Chapter 1

Complex Numbers

1.1. Sums and Products

A *complex number* z is a quantity of the form

$$z = x + iy, \text{ where } x, y \in \mathbb{R} \text{ and } i = \sqrt{-1}; \quad (\text{i.e.} i \text{ satisfies } i^2 = -1).$$

The real numbers x and y are called the *real part* and the *imaginary part* of z, and are denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ respectively. Whenever we have a constant complex number, for example $z = 3 + i4$, we write $z = 3 + 4i$. In this case $\mathrm{Re}(z) = 3$ and $\mathrm{Im}(z) = 4$. The set of all complex numbers is denoted by $\mathbb{C}$, i.e.

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}.$$

Any real number $x = x + 0i$ is a complex number with $\mathrm{Im}(x) = 0$; i.e. $\mathbb{R} \subseteq \mathbb{C}$ and a complex number $z = 0 + yi$ with $\mathrm{Re}(z) = 0$ is called a *pure imaginary number*. To visualize a complex number $z = x + iy$, we plot the point (x, y) in the $xy-$plane. Whenever, we use $xy-$plane to visualize complex numbers, it is called the *complex plane*. In this case, the $x-$axis and the $y-$axis are called the *real axis* and the *imaginary axis* respectively.

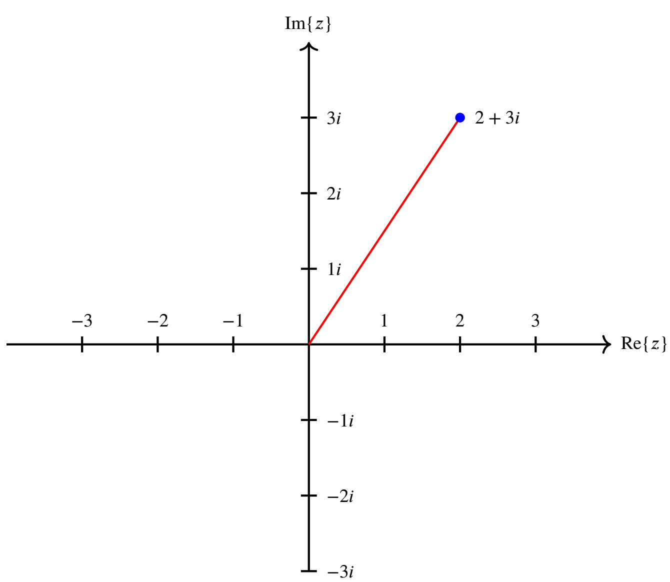


Figure 1.1: Complex Number $2 + 3i$ on an Argand Diagram

A complex number $z = x + iy$ is also written as $z = (x, y)$, the context will determine whether (x, y) represents an order pair or a complex number. So, the complex number $1 = (1, 0)$ and $i = (0, 1)$.

1.4 Vectors and Modulus :

The *modulus* of a complex number $z = x + iy$ is denoted by $|z|$ and is defined as

$$|z| = \sqrt{x^2 + y^2} = \sqrt{[\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2}.$$

Geometrically, the modulus of $z = x + iy$ represents the length of the line segment from the origin to the point (x, y).

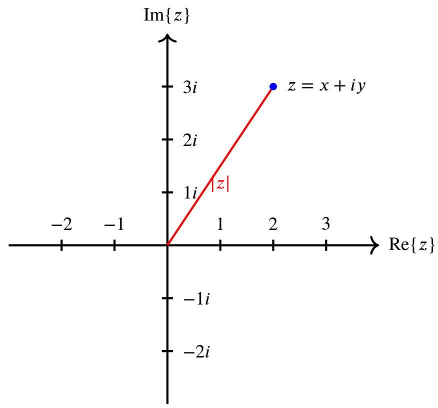


Figure 1.2: Complex Number z and its modulus $|z|$

Lets associate the complex number $z = x + iy$ with the vector $\langle x, y\rangle$, the directed line joining the origin and the point (x, y), then $|z|$ geometrically represents the norm of the vector $\langle x, y\rangle$.

1.6: Conjugate:

The *conjugate* of a complex number $z = x + iy$ is denoted by $\bar{z}$ and is defined as $\bar{z} = x - iy$. Geometrically, z and $\bar{z}$ are reflection of each other across the x−axis (the real axis).

For example: $\overline{2 + 3i} = 2 - 3i$, $\overline{6 - 2i} = 6 + 2i$, $\overline{2i} = -2i$, $\overline{6} = 6$.

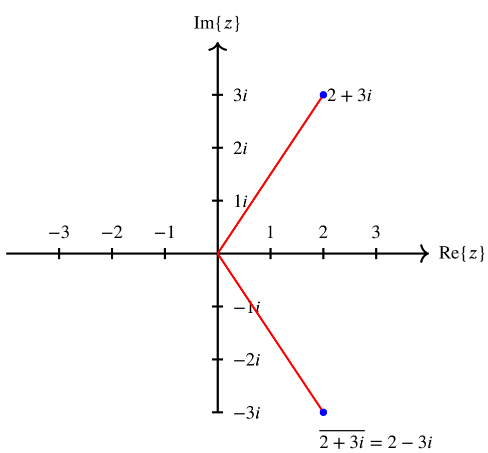


Figure 1.3: Complex number $2 + 3i$ and its conjugate $2 - 3i$

Algebra of Complex Numbers: Given two complex numbers $z = a + bi$ and $w = c + di$

1. Equality: $z = w \iff a = b$ and $c = d$.
2. Addition/Subtraction: $z \pm w = (a + bi) \pm (c + di) = (a \pm c) + (c \pm d)i$
3. Multiplication: $zw = (a+bi)(c+di) = ac+adi+cbi+bdi^2 = ac+(ad+bc)i+bd(-1) = (ac-bd)+(ad+bc)i$
4. Division: $\dfrac{z}{w} = \dfrac{z}{w} \cdot \dfrac{\overline{w}}{\overline{w}} = \dfrac{z\overline{w}}{|w|^2} = \dfrac{(a+bi)(c-di)}{\sqrt{c^2+d^2}} = \dfrac{(ac+bd)-i(ad-bc)}{\sqrt{c^2+d^2}} = \dfrac{ac+bd}{\sqrt{c^2+d^2}} - i\dfrac{ad-bc}{\sqrt{c^2+d^2}}$ provided $w \neq 0$.

Properties:

1. $z + w = w + z$ and $zw = wz$ for any $z, w \in \mathbb{C}$
2. $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1 z_2)z_3 = z_1(z_2 z_3)$ for any $z_1, z_2, z_3 \in \mathbb{C}$.
3. $\overline{(z \pm w)} = \overline{z} \pm \overline{w}$
4. $\overline{(zw)} = \overline{z}\overline{w}$
5. $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$
6. $\bar{\bar{z}} = z$
7. $z = \bar{z}$ iff $\operatorname{Im} z = 0$ and $z = -\bar{z}$ iff $\operatorname{Re} z = 0$
8. $z\overline{z} = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2 = |z|^2$
9. $\overline{\overline{z}} = z$
10. $z^{-1} = \dfrac{1}{z} = \dfrac{1}{z} \cdot \dfrac{\overline{z}}{\overline{z}} = \dfrac{\overline{z}}{|z|^2} = \dfrac{\overline{z}}{|z|^2}$ for any complex number $z \neq 0$.
11. $\operatorname{Re} z = \dfrac{z + \overline{z}}{2}$ because $z + \bar{z} = (x + iy) + (x - iy) = 2x = 2\operatorname{Re}(z)$
12. $\operatorname{Im} z = \dfrac{z - \overline{z}}{2i}$ because $z - \overline{z} = (x + iy) - (x - iy) = 2iy = 2i\operatorname{Im}(z)$.

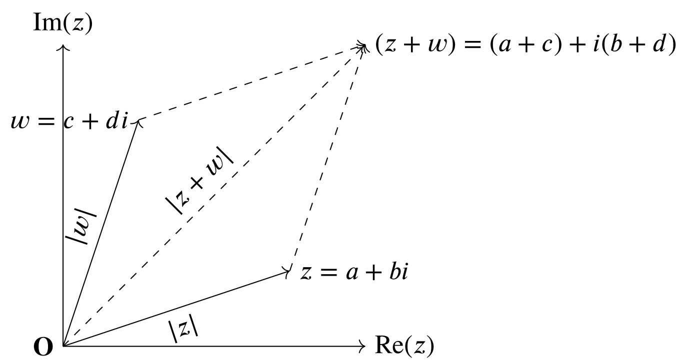
1.5. Triangle Inequality

Let $z = a + ib$ and $w = c + id$ be two complex numbers. Then the sum $z + w = (a + c) + i(b + d)$ represents the end point of the diagonal of the parallelogram with adjacent sides represented by z and w. Note that $z < w$ does not make sense for complex numbers but $|z| < |w|$ makes sense. Since

$$|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2$$

we have

$$\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z| \text{ and } \operatorname{Im} z \leq |\operatorname{Im} z| \leq |z|.$$



Since, no side of a triangle is longer than the sum of remaining two sides, for any complex numbers z and w, we have

$$|z + w| \leq |z| + |w| \tag{1.1}$$

This is called the *triangle inequality*. An immediate consequence of triangle inequality is

$$|z + w| \geq |z| - |w| \tag{1.2}$$

Since $|z| = |-z|$, we have

$$|z \pm w| \leq |z| + |w|$$

and

$$|z \pm w| \geq |z| - |w|$$

Combining above

$$||z| - |w|| \leq |z \pm w| \leq |z| + |w|. \tag{1.3}$$

The triangle inequality can be generalized to the sum of any finite number of complex numbers:

$$|z_1 + z_2 + \cdots \cdots + z_n| \leq |z_1| + |z_2| + \cdots \cdots + |z_n| \tag{1.4}$$

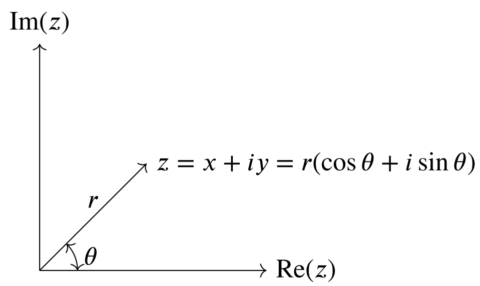
1.7: Exponential Form

Given complex number $z = x + iy$, let the cartesion point (x, y) be (r, θ) in polar coordinates, so that

$$x = r\cos\theta \text{ and } y = r\sin\theta,$$

then the complex number z can be written in *polar form*

$$z = r(\cos\theta + i\sin\theta) \tag{1.5}$$



Then $|z| = r$ and θ represents an angle made by z with the positive $x-$axis (the real axis). Of course, there are infinitely many possibilities for values of θ, including negative and determined by the equation $\tan\theta = \dfrac{y}{x}$. Each value of θ is called an *argument* of z and the set of all arguments of z is denoted by $\arg z$. The *principal value* of $\arg z$, denoted by $\mathrm{Arg}z$ is that unique value Θ such that $-\pi < \Theta \leq \pi$. Therefore

$$\arg z = \mathrm{Arg}z + 2n\pi; \quad n = 0, \pm1, \pm2, \cdots \tag{1.6}$$

For example, if $z = -1 - i$, then $\tan\theta = 1 \implies \arg z = \theta = -\dfrac{3\pi}{4} + 2n\pi$ and $\mathrm{Arg}z = \Theta = -\dfrac{3\pi}{4}$. Note that, it is also true that $\arg z = \theta = \dfrac{5\pi}{4} + 2n\pi; n = 0, \pm1, \pm2, \cdots\cdots$.

The symbol $e^{i\theta}$ is defined by the *Euler's Formula*

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where θ is measured in radians.
This notation allows us to write any complex number $z = x + iy = r(\cos\theta + i\sin\theta)$

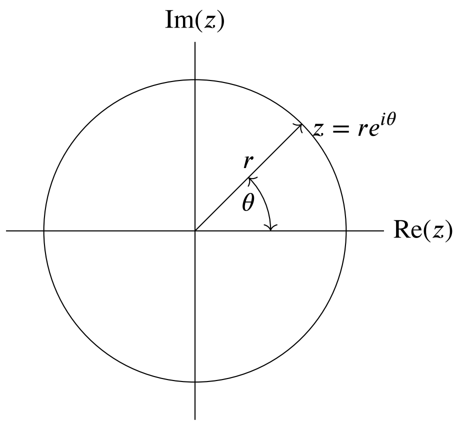
$$z = re^{i\theta}$$

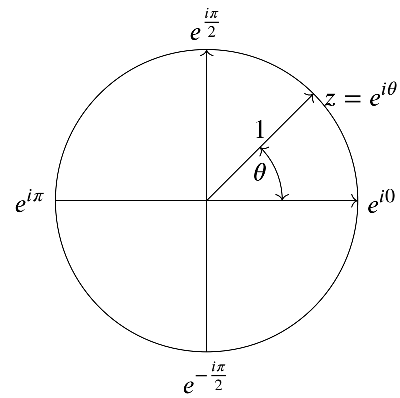
called an *exponential form*. For example, if $z = -1 - i$, then $r = \sqrt{2}$ and $\theta = -\dfrac{3\pi}{4}$. Therefore

$$z = -1 - i = \sqrt{2}e^{i\left(-\frac{3\pi}{4}\right)} = \sqrt{2}e^{i\left(-\frac{3\pi}{4}+2n\pi\right)}, \; n = 0, \pm1, \pm2, \pm3\cdots\cdots.$$

Here are some examples of complex numbers in exponential form:

- $e^{i\pi} = \cos\pi + i\sin\pi = -1.$
- $e^{i\frac{\pi}{2}} = \cos\left(\dfrac{\pi}{2}\right) + i\sin\left(\dfrac{\pi}{2}\right) = i.$





Note that the equation $z = Re^{i\theta},\ 0 \le \theta \le 2\pi$ satisfies $|z| = R$ and hence represents a circle of radius R centered at the origin. Similarly, the equation $z = z_0 + Re^{i\theta},\ 0 \le \theta \le 2\pi$ satisfies $|z - z_0| = R$ and hence represents a circle of radius R centered at z_0.

1.8: Products and Powers in Exponential Form:

1. Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ be two complex numbers, then

$$\begin{aligned}
z_1 z_2 &= \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) \\
&= (r_1 r_2) e^{i\theta_1} e^{i\theta_2} \\
&= (r_1 r_2)(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\
&= (r_1 r_2)(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2) \\
&= (r_1 r_2)\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\right] \\
&= (r_1 r_2)\left[\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right] \\
&= (r_1 r_2) e^{i(\theta_1 + \theta_2)}
\end{aligned}$$

2. Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ be two complex numbers with $z_2 \neq 0$, then

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} \\
&= \frac{r_1}{r_2}\frac{e^{i\theta_1}}{e^{i\theta_2}} \cdot \frac{e^{-i\theta_2}}{e^{-i\theta_2}} \\
&= \frac{r_1}{r_2}\frac{e^{i(\theta_1 - \theta_2)}}{e^{i0}} \\
&= \left(\frac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}
\end{aligned}$$

3. For any non-zero complex number $z = re^{i\theta}$, the inverse is

$$z^{-1} = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1e^{i0}}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}.$$

4. For any non-zero complex number $z = re^{i\theta}$ (can be proved by induction)

$$z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta}$$

For example, if $z = \sqrt{3} + i$, then $r = 2$ and $\theta = \dfrac{\pi}{6}$. Therefore

$$
\begin{aligned}
z^7 &= \left(\sqrt{3} + i\right)^7 \\
&= \left(2e^{i\frac{\pi}{6}}\right)^7 \\
&= 2^7 e^{i\frac{7\pi}{6}} \\
&= \left(2^6 e^{i\frac{6\pi}{6}}\right)\left(2e^{i\frac{\pi}{6}}\right) \\
&= \left(2^6 e^{i\pi}\right)\left(2e^{i\frac{\pi}{6}}\right) \\
&= 64(\cos\pi + i\sin\pi)\left(\sqrt{3} + i\right) \\
&= 64(-1)\left(\sqrt{3} + i\right) \\
&= -64\left(\sqrt{3} + i\right)
\end{aligned}
$$

Note: $\left(e^{i\theta}\right)^n = e^{in\theta}$ implies

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad n = 0, \pm 1, \pm 2, \cdots\cdots. \tag{1.7}$$

The formula in equation (1.7) is known as **DeMoivre's formula**.

1.9: Arguments of Products and Quotients

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$

which also tells us that

$$\arg(z_1 z_2) = (\theta_1 + \theta_2) = \arg(z_1) + \arg(z_2).$$

Now if θ_1 and θ_2 denote any values of $\arg z_1$ and $\arg z_2$ respectively, then the full set of $\arg(z_1 z_2)$ is given by

$$\arg(z_1 z_2) = (\theta_1 + \theta_2) + 2n\pi; \quad n = 0, \pm 1, \pm 2, \cdots\cdots.$$

If $z = re^{i\theta}$, then $\dfrac{1}{z} = z^{-1} = \dfrac{1}{r} e^{-i\theta}$. Therefore

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2).$$

Similar to above, if θ_1 and θ_2 denote any values of $\arg z_1$ and $\arg z_2$ respectively, then the full set of $\arg\left(\dfrac{z_1}{z_2}\right)$ is given by

$$\arg\left(\frac{z_1}{z_2}\right) = (\theta_1 - \theta_2) + 2n\pi; \quad n = 0, \pm 1, \pm 2, \cdots\cdots.$$

Example 1. Let $z = 1 - \sqrt{3}i$ and $w = \sqrt{3} + i$, find $\arg(zw)$, $\arg\left(\frac{z}{w}\right)$, $\operatorname{Arg}(zw)$, and $\operatorname{Arg}\left(\frac{z}{w}\right)$.

1.10. Roots of Complex Numbers

We know, two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are equal to each other if and only if

$$r_1 = r_2 \text{ and } \theta_1 = \theta_2 + 2k\pi, \quad \text{for some} k = 0, \pm 1, \pm 2, \cdots \cdots$$

Now, let $z = re^{i\theta}$ be an n^{th} root of complex number $z_0 = r_0 e^{i\theta_0}$, then $z^n = z_0$ which implies

$$r^n e^{in\theta} = r_0 e^{i\theta_0}.$$

Therefore,

$$r^n = r_0 \text{ and } n\theta = \theta_0 + 2k\pi; k = 0, \pm 1, \pm 2, \cdots$$

So,

$$r = \sqrt[n]{r_0} \text{ the positive } n^{\text{th}} \text{ root of } r$$

and

$$\theta = \frac{\theta_0}{n} + \frac{2k\pi}{n},\ k = 0, \pm 1, \pm 2, \cdots.$$

Therefore

$$z = \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right],\ k = 0, \pm 1, \pm 2, \cdots$$

We can see that the roots $\sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right]$, $k = 0, \pm 1, \pm 2, \cdots \cdots$ all lie on the circle $|z| = \sqrt[n]{r_0}$ and are equally spaced every $\frac{2\pi}{n}$ radians starting with the argument $\frac{\theta_0}{n}$ and all of the distinct roots are obtained by taking $k = 0, 1, 2, \cdots, (n-1)$. We use the notation c_k to denote these distinct roots and write

$$c_k = \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right],\ k = 0, 1, 2, \cdots \cdots, (n-1).$$

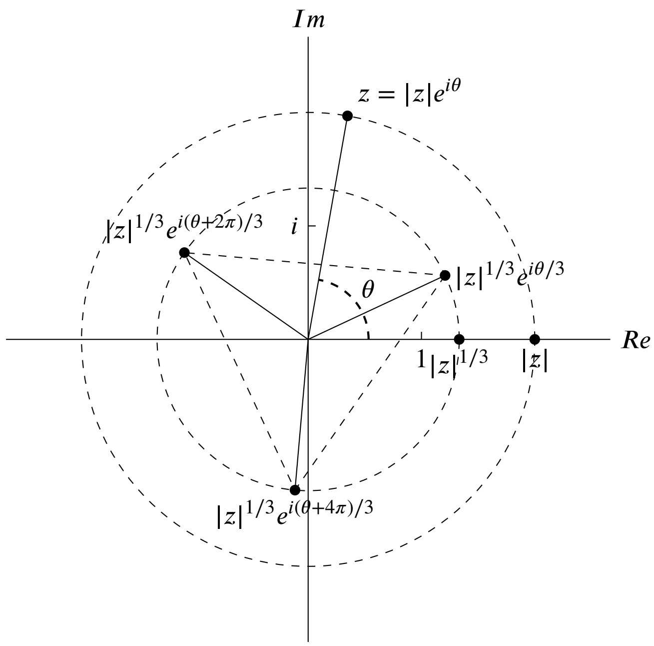
When $\theta_0 \in (-\pi, \pi]$ is the principal value of $\arg z_0$, then

$$c_0 = \sqrt[n]{r_0} \exp\left[\frac{i\theta_0}{n}\right]$$

is called the *principal n^{th} root* of z.

Let us write $\omega_n = \exp\left(i\frac{2\pi}{n}\right)$, so that $\omega_n^k = \exp\left(i\frac{2k\pi}{n}\right)$ $k = 0, 1, 2, \cdots \cdots, (n-1)$.
Since $\left(\omega_n^k\right)^n = \left[\exp\left(i\frac{2k\pi}{n}\right)\right]^n = \exp(i2k\pi) = 1$, we have that ω_n^k, $k = 0, 1, 2, \cdots, (n-1)$. are n^{th} roots of the complex number 1 and hence are called the *n^{th} roots of unity*. Thus we obtain

$$c_k = c_0 \omega_n^k,\ k = 0, 1, 2, \cdots \cdots, (n-1).$$



1.11: Examples

Example 2. Find all 4[th] roots of $-8 - 8\sqrt{3}i$.

Example 3. Find and visualize the n^{th} roots of unity when (a) $n = 3$ (b) $n = 4$ (c) $n = 5$

1.12: Regions in the Complex Plane

The set of all complex numbers z satisfying

$$|z - z_0| < \epsilon$$

is called an $\epsilon-$*neighborhood* of z_0. It consists of all complex numbers that lie inside the circle of radius ϵ centered at z_0 but excludes those on the circle.

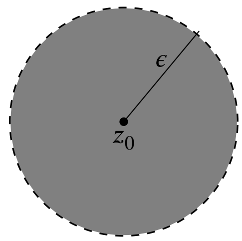


Figure 1.4: $\epsilon-$neighborhood of z_0

The set of all complex numbers z satisfying

$$0 < |z - z_0| < \epsilon$$

is called a *deleted* $\epsilon-$*neighborhood* of z_0. It consists of all complex numbers that lie inside the circle (excluding boundary) of radius ϵ centered at z_0 but excludes the center z_0 of the circle.

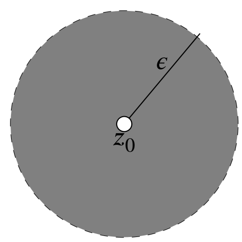


Figure 1.5: Deleted $\epsilon-$neighborhood of z_0

A point z_0 is said to be an *interior point* of a set S if there is a neighborhood of z_0 that lies completely inside S. If there is a neighborhood of z_0 containing no points of S, then z_0 is said to be an *exterior point* of S. If z_0 is neither an interior nor an exterior point of S, then it is called a *boundary point* of S. Each point on the circle $|z| = 1$ is a boundary point of each of the sets

$$|z| < 1 \text{ and } |z| \leq 1.$$

A set S is said to be an *open set* if it does not contain any of its boundary points. In other words, every point of S is its interior point. A set S is said to be *closed* if it contains all of its boundary points. A set containing all points of S and its boundary is called the *closure* of S. The set $S = \{z \in \mathbb{C} : |z| < 1\}$ is open and the set $T = \{z \in \mathbb{C} : |z| \leq 1\}$ is closed. Also, the set T is the closure of S. However the set $A = \{z \in \mathbb{C} : 0 < |z| \leq 1\}$ is neither open nor closed. An open set S is said to be *connected* if each pair of points z_1 and z_2 on it can be connected by finite number of line segments joined end to end, that lie entirely on S. The open set $|z| < 1$ and the annulus $1 < |z| < 2$ are connected.

A non-empty open set that is connected is called a *domain*. A domain together with some or all or none of its boundary points is called a *region.*

A set S is said to be *bounded* if can be contained inside some circle $|z| = R$ of finite radius R, otherwise it is called *unbounded*.

A point z_0 is said to be an *accumulation point* of set S if every deleted neighborhood of z_0 contains at least one point of S. Easy to see that an accumulation point is either an interior point or a boundary point of S. Now, suppose S is closed, then it contains all of its boundary. Therefore, a set S is closed if and only if it contains all of its accumulation points. Note that $(0, 0)$ is the only accumulation point of the set $\left\{ \frac{i}{n} : n = 1, 2, \cdots \cdots \right\}$.

Chapter 2: Analytic Functions

2.13: Functions and Mappings:

Let $S \subseteq \mathbb{C}$. A function f defined on S is a rule that assigns to each $z \in S$ a complex number w. The number w is called the value of f at z and we write $w = f(z)$. The set S is called the *domain of definition* (or domain) of f. When the domain of definition is not given, we assume that it is the largest subset of $\mathbb{C}$ where f is defined. For example the function $f(z) = \frac{1}{z}$ has domain $\mathbb{C} \setminus \{0\}$.

Example 1. Find the domain of (a) $f(z) = \dfrac{z}{z^2+9}$ (b) $f(z) = \dfrac{z}{z+\bar{z}}$

Let $w = u + iv$ be the value of the function f at $z = x + iy$ (i.e. $w = f(z)$), so that

$$u + iv = f(x + iy)$$

then each of the real numbers u and v depends on the real variables x and y, and therefore can be expressed as real valued functions of real variables as

$$u = u(x, y) \text{ and } v = v(x, y)$$

and $f(z)$ can be expressed as complex-valued function of the real variables x and y as

$$f(z) = u(x, y) + iv(x, y).$$

In polar coordinates

$$f(z) = f\left(re^{i\theta}\right) = u(r, \theta) + iv(r, \theta).$$

Example 2. If $f(z) = z^2$, then

$$f(z) = f(x+iy) = (x+iy)^2 = x^2 + 2x(iy) + (iy)^2 = \left(x^2 - y^2\right) + i(2xy)$$

So,

$$u(x,y) = x^2 - y^2 \text{ and } v(x,y) = 2xy.$$

When polar coordinates are used,

$$f(z) = f\left(re^{i\theta}\right) = \left(re^{i\theta}\right)^2 = r^2 e^{2i\theta} = r^2 \cos 2\theta + ir^2 \sin 2\theta$$

So,

$$u(r,\theta) = r^2 \cos 2\theta \text{ and } v(r,\theta) = r^2 \sin 2\theta.$$

Mappings: To graph a function $w = f(z)$, we generally draw two separate complex planes $z-$plane and $w-$plane. In such cases, the function is called a *mapping* or *transformation*. The *image* of z on the domain is the point $w = f(z)$ and the set of images of all points in the set T (subset of $z-$plane) is called the *image* of T.

The terms *translation, rotation,* and *reflection* are used to describe the geometric characteristics of mappings. In such cases, we can consider the z and w planes to be the same. The mapping $w = f(z) = z + 1 = (x+1) + iy$ translates each point z horizontally right by 1 unit and the mapping $w = f(z) = z + i = x + i(y+1)$ translates each point z vertically up by 1 unit.

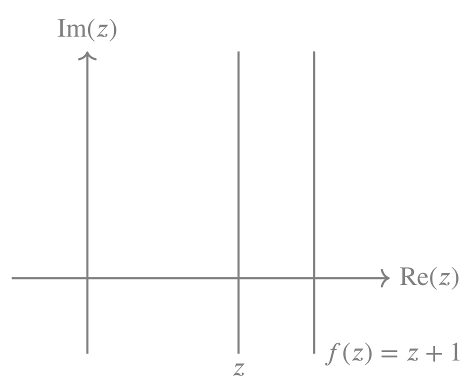


Figure 1: Translation $f(z) = z + 1$

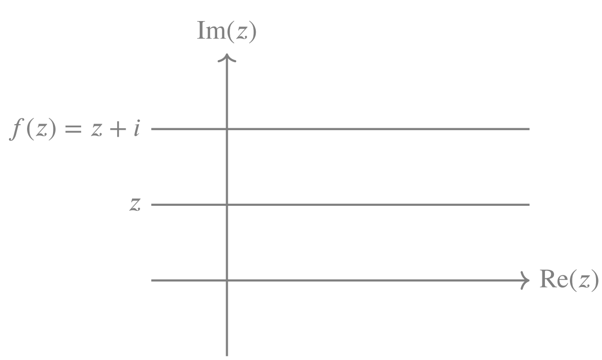


Figure 2: Translation $f(z) = z + i$

The mapping $w = f(z) = \overline{z} = x - iy$ reflects each point across the real axis.

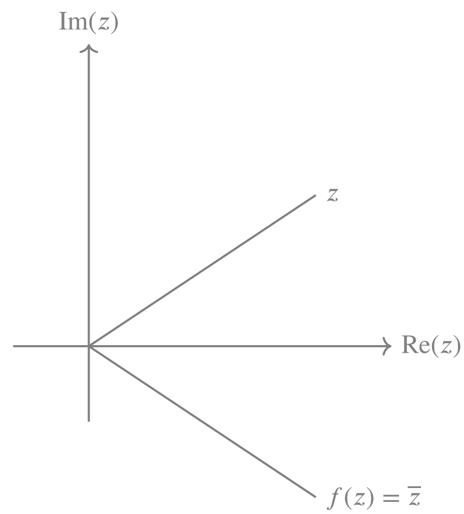


Figure 3: Reflection $f(z) = \overline{z}$

The function $f(z) = iz = i(x + iy) = -y + ix$ is rotation by angle $\frac{\pi}{2}$.

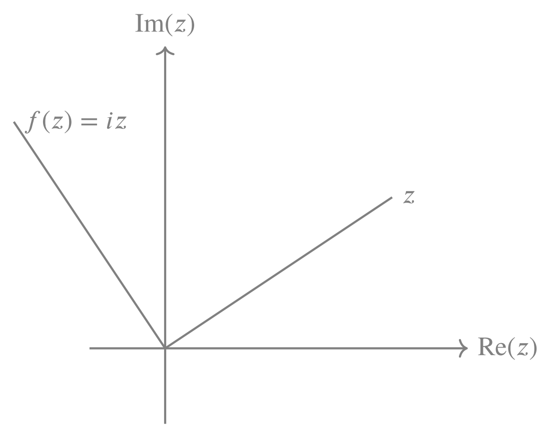


Figure 4: Rotation $f(z) = iz$

2.14: The mapping $w = f(z) = z^2$:

$w = f(z) = z^2 = \left(x^2 - y^2\right) + i(2xy) = u + iv$
Here $u = x^2 - y^2$ and $v = 2xy$. The branch of hyperbola $x^2 - y^2 = 1$ is mapped into

$$u = 1,\ v = \pm 2y\sqrt{1 + y^2}$$

which is a line. The image of right branch of the hyperbola can be can be parametrized as

$$u = 1, v = 2y\sqrt{1 + y^2}$$

As the point (x, y) moves upward on the right branch of the hyperbola, the point (u, v) moves upward on the line $u = 1$. Similarly, the image of the left branch of the hyperbola can be parametrized as

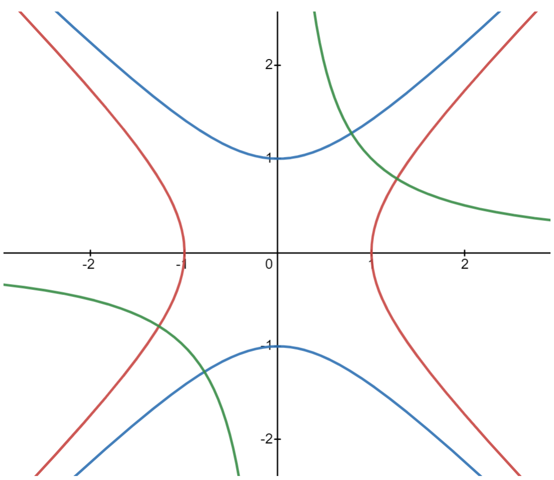
$$u = 1, v = -2y\sqrt{1 + y^2}$$

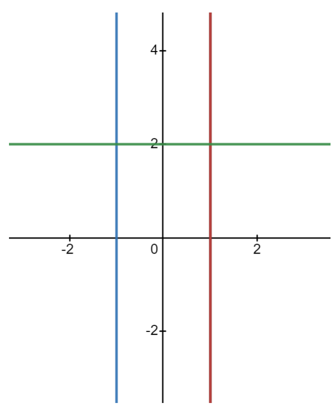
and the point (u, v) moves upward as the point (x, y) moves downward on the hyperbola. Each branch of hyperbola $xy = 1$ is transformed into line

$$v = 2, u = x^2 - \frac{1}{x^2}$$

As (x, y) moves upward on the upper branch of the hyperbola, the point (u, v) moves to the right on the line. Similarly, as (x, y) moves up along the lower branch of the hyperbola, the point (u, v) moves to the right on the line.

The images of hyperbolas: $x^2 - y^2 = 1,\ x^2 - y^2 = -1$ and $xy = 1$ are shown in the figure below.





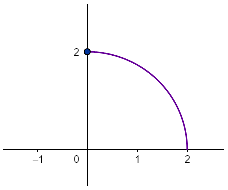
The mapping $w = z^2$ in polar coordinates becomes

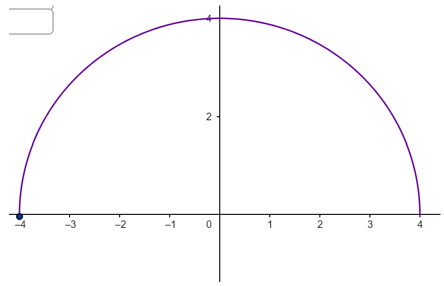
$$w = r^2 e^{i2\theta}$$

This means the image $w = \rho e^{i\phi}$ of any non-zero complex number z is found by squaring the modulus $r = |z|$ and the doubling the argument.

$$\rho = r^2 \text{ and } \phi = 2\theta$$

So, the unit circle centered at origin will be mapped into 2 copies of unit circle. Similarly, a circle $|z| = r$ will be mapped into 2 copies of the circle $|z| = r^2$. A quarter circle $z = 2e^{i\theta}, 0 \leq \theta \leq \dfrac{\pi}{2}$ will be mapped into semi-circle $w = 4e^{i\theta}, 0 \leq \theta\pi$.





2.15 Limits:

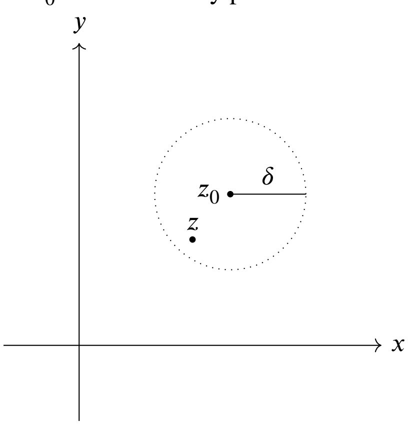
Let f be defined in a deleted neighbourhood of z_0. The function f is said to have the limit w_0 as z approaches to z_0 and we write

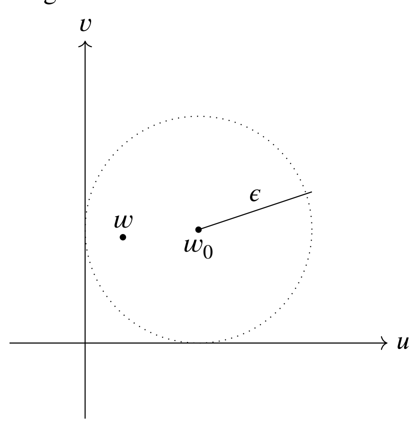
$$\lim_{z \to z_0} f(z) = w_0$$

if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta \tag{1}$$

In other words, for each $\epsilon-$neighborhood $|w - w_0| < \epsilon$ of w_0, there exists a deleted $\delta-$neighborhood $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w in the $\epsilon-$neighborhood.





Theorem 1. Let $\lim_{z\to z_0} f(z) = w_1$ and $\lim_{z\to z_0} f(z) = w_2$, then $w_1 = w_2$. In other words the limit is unique.

Proof. Given $\epsilon > 0$, there exist $\delta_1 > 0, \delta_2 > 0$ such that

$$|f(z) - w_1| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta_1 \tag{2}$$

and

$$|f(z) - w_2| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta_2 \tag{3}$$

Choose $\delta = \min(\delta_1, \delta_2)$, then whenever $0 < |z - z_0| < \delta$, we have,

$$|w_1 - w_2| = |f(z) - w_2 - (f(z) - w_1)| \leq |f(z) - w_2| + |f(z) - w_1| < \epsilon + \epsilon = 2\epsilon.$$

Since this is true for an arbitrary $\epsilon > 0$, we have $|w_1 - w_2| = 0 \implies w_1 = w_2$.

□

Example 3. Verify that: (a) $\lim_{z\to 1}(1 + iz) = 1 + i$ (b) $\lim_{z\to 0} \frac{z}{\bar{z}}$ does not exist.

2.16. Theorems on Limits

Theorem 2. Suppose that

$$z = x + iy \text{ and } f(z) = u(x, y) + iv(x, y)$$

and

$$z_0 = x_0 + iy_0 \text{ and } w_0 = u_0 + iv_0.$$

Then

$$\lim_{z \to z_0} f(z) = w_0$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$$

Theorem 3. Suppose

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} F(z) = W_0$$

then

(a) $\lim_{z \to z_0} [f(z) \pm F(z)] = w_0 \pm W_0$

(b) $\lim_{z \to z_0} [f(z) \cdot F(z)] = w_0 \cdot W_0$

(c) $\lim_{z \to z_0} \left[\frac{f(z)}{F(z)}\right] = \frac{w_0}{W_0}$ if $W_0 \neq 0$

Notes:

1. $\lim_{z \to z_0} c = c$ for any constant complex number c.
2. $\lim_{z \to z_0} z = z_0$.
3. $\lim_{z \to z_0} z^n = z_0^n$
4. $\lim_{z \to z_0} P(z) = P(z_0)$, for any polynomial $P(z)$ of complex variable z.

Remark: As we can see in above theorems and notes, the properties of limit of a function of complex variable are similar to those of function of real variable.

Example 4. Evaluate the following limits

(a) $\lim\limits_{z \to 1+i} \dfrac{z^2 - i}{z^2 - 3z + 1}$

(b) $\lim\limits_{z \to i} \dfrac{z^2 + 1}{z - i}$