FINAL EXAM REVIEW – MATH 4341

1. Proofs of Theorems

(1) A closed subspace of a compact space is compact.
(2) A compact subspace of a Hausdorff space is closed.
(3) Suppose $f : X \to Y$ be a continuous map and X is compact. Then the image $f(X) \subset Y$ is compact. If furthermore Y is Hausdorff and f is a bijection, then f is a homeomorphism.
(4) (The tube lemma) Let X and Y be topological spaces where Y is compact. If N is an open set of $X \times Y$ which contains $\{x_0\} \times Y$ for some $x_0 \in X$, then N contains a tube $W \times Y$, where $W \subset X$ is a neighborhood of x_0.
(5) Let $X_1, \dots, X_n$ be topological spaces. Then $\prod_{i=1}^n X_i$ is compact if and only if X_i is compact for all i.
(6) A topological space X is compact if and only if any collection $\mathcal{C}$ of closed subsets of X with the finite intersection property satisfies $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$.
(7) If X is first countable, then compactness of X implies sequential compactness.
(8) A set $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.
(9) If X is compact and $f : X \to \mathbb{R}$ is continuous, there are x_1 and x_2 with $f(x_1) = \sup f(X)$, $f(x_2) = \inf f(X)$.
(10) The n-sphere S^n is compact.
(11) $\sim_p$ (path homotopy) is an equivalence relation.
(12) Let γ be a path from p to q in some space X, and let γ' be a path from q to r. Then the operation $[\gamma] \star [\gamma'] = [\gamma \star \gamma']$ is well-defined.
(13) The operation $\star$ has the following properties for all paths γ, γ', and γ'' in a topological space X:
   - (i) $[\gamma] \star ([\gamma'] \star [\gamma'']) = ([\gamma] \star [\gamma']) \star [\gamma'']$ when both are defined,
   - (ii) $[\gamma] \star [e_q] = [e_p] \star [\gamma] = [\gamma]$, if γ is a path from p to q,
   - (iii) $[\gamma] \star [\overline{\gamma}] = [e_p]$, $[\overline{\gamma}] \star [\gamma] = [e_q]$, if γ is a path from p to q.

(14) Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps, and let $x \in X$. Then
   - (i) $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is a well-defined homomorphism,
   - (ii) $(g \circ f)_* = g_* \circ f_*$, and if $\mathrm{id} : X \to X$ denotes the identity, then $\mathrm{id}_* : \pi_1(X, x) \to \pi_1(X, x)$ is the identity on $\pi_1(X, x)$.
   - (iii) Finally, if f is a homeomorphism, then f_* is an isomorphism.

(15) We have $\pi_1(S^1) = \mathbb{Z}$, but S^n is simply-connected for $n \geq 2$.
(16) (Brouwer fixed point theorem) Every continuous map $h : D^2 \to D^2$ has a fixed point, that is, a point $x \in D^2$ with $h(x) = x$.
(17) $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.

2. Problems

(1) Examples in lecture notes (chapters 6 and 7).
(2) Homework 9, 10, 11.