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AMATH 503

HOMEWORK 6

Exercises come from *Introduction to Partial Differential Equations by Peter J. Olver* as well as supplemented by instructor provided exercises.

1: Solve the following wave equations by using D'Alambert's formula:

$$u_{tt} - 4u_{xx} = 0, -\infty < x < \infty, t > 0,$$

(a) $u(x,0) = \mathrm{e}^x, u_t(x,0) = \sin(x)$.

Solution:
In order to use D'Alambert's formula we need to identify that

$$\begin{aligned} c &= 2, \\ u(x,0) &= \mathrm{e}^x = f(x), \\ u_t(x,0) &= \sin x = g(x). \end{aligned}$$

Therefore, applying the formula

$$u(x,t) = \frac{1}{2}\Big[f(x-ct) + f(x+ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz$$

we have

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big[f(x-ct) + f(x+ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz \\ &= \frac{1}{2}\Big[\mathrm{e}^{(x-2t)} + \mathrm{e}^{(x+2t)}\Big] + \frac{1}{4}\int_{x-2t}^{x+2t} \sin(z)dz \end{aligned}$$

Let's now calculate the integral on the right

$$\int_{x-ct}^{x+ct} \sin(z)dz = -\cos(z)\Big|_{x-2t}^{x+2t} = -\cos(x+2t) - (-\cos(x-2t)) = \cos(x-2t) - \cos(x+2t)$$

Therefore our final solution is

$$u(x,t) = \frac{1}{2}\Big[\mathrm{e}^{(x-2t)} + \mathrm{e}^{(x+2t)}\Big] + \frac{1}{4}\Big[\cos(x-2t) - \cos(x+2t)\Big]$$

□

(b) $u(x,0) = \sin(x), u_t(x,0) = \cos(2x)$.

Solution:
This time we have $f(x) = \sin(x)$ and $g(x) = \cos(2x)$ while $c = 2$ still. Therefore

the integral we need to calculate is

$$\int_{x-ct}^{x+ct} \cos(2z)dz = \frac{1}{2}\sin(2z)\Big|_{x-2t}^{x+2t}$$
$$= \frac{1}{2}\Big(\sin(2x+4t) - \sin(2x-4t)\Big)$$

Therefore, by D'Alambert's formula we have

$$u(x,t) = \frac{1}{2}\Big[\sin(x-2t) + \sin(x+2t)\Big] + \frac{1}{8}\Big[\sin(2x+4t) - \sin(2x-4t)\Big]$$

$\square$

2: Olver: 2.4.11 (c)
Solve the forced IVP

$$\begin{cases} u_{tt} - 4u_{xx} = \cos 2t, \quad -\infty < x < \infty, t \geq 0 \\ u(0,x) = \sin x, \\ u_t(0,x) = \cos x, \end{cases}$$

Solution:
Similar to problem 1 we want to identify that the functions f, g, and F and the constant c to use **Theorem 2.18** from Olver. This time we also want to identify the force F, all together we have

$$\begin{aligned} c &= 2 \\ f(x) &= \sin x \\ g(x) &= \cos x \\ F(x,t) &= \cos 2t. \end{aligned}$$

Which gives us

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big[f(x-ct) + f(x+ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(y,s)\, dy\, ds \\ &= \frac{1}{2}\Big[\sin(x-2t) + \sin(x+2t)\Big] + \frac{1}{4}\int_{x-2t}^{x+2t} \cos(z)dz + \frac{1}{4}\int_0^t \int_{x-2(t-s)}^{x+2(t-s)} \cos(2s)\, dy\, ds \end{aligned}$$

We will now calculate the necessary integrals beginning first with the integral over $\cos z$

$$\int_{x-2t}^{x+2t} \cos(z)dz = \sin z\Big|_{x-2t}^{x+2t} = \sin(x+2t) - \sin(x-2t)$$

Next the integral over $\cos 2s$

$$\begin{aligned} \int_0^t \int_{x-2(t-s)}^{x+2(t-s)} \cos(2s)\, dy\, ds &= \int_0^t \cos(2s) \int_{x-2(t-s)}^{x+2(t-s)} dy\, ds \\ &= \int_0^t \cos(2s) y\Big|_{x-2(t-s)}^{x+2(t-s)} ds \\ &= \int_0^t \cos(2s)\Big[(x+2(t-s)) - (x-2(t-s))\Big]\, ds \\ &= \int_0^t \cos(2s)\Big[x+2(t-s) - x + 2(t-s)\Big]\, ds \\ &= \int_0^t \cos(2s)4(t-s)\, ds \\ &= 4\left[t\int_0^t \cos(2s)ds - \int_0^t s\cos(2s)ds\right] \\ &= 4\left[\frac{t}{2}\sin(2t) - \int_0^t s\cos(2s)ds\right] \end{aligned}$$

Using integration by parts on the remaining integral we have

$$\begin{aligned}
\int_0^t s\cos(2s)ds &= \frac{1}{2}s\sin(2s)\Big|_0^t - \int_0^t \frac{1}{2}\sin(2s)ds \\
&= \frac{1}{2}t\sin(2t) + \frac{1}{2}\cos(2s)\Big|_0^t \\
&= \frac{1}{2}t\sin(2t) + \frac{1}{2}\cos(2t) - \frac{1}{2} \\
&= \frac{1}{2}\left(t\sin(2t) + \cos(2t) - 1\right).
\end{aligned}$$

Combining these integral back up the chain of equalities we have the final solution

$$u(x,t) = \frac{1}{2}\Big[\sin(x-2t) + \sin(x+2t)\Big] + \frac{1}{4}\Big[\sin(x+2t) - \sin(x-2t)\Big] + \frac{1}{2}\Big[1 - \cos(2t)\Big]$$

$\square$

3: Separation of variables to solve

$$\begin{cases} u_{tt} = u_{xx} + \mathrm{e}^{-t}\sin(x), & 0 < x < \pi, t > 0 \\ u(x,0) = \sin(3x), u_t(x,0) = 0, & 0 < x < \pi, \\ u(0,t) = 1, u(\pi,t) = 0, & t > 0. \end{cases}$$

Solution:
TODO

4: (Bonus question) Solve the following wave equation

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & 0 < x < \infty, 0 < t < \infty \\ u(0,t) = 1, & t > 0, \\ u(x,0) = x, u_t(x,0) = \mathrm{e}^x, & x \geq 0. \end{cases}$$

Solution:
TODO

5: Separation of variables to solve

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < \pi, 0 < y < \pi \\ u(0, y) = u_x(\pi, y) = u(x, 0) = 0 \\ u(x, \pi) = \sin\left(\frac{x}{2}\right) - 2\sin\left(\frac{3x}{2}\right). \end{cases}$$

Solution:

TODO

6: Olver: 4.3.34 (b) Solve the following boundary value problems for the Laplace equation on the annulus $1 < r < 2$ with

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \\ u(1,\theta) = 0, u(2,\theta) = \cos\theta, \\ 1 \le r < 2, 0 \le \theta < 2\pi \end{cases}$$

Is this right?

Solution:
TODO

7: (Bonus) Consider the following Laplace equation

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & 0 \le r < 1, 0 \le \theta < 2\pi \\ u_r(1,\theta) + u(1,\theta) = \cos(2\theta) \end{cases}$$

Use the method of separation of variables to find a solution.

Solution:
TODO