HOMEWORK 9 SOLUTIONS – MATH 4341

Problem 1. Let X be a topological space. Suppose $\{x_n\}_n$ is a sequence in X converging to $x \in X$. Show that the set $A = \{x_n\}_n \cup \{x\}$ is a compact subspace of X.

Proof. Let $\mathcal{U}$ be an arbitrary open cover of $A = \{x_n\}_n \cup \{x\}$. Then there is an open set $U \in \mathcal{U}$ so that $x \in U$. Since $x_n \to x$, U contains the points x_n for all large enough n, say all $n > N$ for some N.

Since $\mathcal{U}$ is an open cover, we can also find open sets $U_1, \dots, U_N \in \mathcal{U}$ so that $x_k \in U_k$ for all $k = 1, \dots, N$. We now see that the collection $U, U_1, \dots, U_N$ together form a finite subcover of A. Hence A is compact. $\square$

Problem 2. Show that any finite union of compact subspaces of a topological space X is a compact subspace of X.

Proof. Suppose $K_1, \dots, K_n$ are compact subspaces of X. Let $\mathcal{U}$ be an open cover of $K_1 \cup \cdots \cup K_n$. For each $i = 1, \dots, n$, $\{U \cap K_i\}_{U \in \mathcal{U}}$ is an open cover of K_i. Since K_i is compact, there exists $U_1^i, \dots, U_{n_i}^i$ such that $K_i = (U_1^i \cap K_i) \cup \cdots \cup (U_{n_i}^i \cap K_i)$. Then

$$\{U_j^i \mid i = 1, \dots, n,\ j = 1, \dots, n_i\} \subset \mathcal{U}$$

is a finite open cover of $K_1 \cup \cdots \cup K_n$. So $K_1 \cup \cdots \cup K_n$ is a compact subspace of X. $\square$

Problem 3. Show that any intersection of compact subspaces of a Hausdorff space X is a compact subspace of X.

Proof. Suppose $\{K_i\}_{i \in I}$ is a family of compact subspaces of a Hausdorff space X. For each i, $K_i \subset X$ is closed since K_i is a compact subspace of a Hausdorff space X. Then $\bigcap_{i \in I} K_i$ is closed. Since $\bigcap_{i \in I} K_i$ is a closed subspace of the compact space K_1, it is compact. $\square$

Problem 4. Suppose $\mathbb{R}$ has the topology consisting of all subsets A such that $\mathbb{R} \setminus A$ is either finite or all of $\mathbb{R}$. Show that every subspace of $\mathbb{R}$ is compact.

Proof. Suppose $A \subset X$ and let $\mathcal{U}$ be an open cover of A. Take any $U_0 \in \mathcal{U}$. Since U_0 is open in A, it covers all but a finite number of points in A. Choose a finite number of sets from $\mathcal{U}$ covering $A \setminus U_0$. These sets and U_0 form a finite subcover, hence A is compact. $\square$

Problem 5. Show that a compact subspace of a topological space is not always closed.

Proof. Consider $\mathbb{R}$ with the topology consisting of all subsets A such that $\mathbb{R} \setminus A$ is either finite or all of $\mathbb{R}$. By problem 4, every subspace of $\mathbb{R}$ is compact. However, not all subspaces of $\mathbb{R}$ are closed, since closed sets in $\mathbb{R}$ are exactly finite sets and $\mathbb{R}$. $\square$