Math 4301 Mathematical Analysis I

Lecture 15

Topic: Riemann Integral

- **Definition of Riemann Integral**
- Let $f : [a, b] \to \mathbb{R}$ be a **bounded function**
- $P = \{x_0, x_1, ..., x_n\}$, where $a = x_0 < x_1 < ... < x_n = b$ – a partition of $[a, b]$.
- $\mathcal{P}([a, b])$ – the set of all partitions of $[a, b]$.
- Points x_i's determine subintervals $[x_{i-1}, x_i]$, for all $i = 1, 2, ..., n$.
- For each subinterval $[x_{i-1}, x_i]$, we denote its length by

$$\Delta x_i = x_i - x_{i-1}$$

- Define

$$\begin{aligned} m_i(f) &= \inf\{f(x) : x \in [x_{i-1}, x_i]\} \text{ and} \\ M_i(f) &= \sup\{f(x) : x \in [x_{i-1}, x_i]\}. \end{aligned}$$

NOTE: Since f is bounded then the numbers $m_i(f)$ and $M_i(f)$ exist.

- Define

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} m_i(f)\,\Delta x_i \text{ and} \\ U(f, P) &= \sum_{i=1}^{n} M_i(f)\,\Delta x_i \end{aligned}$$

and call them the *lower and the upper Darboux sums* with respect to the partition P.

Lower and the upper sums satisfy the following properties:

- **Lemma** Let $f : [a, b] \to \mathbb{R}$ be bounded then

i) $L(f, P) \leq U(f, P)$, for all $P \in \mathcal{P}([a, b])$.

ii) If $c \in [a, b] \setminus P$ and $Q = P \cup \{c\}$ then

$$L(f, P) \leq L(f, Q) \text{ and } U(f, Q) \leq U(f, P).$$

iii) Let $P, Q \in \mathcal{P}([a, b])$, then

$$L(f, P) \leq U(f, Q).$$

Proof. Exercise. ■

- We see that if $P = \{a, b\}$ and $Q \in \mathcal{P}([a, b])$, then

$$\inf\{f(x) : x \in [a, b]\}(b - a) = L(f, P) \leq U(f, Q)$$

- Therefore, the set

$$\{U(f,Q) : Q \in \mathcal{P}([a,b])\}$$

is bounded below and since

$$L(f,Q) \leq U(f,P) = \sup\{f(x) : x \in [a,b]\}(b-a)$$

is bounded above.

- Therefore, both numbers exist

$$\begin{aligned} \underline{\int_a^b} f &= s(f) = \sup\{L(f,P) : P \in \mathcal{P}([a,b])\} \\ \overline{\int_a^b} f &= S(f) = \inf\{U(f,P) : P \in \mathcal{P}([a,b])\} \end{aligned}$$

and we call them *lower* and *upper Darboux integrals*.

- Since

$$L(f,P) \leq U(f,Q),$$

for all $P, Q \in \mathcal{P}([a,b])$, then

$$\underline{\int_a^b} f = s(f) = \sup\{L(f,P) : P \in \mathcal{P}([a,b])\} \leq U(f,Q),$$

hence

$$\begin{aligned} \underline{\int_a^b} f &= s(f) = \sup\{L(f,P) : P \in \mathcal{P}([a,b])\} \\ &\leq \inf\{U(f,P) : P \in \mathcal{P}([a,b])\} = S(f) = \overline{\int_a^b} f. \end{aligned}$$

- Consequently,

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

NOTE: The above inequality can also be sharp.

Example: Let $f : [a,b] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{Q} \cap [a,b] \\ 1 & \text{if} \quad x \in (\mathbb{R} \backslash \mathbb{Q}) \cap [a,b] \end{cases}$$

and let

$$P \in \mathcal{P}([a,b]), \ P = \{x_0, x_1, ..., x_n\},$$

then

$$\begin{aligned} m_i(f) &= \inf\{f(x) : x \in [x_{i-1}, x_i]\} = 0 \\ M_i(f) &= \sup\{f(x) : x \in [x_{i-1}, x_i]\} = 1. \end{aligned}$$

- Consequently,

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{n} m_i(f)\,\Delta x_i = 0 \text{ and} \\ U(f,P) &= \sum_{i=1}^{n} M_i(f)\,\Delta x_i = \sum_{i=1}^{n} \Delta x_i = b-a > 0 \end{aligned}$$

- It follows that

$$\begin{aligned} \underline{\int_a^b} f &= \sup\{L(f,P) : P \in \mathcal{P}([a,b])\} \\ &= \sup\{0\} = 0 \text{ and} \\ \overline{\int_a^b} f &= \inf\{U(f,P) : P \in \mathcal{P}([a,b])\} \\ &= \inf\{b-a\} = b-a > 0, \end{aligned}$$

hence

$$\underline{\int_a^b} f < \overline{\int_a^b} f.$$

We define Riemann intergarbility for a bounded function as follows:

Definition Let $f : [a,b] \to \mathbb{R}$ be bounded.

We say that f is *Riemann integrable* over $[a,b]$ if

$$\overline{\int_a^b} f \leq \underline{\int_a^b} f.$$

We call the number

$$\int_a^b f = \overline{\int_a^b} f = \underline{\int_a^b} f$$

the *Riemann integral* of f over $[a,b]$.

- We see that the function $f : [a,b] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{Q} \cap [a,b] \\ 1 & \text{if} \quad x \in (\mathbb{R} \backslash \mathbb{Q}) \cap [a,b] \end{cases}$$

is not Riemann integrable over $[a,b]$ since $\underline{\int_a^b} f < \overline{\int_a^b} f$.

Theorem A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable over $[a,b]$ if and only if, for every $\epsilon > 0$, there is $P \in \mathcal{P}([a,b])$, such that

$$U(f,P) - L(f,P) < \epsilon.$$

Proof. Assume that f is Riemann integrable.

- By the definition

$$S(f) = s(f),$$

where $S(f)$ and $s(f)$ denote the upper and the lower Darboux integral respectively.

- Take $\epsilon > 0$.
- Since
$$s(f) = \sup \{L(f,Q) \mid Q \in \mathcal{P}([a,b])\},$$
it follows
$$s(f) - \epsilon/2$$
is not an upper bound, so
there is $Q_1 \in \mathcal{P}([a,b])$, such that
$$s(f) - \epsilon/2 < L(f, Q_1).$$
- Analogously, since
$$S(f) = \inf \{U(f,Q) \mid Q \in \mathcal{P}([a,b])\},$$
then
$$S(f) + \epsilon/2$$
is not a lower bound, so
there is $Q_2 \in \mathcal{P}([a,b])$, such that
$$U(f, Q_2) < S(f) + \epsilon/2.$$
- Let $P = Q_1 \cup Q_2$, then
$$Q_i \subseteq P, i = 1, 2,$$
so
$$\begin{aligned} U(f,P) &\leq U(f,Q_2) \text{ and} \\ L(f,P) &\geq L(f,Q_1). \end{aligned}$$
- Since f is Riemann integrable,
$$S(f) = s(f),$$
it follows that
$$\begin{aligned} U(f,P) - L(f,P) &\leq U(f,Q_2) - L(f,Q_1) \\ &< (S(f) + \epsilon/2) - (s(f) - \epsilon/2) \\ &= (S(f) - s(f)) + \epsilon = \epsilon. \end{aligned}$$
- We showed that if f is Riemann integrable then
for any $\epsilon > 0$, there is $P \in \mathcal{P}([a,b])$, such that
$$U(f,P) - L(f,P) < \epsilon.$$
- **Conversely**, notice that, for any $P \in \mathcal{P}([a,b])$,
$$L(f,P) \leq s(f) \text{ and } S(f) \leq U(f,P),$$
so
$$S(f) - s(f) \leq U(f,P) - L(f,P).$$

- Since for $\epsilon > 0$, there is $P \in \mathcal{P}([a,b])$, such that

$$U(f,P) - L(f,P) < \epsilon,$$

it follows that,

$$S(f) - s(f) \leq U(f,P) - L(f,P) < \epsilon$$

for every $\epsilon > 0$:

$$0 \leq S(f) - s(f) < \epsilon.$$

- Consequently,

$$S(f) - s(f) = 0, \text{ so } S(f) = s(f),$$

and we showed that f is Riemann integrable.

This finishes our proof. ■

- **Example** We show that $f : [a,b] \to \mathbb{R}$ given by

$$f(x) = x$$

is Riemann integrable.

- Let $U(f,P)$ and $L(f,P)$ be upper and lower Darboux sums.
- It is sufficient to show that for $\epsilon > 0$, there is a partition

$$P = \{x_0, x_1, ..., x_n\}$$

of $[a,b]$, such that

$$U(f,P) - L(f,P) < \epsilon.$$

- Let $\epsilon > 0$ be given.
- Take $n > \frac{(b-a)^2}{\epsilon}$ and define

$$x_i = a + \frac{b-a}{n} i, \ i = 0, 1, 2, ..., n$$

- Let

$$P_n = \{x_i \mid i = 1, 2, ..., n\}.$$

then

$$\Delta x_i = x_i - x_{i-1} = \frac{b-a}{n}$$

and since f is increasing on $[a,b]$

$$\begin{aligned} M_i(f) &= \sup\{x : x \in [x_{i-1}, x_i]\} = x_i = a + \frac{b-a}{n} i, \\ m_i(f) &= \inf\{x : x \in [x_{i-1}, x_i]\} = x_{i-1} = a + \frac{b-a}{n}(i-1) \end{aligned}$$

- Consequently,

$$\begin{aligned}
U(f,P_n)-L(f,P_n) &= \sum_{i=1}^{n}\left(M_i(f)-m_i(f)\right)\Delta x_i \\
&= \sum_{i=1}^{n}\left(a+\frac{b-a}{n}i-\left(a+\frac{b-a}{n}(i-1)\right)\right)\frac{1}{n}(b-a) \\
&= \sum_{i=1}^{n}\left(\frac{1}{n}(b-a)\right)\frac{1}{n}(b-a)=\frac{1}{n^2}(b-a)^2\sum_{i=1}^{n}1=\frac{1}{n^2}(b-a)^2 n \\
&= \frac{(b-a)^2}{n}<\frac{(b-a)^2}{\frac{(b-a)^2}{\epsilon}}=\epsilon.
\end{aligned}$$

- It follows that f is Riemann integrable over $[a,b]$.
- To find $\int_a^b x$, we need to compute

$$\begin{aligned}
U(f,P_n) &= \sum_{i=1}^{n}M_i(f)\Delta x_i=\sum_{i=1}^{n}\left(a+\frac{b-a}{n}i\right)\frac{1}{n}(b-a) \\
&= \frac{1}{n}(b-a)\sum_{i=1}^{n}\left(a+\frac{b-a}{n}i\right) \\
&= \frac{1}{n}(b-a)\left(a\underbrace{\sum_{i=1}^{n}1}_{n}+\frac{b-a}{n}\underbrace{\sum_{i=1}^{n}i}_{\frac{1}{2}n(n+1)}\right)=\frac{1}{n}(b-a)\left(na+\frac{b-a}{n}\frac{n(n+1)}{2}\right) \\
&= a(b-a)+\frac{(b-a)^2}{n^2}\frac{n(n+1)}{2}.
\end{aligned}$$

- Analogously, we compute

$$\begin{aligned}
L(f,P_n) &= \sum_{i=1}^{n}m_i(f)\Delta x_i=\sum_{i=1}^{n}\left(a+\frac{b-a}{n}(i-1)\right)\frac{1}{n}(b-a) \\
&= \frac{1}{n}(b-a)\sum_{i=1}^{n}\left(a+\frac{b-a}{n}(i-1)\right)=\frac{1}{n}(b-a)\left(na+\frac{b-a}{n}\frac{(n-1)n}{2}\right) \\
&= \frac{1}{n}(b-a)a+\frac{(b-a)^2}{n^2}\frac{n(n-1)}{2}.
\end{aligned}$$

- Since, for all $P\in\mathcal{P}([a,b])$,

$$\begin{aligned}
L(f,P) &\leq \sup\{L(f,Q):Q\in\mathcal{P}([a,b])\}, \text{ so in particular} \\
L(f,P_n) &\leq \underbrace{\sup\{L(f,Q):Q\in\mathcal{P}([a,b])\}}_{\underline{\int_a^b} f(x)dx}, \text{ for all } n\in\mathbb{N}
\end{aligned}$$

and

$$\begin{aligned}
\inf\{U(f,Q):Q\in\mathcal{P}([a,b])\} &\leq U(f,P), \text{ so in particular} \\
\underbrace{\inf\{U(f,Q):Q\in\mathcal{P}([a,b])\}}_{\overline{\int_a^b} f(x)dx} &\leq U(f,P_n), \text{ for all } n\in\mathbb{N}.
\end{aligned}$$

- Therefore, since $\lim_{n\to\infty} L(f, P_n)$ and $\lim_{n\to\infty} U(f, P_n)$ exist and

$$L(f, P_n) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq U(f, P_n),$$

  it follows that

$$\begin{aligned} \lim_{n\to\infty} L(f, P_n) &\leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq \lim_{n\to\infty} U(f, P_n), \text{ so} \\ a(b-a) + \frac{(b-a)^2}{2} &\leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq a(b-a) + \frac{(b-a)^2}{2}, \end{aligned}$$

- hence

$$\begin{aligned} \int_a^b x &= \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} U(f, P_n) \\ &= a(b-a) + \frac{(b-a)^2}{2} = \frac{1}{2}\left(b^2 - a^2\right). \end{aligned}$$

- Notice that using calculus

$$\int_a^b x dx = \frac{1}{2}\, x^2\Big|_a^b = \frac{1}{2}\left(b^2 - a^2\right)$$

  Proposition Let $f : [a, b] \to \mathbb{R}$ be monotone function $(a < b)$.
  Then f is Riemann integrable.

Proof. Clearly, if f is a constant function, then

- f is Riemann integrable since, $f(x) = k$, for all $x \in [a, b]$ and

$$U(f, P) = L(f, P)$$

  for any partition P of $[a, b]$.

- Therefore,

$$U(f, P) - L(f, P) = 0 < \epsilon$$

  for all $\epsilon > 0$.

- Assume that f is *not constant* and non-decreasing.
- Therefore,

$$f(a) < f(b).$$

- Let $\epsilon > 0$ be given and let

$$\delta = \frac{\epsilon}{f(b) - f(a)} > 0.$$

- Consider a partition

$$P = \{x_0, x_1, ..., x_n\}$$

  of $[a,\, b]$, such that

$$\Delta x_i = x_i - x_{i-1} < \delta,\ \ i \in \{1, 2, ..., n\}.$$

- Since f is *non-decreasing*,
  it follows that, for each $i = 1, 2, ..., n$:

$$m_i(f) = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\} = f(x_{i-1})$$

  and

$$M_i(f) = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} = f(x_i).$$

- Therefore,

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{i=1}^{n} M_i(f)\Delta x_i - \sum_{i=1}^{n} m_i(f)\Delta x_i \\
&= \sum_{i=1}^{n} (M_i(f) - m_i(f))\Delta x_i < \sum_{i=1}^{n} (M_i(f) - m_i(f))\delta \\
&= \delta \sum_{i=1}^{n} (M_i(f) - m_i(f)) = \delta \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\
&= \delta (f(x_1) - f(x_0) + f(x_2) - f(x_1) + ... + f(x_n) - f(x_{n-1})) \\
&= \delta (f(x_n) - f(x_0)) = \frac{\epsilon}{f(b) - f(a)} (f(b) - f(a)) = \epsilon.
\end{aligned}$$

- We showed that,
  for any $\epsilon > 0$, there is a partition $P \in \mathcal{P}([a,b])$, such that

$$U(f,P) - L(f,P) < \epsilon.$$

- It follows that whenever f is not constant and non-increasing,
  then f is Riemann integrable over $[a,b]$.

- Analogous argument works for when f is non-increasing.

This finishes our proof. ■

- **Theorem** Let $f : [a,b] \to \mathbb{R}$ be continuous.
  Then f is Riemann integrable.

Proof. Let $\epsilon > 0$ be given.

- We need to show that there is a partition

$$P = \{x_0, x_1, ..., x_n\}$$

  of $[a,b]$, such that

$$U(f,P) - L(f,P) < \epsilon.$$

- Since f is continuous on $[a,b]$ and $[a,b]$ is compact
  ($[a,b]$ is closed and bounded, so it is compact by Heine-Borel Theorem).

- Then by the theorem, f is *uniformly continuous.*

- Thus, there is $\delta > 0$, such that,
for all $x, y \in [a, b]$, if $|x - y| < \epsilon$, then

$$|f(x) - f(y)| < \frac{\epsilon}{b-a}.$$

- Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$, such that

$$\Delta x_i = x_i - x_{i-1} < \delta.$$

- Thus, for all $x, y \in [x_{i-1}, x_i]$,

$$|x - y| \le x_i - x_{i-1} = \Delta x_i < \delta.$$

- Hence,

$$|f(x) - f(y)| < \frac{\epsilon}{b-a}.$$

- Since f is continuous on $[x_{i-1}, x_i]$, by the *extreme value theorem*,
there are x_i^*, $y_i^* \in [x_{i-1}, x_i]$, such that

$$\begin{aligned} f(x_i^*) &= \sup\{f(x) : x \in [x_{i-1}, x_i]\} = M_i(f), \text{ and} \\ f(y_i^*) &= \inf\{f(x) : x \in [x_{i-1}, x_i]\} = m_i(f), \end{aligned}$$

respectively.

- Since, x_i^*, $y_i^* \in [x_{i-1}, x_i]$,
$|x_i^* - y_i^*| < \delta$, so

$$M_i(f) - m_i(f) = f(x_i^*) - f(y_i^*) = |f(x_i^*) - f(y_i^*)| < \frac{\epsilon}{b-a}.$$

- We see that

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{n} (M_i(f) - m_i(f)) \Delta x_i = \sum_{i=1}^{n} (f(x_i^*) - f(y_i^*)) \Delta x_i < \sum_{i=1}^{n} \frac{\epsilon}{b-a} \Delta x_i \\ &= \frac{\epsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \frac{\epsilon}{b-a} (\Delta x_1 + \Delta x_2 + ... + \Delta x_n) \\ &= \frac{\epsilon}{b-a} ((x_1 - x_0) + (x_2 - x_1) + ... + (x_n - x_{n-1})) \\ &= \frac{\epsilon}{b-a} (x_n - x_0) = \frac{\epsilon}{b-a} (b - a) = \epsilon. \end{aligned}$$

- We proved that, for any $\epsilon > 0$,
there is a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$, such that

$$U(f, P) - L(f, P) < \epsilon.$$

- It follows that f is Riemann integrable.

This finishes our proof. ■

- **Theorem** Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable over $[a, b]$ and $\alpha \in \mathbb{R}$. Then

i) αf is Riemann integrable and $\int_a^b \alpha f$ is Riemann integrable and

$$\int_a^b \alpha f = \alpha \int_a^b f$$

ii) $f \pm g$ is Riemann integrable and

$$\int_a^b f \pm g = \int_a^b f \pm \int_a^b g$$

iii) If $f \leq g$ then

$$\int_a^b f \leq \int_a^b g$$

iv) $|f|$ is Riemann integrable and

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

v) For any $c \in [a, b]$, f is Riemann integrable over $[a, c]$ and $[c, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof. We prove only 3) and 4).

- Since f and g are Riemann integrable

$$\begin{aligned} s(f) &= \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx \text{ and} \\ s(g) &= \underline{\int_a^b} g(x)\,dx = \int_a^b g(x)\,dx \end{aligned}$$

- For any partition $P = \{x_0, x_1, ..., x_n\}$, $i = 1, 2, ..., n$ and $x \in [x_{i-1}, x_i]$,

$$\inf \{f(x) : x \in [x_{i-1}, x_i]\} = m_i(f) \leq f(x) \leq g(x), \text{ so}$$

  $m_i(f)$ is a lower bound for the set

$$\{g(x) : x \in [x_{i-1}, x_i]\},$$

  hence

$$m_i(f) \leq m_i(g)$$

  since

$$m_i(g) = \inf \{g(x) : x \in [x_{i-1}, x_i]\}$$

  is the greatest lower bound for

$$\{g(x) : x \in [x_{i-1}, x_i]\}.$$

- It follows, that

$$L(f, P) = \sum_{i=1}^{n} m_i(f)\,\Delta x_i \leq \sum_{i=1}^{n} m_i(g)\,\Delta x_i = L(g, P).$$

- Since
$$L(g,P) \le \sup\{L(g,Q) : Q \in \mathcal{P}([a,b])\} = \underline{\int_a^b} g(x)\,dx$$
the number $\underline{\int_a^b} g(x)\,dx$ is an upper bound for the set
$$\{L(f,R) : R \in \mathcal{P}([a,b])\},$$
so
$$\underline{\int_a^b} f(x)\,dx = \sup\{L(f,R) : R \in \mathcal{P}([a,b])\} \le \underline{\int_a^b} g(x)\,dx,$$
since $\underline{\int_a^b} f(x)\,dx$ is the least upper bound of
$$\{L(f,R) : R \in \mathcal{P}([a,b])\}.$$
- We showed that
$$\underline{\int_a^b} f(x)\,dx \le \underline{\int_a^b} g(x)\,dx.$$
- Since f and g are Riemann integrable,
$$\begin{aligned}\int_a^b f(x)\,dx &= \underline{\int_a^b} f(x)\,dx \le \underline{\int_a^b} g(x)\,dx = \int_a^b g(x)\,dx, \text{ so}\\ \int_a^b f(x)\,dx &\le \int_a^b g(x)\,dx\end{aligned}$$
as claimed.

This finishes our proof for 3). ■

Proof. For 4), let $\epsilon > 0$ be given.

- Since f is Riemann integrable over $[a,b]$,
there is a partition
$$P = \{x_0, x_1, ..., x_n\}$$
of $[a,b]$, such that
$$U(f,P) - L(f,P) < \epsilon.$$
- Notice that, for all $i = 1,2,...,n$, and $x,y \in [x_{i-1}, x_i]$,
$$f(x),\ f(y) \in [m_i(f), M_i(f)],$$
we see that
$$|f(x) - f(y)| \le M_i(f) - m_i(f).$$
- Furthermore, by the *reverse triangle inequality*, i.e.
$$|a| - |b| \le ||a| - |b|| \le |a-b|,\ a,b \in \mathbb{R},$$
for all $x,y \in [x_{i-1}, x_i]$:
$$\begin{aligned}||f|(x) - |f|(y)| &= ||f(x)| - |f(y)|| \le |f(x) - f(y)|\\ &\le M_i(f) - m_i(f).\end{aligned}$$

- In particular, for all $x, y \in [x_{i-1}, x_i]$:

$$|f|(x) - |f|(y) \leq M_i(f) - m_i(f).$$

- Now, let $y \in [x_{i-1}, x_i]$,
  then for all $x \in [x_{i-1}, x_i]$:

$$\begin{aligned} |f|(x) &\leq M_i(f) - m_i(f) + |f|(y), \text{ so} \\ M_i(|f|) &= \sup\{|f|(x) : x \in [x_{i-1}, x_i]\} \\ &\leq M_i(f) - m_i(f) + |f|(y) \end{aligned}$$

- Since $M_i(|f|)$ is the least upper bound of

$$\{|f|(x) : x \in [x_{i-1}, x_i]\}.$$

- Since $y \in [x_{i-1}, x_i]$ is arbitrary, for all $y \in [x_{i-1}, x_i]$

$$\begin{aligned} M_i(|f|) - M_i(f) + m_i(f) &\leq |f|(y), \text{ so} \\ M_i(|f|) - M_i(f) + m_i(f) &\leq \inf\{|f|(y) : y \in [x_{i-1}, x_i]\} = m_i(|f|) \end{aligned}$$

  since $m_i(|f|)$ is the greatest lower bound for

$$\{|f|(y) : y \in [x_{i-1}, x_i]\}.$$

- Therefore,

$$M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f).$$

- Now, we see that

$$\begin{aligned} U(|f|, P) - L(|f|, P) &= \sum_{i=1}^{n} (M_i(|f|) - m_i(|f|)) \Delta x_i \\ &\leq \sum_{i=1}^{n} (M_i(f) - m_i(f)) \Delta x_i \\ &= U(f, P) - L(f, P) < \epsilon. \end{aligned}$$

- We showed that,
  for any $\epsilon > 0$, there is a partition P of $[a, b]$, such that

$$U(|f|, P) - L(|f|, P) < \epsilon,$$

  so by the theorem,

- $|f|$ is Riemann integrable over $[a, b]$.

This finishes our proof for 4). ■

- **Remark**: The **converse statement is not always true:**
- Consider the following function $f : [0, 1] \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ -1 & \text{if } x \in [0,1] \cap (\mathbb{R} \backslash \mathbb{Q}) \end{cases}.$$

- Clearly, for any partition P of $[0,1]$:

$$\begin{aligned} M_i(f,P) &= \sup\{f(x) : x \in [x_{i-1}, x_i]\} \\ &= 1,\ i = 1,2,...,n \end{aligned}$$

- Since each subinterval $[x_{i-1}, x_i]$ contains a rational number.
- Moreover, we see that

$$\begin{aligned} m_i(f,P) &= \inf\{f(x) : x \in [x_{i-1}, x_i]\} \\ &= -1,\ i = 1,2,...,n \end{aligned}$$

- Since every subinterval $[x_{i-1}, x_i]$ contains a irrational number.
- Therefore,

$$U(f,P) = \sum_{i=1}^{n} M_i(f,P)\,\Delta x_i = \sum_{i=1}^{n} 1\Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1$$

and

$$L(f,P) = \sum_{i=1}^{n} m_i(f,P)\,\Delta x_i = \sum_{i=1}^{n} (-1)\,\Delta x_i = -\sum_{i=1}^{n} \Delta x_i = -1$$

- Hence, we see that,

$$U(f,P) = 1 \text{ and } L(f,P) = -1,$$

for every partition P of $[0,1]$.
- Now, we see that

$$\begin{aligned} \overline{\int_0^1} f(x)\,dx &= \inf\{U(f,P) : P \text{ is a partition of } [0,1]\} \\ &= \inf\{1\} = 1 \end{aligned}$$

and

$$\begin{aligned} \underline{\int_0^1} f(x)\,dx &= \sup\{L(f,P) : P \text{ is a partition of } [0,1]\} \\ &= \sup\{-1\} = -1. \end{aligned}$$

- Consequently, we see that

$$\overline{\int_0^1} f(x)\,dx = 1 \neq -1 = \underline{\int_0^1} f(x)\,dx,$$

so f is not Riemann integrable.
- However, $|f| : [0,1] \to \mathbb{R}$, is the function defined by

$$|f|(x) = |f(x)| = 1,$$

for all $x \in [0,1]$.
- Clearly, $|f|$ is Riemann integrable over $[0,1]$.

- As we see, if $|f|$ is Riemann integrable, then f needs not to be Riemann integrable.

Proof. Finally, we show that

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

- As one can verify,
  since f is Riemann integrable,
  $|f|$ and $-|f|$ are also Riemann integrable[1].
- Furthermore, since for all $x \in [a,b]$,

$$-|f(x)| \le f(x) \le |f(x)|,$$

- By one of the properties of the Riemann integral[2] that

$$\begin{aligned} -\int_a^b |f(x)|\,dx &\le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx, \text{ so} \\ \left|\int_a^b f(x)\,dx\right| &\le \int_a^b |f(x)|\,dx. \end{aligned}$$

  Above, we applied inequality $-a \le x \le a$ iff $|x| \le a$.

This finishes our argument. ∎

[1] In Problem #12, we showed exactly that, if f is Riemann integrable, then $|f|$ is also Riemann integrable. Furthermore, since $|f|$ is Riemann integrable, then also its constant multiple $-|f|$ is Riemann integrable.

[2] As we showed in Problem #11, if f and g are Riemann integrable over $[a,b]$ and $f(x) \le g(x)$, for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.