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AMATH 503

HOMEWORK 4

Exercises come from *Introduction to Partial Differential Equations by Peter J. Olver* as well as supplemented by instructor provided exercises.

1: We consider the following IBVP:

$$\begin{cases} u_t = u_{xx}, & x \in (0,1), t > 0, \\ u_x(0,t) + 2u(0,t) = 0, u_x(1,t) - 2u(1,t) = 0, & t > 0, \\ u(x,0) = \phi(x), & x \in (0,1). \end{cases}$$

Solve this IBVP by using separation of variables and analyze the long-term behavior of the solution as $t \to +\infty$

Solution:
We begin by assuming the equation $u(x,t)$ takes on the form

$$u(x,t) = X(x)T(t)$$

hence, we assume

$$X(x)T'(t) = X''(x)T(t)$$
$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

This gives use the equations

$$T'(t) + \lambda T(t) = 0, \tag{1}$$
$$X''(x) + \lambda X(x) = 0. \tag{2}$$

We first solve (2) but have to consider common cases for λ less than, equal to, and greater than 0. We begin with $\lambda = 0$ then we have $X(x) = Ax + B$. Our boundary conditions

$$u_x(0,t) + 2u(0,t) = 0,$$
$$u_x(1,t) - 2u(1,t) = 0,$$

where $t > 0$. Using $A + 2(A(0) + B) = 0 \implies A = -2B$ We also need $A - 2(A(1) + B) = 0 \implies -A - 2B = 0 \implies A = -2B$, again. Hence,

$$X(x) = A\left(x - \frac{1}{2}\right).$$

Continuing on with the case where $\lambda > 0$. Then we have $X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$ which using our same boundary conditions now gives

$$\begin{aligned}
u_x(0,t) + 2u(0,t) &= 0\\
\cancel{-A\sqrt{\lambda}\sin(\sqrt{\lambda}0)} + B\sqrt{\lambda}\cos(\sqrt{\lambda}0) + 2\left(A\cos(\sqrt{\lambda}0) + \cancel{B\sin(\sqrt{\lambda}0)}\right) &= 0\\
B\sqrt{\lambda} + 2A = 0 \implies A \quad &= -\frac{\sqrt{\lambda}}{2}B
\end{aligned}$$

and then for $u_x(1,t) - 2u(1,t) = 0$ we have

$$\begin{aligned}
-A\sqrt{\lambda}\sin(\sqrt{\lambda}) + B\sqrt{\lambda}\cos(\sqrt{\lambda}) - 2\left(A\cos(\sqrt{\lambda}) + B\sin(\sqrt{\lambda})\right) &= 0\\
-A\sqrt{\lambda}\sin(\sqrt{\lambda}) + B\sqrt{\lambda}\cos(\sqrt{\lambda}) - 2A\cos(\sqrt{\lambda}) - 2B\sin(\sqrt{\lambda}) &= 0\\
(-2B - A\sqrt{\lambda})\sin(\sqrt{\lambda}) + (B\sqrt{\lambda} - 2A)\cos(\sqrt{\lambda}) &= 0\\
(B\sqrt{\lambda} - 2A)\cos(\sqrt{\lambda}) &= (2B + A\sqrt{\lambda})\sin(\sqrt{\lambda})\\
\frac{B\sqrt{\lambda} - 2A}{2B + A\sqrt{\lambda}} &= \tan(\sqrt{\lambda})\\
\frac{2B\sqrt{\lambda}}{2B + A\sqrt{\lambda}} &= \tan(\sqrt{\lambda})\\
\frac{2B\sqrt{\lambda}}{2B - \frac{1}{2}B\lambda} &= \tan(\sqrt{\lambda})\\
\frac{2\sqrt{\lambda}}{2 - \frac{1}{2}\lambda} &= \tan(\sqrt{\lambda})\\
\frac{4\sqrt{\lambda}}{4 - \lambda} &= \tan(\sqrt{\lambda}).
\end{aligned}$$

Now for the final case where $\lambda < 0$ we have $X(x) = A\,\mathrm{e}^{\sqrt{\lambda}x} + B\,\mathrm{e}^{-\sqrt{\lambda}x}$. Now we have to use our boundary conditions so

$$\begin{aligned}
\sqrt{\lambda}A\,\mathrm{e}^{\sqrt{\lambda}0} - \sqrt{\lambda}B\,\mathrm{e}^{-\sqrt{\lambda}0} + 2\left(A\,\mathrm{e}^{\sqrt{\lambda}0} + B\,\mathrm{e}^{-\sqrt{\lambda}0}\right) &= 0\\
\sqrt{\lambda}A - \sqrt{\lambda}B + 2(A+B) &= 0\\
(2+\sqrt{\lambda})A + (2-\sqrt{\lambda})B &= 0\\
A = \frac{\sqrt{\lambda}-2}{\sqrt{\lambda}+2}B.&
\end{aligned}$$

Next we get

$$\begin{aligned}
\sqrt{\lambda}A\,\mathrm{e}^{\sqrt{\lambda}} - \sqrt{\lambda}B\,\mathrm{e}^{-\sqrt{\lambda}} - 2\left(A\,\mathrm{e}^{\sqrt{\lambda}} + B\,\mathrm{e}^{-\sqrt{\lambda}}\right) &= 0\\
(\sqrt{\lambda}-2)A\,\mathrm{e}^{\sqrt{\lambda}} &= (\sqrt{\lambda}+2)B\,\mathrm{e}^{-\sqrt{\lambda}}\\
(\sqrt{\lambda}-2)\frac{\sqrt{\lambda}-2}{\sqrt{\lambda}+2}B\,\mathrm{e}^{\sqrt{\lambda}} &= (\sqrt{\lambda}+2)B\,\mathrm{e}^{-\sqrt{\lambda}}\\
(\sqrt{\lambda}-2)^2B\,\mathrm{e}^{\sqrt{\lambda}} &= (\sqrt{\lambda}+2)^2B\,\mathrm{e}^{-\sqrt{\lambda}}
\end{aligned}$$

Next we carefully FOIL out the quadratic terms and move everything to one side

$$\begin{aligned}
(\sqrt{\lambda}-2)^2 B\,\mathrm{e}^{\sqrt{\lambda}} &= (\sqrt{\lambda}+2)^2 B\,\mathrm{e}^{-\sqrt{\lambda}} \\
(\lambda - 4\sqrt{\lambda}+4) B\,\mathrm{e}^{\sqrt{\lambda}} &= (\lambda + 4\sqrt{\lambda}+4) B\,\mathrm{e}^{-\sqrt{\lambda}} \\
(\lambda - 4\sqrt{\lambda}+4) B\,\mathrm{e}^{\sqrt{\lambda}} - (\lambda + 4\sqrt{\lambda}+4) B\,\mathrm{e}^{-\sqrt{\lambda}} &= 0 \\
\lambda B\,\mathrm{e}^{\sqrt{\lambda}} - \lambda B\,\mathrm{e}^{-\sqrt{\lambda}} - 4\sqrt{\lambda} B\,\mathrm{e}^{\sqrt{\lambda}} - 4\sqrt{\lambda} B\,\mathrm{e}^{-\sqrt{\lambda}} + 4B\,\mathrm{e}^{\sqrt{\lambda}} - 4B\,\mathrm{e}^{-\sqrt{\lambda}} &= 0 \\
\lambda B 2\sinh\sqrt{\lambda} - 4\sqrt{\lambda} B(2\cosh\sqrt{\lambda}) + 4B(2\sinh\sqrt{\lambda}) &= 0 \\
\lambda B 2\sinh\sqrt{\lambda} + 4B(2\sinh\sqrt{\lambda}) &= 4\sqrt{\lambda} B(2\cosh\sqrt{\lambda}) \\
(\lambda+4) B 2\sinh\sqrt{\lambda} &= 4\sqrt{\lambda}(B 2\cosh\sqrt{\lambda}) \\
\frac{B2\sinh\sqrt{\lambda}}{B2\cosh\sqrt{\lambda}} &= \frac{4\sqrt{\lambda}}{\lambda+4} \\
\tanh\sqrt{\lambda} &= \frac{4\sqrt{\lambda}}{\lambda+4}.
\end{aligned}$$

2: (a) Consider the following IBVP:

$$\begin{cases} (x^2\phi')' + \lambda\phi = 0, & 1 < x < 2, \\ \phi(1) = 0 = \phi(2). \end{cases}$$

Figure out p, q, w, h_1 and h_2. Write down the properties satisfied by eigenvalues and eigenfunctions by Sturm-Liouville theorem. (e.g. orthogonality of eigenfunctions, completeness of basis, etc.) Solve the eigenpairs $\{(\lambda_k, \phi_k\}$.

Solution:
TODO:

(b) Then, use the eigenpairs to solve the following IBVP:

$$\begin{cases} u_t = (x^2 u_x)_x - u, & 1 < x < 2, t > 0, \\ u(1,t) = u(2,t) = 0, & t > 0, \\ u(x,0) = f(x), & x \in (1,2) \end{cases}$$

(Hint: for Euler's ODE: $aX'' + bX' + cX = 0$, we have the ansatz $X = x^r$ and the characteristic root equation $ar(r-1) + br + x = 0$. If $r_1 \neq r_2$, then $X = c_1 x^{r_1} + c_2 x^{r_2}$; if $r_1 = r_2 = r$, then $X = c_1 x^r + c_2 x^r \log x$; if $r = \nu + \mathrm{i}\mu$ is a complex root, then $X = c_1 x^\nu \cos(\mu \log x) + c_2 x^\nu \sin(\mu \log x)$.)

Solution:
TODO

3: Consider the following BVP:

$$\begin{cases} x^2y'' + xy' + (x^2\lambda^2 - n^2)y = 0, & x \in (0, L), \\ y'(0) = 0 or y(0) = 0, y(L) = 0, \end{cases}$$

where λ and n are real numbers. $L > 0$ is a constant, as well.

(a) rewrite the BVP as the sturm-liouville form and write down the definition of p, q, w, h_1, and h_2:

Solution:
TODO:

(b) write down the orthogonality conditions satisfied by the eigenfunction.

Solution:
TODO:

(c) We have the fact that

$$J_n(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n-k)!} \left(\frac{x}{2}\right)^{n+2k}, \quad n = 0, 1, ...$$

are solutions to

$$\begin{cases} xy'' + y' + (x - \frac{n^2}{x})y = 0, & x \in (0, \infty), \\ y(\infty) = 0. \end{cases}$$

J_n is named the n-th order Bessel function. Plot the figures of J_0, J_1, and J_2 by matlab or python. We have the facts that $J_n(x)$ has infinitely many zeros $\nu_{nm}, m = 1, ...$ and J_n is bounded as $r \to 0$. By using $J_n(x)$ and nu_{nm} to find all eigenpairs $\left\{\lambda_{nm}^2, y_{nm}(x)\right\}_{m=1}^{\infty}$ to the given BVP.

Solution:
TODO:

(d) Solve the following IBVP:

$$\begin{cases} u_t = \Delta u, & (x, y) \in B_a(0), t > 0, \\ u(x, t) = 0, & (x, y) \in \partial B_a(0), t > 0, \\ u(x, 0) = u_0(x, y), & (x, y) \in B_a(0), \end{cases}$$

where $B_a(0) \subset \mathbb{R}^2$ is the disc with radius $a > 0$. (Hint: for (d), recall in the polar coordinate (r, θ), $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$. Use separation of variables $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$ to solve the IBVP. Θ mode is 2π-periodic.)

Solution:
TODO:

4: Solve the following signaling problem:

$$\begin{cases} u_t + cu_x = 0, & 0 < x < +\infty, \\ u(0,t) = g(t), u(x,0) = 0, & x \geq 0, \end{cases}$$

where $c > 0$ is a constant.

Solution:
TODO:

5: Olver 2.2.17

Solution:
TODO:

6: Olver 2.2.31

Solution:
TODO:

7: (a) Solve the ODE:

$$\frac{du}{ds} + u = 2\,\mathrm{e}^x$$

by the integral factor method. And use the same technique to solve the following damping heat equation:

$$\begin{cases} \nu_t = \nu_{xx} = \nu, & x \in (0,\pi), t > 0, \\ \nu(0,t) = \nu(\pi,t) = 0. & t > 0, \\ \nu(x,0) = \nu_0(x), & x \in (0,\pi). \end{cases}$$

(Hint: test the BHS of ODE against e^x and then integrate to solve it, where e^x is called the integral factor.)

Solution:
TODO:

(b) Solving the following transport equation:

$$u_t + tu_x = u, -\infty < x < +\infty, t > 0,$$

with initial condition

$$u(x,1) = f(x), 0 \le x \le 1,$$

where f is continuous. Compute $u(x,t)$ by the method of characteristics and find the subregion in $-\infty < x < +\infty, t > 0$, where the data on $t = 1$ determines this solution. Plot this subregion.

Solution:
TODO: