Hunter Lybbert
Student ID: 2426454
03-05-25
AMATH 502

HOMEWORK 7

Exercises come from the assignment sheet provided by the professor on canvas.

1: A powerful tool for numerically finding the roots of an equation $g(x) = 0$ is *Newton's Method*. Newton's method says to construct a map $x_{n+1} = f(x_n)$, where

$$f(x_n) = x_n - \frac{g(x_n)}{g'(x_n)}$$

.

(a) A simple root of the function $g(x)$ is defined as a value x for which $g(x) = 0$ and $g'(x) \neq 0$. Show that the simple roots of $g(x)$ are fixed points of the Newton Map.

Solution:
Let's first assume x^* is a simple root. Therefore, $g(x^*) = 0$ and $g'(x^*) \neq 0$, for notation let $g'(x^*) = a$ where $a \neq 0$. This also implies that

$$\begin{aligned} f(x^*) &= x^* - \frac{g(x^*)}{g'(x^*)} \\ f(x^*) &= x^* - \frac{0}{a} \\ f(x^*) &= x^*. \end{aligned} \tag{1}$$

Notice, the definition of a fixed point in a discrete time system is $f(x_n) = x_n$ which is exactly what we are left with in (1). Therefore, x^* is a fixed point.

□

(b) Show that these fixed points are *superstable*, which means that the linear stability analysis shows *zero* growth for perturbations ($f'(x^*) = 0$).

Solution:
Let's begin by calculating $f'(x^*)$ we have

$$\begin{aligned} \frac{d}{dx_n} f(x_n) &= \frac{d}{dx_n}\left(x_n - \frac{g(x_n)}{g'(x_n)}\right) \\ f'(x_n) &= 1 - \frac{g'(x_n)g'(x_n) - g(x_n)g''(x_n)}{g'(x_n)^2} \\ f'(x_n) &= 1 - \frac{g'(x_n)^2 - g(x_n)g''(x_n)}{g'(x_n)^2}. \end{aligned}$$

Plugging in x^* we have

$$f'(x^*) = 1 - \frac{g'(x^*)^2 - g(x^*)g''(x^*)}{g'(x^*)^2}$$
$$f'(x^*) = 1 - \frac{a^2 - 0}{a^2}$$
$$f'(x^*) = 1 - 1 = 0.$$

Therefore, the fixed point x^* is superstable.

$\square$

2: Consider the map $x_{n+1} = 3x_n - x_n^3$. This well-studied map is an example of a cubic map and is known to exhibit chaos.

(a) Find all the fixed points and classify their stability.

Solution:
To find the fixed points let's consider finding x_n where

$$\begin{aligned} x_n &= 3x_n - x_n^3 \\ 0 &= 2x_n - x_n^3 \\ 0 &= x_n(2 - x_n^2). \end{aligned}$$

Therefore, $x_n^* = 0, \pm\sqrt{2}$ are the fixed points of the map. Now we need to classify their stabilities, for notational convenience let's allow $f(x_n) = 3x_n - x_n^3$ and thus $f'(x_n) = 3 - 3x_n^2$. If $|f'(x_n^*)| < 1$, then the x_n^* is stable.

$$\begin{aligned} x_n^* = 0: &\quad |f'(0)| = |3 - 3(0)^2| = 3 \not< 1 \implies \text{unstable} \\ x_n^* = -\sqrt{2}: &\quad |f'(-\sqrt{2})| = |3 - 3(-\sqrt{2})^2| = |3 - 6| = 3 \not< 1 \implies \text{unstable} \\ x_n^* = \sqrt{2}: &\quad |f'(\sqrt{2})| = |3 - 3(\sqrt{2})^2| = |3 - 6| = 3 \not< 1 \implies \text{unstable.} \end{aligned}$$

Thus, each of the fixed points are unstable.

□

(b) In Figure 1, you are given the cobweb diagrams for $x_0 = 1.9$ and $x_0 = 2.1$. Show analytically that if $|x| \leq 2$, then $|f(x)| \leq 2$, where $f(x) = 3x - x^3$. Then show that if $|x| > 2$, $|f(x)| > |x|$. Use this to explain the behavior in cobweb diagrams for $x_0 = 1.9$ and $x_0 = 2.1$.

Solution:
Let's begin by calculating where the extrema occur for $f(x) = 3x - x^3$. They occur where $f'(x) = 3 - 3x^2 = 0$ which is at $x = \pm 1$ and possibly at the boundaries of our interval thus we need to check if $|f(x)| \leq 2$ holds for $x = \pm 1, \pm 2$. Notice,

$$\begin{aligned} f(-2) &= 3(-2) - (-2)^3 = -6 + 8 = 2 \\ f(-1) &= 3(-1) - (-1)^3 = -3 + 1 = -2 \\ f(1) &= 3(1) - (1)^3 = 3 - 1 = 2 \\ f(2) &= 3(2) - (2)^3 = 6 - 8 = -2. \end{aligned}$$

Therefore, since these values represent the min and max of the function $f(x) = 3x - x^3$ over the interval $|x| \leq 2$, then we can conclude $|f(x)| \leq 2$ over this same interval.

Next, we need to verify that when $|x| > 2$ we have that $|f(x)| > |x|$. Let's do this one at a time, beginning with $x > 2$. We want to determine if

$$\begin{aligned} |3x - x^3| &\overset{?}{>} |x| \\ |3x - x^3| - |x| &\overset{?}{>} 0 \end{aligned}$$

Plugging in x = 2 as a lower bound we have

$$\begin{aligned}
|3(2)-(2)^3|-|2| &\overset{?}{>} 0\\
|6-8|-2 &\overset{?}{>} 0\\
|-2|-2 &\overset{?}{>} 0\\
2-2 &\overset{?}{>} 0\\
0 &\overset{?}{>} 0.
\end{aligned}$$

Therefore a lower bound for $|3x-x^3|-|x|>0$ and thus $|3x-x^3|>|x|$. Now for when $x<-2$ we have

$$\begin{aligned}
|3x-x^3| &\overset{?}{>} |x|\\
|3x-x^3|-|x| &\overset{?}{>} 0
\end{aligned}$$

Plugging in x = -2 as an upper bound we have

$$\begin{aligned}
|3(-2)-(-2)^3|-|-2| &\overset{?}{>} 0\\
|-6+8|-2 &\overset{?}{>} 0\\
|2|-2 &\overset{?}{>} 0\\
2-2 &\overset{?}{>} 0\\
0 &\overset{?}{>} 0.
\end{aligned}$$

Therefore a lower bound for $|3x-x^3|-|x|>0$ and thus $|3x-x^3|>|x|$ in any case within the constraint $|x|>2$. We can use this to explain the behavior in the cobweb diagrams for $x_0=1.9$ and $x_0=2.1$ because... □

(c) Show that (2, -2) (repeating) is a 2 cycle. This 2 cycle is analogous to a boundary that we defined when we were doing phase-plane analysis. What would you call this 2-cycle? (Not a limit cycle or a periodic orbit).

Solution:
Since

$$f\Big(f(-2)\Big)=f\Big(3(-2)-(-2)^3\Big)=f(-6+8)=f(2)=3(2)-2^3=-2$$

and

$$f\Big(f(2)\Big)=f\Big(3(2)-(2)^3\Big)=f(6-8)=f(-2)=3(-2)-(-2)^3=2$$

(2, -2) is a 2-cycle. This 2-cycle is analogous to a separatrice, dividing the basins of attraction.

□

3: Consider a 1D ODE

(2) $$\dot{x} = f(x), \quad x \in \mathbb{R}.$$

The most basic method for solving this ODE numerically is to use the Forward Euler method,

(3) $$x_{n+1} = x_n + hf(x_n),$$

where $h > 0$ is a chosen step size. This method comes from discretizing the derivative, as discussed in class.

(a) Show that fixed points of the ODE (2) correspond to fixed points of the Forward Euler map (3).

Solution:
Consider the fixed points X^* of the ODE (2), these occur where $\dot{x} = 0$ implying $f(x^*) = 0$. Thus we have

$$x_{n+1} = x_n^* + hf(x_n^*) = x_n^* + h0 = x_n^*$$

which shows that x^* is also a fixed point of the Forward Euler map, since applying the map to x^* simply returns x^* back.

□

(b) Show that stability of the fixed points of the ODE (2) do not necessarily agree with the stability of the fixed points of the Forward Euler map (3).

Solution:
Using Linear Stability Analysis, in order for the fixed point x^* to be stable for the ODE (2) we need $f'(x^*) < 0$. We don't currently have enough information to conclude the stability of the fixed point x^* for the ODE (2), however we can say it is stable if $f'(x^*) < 0$. Now for the stability of the fixed point of the Forward Euler map we need

$$\begin{aligned} \left| \frac{d}{dx_n} \left[x_n + hf(x_n) \right] \Big|_{x_n^*} \right| &< 1 \\ \left| 1 + hf'(x_n^*) \right| &< 1 \\ -1 < 1 + hf'(x_n^*) &< 1 \\ -2 < hf'(x_n^*) &< 0. \end{aligned}$$

Therefore, given this condition the Forward Euler map would be stable depending on the value of h.

□

(c) Give a condition which guarantees stability of fixed points of the Forward Euler map (2). Comment on this condition: how must we generally choose the step size h in order to find equilibrium solutions of the ODE (3) using the Forward Euler method?

Solution:
From part (b) we assume $f'(x_n^*) < 0$ for the fixed point to be stable for the ODE and we need $-2 < hf'(x_n^*) < 0$. Which assuming $h > 0$ and $f'(x_n^*) < 0$ ensures

the right hand side $hf'(x_n^*) < 0$, but we need to solve for h in order to guarantee the left inequality holds $-2 < hf'(x_n^*)$. Solving for h we get $-\frac{2}{f'(x_n^*)} > h$ (note the inequality flips because we divided by a negative number, $f'(x_n^*)$)

□

(d) It is common to see the Forward Euler solution oscillating about the true solution when solving numerically. Give a condition involving $f'(x)$ and h for which the numerical solution oscillates about a fixed point of the ODE (2) (hint: when did we have oscillations for the linear discrete-time dynamical systems?). Given this condition, why is it common to see oscillations in the Forward-Euler solution (hint: see above problem)?

Solution:
A condition for which we would see such oscillations would be if $f'(x_n^*) = -\frac{2}{h}x_n$. This would imply that $f(x_n) \sim -\frac{2}{h}x_n$ and the Forward Euler map would be

$$\begin{aligned} x_{n+1} &= x_n + hf(x_n) \\ x_{n+1} &= x_n - h\frac{2}{h}x_n \\ x_{n+1} &= x_n - 2x_n \\ x_{n+1} &= -x_n. \end{aligned}$$

Which is analogous to oscillations that we are looking for. I would say it is common to see oscillations with Forward Euler because we are using it to look for a fixed point, but the h that we are choosing depends on the value of the derivative evaluated at that fixed point so it's kind of a chicken or the egg thing. We can't find the precise value of h we should use to guarantee the stability of the fixed point because we don't know where the fixed point is yet.

□

(e) Consider a linear ODE,

$$\dot{x} = kx, \quad k \in \mathbb{R}. \tag{4}$$

Give a condition on h and k for which 2-cycles (the non-fixed point 2 cycles) exist for the Forward-Eualer map when solving this ODE. Show that these 2 cycles are neutrally stable. Comment on your results (in particular, when h and k match your condition, what happens to the numerical solution for any initial condition you use?).

Solution:
TODO