Math 4302 Mathematical Analysis I

Lecture 2

Topic: Completeness & Real Numbers

Recall, the least upper bound property for an ordered field $\mathbb{F}$:

- **The least upper bound property (LUB)**

  Every nonempty and bounded above subset $S \subseteq F$ has the least upper bound, that is,

  there is $\beta \in \mathbb{F}$, such that
$$\beta = \sup S.$$

- **Definition** An ordered field $\mathbb{F}$ is called *complete* it satisfies the least upper bound property.

- We proved last time:

  Theorem *Every complete ordered field F is Archimedean.*

- We show later that
$$S = \left\{x \in \mathbb{Q} : 0 < x < \sqrt{2}\right\} \subset \mathbb{Q}$$
  is non-empty and bounded, but it has no least upper bound in $\mathbb{Q}$.

- Therefore, $\mathbb{Q}$ is not complete.

- Note that $\mathbb{Q}$ is Archimedean field.

The following theorem tells us that an ordered field $\mathbb{F}$ that is complete is unique.

- **Theorem** *There exists a unique (up to an isomorphism of ordered fields) a complete ordered field called the field of real numbers and we denote it by $\mathbb{R}$.*

Proof. See any textbook with a construction of $\mathbb{R}$. ■

- **Proposition** *Let F be a complete ordered field.*

  Then every nonempty and bounded below subset $S \subseteq F$ has the greatest lower bound.

  That is, there is $\alpha \in F$, such that
$$\alpha = \inf S.$$
  In fact $\alpha = -\sup(-S)$, where $-S = \{-x \in \mathbb{F} \ : \ x \in S\}$.

Proof. Since S is bounded below,

- there is $m \in \mathbb{F}$ such that
$$m \leq x, \text{ for all } x \in S.$$

- Thus,
$$-x \leq -m, \text{ for all } x \in S.$$

- Therefore, $y \leq -m$, for all $y \in (-S)$,

  i.e. $-m$ is an upper bound of $-S$.

- Since $x \in S$ iff $-x \in (-S)$ and $S \neq \emptyset$,
$$-S \neq \emptyset$$

- Therefore,
  $-S$ is non-empty and bounded above.
- Since $\mathbb{F}$ is complete, there is $\beta \in \mathbb{F}$ such that

$$\beta = \sup(-S).$$

- Since, for all $y \in (-S)$,

$$y \leq \beta,$$

  then for all $y \in (-S)$,

$$\begin{aligned} -\beta &\leq -y, \text{ so} \\ -\beta &\leq x, \end{aligned}$$

  for all $x \in S = -(-S)$,
  i.e. $-\beta$ is a lower bound of S.
- Since $\beta = \sup(-S)$,
  if $\epsilon > 0$, then

$$\begin{aligned} \beta - \epsilon &< x, \text{ for some } x \in (-S), \text{ thus} \\ -x &< -\beta + \epsilon \end{aligned}$$

  Since $-x \in S$, there is $y = -x$, such that

$$\begin{aligned} y &< -\beta + \epsilon, \\ &\quad \text{i.e. } -\beta + \epsilon \text{ is not a lower bound for } S, \text{ so} \\ -\beta &= \inf(S), \text{ i.e.} \\ -\sup(-S) &= \inf(S). \end{aligned}$$

- Hence, $\mathbb{F}$ has the greatest lower bound property.

This finishes our proof. ∎

- **Remark** We observe that we can define a *complete ordered field* as a field $\mathbb{F}$ in which *every nonempty and bounded below subset $S \subseteq F$ has the greatest lower bound*,
  i.e. there is $\alpha \in \mathbb{F}$, such that $\alpha = \inf S$.
  Exercise: Shows that

$$\sup(S) = -\inf(-S),$$

  where $-S = \{-x \in \mathbb{F} \ : \ x \in S\}$.
  Exercise: Let

$$S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \subset \mathbb{R}.$$

  Show that $\inf S = 0$.
- Since $n > 0$, then $\frac{1}{n} > 0$, so for all $x \in S$,

$$0 \leq x,$$

  i.e. 0 is a lower bound for S.
- We show that 0 is the greatest lower bound of S.

- Indeed, let $\epsilon > 0$.
- Since $\mathbb{R}$ is *Archimedean* (every complete ordered field is *Archimedean*),
- by theorem since $\epsilon > 0$, there is $n \in \mathbb{N}$, such that

$$0 < \frac{1}{n} < \epsilon = \epsilon + 0 \text{ and } \frac{1}{n} \in S,$$

- We showed that:
  For every $\epsilon > 0$, there is $x \in S$, such that,

$$x < 0 + \epsilon,$$

- Consequently $0 + \epsilon$ is not a lower bound.
- It follows that

$$0 = \inf S.$$

Exercise Let $a < b$ and

$$S = (a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

Show that $b = \sup S$.

- For every $x \in S$,

$$x \leq b,$$

  so b is an upper bound of S.
- Now, we show that b is the least upper bound for S, i.e.
  we show that, for any $\epsilon > 0$

$$b - \epsilon$$

  is not an upper bound of S.
- Let

$$x = \max\left\{b - \frac{\epsilon}{2}, \frac{a+b}{2}\right\}.$$

- Since $a < b$

$$a = \frac{a+a}{2} < \frac{a+b}{2} < \frac{b+b}{2} = b,$$

  so

$$\frac{a+b}{2} \in S.$$

- Therefore,

$$a < \frac{a+b}{2} \leq \max\left\{b - \frac{\epsilon}{2}, \frac{a+b}{2}\right\} = x$$

- Since $b - \frac{\epsilon}{2} < b$ and $\frac{a+b}{2} < b$,

$$x = \max\left\{b - \frac{\epsilon}{2}, \frac{a+b}{2}\right\} < b.$$

- It follows that

$$a < x < b, \text{ so } x \in S.$$

- Since
$$b-\epsilon < b-\frac{\epsilon}{2} \leq \max\left\{b-\frac{\epsilon}{2}, \frac{a+b}{2}\right\} = x,$$

- it follows that
$$b-\epsilon < x \text{ for some } x \in S$$

- Therefore, $b-\epsilon$ is not an upper bound of S.

- It follows that, b is the least upper bound of S, i.e.
$$b = \sup S.$$

Exercise Let $a < b$ and
$$S = (a,b) = \{x \in \mathbb{R} : a < x < b\}.$$

Show that
$$a = \inf S.$$

Exercise Suppose that $A \subseteq B \subseteq \mathbb{R}$, $A \neq \emptyset$ and B is bounded.
Show that
$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

Properties Of Real Numbers

Proposition $\mathbb{Q} \subset \mathbb{R}$ is dense in $\mathbb{R}$.

That is,

a. If $x, y \in \mathbb{R}$ and $x < y$, then there is an $r \in \mathbb{Q}$, such that
$$x < r < y.$$

b. If $x \in \mathbb{R}$, $\epsilon > 0$, then there is an $r \in \mathbb{Q}$ with
$$|x-r| < \epsilon.$$

Proof. Suppose that $0 < x < y$.

- Since $(y-x) > 0$, so by the *Archemedean property* of $\mathbb{R}$, there is $n \in \mathbb{N}$, such that
$$0 < \frac{1}{n} < (y-x).$$

- By the Archemedaean property of $\mathbb{R}$, there is $m \in \mathbb{N}$, such that
$$nx < m,$$
that is,
$$x < \frac{m}{n}.$$

- Define
$$S = \left\{m \in \mathbb{N} : x < \frac{m}{n}\right\} \subseteq \mathbb{N}.$$

- Since $S \neq \emptyset$ and $\mathbb{N}$ is well-ordered,
  there is
  $$k = \min S.$$
- Notice that
  $$\frac{k-1}{n} \leq x < \frac{k}{n}.$$
- Therefore,
  $$\begin{aligned} x &< \frac{k}{n} = \frac{(k-1)+1}{n} \\ &= \frac{k-1}{n} + \frac{1}{n} \\ &\leq x + \frac{1}{n} \\ &< x + (y-x) \\ &= y. \end{aligned}$$
- We showed that for $0 < x < y$,
  there is $\frac{k}{n} \in \mathbb{Q}$, such that:
  $$x < \frac{k}{n} < y.$$
- If $x < 0 < y$, then clearly $r = 0$ is a rational number between x and y.
- If $x < y < 0$, then
  $$0 < -y < -x$$
  and by previous case, there is
  $$\frac{k}{n} \in \mathbb{Q},$$
  such that
  $$-y < \frac{k}{n} < -x, \text{ so } x < -\frac{k}{n} < y.$$
- For 2), we take
  $$y = x + \epsilon$$
  and then by previous part,
  there is a rational number $r \in \mathbb{Q}$, such that
  $$x < r < x + \epsilon, \text{ so}$$
  $$\begin{aligned} |x - r| &= r - x \\ &< x + \epsilon - x \\ &= \epsilon, \end{aligned}$$
- So for every $x \in \mathbb{R}$ and $\epsilon > 0$,
  there is $r \in \mathbb{Q}$, such that
  $$|x - r| < \epsilon.$$

This finishes our proof. ■

- **Exercise** Show that the equation
$$x^2 = 2$$
has no solutions in $\mathbb{Q}$.

- Suppose this is true, so there are $m, n \in \mathbb{N}$, such that
$$\left(\frac{m}{n}\right)^2 = 2, \text{ and } \gcd(n, m) = 1.$$

- Therefore,
$$m^2 = 2n^2.$$

- Notice that this means that 2 divides m^2.

- Since 2 is prime,
if 2 divides m^2, then 2 divides m, so
$$m = 2k.$$

- Therefore,
$$\begin{aligned} (2k)^2 &= 2n^2, \\ 2k^2 &= n^2. \end{aligned}$$

- So, 2 divides n.

- Since 2 divides both n and m,
then 2 divides $\gcd(m, n) = 1$, a contradiction.
Exercise Show that there is a real number $\alpha > 0$ such that
$$\alpha^2 = 2.$$
We call such a solution the *square root of* 2 and we denote it by
$$\alpha = \sqrt{2}.$$

- We show that:
There is $\alpha \in \mathbb{R}$, $\alpha > 0$ such that
$$\alpha^2 = 2.$$

- Let
$$S = \left\{x \in \mathbb{R} : x > 0 \text{ and } x^2 < 2\right\}.$$

- We show that
$$\alpha = \sup S$$
satisfies the equation
$$x^2 = 2.$$

- We see that $S \neq \emptyset$.
This is because:
$1 \in S$ $(0 < 1$ and $1^2 = 1 < 2)$.

- We show that S is bounded above:
  Notice that if $x > 2$, then
  $$x^2 > 2x > 2 \cdot 2 = 4, \text{ so}$$
  if $x > 2$, then
  $$x \notin S,$$
  thus 2 is an upper bound of S.
- By *completeness property* of $\mathbb{R}$:
  There is a unique $\alpha \in \mathbb{R}$, such that,
  $$\alpha = \sup S.$$
- Since, for all $x \in S$,
  $$x > 0$$
  and $1 \in S$,
  $$\alpha \geq 1 > 0.$$
- We show that
  $$\alpha^2 < 2 \text{ and } \alpha^2 > 2$$
  is impossible.
  So by properties of real numbers, it must be
  $$\alpha^2 = 2.$$
- Suppose that $\alpha^2 < 2$. Then
  $$2 - \alpha^2 > 0.$$
  By Archimedean property of $\mathbb{R}$:
  There is $n \in \mathbb{N}$, such that
  $$\frac{2\alpha + 1}{2 - \alpha^2} < n.$$
- Therefore,
  $$\frac{2\alpha + 1}{n} < 2 - \alpha^2,$$
  so
  $$\begin{aligned} \left(\alpha + \frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &\leq \alpha^2 + \frac{2\alpha + 1}{n} \\ &< \alpha^2 + 2 - \alpha^2 \\ &= 2 \end{aligned}$$
- We showed that
  $$\alpha < \alpha + \frac{1}{n}$$
  and, since $\left(\alpha + \frac{1}{n}\right)^2 < 2$, by the definition of S,
  $$\left(\alpha + \frac{1}{n}\right) \in S$$
  A **contradiction** since α is an upper bound.

- Suppose that $\alpha^2 > 2$, then
$$\alpha^2 - 2 > 0.$$
- By the Archimedean property of $\mathbb{R}$:
There is $n \in \mathbb{N}$, such that
$$\frac{2\alpha}{\alpha^2 - 2} < n,$$
i.e.
$$-\frac{2\alpha}{n} > -\left(\alpha^2 - 2\right).$$
- Thus,
$$\begin{aligned}\left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2}\\ &> \alpha^2 - \left(\alpha^2 - 2\right) + \frac{1}{n^2}\\ &= 2 + \frac{1}{n^2}\\ &> 2.\end{aligned}$$
so $\left(\alpha - \frac{1}{n}\right) \notin S$.
- Since $\left(\alpha - \frac{1}{n}\right) > 0$ and $\left(\alpha - \frac{1}{n}\right) \notin S$,
$\left(\alpha - \frac{1}{n}\right)$ is an upper bound for S, but
$$\alpha - \frac{1}{n} < \alpha$$
- Hence, $\left(\alpha - \frac{1}{n}\right)$ is an upper bound of S that is smaller than α,
a contradiction since α *is the least upper bound*.
- By the trichotomy law, we see
$$\alpha^2 = 2.$$
- This finishes our argument.
Remark One shows that:
For any $a > 0$ there is a unique $\alpha \in \mathbb{R}$, $\alpha > 0$, such that,
$$\alpha^n = a$$
This real number is denoted by
$$a^{\frac{1}{n}} = \sqrt[n]{a}$$
and we call it *the nth root of* a, i.e.
$$\alpha = a^{\frac{1}{n}}.$$
Exercise Show that $\mathbb{Q}$ is *not a complete ordered field.*
- Indeed, we take
$$S = \left\{q \in \mathbb{Q} : q \in \left(0, \sqrt{2}\right)\right\}.$$

- Clearly,
$$\frac{1}{2} \in S,$$
so $S \neq \emptyset$.

- Since, for all $x \in S$,
$$x \leq 2,$$
then S is also bounded above.

- Suppose that
$$\beta = \sup S,$$
where $\beta \in \mathbb{Q}$.

- By the trichotomy law:
$$\beta > \sqrt{2} \text{ or } \beta < \sqrt{2} \text{ or } \beta = \sqrt{2}.$$

- Since $\sqrt{2} \notin \mathbb{Q}$ (we showed that $\alpha^2 = 2$ has no solution in $\mathbb{Q}$),
  it follows that
$$\beta > \sqrt{2} \text{ or } \beta < \sqrt{2}.$$

- If $\beta > \sqrt{2}$, then there is $r \in \mathbb{Q}$, such that
$$\sqrt{2} < r < \beta, \text{ so}$$
β cannot be the least upper bound of S.

- If $\beta < \sqrt{2}$ then again $r \in \mathbb{Q}$, such that
$$\beta < r < \sqrt{2}, \text{ so}$$
$r \in S$ and $\beta < r$,
Thus, β not an upper bound of S.

- It follows that, **there is NO** $\beta \in \mathbb{Q}$, such that
$$\beta = \sup S$$

- Consequently, $\mathbb{Q}$ **is not a complete ordered field.**
  Definition Let $S \subseteq \mathbb{R}$ we say that M is the *maximum* of S if

i) For all $x \in S$, $x \leq M$,
   i.e. M is an upper bound of S.

ii) $M \in S$.

- We denote the maximum of S by $\max S$, i.e.
$$\max S = M.$$

- Analogously, we define the *minimum* of S, i.e.

$$.m = \min S,$$

if m is a lower bound of S that belongs to S.

Exercise Let $S \subseteq \mathbb{R}$ and let

$$M = \sup S.$$

Show that $\max S = M$ iff $M \in S$.

Exercise For each of the following sets S,

find $\sup S$, $\max S$ and $\inf S$, $\min S$ if they exist.

1. $S = \left\{x \in \mathbb{R} : x^2 < 5\right\}$

- Notice that

$$S = \left(-\sqrt{5}, \sqrt{5}\right).$$

- **Claim:**

$$\inf S = -\sqrt{5}.$$

We show that:

- **i**) $-\sqrt{5}$ is a lower bound of S, i.e.

$$\forall x \in S, \ -\sqrt{5} \leq x,$$

- **ii**) For all $\epsilon > 0$

$$-\sqrt{5} + \epsilon$$

is not a lower bound of S, i.e. we prove that:

$$\forall \epsilon > 0, \ \exists x \in S \ni -\sqrt{5} + \epsilon > x.$$

- For **i**): We see that:

if $x \in S$, then

$$-\sqrt{5} < x < \sqrt{5},$$

so

$$x \geq -\sqrt{5},$$

for all $x \in S$.

- Notice that if

$$0 < \epsilon < 2\sqrt{5},$$

then

$$\left(-\sqrt{5}, -\sqrt{5} + \epsilon\right) \subseteq S, \text{ so}$$

if we take

$$\begin{aligned} x &= \frac{-\sqrt{5} + \left(-\sqrt{5} + \epsilon\right)}{2} \\ &= -\sqrt{5} + \frac{\epsilon}{2} \in \left(-\sqrt{5}, -\sqrt{5} + \epsilon\right) \subseteq S \end{aligned}$$

then $x \in S$ and $x < -\sqrt{5} + \epsilon$.

Therefore, $-\sqrt{5} + \epsilon$ is not a lower bound of S.

- We showed that
$$-\sqrt{5} = \inf S.$$
- Analogously, one shows that
$$\sqrt{5} = \sup S.$$
- Notice that
$$\pm\sqrt{5} \notin S,$$
so S has no $\min S$ and $\max S$.

2. $\left\{x \in \mathbb{R} : x^2 > 5\right\}$

- Since
$$\begin{aligned} S &= \left\{x \in \mathbb{R} : x^2 > 5\right\} \\ &= \left(-\infty, -\sqrt{5}\right) \cup \left(\sqrt{5}, \infty\right), \end{aligned}$$
S is not bounded above and below,
- Thus, $\sup S$ and $\inf S$ do not exist.
- However, by our conventions
$$\begin{aligned} \sup S &= +\infty \text{ and} \\ \inf S &= -\infty. \end{aligned}$$
- Since $\pm\infty \notin S$,
$\min S$ and $\max S$ do not exist.

3. $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

- **Claim**:
$$\sup S = \max S = 1.$$
Since for $n \geq 1$,
$$\frac{1}{n} \leq 1$$
we see that, for all $x \in S$,
$$x \leq 1.$$
This implies that 1 is an upper bound of S.
- Let $\epsilon > 0$ is given.
Since $1 \in S$ and $1 - \epsilon < 1$,
$1 - \epsilon$ is not an upper bound of S.
Hence,
$$1 = \sup S$$
Since $1 \in S$,
$$\max S = 1.$$

- We show that:
$$0 = \inf S.$$
Since for all $n \in \mathbb{N}$,
$$\frac{1}{n} > 0.$$
So $0 \leq x$, for all $x \in S$.
Therefore, 0 is a lower bound of S.
- Let $\epsilon > 0$.
It is sufficient to show that
$$0 + \epsilon$$
is not a lower bound of S.
- Since $\epsilon > 0$,
by Archimedean property of $\mathbb{R}$, there is $n \in \mathbb{N}$, such that
$$\frac{1}{n} < \epsilon.$$
- But $\frac{1}{n} \in S$, so
$$\frac{1}{n} < 0 + \epsilon,$$
i.e. $0 + \epsilon$ is not a lower bound of S.
- Therefore,
$$0 = \inf S.$$

Proposition *The unit interval* $(0,1)$ *is uncountable, so is* $\mathbb{R}$.

Proof. Suppose that $(0,1)$ is countable.

- Then there is a bijection
$$f : \mathbb{N} \to (0,1)$$
Let
$$f(n) = 0.a_{n1}a_{n2}a_{n3}...$$
for $n \in \mathbb{N}$.
- Define $b = 0.b_1b_2b_3...$ as follows:
$$b_n = \begin{cases} 5 & if \quad a_{nn} \neq 5 \\ 2 & if \quad a_{nn} = 5 \end{cases}$$
- We see that, for instance,
$$f(1) = 0.a_{11}a_{12}a_{13}... \neq b$$
because $a_{11} \neq b_1$
- Clearly,
$$b \neq f(n),$$
for all $n \in \mathbb{N}$, so
$$b \notin f(\mathbb{N}) = (0,1).$$

- Contradiction, since
$$f(\mathbb{N}) = (0,1)$$
as f is bijective.
- Since $(0,1) \subset \mathbb{R}$ and $(0,1)$ is uncountable,
hence $\mathbb{R}$ is also uncountable.

This completes our proof. ■

- **Proposition** *If $x, y \in \mathbb{R}$ and $x < y$, then the interval (x, y) contains countably many rational numbers and uncountably many irrational numbers.*

Proof. Note that (x, y) is equinumerous to $(0,1)$.

- Indeed, take
$$\begin{aligned} f &: (0,1) \to (x,y), \\ f(t) &= (y-x)t + x \end{aligned}$$
- Clearly, f is a bijection between $(0,1)$ and (x, y).
- Let $A = \mathbb{Q} \cap (x, y)$.
Since $A \subset \mathbb{Q}$ and $\mathbb{Q}$ is countable,
A is countable as a subset of a countable set.
- Clearly,
$$(x,y) = A \cup (x,y) \setminus A,$$
if $(x, y) \setminus A$ is countable, then
(x, y) is countable as a union of countable sets.
A contradiction since (x, y) is equinumerous to $(0,1)$ and,
as we showed, $(0,1)$ is not countable.

This finishes our proof. ■

- **Convergence in an Ordered Field**
- We will be mostly interested in the field of real numbers $\mathbb{R}$.
- However, for the purpose of a generality,
we will consider an arbitrary ordered field $\mathbb{F}$.
- This, in turn, will allow us to formulate *completeness of an ordered field in terms of monotonic and bounded sequences.*

Let $\mathbb{F}$ be an ordered field.

- **Definition** Let $\{x_n\}$ be a sequence in $\mathbb{F}$ and $x \in \mathbb{F}$.
  We say that $\{x_n\}$ *converges to* x if
  for every $\epsilon > 0$, there is $N \in \mathbb{N}$, such that,
  for all $n \in \mathbb{N}$, if $n \geq N$, then

$$|x_n - x| < \epsilon.$$

  We write

$$\lim_{n\to\infty} x_n = x \text{ or } x_n \to x \text{ as } n \to \infty.$$

  Example: Let $x_n = 1$, for all $n \in \mathbb{N}$ converges to $x = 1$ as $n \to \infty$.
  Proposition *Let F be an ordered field and suppose that*

i) *$x_n \to x$ and $y_n \to x$ as $n \to \infty$ and,*

ii) *for all $n \in \mathbb{N}$,*

$$x_n \leq z_n \leq y_n.$$

Then $z_n \to x$ as $n \to \infty$.

Proof. Let $\epsilon > 0$ be given.

- Since $x_n \to x$ and $y_n \to x$,
  there are $N_1, N_2 \in \mathbb{N}$, such that,
  for $n > N_1$:

$$x - \epsilon < x_n < x + \epsilon$$

  and for $n > N_2$:

$$x - \epsilon < y_n < x + \epsilon.$$

- Therefore, if

$$n > \underbrace{\max\{N_1, N_2\}}_{N},$$

  then

$$\begin{aligned} x - \epsilon &< x_n \leq z_n \leq y_n < x + \epsilon, \text{ so} \\ x - \epsilon &< z_n < x + \epsilon, \text{ i.e.} \\ |z_n - x| &< \epsilon. \end{aligned}$$

- Hence $z_n \to x$ as $n \to \infty$.

This finishes our proof. ∎

- **Proposition** *Let F be an ordered field and assume that*

i) *$a \leq x_n \leq b$ and*

ii) *$x_n \to x$ as $n \to \infty$.*
   Then $a \leq x \leq b$.
   Proof Exercise
   Exercise: Let $\left\{\frac{1-n^2}{2n^2-n+1}\right\}$ be a sequence in $\mathbb{R}$.
   Show that

$$\lim_{n\to\infty} \frac{1-n^2}{2n^2-n+1} = -\frac{1}{2}.$$

- Let $\epsilon > 0$ be given.
- For $n \geq 3$

$$\begin{aligned}
\left| \frac{1-n^2}{2n^2-n+1} - \left(-\frac{1}{2}\right) \right| &= \left| -\frac{n-3}{4n^2-2n+2} \right| \\
&= \left| \frac{n-3}{4n^2-2n+2} \right| = \frac{|n-3|}{|4n^2-2n+2|} \\
&= \frac{n-3}{2\left(n^2-n+1\right)} \leq \frac{n}{2\left(n^2-n+1\right)} \\
&\leq \frac{n}{2\left(n^2-n\right)} \\
&\leq \frac{n}{2\left(n^2-\frac{1}{2}n^2\right)}, \text{ since } n \leq \frac{1}{2}n^2, \text{ for } n \geq 3, \text{ so} \\
&\leq \frac{1}{n}.
\end{aligned}$$

- Since $\epsilon > 0$, by Archimedean property of $\mathbb{R}$,
  there is $N \in \mathbb{N}$, such that
$$0 < \frac{1}{N} < \epsilon.$$
- Take $n > \max\{N, 3\}$, then
$$\left| \frac{1-n^2}{2n^2-n+1} + \frac{1}{2} \right| \leq \frac{1}{n} < \frac{1}{N} < \epsilon.$$
- It follows that
$$\lim_{n\to\infty} \frac{1-n^2}{2n^2-n+1} = -\frac{1}{2}.$$

Proposition *In an ordered field $\mathbb{F}$,*
if $x_n \to x$ and $x_n \to y$ as $n \to \infty$, then
$$x = y.$$

Proof. Suppose that $x \neq y$, then $|x-y| > 0$.

- Take $\epsilon = \frac{1}{2}|x-y| > 0$.
- Since $x_n \to x$ and $x_n \to y$, there are
$$N_1, N_2 \in \mathbb{N},$$
  such that, for $n > N_1$:
$$|x_n - x| < \epsilon$$
  and for $n > N_2$:
$$|x_n - y| < \epsilon.$$
- Let $n > \max\{N_1, N_2\}$.
- Then

$$\begin{aligned}
|x-y| &= |(x-x_n)+(x_n-y)| \\
&\leq |x_n-x| + |x_n-y| < 2\epsilon \\
&= 2\cdot\left(\frac{1}{2}|x-y|\right) \\
&= |x-y|,
\end{aligned}$$

- Hence
$$|x-y| < |x-y|, \text{ a contradiction.}$$
- It must be then
$$x = y.$$

This completes our proof. ■

- Recall, a sequence $\{x_n\}$ in an ordered field $\mathbb{F}$ is **bounded** if,
  there is $M \in \mathbb{F}$, such that
  $$|x_n| \leq M,$$
  for all $n \in \mathbb{N}$.
  Proposition *In an ordered field F a convergent sequence is bounded.*

Proof. Let $x_n \to x$.

- Take $\epsilon = 1$.
  Then by the definition,
  there is $N \in \mathbb{N}$, such that, for $n > N$
  $$|x_n - x| < 1.$$
- Therefore, if $n > N$,
$$|x_n| = |(x_n - x) + x| \leq |x_n - 1| + |x| < 1 + |x|.$$
- If
$$K = \max\{|x_n| : n \leq N\}, \text{ then } |x_n| \leq K, \ n = 1, 2, ..., N.$$
  Thus, for all $n \in \mathbb{N}$
  $$|x_n| \leq \underbrace{\max\{1 + |x|, K\}}_{M}.$$
- Therefore, $\{x_n\}$ is bounded.

This finishes our proof. ■

- **Theorem** *Let F be an ordered field, $\alpha \in \mathbb{F}$, and assume that:*
  $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in $\mathbb{F}$
  Then

1. $\alpha x_n \to \alpha x$
2. $x_n + y_n \to x + y$
3. $x_n y_n \to xy$
4. If $x_n, x \neq 0$ then
$$\frac{1}{x_n} \to \frac{1}{x}.$$

Proof. We prove 4).

- Let $\epsilon > 0$ be given.

- Since $x_n \to x$,
  there is $N_1 \in \mathbb{N}$, such that, for $n > N_1$

$$|x_n - x| < \frac{|x|}{2}.$$

- It follows that, for $n > N_1$,

$$\begin{aligned} |x_n| &= |(x_n - x) + x| \geq ||x_n - x| - |x|| \\ &= |x| - |x_n - x| \\ &> |x| - \frac{|x|}{2} \\ &= \frac{|x|}{2} > 0. \end{aligned}$$

  That is, we showed that, if $n > N_1$,

$$|x_n| > \frac{|x|}{2} > 0$$

- Since $x_n \to x$,
  there is $N_2 \in \mathbb{N}$, such that, for $n > N_2$:

$$|x_n - x| < \frac{|x|^2 \epsilon}{2}.$$

- Take $n > \underbrace{\max\{N_1, N_2\}}_{N}$.

  Then

$$\begin{aligned} \left|\frac{1}{x_n} - \frac{1}{x}\right| &= \left|\frac{x - x_n}{x x_n}\right| = \frac{|x_n - x|}{|x_n|\,|x|} \\ &< \frac{|x_n - x|}{\frac{|x|}{2}\,|x|} < \frac{\frac{|x|^2 \epsilon}{2}}{\frac{|x|}{2}\,|x|} = \epsilon, \end{aligned}$$

- It follows that

$$\frac{1}{x_n} \to \frac{1}{x} \text{ as } n \to \infty.$$

This finishes our proof. ■