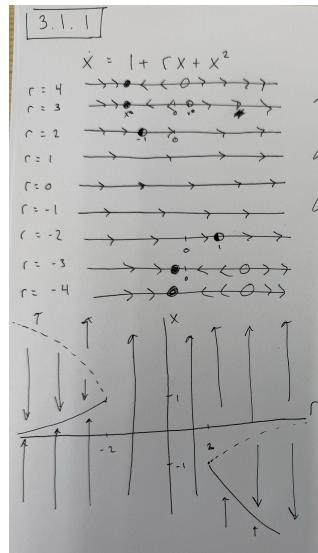


FIGURE 1. We have sketched the vector field of x for various values of r . Additionally we have sketched the bifurcation diagram.

□

3: 3.1.5 (Unusual bifurcations) In discussing the normal form of the saddle-node bifurcation, we mentioned the assumption that $a = \partial f / \partial r|_{(x^*, r_c)} \neq 0$. To see what can happen if $a = \partial f / \partial r|_{(x^*, r_c)} = 0$, sketch the vector fields for the following examples, and then plot the fixed points as a function of r .

(a) $\dot{x} = r^2 - x^2$ *Solution:*

We know that the rhs is equal to 0 when $x^* = \pm r$. So we plot a series of the vector fields for various values of r . See Figure ??.

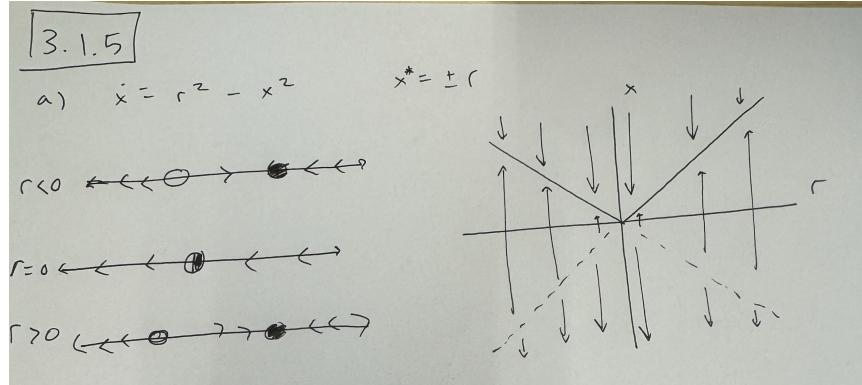


FIGURE 2. We have sketched the vector field of x for various values of r . Additionally we have sketched the bifurcation diagram.

(b) $\dot{x} = r^2 + x^2$

Solution: In this instance there is only one fixed point when $r = 0$ and no fixed points for any other value of r . See Figure ??.

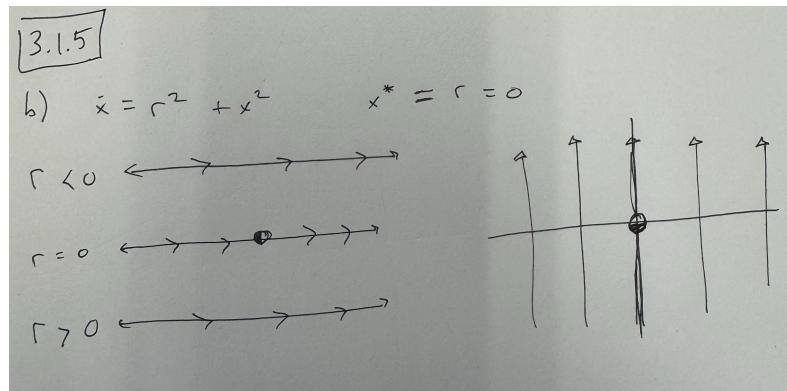


FIGURE 3. We have sketched the vector field of x for various values of r . Additionally we have sketched the bifurcation diagram.

□

4: 3.2.3 $\dot{x} = x - rx(1 - x)$

Sketch all the qualitatively different vector fields that occur as r is varied. Show that transcritical bifurcation occurs at a critical value of r , to be determined. Finally, sketch the bifurcation diagram of fixed points x^* vs. r .

Solution:

Notice we have

$$\begin{aligned}\dot{x} &= x - rx(1 - x) \\ \dot{x} &= (1 - r)x(1 - x)\end{aligned}$$

Therefore we have fixed points at $x^* = 0, 1$ independent of values of r . However, if $r = 1$, then $\dot{x} = 0$ trivially

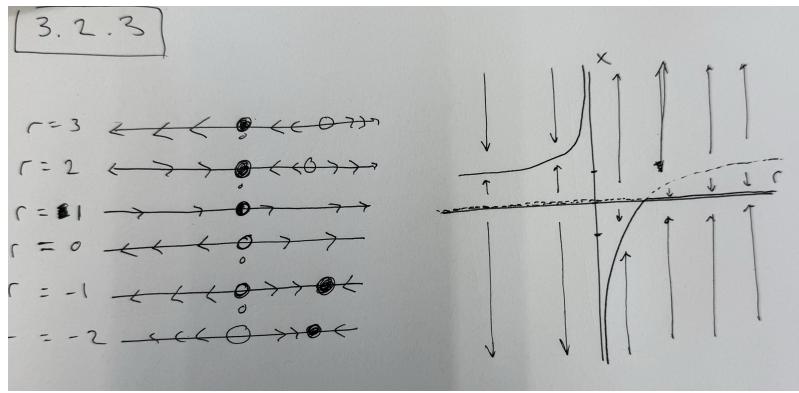


FIGURE 4. We have sketched the vector field of x for various values of r . Additionally we have sketched the bifurcation diagram.

□

5: 3.4.11 (An interesting bifurcation diagram) Consider $\dot{x} = rx - \sin x$

- (a) For the case $r = 0$, find and classify all the fixed points, and sketch the vector field.

Solution:

When $r = 0$, then we have $\dot{x} = -\sin x$. Therefore we have fixed points at $x^* = k\pi$ for $k \in \mathbb{Z}$. See Figure ?? below.

□

- (b) Show that when $r > 1$, there is only one fixed point. What kind of fixed point is it?

Solution:

There is only one fixed point at $x^* = 0$ and it is unstable. See Figure ?? below.

□

- (c) As r decreases from ∞ to 0, classify *all* the bifurcations that occur.

Solution:

As r decreases from ∞ to 0 there is only one fixed point until $r < 1$ then rapidly more fixed points and bifurcations occur. Additionally when $r < 1$ the stability of the fixed point at 0 changes and becomes stable. They are added in pairs on both sides of 0 each pair alternating from stable and unstable.

□

- (d) For $0 < r \ll 1$, find an approximate formula for values of r at which bifurcations occur.

Solution:

We begin by looking at when $rx - \sin x = 0$ as well as when $r - \cos x = 0$. These in turn give us $r = \cos x$ and $r = \frac{\sin x}{x}$. Furthermore, if r is really small and nearly 0, then we basically have fixed points when $\cos x = 0$ in other words at $x^* = \frac{\pi}{2} + 2\pi k$ for $k \in \mathbb{Z}$. Combining this with the information we have from before we then have

$$r = \frac{\sin\left(\frac{\pi}{2} + 2\pi k\right)}{\frac{\pi}{2} + 2\pi k} = \frac{1}{\frac{\pi}{2} + 2\pi k}.$$

□

- (e) Now classify all the bifurcations that occur as r decreases from 0 to $-\infty$.

Solution:

Similar to going from ∞ to 0 we have only one stable fixed point at $x^* = 0$ as r goes from $-\infty$ to some small negative value approximately close to $r \approx -.224$ then as r gets closer and closer to 0 additional fixed points are added to the system. They are added in a similar manner as in part c) in pairs and alternating stability as you go out to negative and positive infinity.

□

- (f) Plot the Bifurcation diagram for $-\infty < r < \infty$, and indicate the stability of the various branches of fixed points.

Solution:

See Figure ??, note the dotted and solid lines indicating the stability of the branches of fixed points.

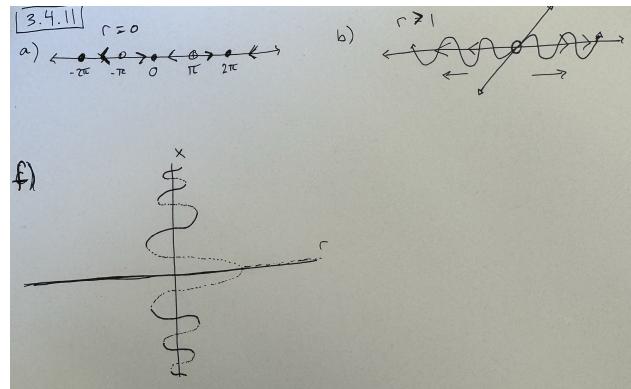


FIGURE 5. We have sketched the vector field of x for various values of r . Additionally we have sketched the bifurcation diagram.

□

- 6:** 3.5.8 (Nondimensionalizing the subcritical pitchfork) The first-order system $\dot{u} = au + bu^3 - cu^5$, where $b, c > 0$, has a subcritical pitchfork bifurcation at $a = 0$. Show that this equation can be rewritten as

$$\frac{dx}{d\tau} = rx + x^3 - x^5$$

where $x = u/U$, $\tau = t/T$, and U , T , and r are to be determined in terms of a , b , and c .

Solution:

We are using the substitutions provided so we can also note that they result in $Udx = du$ and $Td\tau = dt$. Then we have

$$(1) \quad \begin{aligned} \dot{u} &= au + bu^3 - cu^5 \\ \frac{du}{dt} &= au + bu^3 - cu^5 \\ \frac{Udx}{Td\tau} &= a(Ux) + b(Ux)^3 - c(Ux)^5 \\ \frac{dx}{d\tau} &= Tax + TbU^2x^3 - TcU^4x^5. \end{aligned}$$

In order for this to give us the final equation we are looking for we need the following to all hold

$$Ta = r, \quad TbU^2 = 1, \quad TcU^4 = 1.$$

Let's solve these so we have U , T , and r in terms of a , b , and c . Beginning with the second equation we have $T = 1/bU^2$ and plugging that into the third equation we get $\frac{1}{bU^2}cU^4 = 1$ then simplifying gives us

$$\begin{aligned} \frac{c}{b}U^2 &= 1 \\ U^2 &= \frac{b}{c} \\ U &= \sqrt{\frac{b}{c}}. \end{aligned}$$

Thus we also have

$$T = \frac{1}{bU^2} = \frac{1}{b\frac{b}{c}} = \frac{c}{b^2}.$$

Hence

$$T = \frac{c}{b^2}, \quad U = \sqrt{\frac{b}{c}}, \quad r = \frac{ac}{b^2}.$$

Finally we can combine this with (??) to see

$$\begin{aligned} \dot{u} &= au + bu^3 - cu^5 \\ \frac{dx}{d\tau} &= Tax + TbU^2x^3 - TcU^4x^5 \\ \frac{dx}{d\tau} &= \frac{ac}{b^2}ax + \frac{cb}{b^2}\sqrt{\frac{b}{c}}x^3 - \frac{c^2}{b^2}\sqrt{\frac{b}{c}}x^5 \\ \frac{dx}{d\tau} &= rx + x^3 - x^5. \end{aligned}$$

□

7: 3.7.3 (A model of a fishery) The equation

$$\dot{N} = rN \left(1 - \frac{N}{K}\right) - H$$

provides an extremely simple model of a fishery. In the absence of fishing, the population is assumed to grow logistically. The effects of fishing are modeled by the term $-H$, which says that fish are caught or “harvested” at a constant rate $H > 0$, independent of their population N . (This assumes that the fisherman aren’t worried about fishing the population dry—they simply catch the same number of fish every day.)

- (a) Show that the system can be rewritten in dimensionless form as

$$\frac{dx}{d\tau} = x(1 - x) - h$$

for suitably defined dimensionless quantities x , τ , and h

Solution:

Let’s use the substitutions $Mx = N$ and $T\tau = t$ and their derivative forms $Mdx = dN$ and $Td\tau = dt$. Now we can use these as follows

$$\begin{aligned}\dot{N} &= rN \left(1 - \frac{N}{K}\right) - H \\ \frac{dN}{dt} &= rN \left(1 - \frac{N}{K}\right) - H \\ \frac{Mdx}{Td\tau} &= rMx \left(1 - \frac{Mx}{K}\right) - H \\ \frac{dx}{d\tau} &= Trx \left(1 - \frac{Mx}{K}\right) - \frac{TH}{M}.\end{aligned}$$

Then we need the following to hold

$$Tr = 1, \quad \frac{M}{K} = 1, \quad \frac{TH}{M} = h.$$

This implies we have to choose $M = K$, $T = \frac{1}{r}$, and $h = \frac{H}{Kr}$. Therefore,

$$\begin{aligned}\frac{dx}{d\tau} &= Trx \left(1 - \frac{Mx}{K}\right) - \frac{TH}{M} \\ \frac{dx}{d\tau} &= \frac{1}{r}rx \left(1 - \frac{Kx}{K}\right) - \frac{H}{Kr} \\ \frac{dx}{d\tau} &= x(1 - x) - h\end{aligned}$$

□

- (b) Plot the vector field for different values of h

Solution:

Notice in this form we have fixed points at

$$x^* = \frac{-1 \pm \sqrt{1 - 4h}}{-2} = \frac{1 \mp \sqrt{1 - 4h}}{2}.$$

In Figure ?? we can see the different vector fields for various values of h

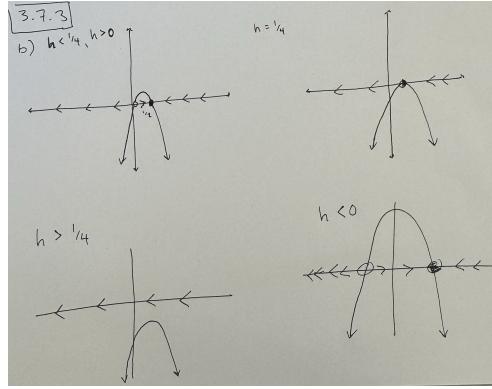


FIGURE 6. We have sketched the vector field of x for various values of r . Additionally we have sketched the bifurcation diagram.

□

- (c) Show that a bifurcation occurs at a certain value of h_c , and classify this bifurcation.

Solution: The bifurcation occurs at $h_c = \frac{1}{4}$. And it is a saddle node bifurcation. For further evidence, please refer to Figure ??.

□

- (d) Discuss the long-term behavior of the fish population for $h < h_c$ and $h > h_c$. Give the biological interpretation in each case.

Solution: For $h > h_c$ we would have overfishing or fishing at a rate faster than they can be born or replenished and thus would result in an extinction. Furthermore, if $h < h_c$ the fishing rate and birth rates would arrive to an equilibrium of sorts and reach a steady state solution at $x^* = \frac{1+\sqrt{1-4h}}{2}$.

□