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Homework 4

$\boxed{1}$

a) $\dot{x} = x - y, \quad \dot{y} = x^2 - 4$

Find fixed points

$$\begin{aligned} \dot{x} &= x - y = 0 \\ \dot{y} &= x^2 - 4 = 0 \end{aligned} \Rightarrow \begin{aligned} x &= y \\ x^2 &= 4 \end{aligned} \Rightarrow x, y = \pm 2.$$

More precisely the two fixed points are at $(x^*, y^*) = (-2,-2), (2,2)$.

Next lets compute the Jacobian. Note $\dot{x} = f(x,y), \dot{y} = g(x,y)$.

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2x & 0 \end{pmatrix}.$$

Now lets evaluate the Jacobian at the first fixed point $(-2,-2)$. Then we have the matrix that defines our linearized system:

$$\begin{pmatrix} 1 & -1 \\ -4 & 0 \end{pmatrix}. \Rightarrow$$ let's get the eigenvalues

$$(1-\lambda)(-\lambda) - 4 = 0$$

$$\lambda^2 - \lambda - 4 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1-4(-4)}}{2} = \frac{1}{2} \pm \frac{\sqrt{17}}{2}$$

1a continued:

Now lets solve for the eigenvectors. starting with $\lambda = \frac{1+\sqrt{17}}{2}$ we have $\left(\begin{array}{cc|c} 1 - \frac{1-\sqrt{17}}{2} & -1 & 0 \\ -4 & -\frac{1-\sqrt{17}}{2} & 0 \end{array}\right)$

$\Rightarrow \frac{2-1-\sqrt{17}}{2} x = y$ $\quad -4x = \frac{1+\sqrt{17}}{2} y$

$\frac{1-\sqrt{17}}{2} x = y$ $\quad \Rightarrow -4x = \frac{1+\sqrt{17}}{2}\left(\frac{1-\sqrt{17}}{2} x\right)$

$-4x = \frac{1-17}{4} x = -\frac{16}{4} x = -4x$. This is always true so we choose $\Rightarrow x = 1$.

Therefore, $y = x\,\frac{1-\sqrt{17}}{2} = \frac{1-\sqrt{17}}{2}$. Thus the eigenvector is $\begin{pmatrix} 1 \\ \frac{1-\sqrt{17}}{2} \end{pmatrix}$. Furthermore, for

$\lambda = \frac{1-\sqrt{17}}{2}$ we have $\left(\begin{array}{cc|c} \frac{2-1+\sqrt{17}}{2} & -1 & 0 \\ -4 & \frac{-1+\sqrt{17}}{2} & 0 \end{array}\right)$

$\Rightarrow \frac{1+\sqrt{17}}{2} x = y, \quad -4x = \frac{1-\sqrt{17}}{2} y$

$\Rightarrow -4x = \frac{1-\sqrt{17}}{2}\left(\frac{1+\sqrt{17}}{2} x\right) = \frac{1-17}{4} x = -\frac{16}{4} x = -4x$ which is always true so we pick $x = 1$. Then $y = \frac{1+\sqrt{17}}{2}$. thence we have the second eigenvector $\begin{pmatrix} 1 \\ \frac{1+\sqrt{17}}{2} \end{pmatrix}$.

On the next page we will classify this fixed point.

In summary for the fixed point $(x^*, y^*) = (-2,-2)$ we have the following eigenvalue eigenvector pairs:

$$\lambda_1 = \frac{1+\sqrt{17}}{2}, \begin{pmatrix} 1 \\ \frac{1-\sqrt{17}}{2} \end{pmatrix} \text{ and } \lambda_2 = \frac{1-\sqrt{17}}{2}, \begin{pmatrix} 1 \\ \frac{1+\sqrt{17}}{2} \end{pmatrix}$$

Since we have two real valued eigenvalues with opposite signs this fixed point is a saddle node. I will draw it's behavior after I do the other fixed point as well. Now ~~we~~ need to evaluate the jacobian at $(x^*, y^*) = (2,2)$ to get $\begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix}$.

The characteristic equation to solve for eigenvalues is

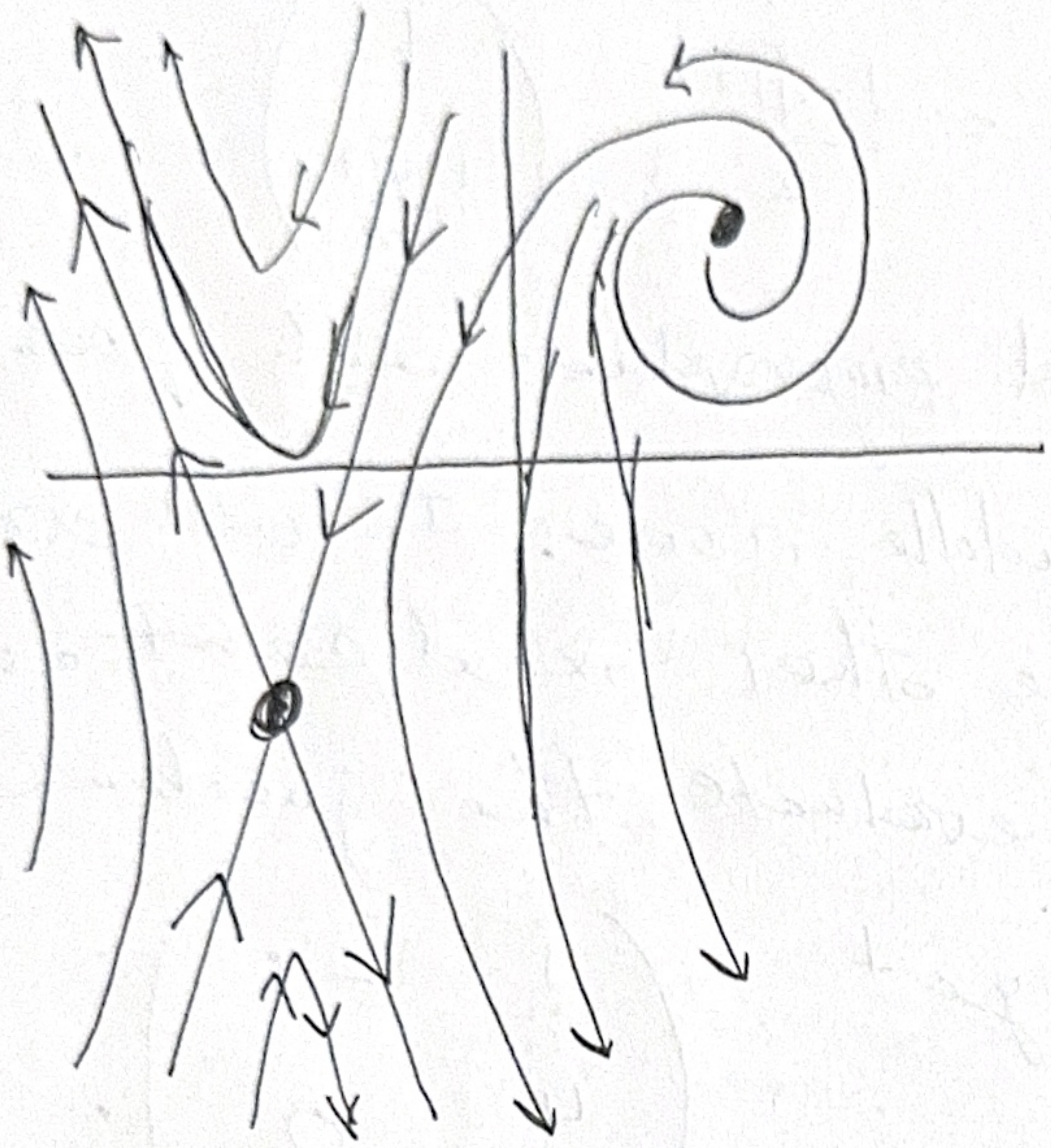
$$(1-\lambda)(-\lambda) + 4 = 0$$
$$\lambda^2 - \lambda + 4 = 0$$
$$\Rightarrow \lambda = \frac{1 \pm \sqrt{1-4(4)}}{2}$$
$$\lambda = \frac{1 \pm \sqrt{-15}}{2}$$
$$\lambda = \frac{1 \pm i\sqrt{15}}{2}.$$

Now lets solve for eigenvectors. we notice the algebra will be nearly identical for these cases as there was before. After performing the same algebra we arrive at the conclusion

$$\lambda_1 = \frac{1+i\sqrt{15}}{2}, \begin{pmatrix} 1 \\ \frac{1-i\sqrt{15}}{2} \end{pmatrix} \text{ and } \lambda_2 = \frac{1-i\sqrt{15}}{2}, \begin{pmatrix} 1 \\ \frac{1+i\sqrt{15}}{2} \end{pmatrix}.$$

Since both eigenvalues have positive real parts and also imaginary parts ~~we~~ have a spiralling <u>Source</u> or <u>Repeller</u> which is <u>unstable</u> f.p. (2,2).

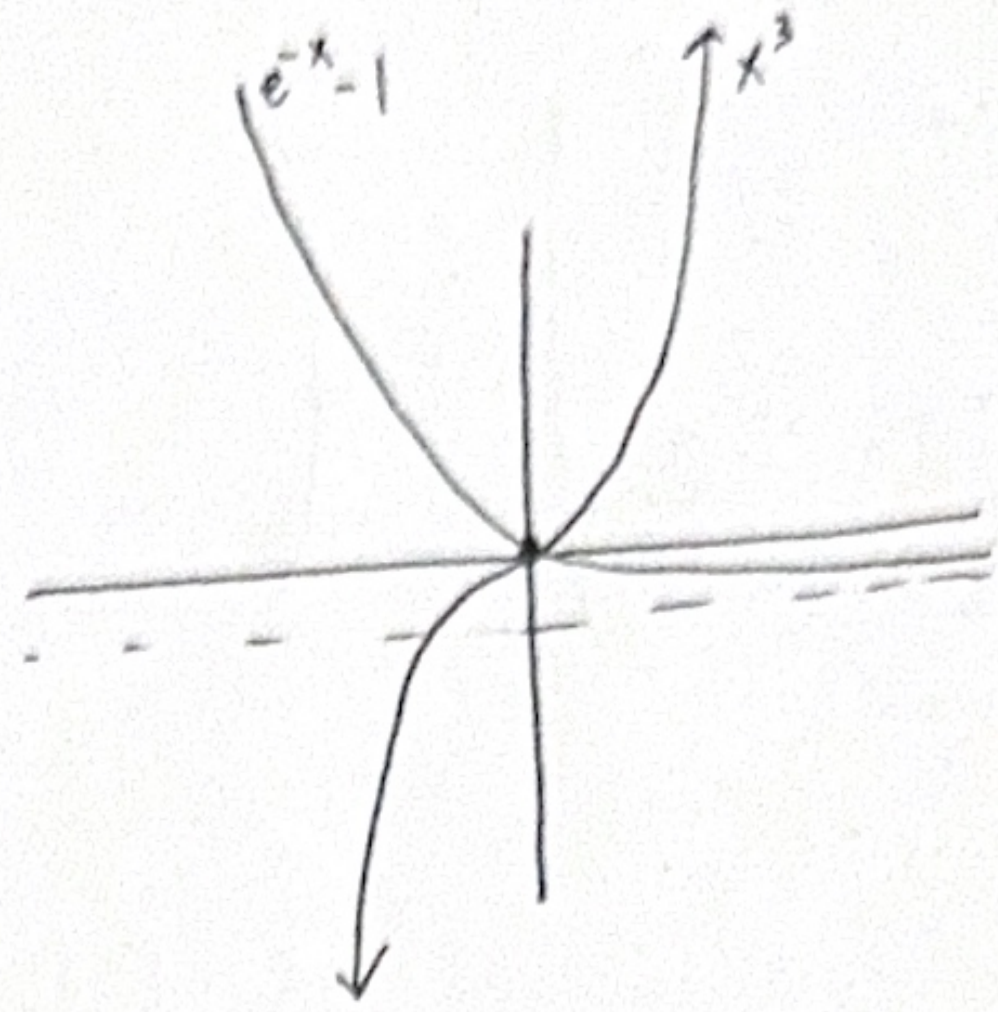
Now the phase portrait would qualitatively behave as follows



fixed points are bolded for emphasis not shaded to depict stability.

$\boxed{1b}$ $\dot{x} = 1 + y - e^{-x}$, $\dot{y} = x^3 - y$

$\dot{x} = f(x,y)$ $\dot{y} = g(x,y)$



Fixed points

$1 + y - e^{-x} = 0$ $\Rightarrow$ $1 + x^3 - e^{-x} = 0$

$x^3 - y = 0$ alternatively

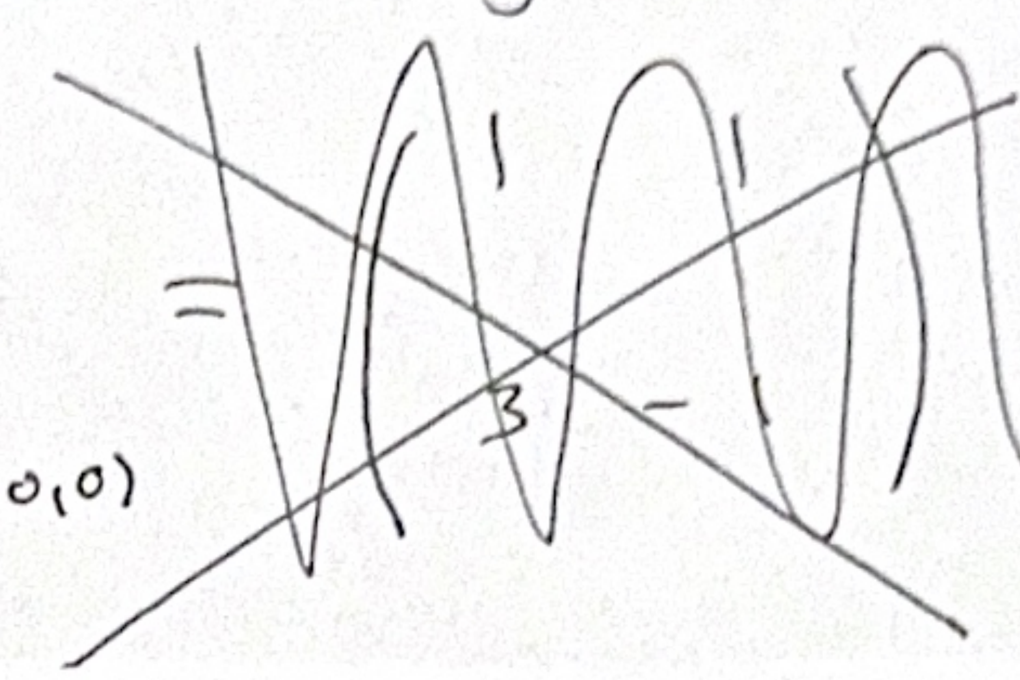
$\Rightarrow x^3 = y$

$x^3 = e^{-x} - 1$

$x = 0, y = 0$ ✓

we only have one fixed point at $(x^*, y^*) = (0,0)$.

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \Rightarrow \left.\begin{pmatrix} e^{-x} & 1 \\ 3x^2 & -1 \end{pmatrix}\right|_{(x^*,y^*)=(0,0)} =$$



$$= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \Rightarrow$$

Characteristic equation

$(1-\lambda)(-1-\lambda) - 0 = 0$

$\lambda^2 \cancel{+\lambda - \lambda} - 1 = 0$

$\lambda^2 - 1 = 0$

$\lambda = \pm 1$

eigenvectors:

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & -2 & 0 \end{array}\right) \Rightarrow \begin{array}{l} 0x + y = 0 \\ 0x - 2y = 0 \end{array}$$

$y = 0$ $x =$ anything

choose

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} 2x + y = 0 \\ 2x = -y \\ x = -\frac{1}{2}y \end{array} \Rightarrow \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}.$$

Once again we have a saddle node f.p. since the eigenvalues are real and opposite signs.

$\boxed{2}$ consider ODE $\quad \dot{x} = y - x^2 + 2, \quad \dot{y} = 2y^2 - 2xy$

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \Bigg($$

$$y - x^2 + 2 = 0$$

$$2y^2 - 2xy = 0$$

$$2(x^2-2)^2 - 2x(x^2-2) = 0$$

$$2(x^4 - 4x^2 + 4) - 2x^3 + 4x = 0$$

$$2x^4 - 8x^2 + 8 - 2x^3 + 4x = 0$$

$$2(x^4 - x^3 + 4x^2 + 2x + 4) = 0.$$