Math 4301 Mathematical Analysis I
<u>Lecture 16</u>
Topic: Properties of Riemann Integral.

- **Definition** Let $f : [a, b] \to \mathbb{R}$ be bounded. We say that f is *Riemann integrable over* $[a, b]$ if

$$\overline{\int_a^b} f \leq \underline{\int_a^b} f.$$

In such a case, the number

$$\int_a^b f = \overline{\int_a^b} f = \underline{\int_a^b} f$$

is called the *Riemann integral* of f over $[a, b]$.
Theorem A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable over $[a, b]$ if and only if, for every $\epsilon > 0$, there is $P \in \mathcal{P}([a, b])$, such that

$$U(f, P) - L(f, P) < \epsilon.$$

- From calculus we know that
if F is an antiderivative of f on the interval $[a, b]$,
then we can find the value of $\int_a^b f(x)\, dx$ in a simple way, namely

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Recall, for a function $f : [a, b] \to \mathbb{R}$ a function

$$F : [a, b] \to \mathbb{R}$$

is called *antiderivative of* f if

$$\frac{d}{dx} F(x) = f(x), \text{ for all } x \in [a, b].$$

Theorem (*Fundamental Theorem of Calculus*) Let $f : [a, b] \to \mathbb{R}$ be continuous.
Then f has an antiderivative F and

$$\int_a^b f = F(b) - F(a).$$

If G is an antiderivative of f, then

$$\int_a^b f = G(b) - G(a).$$

Proof. Let $F : [a, b] \to \mathbb{R}$ be given by

$$F(x) = \int_a^x f(t)\, dt$$

- We show that F is differentiable.
- Since f is integrable, for all $x \in [a,b]$,
  $\int_a^x f(t)\, dt$ exists, so $F(x)$ is defined for all $x \in [a,b]$.
- It is sufficient to show that
$$\frac{d}{dx} F(x) = f(x)\,, \text{ for all } x \in (a,b)\,.$$
- Let $x \in (a,b)$ and $h > 0$ be sufficiently small, so that
$$[x, x+h] \subset [a,b]\,.$$
- Then
$$\begin{aligned}
\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| &= \left|\frac{\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt}{h} - f(x)\right| \\
&= \left|\frac{\int_a^x f(t)\, dt + \int_x^{x+h} f(t)\, dt - \int_a^x f(t)\, dt}{h} - f(x)\right| \\
&= \left|\frac{\int_x^{x+h} f(t)\, dt}{h} - f(x)\right| \\
&= \left|\frac{\int_x^{x+h} f(t)\, dt}{h} - \frac{f(x)\, h}{h}\right| = \left|\frac{\int_x^{x+h} f(t)\, dt}{h} - \frac{f(x) \int_x^{x+h}\, dt}{h}\right| \\
&= \left|\frac{\int_x^{x+h} f(t)\, dt - \int_x^{x+h} f(x)\, dt}{h}\right| \\
&= \frac{\left|\int_x^{x+h} f(t) - f(x)\, dt\right|}{h} \leq \frac{\int_x^{x+h} |f(t) - f(x)|\, dt}{h}.
\end{aligned}$$
- Since f is continuous on $[a,b]$,
  for $\epsilon > 0$ there is $h > 0$, such that, for all $t \in [x, x+h] \subset [a,b]$,
$$|f(t) - f(x)| < \epsilon.$$
- Thus,
$$\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| \leq \frac{\int_x^{x+h} |f(t) - f(x)|\, dt}{h} \leq \frac{\epsilon h}{h} = \epsilon.$$
- This shows that
$$\lim_{h \to 0^+} \frac{F(x+h) - F(x)}{h} = f(x)\,.$$
- Analogously, one shows that
$$\lim_{h \to 0^-} \frac{F(x+h) - F(x)}{h} = f(x)\,.$$
- We showed that
$$\frac{d}{dx} F(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)\,.$$

- Since $F(x) = \int_a^x f(t)\,dt$, so
$$F(a) = \int_a^a f(t)\,dt = 0$$
and
$$F(b) = \int_a^b f(t)\,dt,$$
so
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

- Furthermore, since
$$G'(x) = f(x) = F'(x), \text{ for all } x \in (a,b)$$
it follows that
$$G(x) = F(x) + C,$$
so
$$\begin{aligned}\int_a^b f(x)\,dx &= F(b) - F(a) = (G(b) - C) - (G(a) - C) \\ &= G(b) - G(a).\end{aligned}$$

This finishes our proof. ■

- **Remark:** For instance $f : [0,1] \to \mathbb{R}$, $f(x) = x$ is continuous on $[0,1]$.
Thus, f has an antiderivative
$$F(x) = \int_0^x t\ dt = \frac{1}{2}x^2, \text{ for all } x \in [0,1],$$
so
$$\int_0^1 f(x)\,dx = F(1) - F(0) = \frac{1}{2} \cdot 1^2 - \frac{1}{2} \cdot 0^2 = \frac{1}{2}.$$
Notice that $G(x) = \frac{1}{2}x^2 + 1$ is also antiderivative of f since
$$\frac{d}{dx}G(x) = x = f(x), \text{ for all } x \in [0,1].$$
By the *Fundamental Theorem of Calculus*, we can use also G to compute
$$\int_0^1 f(x)\,dx = G(1) - G(0) = \frac{3}{2} - 1 = \frac{1}{2}.$$
Remark: Notice that for $a > 0$, $f : [0,a] \to \mathbb{R}$,
$$f(x) = \exp\left(-x^2\right)$$
is continuous. Therefore, by the Fundamental Theorem of Calculus,
f has antiderivative, for instance
$$F(x) = \int_0^x \exp\left(-t^2\right) dt, \ x \in [0,a]$$

notice that an antiderivative of f cannot be expressed in terms of elementary functions.

Remark Let $f:[a,b]\to\mathbb{R}$ be continuous and

$$F(x)=\int_a^{u(x)} f(t)\,dt$$

where $u(x)\in[a,b]$, for each $x\in[a,b]$ and u is differentiable. Then by the chain rule

$$\frac{d}{dx}F(x)=f(u(x))\,u'(x).$$

Integral over a bounded subset $A\subseteq\mathbb{R}$

- Let $A\subseteq\mathbb{R}$ be bounded and $f:A\to\mathbb{R}$ be bounded.
- We extend f to $[a,b]$ where $A\subseteq[a,b]$ by setting

$$\widetilde{f}(x)=\begin{cases} f(x) & \text{if} \quad x\in A \\ 0 & \text{if} \quad x\in[a,b]\setminus A \end{cases}.$$

and define

$$\int_A f(x)\,dx=\int_a^b \widetilde{f}(x)\,dx.$$

Remark The definition of $\int_A f(x)\,dx$ does not depend on the interval $[a,b]$, such that $A\subseteq[a,b]$. In particular $\int_A f(x)\,dx$ is well-defined.

- We say that a bounded subset A of $\mathbb{R}$ has *volume* *(or A is Jordan measurable)*, if the characteristic function

$$\chi_A:\mathbb{R}\to\mathbb{R}$$

of A defined by

$$\chi_A(x)=\begin{cases} 1 & \text{if} \quad x\in A \\ 0 & \text{if} \quad x\in\mathbb{R}\setminus A \end{cases}$$

is Riemann integrable over A.

- Define

$$v(A)=\int_A \chi_A(x)\,dx.$$

and we call it the *volume of* A if χ_A is integrable.

Example Let $A=\mathbb{Q}\cap[0,1]$. We observe that the characteristic function

$$\chi_A:\mathbb{R}\to\mathbb{R}$$

of A is defined by

$$\chi_A(x)=\begin{cases} 1 & \text{if} \quad x\in\mathbb{Q}\cap[0,1] \\ 0 & \text{if} \quad x\in\mathbb{R}\setminus(\mathbb{Q}\cap[0,1]) \end{cases}$$

- Clearly, χ_A is not Riemann integrable (as we checked it before).

- Therefore, A has no volume.

  Example Let $A = [a, b] \subset \mathbb{R}$, where $a < b$.

  Then
  $$v(A) = \int_A \chi_A(x)\, dx = b - a,$$
  so A has volume.

- We say that $A \subseteq \mathbb{R}$ has *volume zero* if $v(A) = 0$,

- That is, for any $\epsilon > 0$, there are intervals $R_1, R_2, ..., R_m$, such that
  $$A \subseteq \bigcup_{j=1}^{m} R_j \text{ and } \sum_{j=1}^{m} v(R_j) < \epsilon.$$

- Let $A = \{x_1, x_2, ..., x_m\} \subset \mathbb{R}$ and $\epsilon > 0$ be given.

- WLOG we may assume that $x_1 < x_2 < ... < x_m$.

- Define
  $$R_i = \left[x_i - \frac{\epsilon}{4m}, x_i + \frac{\epsilon}{4m}\right].$$

- Then
  $$A \subseteq \bigcup_{j=1}^{m} R_j$$
  and
  $$\begin{aligned}\sum_{j=1}^{m} v(R_j) &= \sum_{j=1}^{m} \left(x_i + \frac{\epsilon}{4m} - \left(x_i - \frac{\epsilon}{4m}\right)\right) \\ &= \frac{1}{2m} \sum_{j=1}^{m} \epsilon = \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

- It follows that A has volume zero.

  Exercise Give examples of bounded subsets of $\mathbb{R}$ that have and don't have volume.

a. One of the examples of a subset of $\mathbb{R}$ which has no volume is $A = \mathbb{Q} \cap [0, 1]$.

- We observe that the characteristic function $\chi_A : \mathbb{R} \to \mathbb{R}$ of A defined by
  $$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{if} \quad x \in \mathbb{R} \backslash (\mathbb{Q} \cap [0, 1]) \end{cases}$$
  is not Riemann integrable (as we checked it before).

b. Let
  $$A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subset [0, 1],$$
  then its characteristic function $\chi_A : \mathbb{R} \to \mathbb{R}$, is given by
  $$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \in \mathbb{R} \backslash A \end{cases}$$

- We show that χ_A is Riemann integrable.

  Solution: We see that $A \subset [0,1]$ so by the definition

$$v(A) = \int_A \chi_A = \int_0^1 \widetilde{\chi_A}(x)\,dx$$

  so we want to show that $\int_0^1 \widetilde{\chi_A}(x)\,dx$ exists.

- Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[0,1]$, such that

$$\Delta x_1 = x_1 - x_0 = \frac{1}{N}$$

  and for $j > 1$,

$$\Delta x_j = x_j - x_{j-1} < \frac{1}{N^2}.$$

- We see that

$$L(\chi_A, P) = 0$$

  and

$$\begin{aligned} U(\chi_A, P) &\leq 1 \cdot \frac{1}{N} + \sum_{j=1}^{N-1} 1 \cdot \frac{1}{N^2} = \frac{1}{N} + \frac{N-1}{N^2} \\ &= \frac{2N-1}{N^2} \to 0 \text{ as } N \to \infty \end{aligned}$$

- Now, we see that, for all $N \in \mathbb{N}$,

$$0 \leq L(\chi_A, P) \leq \underline{\int_0^1} \chi_A(x)\,dx \leq \overline{\int_0^1} \chi_A(x)\,dx \leq U(\chi_A, P) \leq \frac{2N-1}{N^2}.$$

- Therefore,

$$\underline{\int_0^1} \chi_A(x)\,dx = \overline{\int_0^1} \chi_A(x)\,dx = 0,$$

  so χ_A is Riemann integrable and

$$\int_0^1 \chi_A(x)\,dx = 0.$$

- It follows that A has volume and

$$v(A) = 0.$$

  Definition A subset $A \subseteq \mathbb{R}$ is said to be of *Lebesgue measure zero* if,
  for every $\epsilon > 0$, there is a countable family of intervals I_1, I_2, ... in $\mathbb{R}$, such that

$$A \subseteq \bigcup_{j=1}^{\infty} I_j \text{ and } \sum_{j=1}^{\infty} v(I_j) < \epsilon.$$

- Let

$$A = \mathbb{N}$$

  and $\epsilon > 0$ be given.

- Define, for each $j \in \mathbb{N}$,

$$I_j = \left[j - \frac{\epsilon}{2^{j+2}}, j + \frac{\epsilon}{2^{j+2}}\right].$$

- Clearly, $A \subseteq \bigcup_{j=1}^{\infty} I_j$ and since

$$v(I_j) = j + \frac{\epsilon}{2^{j+2}} - \left(j - \frac{\epsilon}{2^{j+2}}\right) = \frac{\epsilon}{2^{j+1}},$$

- Hence

$$\begin{aligned} \sum_{j=1}^{\infty} v(I_j) &= \sum_{j=1}^{\infty} \frac{\epsilon}{2^{j+1}} = \frac{\epsilon}{2^2} \sum_{j=1}^{\infty} \frac{1}{2^{j-1}} \\ &= \frac{\epsilon}{2^2} \frac{1}{1-\frac{1}{2}} = \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

- It follows that A has Lebesgue measure zero.

  Theorem Let A_i have Lebesgue measure zero, for $i = 1, 2, ...$

  Then

$$A = \bigcup_{j=1}^{\infty} A_j$$

  has measure zero.

Proof. Let $\epsilon > 0$ be given.

- Since A_i has Lebesgue measure zero there are intervals $R_{1,i}, R_{2,i}, ...$ such that

$$A_i \subseteq \bigcup_{j=1}^{\infty} R_{j,i}$$

  and

$$\sum_{j=1}^{\infty} v(R_{j,i}) < \frac{\epsilon}{2^i}.$$

- Clearly,

$$A \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} R_{j,i}$$

  and

$$\begin{aligned} v(A) &\leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} v(R_{j,i}) < \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} \\ &= \frac{\epsilon}{2} \frac{1}{1-\frac{1}{2}} = \epsilon. \end{aligned}$$

- It follows that $A = \bigcup_{j=1}^{\infty} A_j$ has Lebesgue measure zero.

This finishes our argument. ■

- **Proposition** Let $F \subset \mathbb{R}$ be a set of measure zero and $H \subseteq F$.
  Then H has measure zero.

Proof. Let $\epsilon > 0$ be given.

- Since F has measure zero,
- There is a countable family of intervals $\{I_i\}$ in $\mathbb{R}$, such that,
- $F \subseteq \bigcup_{n=1}^{\infty} I_n$ and
$$\sum_{n=1}^{\infty} v(I_n) < \epsilon.$$
- Since
$$H \subseteq F \subseteq \bigcup_{n=1}^{\infty} I_n,$$
- Then H has also measure zero.

This finishes our proof. ■

- **Proposition** Let $F \subset \mathbb{R}$ be a set of measure zero.
  Then, for every $\epsilon > 0$, there is a countable family of intervals $\{I_j\}$ such that
$$F \subseteq \bigcup_{n=1}^{\infty} \operatorname{Int}(I_n)$$
  and
$$\sum_{n=1}^{\infty} v(I_n) < \epsilon.$$

Proof. Since $F \subset \mathbb{R}$ be a set of measure zero,

- There is a countable family of intervals $\{R_j\}$, such that
$$F \subseteq \bigcup_{n=1}^{\infty} R_n$$
  and
$$\sum_{n=1}^{\infty} v(R_n) < \frac{\epsilon}{2}.$$
- Consider interval
$$R_j = [a_j, b_j]$$
  and let
$$I_j = \left(a_j - \frac{\epsilon}{2^{j+2}}, b_j + \frac{\epsilon}{2^{j+2}}\right).$$
  Then
$$v(I_j) = v(R_j) + \frac{\epsilon}{2^{j+1}}.$$

- Clearly I_j is open,

$$F \subseteq \bigcup_{n=1}^{\infty} I_n$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} v(I_n) &= \sum_{n=1}^{\infty} \left(v(R_n) + \frac{\epsilon}{2^{n+1}} \right) \\ &= \sum_{n=1}^{\infty} v(R_n) + \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

- Since $I_j \subset \overline{I_j}$

$$F \subseteq \bigcup_{n=1}^{\infty} \overline{I_n}$$

and

$$v(I_j) = v\left(\overline{I_j}\right),$$

- Then

$$\sum_{n=1}^{\infty} v\left(\overline{I_n}\right) < \epsilon.$$

- Moreover,

$$\operatorname{Int}\left(\overline{I_j}\right) = I_j,$$

so

$$F \subseteq \bigcup_{n=1}^{\infty} \operatorname{Int}\left(\overline{I_n}\right).$$

- Therefore, for the family $\left\{\overline{I_j}\right\}$:

$$F \subseteq \bigcup_{n=1}^{\infty} \operatorname{Int}\left(\overline{I_n}\right) \text{ and } \sum_{n=1}^{\infty} v\left(\overline{I_n}\right) < \epsilon$$

as desired.

This finishes our proof. ■

- **Theorem** (*Lebesgue's Theorem*) Let $A \subseteq \mathbb{R}$ be bounded and $f : A \to \mathbb{R}$ be a bounded function. Define $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{if } x \in \mathbb{R} \backslash A \end{cases}.$$

Function f is Riemann integrable over A if and only if the set

$$D_{\widetilde{f}} = \left\{ x \in \mathbb{R} : \omega_{\widetilde{f}}(x) > 0 \right\}$$

of discontinuities of $\widetilde{f}$ has Lebesgue measure zero.

Proof. We give a proof of this theorem in M 4302. ■

- **Example** Let us consider the function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{1}{q} & if \quad x = \frac{p}{q},\ \gcd(p,q) = 1 \\ 0 & if \quad x \in [0,1] \cap (\mathbb{R} \backslash \mathbb{Q}) \end{cases}$$

  This function is known as *Riemann function.*

  We show that the Riemann function is Riemann integrable.

- Using theorem above,

  it suffices to show that the set of all points,

  where f is discontinuous has measure zero in $\mathbb{R}$.

- Recall, for a function

$$f : A \subseteq \mathbb{R} \to \mathbb{R}$$

  and $x_0 \in A$,

  define the oscillation of f on an open disk $D(x_0, \delta)$, $\delta > 0$ as follows

$$\omega_f(D(x_0,\delta)) = \sup\{|f(x) - f(y)| : x, y \in D(x_0,\delta) \cap A\}$$

  and

$$\omega_f(x_0) = \inf\{\omega_f(D(x_0,\delta)) : \delta > 0\}.$$

- As we showed f is continuous at $x_0 \in A$ iff

$$\omega_f(x_0) = 0.$$

- Therefore, f is continuous at x_0 if and only if

$$x_0 \in [0,1] \cap (\mathbb{R} \backslash \mathbb{Q}).$$

- From the above, we see that, the set D_f of all points where f is discontinuous is

$$D_f = [0,1] \cap \mathbb{Q}.$$

- Since D_f is countable (the set of all rational numbers in $[0,1]$ is countable),

  D_f has measure zero.

- From the Lebesgue theorem it follows that f is Riemann integrable.

- In addition, one may also show that

$$\int_{[0,1]} f = 0.$$

  Corollary A bounded set $A \subset \mathbb{R}$ has volume iff ∂A has measure zero.

Proof. Exercise ■

- **Corollary** Let $A \subset \mathbb{R}$ be bounded and assume that A has volume.

  A bounded function $f : A \to \mathbb{R}$ with a countable number of discontinuities is Riemann integrable over A.

Proof. Exercise ■

- **Remark** We note that a subset $F \subset \mathbb{R}$ of measure zero cannot contain a non-trivial interval, i.e.

$$I = [a, b],$$

such that $a < b$.

Theorem Let $A \subset \mathbb{R}$ be bounded of Lebesgue measure zero, $f : A \to \mathbb{R}$ be a bounded and Riemann integrable over A. Then

$$\int_A f(x)\,dx = 0.$$

Proof. Let $I = [a, b]$ be an interval such that $A \subseteq I$ and $\widetilde{f} : I \to \mathbb{R}$

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \in I \backslash A \end{cases}$$

be an extension of f.

- Since f is bounded, so is $\widetilde{f}$.
- Hence there is $M \geq 0$, such that

$$\left|\widetilde{f}(x)\right| \leq M.$$

- Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of I.
- Since for all $x \in A$,

$$\widetilde{f}(x) \leq M \cdot \chi_A(x),$$

it follows that

$$L(f, P) = \sum_{i=1}^{n} m_i\left(\widetilde{f}\right)\Delta x_i \leq M \sum_{i=1}^{n} m_i(\chi_A)\,\Delta x_i.$$

- Suppose that

$$m_i(\chi_A) \neq 0,$$

for some i, and let $I_i = [x_{i-1}, x_i]$, $x_{i-1} < x_i$

- Since $m_i(\chi_A) \neq 0$, $I_i \subseteq A$, $v(I_i) > 0$, A does not have Lebesgue measure zero. This gives us a contradiction.
- It follows that, for all $i = 1, 2, ..., n$:

$$m_i(\chi_A) = 0.$$

so

$$L\left(\widetilde{f}, P\right) \leq M \sum_{i=1}^{n} m_i(\chi_A)\,\Delta x_i = 0.$$

- Now, we see that

$$M_i\left(\widetilde{f}\right) = -m_i\left(-\widetilde{f}\right), \text{ so}$$

$$\begin{aligned} U\left(\widetilde{f}, P\right) &= \sum_{i=1}^{n} M_i\left(\widetilde{f}\right)\Delta x_i \\ &= -\sum_{i=1}^{n} m_i\left(-\widetilde{f}\right)\Delta x_i \\ &= -L\left(-\widetilde{f}, P\right) \geq 0. \end{aligned}$$

- It follows that, for any partition Q of I :

$$\begin{aligned} U\left(\widetilde{f}, Q\right) &\geq 0 \geq L\left(\widetilde{f}, Q\right), \text{ so} \\ \overline{\int_A} f &\geq 0 \geq \underline{\int_A} f \end{aligned}$$

  and since f is Riemann integrable,

$$\overline{\int_A} f = \underline{\int_A} f,$$

  so

$$\int_A f = \underline{\int_A} f = 0.$$

This finishes our proof. ∎

- **Theorem** Let $A \subseteq \mathbb{R}$ be bounded, $f : A \to \mathbb{R}$ be bounded, Riemann integrable,

$$f(x) \geq 0,$$

  for all $x \in A$, and

$$\int_A f(x)\, dx = 0.$$

  Then the set

$$\{x \in A : f(x) \neq 0\}$$

  has Lebesgue measure zero.

Proof. Let

$$A_m = \left\{x \in A : f(x) > \frac{1}{m}\right\}$$

and $\epsilon > 0$ be given.

- Let $I = [a, b] \subseteq \mathbb{R}$ be an interval such that $A \subseteq I$.
- Define $\widetilde{f} : I \to \mathbb{R}$, by

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \in I \backslash A \end{cases}.$$

- Since

$$\int_A f(x)\,dx = 0,$$

and

$$\int_A f(x)\,dx = \overline{\int_a^b}\widetilde{f} = \inf\left\{U\left(\widetilde{f},P\right) : P \text{ is a partition of } I\right\},$$

there is a partition $P = \{x_0, x_1, ..., x_n\}$ of I, such that

$$0 \leq U\left(\widetilde{f},P\right) < \frac{\epsilon}{m}.$$

- Let $R_i = [x_{i-1}, x_i]$ be subinterval determined by P, such that

$$A_m \cap R_i \neq \emptyset.$$

- Since $mM_i\left(\widetilde{f}\right) > 1$, it follows that

$$\sum_{i=1}^{n} v(R_i) = \sum_{i=1}^{n} \Delta x_i \leq \sum_{i=1}^{n} mM_i\left(\widetilde{f}\right)\Delta x_i < m \cdot \frac{\epsilon}{m} = \epsilon.$$

- Moreover,

$$A_m \subseteq \bigcup_{i=1}^{n} R_i,$$

so A_m has volume zero (hence also Lebesgue measure zero).

- Furthermore, we notice that

$$S = \{x \in A : f(x) \neq 0\} = \bigcup_{n=1}^{\infty} A_n$$

- Then S is a countable union of sets of measure zero, so
S has measure zero.

This finishes our argument. ■