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AMATH 502

HOMEWORK 2

Exercises come from *Nonlinear Dynamics and Chaos by Steven H. Strogatz*

1: 2.6.1 Explain this paradox: a simple harmonic oscillator $m\ddot{x} = -kx$ is a system which oscillates in one dimension (along the x-axis). But the text says one-dimensional systems can't oscillate.

Solution:
Not a formal method of finding a solution but I can see that if $x = \sin\left(\frac{k}{m}t\right)$ then

$$\begin{aligned}\dot{x} &= \frac{k}{m}\cos\left(\frac{k}{m}t\right)\\ \ddot{x} &= -\frac{k^2}{m^2}\sin\left(\frac{k}{m}t\right)\\ \ddot{x} &= -\frac{k^2}{m^2}x\\ m^2\ddot{x} &= -k^2x\end{aligned}$$

Therefore to adjust the constants k and m so they agree with the original solution we need to actually have $x = \sin\left(\sqrt{k/m}\,t\right)$. Furthermore, we could similarly have arrived at a similar solution with $x = \cos\left(\sqrt{k/m}\,t\right)$. Therefore, we have $x = c_1\sin\left(\sqrt{k/m}\,t\right) + c_2\cos\left(\sqrt{k/m}\,t\right)$. Now as for how this helps me determine that this is not a contradiction I have no idea.
TODO: Finish. Perhaps it's because ... nah idk.

2: 3.1.1 $\dot{x} = 1 + rx + x^2$
Sketch all the qualitatively different vector fields that occur as r is varied. Show that a saddle-node bifurcation occurs at a critical value of r, to be determined. Finally sketch the bifurcation diagram of fixed points x^* vs. r.

Solution:
Well we know that we have fixed points at

$$x = \frac{-r \pm \sqrt{r^2 - 4}}{2}$$

TODO

3: 3.1.5 (Unusual bifurcations) In discussing the normal form of the saddle-node bifurcation, we mentioned the assumption that $a = \partial f/\partial r|_{(x^*, r_c)} \neq 0$. To see what can happen if $a = \partial f/\partial r|_{(x^*, r_c)} = 0$, sketch the vector fields for the following examples, and then plot the fixed points as a function of r.

(a) $\dot{x} = r^2 - x^2$
Solution:
TODO

(b) $\dot{x} = r^2 + x^2$
Solution:
TODO

4: 3.2.3 $\dot{x} = x - rx(1 - x)$
Sketch all the qualitatively different vector fields that occur as r is varied. Show that transcritical bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x^* vs. r.

Solution:
TODO

5: 3.4.11 (An interesting bifurcation diagram) Consider $\dot{x} = rx - \sin x$

(a) For the case $r = 0$, find and classify all the fixed points, and sketch the vector field.

Solution:
TODO

(b) Show that when $r > 1$, there is only one fixed point. What kind of fixed point is it?

Solution:
TODO

(c) As r decreases from ∞ to 0, classify *all* the bifurcations that occur.

Solution:
TODO

(d) For $0 < r << 1$, find an approximate formula for values of r at which bifurcations occur.

Solution:
TODO

(e) Now classify all the bifurcations that occur as r decreases from 0 to $-\infty$.

Solution:
TODO

(f) Plot the Bifurcation diagram for $-\infty < r < \infty$, and indicate the stability of the various branches of fixed points.

Solution:
TODO

6: 3.5.8 (Nondimensionalizing the subcritical pitchfork) The first-order system $\dot{u} = au + bu^3 - cu^5$, where $b, c > 0$, has a subcritical pitchfork bifurcation at $a = 0$. Show that this equation can be rewritten as

$$\frac{dx}{d\tau} = rx + x^3 - x^5$$

where $x = u/U$, $\tau = t/T$, and U, T, and r are to be determined in terms of a, b, and c.

Solution:
TODO

7: 3.7.3 (A model of a fishery) The equation

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - H$$

provides an extremely simple model of a fishery. In the absence of fishing, the population is assumed to grow logistically. The effects of fishing are modeled by the term $-H$, which says that fish are caught or “harvested” at a constant rate $H > 0$, independent of their population N. (This assumes that the fisherman aren't worried about fishing the population dry–they simply catch the same number of fish every day.)

(a) Show that the system can be rewritten in dimensionless form as

$$\frac{dx}{d\tau} = x(1 - x) - h$$

for suitably defined dimensionless quantities x, τ, and h

Solution:
TODO

(b) Plot the vector field for different values of h

Solution:
TODO

(c) Show that a bifurcation occurs at a certain value of h_c, and classify this bifurcation.

Solution:
TODO

(d) Discuss the long-term behavior of the fish population for $h < h_c$ and $h > h_c$. Give the biological interpretation in each case.

Solution:
TODO