HOMEWORK 8 SOLUTIONS – MATH 4341

Problem 1. Consider $\mathbb{R}$ with the topology $\mathcal{T} = \{\emptyset, \mathbb{Q}, \mathbb{I}, \mathbb{R}\}$, where $\mathbb{Q}$ is the set of all rational numbers and $\mathbb{I}$ is the set of all irrational numbers.

(a) Is $(\mathbb{R}, \mathcal{T})$ connected?

(b) Is $(\mathbb{R}, \mathcal{T})$ path connected?

Proof. (a) $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$ is a separation (why?), so $(\mathbb{R}, \mathcal{T})$ is not connected.

(b) If $(\mathbb{R}, \mathcal{T})$ is path connected, then it will be connected. This contradicts part (a), hence $(\mathbb{R}, \mathcal{T})$ is not path connected. □

Problem 2. Let X be a topological space. Define a relation $\sim_p$ on X by declaring that $x \sim_p y$ if and only if there is a path connected set $A \subset X$ such that $x, y \in A$.

(a) Show that $\sim_p$ is an equivalence relation.

(b) The equivalence classes of $\sim_p$ are called the path connected components of X. Show that any path connected component is a path connected subset of X.

Proof. (a) Reflexive: $x \sim_p x$ since x is contained in the path connected set $\{x\}$.

Symmetric: obvious.

Transitive: Suppose $x \sim_p y$ and $y \sim_p z$. Then there exist two path connected sets $A, B \subset X$ such that $x, y \in A$ and $y, z \in B$. Since $A \cup B$ is path connected (why?) and $x, z \in A \cup B$, we have $x \sim_p z$.

(b) Let $[x]$ be a path connected component. For any $y, z \in [x]$, we have $y \sim_p z$, so there exists a path connected set A containing y, z. We claim that $A \subset [x]$. Indeed, for any $w \in A$ we have $w \sim_p y$. This implies that $w \in [y] = [x]$. Hence $A \subset [x]$.

Since A is path connected, there exists a path in A connecting y and z. This path is also in $[x]$, so $[x]$ is path connected. □

Problem 3. Let $\mathcal{I}$ be the subspace of $\mathbb{R}$ consisting of all irrational numbers.

(a) Find all connected components of $\mathcal{I}$.

(b) Find all path connected components of $\mathcal{I}$.

Proof. (a) Any connected component is connected. Suppose $A \subset \mathcal{I}$ is a connected subset. If A contains at least two elements, say $a < b$, then by taking any $c \in (a, b) \cap \mathbb{Q}$ we have $A = (A \cap (\infty, c)) \cup (A \cap (c, \infty))$. This is a separation of A (note that $a \in A \cap (\infty, c)$ and $b \in A \cap (c, \infty)$), which is a contradiction. Hence all connected subsets of $\mathcal{I}$ are $\{x\}$, $x \in \mathcal{I}$. These are all connected components of $\mathcal{I}$.

(b) Any path connected component is path connected and hence is connected. Since all connected subsets of $\mathcal{I}$ are $\{x\}$, $x \in \mathcal{I}$, they are all path connected components of $\mathcal{I}$. □

Problem 4. Let $n \geq 2$ be an integer.

(a) Show that $\mathbb{R}^n \setminus \{\mathbf{0}\}$ is path connected, where $\mathbf{0}$ is the origin in $\mathbb{R}^n$.

(b) Show that $\mathbb{R}^n \not\cong \mathbb{R}$.

Proof. (a) Let $X = \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $x, y \in X$. Consider the straight line from x to y in $\mathbb{R}^n$: $f(t) = (1-t)x + ty$ for $t \in [0, 1]$. If this does not pass through $\mathbf{0}$, then it is a path from x to y in X.

If the line does pass through $\mathbf{0}$, then take a third point z which is not on the line itself. (This is possible since $n \geq 2$). Now take the line from x to z and combine it with the line from z to y to get a path from x to y in X. This path can be defined as

$$f(t) = \begin{cases} (1-2t)x + 2tz & \text{if } t \in [0, \frac{1}{2}], \\ (2-2t)z + (2t-1)y & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

(b) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a homeomorphism. Then $f : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R} \setminus \{f(\mathbf{0})\}$ is also a homeomorphism. By (a), $\mathbb{R}^n \setminus \{\mathbf{0}\}$ is path connected and hence is connected. This implies that $\mathbb{R} \setminus \{f(\mathbf{0})\}$ is connected. However $\mathbb{R} \setminus \{f(\mathbf{0})\} = (-\infty, f(\mathbf{0})) \cup (f(\mathbf{0}), \infty)$ is a separation, so we obtain a contradiction. This proves that $\mathbb{R}^n \not\cong \mathbb{R}$. $\square$