Math 4301 Mathematical Analysis I
Lecture 18
Topic: Sequences and series of functions

Consider a sequence of functions $f_n : A \subseteq \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$ and $f : A \to \mathbb{R}$.

- We say that $f_n(x) \to f(x)$ pointwise, **if for all** $x \in A$, $\{f_n(x)\}$ converges to $f(x)$, i.e.
$$f(x) = \lim f_n(x)$$
In $\epsilon - \delta$ language we write:
- $f_n(x) \to f(x)$ pointwise if for every $x \in A$ and $\epsilon > 0$, there is $N_x \in \mathbb{N}$, such that for all $n > N_x$,
$$|f_n(x) - f(x)| < \epsilon$$
- We say that $f_n \to f$ uniformly, if for all $\epsilon > 0$ there is $N \in \mathbb{N}$, such that for all $n > N$ and **all** $x \in A$,
$$|f_n(x) - f(x)| < \epsilon.$$
- **Important**: If $f_n \to f$ *uniformly*, then $f_n(x) \to f(x)$ *pointwise*
- **Important:** If f_n is continuous for all n and $f_n \to f$ uniformly, then f is continuous.
Consider a sequence of functions $f_n : A \subseteq \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$ and $f : A \to \mathbb{R}$.
We define a new sequence (called the sequence of partial sums)
$$s_n(x) = \sum_{k=1}^{n} f_k(x), \ x \in A, \ n = 1, 2, ...$$
- We say that series $\sum_{n=1}^{\infty} f_n(x)$ *converges pointwise* to f, we write
$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$
if the sequence of partial sums $s_n(x) \to f(x)$ pointwise.
- We say that series $\sum_{n=1}^{\infty} f_n$ converges uniformly to f, we write
$$f = \sum_{n=1}^{\infty} f_n$$
if the sequence of partial sums $s_n \to f$ uniformly.
Tests for Convergence

- We formulate the following two important tests for the uniform convergence of series of functions.

  Lemma For two sequences (a_n) and (b_n) if

$$s_n = \sum_{k=1}^{n} a_k$$

then

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= s_n b_{n+1} - \sum_{k=1}^{n} s_k \left(b_{k+1} - b_k\right) \\ &= s_n b_1 + \sum_{k=1}^{n} \left(s_n - s_k\right)\left(b_{k+1} - b_k\right).\end{aligned}$$

Proof. Let $s_0 = 0$, since $a_n = s_n - s_{n-1}$,

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} \left(s_k - s_{k-1}\right) b_k = \sum_{k=1}^{n} s_k b_k - \sum_{k=1}^{n} s_{k-1} b_k.$$

- Now, we see that

$$\sum_{k=1}^{n} s_{k-1} b_k = \sum_{k=1}^{n} s_k b_{k+1} - s_n b_{n+1},$$

- Hence

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= \sum_{k=1}^{n} s_k b_k - \sum_{k=1}^{n} s_{k-1} b_k = \sum_{k=1}^{n} s_k b_k - \left(\sum_{k=1}^{n} s_k b_{k+1} - s_n b_{n+1}\right) \\ &= \sum_{k=1}^{n} s_k b_k - \sum_{k=1}^{n} s_k b_{k+1} + s_n b_{n+1} = \sum_{k=1}^{n} \left(s_k b_k - s_k b_{k+1}\right) + s_n b_{n+1} \\ &= \sum_{k=1}^{n} s_k \left(b_k - b_{k+1}\right) + s_n b_{n+1} = s_n b_{n+1} - \sum_{k=1}^{n} s_k \left(b_{k+1} - b_k\right).\end{aligned}$$

- For the second equality, we observe that

$$b_{n+1} = \sum_{k=1}^{n} \left(b_{k+1} - b_k\right) + b_1$$

- so

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= s_n b_{n+1} - \sum_{k=1}^{n} s_k \left(b_{k+1} - b_k\right) \\ &= \sum_{k=1}^{n} s_n \left(b_{k+1} - b_k\right) + s_n b_1 - \sum_{k=1}^{n} s_k \left(b_{k+1} - b_k\right) \\ &= s_n b_1 + \sum_{k=1}^{n} \left(s_n \left(b_{k+1} - b_k\right) - s_k \left(b_{k+1} - b_k\right)\right) \\ &= s_n b_1 + \sum_{k=1}^{n} \left(s_n - s_k\right)\left(b_{k+1} - b_k\right)\end{aligned}$$

This finishes our proof. ■

- **Theorem (***Abel's Test***)** Let $A \subseteq \mathbb{R}$ and $\varphi_n : A \to \mathbb{R}$ be **decreasing sequence of functions**; That is, for all $n \in \mathbb{N}$,

$$\varphi_{n+1}(x) \leq \varphi_n(x), \text{ for each } x \in A$$

and assume that there is a constant $M > 0$, such that, for all $n \in \mathbb{N}$,

$$|\varphi_n(x)| \leq M, \text{ for all } x \in A.$$

If $\displaystyle\sum_{n=1}^{\infty} f_n$ **converges uniformly** on A, then

$$\sum_{n=1}^{\infty} \varphi_n f_n$$

is uniformly convergent on A.

Example Use Abel's Test to show that

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} e^{-nx}, \ 0 \leq x \leq 1$$

converges uniformly.

Solution Let

$$\begin{aligned} \varphi_n &: [0,1] \to \mathbb{R}, \\ \varphi_n(x) &= e^{-nx} \end{aligned}$$

and

$$\begin{aligned} f_n &: [0,1] \to \mathbb{R}, \\ f_n(x) &= \frac{x^n}{n!}. \end{aligned}$$

- By the Waierstrass M-test, since for all $x \in [0,1]$,

$$\frac{x^n}{n!} \leq \frac{1}{n!}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n!} \text{ converges}$$

it follows that $\sum_{n=1}^{\infty} f_n$ is converges uniformly on $[0,1]$.

- Since $e^x \geq 1$, for $x \in [0,1]$, then

$$e^{nx} \cdot e^x \geq e^{nx} \cdot 1$$
$$e^{(n+1)x} \geq e^{nx}, \text{ for all } n \in \mathbb{N} \text{ and } x \in [0,1].$$

- Therefore,

$$0 \leq \varphi_{n+1}(x) = e^{-(n+1)x} \leq e^{-nx} = \varphi_n(x), \text{ for all } n \in \mathbb{N}, \ x \in [0,1].$$

- Furthermore, since $e^x \geq 1$, $e^{nx} \geq 1$, and

$$|\varphi_n(x)| = e^{-nx} \leq 1,$$

for all $n \in \mathbb{N}$, $x \in [0,1]$, functions φ_n and f_n satisfy assumptions of Abel's Test, hence

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} e^{-nx}$$

converges uniformly on $[0,1]$.

Proof. We let

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \text{ and } r_n(x) = \sum_{k=1}^{n} \varphi_n(x) f_k(x)$$

- Therefore, for $n \geq m$, by the Lemma,

$$\begin{aligned} r_n(x) - r_m(x) &= s_n(x)\varphi_1(x) + \sum_{k=1}^{n}(s_n(x) - s_k(x))\big(\varphi_{k+1}(x) - \varphi_k(x)\big) \\ &\quad - \left(s_m(x)\varphi_1(x) + \sum_{k=1}^{m}(s_m(x) - s_k(x))\big(\varphi_{k+1}(x) - \varphi_k(x)\big)\right) \\ &= (s_n(x) - s_m(x))\varphi_1(x) + \sum_{k=1}^{n}(s_n(x) - s_k(x))\big(\varphi_{k+1}(x) - \varphi_k(x)\big) \\ &\quad - \sum_{k=1}^{m}(s_m(x) - s_k(x))\big(\varphi_{k+1}(x) - \varphi_k(x)\big) \end{aligned}$$

- Hence,

$$\begin{aligned} r_n(x) - r_m(x) &= (s_n(x) - s_m(x))\varphi_1(x) \\ &\quad + \sum_{k=1}^{m}(s_n(x) - s_k(x))\big(\varphi_{k+1}(x) - \varphi_k(x)\big) \\ &\quad + \sum_{k=m+1}^{n}(s_n(x) - s_k(x))\big(\varphi_{k+1}(x) - \varphi_k(x)\big) \\ &\quad - \sum_{k=1}^{m}(s_m(x) - s_k(x))\big(\varphi_{k+1}(x) - \varphi_k(x)\big) \end{aligned}$$

so

$$\begin{aligned} r_n(x) - r_m(x) &= (s_n(x) - s_m(x))\varphi_1(x) \\ &\quad + \sum_{k=1}^{m}(s_n(x) - s_k(x))\big(\varphi_{k+1}(x) - \varphi_k(x)\big) \\ &\quad - \sum_{k=1}^{m}(s_m(x) - s_k(x))\big(\varphi_{k+1}(x) - \varphi_k(x)\big) \\ &\quad + \sum_{k=m+1}^{n}(s_n(x) - s_k(x))\big(\varphi_{k+1}(x) - \varphi_k(x)\big) \end{aligned}$$

- Therefore,

$$\begin{aligned} r_n(x) - r_m(x) &= (s_n(x) - s_m(x))\,\varphi_1(x) \\ &\quad + \sum_{k=1}^{m} (s_n(x) - s_m(x))\big(\varphi_{k+1}(x) - \varphi_k(x)\big) \\ &\quad + \sum_{k=m+1}^{n} (s_n(x) - s_k(x))\big(\varphi_{k+1}(x) - \varphi_k(x)\big) \\ &= (s_n(x) - s_m(x))\left(\varphi_1(x) + \sum_{k=1}^{m}\big(\varphi_{k+1}(x) - \varphi_k(x)\big)\right) \\ &\quad + \sum_{k=m+1}^{n} (s_n(x) - s_k(x))\big(\varphi_{k+1}(x) - \varphi_k(x)\big) \\ &= (s_n(x) - s_m(x))\,\varphi_{m+1}(x) \\ &\quad + \sum_{k=m+1}^{n} (s_n(x) - s_k(x))\big(\varphi_{k+1}(x) - \varphi_k(x)\big). \end{aligned}$$

- Hence, for every $x \in A$,

$$\begin{aligned} |r_n(x) - r_m(x)| &\leq |s_n(x) - s_m(x)|\,\big|\varphi_{m+1}(x)\big| \\ &\quad + \sum_{k=m+1}^{n} |s_n(x) - s_k(x)|\,\big|\varphi_{k+1}(x) - \varphi_k(x)\big|. \end{aligned}$$

- Since $\varphi_{k+1}(x) \leq \varphi_k(x)$,

$$\big|\varphi_{k+1}(x) - \varphi_k(x)\big| = \varphi_k(x) - \varphi_{k+1}(x).$$

- Moreover, since $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A,
  given $\epsilon > 0$, there is N, such that $n, m \geq N$, for all $x \in A$,

$$|s_n(x) - s_m(x)| < \frac{\epsilon}{3M}.$$

- Since, for all $n \in \mathbb{N}$,

$$|\varphi_n(x)| \leq M, \text{ for all } x \in A,$$

$$\begin{aligned} |r_n(x) - r_m(x)| &\leq \frac{\epsilon}{3M}M + \sum_{k=m+1}^{n} \frac{\epsilon}{3M}\big(\varphi_k(x) - \varphi_{k+1}(x)\big) \\ &= \frac{\epsilon}{3} + \frac{\epsilon}{3M}\big(\varphi_{m+1}(x) - \varphi_{n+1}(x)\big) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3M}\big(\big|\varphi_{m+1}(x)\big| + \big|\varphi_{n+1}(x)\big|\big) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3M}(M + M) = \epsilon. \end{aligned}$$

- Therefore, the sequence (r_n) is uniformly Cauchy, so the series

$$\sum_{n=1}^{\infty} \varphi_n(x)\, f_n(x)$$

  converges uniformly on A.

This finishes our argument. ■

- **Remark**: Define $f : [0, 1] \to \mathbb{R}$, by
$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} e^{-nx}$$
- Since the series $\sum_{n=1}^{\infty} \frac{x^n}{n!} e^{-nx}$ is uniformly convergent to f, is well-defined.
- What can we say about properties of such function?
- We can show, for instance, that f is continuous.
Indeed, let $g_n : [0, 1] \to \mathbb{R}$,
$$g_n(x) = \frac{x^n}{n!} e^{-nx}.$$
Notice that g_n is continuous on $[0, 1]$, so
$$\begin{aligned} s_n &: [0, 1] \to \mathbb{R}, \\ s_n(x) &= \sum_{k=1}^{n} g_k(x) \end{aligned}$$
is continuous on $[0, 1]$ as a finite sum of continuous functions.
- Since $\sum_{n=1}^{\infty} g_k$ converges uniformly to f, and each s_n is continuous, then by theorem, $f = \sum_{n=1}^{\infty} g_k$ is also continuous.
Theorem (*Dirichlet's Test*) Let $A \subseteq \mathbb{R}^m$, $f_n : A \to \mathbb{R}$ be sequence of functions and
$$s_n(x) = \sum_{k=1}^{n} f_k(x)$$
be the sequence of its partial sums.
Assume that, there is $M > 0$, such that, for all $n \in \mathbb{N}$ and for all $x \in A$
$$|s_n(x)| \leq M.$$
Let $g_n : A \to \mathbb{R}$ be a sequence of functions such that
$$0 \leq g_{n+1}(x) \leq g_n(x), \text{ for each } x \in A$$
and $g_n \underset{n\to\infty}{\to} 0$ uniformly.
Then $\displaystyle\sum_{n=1}^{\infty} f_n g_n$ converges uniformly on A.
Exercise Show
$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$$
converges uniformly on $[\delta, 2\pi - \delta]$, $\delta > 0$.

Remark: We see that if

$$\begin{aligned} f_n &: [\delta, 2\pi - \delta] \to \mathbb{R}, \\ f_n(x) &= \frac{\sin(nx)}{n}, \end{aligned}$$

then of course,

$$|f_n(x)| = \left| \frac{\sin(nx)}{n} \right| \leq \frac{1}{n},$$

for all $x \in [\delta, 2\pi - \delta]$.

- However, **we cannot apply Weierstrass M-test** for $M_n = \frac{1}{n}$ since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

  Solution We apply Dirichlet's Test:

- Let $f_n : [\delta, 2\pi - \delta] \to \mathbb{R}$, be given by

  $$f_n(x) = \sin(nx)$$

  and $g_n : [\delta, 2\pi - \delta] \to \mathbb{R}$, be given by

  $$g_n(x) = \frac{1}{n}.$$

- Clearly,

  $$0 \leq g_{n+1}(x) = \frac{1}{n+1} \leq \frac{1}{n} = g_n(x),$$

  for all $x \in [\delta, 2\pi - \delta]$ and $g_n \underset{n\to\infty}{\to} 0$ uniformly since for $\epsilon > 0$, if $N > \frac{1}{\epsilon}$,
  then for $n \geq N$ and for all $x \in [\delta, 2\pi - \delta]$:

  $$|g_n(x) - 0| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

- Now, it is sufficient to show that

  $$s_n(x) = \sum_{k=1}^{n} f_k(x) = \sum_{k=1}^{n} \sin(kx)$$

  is bounded for all $n \in \mathbb{N}$ and $x \in [\delta, 2\pi - \delta]$.

- Indeed, since

  $$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2},$$

- Take

  $$\alpha = \left(k - \frac{1}{2}\right)x \text{ and } \beta = \left(k + \frac{1}{2}\right)x,$$

  thus

  $$\frac{\alpha+\beta}{2} = kx \text{ and } \frac{\alpha-\beta}{2} = -x/2.$$

- Therefore,

$$2\sin(kx)\sin(x/2) = \cos\left(k-\frac{1}{2}\right)x - \cos\left(k+\frac{1}{2}\right)x.$$

- Hence,

$$\begin{aligned}
\sum_{k=1}^{n} 2\sin(kx)\sin(x/2) &= \sum_{k=1}^{n}\left(\cos\left(k-\frac{1}{2}\right)x - \cos\left(k+\frac{1}{2}\right)x\right) \\
2\sin(x/2)\underbrace{\sum_{k=1}^{n}\sin(kx)}_{s_n=\sum_{k=1}^{n} f_k} &= \cos(x/2) - \cos\left(n+\frac{1}{2}\right)x
\end{aligned}$$

and since $x \in [\delta, 2\pi-\delta]$, then

$$\frac{x}{2} \in \left[\frac{\delta}{2}, \pi - \frac{\delta}{2}\right].$$

- Therefore,

$$\sin(x/2) \geq \min\{\sin(\delta/2), \sin(\pi - \delta/2)\} = K > 0.$$

- In particular,

$$\sin(x/2) \neq 0,$$

for $x \in [\delta, 2\pi-\delta]$, and

$$s_n(x) = \sum_{k=1}^{n} f_k(x) = \sum_{k=1}^{n}\sin(kx) = \frac{\cos(x/2) - \cos\left(n+\frac{1}{2}\right)x}{2\sin(x/2)}.$$

- Since $\sin(x/2) \geq K > 0$, for all $x \in [\delta, 2\pi-\delta]$, it follows that

$$\begin{aligned}
|s_n(x)| &= \left|\frac{\cos(x/2) - \cos\left(n+\frac{1}{2}\right)x}{2\sin(x/2)}\right| \\
&= \frac{\left|\cos(x/2) - \cos\left(n+\frac{1}{2}\right)x\right|}{2\sin(x/2)} \leq \frac{|\cos(x/2)| + \left|\cos\left(n+\frac{1}{2}\right)x\right|}{2\sin(x/2)} \\
&\leq \frac{1+1}{2\sin(x/2)} \\
&= \frac{2}{2\sin(x/2)} \leq \frac{1}{K}, \text{ for all } x \in [\delta, 2\pi-\delta] \text{ and } n \in \mathbb{N}.
\end{aligned}$$

- Therefore, $\{f_n\}$ and $\{g_n\}$ satisfy assumptions of Dirichlet's Test.
- Thus the series

$$\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$$

is convergent uniformly on $[\delta, 2\pi-\delta]$.

Proof. Since for any real sequences (a_n) and (b_n), by Lemma,

$$\sum_{k=1}^{n} a_k b_k = s_n b_{n+1} - \sum_{k=1}^{n} s_k (b_{k+1} - b_k).$$

- Let

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \text{ and } r_n(x) = \sum_{k=1}^{n} g_n(x) f_k(x)$$

  and, for $n \geq m$:

$$\begin{aligned} r_n(x) - r_m(x) &= s_n(x) g_{n+1}(x) - \sum_{k=1}^{n} s_k(x)(g_{k+1}(x) - g_k(x)) \\ &\quad - s_m(x) g_{m+1}(x) + \sum_{k=1}^{m} s_k(x)(g_{k+1}(x) - g_k(x)) \\ &= s_n(x) g_{n+1}(x) - s_m(x) g_{m+1}(x) - \sum_{k=m+1}^{n} s_k(x)(g_{k+1}(x) - g_k(x)). \end{aligned}$$

- Therefore,

$$\begin{aligned} |r_n(x) - r_m(x)| &\leq |s_n(x) g_{n+1}(x) - s_m(x) g_{m+1}(x)| \\ &\quad + \sum_{k=m+1}^{n} |s_k(x)| |g_{k+1}(x) - g_k(x)|. \end{aligned}$$

- Since $|s_n(x)| \leq M$ and

$$0 \leq g_{n+1}(x) \leq g_n(x)$$

  for all $n \in \mathbb{N}$ and $x \in A$, and

$$\begin{aligned} |r_n(x) - r_m(x)| &\leq |s_n(x)| |g_{n+1}(x)| + |s_m(x)| |g_{m+1}(x)| \\ &\quad + \sum_{k=m+1}^{n} |s_k(x)| |g_{k+1}(x) - g_k(x)| \\ &\leq M(g_{n+1}(x) + g_{m+1}(x)) + M \sum_{k=m+1}^{n} (g_k(x) - g_{k+1}(x)) \\ &= M(g_{n+1}(x) + g_{m+1}(x)) + M(g_{m+1}(x) - g_{n+1}(x)) \\ &= 2M g_{m+1}(x). \end{aligned}$$

- Since $g_n \underset{n\to\infty}{\to} 0$ uniformly,
  for $\epsilon > 0$, there is $N \in \mathbb{N}$, such that,
  for $m > N$ and all $x \in A$,

$$0 \leq g_m(x) < \frac{\epsilon}{2M}.$$

- Therefore, for $m, n > N$ and for all $x \in A$

$$|r_n(x) - r_m(x)| \leq 2M g_{m+1}(x) < 2M \frac{\epsilon}{2M} = \epsilon.$$

- It follows that the sequence (r_n) is uniformly Cauchy,

  hence $\sum_{n=1}^{\infty} f_n(x) g_n(x)$ converges uniformly on A.

This finishes our argument. ■

- **Example** Test the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}, \ 0 \leq x \leq 1,$$

  for convergence and uniform convergence

- **Solution** Let

$$f_n : [0,1] \to \mathbb{R}, \ f_n(x) = (-1)^n$$

  and

$$g_n : [0,1] \to \mathbb{R}, \ g_n(x) = \frac{x^n}{n}.$$

- Since, for all $x \in [0,1]$

$$g_n(x) \underset{n\to\infty}{\to} 0,$$

  let $g(x) = 0$, for all $x \in [0,1]$.

- We show that $g_n \underset{n\to\infty}{\to} g$ uniformly.

- Indeed, if $n > N$ and $x \in [0,1]$, then

$$|g_n(x) - g(x)| = \frac{x^n}{n} \leq \frac{1}{n} < \frac{1}{N}$$

- For given $\epsilon > 0$, by Archimedean property,
  there is $N \in \mathbb{N}$, such that $\frac{1}{N} < \epsilon$.

- Therefore, for all $n \geq N$ and for all $x \in [0,1]$:

$$|g_n(x) - g(x)| = \frac{x^n}{n} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

- Hence, $g_n \underset{n\to\infty}{\to} g$ uniformly.

- Moreover, for all $n \in \mathbb{N}$ and $x \in [0,1]$:

$$0 \leq g_{n+1}(x) = \frac{x^{n+1}}{n+1} \leq \frac{x^n}{n+1} \leq \frac{x^n}{n} = g_n(x),$$

  since $x^{n+1} \leq x^n$, $\frac{1}{n+1} \leq \frac{1}{n}$, for all $n \in \mathbb{N}$, $x \in [0,1]$.

- Let

$$s_n(x) = \sum_{k=1}^{n} f_k(x) = \sum_{k=1}^{n} (-1)^k = \begin{cases} -1 & \text{if} \quad n \text{ is odd} \\ 0 & \text{if} \quad n \text{ is even} \end{cases}.$$

- Clearly, for all $n \in \mathbb{N}$ and $x \in [0,1]$:

$$|s_n(x)| \leq 1,$$

- so f_n and g_n satisfy the assumptions of Dirichlet's Test.
- It follows that the series
$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$$
is uniformly convergent on $[0,1]$.
Remark Notice that if $h_n : [0,1] \to \mathbb{R}$,
$$h_n(x) = \frac{(-1)^n x^n}{n}$$
and
$$f(x) = \sum_{n=1}^{\infty} h_n(x), \ x \in [0,1],$$
then $\sum_{n=1}^{\infty} h_n$ uniformly convergent to f.
- Notice that h_n is a continuous function, so
$$s_n = \sum_{k=1}^{n} h_k$$
is continuous for all n as a finite sum of continuous functions.
- Since $\sum_{n=1}^{\infty} h_n$ uniformly convergent to f,
the sequence $\{s_n\}$ converges uniformly to f.
Since each s_n is continuous, by theorem form the previous lecture,
f is also continuous.
- We showed $f : [0,1] \to \mathbb{R}$ defined by
$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$$
is continuous.
- Note that
$$f(1) = \sum_{n=1}^{\infty} \frac{(-1)^n 1^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$
which is an alternating series and, as we know, it converges.
Exercise Compute
$$f(1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = ...?$$
- As we showed
$$\sum_{n=1}^{\infty} \frac{x^n}{n!} e^{-nx}$$
converges uniformly on $[0,1]$, and let
$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} e^{-nx}, \ x \in [0,1].$$

- **Question**: Is f differentiable on $(0,1)$?
  If f is differentiable, what is $f'(x)$, for each $x \in (0,1)$?
- **Question:** Is f Riemann integrable over $[0,1]$.
- If so, can we, for instance, compute

$$\int_0^1 f(x)\,dx = \int_0^1 \left(\sum_{n=1}^{\infty} \frac{x^n}{n!} e^{-nx}\right) dx?$$

Example Consider $f_n : \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \frac{\sin(n^2 x)}{n}$$

and $g_n : [0,1] \to \mathbb{R}$,

$$g_n(x) = \frac{x^{n+1}}{n+1}.$$

Show that both $\{f_n\}$ and $\{g_n\}$ converge uniformly,
but $\{f_n'\}$ and $\{g_n'\}$ are not uniformly convergent.

- Let us consider the first sequence: $f_n : \mathbb{R} \to \mathbb{R}$,

$$f_n(x) = \frac{\sin(n^2 x)}{n}.$$

We see that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin(n^2 x)}{n} = 0.$$

- Indeed, since given $\epsilon > 0$, we take $N > \frac{1}{\epsilon}$ so for $n > N$, and then for all $x \in \mathbb{R}$,

$$|f_n(x) - 0| = \left|\frac{\sin(n^2 x)}{n}\right| \le \frac{1}{n} < \frac{1}{N} < \epsilon.$$

- We showed that $f_n \to f$ uniformly, where $f(x) = 0$, for all $x \in \mathbb{R}$.
- We find

$$\begin{aligned}\int_0^1 f_n(x)\,dx &= \int_0^1 \frac{\sin(n^2 x)}{n} dx = \frac{1}{n}\int_0^1 \sin(n^2 x)\,dx \\ &= -\frac{1}{n}\left[\frac{\cos(n^2 x)}{n^2}\right]_0^1 \\ &= -\frac{1}{n}\left(\frac{\cos(n^2) - 1}{n^2}\right) = \frac{1-\cos(n^2)}{n^3}\end{aligned}$$

and we see that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\left(\frac{1-\cos(n^2)}{n^3}\right) = 0 = \int_0^1 f(x)\,dx$$

- We see that
$$\lim_{n\to\infty}\int_0^1 f_n\left(x\right)dx=\int_0^1\lim_{n\to\infty}f_n\left(x\right)dx=\int_0^1 f\left(x\right)dx$$
for our sequence $\{f_n\}$.
- Also,
$$f_n'\left(x\right)=\frac{d}{dx}\left(\frac{\sin\left(n^2x\right)}{n}\right)=n\cos n^2x$$
and
$$\lim_{n\to\infty}f_n'\left(0\right)=\lim_{n\to\infty}n\cos\left(n^2\cdot 0\right)=\lim_{n\to\infty}n=\infty.$$
- Moreover, if
$$n^2x\neq\pm\frac{\pi}{2}+m\pi, m\in\mathbb{Z},$$
then
$$\lim_{n\to\infty}f_n'\left(x\right)$$
does not exist.
- Therefore, $\{f_n'\}$ is not even pointwise convergent.
In summary, we see that
$$0=\frac{d}{dx}\left(f\left(x\right)\right)=\frac{d}{dx}\left(\lim_{n\to\infty}f_n\left(x\right)\right)\neq\lim_{n\to\infty}\frac{d}{dx}\left(f_n\left(x\right)\right)\text{ DNE}$$
- **Question**: Which conditions on $\{f_n\}$ assure
$$\frac{d}{dx}\left(\lim_{n\to\infty}f_n\left(x\right)\right)=\lim_{n\to\infty}\frac{d}{dx}\left(f_n\left(x\right)\right)$$
- We see from the above example that
the uniform convergence of the sequence $\{f_n\}$ and
differentiability of each f_n is not sufficient.
Exercise: For the sequence $g_n:[0,1]\to\mathbb{R}$,
$$g_n\left(x\right)=\frac{x^{n+1}}{n+1},$$
find
$$g\left(x\right)=\lim_{n\to\infty}g_n\left(x\right),\ x\in[0,1]$$
and check if
$$\frac{d}{dx}\left(\lim_{n\to\infty}g_n\left(x\right)\right)=\lim_{n\to\infty}\frac{d}{dx}\left(g_n\left(x\right)\right),\ x\in(0,1)$$
and
$$\lim_{n\to\infty}\int_0^1 g_n\left(x\right)dx=\int_0^1\lim_{n\to\infty}g_n\left(x\right)dx=\int_0^1 g\left(x\right)dx.$$

- Since, for each $x \in [0,1]$,
$$0 \leq x^{n+1} \leq 1,$$
it follows that
$$0 \leq \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{x^{n+1}}{n+1} \leq \lim_{n\to\infty} \frac{1}{n+1} = 0,$$
then
$$\lim_{n\to\infty} \frac{x^{n+1}}{n+1} = 0.$$
- Let
$$g : [0,1] \to \mathbb{R},\ g(x) = \lim_{n\to\infty} g_n(x) = 0$$
be the pointwise limit of (g_n).
- We show that $g_n \to g$ uniformly.
- Indeed, let $\epsilon > 0$ be given,
for $n > N$ and $x \in [0,1]$, then
$$|g_n(x) - g(x)| = \frac{x^{n+1}}{n+1} \leq \frac{1}{n+1} < \frac{1}{n} < \frac{1}{N}.$$
- By the Archimedean property,
there is $N \in \mathbb{N}$ such that
$$\frac{1}{N} < \epsilon,$$
hence for all $n > N$ and all $x \in [0,1]$,
$$|g_n(x) - g(x)| = \frac{x^{n+1}}{n+1} \leq \frac{1}{n+1} < \frac{1}{n} < \frac{1}{N} < \epsilon.$$
- It follows that $g_n \to g$ uniformly.
- Consider the sequence (g_n'), where
$$g_n' : [0,1] \to \mathbb{R},\ g_n'(x) = x^n.$$
- We see that,
$$\lim_{n\to\infty} g_n'(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & if \quad x \neq 1 \\ 1 & if \quad x = 1 \end{cases}.$$
- Indeed, if $x = 0$, then
$$\lim_{n\to\infty} g_n'(0) = 0,$$
and for
$$0 < x < 1,$$
if $\epsilon > 0$ is given, and $n > N$,
$$|g_n'(x) - 0| = x^n < x^N$$
- By the Archimedean property,
there is $N > \log_x \epsilon$.

- We see that if $n > N$, then
$$\left|g_n'(x) - 0\right| = x^n < x^N < x^{\log_x \epsilon} = \epsilon.$$
- Therefore, $g_n'(x) \to 0$ as $n \to \infty$.
- Finally, for $x = 1$,
$$g_n'(1) = 1,$$
for all $n \in \mathbb{N}$.
- Therefore, $g_n'(1) \to 1$ as $n \to \infty$.
- Let $h : [0,1] \to \mathbb{R}$ be defined by
$$h(x) = \lim_{n\to\infty} g_n'(x),$$
i.e. h is a pointwise limit of the sequence (g_n').
- Suppose that $g_n' \to h$ uniformly.
- For each $n \in \mathbb{N}$,
$$\begin{aligned} g_n' &: [0,1] \to \mathbb{R}, \\ g_n'(x) &= x^n \end{aligned}$$
is continuous, thus by theorem,
h must also be continuous on $[0,1]$.
- Contradiction, since $h : [0,1] \to \mathbb{R}$, given by
$$h(x) = \begin{cases} 0 & if \quad x \neq 1 \\ 1 & if \quad x = 1 \end{cases}$$
is not continuous at $x = 1$.
- It follows that the sequence (g_n') is not uniformly convergent.
Remark From the above two examples, we see that, it is not true
$$\frac{d}{dx}\left(\lim_{n\to\infty} f(x)\right) = \lim_{n\to\infty} \frac{d}{dx} f(x)$$