DSP Chapter 1 Notes

Hunter Mills

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1 Classification of Signals

1.1 Multichannel and Multidimensional Signals

A signal is described by a function of one or more independent variables. The value of the function can be real, complex, or a vector. For example, a earthquake creates primary, secondary and surface waves. These can be described by a **multichannel signal**, $S_3(t)$.

$$S_3(t) = \begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{bmatrix} \tag{1}$$

If the signal is a function of a single independent variable, it is a **one-dimensional signal**, whereas a signal is a **M-dimensional signal** if it is a function of M independent variables. A TV would be a 3-Dimensional and 3 Channel signal with $I_r(x, y, z), I_g(x, y, z)$ and $I_b(x, y, z)$.

$$I(x, y, t) = \begin{bmatrix} i_r(x, y, x) \\ i_b(x, y, x) \\ i_g(x, y, x) \end{bmatrix} \tag{2}$$

2 Frequency in CT and DT signals

2.1 CT Sinusoidal Signals

An example of a time domain sinusoidal signal is

$$x_a(t) = A\cos(\Omega t + \theta), \quad -\infty < t < \infty \tag{3}$$

The subscript a denotes an analog signal. This function is defined by amplitude (A), frequency(Ω in rad/sec, or Hz) and phase (θ). Instead of using Ω, F is also used as frequency meaning cycles per second or hertz.

$$\Omega = 2\pi F \tag{4}$$

An analog sinusoidal signal is characterized by the following properties:

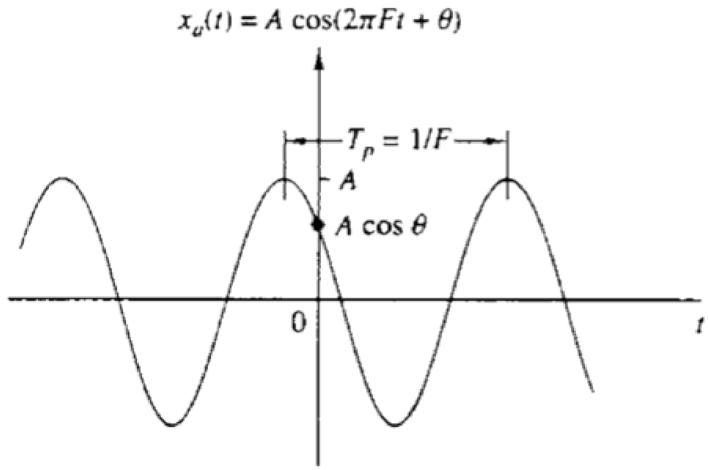

Figure 1: Example of analog sine wave

A1. For every fixed value of the frequency F, the sine wave will be periodic.

$$x_a(t + T_p) = x_a(t) \tag{5}$$

where $T + p = 1/F$ is the fundamental period of the sine wave. If a function of multiple sine waves added together, one must find the lowest common multiple of the frequencies. If there is none, the function is not periodic.

A2. CT sine waves with different frequencies are themselves distinct.

A3. Increasing the frequency F results in an increase in the rate of oscillation of the signal. Euler's Identity:

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi \tag{6}$$

A sinusoidal signal can also be represented as:

$$A\cos(\Omega t + \theta) = .5Ae^{j(\Omega t+\theta)} + .5Ae^{-j(\Omega t+\theta)} \tag{7}$$

2.2 Discrete Time Sinusoidal Signals

A DT Sinusoid can be expressed as

$$x[n] = A\cos(\omega n + \theta), \quad -\infty < t < \infty \tag{8}$$

where n in the sample value (integer) and ω is the frequency (radians/sample or cycles/sample).

$$\omega = 2\pi f \tag{9}$$

DT sinusoids are characterized by the following properties:

B1. A DT sinusiod is periodic only if its frequency f is a rational number.

$$x[n+N] = x[n], \quad for\ all\ n \tag{10}$$

The smallest number of N for which the previous equation is true if the fundamental period. Proof of periodicity:

$$\cos[2\pi f_0(N+n)+\theta] = \cos[2\pi f_0 n + \theta] \tag{11}$$

This relation is true if and only if there exists an integer k such that

$$2\pi f_0 N = 2k\pi \tag{12}$$

or

$$f_0 = \frac{k}{N} \tag{13}$$

B2. DT sinusiods whose frequencies are sparated by an integer multiple of 2π are identical. For all frequencies $|\omega| > \pi$ or $|f| > .5$ are aliases. The difference between CT and DT signals are that while any frequency of CT signal is distinct not different frequencies of a DT signal are distinct.

B3. The highest rate of oscillation in a DT sinusoid is attained when $\omega = \pm\pi$ or f $= \pm.5$

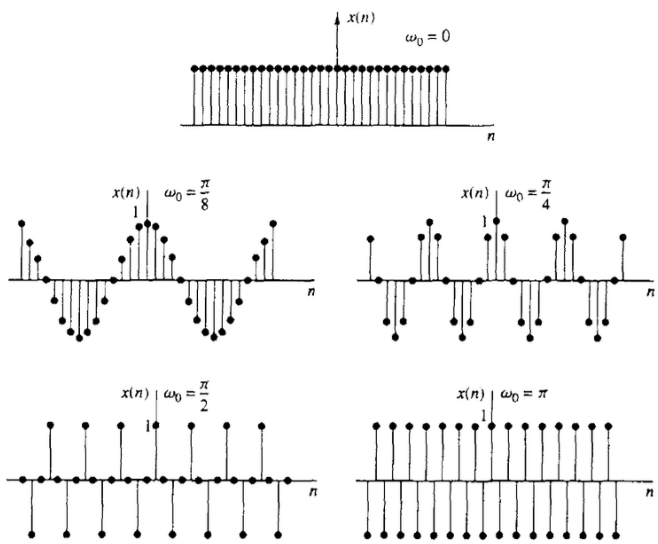

Figure 2: Signal showing a DT sine wave with different frequency values.

3 ADC and DAC

Analog to digital conversion is done in the order of sampling, quantizing and encoding.

Sampling: This is the conversion of a CT signal into a DT signal by taking samples of the CT wave at distinct time instants.
Quantization: Rounding the analog signal to a discrete value.
Coding: Converting from the quantized value to a binary value.

3.1 Sampling of Analog Signals

In this book the only type of sampling discussed is uniform sampling and is described by:

$$x[n] = x_a(nT), \quad -\infty < n < \infty \tag{14}$$

Periodic sampling establishes a relationship between time (t) and sample (n). The sample period T from the previous equation is related to the sample rate $F_s = 1/T$.

$$t = nT = \frac{n}{F_s} \tag{15}$$

As a consequence of equation 15, there exists a relationship between the frequency variable F (or Ω) of analog signals and the frequency variable f or (ω) of DT signals. These following equations show the relationship. When

$$x_a(t) = A\cos(2\pi Ft + \theta) \tag{16}$$

is sampled at a rate $F_s = 1/T$ it produces

$$x_a(nT) = x[n] = A\cos(2\pi FnT + \theta) \tag{17}$$

$$x[n] = A\cos(\frac{2\pi nF}{F_s} + \theta) \tag{18}$$

The frequency variables F (CT) and f (DT) are linearly related as:

$$f = \frac{F}{F_s} \tag{19}$$

or

$$\omega = \Omega T \tag{20}$$

DT frequency, f, can be referred to as **relative or normalized**.

3.2 Sampling Theory

The fundamental difference between CT and DT signals is the range of values. Periodic sampling of a CT signal implies a mapping of the infinite frequency range, F (or Ω), to a finite frequency rage, f (or ω), for DT signals. Since the highest frequency in a DT signal is $\omega = pi$ or f = .5, it follows that, with sampling at F_s, the corresponding highest values or F and Ω are

$$F_{\max} = \frac{F_s}{2} = \frac{1}{2T} \tag{21}$$

Continuous-time signals		Discrete-time signals
$\Omega = 2\pi F$		$\omega = 2\pi f$
$\frac{\text{radians}}{\text{sec}}$ Hz		$\frac{\text{radians}}{\text{sample}}$ $\frac{\text{cycles}}{\text{sample}}$
	$\omega = \Omega T, f = F/F_s$ →	$-\pi \le \omega \le \pi$ $-\frac{1}{2} \le f \le \frac{1}{2}$
	← $\Omega = \omega/T, F = f \cdot F_s$	
$-\infty < \Omega < \infty$ $-\infty < F < \infty$		$-\pi/T \le \Omega \le \pi/T$ $-F_s/2 \le F \le F_s/2$

Figure 3: Relations between DT and CT frequencies

$$\Omega_{\max} = \pi F_s = \frac{\pi}{T} \tag{22}$$

Sampling introduces an ambiguity since the highest frequency in a CT signal that can be uniquely distinguished when sampled at F_s is $F_s/2$. This is the **nyquist theorem** which states if the sampling rate is not twice the highest frequency in the signal, aliasing will occur.

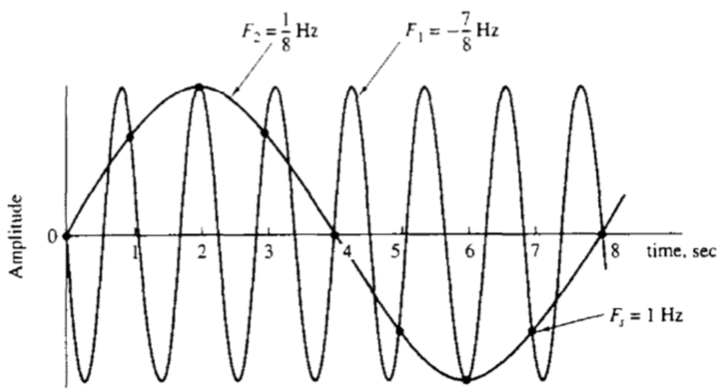

Figure 4: Illustration of aliasing