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AMATH 503
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Homework 7

problem 1 Let $u = u(x)$ and define $Lu := u'' - k^2 u$ on $0 < x < 1$, where $k > 0$ is a constant

(a) Find Green's function for Lu with $u(0) = 0$ and $u(1) = 0$.
We want to solve $Lu = \delta(x - \xi)$ to find Green's Function, but let's first solve the Homogeneous part

$$Lu = 0$$
$$u'' - k^2 u = 0$$

Since we know $k > 0$ we have the solution

$$u(x) = \begin{cases} A(\xi)\cosh(kx) + B(\xi)\sinh(kx) & 0 < x < \xi \\ C(\xi)\cosh(kx) + D(\xi)\sinh(kx) & \xi < x < 1 \end{cases}$$

Now we want to apply our B.C.

$$u(0) = \begin{cases} A(\xi) = 0 \\ \end{cases}$$

$$u(1) = C(\xi)\cosh(k) + D(\xi)\sinh(k) = 0 \qquad (*)$$

Thus $u(x) = \begin{cases} B(\xi)\sinh(kx) & 0 < x < \xi \\ C(\xi)\cosh(kx) + D(\xi)\sinh(kx) & \xi < x < 1 \end{cases}$

with constraint $(*)$.
We also want continuity so we need at $x = \xi$ for

$$(**) \quad B(\xi)\sinh(k\xi) = C(\xi)\cosh(k\xi) + D(\xi)\sinh(k\xi)$$ and then in the derivative we need

~~$B(\xi)k\cosh(k\xi)$ $C(\xi)k\sinh(k\xi)$~~

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In the derivative we need the jump ~~[illegible]~~ equal to 1 at $x = \xi$

$$C(\xi)K\sinh(K\xi) + D(\xi)K\cosh(K\xi) - B(\xi)K\cosh(K\xi) = 1$$

$$(***)\ C(\xi)\sinh(K\xi) + D(\xi)\cosh(K\xi) - B(\xi)\cosh(K\xi) = \frac{1}{K}.$$

Now we have the system of equations $(*)$, $(**)$, and $(***)$

We need to solve this system for $B(\xi)$, $C(\xi)$, and $D(\xi)$. We have

$$\begin{cases} C(\xi)\cosh(K) + D(\xi)\sinh(K) = 0 \\ C(\xi)\cosh(K\xi) + D(\xi)\sinh(K\xi) = B(\xi)\sinh(K\xi) \\ C(\xi)\sinh(K\xi) + D(\xi)\cosh(K\xi) - B(\xi)\cosh(K\xi) = \frac{1}{K} \end{cases}$$

From $(**)$

$$B(\xi) = \frac{C(\xi)\cosh(K\xi) + D(\xi)\sinh(K\xi)}{\sinh(K(\xi)}$$

$$= C(\xi)\frac{\cosh(K\xi)}{\sinh(K\xi)} + D(\xi)$$

Plugging this info $(***)$

$$C(\xi)\sinh(K\xi) + D(\xi)\cosh(K\xi) - \left(C(\xi)\frac{\cosh(K\xi)}{\sinh(K\xi)} + D(\xi)\right)\cosh(K\xi) = \frac{1}{K}$$

~~[illegible]~~

$$C(\xi)\sinh(K\xi) - C(\xi)\frac{\cosh^2(K\xi)}{\sinh(K\xi)} = \frac{1}{K}$$

$$C(\xi) = \frac{1}{K\left(\sinh(K\xi) - \frac{\cosh^2(K\xi)}{\sinh(K\xi)}\right)} = \frac{1}{K\left(\frac{\sinh^2(K\xi) - \cosh^2(K\xi)}{\sinh(K\xi)}\right)}$$

$$= -\frac{\sinh(K\xi)}{K}$$

plugging $C(\xi)$ into $(*)$ we have

$$-\frac{\sinh(k\xi)}{k}\cosh(k) + D(\xi)\sinh(k) = 0$$

$$\Rightarrow D(\xi) = \frac{\sinh(k\xi)\cosh(k)}{k\sinh(k)}$$

Finally plug both $C(\xi)$ and $D(\xi)$ into the expression for $B(\xi)$ we get

$$B(\xi) = -\frac{\sinh(k\xi)}{k}\frac{\cosh(k\xi)}{\sinh(k\xi)} + \frac{\sinh(k\xi)\cosh(k)}{k\sinh(k)}$$

$$B(\xi) = \frac{1}{k}\left[\frac{\sinh(k\xi)\cosh(k)}{\sinh(k)} - \cosh(k\xi)\right]$$

Thus

$$u(x) = \begin{cases} \frac{\sinh(kx)}{k}\left[\frac{\sinh(k\xi)\cosh(k)}{\sinh(k)} - \cosh(k\xi)\right] & 0<x<\xi \\ -\frac{\sinh(k\xi)}{k}\cosh(kx) + \frac{\sinh(k\xi)\cosh(k)}{k\sinh(k)}\sinh(kx) & \xi<x<1 \end{cases}$$

This is our green's Function. $\square$

b) Find the integral representation for the solution to
$Lu = f(x),\ 0<x<1,\ u(0)=1,\ u(1)=0$

Let $v = u-(1-x)$ thus $v(0)=v(1)=0$ then
$Lv = v'' - k^2 v = u'' - k^2 u + k^2(1-x) = f(x) + k^2(1-x)$

Thus
$$Lv = f(x) + k^2(1-x)$$

Using Green's function, we have

$$v(x) = \int_0^1 G(x,\xi)\left[f(\xi) + k^2(1-\xi)\right]d\xi$$

implying $$u(x) = 1 - x + \int_0^1 G(x,\xi)\left[f(\xi) + k^2(1-\xi)\right]d\xi$$

c) Let $f(x) = \frac{1}{\varepsilon}\left[\frac{1}{\cosh^2\left(\frac{x-1/2}{\varepsilon}\right)}\right]$

as $\varepsilon \to 0^+$ we have

$$f(x) \to 2\,\delta(x - 1/2)$$

Thus
$$u(x) = 2\,G(x, 1/2)$$

$\square$

$\boxed{\text{problem 2)}}$

$u = u(x)$, $Lu = u'' + k^2 u$ on $0 < x < a$, $k > 0$, $a > 0$

a) Green's function for Lu where $u'(0) = u'(a) = 0$

Similar to problem 1a) we want to first solve the Homogeneous part $Lu = 0$ then later get conditions for the Green function.

$Lu = 0$

$u'' + k^2 u = 0$

$u'' = -k^2 u$

$$\Rightarrow u(x) = \begin{cases} A(\xi)\cos(kx) + B(\xi)\sin(kx) & 0 < x < \xi \\ C(\xi)\cos(kx) + D(\xi)\sin(kx) & \xi < x < a \end{cases}$$

<u>Apply BC</u>

$$u'(x) = \begin{cases} -A(\xi)k\sin(kx) + B(\xi)k\cos(kx) & \cdots \\ -C(\xi)k\sin(kx) + D(\xi)k\cos(kx) & \cdots \end{cases}$$

$u'(0) = \cancel{-A(\xi)k\sin(0)} + B(\xi)k\cos(0) = 0$

$B(\xi)k = 0$

$B(\xi) = 0$

$u'(a) = -C(\xi)k\sin(ka) + D(\xi)k\cos(ka) = 0$

$\Rightarrow D(\xi)k\cos(ka) = C(\xi)k\sin(ka)$

$D(\xi)\cos(ka) = C(\xi)\sin(ka)$ $\quad (*)$

Thus

$$u(x) = \begin{cases} A(\xi)\cos(kx) & 0 < x < \xi \\ C(\xi)\cos(kx) + D(\xi)\sin(kx) & \xi < x < a \end{cases}$$

Subject to $(*)$.

We now want continuity at $x=\xi$ thus we also need

$$A(\xi)\cos(k\xi) = C(\xi)\cos(k\xi) + D(\xi)\sin(k\xi) \qquad (**)$$

Finally we want a jump in the derivative of 1 at $x=\xi$

$$-kC(\xi)\sin(k\xi) + kD(\xi)\cos(k\xi) - (-A(\xi)k\sin(k\xi)) = 1 \qquad (***)$$

$$A(\xi)\sin(k\xi) - C(\xi)\sin(k\xi) + D(\xi)\cos(k\xi) = \frac{1}{k}$$

$$[A(\xi) - C(\xi)]\sin(k\xi) + D(\xi)\cos(k\xi) = \frac{1}{k}.$$

Manipulating $(*)$ we have

$$\frac{D(\xi)}{C(\xi)} = \frac{\sin(ka)}{\cos(ka)} \Rightarrow D(\xi) = C(\xi)\tan(ka)$$

subbing this into $(**)$ we have

$$A(\xi)\cos(k\xi) = C(\xi)\cos(k\xi) + C(\xi)\tan(ka)\sin(k\xi)$$
$$A(\xi)\cos(k\xi) = C(\xi)[\cos(k\xi) + \sin(k\xi)\tan(ka)]$$
$$A(\xi) = C(\xi)[1 + \tan(k\xi)\tan(ka)].$$

plugging this and the one before into $(***)$ and solve for $C(\xi)$,

$$[C(\xi)[1+\tan(k\xi)\tan(ka)] - C(\xi)]\sin(k\xi) + C(\xi)\tan(ka) = \frac{1}{k}$$

$$C(\xi)\left[[(\cancel{1+}\tan(k\xi)\tan(ka)) \cancel{-1}]\sin(k\xi) + \tan(ka)\right] = \frac{1}{k}$$

$$C(\xi) = \frac{1}{k[\tan(ka)(\tan(k\xi)\sin(k\xi)+1)]}$$

Hence,

$$D(\xi) = \frac{1}{k} \frac{1}{(\tan(k\xi)\sin(k\xi)+1)}$$

Finally

$$A(\xi) = \frac{1}{k[\tan(ka)(\tan(k\xi)\sin(k\xi)+1)]}\left[1 + \tan(k\xi)\tan(ka)\right]$$

Thus

$$u(x) = \begin{cases} \dfrac{1 + \tan(k\xi)\tan(ka)}{k\tan(ka)(\tan(k\xi)\sin(k\xi)+1)} \cos(kx) & 0 < x < \xi \\[2ex] \dfrac{1}{k\tan(ka)(\tan(k\xi)\sin(k\xi)+1)} \cos(kx) + \dfrac{1}{k(\tan(k\xi)\sin(k\xi)+1)} \sin(kx) & \xi < x < a \end{cases}$$

b) Which is our G.F. when $k = \frac{n\pi}{a}$ for $n \in$ integers we note

$$\tan(ka) \Rightarrow \tan(n\pi) = \frac{\sin(n\pi)}{\cos(n\pi)} = \frac{0}{(-1)^n} = 0$$

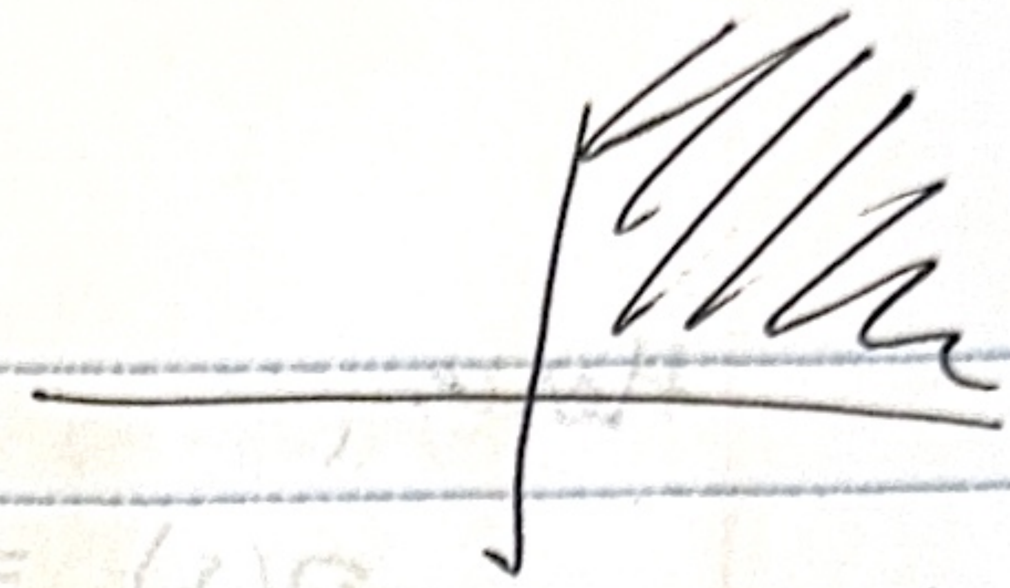
Thus the G.F. in a) d.n.e. when $k = \frac{n\pi}{a}$ since $\tan(ka) = 0$ would imply multiple divide by zero issues.

c) $u'' + k^2 u = f(x)$, $0 < x < a$, $u'(0) = u'(a) = 0$ and $k = \frac{n\pi}{a}$ for a specific $n \in$ integers positive

The condition on $f(x)$ is that it needs to be continuously differentiable.

$\square$

d) TBD . . .



problem 3

let Ω be quarter plane defined by
$\Omega = \{(\xi,\eta) \mid 0 \leq \xi < +\infty,\ 0 \leq \eta < \infty\}$ with Boundary
$\partial\Omega$. Let (x,y) be some point in Ω

(i) Meth. of Images

$$\begin{cases} G_{\xi\xi} + G_{\eta\eta} = \delta(\xi - x)\delta(\eta - y) \in \Omega; \\ G = 0 \text{ on } \xi = 0,\quad G = 0 \text{ on } \eta = 0. \end{cases}$$

Step 1: the free space Green's function for the 2D Laplacian is

$$G_0(\xi,\eta,x,y) = -\frac{1}{2\pi}\log\left(\sqrt{(\xi-x)^2+(\eta-y)^2}\right)$$
$$= -\frac{1}{4\pi}\log\left((\xi-x)^2+(\eta-y)^2\right).$$

Solving $\nabla^2 G_0 = \delta(\xi-x)\delta(\eta-y)$ in $\mathbb{R}^2$.

Now we want to modify G_0 s.t. the total Greens Func satisfies our DBC. and the Laplace eq is satisfied away from our singularity (x,y).

We use a reflection strategy placing image charges or delta sources outside the domain to enforce b.c. The original source loc denoted as

$P_1 = (x,y)$ Now we create 3 image charges

$P_2 = (-x,y)$, $P_3 = (x,-y)$, $P_4 = (-x,-y)$

placing this into our Green's Func we have

$$G(\xi,\eta,x,y) = -\frac{1}{4\pi}\Big[\log\left((\xi-x)^2+(\eta-y)^2\right) - \log\left((\xi+x)^2+(\eta-y)^2\right)$$
$$- \log\left((\xi-x)^2+(\eta+y)^2\right) + \log\left((\xi+x)^2+(\eta+y)^2\right)\Big]$$
$$= -\frac{1}{4\pi}\log\left(\frac{\left((\xi-x)^2+(\eta-y)^2\right)\left((\xi+x)^2+(\eta+y)^2\right)}{\left((\xi-x)^2+(\eta+y)^2\right)\left((\xi+x)^2+(\eta-y)^2\right)}\right)$$

$\square$

(ii) Green's second identity write solution to PDE below

$$\begin{cases} u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0 & 0<r<\infty,\ 0<\theta<\pi/2 \\ u \text{ is bounded as } r\to 0 \ \& \ r\to\infty \\ u(r,0)=F(r),\ u(r,\pi/2)=0 \end{cases}$$

lets turn this into a cartesian problem so we can use the previous result from (i). we know

$$x = r\cos\theta,\quad y = r\sin\theta,\quad \theta = \tan^{-1}\left(\frac{x}{y}\right)$$

The domain becomes $\Omega = \{x>0, y>0\}$ as before.
The PDE is now $\Delta u = 0$ in Ω with BC's

$\theta = 0 \Rightarrow y=0 \Rightarrow u(x,0) = f(x)$
$\theta = \pi/2 \Rightarrow x=0 \Rightarrow u(0,y)=0.$

Now Green's second identity

$$\int_\Omega (u\Delta G - G\Delta u)\,dA = \int_{\partial\Omega}\left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right)ds$$

Since $\Delta u = 0$ and $\Delta G = \delta(\xi-x)\delta(\eta-y)$ we arrive at

$$u(x,y) = \int_{\partial\Omega}\left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right)ds.$$

Our BC's imply only the bottom edge of our boundary contributes $u(x,0)=f(x)$ however the $u(0,y)=0$ therefore

on $y=0$ we have $\eta=0$, $\xi\in(0,\infty)$

outward normal is $n=-\hat{y}$ so $\frac{\partial G}{\partial n}=-\frac{\partial G}{\partial \eta}$

Thus

$$u(x,y)=\int_0^\infty f(\xi)\left(-\frac{\partial G}{\partial \eta}(\xi,0;x,y)\right)d\xi$$

Thus our full solution is

$$u(x,y)=\frac{1}{4\pi}\int_0^\infty f(\xi)\cdot\frac{\partial}{\partial\eta}\left[\log\left(\frac{(\xi-x)^2+\eta^2}{(\xi+x)^2+\eta^2}\cdot\frac{(\xi+x)^2+(2y-\eta)^2}{(\xi-x)^2+(2y-\eta)^2}\right)\right]_{\eta=0}d\xi$$

where $x=r\cos\theta$, $y=r\sin\theta$.

$\square$

Problem 5 Olver 7.3.4

Find a solution to the diff eq $-\frac{d^2u}{dx^2} + 4u = \delta(x)$ by using the F.T.

transform both sides we get

$$-(-k^2\hat{u}(k)) + 4\hat{u}(k) = 1$$

$$k^2\hat{u}(k) + 4\hat{u}(k) = 1$$

$$(k^2+4)\hat{u}(k) = 1$$

$$\hat{u}(k) = \frac{1}{k^2+4}$$

Now inverse F.T. gives

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{k^2+4}\,dk = \frac{1}{2}e^{-2|x|}$$

$\square$

problem 6

Use Fourier transform to solve the following Laplace Eq

$$\begin{cases} u_{xx} + u_{yy} = 0 & -\infty < x < \infty,\ y > 0 \\ u(x,0) = h(x) & -\infty < x < \infty \\ u \text{ is bounded as } y \to \infty \text{ and } |x| \to +\infty \end{cases}$$

Taking the FT in x we have

$$\hat{u}(k,y) = F_x[u(x,y)] = \int_{-\infty}^{\infty} u(x,y) e^{-ikx}\, dx$$

$$F_x[u_{xx} + u_{yy}] = F_x[0]$$

$$\Rightarrow -k^2 \hat{u}(k,y) + \partial_{yy} \hat{u}(k,y) = 0 \quad \Rightarrow \hat{u}_{yy} - k^2 \hat{u} = 0$$

Solving the resulting diff eq we have

$$\hat{u}(k,y) = A(k) e^{|k|y} + B(k) e^{-|k|y}$$

$A(k) = 0$ so the exponential term doesn't blow up.

Thus

$$\hat{u}(k,y) = B(k) e^{-|k|y}$$

$$u(x,0) = h(x) \quad \Rightarrow \quad \hat{u}(k,0) = \hat{h}(k) = B(k)$$

so $\hat{u}(k,y) = \hat{h}(k) e^{-|k|y}$.

The inverse F.T. gives

$$u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{h}(k) e^{-|k|y} e^{ikx}\, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} h(\xi)\, d\xi$$

$\square$