7. Introduction to Homotopy Theory

Math 4341 (Topology)

Homotopy

- In this chapter we will introduce a powerful topological invariant called the fundamental group.

Homotopy

- In this chapter we will introduce a powerful topological invariant called the fundamental group.
- **Definition.** Let X and Y be topological spaces, and let $f, g : X \to Y$ be continuous maps. We say that f is *homotopic* to g if there exists a continuous map $F : X \times [0,1] \to Y$ such that

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = g(x)$$

for all $x \in X$.

Homotopy

- In this chapter we will introduce a powerful topological invariant called the fundamental group.
- **Definition.** Let X and Y be topological spaces, and let $f, g : X \to Y$ be continuous maps. We say that f is *homotopic* to g if there exists a continuous map $F : X \times [0,1] \to Y$ such that

  $$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = g(x)$$

  for all $x \in X$.
  - The map F is called a *homotopy* from f to g, and we write $f \sim g$.

Homotopy

- In this chapter we will introduce a powerful topological invariant called the fundamental group.
- **Definition.** Let X and Y be topological spaces, and let $f, g : X \to Y$ be continuous maps. We say that f is *homotopic* to g if there exists a continuous map $F : X \times [0,1] \to Y$ such that

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = g(x)$$

  for all $x \in X$.
  - The map F is called a *homotopy* from f to g, and we write $f \sim g$.
  - If $f \sim g$ where g is a constant map, we say that f is *null-homotopic*.

Path homotopy

- We will be interested in the special case where the maps f and g are paths that start and end at the same point. In this case, we will furthermore require that the homotopy fixes the two end-points of the paths.

Path homotopy

- We will be interested in the special case where the maps f and g are paths that start and end at the same point. In this case, we will furthermore require that the homotopy fixes the two end-points of the paths.
- **Definition.** Two $\gamma, \gamma' : [0,1] \to X$ be two paths from p to q in a topological space X. We say that γ is *path homotopic* to γ' if there is a homotopy $F : [0,1] \times [0,1] \to X$ from γ to γ' so that
$$F(0,t) = p, \quad F(1,t) = q$$
for all $t \in [0,1]$.

Path homotopy

- We will be interested in the special case where the maps f and g are paths that start and end at the same point. In this case, we will furthermore require that the homotopy fixes the two end-points of the paths.
- **Definition.** Two $\gamma, \gamma' : [0,1] \to X$ be two paths from p to q in a topological space X. We say that γ is *path homotopic* to γ' if there is a homotopy $F : [0,1] \times [0,1] \to X$ from γ to γ' so that
$$F(0,t) = p, \quad F(1,t) = q$$
for all $t \in [0,1]$.
  - The map F is called a *path homotopy*, and we write $\gamma \sim_p \gamma'$.

- **Lemma 7.1.** $\sim$ and $\sim_p$ are equivalence relations.

Homotopy and path homotopy

- **Lemma 7.1.** $\sim$ and $\sim_p$ are equivalence relations.
- *Proof.* Let $f, g, h : X \to Y$ be continuous maps.

Homotopy and path homotopy

- **Lemma 7.1.** $\sim$ and $\sim_p$ are equivalence relations.
- *Proof.* Let $f, g, h : X \to Y$ be continuous maps.
  - To see reflexivity, define $F : X \times [0,1] \to Y$ by $F(x,t) = f(x)$. Then F is a homotopy from f to f, and hence $f \sim f$. If f is a path, then F is a path homotopy, so $f \sim_p f$.

Homotopy and path homotopy

- **Lemma 7.1.** $\sim$ and $\sim_p$ are equivalence relations.
- *Proof.* Let $f, g, h : X \to Y$ be continuous maps.
  - To see reflexivity, define $F : X \times [0,1] \to Y$ by $F(x,t) = f(x)$. Then F is a homotopy from f to f, and hence $f \sim f$. If f is a path, then F is a path homotopy, so $f \sim_p f$.
  - For symmetry, suppose $f \sim g$. Then there is a homotopy $F : X \times [0,1] \to Y$ from f to g. Define $G(x,t) = F(x, 1-t)$. Then G is a homotopy from g to f, so $g \sim f$. If f and g are paths, then G is a path homotopy, so $f \sim_p g$ implies $g \sim_p f$.

- **Lemma 7.1.** $\sim$ and $\sim_p$ are equivalence relations.
- *Proof.* Let $f, g, h : X \to Y$ be continuous maps.
  - To see reflexivity, define $F : X \times [0,1] \to Y$ by $F(x,t) = f(x)$. Then F is a homotopy from f to f, and hence $f \sim f$. If f is a path, then F is a path homotopy, so $f \sim_p f$.
  - For symmetry, suppose $f \sim g$. Then there is a homotopy $F : X \times [0,1] \to Y$ from f to g. Define $G(x,t) = F(x, 1-t)$. Then G is a homotopy from g to f, so $g \sim f$. If f and g are paths, then G is a path homotopy, so $f \sim_p g$ implies $g \sim_p f$.
  - Finally, for transitivity, if $f \sim g$ and $g \sim h$, let F be a homotopy from f to g, and let G be a homotopy from g to h. Define a function $H : X \times [0,1] \to Y$ by

$$H(x,t) = \begin{cases} F(x, 2t), & \text{if } t \in [0, 1/2], \\ G(x, 2t-1), & \text{if } t \in [1/2, 1]. \end{cases}$$

    Then H is a homotopy from f to h, so $f \sim h$. If F and G are path homotopies, then so is H.

Homotopy and path homotopy

- **Example.** Let $f, g : X \to \mathbb{R}^n$ be two continuous functions. Then the map $F : X \times [0,1] \to \mathbb{R}^n$ given by

$$F(x,t) = (1-t)f(x) + tg(x)$$

is a homotopy from f to g.

- **Example.** Let $f, g : X \to \mathbb{R}^n$ be two continuous functions. Then the map $F : X \times [0,1] \to \mathbb{R}^n$ given by

$$F(x,t) = (1-t)f(x) + tg(x)$$

  is a homotopy from f to g.
  - That is, all functions into $\mathbb{R}^n$ are homotopic. In other words, there is only one homotopy equivalence class.

- **Example.** Let $f, g : X \to \mathbb{R}^n$ be two continuous functions. Then the map $F : X \times [0,1] \to \mathbb{R}^n$ given by

$$F(x,t) = (1-t)f(x) + tg(x)$$

  is a homotopy from f to g.
  - That is, all functions into $\mathbb{R}^n$ are homotopic. In other words, there is only one homotopy equivalence class.
- Likewise, if γ and γ' are paths from p to q in $\mathbb{R}^n$, then γ and γ' are homotopic: there is only a single equivalence class of path homotopy. Indeed, the path homotopy is obtained in exactly this way. In the special case where $p = q$, this means that all paths are null-homotopic.

- **Example.** Let γ and γ' be the paths from $(1, 0)$ to $(-1, 0)$ given by

$$\gamma(x) = (\cos(\pi x), \sin(\pi x)), \quad \gamma'(x) = (\cos(\pi x), -\sin(\pi x)).$$

Then γ and γ' are path homotopic as paths in $\mathbb{R}^2$ by the previous example, but they are *not* path homotopic as paths in $\mathbb{R}^2 \setminus \{(0, 0)\}$.

- **Example.** Let γ and γ' be the paths from $(1,0)$ to $(-1,0)$ given by

$$\gamma(x) = (\cos(\pi x), \sin(\pi x)), \quad \gamma'(x) = (\cos(\pi x), -\sin(\pi x)).$$

Then γ and γ' are path homotopic as paths in $\mathbb{R}^2$ by the previous example, but they are *not* path homotopic as paths in $\mathbb{R}^2 \setminus \{(0,0)\}$.

  - This is a non-trivial fact though, but for instance, the homotopy from the previous example does not work since

$$F(\tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{1}{2}(\gamma(\tfrac{1}{2}) + \gamma'(\tfrac{1}{2})) = (0,0).$$

.

Concatenation and reverse

- **Definition.** For any path $\gamma : [0, 1] \to X$, define the *reverse* of γ, denoted $\overline{\gamma}$, by $\overline{\gamma}(x) = \gamma(1 - x)$. Then $\overline{\gamma}$ is continuous, and if γ is a path from p to q, then $\overline{\gamma}$ is a path from q to p.

Concatenation and reverse

- **Definition.** For any path $\gamma : [0,1] \to X$, define the *reverse* of γ, denoted $\overline{\gamma}$, by $\overline{\gamma}(x) = \gamma(1-x)$. Then $\overline{\gamma}$ is continuous, and if γ is a path from p to q, then $\overline{\gamma}$ is a path from q to p.
- **Definition.** Let $\gamma, \gamma' : [0,1] \to X$ be two paths so that $\gamma(1) = \gamma'(0)$, so that γ is a path from p to q, and γ' is a path from q to r. We then form a path from p to r as follows: define the *concatenation* $\gamma \star \gamma' : [0,1] \to X$ by

$$\gamma \star \gamma'(x) = \begin{cases} \gamma(2x), & x \in [0,1/2], \\ \gamma'(2x-1), & x \in [1/2,1]. \end{cases}$$

Homotopy classes

- If γ is a path, denote by $[\gamma]$ its path homotopy equivalence class or in short, its *homotopy class*.

Homotopy classes

- If γ is a path, denote by $[\gamma]$ its path homotopy equivalence class or in short, its *homotopy class*.
- **Proposition 7.2.** Let γ be a path from p to q in some space X, and let γ' be a path from q to r. Then the operation

$$[\gamma] \star [\gamma'] = [\gamma \star \gamma']$$

is well-defined.

Homotopy classes

- If γ is a path, denote by $[\gamma]$ its path homotopy equivalence class or in short, its *homotopy class*.
- **Proposition 7.2.** Let γ be a path from p to q in some space X, and let γ' be a path from q to r. Then the operation

$$[\gamma] \star [\gamma'] = [\gamma \star \gamma']$$

  is well-defined.
- *Proof.* Suppose that F is a path homotopy from γ to some other curve $\tilde{\gamma}$ and that G is a path homotopy from γ' to $\widetilde{\gamma'}$. The claim that the operation is well-defined is then the claim that $\gamma \star \gamma' \sim_p \tilde{\gamma} \star \widetilde{\gamma'}$.

Homotopy classes

- If γ is a path, denote by $[\gamma]$ its path homotopy equivalence class or in short, its *homotopy class*.
- **Proposition 7.2.** Let γ be a path from p to q in some space X, and let γ' be a path from q to r. Then the operation

$$[\gamma] \star [\gamma'] = [\gamma \star \gamma']$$

is well-defined.
- *Proof.* Suppose that F is a path homotopy from γ to some other curve $\tilde{\gamma}$ and that G is a path homotopy from γ' to $\widetilde{\gamma}'$. The claim that the operation is well-defined is then the claim that $\gamma \star \gamma' \sim_p \tilde{\gamma} \star \widetilde{\gamma}'$.
  - Define $H : [0,1] \times [0,1] \to X$ by

$$H(x,t) = \begin{cases} F(2x,t), & \text{if } x \in [0,1/2], \\ G(2x-1,t), & \text{if } x \in [1/2,1]. \end{cases}$$

  Then H is a path homotopy from $\gamma \star \gamma'$ to $\tilde{\gamma} \star \widetilde{\gamma}'$.

Homotopy classes

- For a point $p \in X$ in a topological space, let $e_p : [0,1] \to X$ denote the constant path $e_p(x) = p$, for $x \in [0,1]$.

Homotopy classes

- For a point $p \in X$ in a topological space, let $e_p : [0,1] \to X$ denote the constant path $e_p(x) = p$, for $x \in [0,1]$.
- **Theorem 7.3.** The operation $\star$ has the following properties for all paths γ, γ', and γ'' in a topological space X:

Homotopy classes

- For a point $p \in X$ in a topological space, let $e_p : [0,1] \to X$ denote the constant path $e_p(x) = p$, for $x \in [0,1]$.
- **Theorem 7.3.** The operation $\star$ has the following properties for all paths γ, γ', and γ'' in a topological space X:
  - $[\gamma] \star ([\gamma'] \star [\gamma'']) = ([\gamma] \star [\gamma']) \star [\gamma'']$ when both are defined,

Homotopy classes

- For a point $p \in X$ in a topological space, let $e_p : [0,1] \to X$ denote the constant path $e_p(x) = p$, for $x \in [0,1]$.
- **Theorem 7.3.** The operation $\star$ has the following properties for all paths γ, γ', and γ'' in a topological space X:
  - $[\gamma] \star ([\gamma'] \star [\gamma'']) = ([\gamma] \star [\gamma']) \star [\gamma'']$ when both are defined,
  - $[\gamma] \star [e_q] = [e_p] \star [\gamma] = [\gamma]$, if γ is a path from p to q,

Homotopy classes

- For a point $p \in X$ in a topological space, let $e_p : [0,1] \to X$ denote the constant path $e_p(x) = p$, for $x \in [0,1]$.
- **Theorem 7.3.** The operation $\star$ has the following properties for all paths γ, γ', and γ'' in a topological space X:
  - $[\gamma] \star ([\gamma'] \star [\gamma'']) = ([\gamma] \star [\gamma']) \star [\gamma'']$ when both are defined,
  - $[\gamma] \star [e_q] = [e_p] \star [\gamma] = [\gamma]$, if γ is a path from p to q,
  - $[\gamma] \star [\overline{\gamma}] = [e_p]$, $[\overline{\gamma}] \star [\gamma] = [e_q]$, if γ is a path from p to q.

The fundamental group

- The idea in this section will be to use the operation $\star$ on path homotopy classes to associate an algebraic structure to any pair (X, p) for X a topological space and $p \in X$.

The fundamental group

- The idea in this section will be to use the operation $\star$ on path homotopy classes to associate an algebraic structure to any pair (X, p) for X a topological space and $p \in X$.
- If γ is a path from p to p, we say that γ is a *loop* based at p.

The fundamental group

- The idea in this section will be to use the operation $\star$ on path homotopy classes to associate an algebraic structure to any pair (X, p) for X a topological space and $p \in X$.
- If γ is a path from p to p, we say that γ is a *loop* based at p.
- **Definition.** Let X be a topological space, and let $p \in X$. Then the *fundamental group* $\pi_1(X, p)$ is the set of all path homotopy classes of loops based at p.

- The idea in this section will be to use the operation $\star$ on path homotopy classes to associate an algebraic structure to any pair (X, p) for X a topological space and $p \in X$.
- If γ is a path from p to p, we say that γ is a *loop* based at p.
- **Definition.** Let X be a topological space, and let $p \in X$. Then the *fundamental group* $\pi_1(X, p)$ is the set of all path homotopy classes of loops based at p.
- To make sense of the terminology, let us recall a few basic notions from abstract algebra.

Groups

- **Definition.** A *group* is a set G with an operation $G \times G \to G$, denoted $(g, h) \mapsto g \cdot h$, an element $e \in G$ called a unit, and a bijection $G \to G$ denoted $x \mapsto x^{-1}$ called the inverse, so that

Groups

- **Definition.** A *group* is a set G with an operation $G \times G \to G$, denoted $(g, h) \mapsto g \cdot h$, an element $e \in G$ called a unit, and a bijection $G \to G$ denoted $x \mapsto x^{-1}$ called the inverse, so that
  - $g \cdot (h \cdot k) = (g \cdot h) \cdot k$ for all $g, h, k \in G$,

Groups

- **Definition.** A *group* is a set G with an operation $G \times G \to G$, denoted $(g, h) \mapsto g \cdot h$, an element $e \in G$ called a unit, and a bijection $G \to G$ denoted $x \mapsto x^{-1}$ called the inverse, so that
  - $g \cdot (h \cdot k) = (g \cdot h) \cdot k$ for all $g, h, k \in G$,
  - $e \cdot g = g = g \cdot e$ for all $g \in G$, and

- **Definition.** A *group* is a set G with an operation $G \times G \to G$, denoted $(g, h) \mapsto g \cdot h$, an element $e \in G$ called a unit, and a bijection $G \to G$ denoted $x \mapsto x^{-1}$ called the inverse, so that
  - $g \cdot (h \cdot k) = (g \cdot h) \cdot k$ for all $g, h, k \in G$,
  - $e \cdot g = g = g \cdot e$ for all $g \in G$, and
  - $g \cdot g^{-1} = g^{-1} \cdot g = e$ for all $g \in G$.

Groups

- **Definition.** A *group* is a set G with an operation $G \times G \to G$, denoted $(g, h) \mapsto g \cdot h$, an element $e \in G$ called a unit, and a bijection $G \to G$ denoted $x \mapsto x^{-1}$ called the inverse, so that
  - $g \cdot (h \cdot k) = (g \cdot h) \cdot k$ for all $g, h, k \in G$,
  - $e \cdot g = g = g \cdot e$ for all $g \in G$, and
  - $g \cdot g^{-1} = g^{-1} \cdot g = e$ for all $g \in G$.
- If G and H are groups, then a map $\phi : G \to H$ is called a *homomorphism* if $\phi(g \cdot h) = \phi(g) \cdot \phi(h)$ for all $g, h \in G$. A bijective group homomorphism is called an *isomorphism*.

Groups

- **Example.** The one-point set $\{e\}$ is a group under the operation $(e, e) \mapsto e$. This group is called the *trivial group*.

Groups

- **Example.** The one-point set $\{e\}$ is a group under the operation $(e, e) \mapsto e$. This group is called the *trivial group*.
- **Example.** The integers form a group under the operation $(g, h) \mapsto g + h$. The unit is $0 \in \mathbb{Z}$, and if $n \in \mathbb{Z}$, then the inverse of n is $-n$.

- **Example.** The one-point set $\{e\}$ is a group under the operation $(e, e) \mapsto e$. This group is called the *trivial group*.
- **Example.** The integers form a group under the operation $(g, h) \mapsto g + h$. The unit is $0 \in \mathbb{Z}$, and if $n \in \mathbb{Z}$, then the inverse of n is $-n$.
- **Example.** The set $\mathbb{R} \setminus \{0\}$ is a group with operation $(g, h) \mapsto gh$. The unit is 1, and the inverse of $x \in \mathbb{R} \setminus \{0\}$ is $1/x$.

Groups

- **Example.** The one-point set $\{e\}$ is a group under the operation $(e,e) \mapsto e$. This group is called the *trivial group*.
- **Example.** The integers form a group under the operation $(g,h) \mapsto g+h$. The unit is $0 \in \mathbb{Z}$, and if $n \in \mathbb{Z}$, then the inverse of n is $-n$.
- **Example.** The set $\mathbb{R} \setminus \{0\}$ is a group with operation $(g,h) \mapsto gh$. The unit is 1, and the inverse of $x \in \mathbb{R} \setminus \{0\}$ is $1/x$.
- **Example.** The set $\mathrm{GL}(n,\mathbb{R})$ of invertible $(n \times n)$-matrices with entries in $\mathbb{R}$ is a group under matrix multiplication. The unit is the unit matrix.

The fundamental group

- **Proposition 7.4.** The fundamental group $\pi_1(X, p)$ is a group under the operation $\star$ on homotopy classes of loops for any topological space X and any $p \in X$.

The fundamental group

- **Proposition 7.4.** The fundamental group $\pi_1(X, p)$ is a group under the operation $\star$ on homotopy classes of loops for any topological space X and any $p \in X$.
- *Proof*. This follows immediately from Theorem 8.3.

- **Proposition 7.4.** The fundamental group $\pi_1(X, p)$ is a group under the operation $\star$ on homotopy classes of loops for any topological space X and any $p \in X$.
- *Proof.* This follows immediately from Theorem 8.3.
- **Example.** In a previous example we saw that any two given paths in $\mathbb{R}^n$ between the same points were homotopic. This, in particular, implies that any loop based at a point $p \in \mathbb{R}^n$ is null-homotopic; that is, homotopic to e_p. In other words,

$$\pi_1(\mathbb{R}^n, p) = \{[e_p]\},$$

the trivial group, for all $p \in \mathbb{R}^n$.

- **Theorem 7.5.** Let X be a topological space, and let α be a path from x to y in X. Define a map $\hat{\alpha} : \pi_1(X, x) \to \pi_1(X, y)$ by
$$\hat{\alpha}([\gamma]) = [\overline{\alpha}] \star [\gamma] \star [\alpha].$$
Then $\hat{\alpha}$ is well-defined and an isomorphism.

- **Theorem 7.5.** Let X be a topological space, and let α be a path from x to y in X. Define a map $\hat{\alpha} : \pi_1(X, x) \to \pi_1(X, y)$ by
$$\hat{\alpha}([\gamma]) = [\overline{\alpha}] \star [\gamma] \star [\alpha].$$
Then $\hat{\alpha}$ is well-defined and an isomorphism.
- *Proof.* That $\hat{\alpha}$ is well-defined means that $\hat{\alpha}([\gamma]) = \hat{\alpha}([\gamma'])$ whenever $[\gamma] = [\gamma']$, i.e. whenever $\gamma \sim_p \gamma'$.

- **Theorem 7.5.** Let X be a topological space, and let α be a path from x to y in X. Define a map $\hat{\alpha} : \pi_1(X, x) \to \pi_1(X, y)$ by
$$\hat{\alpha}([\gamma]) = [\overline{\alpha}] \star [\gamma] \star [\alpha].$$
Then $\hat{\alpha}$ is well-defined and an isomorphism.
- *Proof.* That $\hat{\alpha}$ is well-defined means that $\hat{\alpha}([\gamma]) = \hat{\alpha}([\gamma'])$ whenever $[\gamma] = [\gamma']$, i.e. whenever $\gamma \sim_p \gamma'$.
  - Indeed, if $F : [0,1] \times [0,1] \to X$ is a path homotopy from γ to γ', then $G : [0,1] \times [0,1] \to X$, defined by
$$G(s,t) = (\overline{\alpha} \star F(\cdot, t) \star \alpha)(s)$$
is a path homotopy from $\overline{\alpha} \star \gamma \star \alpha$ to $\overline{\alpha} \star \gamma' \star \alpha$, so $\hat{\alpha}$ is well-defined.

- To see that $\hat{\alpha}$ is an homomorphism, notice that for any $[\gamma], [\gamma'] \in \pi_1(X, x)$, we have

$$\begin{aligned}\hat{\alpha}([\gamma]) \star \hat{\alpha}([\gamma']) &= [\overline{\alpha}] \star [\gamma] \star [\alpha] \star [\overline{\alpha}] \star [\gamma'] \star [\alpha] \\ &= [\overline{\alpha}] \star ([\gamma] \star [\gamma']) \star [\alpha] = \hat{\alpha}([\gamma] \star [\gamma']).\end{aligned}$$

The fundamental group

- To see that $\hat{\alpha}$ is an homomorphism, notice that for any $[\gamma], [\gamma'] \in \pi_1(X, x)$, we have

$$\begin{aligned}\hat{\alpha}([\gamma]) \star \hat{\alpha}([\gamma']) &= [\overline{\alpha}] \star [\gamma] \star [\alpha] \star [\overline{\alpha}] \star [\gamma'] \star [\alpha] \\ &= [\overline{\alpha}] \star ([\gamma] \star [\gamma']) \star [\alpha] = \hat{\alpha}([\gamma] \star [\gamma']).\end{aligned}$$

- To see that $\hat{\alpha}$ is a bijection, notice that $\widehat{\overline{\alpha}} \circ \hat{\alpha}$ is the identity on $\pi_1(X, x)$ since for any $[\gamma] \in \pi_1(X, x)$, we have

$$\begin{aligned}(\widehat{\overline{\alpha}} \circ \hat{\alpha})[\gamma] &= \widehat{\overline{\alpha}}([\overline{\alpha}] \star [\gamma] \star [\alpha]) \\ &= [\alpha] \star [\overline{\alpha}] \star [\gamma] \star [\alpha] \star [\overline{\alpha}] = [\gamma],\end{aligned}$$

and $\hat{\alpha} \circ \widehat{\overline{\alpha}}$ is the identity on $\pi_1(X, y)$ by the same reason, so $\hat{\alpha}$ is a bijection and thus a group isomorphism.

The fundamental group

- **Corollary 7.6.** If X is a path-connected topological space, then $\pi_1(X, x)$ is independent of $x \in X$ up to isomorphism.

- **Corollary 7.6.** If X is a path-connected topological space, then $\pi_1(X, x)$ is independent of $x \in X$ up to isomorphism.
- Because of this result, one often writes $\pi_1(X) = \pi_1(X, x)$ for any $x \in X$ when X is path-connected. It is then understood that the equality is really up to isomorphism.

- **Corollary 7.6.** If X is a path-connected topological space, then $\pi_1(X, x)$ is independent of $x \in X$ up to isomorphism.
- Because of this result, one often writes $\pi_1(X) = \pi_1(X, x)$ for any $x \in X$ when X is path-connected. It is then understood that the equality is really up to isomorphism.
- **Definition.** A topological space X is called *simply-connected* if it is path-connected and $\pi_1(X)$ consists of a single element.

The fundamental group

- **Corollary 7.6.** If X is a path-connected topological space, then $\pi_1(X, x)$ is independent of $x \in X$ up to isomorphism.
- Because of this result, one often writes $\pi_1(X) = \pi_1(X, x)$ for any $x \in X$ when X is path-connected. It is then understood that the equality is really up to isomorphism.
- **Definition.** A topological space X is called *simply-connected* if it is path-connected and $\pi_1(X)$ consists of a single element.
- **Example.** $\mathbb{R}^n$ is simply-connected.

The fundamental group

- The next result says that for path-connected spaces, π_1 is a topological invariant. As preparation, suppose that $f : X \to Y$ is a continuous map, and let $x \in X$. Define a map

$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$$

by

$$f_*([\gamma]) = [f \circ \gamma].$$

The fundamental group

- The next result says that for path-connected spaces, π_1 is a topological invariant. As preparation, suppose that $f : X \to Y$ is a continuous map, and let $x \in X$. Define a map
$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$$
by
$$f_*([\gamma]) = [f \circ \gamma].$$
- **Theorem 7.7.** Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps, and let $x \in X$. Then

The fundamental group

- The next result says that for path-connected spaces, π_1 is a topological invariant. As preparation, suppose that $f : X \to Y$ is a continuous map, and let $x \in X$. Define a map

$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$$

by

$$f_*([\gamma]) = [f \circ \gamma].$$

- **Theorem 7.7.** Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps, and let $x \in X$. Then
  - (i) $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is a well-defined homomorphism,

The fundamental group

- The next result says that for path-connected spaces, π_1 is a topological invariant. As preparation, suppose that $f : X \to Y$ is a continuous map, and let $x \in X$. Define a map

$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$$

by

$$f_*([\gamma]) = [f \circ \gamma].$$

- **Theorem 7.7.** Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps, and let $x \in X$. Then
  - (i) $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is a well-defined homomorphism,
  - (ii) $(g \circ f)_* = g_* \circ f_*$, and if $\mathrm{id} : X \to X$ denotes the identity, then $\mathrm{id}_* : \pi_1(X, x) \to \pi_1(X, x)$ is the identity on $\pi_1(X, x)$.

- The next result says that for path-connected spaces, π_1 is a topological invariant. As preparation, suppose that $f : X \to Y$ is a continuous map, and let $x \in X$. Define a map

$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$$

by

$$f_*([\gamma]) = [f \circ \gamma].$$

- **Theorem 7.7.** Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps, and let $x \in X$. Then
  - (i) $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is a well-defined homomorphism,
  - (ii) $(g \circ f)_* = g_* \circ f_*$, and if $\mathrm{id} : X \to X$ denotes the identity, then $\mathrm{id}_* : \pi_1(X, x) \to \pi_1(X, x)$ is the identity on $\pi_1(X, x)$.
  - (iii) Finally, if f is a homeomorphism, then f_* is an isomorphism.

The fundamental group

- That f_* is well-defined means that $f \circ \gamma \sim_p f \circ \gamma'$ whenever $\gamma \sim_p \gamma'$. This is the case since if F is a homotopy from γ to γ', then $f \circ F$ is a homotopy from $f \circ \gamma$ to $f \circ \gamma'$.

- That f_* is well-defined means that $f \circ \gamma \sim_p f \circ \gamma'$ whenever $\gamma \sim_p \gamma'$. This is the case since if F is a homotopy from γ to γ', then $f \circ F$ is a homotopy from $f \circ \gamma$ to $f \circ \gamma'$.
- To see that f_* is a homomorphism, let $[\gamma], [\gamma'] \in \pi_1(X, x)$ be arbitrary homotopy classes. We first notice that by definition of concatenation, we have

$$f \circ (\gamma \star \gamma') = (f \circ \gamma) \star (f \circ \gamma'),$$

from which it follows that

$$\begin{aligned} f_*([\gamma] \star [\gamma']) = f_*([\gamma \star \gamma']) = [f \circ (\gamma \star \gamma')] = [(f \circ \gamma) \star (f \circ \gamma')] \\ = [f \circ \gamma] \star [f \circ \gamma'] = f_*([\gamma]) \star f_*([\gamma']), \end{aligned}$$

so f_* is a homomorphism, which shows (i).

- Similarly,

$$(g_* \circ f_*)([\gamma]) = g_*([f \circ \gamma]) = [g \circ f \circ \gamma] = (g \circ f)_*([\gamma]),$$

which shows the first part of (ii). The last part of (ii) is obvious.

- Similarly,

$$(g_* \circ f_*)([\gamma]) = g_*([f \circ \gamma]) = [g \circ f \circ \gamma] = (g \circ f)_*([\gamma]),$$

which shows the first part of (ii). The last part of (ii) is obvious.

- Finally, (iii) follows from (ii) as it follows that $(f^{-1})_*$ satisfies that both $f_* \circ (f^{-1})_*$ and $(f^{-1})_* \circ f_*$ are the identity homomorphisms. Thus f_* is a bijection and therefore an isomorphism.

The fundamental group

- If G and H are two groups, then their Cartesian product $G \times H$ is a group with the group operation

$$(g, h) \cdot (g', h') = (g \cdot g', h \cdot h').$$

The fundamental group

- If G and H are two groups, then their Cartesian product $G \times H$ is a group with the group operation

$$(g, h) \cdot (g', h') = (g \cdot g', h \cdot h').$$

- **Proposition 7.8.** Let X and Y be topological spaces, and let $x \in X$, $y \in Y$. Then $\pi_1(X \times Y, (x, y))$ is isomorphic to $\pi_1(X, x) \times \pi_1(Y, y)$.

The fundamental group

- If G and H are two groups, then their Cartesian product $G \times H$ is a group with the group operation

$$(g, h) \cdot (g', h') = (g \cdot g', h \cdot h').$$

- **Proposition 7.8.** Let X and Y be topological spaces, and let $x \in X$, $y \in Y$. Then $\pi_1(X \times Y, (x, y))$ is isomorphic to $\pi_1(X, x) \times \pi_1(Y, y)$.
- **Theorem 7.9.** We have $\pi_1(S^1) = \mathbb{Z}$, but S^n is simply-connected for $n \geq 2$.

The fundamental group

- If G and H are two groups, then their Cartesian product $G \times H$ is a group with the group operation

$$(g, h) \cdot (g', h') = (g \cdot g', h \cdot h').$$

- **Proposition 7.8.** Let X and Y be topological spaces, and let $x \in X$, $y \in Y$. Then $\pi_1(X \times Y, (x, y))$ is isomorphic to $\pi_1(X, x) \times \pi_1(Y, y)$.
- **Theorem 7.9.** We have $\pi_1(S^1) = \mathbb{Z}$, but S^n is simply-connected for $n \geq 2$.
- **Definition.** The m-torus $\mathbb{T}^m$ is defined to be the product $\mathbb{T}^m = S^1 \times \cdots \times S^1$ of m copies of S^1.

The fundamental group

- If G and H are two groups, then their Cartesian product $G \times H$ is a group with the group operation
$$(g, h) \cdot (g', h') = (g \cdot g', h \cdot h').$$
- **Proposition 7.8.** Let X and Y be topological spaces, and let $x \in X$, $y \in Y$. Then $\pi_1(X \times Y, (x, y))$ is isomorphic to $\pi_1(X, x) \times \pi_1(Y, y)$.
- **Theorem 7.9.** We have $\pi_1(S^1) = \mathbb{Z}$, but S^n is simply-connected for $n \geq 2$.
- **Definition.** The m-torus $\mathbb{T}^m$ is defined to be the product $\mathbb{T}^m = S^1 \times \cdots \times S^1$ of m copies of S^1.
- **Corollary 7.10.** We have $S^n \simeq \mathbb{T}^m$ if and only if $n = m = 1$.

The fundamental group

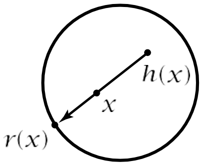
- If G and H are two groups, then their Cartesian product $G \times H$ is a group with the group operation
$$(g, h) \cdot (g', h') = (g \cdot g', h \cdot h').$$
- **Proposition 7.8.** Let X and Y be topological spaces, and let $x \in X$, $y \in Y$. Then $\pi_1(X \times Y, (x, y))$ is isomorphic to $\pi_1(X, x) \times \pi_1(Y, y)$.
- **Theorem 7.9.** We have $\pi_1(S^1) = \mathbb{Z}$, but S^n is simply-connected for $n \geq 2$.
- **Definition.** The m-torus $\mathbb{T}^m$ is defined to be the product $\mathbb{T}^m = S^1 \times \cdots \times S^1$ of m copies of S^1.
- **Corollary 7.10.** We have $S^n \simeq \mathbb{T}^m$ if and only if $n = m = 1$.
- *Proof.* $\pi_1(\mathbb{T}^m) = \pi_1(S^1 \times \cdots \times S^1)$ is the product of m copies of $\mathbb{Z}$. Since $\mathbb{Z}^m$ is not isomorphic to $\mathbb{Z}$ if $m > 1$, $\pi_1(\mathbb{T}^m)$ can only be isomorphic to $\pi_1(S^n)$ if $n = m = 1$.

Application: Brouwer fixed point theorem

- **Theorem 7.11.** Every continuous map $h : D^2 \to D^2$ has a fixed point, that is, a point $x \in D^2$ with $h(x) = x$.

Application: Brouwer fixed point theorem

- **Theorem 7.11.** Every continuous map $h : D^2 \to D^2$ has a fixed point, that is, a point $x \in D^2$ with $h(x) = x$.
- *Proof.* Suppose $h(x) \neq x$ for all $x \in D^2$. Then we can define a map $r : D^2 \to S^1$ by letting $r(x)$ be the point of S^1 where the ray in $\mathbb{R}^2$ starting at $h(x)$ and passing through x leaves D^2.



Application: Brouwer fixed point theorem

- r is continuous, since small perturbations of x produce small perturbations of $h(x)$, hence also small perturbations of the ray through these two points.

Application: Brouwer fixed point theorem

- r is continuous, since small perturbations of x produce small perturbations of $h(x)$, hence also small perturbations of the ray through these two points.
- Note that $r(x) = x$ if $x \in S^1$. That is $r \circ \iota = \mathrm{id}_{S^1}$ where $\iota : S^1 \to D^2$ is the inclusion map.

Application: Brouwer fixed point theorem

- r is continuous, since small perturbations of x produce small perturbations of $h(x)$, hence also small perturbations of the ray through these two points.
- Note that $r(x) = x$ if $x \in S^1$. That is $r \circ \iota = \mathrm{id}_{S^1}$ where $\iota : S^1 \to D^2$ is the inclusion map.
- Then $(r \circ \iota)_* = (\mathrm{id}_{S^1})_* = \mathrm{id}_{\pi_1(S^1)}$. For $[\gamma] \in \pi_1(S^1)$ we have

$$[\gamma] = (r \circ \iota)_*[\gamma] = r_*(\iota_*[\gamma]).$$

Application: Brouwer fixed point theorem

- r is continuous, since small perturbations of x produce small perturbations of $h(x)$, hence also small perturbations of the ray through these two points.
- Note that $r(x) = x$ if $x \in S^1$. That is $r \circ \iota = \mathrm{id}_{S^1}$ where $\iota : S^1 \to D^2$ is the inclusion map.
- Then $(r \circ \iota)_* = (\mathrm{id}_{S^1})_* = \mathrm{id}_{\pi_1(S^1)}$. For $[\gamma] \in \pi_1(S^1)$ we have

$$[\gamma] = (r \circ \iota)_*[\gamma] = r_*(\iota_*[\gamma]).$$

- Since $\iota_*[\gamma] = 1$ in $\pi_1(D^2) = \{1\}$, we obtain $[\gamma] = r_*(1) = 1$. Hence $\pi_1(S^1) = \{1\}$, which contradicts $\pi_1(S^1) \cong \mathbb{Z}$.

Application

- **Theorem 7.12.** $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.

Application

- **Theorem 7.12.** $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.
- *Proof.* Suppose $f : \mathbb{R}^n \to \mathbb{R}^2$ is a homeomorphism. Then

$$\mathbb{R}^2 \setminus \{\vec{0}\} \simeq \mathbb{R}^n \setminus \{f(\vec{0})\}.$$

Application

- **Theorem 7.12.** $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.
- *Proof.* Suppose $f : \mathbb{R}^n \to \mathbb{R}^2$ is a homeomorphism. Then

$$\mathbb{R}^2 \setminus \{\vec{0}\} \simeq \mathbb{R}^n \setminus \{f(\vec{0})\}.$$

- The case n $= 1$ was done by using connectedness.

- **Theorem 7.12.** $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.
- *Proof.* Suppose $f : \mathbb{R}^n \to \mathbb{R}^2$ is a homeomorphism. Then

$$\mathbb{R}^2 \setminus \{\vec{0}\} \simeq \mathbb{R}^n \setminus \{f(\vec{0})\}.$$

- The case n $= 1$ was done by using connectedness.
- Suppose $n \geq 3$. Note that $\mathbb{R}^k \setminus \{x\} \simeq S^{k-1} \times (0, \infty)$, so

$$\pi_1(\mathbb{R}^k \setminus \{x\}) \cong \pi_1(S^{k-1}) \times \pi_1((0, \infty)) \cong \pi_1(S^{k-1}).$$

Application

- **Theorem 7.12.** $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.
- *Proof.* Suppose $f : \mathbb{R}^n \to \mathbb{R}^2$ is a homeomorphism. Then

$$\mathbb{R}^2 \setminus \{\vec{0}\} \simeq \mathbb{R}^n \setminus \{f(\vec{0})\}.$$

- The case n $= 1$ was done by using connectedness.
- Suppose $n \geq 3$. Note that $\mathbb{R}^k \setminus \{x\} \simeq S^{k-1} \times (0, \infty)$, so

$$\pi_1(\mathbb{R}^k \setminus \{x\}) \cong \pi_1(S^{k-1}) \times \pi_1((0, \infty)) \cong \pi_1(S^{k-1}).$$

- Hence $\pi_1(\mathbb{R}^k \setminus \{x\})$ is $\mathbb{Z}$ if $k = 2$ and trivial for $k \geq 3$.

Application

- **Theorem 7.12.** $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.
- *Proof.* Suppose $f : \mathbb{R}^n \to \mathbb{R}^2$ is a homeomorphism. Then

$$\mathbb{R}^2 \setminus \{\vec{0}\} \simeq \mathbb{R}^n \setminus \{f(\vec{0})\}.$$

- The case n $=$ 1 was done by using connectedness.
- Suppose $n \geq 3$. Note that $\mathbb{R}^k \setminus \{x\} \simeq S^{k-1} \times (0, \infty)$, so

$$\pi_1(\mathbb{R}^k \setminus \{x\}) \cong \pi_1(S^{k-1}) \times \pi_1((0, \infty)) \cong \pi_1(S^{k-1}).$$

- Hence $\pi_1(\mathbb{R}^k \setminus \{x\})$ is $\mathbb{Z}$ if $k = 2$ and trivial for $k \geq 3$.
- Remark: $\mathbb{R}^k \setminus \{x\} \simeq S^{k-1} \times (0, \infty)$ via the map

$$y \mapsto \left(\frac{y - x}{||y - x||}, ||y - x|| \right).$$

Proof of Theorem 7.9 for $n \geq 2$: S^n is simply-connected

- **Lemma 7.13.** Suppose X is the union of a collection of path-connected open sets A_α each containing the base point $x_0 \in X$ and each intersection $A_\alpha \cap A_\beta$ is path-connected. Then every loop in X based at x_0 is homotopic to a product of loops each of which is contained in a single A_α.

Proof of Theorem 7.9 for $n \geq 2$: S^n is simply-connected

- **Lemma 7.13.** Suppose X is the union of a collection of path-connected open sets A_α each containing the base point $x_0 \in X$ and each intersection $A_\alpha \cap A_\beta$ is path-connected. Then every loop in X based at x_0 is homotopic to a product of loops each of which is contained in a single A_α.
- *Proof.* Given a loop $f : [0,1] \to X$ based at x_0, we claim that there is a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ such that each subinterval $[s_{i-1}, s_i]$ is mapped by f to a single A_α.

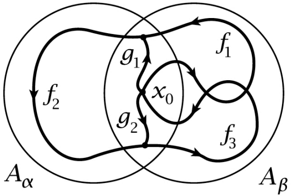
Proof of Theorem 7.9 for $n \geq 2$: S^n is simply-connected

- **Lemma 7.13.** Suppose X is the union of a collection of path-connected open sets A_α each containing the base point $x_0 \in X$ and each intersection $A_\alpha \cap A_\beta$ is path-connected. Then every loop in X based at x_0 is homotopic to a product of loops each of which is contained in a single A_α.
- *Proof.* Given a loop $f : [0,1] \to X$ based at x_0, we claim that there is a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ such that each subinterval $[s_{i-1}, s_i]$ is mapped by f to a single A_α.
- In fact, since f is continuous, each $s \in [0,1]$ has an open nbh $V_s \subset [0,1]$ mapped by f to some A_α. We may take V_s to be an interval whose closure is mapped to a single A_α.

Proof of Theorem 7.9 for $n \geq 2$: S^n is simply-connected

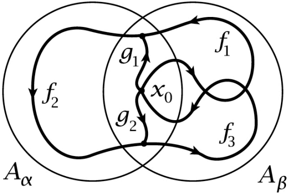
- **Lemma 7.13.** Suppose X is the union of a collection of path-connected open sets A_α each containing the base point $x_0 \in X$ and each intersection $A_\alpha \cap A_\beta$ is path-connected. Then every loop in X based at x_0 is homotopic to a product of loops each of which is contained in a single A_α.
- *Proof.* Given a loop $f : [0,1] \to X$ based at x_0, we claim that there is a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ such that each subinterval $[s_{i-1}, s_i]$ is mapped by f to a single A_α.
- In fact, since f is continuous, each $s \in [0,1]$ has an open nbh $V_s \subset [0,1]$ mapped by f to some A_α. We may take V_s to be an interval whose closure is mapped to a single A_α.
- Compactness of $[0,1]$ implies that a finite number of these intervals cover $[0,1]$. The endpoints of this finite set of intervals then define the desired partition of $[0,1]$.

Proof of Theorem 7.9 for $n \geq 2$: S^n is simply-connected



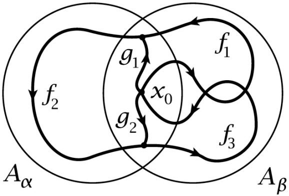
- Denote the A_α containing $f([s_{i-1}, s_i])$ by B_i, and let f_i be the path obtained by restricting f to $[s_{i-1}, s_i]$. Then $f = f_1 \star f_2 \star \cdots \star f_m$ with f_i a path in B_i.

Proof of Theorem 7.9 for $n \geq 2$: S^n is simply-connected



- Denote the A_α containing $f([s_{i-1}, s_i])$ by B_i, and let f_i be the path obtained by restricting f to $[s_{i-1}, s_i]$. Then $f = f_1 \star f_2 \star \cdots \star f_m$ with f_i a path in B_i.
- Since we assume $B_i \cap B_{i+1}$ is path-connected, we may choose a path g_i in $B_i \cap B_{i+1}$ from x_0 to the point $f(s_i) \in B_i \cap B_{i+1}$.

Proof of Theorem 7.9 for $n \geq 2$: S^n is simply-connected



- Denote the A_α containing $f([s_{i-1}, s_i])$ by B_i, and let f_i be the path obtained by restricting f to $[s_{i-1}, s_i]$. Then $f = f_1 \star f_2 \star \cdots \star f_m$ with f_i a path in B_i.
- Since we assume $B_i \cap B_{i+1}$ is path-connected, we may choose a path g_i in $B_i \cap B_{i+1}$ from x_0 to the point $f(s_i) \in B_i \cap B_{i+1}$.
- Then the loop

$$(f_1 \star \overline{g_1}) \star (g_1 \star f_2 \star \overline{g_2}) \star (g_2 \star f_3 \star \overline{g_3}) \star \cdots \star (g_{m-1} \star f_m)$$

is homotopic to f. This loop is the product of loops each lying in a single B_i.

Proof of Theorem 7.9 for $n \geq 2$: S^n is simply-connected

- *Proof*. We can express S^n as the union of two open sets A_1 and A_2 each homeomorphic to $\mathbb{R}^n$ such that $A_1 \cap A_2$ is homeomorphic to $S^{n-1} \times \mathbb{R}$, for example by taking A_1 and A_2 to be the complements of two antipodal points in S^n.

Proof of Theorem 7.9 for $n \geq 2$: S^n is simply-connected

- *Proof*. We can express S^n as the union of two open sets A_1 and A_2 each homeomorphic to $\mathbb{R}^n$ such that $A_1 \cap A_2$ is homeomorphic to $S^{n-1} \times \mathbb{R}$, for example by taking A_1 and A_2 to be the complements of two antipodal points in S^n.
- If $n \geq 2$ then $A_1 \cap A_2$ is path-connected. The lemma then applies to say that every loop in S^n based at x_0 is homotopic to a product of loops in A_1 or A_2.

Proof of Theorem 7.9 for $n \geq 2$: S^n is simply-connected

- *Proof*. We can express S^n as the union of two open sets A_1 and A_2 each homeomorphic to $\mathbb{R}^n$ such that $A_1 \cap A_2$ is homeomorphic to $S^{n-1} \times \mathbb{R}$, for example by taking A_1 and A_2 to be the complements of two antipodal points in S^n.
- If $n \geq 2$ then $A_1 \cap A_2$ is path-connected. The lemma then applies to say that every loop in S^n based at x_0 is homotopic to a product of loops in A_1 or A_2.
- Both A_1 or A_2 are simply-connected since they are homeomorphic to $\mathbb{R}^n$. Hence S^n is also simply-connected.

Proof of Theorem 7.9 for $n = 1$: $\pi_1(S^1) = \mathbb{Z}$

- We will prove the theorem by studying how paths in S^1 lift to paths in $\mathbb{R}$ via the covering map $p : \mathbb{R} \to S^1$ given by

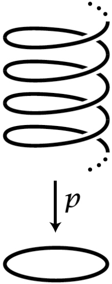
$$p(s) = (\cos 2\pi s, \sin 2\pi s).$$

Proof of Theorem 7.9 for $n = 1$: $\pi_1(S^1) = \mathbb{Z}$

- We will prove the theorem by studying how paths in S^1 lift to paths in $\mathbb{R}$ via the covering map $p : \mathbb{R} \to S^1$ given by

$$p(s) = (\cos 2\pi s, \sin 2\pi s).$$

- This map can be visualized geometrically by embedding $\mathbb{R}$ in $\mathbb{R}^3$ as the helix parametrized by $s \mapsto (\cos 2\pi s, \sin 2\pi s, s)$, and then p is the restriction to the helix of the projection of $\mathbb{R}^3$ onto $\mathbb{R}^2$, $(x, y, z) \mapsto (x, y)$.



Proof of Theorem 7.9 for $n = 1$: $\pi_1(S^1) = \mathbb{Z}$

- Given a space X, a **covering space** of X consists of a space $\widetilde{X}$ and a map $p : \widetilde{X} \to X$ satisfying the following condition: (*) For each point $x \in X$ there is an open neighborhood U of x in X such that $p^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by p.

Proof of Theorem 7.9 for $n = 1$: $\pi_1(S^1) = \mathbb{Z}$

- Given a space X, a **covering space** of X consists of a space $\widetilde{X}$ and a map $p : \widetilde{X} \to X$ satisfying the following condition: (*) For each point $x \in X$ there is an open neighborhood U of x in X such that $p^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by p.
- Such a U will be called **evenly covered**. For example, for the previously defined map $p : \mathbb{R} \to S^1$ any open arc in S^1 is evenly covered.

Proof of Theorem 7.9 for $n = 1$: $\pi_1(S^1) = \mathbb{Z}$

- Given a space X, a **covering space** of X consists of a space $\widetilde{X}$ and a map $p : \widetilde{X} \to X$ satisfying the following condition: (*) For each point $x \in X$ there is an open neighborhood U of x in X such that $p^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by p.
- Such a U will be called **evenly covered**. For example, for the previously defined map $p : \mathbb{R} \to S^1$ any open arc in S^1 is evenly covered.
- To prove the theorem we will need just the following two facts about covering spaces $p : \widetilde{X} \to X$:

Proof of Theorem 7.9 for $n = 1$: $\pi_1(S^1) = \mathbb{Z}$

- Given a space X, a **covering space** of X consists of a space $\widetilde{X}$ and a map $p : \widetilde{X} \to X$ satisfying the following condition: (*) For each point $x \in X$ there is an open neighborhood U of x in X such that $p^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by p.
- Such a U will be called **evenly covered**. For example, for the previously defined map $p : \mathbb{R} \to S^1$ any open arc in S^1 is evenly covered.
- To prove the theorem we will need just the following two facts about covering spaces $p : \widetilde{X} \to X$:
  - (a) For each path $f : [0, 1] \to X$ starting at a point $x_0 \in X$ and each $\widetilde{x_0} \in p^{-1}(x_0)$ there is a unique lift $\widetilde{f} : [0, 1] \to \widetilde{X}$ starting at $\widetilde{x_0}$.

Proof of Theorem 7.9 for $n = 1$: $\pi_1(S^1) = \mathbb{Z}$

- Given a space X, a **covering space** of X consists of a space $\widetilde{X}$ and a map $p : \widetilde{X} \to X$ satisfying the following condition: (*) For each point $x \in X$ there is an open neighborhood U of x in X such that $p^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by p.
- Such a U will be called **evenly covered**. For example, for the previously defined map $p : \mathbb{R} \to S^1$ any open arc in S^1 is evenly covered.
- To prove the theorem we will need just the following two facts about covering spaces $p : \widetilde{X} \to X$:
  - (a) For each path $f : [0, 1] \to X$ starting at a point $x_0 \in X$ and each $\widetilde{x}_0 \in p^{-1}(x_0)$ there is a unique lift $\widetilde{f} : [0, 1] \to \widetilde{X}$ starting at $\widetilde{x}_0$.
  - (b) For each homotopy $f_t : [0, 1] \to X$ of paths starting at x_0 and each $\widetilde{x}_0 \in p^{-1}(x_0)$ there is a unique lifted homotopy $\widetilde{f}_t : [0, 1] \to \widetilde{X}$ of paths starting at $\widetilde{x}_0$.

Proof of Theorem 7.9 for $n = 1$: $\pi_1(S^1) = \mathbb{Z}$

- Let $x_0 = (1, 0)$. We will show that $\pi(S^1, x_0)$ is an infinite cyclic group generated by the homotopy class of the loop $\omega(s) = (\cos 2\pi s, \sin 2\pi s)$.

Proof of Theorem 7.9 for $n = 1$: $\pi_1(S^1) = \mathbb{Z}$

- Let $x_0 = (1, 0)$. We will show that $\pi(S^1, x_0)$ is an infinite cyclic group generated by the homotopy class of the loop $\omega(s) = (\cos 2\pi s, \sin 2\pi s)$.
- Note that $[\omega]^n = [\omega_n]$ where $\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$ for $n \in \mathbb{Z}$. The theorem is therefore equivalent to the statement that every loop in S^1 based at x_0 is homotopic to ω_n for a unique $n \in \mathbb{Z}$.

Proof of Theorem 7.9 for $n = 1$: $\pi_1(S^1) = \mathbb{Z}$

- Let $x_0 = (1, 0)$. We will show that $\pi(S^1, x_0)$ is an infinite cyclic group generated by the homotopy class of the loop $\omega(s) = (\cos 2\pi s, \sin 2\pi s)$.
- Note that $[\omega]^n = [\omega_n]$ where $\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$ for $n \in \mathbb{Z}$. The theorem is therefore equivalent to the statement that every loop in S^1 based at x_0 is homotopic to ω_n for a unique $n \in \mathbb{Z}$.
- Let $f : [0, 1] \to S^1$ be a loop at the basepoint $x_0 = (1, 0)$, representing a given element of $\pi_1(S^1, x_0)$. By (a) there is a lift $\widetilde{f}$ starting at 0. This path $\widetilde{f}$ ends at some integer n since $p\widetilde{f}(1) = f(1) = x_0$ and $p^{-1}(x_0) = \mathbb{Z} \subset \mathbb{R}$.

Proof of Theorem 7.9 for $n = 1$: $\pi_1(S^1) = \mathbb{Z}$

- Let $x_0 = (1, 0)$. We will show that $\pi(S^1, x_0)$ is an infinite cyclic group generated by the homotopy class of the loop $\omega(s) = (\cos 2\pi s, \sin 2\pi s)$.
- Note that $[\omega]^n = [\omega_n]$ where $\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$ for $n \in \mathbb{Z}$. The theorem is therefore equivalent to the statement that every loop in S^1 based at x_0 is homotopic to ω_n for a unique $n \in \mathbb{Z}$.
- Let $f : [0, 1] \to S^1$ be a loop at the basepoint $x_0 = (1, 0)$, representing a given element of $\pi_1(S^1, x_0)$. By (a) there is a lift $\widetilde{f}$ starting at 0. This path $\widetilde{f}$ ends at some integer n since $p\widetilde{f}(1) = f(1) = x_0$ and $p^{-1}(x_0) = \mathbb{Z} \subset \mathbb{R}$.
- Another path in $\mathbb{R}$ from 0 to n is $\widetilde{\omega}_n$, and $\widetilde{f} \simeq \widetilde{\omega}_n$ via the linear homotopy $(1 - t)\widetilde{f} + t\widetilde{\omega}_n$. Composing this homotopy with p gives a homotopy $f \simeq \omega_n$ so $[f] = [\omega_n]$.

Proof of Theorem 7.9 for $n = 1$: $\pi_1(S^1) = \mathbb{Z}$

- To show that n is uniquely determined by $[f]$, suppose that $f \simeq \omega_n$ and $f \simeq \omega_m$, so $\omega_m \simeq \omega_n$. Let g_t be a homotopy from $\omega_m = g_0$ to $\omega_n = g_1$. By (b) this homotopy lifts to a homotopy $\widetilde{g}_t$ of paths starting at 0.

Proof of Theorem 7.9 for $n = 1$: $\pi_1(S^1) = \mathbb{Z}$

- To show that n is uniquely determined by $[f]$, suppose that $f \simeq \omega_n$ and $f \simeq \omega_m$, so $\omega_m \simeq \omega_n$. Let g_t be a homotopy from $\omega_m = g_0$ to $\omega_n = g_1$. By (b) this homotopy lifts to a homotopy $\widetilde{g}_t$ of paths starting at 0.
- The uniqueness part of (a) implies that $\widetilde{g}_0 = \widetilde{\omega}_m$ and $\widetilde{g}_1 = \widetilde{\omega}_n$. Since $\widetilde{g}_t$ is a homotopy of paths, the endpoint $\widetilde{g}_t(1)$ is independent of t. For $t = 0$ this endpoint is m and for $t = 1$ it is n, so $m = n$.