HOMEWORK 2 SOLUTIONS – MATH 4341

Problem 1. Suppose X is a finite set of cardinality n. Show that $\mathcal{P}(X)$, the power set of X, is a finite set of cardinality 2^n.

Proof. The cardinality of $\mathcal{P}(X)$ is equal to the number of subsets of X.

Method 1: If A is a subset of X, then each $x \in X$ has 2 possibilities: either $x \in A$ or $x \notin A$. Since X has n elements, the number of ways to choose a subset A of X is 2^n. Hence $|\mathcal{P}(X)| = 2^n$.

Method 2: The number of subsets of X of cardinality m, where $0 \leq m \leq n$, is equal to the number of ways to choose m elements from n elements of X which is equal to $\binom{n}{m}$. Hence, by the binomial theorem we have

$$|\mathcal{P}(X)| = \sum_{m=0}^{n} \binom{n}{m} = (1+1)^n = 2^n.$$

Note that $(a+b)^n = \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m}$. $\square$

Problem 2. Describe all possible topologies on the set $X = \{a, b, c\}$. Justify your answer.

Proof. All subsets of X are $\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}$ and X. The trivial and discrete topologies are always topologies.

If a topology $\mathcal{T}$ contains exactly three 2-elements subsets then it is discrete.

If $\mathcal{T}$ contains exactly two 2-element sets, say $\{a, b\}$ and $\{b, c\}$, then it contains $\{b\}$. In this case $\mathcal{T}$ contains at most one of the 1-element sets $\{a\}$ and $\{c\}$. Hence $\mathcal{T}$ is either $\{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}\}$ or $\{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}, \{x\}\}$ where $x \in \{a, c\}$. The other 6 topologies which contain exactly two 2-element sets are obtained in the same way.

If $\mathcal{T}$ contains exactly one 2-element set, say $\{a, b\}$, then it contains either no 1-element set, exactly one 1-element set, or exactly two 1-element sets $\{a\}$ and $\{b\}$. Hence $\mathcal{T}$ is either $\{\emptyset, X, \{a, b\}\}$, $\{\emptyset, X, \{a, b\}, \{x\}\}$, or $\{\emptyset, X, \{a, b\}, \{a\}, \{b\}\}$, where $x \in \{a, b, c\}$. The other 10 topologies which contain exactly one 2-element set are obtained in the same way.

If $\mathcal{T}$ contains no 2-element set, then it contains at most one 1-element set. In this case $\mathcal{T}$ is either trivial or $\{\emptyset, X, \{x\}\}$, where $x \in \{a, b, c\}$.

There are totally $2 + 9 + 15 + 3 = 29$ topologies on $X = \{a, b, c\}$. $\square$

Problem 3. Let $\mathcal{I}$ be the set of all irrational numbers. We define $\mathcal{T}$ to be the collection of all subsets U of $\mathcal{I}$ such that either $U = \emptyset$ or $\mathcal{I} \setminus U$ is countable. Show that $\mathcal{T}$ is a topology on $\mathcal{I}$.

Proof. We will check 3 conditions for $\mathcal{T}$ to be a topology.

– (T1): $\emptyset \in \mathcal{T}$ by definition. $\mathcal{I} \in \mathcal{T}$ since $\mathcal{I} \setminus \mathcal{I} = \emptyset$ is a countable set.

– (T2): Suppose $\{U_i\}_{i \in I}$ is an indexed family of nonempty elements of $\mathcal{T}$. Since $U_i \in \mathcal{T}$ and $U_i \neq \emptyset$, we have $\mathcal{I} \setminus U_i$ is countable. This implies that $\bigcap_{i \in I} (\mathcal{I} \setminus U_i)$ is also countable. But $\bigcap_{i \in I} (\mathcal{I} \setminus U_i) = \mathcal{I} \setminus \bigcup_{i \in I} U_i$, hence $\bigcup_{i \in I} U_i \in \mathcal{T}$.

– (T3): Suppose $U_1, \cdots, U_n$ are nonempty elements of $\mathcal{T}$. Since $U_i \in \mathcal{T}$ and $U_i \neq \emptyset$, we have $\mathcal{I} \setminus U_i$ is countable for all $i = 1, \cdots, n$. This implies that $\bigcup_{i=1}^{n} (\mathcal{I} \setminus U_i)$ is also countable. But $\bigcup_{i=1}^{n} (\mathcal{I} \setminus U_i) = \mathcal{I} \setminus \bigcap_{i=1}^{n} U_i$, hence $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$. $\square$

Problem 4. Let
$$\mathcal{B}_\ell = \{[a,b) \mid a,b \in \mathbb{R}\}.$$
Show that $\mathcal{B}_\ell$ is a basis for a topology on $\mathbb{R}$.

Proof. We will check 2 conditions for $\mathcal{B}_\ell$ to be a basis.

– (B1): Suppose $x \in \mathbb{R}$. Let $B = [x, x+1) \in \mathcal{B}_\ell$. Then $x \in B$.

– (B2): Suppose $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}_\ell$. We write $B_1 = [a,b)$ and $B_2 = [c,d)$. Take $B_3 = B_1 \cap B_2 = [\max\{a,c\}, \min\{b,d\}) \in \mathcal{B}_\ell$. Then $x \in B_3 \subset B_1 \cap B_2$. $\square$

Problem 5. Let $K = \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$ and let
$$\mathcal{B}_K = \{(a,b) \mid a,b \in \mathbb{R}\} \bigcup \{(a,b) \setminus K \mid a,b \in \mathbb{R}\}.$$
Show that $\mathcal{B}_K$ is a basis for a topology on $\mathbb{R}$.

Proof. We will check 2 conditions for $\mathcal{B}_K$ to be a basis.

– (B1): Suppose $x \in \mathbb{R}$. Let $B = (x-1, x+1) \in \mathcal{B}_K$. Then $x \in B$.

– (B2): Suppose $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}_\ell$. We write $B_1 = (a,b)$ or $B_1 = (a,b) \setminus K$, and $B_2 = (c,d)$ or $B_2 = (c,d) \setminus K$. Take $B_3 = B_1 \cap B_2$. Then $x \in B_3 \subset B_1 \cap B_2$. Note that $B_3 = (\max\{a,c\}, \min\{b,d\})$ or $B_3 = (\max\{a,c\}, \min\{b,d\}) \setminus K$, hence in both cases we have $B_3 \in \mathcal{B}_K$. $\square$