MIDTERM 2 REVIEW – MATH 4341

1. Proofs of Theorems

(1) Let Y be a subset of a topological space X. Then
   (a) $\partial Y = X \setminus (\mathrm{Int}Y \cup \mathrm{Int}(X \setminus Y)) = \overline{Y} \cap \overline{X \setminus Y}$,
   (b) $\overline{Y} = Y \cup \partial Y$,
   (c) $\overline{Y} = Y \cup Y'$,
   (d) $x \in \overline{Y}$ if and only if every neighborhood of x intersects Y.

(2) A topological space X is T_1 if and only if $\{x\}$ is closed for all $x \in X$.

(3) If $\{y_n\}$ is a subsequence of $\{x_n\}$ and $x_n \to x$, then $y_n \to x$.

(4) Let (X, d) be a metric space with the metric topology. Then a sequence $\{x_n\}$ in X converges to $x \in X$ if and only if

$$\forall \epsilon > 0, \exists N > 0 : n > N \Rightarrow d(x_n, x) < \epsilon.$$

(5) Let X be Hausdorff. If $x_n \to x$ and $x_n \to y$ in X, then $x = y$.

(6) (The sequence lemma) Let $A \subset X$. If there is a sequence in A that converges to x then $x \in \bar{A}$. The converse holds if X is first-countable.

(7) Let X and Y be topological spaces. If $f : X \to Y$ be continuous, then $x_n \to x$ in X implies that $f(x_n) \to f(x)$ in Y. The converse holds if X is first-countable; that is, if $x_n \to x$ implies that $f(x_n) \to f(x)$ for all convergent sequences $\{x_n\}$, then f is continuous.

(8) Let $p = (0, 0, \ldots, 0, 1) \in S^n$ be the "north pole". Then $S^n \setminus \{p\} \simeq \mathbb{R}^n$.

(9) A topological space X is connected if and only if $\emptyset$ and X are the only subsets of X that are both open and closed.

(10) Let $X = U \cup V$ for disjoint open sets U and V, and let $Y \subset X$. If Y is connected, then $Y \subset U$ or $Y \subset V$.

(11) Let $\{A_i\}_{i \in I}$ be a collection of connected subspaces of a topological space X with a common point $x \in X$; i.e. $x \in A_i$ for all $i \in I$. Then $\bigcup_{i \in I} A_i$ is connected.

(12) Let $A \subset X$ be connected. If a subset $B \subset X$ satisfies $A \subset B \subset \bar{A}$, then B is also connected. In particular, $\bar{A}$ is connected whenever A is.

(13) Let $f : X \to Y$ be a continuous map between topological spaces. If X is connected, then $f(X)$ is also connected.

(14) (Intermediate value theorem) Let $f : X \to \mathbb{R}$ be continuous and assume that X is connected. If there is an $r \in \mathbb{R}$ and $x, y \in X$ so that $f(x) < r < f(y)$, then there is a $z \in X$ with $f(z) = r$.

(15) Let $X_1, \cdots, X_n$ be topological spaces. Then $X_1 \times \cdots \times X_n$ is connected if and only if every X_i is.

(16) $\mathbb{R}$ is connected.

(17) S^n is connected for all $n \geq 1$.

(18) A path-connected space is connected.

(19) Let $\{C_i\}_{i \in I}$ be the set of connected components of a topological space X. Then
   (a) $X = \bigcup_{i \in I} C_i$ and the C_i are pairwise disjoint,

(b) if $Y \subset X$ is connected, then $Y \subset C_i$ for some $i \in I$,
(c) $C_i \subset X$ is connected for each $i \in I$, and
(d) C_i is closed for all $i \in I$.

2. Problems

(1) Examples in lecture notes.
(2) Homework 5, 6, 7, 8.