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AMATH 503

HOMEWORK 4

Exercises come from *Introduction to Partial Differential Equations by Peter J. Olver* as well as supplemented by instructor provided exercises.

1: We consider the following IBVP:

$$\begin{cases} u_t = u_{xx}, & x \in (0,1), t > 0, \\ u_x(0,t) + 2u(0,t) = 0, u_x(1,t) - 2u(1,t) = 0, & t > 0, \\ u(x,0) = \phi(x), & x \in (0,1). \end{cases}$$

Solve this IBVP by using separation of variables and analyze the long-term behavior of the solution as $t \to +\infty$

Solution:
TODO

2: (a) Consider the following IBVP:

$$\begin{cases} (x^2\phi')' + \lambda\phi = 0, & 1 < x < 2, \\ \phi(1) = 0 = \phi(2). \end{cases}$$

Figure out p, q, w, h_1 and h_2. Write down the properties satisfied by eigenvalues and eigenfunctions by Sturm-Liouville theorem. (e.g. orthogonality of eigenfunctions, completeness of basis, etc.) Solve the eigenpairs $\{(\lambda_k, \phi_k\}$.

Solution:
TODO:

(b) Then, use the eigenpairs to solve the following IBVP:

$$\begin{cases} u_t = (x^2 u_x)_x - u, & 1 < x < 2, t > 0, \\ u(1,t) = u(2,t) = 0, & t > 0, \\ u(x,0) = f(x), & x \in (1,2) \end{cases}$$

(Hint: for Euler's ODE: $aX'' + bX' + cX = 0$, we have the ansatz $X = x^r$ and the characteristic root equation $ar(r-1) + br + x = 0$. If $r_1 \neq r_2$, then $X = c_1 x^{r_1} + c_2 x^{r_2}$; if $r_1 = r_2 = r$, then $X = c_1 x^r + c_2 x^r \log x$; if $r = \nu + \mathrm{i}\mu$ is a complex root, then $X = c_1 x^\nu \cos(\mu \log x) + c_2 x^\nu \sin(\mu \log x)$.)

Solution:
TODO

3: Consider the following BVP:

$$\begin{cases} x^2y'' + xy' + (x^2\lambda^2 - n^2)y = 0, & x \in (0, L), \\ y'(0) = 0 or y(0) = 0, y(L) = 0, \end{cases}$$

where λ and n are real numbers. $L > 0$ is a constant, as well.

(a) rewrite the BVP as the sturm-liouville form and write down the definition of p, q, w, h_1, and h_2:

Solution:
TODO:

(b) write down the orthogonality conditions satisfied by the eigenfunction.

Solution:
TODO:

(c) We have the fact that

$$J_n(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n-k)!} \left(\frac{x}{2}\right)^{n+2k}, \quad n = 0, 1, ...$$

are solutions to

$$\begin{cases} xy'' + y' + (x - \frac{n^2}{x})y = 0, & x \in (0, \infty), \\ y(\infty) = 0. \end{cases}$$

J_n is named the n-th order Bessel function. Plot the figures of J_0, J_1, and J_2 by matlab or python. We have the facts that $J_n(x)$ has infinitely many zeros $\nu_{nm}, m = 1, ...$ and J_n is bounded as $r \to 0$. By using $J_n(x)$ and nu_{nm} to find all eigenpairs $\left\{\lambda_{nm}^2, y_{nm}(x)\right\}_{m=1}^{\infty}$ to the given BVP.

Solution:
TODO:

(d) Solve the following IBVP:

$$\begin{cases} u_t = \Delta u, & (x, y) \in B_a(0), t > 0, \\ u(x, t) = 0, & (x, y) \in \partial B_a(0), t > 0, \\ u(x, 0) = u_0(x, y), & (x, y) \in B_a(0), \end{cases}$$

where $B_a(0) \subset \mathbb{R}^2$ is the disc with radius $a > 0$. (Hint: for (d), recall in the polar coordinate (r, θ), $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$. Use separation of variables $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$ to solve the IBVP. Θ mode is 2π-periodic.)

Solution:
TODO:

4: Solve the following signaling problem:

$$\begin{cases} u_t + cu_x = 0, & 0 < x < +\infty, \\ u(0,t) = g(t), u(x,0) = 0, & x \geq 0, \end{cases}$$

where $c > 0$ is a constant.

Solution:
TODO:

5: Olver 2.2.17

Solution:
TODO:

6: Olver 2.2.31

Solution:
TODO:

7: (a) Solve the ODE:

$$\frac{du}{ds} + u = 2\,\mathrm{e}^x$$

by the integral factor method. And use the same technique to solve the following damping heat equation:

$$\begin{cases} \nu_t = \nu_{xx} = \nu, & x \in (0,\pi), t > 0, \\ \nu(0,t) = \nu(\pi,t) = 0. & t > 0, \\ \nu(x,0) = \nu_0(x), & x \in (0,\pi). \end{cases}$$

(Hint: test the BHS of ODE against e^x and then integrate to solve it, where e^x is called the integral factor.)

Solution:
TODO:

(b) Solving the following transport equation:

$$u_t + tu_x = u, -\infty < x < +\infty, t > 0,$$

with initial condition

$$u(x,1) = f(x), 0 \leq x \leq 1,$$

where f is continuous. Compute $u(x,t)$ by the method of characteristics and find the subregion in $-\infty < x < +\infty, t > 0$, where the data on $t = 1$ determines this solution. Plot this subregion.

Solution:
TODO: