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04-24-25
AMATH 503

HOMEWORK 3

Exercises come from *Introduction to Partial Differential Equations by Peter J. Olver* as well as supplemented by instructor provided exercises.

1: Olver: 3.2.6 (a,c,e)

Solution:
TODO:

2: Olver: 3.3.2 and 3.3.3

Solution:
TODO:

3: Olver: 3.2.55

(a) Find the complex Fourier series for $x\,\mathrm{e}^{\mathrm{i}x}$

Solution:
First of all we define the complex Fourier series for a piecewise continuous real or complex function f is the doubly infinite series

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k\,\mathrm{e}^{\mathrm{i}kx}$$

where the c_k are given by

$$c_k = \langle f, \mathrm{e}^{\mathrm{i}kx}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,dx.$$

Therefore, the bulk of our work here is to establish what the coefficients c_k need to be. In other words we need to calculate

$$\begin{aligned} c_k = \langle x\,\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{\mathrm{i}kx}\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,\mathrm{e}^{\mathrm{i}x}\,\mathrm{e}^{-\mathrm{i}kx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,\mathrm{e}^{\mathrm{i}x(1-k)}\,dx. \end{aligned}$$

I believe integration by parts would be useful. Let $u = x$ and let $dv = \mathrm{e}^{\mathrm{i}x(1-k)}\,dx$ these then also give rise to $du = dx$ and $v = \frac{1}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}x(1-k)}$, respectively. Then we have

$$\int udv = uv - \int vdu$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x\,\mathrm{e}^{\mathrm{i}x(1-k)}\,dx = \frac{1}{2\pi}\left[\frac{x}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}x(1-k)}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{1}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}x(1-k)}\,dx\right]$$

Let's take the right hand piece by piece in order to keep the calculations clean. First with the uv term

$$\begin{aligned} uv &= \frac{x}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}x(1-k)}\Big|_{-\pi}^{\pi} \\ &= \frac{\pi}{\mathrm{i}(1-k)}\,\mathrm{e}^{\mathrm{i}\pi(1-k)} - \frac{(-\pi)}{\mathrm{i}(1-k)}\,\mathrm{e}^{-\mathrm{i}\pi(1-k)} \\ &= \frac{\pi}{\mathrm{i}(1-k)}\left(\mathrm{e}^{\mathrm{i}\pi(1-k)} + \mathrm{e}^{-\mathrm{i}\pi(1-k)}\right) \\ &= \frac{2\pi\cos(\pi(1-k))}{\mathrm{i}(1-k)} \\ &= -\frac{2\pi\mathrm{i}\cos(\pi(1-k))}{1-k}. \end{aligned}$$

Now we proceed with the integral on the right hand side

$$\begin{aligned}
\int v du &= \int_{-\pi}^{\pi} \frac{1}{\mathrm{i}(1-k)}\, \mathrm{e}^{\mathrm{i}x(1-k)}\, dx \\
&= \frac{1}{(\mathrm{i}(1-k))^2}\, \mathrm{e}^{\mathrm{i}x(1-k)} \Big|_{-\pi}^{\pi} \\
&= \frac{1}{(\mathrm{i}(1-k))^2}\, \mathrm{e}^{\mathrm{i}\pi(1-k)} - \frac{1}{(\mathrm{i}(1-k))^2}\, \mathrm{e}^{-\mathrm{i}\pi(1-k)} \\
&= \frac{1}{\mathrm{i}^2(1-k)^2} \left(\mathrm{e}^{\mathrm{i}\pi(1-k)} - \mathrm{e}^{-\mathrm{i}\pi(1-k)} \right) \\
&= -\frac{1}{(1-k)^2} \left(2\mathrm{i}\sin(\pi(1-k)) \right) \\
&= 0
\end{aligned}$$

where the final equality holds due to the fact that $\sin(\pi(1-k))$ is always 0 since $1-k$ is an integer. Using these in our initial IBP step we have

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} x\, \mathrm{e}^{\mathrm{i}x(1-k)}\, dx &= \frac{1}{2\pi}\left[\frac{x}{\mathrm{i}(1-k)}\, \mathrm{e}^{\mathrm{i}x(1-k)} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{1}{\mathrm{i}(1-k)}\, \mathrm{e}^{\mathrm{i}x(1-k)}\, dx \right] \\
&= \frac{1}{2\pi}\left[-\frac{2\pi \mathrm{i}\cos(\pi(1-k))}{1-k} \right] \\
&= -\frac{\mathrm{i}\cos(\pi(1-k))}{1-k}.
\end{aligned}$$

Thus we have calculated the c_k to be

$$c_k = -\frac{\mathrm{i}\cos(\pi(1-k))}{1-k}$$

and thus our Fourier series of the function $x\,\mathrm{e}^{\mathrm{i}x}$ is given by

$$f(x) \sim \sum_{k=-\infty}^{\infty} -\frac{\mathrm{i}\cos(\pi(1-k))}{1-k}\, \mathrm{e}^{\mathrm{i}kx}$$

□

(b) Use your result to write down the real Fourier series for $x\cos x$ and $x\sin s$

Solution:

I am not confident on how to go from part a) to the results that are requested here but I do think it will be helpful to know $a_k = c_k + c_{-k}$ and $b_k = \mathrm{i}(c_k - c_{-k})$.

4: Olver: 3.4.6

Solution:
TODO:

5: Olver: 3.5.29

Solution:
TODO:

6: Olver: 3.5.43

Solution:
TODO:

7: We consider the complex orthonormal basis

$$\varphi_n = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{inx}$$

where $n = 0, 1, -1, 2, -2, \ldots.$ Consider the function $f_a(x) = \mathrm{e}^{ax}$ with real number $a \neq 0$ and compute the Fourier coefficient

$$\langle f_a, \varphi_n \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f_a(x)\,\mathrm{e}^{-inx}\,dx.$$

Then prove the formula

$$\sum_{n=1}^{\infty} \frac{1}{a^2 + n^2} = \frac{\pi}{2a}\coth(\pi a) - \frac{1}{2a^2}$$

(Hint: Plancherel's formula: the relation between L^2 norm of coefficients and $\langle f_a, f_a \rangle$.)

Solution:
TODO: