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02-20-25

AMATH 502

Assignment 6

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$\boxed{\text{Problem 1}}$ Consider the following weakly nonlinear oscillator:

$$\ddot{x} + x + \varepsilon(x^2 - 1)\dot{x}^3 = 0$$

where $0 < \varepsilon \ll 1$ (really small). Use two timing to find the location of closed trajectories for this eq.

(a) Define the two time scales $\underline{\tau = t}$ and $\underline{T = \varepsilon t}$

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$$

using

$$\dot{x} = \frac{\partial x_0}{\partial \tau} + \varepsilon \frac{\partial x_0}{\partial T} + \varepsilon \frac{\partial x_1}{\partial \tau} + O(\varepsilon^2)$$

$$\ddot{x} = \frac{\partial^2 x_0}{\partial \tau^2} + 2\varepsilon \frac{\partial^2 x_0}{\partial \tau \partial T} + \varepsilon \frac{\partial^2 x_1}{\partial \tau^2} + O(\varepsilon^2)$$

we have

$$\frac{\partial^2 x_0}{\partial \tau^2} + 2\varepsilon \frac{\partial^2 x_0}{\partial \tau \partial T} + \varepsilon \frac{\partial^2 x_1}{\partial \tau^2} + O(\varepsilon^2) + x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$$

supposed to be $\dot{x}$

$$+ \varepsilon\left((x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3))^2 - 1\right)\left(\frac{\partial^2 x_0}{\partial \tau^2} \cancel{+ 2\varepsilon \frac{\partial^2 x_0}{\partial \tau \partial T} + \varepsilon} \frac{\partial^2 x_1}{\partial \tau} + O(\varepsilon^2)\right)^3$$

$$= 0$$

1 a) continued

$$\frac{\partial^2 x_0}{\partial \tau^2} + 2\varepsilon \frac{\partial^2 x_0}{\partial \tau \partial T} + \varepsilon \frac{\partial^2 x_1}{\partial \tau^2} + O(\varepsilon^2) + x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$$

$$+ \varepsilon \left[\left(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)\right)^2 - 1 \right] \left(\frac{\partial x_0}{\partial \tau} + \varepsilon \frac{\partial x_0}{\partial T} + \varepsilon \frac{\partial x_1}{\partial \tau} + O(\varepsilon^2) \right)^3 = 0$$

$(*)$

Gathering the ε^0 terms we have

$$\frac{\partial^2 x_0}{\partial \tau^2} + x_0 = 0 \qquad x_0(t) = C_0 \sin(t) + C_1 \cos(t)$$

$$\Rightarrow$$

$$\ddot{x}_0 + x_0 = 0$$

$$\ddot{x}_0 = -x_0$$

Now for (b) ε^1 terms we have

$$\frac{2\partial x_0}{\partial \tau \partial T} + \frac{\partial^2 x_1}{\partial \tau^2} + x_1 + x_0^2 \left(\frac{\partial x_0}{\partial \tau}\right)^3 - \left(\frac{\partial x_0}{\partial \tau}\right)^3 = 0.$$

where the $\left(\frac{\partial x_0}{\partial \tau}\right)^3$ terms come from expanding $(*)$ from our original D.S. Notice, we also have

$$2\frac{\partial x_0}{\partial \tau \partial T} + \frac{\partial^2 x_1}{\partial \tau^2} + x_1 + \left(x_0^2 - 1\right)\left(\frac{\partial x_0}{\partial \tau}\right)^3 = 0.$$

2 D.S. bifurcation at $b=a$.

$$\dot{x} = x(y-1)$$
$$\dot{y} = -y(x+a)+b$$

f.p. @ $\left\{ \begin{array}{l} (0, \frac{b}{a}) \\ (b-a, 1) \end{array} \right\}$

$$J = \begin{pmatrix} y-1 & x \\ -y & x+a \end{pmatrix} \Bigg|_{(0,\frac{b}{a})} \Rightarrow \begin{pmatrix} \frac{b}{a}-1 & 0 \\ \frac{b}{a} & a \end{pmatrix}$$

$$\Rightarrow \left(\frac{b}{a}-1-\lambda\right)(a-\lambda) - \frac{b}{a}\cdot 0 = 0$$

$$\left(\frac{b}{a}-1\right)a + \lambda^2 - a\lambda - \lambda\left(\frac{b}{a}-1\right) = 0$$

$$\lambda^2 - \lambda\left(a+\frac{b}{a}-1\right) + \left(\frac{b}{a}-1\right)a = 0$$

$$\lambda = \frac{-a-\frac{b}{a}+1 \pm \sqrt{\left(a+\frac{b}{a}-1\right)^2 - 4\left(\frac{b}{a}-1\right)a}}{2}$$

to classify we need to know what happens as $b<a$ vs. $b=a$ and $b>a$.

If I had time, I would classify the stability of the f.p. for various values of a,b! After classifying these in each case, I will be able to see that before and after $a=b$ the f.p. switch stabilities which we call a ~~[illegible]~~ transcritical fixedpoint.

$\boxed{3.}$

$$\dot{x} = -ax + y + x(x^2+y^2) - a\frac{x^3}{\sqrt{x^2+y^2}}$$

$$\dot{y} = -x - ay + y(x^2+y^2) - a\frac{x^2 y}{\sqrt{x^2+y^2}}$$

a) f.p. @ $(0,0)$

$$J = \begin{pmatrix} -a + 3x^2 + y^2 - a\left(\frac{3x^2\sqrt{x^2+y^2} - x^3\left(\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}2x\right)}{x^2+y^2}\right), & 1 + 2xy + \frac{a}{2}\frac{x^3}{(x^2+y^2)^{3/2}}2y \\ -1 + 2xy - a\left(\frac{2xy\sqrt{x^2+y^2} - x^2y\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}2x}{x^2+y^2}\right), & -a + x^2 + 3y^2 - a\left(\frac{x^2\sqrt{x^2+y^2} - x^2y\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}2y}{x^2+y^2}\right) \end{pmatrix}_{(0,0)}$$

$$= \begin{pmatrix} -a & 1 \\ -1 & -a \end{pmatrix}$$

$$(-a-\lambda)(-a-\lambda) + 1 = 0$$

$$a^2 + 2\lambda a + \lambda^2 + 1 = 0$$

$$\lambda^2 + 2a\lambda + a^2 + 1 = 0$$

$$\lambda = \frac{-2a \pm \sqrt{4a^2 - 4(a^2+1)}}{2}$$

$$= \frac{-2a \pm \sqrt{4(a^2-(a^2+1))}}{2}$$

$$= -a \pm \sqrt{\cancel{a^2} - \cancel{a^2} + 1}$$

$$= -a$$