

# Communication Systems

## Chapter 8 Random Processes and Spectral Analysis

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April 10, 2024

A **Random Process** is a natural extension from random variables, such that a random variable  $\chi$  that is a function of time  $\chi(t)$  is a Random Process or **Stochastic Process**.

### 1 From Random Variable to Random Process

To specify an RV  $\chi$ , one must run multiple trials of the experiment and estimate  $p_\chi(x)$  from the outcomes. Similarly, for the random process  $\chi(t)$  one must repeat trails at each  $t$ . The collection of all possible waveforms if known as the **ensemble** of the RP  $\chi(t)$ . A waveform in this collection is a **sample function** ( $x(t, \zeta_i)$ ) of the RP.

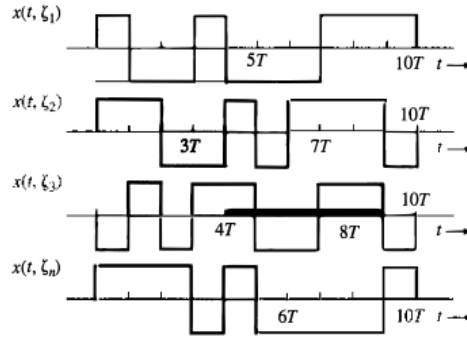


Figure 1: Ensemble of a Random Process

We can view a random process as the outcome of an experiment where instead of being a sample point (RV) the outcome is a sample function (RP).

One important point is that the sample functions in the ensemble are not random. They have occurred and are therefor deterministic. Randomness in this situation is associated not with the waveform but with the uncertainty regarding which waveform would occur in a given trial.

### 2 Autocorrelation Function of a Random Process

An important characteristic of a RP is its **Autocorrelation Function** which is related to the spectral information of the RP. The spectral content of a process depends on the rapidity of the amplitude change over time. This can be done by correlating the amplitudes at  $t_1$  and  $t_1 + \tau$ . We can use the correlation to measure the similarity of amplitudes at  $t_1$  and  $t_2 = t_1 + \tau$ . If the RV's  $\chi(t_1)$  and  $\chi(t_2)$  are denoted as  $\chi_1$  and  $\chi_2$  then the autocorrelation of a real RP  $R_\chi(t_1, t_2)$  instead

$$R_\chi(t_1, t_2) = \overline{\chi(t_1)\chi(t_2)} = \overline{\chi_1\chi_2} \quad (1)$$

$$R_\chi(t_1, t_2) = \int \int x_1 x_2 p_\chi(x_1, x_2; t_1, t_2) dx_1 dx_2 \quad (2)$$

### 3 Classification of Random Processes

#### 3.1 Stationary and Non-stationary Random Processes

A RP whose statistical characteristics do not change with time is classified as a **stationary random process**. For a stationary RP, a shift of the time origin will be impossible to detect; the process will appear the same. If the time origin is shifted for a RP from  $t_1$  to  $t_2 = t_1 + t_0$  then the PDF's at  $t_1$  and  $t_2$  must be the same. This is only possible if  $p_\chi(x; t)$  is independent of  $t$ . This the first order density of a RP can be expresses as

$$p_\chi(x; t) = p_\chi(x) \quad (3)$$

Similarly for a stationary RP, the autocorrelation function  $R_\chi(t_1, t_2)$  must depend on  $t_1$  and  $t_2$  only through the difference  $t_2 - t_1$ .

$$R_\chi(t_1, t_2) = R_\chi(t_2 - t_1) \quad (4)$$

$$R_\chi(\tau) = \overline{\chi(t)\chi(t+\tau)} \quad (5)$$

#### 3.2 Wide-Sense (Weakly) Stationary Processes

If the mean and autocorrelation of a RP are independent of time (but not strictly stationary) they are called **Wide-Sense Stationary**. This conditions are

$$\overline{\chi(t)} = A \quad (\text{constant}) \quad (6)$$

and

$$R_\chi(t_1, t_2) = R_\chi(\tau) \quad \tau = t_2 - t_1 \quad (7)$$

#### 3.3 Ergodic Wide-Sense Stationary Process

The time average of a sample function is defined as

$$\widetilde{x(t)} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t, \zeta_i) dt \quad (8)$$

and the time autocorrelation of that sample function is

$$\mathcal{R}_\chi(\tau) = (x(t)\widetilde{x(t+\tau)}) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t, \zeta_i)x(t+\tau, \zeta_i)dt \quad (9)$$

For **ergodic wide sense stationary processes**, these must be true:

$$\overline{\chi(t)} = \widetilde{x(t)} \quad (10)$$

and

$$R_\chi(\tau) = \mathcal{R}_\chi(\tau) \quad (11)$$

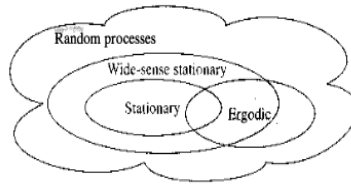


Figure 2: Relation Among different Random Processes

## 4 Power Spectral Density

For WSS RP's there is a meaningful PSD but if it is non-stationary it may not have a meaningful PSD. The PSD  $S_\chi(f)$  of a random process  $\chi(t)$  is the ensemble average of the PSD's of all sample functions.

$$S_\chi(f) = \lim_{T \rightarrow \infty} \left[ \frac{|\chi_T(f)|^2}{T} \right] \quad \text{W/Hz} \quad (12)$$

where  $\chi_T(f)$  is the Fourier transform of the time truncated RP:

$$\chi_T(t) = \chi(t)\Pi(t/T) \quad (13)$$

The PSD is the Fourier transform of the autocorrelation function  $R_\chi(\tau)$ , **Wiener Khintchine theorem**

$$R_\chi(\tau) \leftrightarrow S_\chi(f) \quad (14)$$

Some useful formulas from the previous derivation:

$$R_\chi(-\tau) = R_\chi(\tau) \quad (15)$$

$$R_\chi(0) = \overline{\chi(t)\chi^*(t)} = \overline{|\chi(t)|^2} = \overline{|\chi|^2} \quad (16)$$

The mean square value of the random process is  $R_\chi(0)$ .

### 4.1 The Power of a Random Process

The power  $P_\chi$  of a WSS RP  $\chi(t)$  is

$$P_\chi = \overline{|\chi|^2} = 2 \int_0^\infty S_\chi(f) df \quad (17)$$

## 5 Multiple Random Processes

For two random processes  $\chi(t)$  and  $\gamma(t)$  the **cross correlation** is

$$R_{\chi\gamma}(t_1, t_2) = \overline{x(t_1)y(t_2)} \quad (18)$$

The two processes are said to be **jointly stationary** (WSS) if each process is WSS and

$$R_{\chi\gamma}(t_1, t_2) = R_{\chi\gamma}(t_2 - t_1) = R_{\chi\gamma}(\tau) \quad (19)$$

### 5.1 Uncorrelated, Orthogonal (Incoherent), and Independent Processes

Two processes are said to be **uncorrelated** if their cross correlation is equal to the product of their means,

$$R_{\chi\gamma}(\tau) = \overline{\chi(t)\gamma(t+\tau)} = \bar{\chi}\bar{\gamma} \quad (20)$$

And are incoherent/orthogonal If

$$R_{\chi\gamma}(\tau) = 0- \quad (21)$$

Finally the two processes are independent if  $\chi(t_1)$  and  $\gamma(t_2)$  are independent for all  $t_1$  and  $t_2$ .

### 5.2 Cross Power Spectral Density

The cross power spectral density  $S_{\chi\gamma}(f)$  is defined as

$$S_{\chi\gamma}(f) = \lim_{T \rightarrow \infty} \frac{\overline{X_T^*(f)Y_T(f)}}{T} \quad (22)$$

where  $X_T(f)$  and  $Y_T(f)$  are fourier transforms of the truncated process.

$$R_{\chi\gamma}(\tau) \leftrightarrow S_{\chi\gamma}(f) \quad (23)$$

For real random processes

$$R_{\chi\gamma}(\tau) = R_{\gamma\chi}(-\tau) \quad (24)$$

$$S_{\chi\gamma}(f) = S_{\gamma\chi}(-f) \quad (25)$$

## 6 Transmission of Random Processes Through Linear Systems

If a RP  $\chi(t)$  is an input to a stable LTI system with transfer function  $H(f)$  we can determine the autocorrelation function and output PSD of  $\gamma(t)$ .

$$R_\gamma(\tau) = h(\tau) * h(-\tau) * R_\chi(\tau) \quad (26)$$

$$S_\gamma(f) = |H(f)|^2 S_\chi(f) \quad (27)$$

### 6.1 Sum of Random Processes

If two stationary (at least WSS) are added

$$\zeta(t) = \chi(t) + \gamma(t) \quad (28)$$

then

$$R_\zeta(\tau) = R_\chi(\tau) + R_\gamma(\tau) + R_{\chi\gamma}(\tau) + R_{\gamma\chi}(\tau) \quad (29)$$

If the two processes are uncorrelated then

$$R_\zeta(\tau) = R_\chi(\tau) + R_\gamma(\tau) + 2\bar{\chi}\bar{\gamma} \quad (30)$$

Most processes in communication systems are zero mean. If the two processes are uncorrelated with one of the means of 0 then

$$R_\zeta(\tau) = R_\chi(\tau) + R_\gamma(\tau) \quad (31)$$

and

$$\bar{\zeta}^2 = \bar{\chi}^2 + \bar{\gamma}^2 \quad (32)$$

Hence the mean square of a sum of incoherent processes is equal to the sum of the mean squares of these processes.