Hunter Lybbert
Student ID: 2426454
04-09-25
AMATH 503

HOMEWORK 1

Exercises come from *Introduction to Partial Differential Equations by Peter J. Olver* as well as supplemented by instructor provided exercises.

1: Olver 1.1
Solution:
(a) $\frac{du}{dx} + xu = 1$: Ordinary equilibrium differential equation of the first order.
(b) $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = x$: Partial dynamic differential equation of the first order.
(c) $u_{tt} = 9u_{xx}$: Partial dynamic differential equation of the second order.
(d) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$: Partial dynamic differential equation of the second order.
(e) $-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = x^2 + y^2$: Partial equilibrium differential equation of the second order.
(f) $\frac{\partial^2 u}{\partial t^2} + 3u = \sin t$: Ordinary equilibrium differential equation of the second order.
(g) $u_{xx} + u_{yy} + u_{zz} + (x^2 + y^2 + z^2)u = 0$: Partial equilibrium differential equation of the second order.
(h) $u_{xx} = x + u^2$: Ordinary equilibrium differential equation of the second order.
(i) $\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x} = 0$: Partial dynamic differential equation of the third order.
(j) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y \partial z} = u$: Partial equilibrium differential equation of the second order.
(k) $u_{tt} = u_{xxxx} + 2u_{xxyy} + u_{yyyy}$: Partial dynamic differential equation of the fourth order.

$\square$

2: Olver 1.17
Solution:
(a) $u_t = x^2 u_{xx} + 2xu_x$: homogeneous linear
(b) $-u_{xx} = u_{yy} = \sin u$: nonlinear
(c) $u_{xx} + 2yu_{yy} = 3$: inhomogeneous linear
(d) $u_t + uu_x = 3u$: nonlinear
(e) $\mathrm{e}^y\, u_x = \mathrm{e}^x\, u_y$: homogeneous linear
(f) $u_t = 5u_{xxx} + x^2 u + x$: inhomogeneous linear

$\square$

3: Olver 1.22

(a) Prove that the Laplacian $\Delta = \partial_x^2 + \partial_y^2$ defines a linear differential operator.

Solution: We need to show that for some appropriate functions u, v and two scalars $a, b \in \mathbb{R}$

$$\Delta[au + bv] = a\Delta[u] + b\Delta[v].$$

We will do this directly,

$$\begin{aligned}
\Delta[au + bv] = (\partial_x^2 + \partial_y^2)(au + bv) &= (\partial_x^2 + \partial_y^2)au + (\partial_x^2 + \partial_y^2)bv \\
&= \partial_x^2 au + \partial_y^2 au + \partial_x^2 bv + \partial_y^2 bv \\
&= a\partial_x^2 u + a\partial_y^2 u + b\partial_x^2 v + b\partial_y^2 v \\
&= au_{xx} + au_{yy} + bv_{xx} + bv_{yy} \\
&= a(u_{xx} + u_{yy}) + b(v_{xx} + v_{yy}) \\
&= a(\partial_x^2 u + \partial_y^2 u) + b(\partial_x^2 v + \partial_y^2 v) \\
&= a(\partial_x^2 + \partial_y^2)u + b(\partial_x^2 + \partial_y^2)v \\
&= a\Delta[u] + b\Delta[v].
\end{aligned}$$

□

(b) Write out the Laplace equation $\Delta[u] = 0$ and the Poisson equation $-\Delta[u] = f$.

Solution: The Laplace equation is

$$\Delta[u] = (\partial_x^2 + \partial_y^2)u = u_{xx} + u_{yy} = 0$$

and the Poisson equation is

$$-\Delta[u] = -(\partial_x^2 + \partial_y^2)u = -u_{xx} - u_{yy} = f.$$

□

4: We derive the advection-diffusion equation from the microscopic view. Define $u(x,t)$ as the density of the particles at location x and time t. Define the probability of jumping from the left as $p(x-\Delta x \to x, t) \approx \frac{1}{2} + \Delta x$ when Δx is small, and the probability of jumping from the right as $q(x+\Delta x \to x, t) \approx \frac{1}{2} - \Delta x)$ with small Δx. Assume $D := \lim_{\Delta x, \Delta t \to 0} \frac{(\Delta x)^2}{\Delta t}$. Establish the equation of $u(x,t)$ in the continuum limit.

Solution:

We begin by taylor expanding $u(x,t+\Delta t), u(x-\Delta x,t),$ and $u(x+\Delta x,t)$

$$
\begin{aligned}
u(x,t+\Delta t) &= u(x,t) + u_t \Delta t + \mathcal{O}\big((\Delta t)^2\big) \\
u(x-\Delta x,t) &= u(x,t) - u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}\big((\Delta x)^3\big) \\
u(x+\Delta x,t) &= u(x,t) + u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}\big((\Delta x)^3\big).
\end{aligned}
$$

Additionally, we have the following relationship for the evolution of the system in one time step

$$
\begin{aligned}
u(x,t+\Delta t) &= q(x+\Delta x \to x,t)u(x+\Delta x,t) + p(x-\Delta x \to x,t)u(x-\Delta x,t) \\
u(x,t+\Delta t) &\approx \left(\frac{1}{2} - \Delta x\right) u(x+\Delta x,t) + \left(\frac{1}{2} + \Delta x\right) u(x-\Delta x,t).
\end{aligned}
$$

Combining this with the taylor expansions from earlier we have

$$
\begin{aligned}
u(x,t) + u_t \Delta t &\approx \left(\frac{1}{2} - \Delta x\right) \left(u(x,t) + u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}\big((\Delta x)^3\big)\right) \\
&\quad + \left(\frac{1}{2} + \Delta x\right) \left(u(x,t) - u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}\big((\Delta x)^3\big)\right) \\
u(x,t) + u_t \Delta t &\approx \frac{1}{2} \left(u(x,t) + u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}\big((\Delta x)^3\big)\right) - \Delta x \left(u(x,t) + u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}\big((\Delta x)^3\big)\right) \\
&\quad + \frac{1}{2} \left(u(x,t) - u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}\big((\Delta x)^3\big)\right) + \Delta x \left(u(x,t) - u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}\big((\Delta x)^3\big)\right) \\
u_t \Delta t &\approx \frac{1}{2} u_{xx} (\Delta x)^2 - \Delta x \left(u(x,t) + u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}\big((\Delta x)^3\big)\right) \\
&\quad + \Delta x \left(u(x,t) - u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + \mathcal{O}\big((\Delta x)^3\big)\right) \\
u_t \Delta t &\approx \frac{1}{2} u_{xx} (\Delta x)^2 - u_x (\Delta x)^2 \\
u_t &\approx \left(\frac{1}{2} u_{xx} - u_x\right) \frac{(\Delta x)^2}{\Delta t} \\
u_t &= D \left(\frac{1}{2} u_{xx} - u_x\right).
\end{aligned}
$$

This is the differential equation for the equation $u(x,t)$ in the continuum limit.

$\square$

5: (a) Consider the following boundary value problem (BVP).

$$\begin{cases} X''(x) + \lambda X = 0, & x \in (0, L) \\ X(0) = X(L) = 0, \end{cases}$$

where $L > 0$ is a constant. Solve the eigenpair:

$$(X_k, \lambda_k) = \left\{ \sin\left(\frac{k\pi x}{L}\right), \left(\frac{k\pi}{L}\right)^2 \right\}_{k=1}^{\infty}$$

Solution:

I begin by rewriting the second order ODE as a 2D first order ODE. Let $X_1 = X$ and $X_2 = X'$ then we have

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}' = \begin{bmatrix} X_2 \\ -\lambda X_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\lambda & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

Using this system we can solve for the eigenvalues of this system, that is we want to solve for γ (since λ is already in use in this function we choose a stand in variable) in the following equation

$$(-\gamma)^2 - (-\lambda \cdot 1) = \gamma^2 + \lambda = 0$$

Therefore the eigenvalues are $\gamma = \pm i\sqrt{\lambda}$. Since the eigenvalues are both imaginary we know that the solution will be of the form

$$X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x).$$

Now using the boundary values we can determine

$$\begin{aligned} X(0) &= C_1 \cos(\sqrt{\lambda}0) + C_2 \sin(\sqrt{\lambda}0) \\ 0 &= C_1. \end{aligned}$$

Furthermore, we need

$$\begin{aligned} X(L) &= C_2 \sin(\sqrt{\lambda}L) \\ 0 &= C_2 \sin(\sqrt{\lambda}L) \\ 0 &= \sin(\sqrt{\lambda}L) \end{aligned}$$

assuming $C_2 \neq 0$ to avoid arriving at the uninteresting trivial solution. We know $\sin(x) = 0$ where $x = \pi k$ for $k \in \mathbb{Z}$. Thus we need $\sqrt{\lambda}L = \pi k$ which gives us $\lambda = \left(\frac{\pi k}{L}\right)^2$. Finally, our general solution currently is the following

$$X_k(x) = \sin\left(\frac{\pi k x}{L}\right)$$

where $\lambda_k = \left(\frac{\pi k}{L}\right)^2$.

□

(b) Consider the following boundary value problem (BVP).

$$\begin{cases} X''(x) + \lambda X = 0, & x \in (0, L) \\ X'(0) = X'(L) = 0, \end{cases}$$

where $L > 0$ is a constant. Solve the eigenpair:

$$(X_k, \lambda_k) = \left\{ \cos\left(\frac{k\pi x}{L}\right), \left(\frac{k\pi}{L}\right)^2 \right\}_{k=0}^{\infty}$$

Solution:

We reuse much of the work for $X(x)$ from part (a), however the final steps using the boundary values will vary slightly. Beginning from the solution following the form

$$X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x),$$

we now apply boundary conditions. We first need this time to calculate

$$X'(x) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

Plugging in our boundary conditions we have

$$\begin{aligned} X'(0) &= -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}0) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda}0) \\ 0 &= C_2. \end{aligned}$$

Once more, we have

$$\begin{aligned} X'(L) &= -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) \\ 0 &= -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) \end{aligned}$$

which only holds when $\sqrt{\lambda} = \frac{\pi k}{L}$ for $k \in \mathbb{Z}$. This in total gives the solution

$$X(x) = C_1 \cos\left(\frac{\pi k x}{L}\right)$$

where $\lambda = \left(\frac{\pi k}{L}\right)^2$. $\square$

6: Consider the following IBVP in a rectangle:

$$\begin{cases} u_t = \Delta u, & (x,y) \in (0,L_1) \times (0,L_2), t > 0 \\ \partial_{\boldsymbol{n}} u(x,y,t) = 0, & (x,y) \in \partial\big((0,L_1) \times (0,L_2)\big), t > 0 \\ u(x,y,0) = u_0(x,y) \geq 0, \not\equiv 0 & (x,y) \in (0,L_1) \times (0,L_2) \end{cases}$$

where $\boldsymbol{n}$ denotes the unit outer normal derivative and $L_1, L_2 > 0$ are given constants. Solve to get the general solution. Recall that $\Delta = \partial_{xx} + \partial_{yy}$.

Solution:

We have

$$u_t = u_{xx} + u_{yy},$$

assuming we have the ability to use separation of variables we let $u(x,y,t)$ take the form $u(x,y,t) = X(x)Y(y)T(t)$. Then plugging this in we have:

$$X(x)Y(y)T'(t) = X''(x)Y(y)T(t) + X(x)Y''(y)T(t)$$
$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\lambda.$$

Where the $-\lambda$ came from following the example in class.

TODO: Come back if time allows