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AMATH 502

HOMEWORK 1

Exercises come from *Nonlinear Dynamics and Chaos by Steven H. Strogatz*

1: 2.2.3
Solution:
TODO

2: 2.2.7
Solution:
TODO

3: 2.2.8
Solution:
TODO

4: 2.2.10
Solution:
TODO

5: 2.2.13 (a,b,c,d)

(a) $m\dot{v} = mg - kv^2$ with initial condition $v(0) = 0$

Solution:
Let's begin by dividing by m and factoring out a g term.

$$\begin{aligned} m\dot{v} &= mg - kv^2 \\ \dot{v} &= g - \frac{kv^2}{m} \\ \dot{v} &= g\left(1 - \frac{kv^2}{gm}\right). \end{aligned}$$

Now rewriting $\dot{v}$ as the derivative of v with respect to t we have

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}t} &= g\left(1 - \frac{kv^2}{gm}\right) \\ \frac{1}{\left(1 - \frac{kv^2}{gm}\right)}\mathrm{d}v &= g\mathrm{d}t \\ \int \frac{1}{\left(1 - \sqrt{\frac{k}{gm}}v\right)\left(1 + \sqrt{\frac{k}{gm}}v\right)}\mathrm{d}v &= \int g\mathrm{d}t. \end{aligned}$$

Now, we can do partial fractions on the left and integrate both sides

$$\int \frac{1}{\left(1-\sqrt{\frac{k}{gm}}v\right)\left(1+\sqrt{\frac{k}{gm}}v\right)}\mathrm{d}v = \int g\mathrm{d}t$$

$$\int \frac{1/2}{\left(1-\sqrt{\frac{k}{gm}}v\right)}\mathrm{d}v + \int \frac{1/2}{\left(1+\sqrt{\frac{k}{gm}}v\right)}\mathrm{d}v = gt + C$$

$$-\frac{1}{2}\sqrt{\frac{gm}{k}}\log\left(1-\sqrt{\frac{k}{gm}}v\right) + \frac{1}{2}\sqrt{\frac{gm}{k}}\log\left(1+\sqrt{\frac{k}{gm}}v\right) = gt + C$$

$$\frac{1}{2}\sqrt{\frac{gm}{k}}\left(\log\left(1+\sqrt{k/(gm)}\,v\right) - \log\left(1-\sqrt{k/(gm)}\,v\right)\right) = gt + C$$

Interesting ... now I need to input the initial condition

$$\frac{1}{2}\sqrt{\frac{gm}{k}}\left(\log\left(1+\sqrt{k/(gm)}\,0\right) - \log\left(1-\sqrt{k/(gm)}\,0\right)\right) = g0 + C$$

$$\frac{1}{2}\sqrt{\frac{gm}{k}}\left(\log(1) - \log(1)\right) = C$$

$$0 = C.$$

Plugging this in and simplifying we have

$$\frac{1}{2}\sqrt{\frac{gm}{k}}\left(\log\left(1+\sqrt{k/(gm)}\,v\right) - \log\left(1-\sqrt{k/(gm)}\,v\right)\right) = gt$$

$$\log\left(\frac{1+\sqrt{k/(gm)}\,v}{1-\sqrt{k/(gm)}\,v}\right) = 2gt\sqrt{k/(gm)}.$$

Now we can exponentiate and solve for v

$$\frac{1+\sqrt{k/(gm)}\,v}{1-\sqrt{k/(gm)}\,v} = \mathrm{e}^{2gt\sqrt{k/(gm)}}$$

$$1+\sqrt{k/(gm)}\,v = \mathrm{e}^{2gt\sqrt{k/(gm)}}(1-\sqrt{k/(gm)}\,v)$$

$$1+\sqrt{k/(gm)}\,v = \mathrm{e}^{2gt\sqrt{k/(gm)}} - \sqrt{k/(gm)}\,v\,\mathrm{e}^{2gt\sqrt{k/(gm)}}$$

$$\sqrt{k/(gm)}\,v + \sqrt{k/(gm)}\,v\,\mathrm{e}^{2gt\sqrt{k/(gm)}} = \mathrm{e}^{2gt\sqrt{k/(gm)}} - 1$$

$$v = \frac{\mathrm{e}^{2gt\sqrt{k/(gm)}} - 1}{\sqrt{k/(gm)} + \sqrt{k/(gm)}\,\mathrm{e}^{2gt\sqrt{k/(gm)}}}.$$

Some final simplifications gives us

$$v = \frac{1}{\sqrt{k/(gm)}}\frac{\mathrm{e}^{2gt\sqrt{k/(gm)}} - 1}{1+\mathrm{e}^{2gt\sqrt{k/(gm)}}}$$

$$v = \sqrt{\frac{gm}{k}}\left(\frac{\mathrm{e}^{2gt\sqrt{k/(gm)}} - 1}{\mathrm{e}^{2gt\sqrt{k/(gm)}} + 1}\right).$$

Therefore our final analytical solution is

$$v = \sqrt{\frac{gm}{k}}\left(\frac{\mathrm{e}^{2gt\sqrt{k/(gm)}} - 1}{\mathrm{e}^{2gt\sqrt{k/(gm)}} + 1}\right).$$

□

(b) Determine the limit of $v(t)$ as $t \to \infty$. We will need to utilize L'Hôpital's rule since the numerator and the denominator go to infinity.

$$\begin{aligned}
\lim_{t\to\infty} \sqrt{\frac{gm}{k}}\left(\frac{\mathrm{e}^{2gt\sqrt{k/(gm)}} - 1}{\mathrm{e}^{2gt\sqrt{k/(gm)}} + 1}\right) &= \lim_{t\to\infty} \sqrt{\frac{gm}{k}}\left(\frac{2gt\sqrt{k/(gm)}\,\mathrm{e}^{2gt\sqrt{k/(gm)}}}{2gt\sqrt{k/(gm)}\,\mathrm{e}^{2gt\sqrt{k/(gm)}}}\right) \\
&= \lim_{t\to\infty} \sqrt{\frac{gm}{k}} \\
&= \sqrt{\frac{gm}{k}}.
\end{aligned}$$

Therefore, the terminal velocity is $\sqrt{\frac{gm}{k}}$.

□

(c) **TODO:** insert pictures.

(d)

6: 2.3.2

Solution:

TODO

7: 2.3.6

Solution:

TODO

8: 2.4.7

Solution:

TODO