Hunter Lybbert
Student ID: 2426454
12-13-24
AMATH 561

FINAL EXAM

Note: Submit electronically to Canvas.

Directions: You are to work alone on this exam. You may use everything that is on the course website (lectures, homework solutions) and Lorig lecture notes. You may *not* use the internet, or discuss the exam with others. You may use Mathematica, or any other computational tool you find helpful. Good luck!

1. Consider two continuous time Markov chains $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ that evolve independently on the same state space $S = \{1, 2, ..., N+1\}$. After X arrives in any state, it remains there for a random amount of time, which is exponentially distributed with parameter μ. When X leaves a state, it jumps to any other state with equal probability (i.e. the probability that X jumps from i to j is $1/N$ for $j \neq i$). After Y arrives in a state, it remains there for a random amount of time, which is exponentially distributed with parameter λ. When Y leaves a state, it can jump to any other state with equal probability.

(a) Write the generator G of X.

Solution:
As the Markov Chain X is described, given a small amount of time Δt we can say

$$p_{\Delta t}(i,j) = P(X_{t+\Delta t} = j | X_t = i) = \frac{1}{N}, \qquad \text{for } i \neq j.$$

Recall that we have

$$p_{\Delta t}(i,j) = g(i,j)\Delta t + \mathcal{O}(\Delta t)$$

and

$$\begin{aligned} p_{\Delta t}(i,i) &= 1 + g(i,i)\Delta t + \mathcal{O}(\Delta t) \\ -1 + p_{\Delta t}(i,i) &= g(i,i)\Delta t + \mathcal{O}(\Delta t) \\ p_{\Delta t}(i,i) - 1 &= -g(i,i)\Delta t - \mathcal{O}(\Delta t). \end{aligned}$$

Furthermore, with the state space $S = \{1, 2, ..., N+1\}$, the generator can be denoted as

$$\boldsymbol{G}_X = \begin{bmatrix} -\mu & 1/N & 1/N & \dots & 1/N \\ 1/N & -\mu & 1/N & \dots & 1/N \\ 1/N & 1/N & -\mu & \dots & 1/N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/N & 1/N & 1/N & \dots & -\mu \end{bmatrix}.$$

TODO: Moreover, I am speculating that $\mu = 1 - \frac{1}{N}$, but my only hesitation is that we don't necessarily know that $p_{\Delta t}(i,i) = 1/N$ as well.

(b) Now, consider a Markov chain $Z = (Z_t)_{t\geq 0}$, defined as follows:

$$Z_t = \mathbb{1}_{\{X_t=Y_t\}} + 2\mathbb{1}_{\{X_t\neq Y_t\}}.$$

Write the generator H of Z.
Solution:
Notice that there are only two states for the Markov chain Z and they are $S = \{1, 2\}$. The nature of defining it by these indicator functions is equivalent to saying

$$Z_t = \begin{cases} 1 & \text{if } X_t = Y_t \\ 2 & \text{if } X_t \neq Y_t \end{cases}$$

We now need to think through the possible ways this occurs. Suppose at time s (which could be 0) we have $Z_s = 2$ meaning $X_s \neq Y_s$. For notational assistance suppose $X_s = i$ and $Y_s = j$ where $i \neq j$. The means by which $Z_{s+t} = 1$ are the following scenarios (I recognize the short coming of my notation in this interim section while I am still thinking through the way these probabilities will work together)

(1) X does not change states but Y changes to state i. That is $X_{s+t} = i$ and $Y_{s+t} = i$. This can occur w.p. (using the fact that X and Y are independent Markov chains)

$$\begin{aligned} P(Z_{s+t} = 1|Z_s = 2) &= P(X_{s+t} = i, Y_{s+t} = i|X_s = i, Y_s = j) \\ &= P(X_{s+t} = i|X_s = i)P(Y_{s+t} = i|Y_s = j) \\ &= p_X(t, i; s, i)\frac{1}{N}. \end{aligned}$$

(2) The opposite outcome where Y does not change states but X changes to state j. That is $X_{s+t} = j$ and $Y_{s+t} = j$. This can occur w.p. (using the fact that X and Y are independent Markov chains)

$$\begin{aligned} P(Z_{s+t} = 1|Z_s = 2) &= P(X_{s+t} = j, Y_{s+t} = j|X_s = i, Y_s = j) \\ &= P(X_{s+t} = j|X_s = i)P(Y_{s+t} = j|Y_s = j) \\ &= p_X(t, j; s, i)p_Y(t, j; s, j) \\ &= \frac{1}{N}p_Y(t, j; s, j). \end{aligned}$$

(3) Lastly, it is possible that X and Y both change to the same new state k. That is $X_{s+t} = k$ and $Y_{s+t} = k$. This can occur w.p. (using the fact that X and Y are independent Markov chains)

$$\begin{aligned} P(Z_{s+t} = 1|Z_s = 2) &= P(X_{s+t} = k, Y_{s+t} = k|X_s = i, Y_s = j) \\ &= P(X_{s+t} = k|X_s = i)P(Y_{s+t} = k|Y_s = j) \\ &= p_X(t, k; s, i)p_Y(t, k; s, j) \\ &= \frac{1}{N}\frac{1}{N} \\ &= \frac{1}{N^2}. \end{aligned}$$

$$\boldsymbol{H}_Z = \begin{bmatrix} -\mu & 1/N & 1/N & \dots & 1/N \\ 1/N & -\mu & 1/N & \dots & 1/N \\ 1/N & 1/N & -\mu & \dots & 1/N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/N & 1/N & 1/N & \dots & -\mu \end{bmatrix}.$$

(c) Let $\boldsymbol{\mu}_t$ be the distribution of Z_t. Compute $\boldsymbol{\mu}_t$ assuming $X_0 = i$ and $Y_0 = j$ where $i \neq j$.

(d) Let $\boldsymbol{\pi}$ be the stationary distribution of the process Z. Compute $\boldsymbol{\pi}$ and show that $\lim_{t\to\infty} \boldsymbol{\mu}_t = \boldsymbol{\pi}$.

2. Consider a discrete-time Markov chain with the $N+1$ states $0, 1, ..., N$ and one-step transition probabilities

$$p_{ij} = \binom{N}{j} \pi_i^j (1-\pi_i)^{N-j}, \quad 0 \le i, j \le N,$$

$$\pi_i = \frac{1 - e^{-2ai/N}}{1 - e^{-2a}}, \quad a > 0.$$

Note that 0 and N are absorbing states.

(a) Verify that $\exp(-2aX_n)$ is a martingale.

(b) Using the martingale property from (a), show that the probability $P_N(k)$ of absorbing into state N starting at state k (i.e. given $X_0 = k$) is given by

$$P_N(k) = \frac{1 - e^{-2ak}}{1 - e^{-2aN}}.$$

3. Let $\boldsymbol{Y}_0, \boldsymbol{Y}_1, \boldsymbol{Y}_2,,,$ be a sequence of i.i.d. unsigned 32 bit integers (i.e. $\boldsymbol{Y}_i = (y_{i,1}, y_{i,2}, ..., y_{i,31}, y_{i,32})$, $y_{i,k} = 0$ or 1, every value of $\boldsymbol{Y}_i$ equally likely).

For the sequence $\boldsymbol{X}_i$ the following recursion is given:

$$\boldsymbol{X}_0 = \mathbf{0}, \quad \mathbf{0} \equiv (0, 0, ...0, 0),$$

$$\boldsymbol{X}_i = \boldsymbol{X}_{i-1} \oplus \boldsymbol{Y}_{i-1},$$

where $\oplus$ is the operator defined by $x_{i-1,k} \oplus y_{i-1,k} = \min(1, x_{i-1,k} + y_{i-1,k})$.

It can be seen that eventually there will be an index N such that $\boldsymbol{X}_i = \mathbf{1}$, $\mathbf{1} \equiv (1, 1, ..., 1, 1)$ (a bit-pattern of all ones), for all $i \geq N$. Find the expected value of N. Please leave your answer in the form $EN = \sum_{k=0}^{\infty} a_k$ (i.e. give an explicit expression for a_k).

4. We are considering a model of bacterial evolution in which the process starts with N_0 sensitive and 0 resistant cells. Sensitive cells grow exponentially with rate 1 and their growth is deterministic. In other words, the number of sensitive cells at time t will be $N_t = N_0 e^t$. Resistant cells are produced in two ways: by mutation from sensitive cells or from division (birth) of currently present resistant cells. Resistant cells are produced by sensitive cells with mutation rate a. This means that in a small time interval $(t, t + \Delta t)$, the chance that a new resistant cell is produced by sensitive cells is $aN_t\Delta t$. In addition, resistant cells will follow a pure birth process with rate 1 (i.e. if there are currently n resistant cells, the chance that they will produce an extra resistant cell in a short time interval Δt is $n\Delta t$).

(a) Derive the partial differential equation for the probability generating function (PGF) of the process describing the number of resistant cells at time t, $X(t)$.

(b) Solve the PDE from part a) to obtain the PGF for the number of resistant cells at time t.

(c) Find the mean and variance of the process, $E[X(t)]$ and $\mathrm{Var}(X(t))$.

(d) How do the mean and variance of the process compare? Which one will be larger for large time t?

Note: if you need to evaluate a function $f(y)$ at a value $y = y_0$ where it is not defined, you can instead evaluate $\lim_{y \to y_0} f(y)$.