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03-05-25
AMATH 502

HOMEWORK 7

Exercises come from the assignment sheet provided by the professor on canvas.

- 1: A powerful tool for numerically finding the roots of an equation $g(x) = 0$ is *Newton's Method*. Newton's method says to construct a map $x_{n+1} = f(x_n)$, where

$$f(x_n) = x_n - \frac{g(x_n)}{g'(x_n)}$$

- (a) A simple root of the function $g(x)$ is defined as a value x for which $g(x) = 0$ and $g'(x) \neq 0$. Show that the simple roots of $g(x)$ are fixed points of the Newton Map.

Solution:

Let's first assume x^* is a simple root. Therefore, $g(x^*) = 0$ and $g'(x^*) \neq 0$, for notation let $g'(x^*) = a$ where $a \neq 0$. This also implies that

$$\begin{aligned} f(x^*) &= x^* - \frac{g(x^*)}{g'(x^*)} \\ f(x^*) &= x^* - \frac{0}{a} \\ f(x^*) &= x^*. \end{aligned} \tag{1}$$

Notice, the definition of a fixed point in a discrete time system is $f(x_n) = x_n$ which is exactly what we are left with in (1). Therefore, x^* is a fixed point. \square

- (b) Show that these fixed points are *superstable*, which means that the linear stability analysis shows *zero* growth for perturbations ($f'(x^*) = 0$).

Solution:

Let's begin by calculating $f'(x^*)$ we have

$$\begin{aligned} \frac{d}{dx_n} f(x_n) &= \frac{d}{dx_n} \left(x_n - \frac{g(x_n)}{g'(x_n)} \right) \\ f'(x_n) &= 1 - \frac{g'(x_n)g'(x_n) - g(x_n)g''(x_n)}{g'(x_n)^2} \\ f'(x_n) &= 1 - \frac{g'(x_n)^2 - g(x_n)g''(x_n)}{g'(x_n)^2}. \end{aligned}$$

Plugging in x^* we have

$$f'(x^*) = 1 - \frac{g'(x^*)^2 - g(x^*)g''(x^*)}{g'(x^*)^2}$$

$$f'(x^*) = 1 - \frac{a^2 - 0}{a^2}$$

$$f'(x^*) = 1 - 1 = 0.$$

Therefore, the fixed point x^* is superstable.

□

2: Consider the map $x_{n+1} = 3x_n - x_n^3$. This well-studied map is an example of a cubic map and is known to exhibit chaos.

(a) Find all the fixed points and classify their stability.

Solution:

To find the fixed points let's consider finding x_n where

$$\begin{aligned}x_n &= 3x_n - x_n^3 \\0 &= 2x_n - x_n^3 \\0 &= x_n(2 - x_n^2).\end{aligned}$$

Therefore, $x_n^* = 0, \pm\sqrt{2}$ are the fixed points of the map. Now we need to classify their stabilities, for notational convenience let's allow $f(x_n) = 3x_n - x_n^3$ and thus $f'(x_n) = 3 - 3x_n^2$. If $|f'(x_n^*)| < 1$, then the x_n^* is stable.

$$\begin{aligned}x_n^* = 0 : & \quad |f'(0)| = |3 - 3(0)^2| = 3 \not< 1 \implies \text{unstable} \\x_n^* = -\sqrt{2} : & \quad |f'(-\sqrt{2})| = |3 - 3(-\sqrt{2})^2| = |3 - 6| = 3 \not< 1 \implies \text{unstable} \\x_n^* = \sqrt{2} : & \quad |f'(\sqrt{2})| = |3 - 3(\sqrt{2})^2| = |3 - 6| = 3 \not< 1 \implies \text{unstable}.\end{aligned}$$

Thus, each of the fixed points are unstable.

□

(b) In Figure 1, you are given the cobweb diagrams for $x_0 = 1.9$ and $x_0 = 2.1$. Show analytically that if $|x| \leq 2$, then $|f(x)| \leq 2$, where $f(x) = 3x - x^3$. Then show that if $|x| > 2$, $|f(x)| > |x|$. Use this to explain the behavior in cobweb diagrams for $x_0 = 1.9$ and $x_0 = 2.1$.

Solution:

Let's begin by calculating where the extrema occur for $f(x) = 3x - x^3$. They occur where $f'(x) = 3 - 3x^2 = 0$ which is at $x = \pm 1$ and possibly at the boundaries of our interval thus we need to check if $|f(x)| \leq 2$ holds for $x = \pm 1, \pm 2$. Notice,

$$\begin{aligned}f(-2) &= 3(-2) - (-2)^3 = -6 + 8 = 2 \\f(-1) &= 3(-1) - (-1)^3 = -3 + 1 = -2 \\f(1) &= 3(1) - (1)^3 = 3 - 1 = 2 \\f(2) &= 3(2) - (2)^3 = 6 - 8 = -2.\end{aligned}$$

Therefore, since these values represent the min and max of the function $f(x) = 3x - x^3$ over the interval $|x| \leq 2$, then we can conclude $|f(x)| \leq 2$ over this same interval.

Next, we need to verify that when $|x| > 2$ we have that $|f(x)| > |x|$. Let's do this one at a time, beginning with $x > 2$. We want to determine if

$$\begin{aligned}|3x - x^3| &\stackrel{?}{>} |x| \\|3x - x^3| - |x| &\stackrel{?}{>} 0\end{aligned}$$

Plugging in $x = 2$ as a lower bound we have

$$\begin{aligned} |3(2) - (2)^3| - |2| &\stackrel{?}{>} 0 \\ |6 - 8| - 2 &\stackrel{?}{>} 0 \\ |-2| - 2 &\stackrel{?}{>} 0 \\ 2 - 2 &\stackrel{?}{>} 0 \\ 0 &\stackrel{?}{>} 0. \end{aligned}$$

Therefore a lower bound for $|3x - x^3| - |x| > 0$ and thus $|3x - x^3| > |x|$. Now for when $x < -2$ we have

$$\begin{aligned} |3x - x^3| &\stackrel{?}{>} |x| \\ |3x - x^3| - |x| &\stackrel{?}{>} 0 \end{aligned}$$

Plugging in $x = -2$ as an upper bound we have

$$\begin{aligned} |3(-2) - (-2)^3| - |-2| &\stackrel{?}{>} 0 \\ |-6 + 8| - 2 &\stackrel{?}{>} 0 \\ |2| - 2 &\stackrel{?}{>} 0 \\ 2 - 2 &\stackrel{?}{>} 0 \\ 0 &\stackrel{?}{>} 0. \end{aligned}$$

Therefore a lower bound for $|3x - x^3| - |x| > 0$ and thus $|3x - x^3| > |x|$ in any case within the constraint $|x| > 2$. We can use this to explain the behavior in the cobweb diagrams for $x_0 = 1.9$ and $x_0 = 2.1$ because... \square

- (c) Show that $(2, -2)$ (repeating) is a 2 cycle. This 2 cycle is analogous to a boundary that we defined when we were doing phase-plane analysis. What would you call this 2-cycle? (Not a limit cycle or a periodic orbit).

Solution:

Since

$$f(f(-2)) = f(3(-2) - (-2)^3) = f(-6 + 8) = f(2) = 3(2) - 2^3 = -2$$

and

$$f(f(2)) = f(3(2) - (2)^3) = f(6 - 8) = f(-2) = 3(-2) - (-2)^3 = 2$$

$(2, -2)$ is a 2-cycle. This 2-cycle is analogous to a separatrix, dividing the basins of attraction. \square

3: Consider a 1D ODE

(2)
$$\dot{x} = f(x), \quad x \in \mathbb{R}.$$

The most basic method for solving this ODE numerically is to use the Forward Euler method,

(3)
$$x_{n+1} = x_n + hf(x_n),$$

where $h > 0$ is a chosen step size. This method comes from discretizing the derivative, as discussed in class.

- (a) Show that fixed points of the ODE (2) correspond to fixed points of the Forward Euler map (3).

Solution:

Consider the fixed points X^* of the ODE (2), these occur where $\dot{x} = 0$ implying $f(x^*) = 0$. Thus we have

$$x_{n+1} = x_n^* + hf(x_n^*) = x_n^* + h0 = x_n^*$$

which shows that x^* is also a fixed point of the Forward Euler map, since applying the map to x^* simply returns x^* back.

□

- (b) Show that stability of the fixed points of the ODE (2) do not necessarily agree with the stability of the fixed points of the Forward Euler map (3).

Solution:

Using Linear Stability Analysis, in order for the fixed point x^* to be stable for the ODE (2) we need $f'(x^*) < 0$. We don't currently have enough information to conclude the stability of the fixed point x^* for the ODE (2). Now for the stability of the fixed point of the Forward Euler map we need

$$\begin{aligned} \left| \frac{d}{dx_n} [x_n + hf(x_n)] \right| &< 1 \\ \left| 1 + hf'(x_n) \right| &< 1 \\ -1 &< 1 + hf'(x_n) < 1 \\ -2 &< hf'(x_n) < 0. \end{aligned}$$

- (c) Give a condition which guarantees stability of fixed points of the Forward Euler map (2). Comment on this condition: how must we generally choose the step size h in order to find equilibrium solutions of the ODE (3) using the Forward Euler method?

Solution:

TODO

- (d) It is common to see the Forward Euler solution oscillating about the true solution when solving numerically. Give a condition involving $f'(x)$ and h for which the numerical solution oscillates about a fixed point of the ODE (2) (hint: when did we have oscillations for the linear discrete-time dynamical systems?). Given this condition, why is it common to see oscillations in the Forward-Euler solution (hint:

see above problem)?

Solution:

TODO

(e) Consider a linear ODE,

(4)
$$\dot{x} = kx, \quad k \in \mathbb{R}.$$

Give a condition on h and k for which 2-cycles (the non-fixed point 2 cycles) exist for the Forward-Euler map when solving this ODE. Show that these 2 cycles are neutrally stable. Comment on your results (in particular, when h and k match your condition, what happens to the numerical solution for any initial condition you use?).

Solution:

TODO