§2. Topological Spaces

Math 4341 (Topology)

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- In the example above, $\mathcal{T}_2$ is strictly coarser than $\mathcal{T}_4$, but $\mathcal{T}_2$ and $\mathcal{T}_3$ are not comparable.

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- If $\mathcal{B}$ is a basis, we define $\mathcal{T}_{\mathcal{B}}$, the *topology generated by* $\mathcal{B}$, by declaring that $U \in \mathcal{T}_{\mathcal{B}}$ if for every $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$.

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    - Take any $y \in \bigcup_{x \in U} B_x$. Then there exists a $z \in U$ so that $y \in B_z$, but by our choices of the basis elements, we have that $B_z \subset U$, so $y \in B_z \subset U$.

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2.2. Basis for a topology

- **Lemma 2.4**. Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{C} \subset \mathcal{T}$ be a collection of open sets on X with the following property: for each set $U \in \mathcal{T}$ and each $x \in U$ there is a $C \in \mathcal{C}$ so that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for $\mathcal{T}$.
- *Proof*. There are two things to show.
  - $\mathcal{C}$ is a basis for a topology.
    - (B1): Let $x \in X$. Since $X \in \mathcal{T}$, by definition of $\mathcal{C}$ we get a $C \in \mathcal{C}$ so that $x \in C \subset X$.
    - (B2): Let $x \in C_1 \cap C_2$ for $C_1, C_2 \in \mathcal{C}$. Since C_1 and C_2 are open sets, so is $C_1 \cap C_2$. Therefore we get a $C \in \mathcal{C}$ so that $x \in C \subset C_1 \cap C_2$.
  - The topology $\mathcal{T}_{\mathcal{C}}$ generated by $\mathcal{C}$ is actually $\mathcal{T}$.
    - Take any $U \in \mathcal{T}_{\mathcal{C}}$. By Lemma 2.3, U is a union of elements of $\mathcal{C}$. Since $\mathcal{C} \subset \mathcal{T}$, it follows from (T2) that $U \in \mathcal{T}$.

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    - Take any $U \in \mathcal{T}$. By definition of $\mathcal{C}$, for any $x \in U$ we can find a $C \in \mathcal{C}$ so that $x \in C \subset U$. Hence $U \in \mathcal{T}_{\mathcal{C}}$.

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- **Proposition 2.5**. The collection

$$\mathcal{B} = \{B(x, r) \mid x \in \mathbb{R}^n, r > 0\}$$

is the basis for a topology on $\mathbb{R}^n$. The resulting topology $\mathcal{T}_{\mathcal{B}}$ is called the standard topology and its open sets are exactly the open sets in analysis/calculus.

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  - Suppose (2) holds. Let $U \in \mathcal{T}$, and let $x \in U$ be any element. Then there is a $B \in \mathcal{B}$ with $x \in B \subset U$, and by (2) we get $B' \in \mathcal{B}'$ with $x \in B' \subset B \subset U$. This implies that $U \in \mathcal{T}'$.

- **Example**. We can define a basis for a topology on $\mathbb{R}$ by letting $\mathcal{B}_\ell$ consist of all sets of the form

$$\{x \in \mathbb{R} \mid a \leq x < b\},$$

where $a, b \in \mathbb{R}$ vary. The topology $\mathcal{T}_\ell$ generated by $\mathcal{B}_\ell$ is called the *lower limit topology* on $\mathbb{R}$, and we write $\mathbb{R}_\ell = (\mathbb{R}, \mathcal{T}_\ell)$.

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- **Example**. Let $K = \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$ and let $\mathcal{B}_K$ consist of all open intervals as well as all sets of the form $(a, b) \setminus K$. Then $\mathcal{B}_K$ is a basis and the topology $\mathcal{T}_K$ that it generates is called the K-topology on $\mathbb{R}$. We write $\mathbb{R}_K = (\mathbb{R}, \mathcal{T}_K)$.

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  - $\mathbb{R}_K \supsetneq \mathbb{R}$: Let $x \in \mathbb{R}$ and let (a, b) contain x. This interval belongs to $\mathcal{B}_K$ so by Lemma 2.6 we have $\mathbb{R}_K \supset \mathbb{R}$. To see that it is strictly finer, consider the set $U = (-1, 1) \setminus K \in \mathcal{T}_K$. Then $0 \in U$ but there is no open interval B so that $0 \in B \subset U$.

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  - $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable with each other: Note that $U \in \mathcal{T}_K$ but $U \notin \mathcal{T}_\ell$, and that $[1, 2) \in \mathcal{T}_\ell$ but $[1, 2) \notin \mathcal{T}_K$.

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- The function d is called a *metric*, and $d(x, y)$ is called the *distance* from x to y.
- For a metric space (X, d) the open *ball* $B_d(x, r)$ centered at x, with radius $r > 0$, with respect to the metric d is defined as

$$B_d(x, r) = \{y \in X \mid d(x, y) < r\}.$$

We will use the open balls to define a topology, called *the metric topology* on any metric space.

- **Proposition 2.8**. If (X, d) is a metric space, then the collection
$$\mathcal{B} = \{B_d(x, r) \mid x \in X, r > 0\}$$
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    - Choose $r = \min\big(r_1 - d(x, y_1), r_2 - d(x, y_2)\big)$. For any $z \in B_d(x, r)$, by the triangle inequality we have
    $$d(z, y_i) \leq d(z, x) + d(x, y_i) < r + d(x, y_i) \leq r_i$$
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- **Remark**. For the case of $\mathbb{R}^n$, we recover the usual condition for a set to be open.

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  - Let $x \in X$ be arbitrary. If $r \leq 1$, then $B_d(x, r) = \{x\}$ while if $r > 1$ then $B_d(x, r) = X$.
  - Thus the basis of open balls is

$$\mathcal{B} = \{\{x\} \mid x \in X\} \cup \{X\}.$$

    Hence the topology induced by d is the discrete topology.

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  - If $f : X \to Y$ and $\{B_i\}_{i \in I}$ is a family of subsets of Y, then
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  - If $B \subset Y$, then $f^{-1}(B^c) = f^{-1}(B)^c$.
  - If $g : Y \to Z$ is another map and $C \subset Z$, then
$$(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C)).$$

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- **Remark**. The notion of "continuity" depends heavily on the topologies on the spaces under consideration.

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- **Example**. Let X be any topological space, and let Y have the trivial topology. Then any map $f : X \to Y$ is continuous, since $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$ which are both open.

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  - Suppose f is continuous at x for all $x \in X$. Let $U \in \mathcal{T}_Y$. We need to show that $f^{-1}(U) \in \mathcal{T}_X$. For each $x \in f^{-1}(U)$, there exists $V_x \in \mathcal{T}_X$ so that $x \in V_x$ and $f(V_x) \subset U$. Now $f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} V_x$ and thus open since each V_x is.

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- Recall from analysis that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *continuous at a point* $x \in \mathbb{R}^n$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\| x - y \| < \delta \Rightarrow \| f(x) - f(y) \| < \epsilon$.
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- **Theorem 2.12**. Let (X, d_X) and (Y, d_Y) be metric spaces with their induced metric topologies. Then a function $f : X \to Y$ is continuous if and only if

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- *Proof*. Theorem 2.12 follows from Theorem 2.11(iii) and Lemma 2.13.
- **Lemma 2.13**. Let (X, d_X) and (Y, d_Y) be metric spaces with the metric topologies. Then a function $f : X \to Y$ is continuous at a point $x \in X$ if and only if

$$\forall \epsilon > 0, \exists \delta > 0 : f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \epsilon). \tag{1}$$

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- Suppose (1) holds for f. Let U be an open set in Y containing $f(x)$.
  - By Proposition 2.9, the openness of U implies that there exists $\epsilon > 0$ so that $B_{d_Y}(f(x), \epsilon) \subset U$.
  - By (1) we then get a $\delta > 0$ with $f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \epsilon)$.

Continuous functions: Proof of Lemma 2.13

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  - There is an open set V in X such that $x \in V$ and $f(V) \subset B_{d_Y}(f(x), \epsilon)$.
  - By Proposition 2.9, the openness of V implies that there is a $\delta > 0$ so that $B_{d_X}(x, \delta) \subset V$.
  - Hence $f(B_{d_X}(x, \delta)) \subset f(V) \subset B_{d_Y}(f(x), \epsilon)$.
- Suppose (1) holds for f. Let U be an open set in Y containing $f(x)$.
  - By Proposition 2.9, the openness of U implies that there exists $\epsilon > 0$ so that $B_{d_Y}(f(x), \epsilon) \subset U$.
  - By (1) we then get a $\delta > 0$ with $f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \epsilon)$.
  - Hence $f(B_{d_X}(x, \delta)) \subset U$. Since $B_{d_X}(x, \delta)$ is open in X and contains x we are done.