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Homework #5

02-14-25

$\boxed{1}$ Decide whether the system

$$\dot{x} = 2x$$
$$\dot{y} = 8y$$

is a gradient system. If so, find V and sketch the phase portrait.

<u>Solution</u>:

From the text we can show the DS is a gradient system if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ where $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$. Notice,

$$\frac{\partial f}{\partial y} = 0 = 0 = \frac{\partial g}{\partial x}$$

Therefore this is a gradient system. Now we need to find V s.t. $\dot{x} = -\partial V/\partial x$ and $\dot{y} = -\partial V/\partial y$. Then we have (using partial integration)

$$\dot{x} = -\frac{\partial V}{\partial x}$$

$$2x = \frac{-\partial V}{\partial x}$$

$$\int 2x \, \partial x = \int -\partial V$$

$$-\int 2x\,\partial x = \int \partial V$$

$$-x^2 + C(y) = V,$$ (where $C(y)$ is a y dependent constant).

we also have,

$$\dot{y} = -\frac{\partial V}{\partial y}$$

$$8y = -\frac{\partial V}{\partial y}$$

$$-\int 8y\,\partial y = \partial V$$

$$-4y^2 + C(x) = V.$$

Combining these, we have

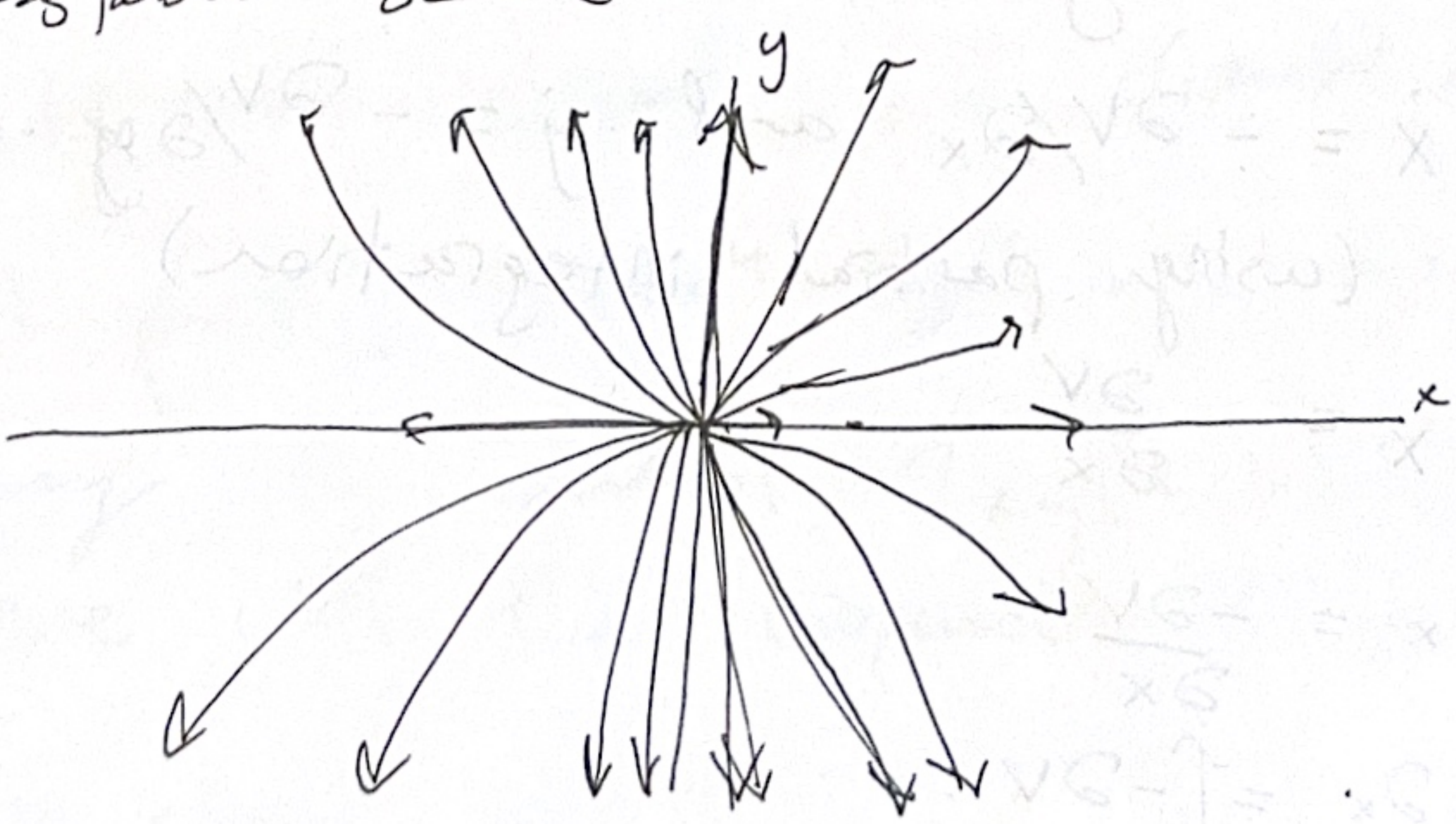
$$V = -x^2 - 4y^2.$$

Note we have one fixed point at $(x^*, y^*) = (0,0)$, since

$$\begin{aligned} \dot{x} &= 2x = 0 \\ \dot{y} &= 8y = 0 \end{aligned} \Rightarrow (x^*, y^*) = (0,0).$$

Computing the Jacobian and evaluating at $(0,0)$ we have

$$\begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \Rightarrow \lambda = 2, 8$$ which means the f.p. is an unstable source.



□

$\boxed{2}$ $\dot{x} = y - x^3$, $\dot{y} = -x - y^3$ has no closed orbits, we will demonstrate this by constructing a Liapunov function $V = ax^2 + by^2$.

<u>Solution</u>: In order for V to be a Liapunov we need $V(\vec{x}) > 0 \;\forall\; x \neq \vec{x}^*$ & $\dot{V} < 0 \;\forall\; x \neq \vec{x}^*$.

This gives us the condition that $a, b > 0$.

Now we need $\dot{V} < 0 \;\forall\; \vec{x} \in \mathbb{R}^2 \setminus \{(0,0)\}$. We calculate

$$\dot{V} = \frac{\partial}{\partial t}\left(a(x(t))^2 + b(y(t))^2\right)$$

$$\dot{V} = 2ax\dot{x} + 2by\dot{y}$$

$$\dot{V} = 2ax(y - x^3) + 2by(-x - y^3)$$

$$\dot{V} = 2axy - 2ax^4 - 2bxy - 2by^4.$$

Thus we need

$$2axy - 2ax^4 - 2bxy - 2by^4 < 0$$

$$2xy(a - b) - 2ax^4 - 2by^4 < 0.$$

let's determine what a, b should be. Since $a, b > 0$ from the first part, $-2ax^4 - 2by^4$ is always less than 0. The only way to control $2xy(a-b) < 0$ is for $a = b$. Therefore with the Liapunov $V = ax^2 + by^2$ then we have no closed orbits. □

$\boxed{3}$ Show that the DS

$$\dot{x} = ye^{-2x} + 3$$
$$\dot{y} = -e^{-2x}(x + y - x^2 - y^2)$$

has no closed traj.

Solution: Using Bendixson's theorem presented in class we need to show that

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \neq 0 \quad \text{where} \quad \begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y). \end{aligned}$$

The requisite partials are

$$\frac{\partial f}{\partial x} = -2ye^{-2x}$$

$$\frac{\partial g}{\partial y} = -e^{-2x} + 2ye^{-2x}$$

Thus

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -e^{-2x} \neq 0$$

Since the exponential is never 0. $\square$

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Consider $\dot{x} = x - y - x(x^2+5y^2)$

$\dot{y} = x + y - y(x^2+y^2)$

a) fixed point at origin: Solution:

Jacobian

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 1-3x^2-5y^2 & -1-10xy \\ 1-2xy & 1-x^2-3y^2 \end{pmatrix}\Bigg|_{(0,0)}$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Characteristic eq.

$(1-\lambda)(1-\lambda)+1 = 0$

$1 - 2\lambda + \lambda^2 + 1 = 0$

$\lambda^2 - 2\lambda + 2 = 0$

$\lambda = \frac{2 \pm \sqrt{4-4(2)}}{2}$

$\lambda = \frac{2}{2} \pm \frac{\sqrt{4-8}}{2}$

$\lambda = 1 \pm i$

Therefore we have that this is an unstable spiral.

b) rewrite the system in polar coords. using $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$.

Solution:

Starting from $x = r\cos\theta$ $y = r\sin\theta$.