Module 6 Solutions: The z-Transform

Digital Signal Processing (EN.525.627.8X)

1. Determine all possible signals $x(n)$ associated with the z-Transform

$$X(z) = \frac{5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(3 - z^{-1})}$$

–

First, use PFE to determine poles at z=1/2 and z=1/3:

$$X(z) = \frac{5z}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)} = \frac{A}{\left(z - \frac{1}{2}\right)} + \frac{B}{\left(z - \frac{1}{3}\right)}$$

Possible equations:

For ROC = $\frac{1}{3} < |z| < \frac{1}{2}$: $x(n) = -\left(\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1)$
This is the stable condition

For ROC = $|z| > \frac{1}{2}$: $x(n) = -\left(\frac{1}{3}\right)^n u(n) + \left(\frac{1}{2}\right)^n u(n)$
This is the causal condition

For ROC = $|z| < \frac{1}{3}$: $x(n) = \left(\frac{1}{3}\right)^n u(-n-1) - \left(\frac{1}{2}\right)^n u(-n-1)$
This is the anti-causal condition

2. Determine the z-Transform of the following signals:

a. $x(n) = \{3, 0, 0, 0, 0, 6, 1, -4\}$

Problem should have marked 6 for n=0.

$$\begin{aligned} X(z) &= \sum_n x(n) z^{-n} \\ &= 3z^5 + 6 + z^{-1} - 4z^{-2} \text{ ROC: } 0 < |z| < \infty \end{aligned}$$

b. $x(n) = \begin{cases} \left(\frac{1}{2}\right)^2 & for \;\; n \geq 5 \\ 0 & for \;\; n \leq 4 \end{cases}$

$$\begin{aligned} X(z) &= \sum_n x(n) z^{-n} \\ &= \sum_{n=5}^{\infty} (\frac{1}{2})^n z^{-n} \\ &= \sum_{n=5}^{\infty} (\frac{1}{2z})^n \\ &= \sum_{m=0}^{\infty} (\frac{1}{2} z^{-1})^{m+5} \\ &= (\frac{z^{-1}}{2})^5 \frac{1}{1 - \frac{1}{2} z^{-1}} \\ &= (\frac{1}{32}) \frac{z^{-5}}{1 - \frac{1}{2} z^{-1}} \text{ ROC: } |z| > \frac{1}{2} \end{aligned}$$

3. Determine the z-Transforms and sketch the ROC of the following signal:

a. $x_1(n) = \begin{cases} \left(\frac{1}{3}\right)^n & for \;\; n \geq 0 \\ \left(\frac{1}{2}\right)^{-n} & for \;\; n < 0 \end{cases}$

$$\begin{aligned} X_1(z) &= \sum_{n=0}^{\infty}(\frac{1}{3})^n z^{-n} + \sum_{n=-\infty}^{0}(\frac{1}{2})^n z^{-n} - 1 \\ &= \frac{1}{1-\frac{1}{3}z^{-1}} + \sum_{n=0}^{\infty}(\frac{1}{2})^n z^n - 1 \\ &= \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z} - 1, \\ &= \frac{\frac{5}{6}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)} \end{aligned}$$

The ROC is $\frac{1}{3} < |z| < 2.$

b. $x_2(n) = \begin{cases} \left(\frac{1}{3}\right)^n - 2^n & for \;\; n \geq 0 \\ 0 & for \;\; n < 0 \end{cases}$

$$\begin{aligned} X_2(z) &= \sum_{n=0}^{\infty}(\frac{1}{3})^n z^{-n} - \sum_{n=0}^{\infty} 2^n z^{-n} \\ &= \frac{1}{1-\frac{1}{3}z^{-1}} - \frac{1}{1-2z^{-1}}, \\ &= \frac{-\frac{5}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})} \end{aligned}$$

The ROC is $|z| > 2.$

c. $x_3(n) = x_1(n+4)$

$$\begin{aligned} X_3(z) &= \sum_{n=-\infty}^{\infty} x_1(n+4)z^{-n} \\ &= z^4 X_1(z) \\ &= \frac{\frac{5}{6}z^4}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)} \end{aligned}$$

The ROC is $\frac{1}{3} < |z| < 2.$

d. $x_4(n) = x_1(-n)$

$$
\begin{aligned}
X_4(z) &= \sum_{n=-\infty}^{\infty} x_1(-n) z^{-n} \\
&= \sum_{m=-\infty}^{\infty} x_1(m) z^{m} \\
&= X_1(z^{-1}) \\
&= \frac{\frac{5}{6}}{(1-\frac{1}{3}z)(1-\frac{1}{2}z^{-1})}
\end{aligned}
$$

The ROC is $\frac{1}{2} < |z| < 3$.

e. $x_4(n) = x_1(-n) + x_2(n)$

f. $x_5(n) = x_2(-2n)$

4. Compute the convolution of the following signals by means of the z-Transforms:

$$x_1(n) = \begin{cases} \left(\frac{1}{3}\right)^n & for\ \ n \geq 0 \\ \left(\frac{1}{2}\right)^{-n} & for\ \ n < 0 \end{cases}$$

$$x_2(n) = \left(\frac{1}{2}\right)^n u(n)$$

-

$$\begin{aligned} X_1(z) &= \sum_{n=0}^{\infty}(\frac{1}{3})^n z^{-n} + \sum_{n=-\infty}^{-1}(\frac{1}{2})^{-n} z^{-n} \\ &= \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z} - 1 \\ &= \frac{\frac{5}{6}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)} \\ X_2(z) &= \sum_{n=0}^{\infty}(\frac{1}{2})^n z^{-n} \\ &= \frac{1}{1-\frac{1}{2}z^{-1}}, \frac{1}{2} < |z| < 2 \end{aligned}$$

$$\begin{aligned} \text{Then,} Y(z) &= \frac{-2}{1-\frac{1}{3}z^{-1}} + \frac{\frac{10}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{-4}{3}}{1-2z^{-1}} \\ \text{Hence,} y(n) &= \begin{cases} -2(\frac{1}{3})^n + \frac{10}{3}(\frac{1}{2})^n, & n \geq 0 \\ \frac{4}{3}(2)^n, & n < 0 \end{cases} \end{aligned}$$

5. Using long division, determine the inverse z-Transform of

$$X(z) = \frac{1 + 2z^{-1}}{1 - 2z^{-1} + z^{-2}}$$

-

6. Determine the causal signal $x(n)$ having the z-Transform:

$$X(z) = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})^2}$$

-*<u>Similar process to Problem #1:</u>*

Use PFE to solve for the time domain signal:

$$X(z) = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})^2} = \frac{A}{\left(1 - \frac{1}{2}z^{-1}\right)} + \frac{B}{(1 - z^{-1})} + \frac{C}{(1 - z^{-1})^2}$$

Solve for C=-2, A=1, B= 16/3 (Z=1, Z=1/3, Z=2)

$$X(z) = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)} + \frac{16}{3(1 - z^{-1})} + \frac{2}{(1 - z^{-1})^2}$$

So, the time-domain signal is: $x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{16}{3}\right) u(n) - 2(n+1)u(n+1)$