Discrete Time Signals and Systems

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1 Frequency Analysis of Signals

Two of the most important mathematical tools in engineering is the Fourier Transform and Fourier Series. In this chapter they will be used to do frequency analysis. This signal representations basically involve the decomposition of the signals into terms of sinusoidal functions. For infinite periodic signals, use the Fourier series and for a finite energy series use the Fourier transform. These decomposition are very important in LTI systems since the response of an LTI signal to a sinusoid is a sinusoid of the same frequency but different amplitude and phase.

1.1 Euler's Identity

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{1}$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{2}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{j2} \tag{3}$$

$$\cos(\Omega_k t) = \frac{1}{2}(e^{j\Omega_k t} + e^{-j\Omega_k t}) \tag{4}$$

$$\sin(\Omega_k t) = \frac{-j}{2}(e^{j\Omega_k t} - e^{-j\Omega_k t}) \tag{5}$$

1.2 Frequency Analysis of Continuous Time Signals

The Fourier Series for CT Periodic Signals

The Fourier Series is a representation of the signal as a linear weighted sum of harmonically related sinusiods or complex exponentials. From chapter 1, we can recall that a linear combination of harmonically related complex exponentials of the form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t} \tag{6}$$

is a periodic signal with fundamental period $T_P = 1/F_0$. Hence we can think of the exponential signals

$$e^{j2\pi kF_0 t}, \qquad k = 0, \pm 1, \pm 2... \tag{7}$$

as the building blocks to construct periodic signals by the proper choice of fundamental frequency and c_k. The Fourier series coefficients can be calculated with:

$$c_k = \frac{1}{T_P}\int_{T_P} x(t)e^{-j2\pi kF_0 t}dt. \tag{8}$$

In general the Fourier coefficients c_k are complex. It is easily shown that if the periodic signal is real, c_k and c_{-k} are complex conjugates. As a result:

$$c_k = |c_k|e^{j\theta_k} \tag{9}$$

then

$$c_{-k} = |c_k|^{-j\theta_k} \tag{10}$$

Consequently the FS can be represented as:

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} |c_k|\cos(2\pi kF_0 t + \theta_k) \tag{11}$$

or

$$Trigonometric : x(t) = a_0 + \sum_{k=1}^{\infty}(a_k \cos(2\pi kF_0 t) - b_k \sin(2\pi kF_0 t)) \tag{12}$$

$$Harmonic : x(t) = A_{DC} + \sum_{k=1}^{\infty} \sqrt{a_k^2 + b_k^2}\cos(\Omega_k t - \theta_k) \tag{13}$$

where $a_0 = c_0$, $a_k = 2|c_k|\cos(\theta_k)$, and $b_k = 2|c_k|\sin(\theta_k)$. These are called the *Continuous Time Fourier Series* (CTFS).

The Dirichlet conditions guarantee convergence and the series converges to x(t) when
1. Signal has a finite number of discontinuities in any period.
2. Signal has a finite maxima and minima in any period.
3. Signal is absolutely summable.

Power Density Spectrum of Periodic Signals

A periodic signal has infinite energy and a fine average power. Using the FS the **Parsevals Relation for Power Signals** becomes clear.

$$P_x = \frac{1}{T_P}\int_{T_P} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2 \tag{14}$$

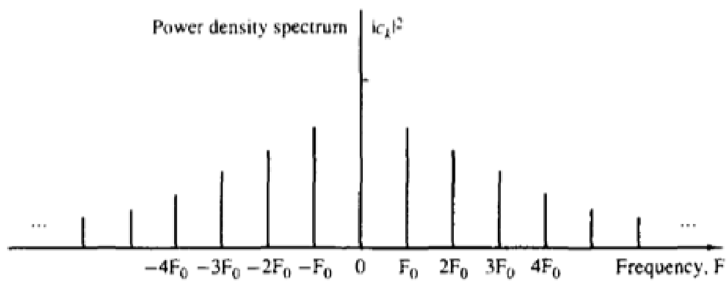

Figure 1: Power Density Spectrum

If we would plot $|c_k|^2$ as a function of the frequencies $kF_0, k = 0, \pm1, \pm2, ...$ then figure 1 shows the *power density spectrum.*

This is called a line spectrum and the spacing between the spectral lines is the fundamental frequency. The total average power can be expressed as

$$P_x = c_0^2 + 2\sum_{k=1}^{\infty} |c_k|^2 \tag{15}$$

$$P_x = a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2) \tag{16}$$

The Fourier Transform for CT Aperiodic Signals

As the period of a wave increases towards infinity, the spacing between the line spectrum decreases towards zero. When the period becomes infinite the spacing becomes zero and the spectrum is continuous. Consider the aperiodic and finite signal x(t)

$$x(t) = \lim_{T_P \to \infty} x_P(t). \tag{17}$$

This implies that we should be able to get the spectrum for x(t) from the spectrum of $x_P(t)$ and by taking the limit as $T_P \to \infty$. The **CT Fourier Transform** is defined as

$$X(F) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt. \tag{18}$$

A relationship between the Fourier transform and the Fourier series is

$$T_P c_k = X(kF_0) = X(\frac{k}{T_P}) \tag{19}$$

The inverse Fourier Transform is defined as

$$x(t) = \int_{-\infty}^{\infty} X(F)e^{j2\pi Ft}dt. \tag{20}$$

This is called the *Continuous Time Fourier Transform* (CTFT). The difference between the Fourier transform and series the former is continuous and hence the synthesis of an aperiodic signal from its spectrum is done by integration not summation. The previous Fourier transform equations can also be represented with its radial frequency, Ω.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega. \tag{21}$$

$$x(\Omega) = \int_{-\infty}^{\infty} x(t)e^{j\Omega t}dt. \tag{22}$$

Dirichlet Condition for Fourier Transform to exist
1. x(t) must have at most a finite number of finite discontinuities
2. x(t) must have at most a finite number of maxima and minima
3. x(t) is absolutely integral that is $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$

Energy Density Spectrum of Aperiodic Signals
Let x(t) be any finite energy signal, its energy is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt. \tag{23}$$

Parsevals relation for finite energy signals is:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(F)|^2 dF \tag{24}$$

and it expresses the principle of conservation of energy in the frequency and time domain. The energy in a signal over a frequency band is,

$$E = \int_{F_1}^{F_1+\delta F} |X(F)|^2 dF \tag{25}$$

If the signal x(t) is real then

$$|X(-F)| = |X(F)| \tag{26}$$

$$\angle X(-F) = -\angle X(F) \tag{27}$$

1.3 Frequency Analysis if DT Signals

In contrast to CT signals which $-\infty < f < \infty$ the frequency range for DT signals is $(-\pi, \pi) or (0, 2\pi)$. A DT signal of fundamental period N can consist of frequency components separated by $2\pi/N$ radians or $f = 1/N$ cycles. Consequently, the Fourier series representation will contain at most N frequency components.

The Fourier Series for DT periodic signals

Suppose we have x(n) with period N. The Fourier series representation for x(n) consists of N harmonically related exponential functions,

$$e^{j2\pi kn/N}, \qquad k = 0, 1, ..., N-1 \tag{28}$$

and is expressed as

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \tag{29}$$

where

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \tag{30}$$

This is called the *Discrete Time Fourier Series* (DTFS). c_k provide the description of x(n) in the frequency domain in the sense that c_k represents the amplitude and phase associated with the frequency component

$$s_k(n) = e^{j2\pi kn/N} = e^{j\omega_k n} \tag{31}$$

where $\omega_k = 2\pi k/N$. The spectrum of a singal x(n), which is periodic N, is a periodic sequence with period N, ie the spectrum repeats.

Power Density Spectrum of Periodic Signals

The average power of a DT periodic signal with period N is

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2. \tag{32}$$

Using the Fourier series coefficients we can calculate the power using Parseval's relation for DT periodic signals.

$$P_x = \sum_{k=0}^{N-1} |c_k|^2 \tag{33}$$

To get the energy in a period multiply the power by N. Since the spectrum is periodic about N, the magnitude and phase properties are shown in the next figure.

Other ways to calculate the DTFS is with

$$x(n) = c_0 + 2\sum_{k=1}^{L} |c_k| \cos(\frac{2\pi}{N}kn + \theta_k) \tag{34}$$

or

$$x(n) = a_0 + \sum_{k=1}^{L}(a_k \cos(\frac{2\pi}{N}kn) - b_k \sin(\frac{2\pi}{N}kn)) \tag{35}$$

$$
\begin{array}{rclrcll}
|c_0| &=& |c_N|, & \measuredangle c_0 &=& -\measuredangle c_N = 0 & \\
|c_1| &=& |c_{N-1}|, & \measuredangle c_1 &=& -\measuredangle c_{N-1} & \\
|c_{N/2}| &=& |c_{N/2}|, & \measuredangle c_{N/2} &=& 0 & \text{if } N \text{ is even} \\
|c_{(N-1)/2}| &=& |c_{(N+1)/2}|, & \measuredangle c_{(N-1)/2} &=& -\measuredangle c_{(N+1)/2} & \text{if } N \text{ is odd}
\end{array}
$$

Figure 2: Magnitude and Phase Properties of DTFS

where $a_0 = c_0, a_k = 2|c_k|\cos\theta_k, b_k = 2|c_k|\sin\theta_k andL = N/2$ if N is even and $L = (N-1)/2$ if odd.

The Fourier Transform of DT Aperiodic Signals

This section heavily parallels the section for CTFT. The Discrete Time Fourier Transform (DTFT) is defined as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{36}$$

The same as the CTFS and DTFS, the DTFT has a frequency range over $(0, 2\pi)$ and is periodic with 2π To go from the frequency domain to the time domain, the inverse Fourier transform is

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega \tag{37}$$

Convergence of the Fourier Transform

If the value $\sum_{n=-\infty}^{\infty}|x(n)| < \infty$ then this condition (absolutely summable) is sufficient for the existence of the DTFT. Some signals are not absolutely summable but they are square summable ($E_x = \sum_{n=-\infty}^{infty}|x(n)|^2 < \infty$). If the signal is absolutely summable there will be *uniform convergence*. If it is square summable then there will be *mean-square convergence* where at jump discontinuities the DTFT converges to a midpoint and demonstrates Gibbs Phenomenon.

Energy Density Spectrum of Aperiodic Signals The energy in a DT signal x(n) is

$$E_x = \sum_{n=-\infty}^{\infty}|x(n)|^2 \tag{38}$$

Using **Parsevals relation for DT aperiodic signals** the energy can be calculated with

$$E_x = \int_{-\infty}^{\infty}|X(\omega)^2 d\omega \tag{39}$$

If x(n) is real, the magnitude has even symmetry and the phase has odd. The symmetry properties let us calculated the magnitude and phase from the spectrum in the range $(0, \pi)$.

Frequency domain classification of Signals: Bandwidth If the frequency

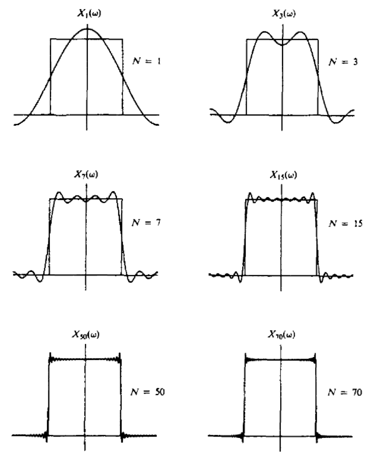

Figure 3: DTFT and Gibbs Phenomenon

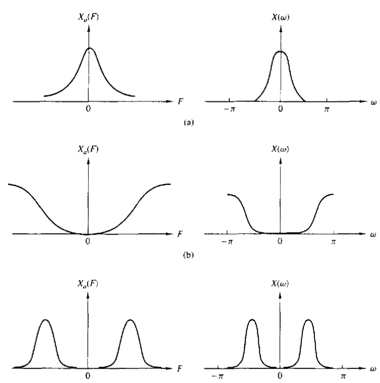

Figure 4: a: Baseband, b: High, c: Bandpass

components are located around zero Hz then the signal is considered baseband and low-frequency. The two types of signals are shown in the next figure.

The relative measurement of how wide the frequency components are is called the *Bandwidth*. A narrowband signal is one where the bandwidth is much smaller (near 10x) than the median frequency otherwise it is wideband. A signal is bandlimited if its spectrum is zero outside the frequency band.

$$|X(\Omega)| = 0 for |F| > B(Bandwidth) \tag{40}$$

Without proving, no signal can be time-limited and bandlimited at the same time. A reciprocal relationship exists between the time duration and the frequency duration.

1.4 Frequency Domain and Time Domain Signal Properties

In the previous section we introduced 4 frequency analysis tools,
1. The CTFS for periodic signals
2. The CTFT for aperiodic signals
3. The DTFS for periodic signals
4. The DTFT for aperiodic signals
Some other properties that can be derived are, CT signals have aperiodic spectra, DT signals have periodic spectra with $2pi$, periodic signals have discrete spectra (the FS is used which results in line spectra) and aperiodic finite energy signals have continuous spectra. The following figure shows the summary of analysis and synthesis formulas.

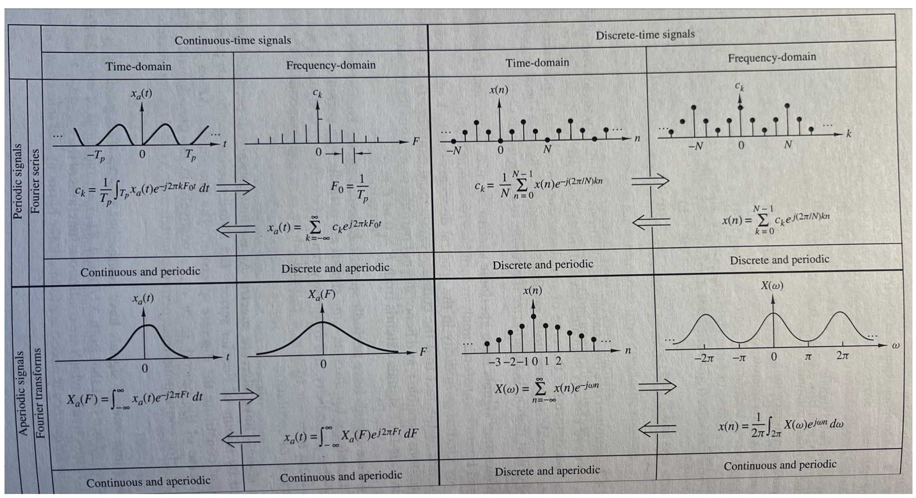

Figure 5: Summary of analysis and synthesis equations

1.5 Properties of the Fourier Transform for DT signals

The Fourier Transform for aperiodic finite-energy signals has a number of properties that can make the complexity of frequency analysis easier. $x(n) and X(\omega)$ are called Fourier transform pairs. In this section, I will not derive the properties, but there will be diagrams.

Symmetry Properties of the Fourier Transform

When a signal satisfies some symmetry properties in the time domain, these properties impose some symmetry conditions in the frequency domain.

Real Signals

When x(n) is a real signal, the magnitude and phase spectra possess the following symmetry properties

$$|X(\omega)| = |X(-\omega)(even) \tag{41}$$

$$\angle X(-\omega) = -\angle X(\omega)(odd) \tag{42}$$

Real and Even Signals

$$X(\omega) = X_R(\omega) = x(0) + 2\sum_{n=1}^{\infty} x(n)\cos(\omega n) \tag{43}$$

If the input x(n) is real and even then the spectra will be real-valued.
Real and Odd Signals

$$X(\omega) = X_I(\omega) = -2\sum_{n=1}^{\infty} x(n)\sin(\omega n), \tag{44}$$

thus real and odd signals produce a purely imaginary spectrum.

Sequence	DTFT
$x(n)$	$X(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x_R(n)$	$X_e(\omega) = \frac{1}{2}[X(\omega) + X^*(-\omega)]$
$jx_I(n)$	$X_o(\omega) = \frac{1}{2}[X(\omega) - X^*(-\omega)]$
$x_e(n) = \frac{1}{2}[x(n) + x^*(-n)]$	$X_R(\omega)$
$x_o(n) = \frac{1}{2}[x(n) - x^*(-n)]$	$jX_I(\omega)$
Real Signals	
Any real signal $x(n)$	$X(\omega) = X^*(-\omega)$ $X_R(\omega) = X_R(-\omega)$ $X_I(\omega) = -X_I(-\omega)$ $\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$ $\measuredangle X(\omega) = -\measuredangle X(-\omega)$
$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$ (real and even)	$X_R(\omega)$ (real and even)
$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$ (real and odd)	$jX_I(\omega)$ (imaginary and odd)

Figure 6: Real Signal FT Properties

Fourier Transform Theorems and Properties
All of these properties are summed up well in the following table. In Parsevals relation, when $x_1(n) = x_2(n) = x(n)$ then

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\omega)|^2 \tag{45}$$

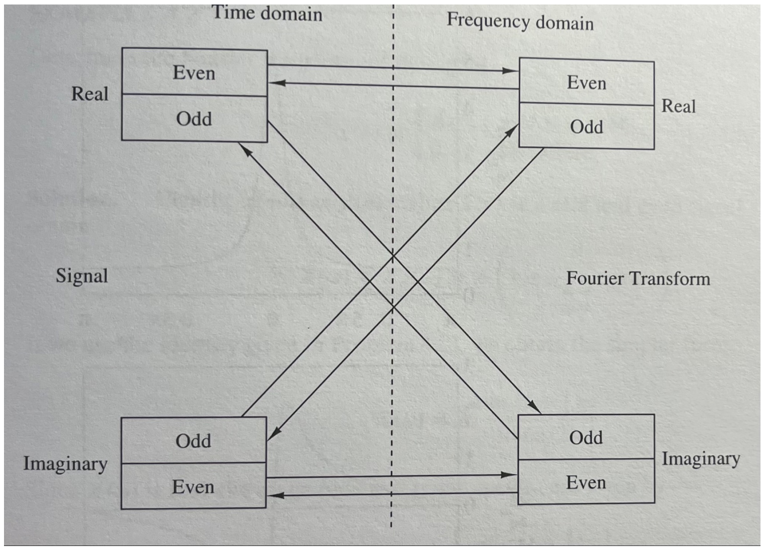

Figure 7: Symmetry Diagram

Property	Time Domain	Frequency Domain
Notation	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_2(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$ $= X_1(\omega)X_2^*(\omega)$ [if $x_2(n)$ is real]
Wiener–Khintchine theorem	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency shifting	$e^{j\omega_0 n}x(n)$	$X(\omega - \omega_0)$
Modulation	$x(n)\cos\omega_0 n$	$\frac{1}{2}X(\omega+\omega_0) + \frac{1}{2}X(\omega-\omega_0)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$
Differentiation in the frequency domain	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$	

Figure 8: FT Properties