Communication Systems

Chapter 3 Analysis and Transmission of Signals

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1 Fourier Transform of Signals

The **Fourier Transform** of $g(t)$ is defined as

$$G(f) = \mathfrak{F}[g(t)] = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt \tag{1}$$

The **Inverse Fourier Transform** is defined as

$$g(t) = \mathfrak{F}^{-1}[G(f)] = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}dt \tag{2}$$

1.1 Conjugate Symmetry Property

If $g(t)$ is real then $G(f)$ and $G(-f)$ are complex conjugates, that is

$$G(-f) = G^*(f) = \int_{-\infty}^{\infty} g(t)e^{j2\pi ft}dt \tag{3}$$

$$|G(-f)| = |G(f)| \tag{4}$$

$$\theta_g(-f) = -\theta_g(f) \tag{5}$$

For real signals $g(t)$ the amplitude spectrum is even and the phase spectrum is odd.

1.2 Existence of the Fourier Transform

If a signal satisfies the following condition then the signal has a fourier transform.

$$\int_{-\infty}^{\infty} |g(t)|dt < 0 \tag{6}$$

There is a Fourier Transform table on page 106 of the textbook.

2 Some Fourier Transform Properties

I will only note specific properties, the others are in a table on page 123 of the text book.

2.1 Time-Frequency Duality

The operations required to go from $g(t)$ to $G(t)$ and then from $G(t)$ to $g(t)$ are very similar, only differing in the signs of the exponential. This is the basis of the duality of time and frequency. For any relationship between $g(t)$ and $G(f)$ there exists a dual relationship obtained by interchanging the roles of $g(t)$ and $G(f)$. For example the time-shifting property

$$g(t - t_0) \leftrightarrow G(f)e^{-j2\pi f t_0} \tag{7}$$

and the dual of this property

$$g(t)e^{-j2\pi f_0 t} \leftrightarrow G(f - f_0) \tag{8}$$

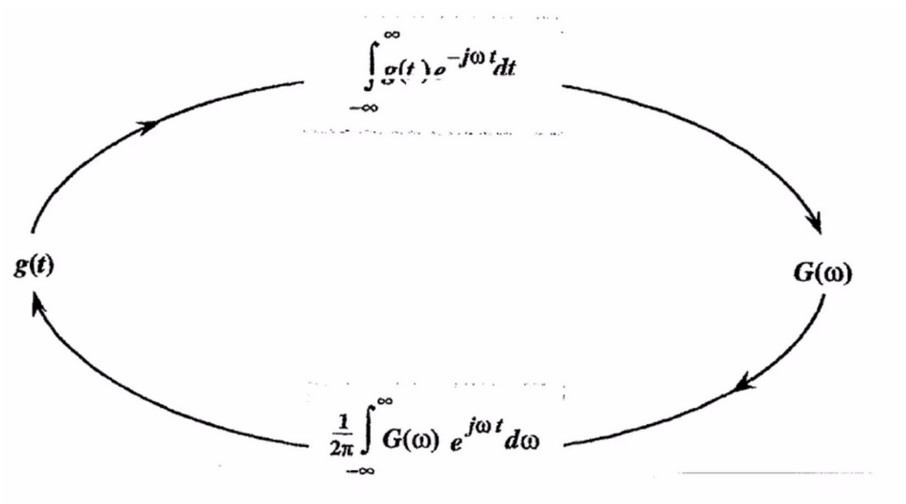

Figure 1: Near symmetry between Fourier and Inverse Fourier Transforms

2.2 Reciprocity of Signal Duration and its Bandwidth

The time-scaling property implies that if $g(t)$ gets wider (time extension) its spectrum gets narrower and the reverse, if $g(t)$ is time compressed the spectrum gets wider. The bandwidth of a signal is inversely proportional to the signal duration.

2.3 Convolution

Convolution is defined as

$$g(t) * w(t) = \int_{-\infty}^{\infty} g(\tau)w(t - \tau)d\tau \tag{9}$$

3 Signal Transmission Through a LTI System

The LTI system model shown in the next figure can be used to characterize communication channels. A stable LTI system can be characterized in the time domain by its impulse response $h(t)$ which is the system response $y(t)$ to a unit impulse $x(t) = \delta(t)$.

$$y(t) = h(t) * x(t) \tag{10}$$

and when $x(t) = \delta(t)$, $y(t) = h(t)$.

$$Y(f) = H(f)X(f) \tag{11}$$

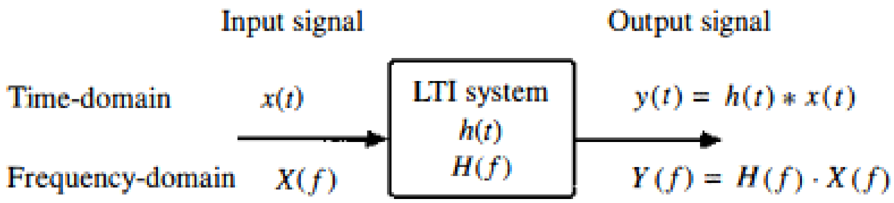

Figure 2: Signal Transmission through a LTI system

The fourier transform of the impulse response is referred to as the **transfer function** or the **frequency response**. In general $H(f)$ is complex and can be written as

$$H(f) = |H(f)|e^{j\theta_h(f)} \tag{12}$$

3.1 Signal Distortion During Transmission

$$|Y(f)| = |X(f)||H(f)| \tag{13}$$

$$\theta_y(f) = \theta_x(f) + \theta_h(f) \tag{14}$$

3.2 Distortionless Transmission

It is of practical interest to determine the characteristics of a system that allow for **Distortionless Transmission**. In distortionless transmission satisfies the following

$$y(t) = kx(t - t_d) \tag{15}$$

The transfer function (frequency response) required for distortionless transmission is

$$|H(f)| = k \tag{16}$$

$$\theta_h(f) = -2\pi f t_d \tag{17}$$

That says the amplitude response must be a constant and the phase response must be linear and go through the origin. If the slope is not linear then components of different frequencies undergo different time delays. The **group delay,** t_g**,** is when the output envelope is the same as the input envelope delayed by t_g.

$$t_g = t_d(f) = -\frac{1}{2\pi}\frac{d\theta_h(f)}{df} \tag{18}$$

4 Ideal vs Practical Filters

Ideal filters allow distortionless transmission over a certain band while completely suppressing signals outside that band.

4.1 Practically Realizable Filters

For physically realizable system $h(t)$ must be casual. This is equivalent to the Paley-Wiener criterion in the frequency domain that states that necessary and sufficient condition for $|H(f)|$ to be the amplitude response of a casual system:

$$\int_{-\infty}^{\infty} \frac{|ln|H(f)||}{1 + (2\pi f)^2} df < \infty \tag{19}$$

This states that $|H(f)|$ cannot be 0 over a band. You can create a causal system by $\hat{h}(t) = h(t)u(t)$ and with sufficient time delay it will be close to the ideal filter. The trade off is the much higher delay in the output. The **cutoff frequency** of a filter is the 3db (half power) bandwidth.

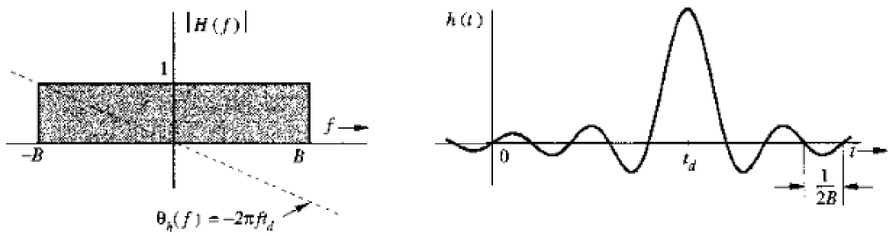

Figure 3: Ideal Low Pass Filter

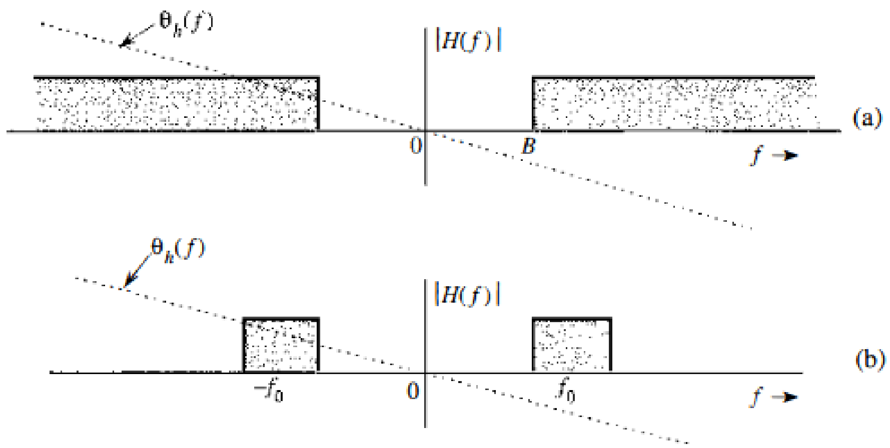

Figure 4: Ideal High and Band Pass Filter

5 Signal Energy and Energy Spectral Density

The energy E_g of a signal $g(t)$ is defined as the area under $|g(t)|^2$.

5.1 Parseval's Theorem

$$E_g = \int_{-\infty}^{\infty} |G(f)|^2 df = \int_{-\infty}^{\infty} g^2(t) dt \tag{20}$$

5.2 Energy Spectral Density (ESD)

We can interpret $|G(f)|^2$ as the energy per unit bandwidth (in Hz) of $g(t)$. The **energy spectral density (ESD, $\Psi_g(f)$)** is defined as

$$\Psi_g(f) = |G(f)|^2 \tag{21}$$

and the energy from Parseval's Theorem can be defined as

$$E_g = \int_{-\infty}^{\infty} \Psi_g(f) df \tag{22}$$

5.3 Time Autocorrelation Function and ESD

There is a very important relationship between the autocorrelation $\psi_g(\tau)$ and its ESD $\Psi_g(f)$. The autocorrelation function and its ESD form a Fourier Transform pair known as the **Wiener-Khintchine Theorem**.

$$\psi_g(\tau) \leftrightarrow \Psi_g(f) = |G(f)|^2 \tag{23}$$

This is also valid for complex signals.

6 Signal Power and Power Spectral Density

The power of a real $g(t)$ is

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g^2(t)dt \tag{24}$$

The relationship between power and energy of non-periodic signals is

$$P_g = \lim_{T\to\infty} \frac{E_{gT}}{T} \tag{25}$$

where E_{gT} is the energy of a truncated $g(t)$ over a period.

6.1 Power Spectral Density (PSD)

The **Power Spectral Density** is defined as

$$S_g(f) = \lim_{T\to\infty} \frac{|G_T(f)|^2}{T} \tag{26}$$

where $G_T(f)$ is the fourier transform of a truncated $g(t)$ over a period.

$$P_g = 2\int_0^{\infty} S_g(f)df \tag{27}$$

6.2 Time Autocorrelation Function of Power Signals

The time autocorrelation function $\mathfrak{R}_g(\tau)$ of a real power signal $g(t)$ is

$$\mathfrak{R}_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)g(t-\tau)dt \tag{28}$$

$\mathfrak{R}_g(\tau)$ is a even function of t so

$$\mathfrak{R}_g(\tau) = \mathfrak{R}_g(-\tau) \tag{29}$$

The Wiener-Khintchine theorem states

$$\mathfrak{R}_g(\tau) \leftrightarrow \lim_{T\to\infty} \frac{|G_T(f)|^2}{T} = S_g(f) \tag{30}$$

$E_g = \int_{-\infty}^{\infty} g^2(t)\,dt$	$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g^2(t)\,dt = \lim_{T\to\infty} \frac{E_{g_T}}{T}$
$\psi_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t+\tau)\,dt$	$\mathcal{R}_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)g(t+\tau)\,dt = \lim_{T\to\infty} \frac{\psi_{g_T}(\tau)}{T}$
$\Psi_g(f) = \lvert G(f)\rvert^2$	$S_g(f) = \lim_{T\to\infty} \frac{\lvert G_T(f)\rvert^2}{T} = \lim_{T\to\infty} \frac{\Psi_{g_T}(f)}{T}$
$\psi_g(\tau) \Longleftrightarrow \Psi_g(f)$	$\mathcal{R}_g(\tau) \Longleftrightarrow S_g(f)$
$E_g = \int_{-\infty}^{\infty} \Psi_g(f)\,df$	$P_g = \int_{-\infty}^{\infty} S_g(f)\,df$

Figure 5: Signal Energy vs Signal Power

6.3 Signal Power is its Mean Square Value

Defining $<>$ as the time average

$$P_g =< g^2(t) >= \lim_{T \to \infty} \int_{-T/2}^{T/2} g^2(t)dt \tag{31}$$

The RMS value is $[g(t)]_{RMS} = \sqrt{P_g}$

$$\Re_g(\tau) =< g(t)g(t \pm \tau) > \tag{32}$$

6.4 Input PSD vs Output PSD

$$S_y(f) = |H(f)|^2 S_g(f) \tag{33}$$