Hunter Lybbert
Student ID: 2426454
05-22-25
AMATH 503

HOMEWORK 6

Exercises come from *Introduction to Partial Differential Equations by Peter J. Olver* as well as supplemented by instructor provided exercises.

1: Solve the following wave equations by using D'Alambert's formula:

$$u_{tt} - 4u_{xx} = 0, -\infty < x < \infty, t > 0,$$

(a) $u(x, 0) = \mathrm{e}^x, u_t(x, 0) = \sin(x)$.

Solution:
In order to use D'Alambert's formula we need to identify that

$$\begin{aligned} c &= 2, \\ u(x, 0) &= \mathrm{e}^x = f(x), \\ u_t(x, 0) &= \sin x = g(x). \end{aligned}$$

Therefore, applying the formula

$$u(x, t) = \frac{1}{2}\Big[f(x - ct) + f(x + ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz$$

we have

$$\begin{aligned} u(x, t) &= \frac{1}{2}\Big[f(x - ct) + f(x + ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz \\ &= \frac{1}{2}\Big[\mathrm{e}^{(x-2t)} + \mathrm{e}^{(x+2t)}\Big] + \frac{1}{4}\int_{x-2t}^{x+2t} \sin(z)dz \end{aligned}$$

Let's now calculate the integral on the right

$$\int_{x-ct}^{x+ct} \sin(z)dz = -\cos(z)\Big|_{x-2t}^{x+2t} = -\cos(x + 2t) - (-\cos(x - 2t)) = \cos(x - 2t) - \cos(x + 2t)$$

Therefore our final solution is

$$u(x, t) = \frac{1}{2}\Big[\mathrm{e}^{(x-2t)} + \mathrm{e}^{(x+2t)}\Big] + \frac{1}{4}\Big[\cos(x - 2t) - \cos(x + 2t)\Big]$$

$\square$

(b) $u(x, 0) = \sin(x), u_t(x, 0) = \cos(2x)$.

Solution:
This time we have $f(x) = \sin(x)$ and $g(x) = \cos(2x)$ while $c = 2$ still. Therefore

the integral we need to calculate is

$$\begin{aligned}\int_{x-ct}^{x+ct} \cos(2z)dz &= \frac{1}{2}\sin(2z)\Big|_{x-2t}^{x+2t} \\ &= \frac{1}{2}\Big(\sin(2x+4t)-\sin(2x-4t)\Big)\end{aligned}$$

Therefore, by D'Alambert's formula we have

$$u(x,t)=\frac{1}{2}\Big[\sin(x-2t)+\sin(x+2t)\Big]+\frac{1}{8}\Big[\sin(2x+4t)-\sin(2x-4t)\Big]$$

$\square$

2: Olver: 2.4.11 (c)
Solve the forced IVP

$$\begin{cases} u_{tt} - 4u_{xx} = \cos 2t, & -\infty < x < \infty, t \geq 0 \\ u(0,x) = \sin x, \\ u_t(0,x) = \cos x, \end{cases}$$

Solution:
Similar to problem 1 we want to identify that the functions f, g, and F and the constant c to use **Theorem 2.18** from Olver. This time we also want to identify the force F, all together we have

$$\begin{aligned} c &= 2 \\ f(x) &= \sin x \\ g(x) &= \cos x \\ F(x,t) &= \cos 2t. \end{aligned}$$

Which gives us

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big[f(x-ct) + f(x+ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(y,s)\, dy\, ds \\ &= \frac{1}{2}\Big[\sin(x-2t) + \sin(x+2t)\Big] + \frac{1}{4}\int_{x-2t}^{x+2t} \cos(z)dz + \frac{1}{4}\int_0^t \int_{x-2(t-s)}^{x+2(t-s)} \cos(2s)\, dy\, ds \end{aligned}$$

We will now calculate the necessary integrals beginning first with the integral over $\cos z$

$$\int_{x-2t}^{x+2t} \cos(z)dz = \sin z\Big|_{x-2t}^{x+2t} = \sin(x+2t) - \sin(x-2t)$$

Next the integral over $\cos 2s$

$$\begin{aligned} \int_0^t \int_{x-2(t-s)}^{x+2(t-s)} \cos(2s)\, dy\, ds &= \int_0^t \cos(2s) \int_{x-2(t-s)}^{x+2(t-s)} dy\, ds \\ &= \int_0^t \cos(2s) y\Big|_{x-2(t-s)}^{x+2(t-s)} ds \\ &= \int_0^t \cos(2s)\Big[(x+2(t-s)) - (x-2(t-s))\Big]\, ds \\ &= \int_0^t \cos(2s)\Big[x + 2(t-s) - x + 2(t-s)\Big]\, ds \\ &= \int_0^t \cos(2s) 4(t-s)\, ds \\ &= 4\left[t\int_0^t \cos(2s)ds - \int_0^t s\cos(2s)ds\right] \\ &= 4\left[\frac{t}{2}\sin(2t) - \int_0^t s\cos(2s)ds\right] \end{aligned}$$

Using integration by parts on the remaining integral we have

$$\begin{aligned}
\int_0^t s\cos(2s)ds &= \frac{1}{2}s\sin(2s)\Big|_0^t - \int_0^t \frac{1}{2}\sin(2s)ds \\
&= \frac{1}{2}t\sin(2t) + \frac{1}{2}\cos(2s)\Big|_0^t \\
&= \frac{1}{2}t\sin(2t) + \frac{1}{2}\cos(2t) - \frac{1}{2} \\
&= \frac{1}{2}\left(t\sin(2t) + \cos(2t) - 1\right).
\end{aligned}$$

Combining these integral back up the chain of equalities we have the final solution

$$u(x,t) = \frac{1}{2}\Big[\sin(x-2t) + \sin(x+2t)\Big] + \frac{1}{4}\Big[\sin(x+2t) - \sin(x-2t)\Big] + \frac{1}{2}\Big[1 - \cos(2t)\Big]$$

□

3: Separation of variables to solve

$$\begin{cases} u_{tt} = u_{xx} + \mathrm{e}^{-t}\sin(x), & 0 < x < \pi, t > 0 \\ u(x,0) = \sin(3x), u_t(x,0) = 0, & 0 < x < \pi, \\ u(0,t) = 1, u(\pi,t) = 0, & t > 0. \end{cases}$$

Solution:
Let's begin by getting homogenous DBC's. We do this by introducing the substitution $u = v + w$. Therefore we have $v = u - w$ and we need

$$v(0,t) = v(\pi,t) = 0$$

which implies we need

$$\begin{aligned} v(0,t) &= v(\pi,t) = 0 \\ u(0,t) - w(0) &= u(\pi,t) - w(\pi) = 0 \\ 1 - w(0) &= -w(\pi) = 0 \end{aligned}$$

Implying that $w(0) = 1$ and $w(\pi) = 0$. One such function which satisfies this is $w(x) = \cos(x/2)$ we could also use $w(x) = 1 - x/\pi$. Let's see where the $\cos(x/2)$ goes right or wrong. Now after this transformation we now have the IBVP with DBC as follows

$$\begin{cases} v_{tt} + w_{tt} = v_{xx} + w_{xx} + \mathrm{e}^{-t}\sin(x), & 0 < x < \pi, t > 0 \\ v(x,0) = \sin(3x) - w(x), v_t(x,0) = 0, & 0 < x < \pi, \\ v(0,t) = v(\pi,t) = 0, & t > 0. \end{cases}$$

Notice, $w_{tt} = 0$ in either case, however $w_{xx} = 0$ only if we choose w to be linear in terms of x rather than trigonometric. Therefore, we actually are motivated to choose

$$w(x) = 1 - x/\pi.$$

Hence, we want to solve

$$\begin{cases} v_{tt} = v_{xx} + \mathrm{e}^{-t}\sin(x), & 0 < x < \pi, t > 0 \\ v(x,0) = \sin(3x) - 1 + \frac{x}{\pi}, v_t(x,0) = 0, & 0 < x < \pi, \\ v(0,t) = v(\pi,t) = 0, & t > 0. \end{cases}$$

Now let's first solve the homogenous portion of this $\tilde{v}_{tt} = \tilde{v}_{xx}$ to help us find the basis to expand our forcing term with respect to. Using separation of variables we have $\tilde{v}(x,t) = X(x)T(t)$ which implies

$$\begin{aligned} \tilde{v}_{tt} &= \tilde{v}_{xx} \\ XT'' &= X''T \\ \frac{T''}{T} &= \frac{X''}{X} = -\lambda \end{aligned}$$

given us both

$$T'' + \lambda T = 0 \quad \text{and} \quad X'' + \lambda X = 0$$

with $X(0) = X(\pi) = 0$. Since we have the DBCs for X we know the portion of our solution basis with respect to X will need to be in terms of the eigenpairs

$$\left\{n^2, \sin\left(nx\right)\right\}_{n=1}^{\infty}$$

Thus, using the table from Olver 141, we have

$$\tilde{v}(x,t) = \sum_{n=1}^{\infty} C_n(t)\sin(nx).$$

We can therefore use $\sin(nx)$ to expand the inhomogeneous forcing term in the IBVP for v. Therefore we have

$$\mathrm{e}^{-t}\sin(x) = \sum_{n=1}^{\infty} D_n(t)\sin(nx),$$

with,

$$D_n(t) = \frac{2}{\pi}\int_0^{\pi} \mathrm{e}^{-t}\sin(x)\sin(nx)dx = \begin{cases} 0, & n \neq 1, \\ \mathrm{e}^{-t}, & \text{otherwise.} \end{cases}$$

We can now put this together in the original IBVP for v to get

$$\begin{aligned} v_{tt} &= v_{xx} + \mathrm{e}^{-t}\sin(x) \\ \left(\sum_{n=1}^{\infty} C_n(t)\sin(nx)\right)_{tt} &= \left(\sum_{n=1}^{\infty} C_n(t)\sin(nx)\right)_{xx} + \sum_{n=1}^{\infty} D_n(t)\sin(nx) \\ \sum_{n=1}^{\infty} C_n''(t)\sin(nx) &= \sum_{n=1}^{\infty} -n^2 C_n(t)\sin(nx) + \sum_{n=1}^{\infty} D_n(t)\sin(nx). \end{aligned}$$

Which implies

$$C_n''(t) = -n^2 C_n(t) + D_n(t).$$

From our various conditions we also need

$$v(x,0) = \sum_{n=1}^{\infty} C_n(0)\sin(nx) = \sin(3x) - 1 + \frac{x}{\pi}$$

which gives rise to

$$C_n(0) = \frac{2}{\pi}\int_0^{\pi}\left(\sin(3x) - 1 + \frac{x}{\pi}\right)\sin(nx)dx.$$

And finally, we want

$$v_t(x,0) = \sum_{n=1}^{\infty} C_n'(0)\sin(nx) = 0 \quad \text{implying} \quad C_n'(0) = \frac{2}{\pi}\int_0^{\pi} 0\sin(nx)dx = 0.$$

Let's calculate $C_n(0)$

$$\begin{aligned} C_n(0) &= \frac{2}{\pi}\int_0^{\pi}\left(\sin(3x) - 1 + \frac{x}{\pi}\right)\sin(nx)dx \\ &= \frac{2}{\pi}\left[\int_0^{\pi}\sin(3x)\sin(nx)dx - \int_0^{\pi}\sin(nx)dx + \int_0^{\pi}\frac{x}{\pi}\sin(nx)dx\right] \\ &= \frac{2}{\pi}\left[I_1 - I_2 + I_3\right]. \end{aligned}$$

First, we have

$$I_1 = \begin{cases} 0, & n \neq 3, \\ \pi/2, & \text{otherwise.} \end{cases}$$

Next, we have

$$I_2 = \int_0^\pi \sin(nx)dx = -\frac{\cos(\pi n)}{n} + \frac{\cos(0)}{n} = \frac{1-(-1)^n}{n} = \begin{cases} 2/n, & n \text{ is odd,} \\ 0, & n \text{ is even.} \end{cases}$$

Finally, IBP on I_3 gives us

$$I_3 = \frac{1}{\pi}\int_0^\pi x\sin(nx)dx = \frac{1}{\pi}\left[-\frac{\pi\cos(n\pi)}{n} + \int_0^\pi \frac{\cos(nx)}{n}dx\right] = \frac{1}{\pi}\left[\frac{\pi(-1)^{n+1}}{n} + \cancel{\frac{\sin(n\pi)}{n^2}} - \cancel{\frac{\sin(0)}{n^2}}\right] = \frac{(-1)^{n+1}}{n}.$$

Notice,

$$\begin{aligned} C_n(0) &= \frac{2}{\pi}\left[I_1 - I_2 + I_3\right] \\ &= \frac{2}{\pi}\left[I_1 - \frac{1-(-1)^n}{n} + \frac{(-1)^{n+1}}{n}\right] \\ &= \frac{2}{\pi}\left[I_1 - \frac{1}{n} + \cancel{\frac{(-1)^n}{n}} - \cancel{\frac{(-1)^n}{n}}\right] \\ &= \begin{cases} \frac{2}{\pi}\left[\pi/2 - \frac{1}{n}\right] & n = 3 \\ \frac{2}{\pi}\left[0 - \frac{1}{n}\right] & n \neq 3 \end{cases} \\ &= \begin{cases} 1 - \frac{2}{3\pi} & n = 3 \\ -\frac{2}{n\pi} & n \neq 3 \end{cases}. \end{aligned}$$

Bringing these things all together, we have the following conditions for our ODE with respect to $C_n(t)$

$$\begin{cases} C_n''(t) = -n^2 C_n(t) + D_n(t) \\ C_n(0) = \begin{cases} 1 - \frac{2}{3\pi} & n = 3 \\ -\frac{2}{n\pi} & n \neq 3 \end{cases} \\ C_n'(0) = 0. \end{cases}$$

Where $D_n(t)$ is defined as above with the condition on $n = 1$ or not. Let's first solve it given the case that $n = 1$, therefore $D_1(t) = \mathrm{e}^{-t}$ and we have

$$\begin{aligned} C_1''(t) &= -C_1(t) + \mathrm{e}^{-t} \\ C_1''(t) + C_1(t) &= \mathrm{e}^{-t}. \end{aligned}$$

Let's assume an ansatz of $C_1(t) = \mu\,\mathrm{e}^{-t}$, then

$$\begin{aligned} C_1''(t) + C_1(t) &= \mathrm{e}^{-t} \\ \mu\,\mathrm{e}^{-t} + \mu\,\mathrm{e}^{-t} &= \mathrm{e}^{-t} \\ 2\mu\,\mathrm{e}^{-t} &= \mathrm{e}^{-t} \\ \mu &= 1/2. \end{aligned}$$

Thus in this case we have $C_1(t) = \frac{1}{2}\mathrm{e}^{-t} + \sigma\cos t + \eta\sin t$. Let's solve for these unknowns using the conditions we were given

$$C_1(0) = \frac{1}{2} + \sigma = -\frac{2}{\pi} \implies \sigma = -\frac{1}{2} - \frac{2}{\pi}.$$

And we also have

$$\begin{aligned}
C_1'(0) &= -\frac{1}{2}\,\mathrm{e}^0 - (-1 - \frac{2}{\pi})\sin 0 + \eta\cos 0 \\
0 &= -\frac{1}{2} + \eta \\
\frac{1}{2} &= \eta
\end{aligned}$$

Thus

$$C_1(t) = \frac{1}{2}\,\mathrm{e}^{-t} - \left(\frac{1}{2} + \frac{2}{\pi}\right)\cos t + \frac{1}{2}\sin t.$$

Now for the case where $n \neq 1$ we know $D_n(t) = 0$ therefore our ODE reduces to

$$C_n''(t) = C_n(t)$$

Which has the common solution of $C_n(t) = A\cos(nt) + B\sin(nt)$. Let's apply the conditions. We get

$$C_n(0) = A = \begin{cases} 1 - \frac{2}{3\pi} & n = 3 \\ -\frac{2}{n\pi} & n \neq 3 \end{cases}$$

and

$$\begin{aligned}
C_n'(0) &= -An\sin(0) - Bn\cos(0) \\
0 &= -Bn \\
0 &= B
\end{aligned}$$

Hence, when $n \neq 1$ we have

$$C_n(t) = \begin{cases} \left(1 - \frac{2}{3\pi}\right)\cos(3t) & n = 3 \\ -\frac{2}{n\pi}\cos(nt) & n \neq 3 \end{cases}.$$

Bringing it all together then we have the solution for $u(x,t)$ is as follows

$$\begin{aligned}
u(x,t) &= w(x) + v(x,t) \\
&= w(x) + \sum_{n=1}^{\infty} C_n(t)\sin(nx) \\
&= w(x) + C_1(t)\sin(x) + C_2(t)\sin(2x) + C_3(t)\sin(3x) + \sum_{n=4}^{\infty} C_n(t)\sin(nx) \\
&= (1 - x/\pi) + \left[\frac{1}{2}\,\mathrm{e}^{-t} - \left(\frac{1}{2} + \frac{2}{\pi}\right)\cos t + \frac{1}{2}\sin t\right]\sin(x) \\
&\quad - \frac{1}{\pi}\cos(nt)\sin(2x) + \left[\left(1 - \frac{2}{3\pi}\right)\cos(3t)\right]\sin(3x) - \sum_{n=4}^{\infty}\frac{2}{n\pi}\cos(nt)\sin(nx).
\end{aligned}$$

$\square$

4: (Bonus question) Solve the following wave equation

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & 0 < x < \infty, 0 < t < \infty \\ u(0,t) = 1, & t > 0, \\ u(x,0) = x, u_t(x,0) = \mathrm{e}^x, & x \geq 0. \end{cases}$$

Solution:

TODO

5: Separation of variables to solve

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < \pi, 0 < y < \pi \\ u(0,y) = u_x(\pi,y) = u(x,0) = 0 \\ u(x,\pi) = \sin\left(\frac{x}{2}\right) - 2\sin\left(\frac{3x}{2}\right). \end{cases}$$

Solution:
In order to use separation of variables we let $u(x,y) = X(x)Y(y)$ and thus we have

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \\ X''Y + XY'' &= 0 \\ \frac{X''}{X} + \frac{Y''}{Y} &= 0 \\ \frac{X''}{X} &= -\frac{Y''}{Y} = -\lambda \end{aligned}$$

which gives us $X'' + \lambda X = 0$ and $Y'' - \lambda Y = 0$. We also have the boundary conditions for X which are $X(0) = X'(\pi) = 0$. Using these conditions to solve for $X(x)$ we have

$$X(x) = A_n \cos(\sqrt{\lambda}x) + B_n \sin(\sqrt{\lambda}x)$$

Using the BCs we have, $X(0) = A = 0$, then $X'(\pi) = B_n\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0$. We want $B_n \neq 0$ such that this is not a trivial solution. Therefore

$$\begin{aligned} \cos(\sqrt{\lambda}\pi) &= 0 \\ \implies \sqrt{\lambda_n}\pi &= \frac{(2n-1)\pi}{2} \\ \sqrt{\lambda_n} &= \frac{2n-1}{2} \\ \lambda_n &= \left(\frac{2n-1}{2}\right)^2 \\ \lambda_n &= \left(n - \frac{1}{2}\right)^2 \end{aligned}$$

Finally, we have the eigenpair

$$\lambda_n = \left(n - \frac{1}{2}\right)^2, \quad X_n(x) = \sin\left(\left(n - \frac{1}{2}\right)x\right)$$

for $n = 1, 2, 3, ...$ Now since $\lambda_n = \left(n - \frac{1}{2}\right)^2 > 0$ we know that when solving

$$Y'' - \lambda Y = 0$$

for Y we will have the solution of the form

$$Y(y) = C_n \cosh(\sqrt{\lambda_n}y) + D_n \sinh(\sqrt{\lambda_n}y).$$

We can apply the condition $u(x,0) = Y(0) = 0$ to get

$$Y(0) = C_n \cosh(0) + D_n \sinh(0) = C_n = 0.$$

Hence,

$$Y(y) \propto \sinh(\sqrt{\lambda_n}y).$$

By Superposition we have

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\left(n - \frac{1}{2}\right)y\right) \sin\left(\left(n - \frac{1}{2}\right)x\right).$$

Now to determine the coefficients B_n let's use our condition on $u(x,\pi)$ and match terms.

$$\sin\left(\frac{x}{2}\right) - 2\sin\left(\frac{3x}{2}\right) = \sum_{n=1}^{\infty} B_n \sinh\left(\left(n - \frac{1}{2}\right)\pi\right) \sin\left(\left(n - \frac{1}{2}\right)x\right)$$

Notice, we see terms on the left which line up with the components from sin when $n = 1$ and $n = 2$. Therefore $B_n = 0$ for all n except those two. Hence, the previous equation reduces to

$$\sin\left(\frac{x}{2}\right) - 2\sin\left(\frac{3x}{2}\right) = B_1 \sinh\left(\frac{\pi}{2}\right) \sin\left(\frac{x}{2}\right) + B_2 \sinh\left(\frac{3\pi}{2}\right) \sin\left(\frac{3x}{2}\right).$$

Now matching coefficients this implies

$$\begin{aligned} 1 &= B_1 \sinh\left(\frac{\pi}{2}\right) \\ \frac{1}{\sinh\left(\frac{\pi}{2}\right)} &= B_1 \end{aligned}$$

and

$$\begin{aligned} -2 &= B_2 \sinh\left(\frac{3\pi}{2}\right) \\ -\frac{2}{\sinh\left(\frac{3\pi}{2}\right)} &= B_2. \end{aligned}$$

Hence, our final solution is

$$u(x,y) = \frac{1}{\sinh\left(\frac{\pi}{2}\right)} \sinh\left(\frac{y}{2}\right) \sin\left(\frac{x}{2}\right) - \frac{2}{\sinh\left(\frac{3\pi}{2}\right)} \sinh\left(\frac{3y}{2}\right) \sin\left(\frac{3x}{2}\right).$$

$\square$

6: Olver: 4.3.34 (b) Solve the following boundary value problems for the Laplace equation on the annulus $1 < r < 2$ with

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \\ u(1,\theta) = 0, u(2,\theta) = \cos\theta, \\ 1 \le r < 2, 0 \le \theta < 2\pi \end{cases} \qquad \textbf{Is this right?}$$

Solution:

We use separation of variables with $u(r,\theta) = \Theta(\theta)R(r)$ giving us

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0 \\ \Theta R'' + \frac{1}{r}\Theta R' + \frac{1}{r^2}\Theta'' R &= 0 \\ \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} &= 0 \\ r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} &= 0 \\ r^2\frac{R''}{R} + r\frac{R'}{R} &= -\frac{\Theta''}{\Theta} = k^2. \end{aligned}$$

Which gives us the following ODEs

$$\begin{aligned} r^2\frac{R''}{R} + r\frac{R'}{R} &= k^2 \\ r^2R'' + rR' &= k^2R \\ r^2R'' + rR' - k^2R &= 0 \end{aligned}$$

and

$$\begin{aligned} -\frac{\Theta''}{\Theta} &= k^2 \\ -\Theta'' &= k^2\Theta \\ -\Theta'' - k^2\Theta &= 0 \\ \Theta'' + k^2\Theta &= 0. \end{aligned}$$

These give rise to the following solutions

$$\Theta_k(\theta) = A_k\cos(k\theta) + B_k\sin(k\theta) \quad \text{and} \quad R_k(r) = C_kr^k + D_kr^{-k}.$$

The general solution is thus

$$u(r,\theta) = A_0 + B_0\log r + \sum_{k=1}^{\infty}\Big(C_kr^k + D_kr^{-k}\Big)\Big(A_k\cos(k\theta) + B_k\sin(k\theta)\Big)$$

Due to our boundary condition relying on $\cos\theta$ we only care about when $k = 1$ so thus $A_k = B_k = 0$ except for $A_1 = 1$ then

$$u(r,\theta) = \big(C_1r + D_1r^{-1}\big)\cos(\theta).$$

Based on our BC we know that we need

$$\begin{aligned} u(1,\theta) &= \big(C_1 + D_1\big)\cos(\theta) \\ 0 &= \big(C_1 + D_1\big)\cos(\theta) \\ 0 &= C_1 + D_1 \\ -D_1 &= C_1. \end{aligned}$$

Furthermore, we have

$$\begin{aligned}
u(2,\theta) &= \left(C_1 2 + D_1 \frac{1}{2}\right)\cos(\theta) \\
\cos\theta &= \left(C_1 2 + D_1 \frac{1}{2}\right)\cos(\theta) \\
1 &= \left(C_1 2 - C_1 \frac{1}{2}\right) \\
1 &= C_1 \frac{3}{2} \\
\frac{2}{3} &= C_1 \implies D_1 = -\frac{2}{3}
\end{aligned}$$

Hence, our final solution is

$$u(r,\theta) = \left(\frac{2}{3}r - \frac{2}{3r}\right)\cos(\theta).$$

$\square$

7: (Bonus) Consider the following Laplace equation

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & 0 \le r < 1, 0 \le \theta < 2\pi \\ u_r(1,\theta) + u(1,\theta) = \cos(2\theta) \end{cases}$$

Use the method of separation of variables to find a solution.

Solution:
TODO