HOMEWORK 6 SOLUTIONS – MATH 4341

Problem 1. (a) Show that all metric spaces (with the metric topology) are Hausdorff.

(b) Show that all metric spaces (with the metric topology) are first-countable.

Proof. Suppose (X, d) is a metric space.

(a) Let $x \neq y$ in X. Then $r = d(x, y) > 0$. Consider $U = B_d(x, r/2)$ and $V = B_d(y, r/2)$. Then U and V are neighborhoods of x and y respectively. We claim that U and V are disjoint. Assume $z \in U \cap V$. Then $z \in B_d(x, r/2)$ and $z \in B_d(x, r/2)$, i.e. $d(z, x) < r/2$ and $d(y, z) < r/2$. This implies that $d(z, x) + d(y, z) < r = d(x, y)$, which contradicts the triangle inequality. Hence U and V are disjoint.

(b) Let $x \in X$ and $B_n = B_d(x, 1/n)$ for all $n \in \mathbb{N}$. We claim that $\{B_n\}_{n\in\mathbb{N}}$ is a countable basis at x. Indeed, for any neighborhood U of x, there exists $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subset U$. Choose $n_0 \in \mathbb{N}$ such that $n_0 > 1/\varepsilon$. Then $B_{n_0} = B_d(x, 1/n_0) \subset B_d(x, \varepsilon) \subset U$. □

Problem 2. (a) Show that $\mathcal{B} = \{(a, \infty) \mid a \in \mathbb{R}\}$ is a basis for some topology on $\mathbb{R}$.

(b) Is $\mathbb{R}$ Hausdorff in the topology generated by $\mathcal{B}$?

Proof. (a) Every $x \in \mathbb{R}$ is an element of $(x - 1, \infty) \in \mathcal{B}$. Suppose $x \in (a_1, \infty) \cap (a_2, \infty)$. Then $x \in (\max\{a_1, a_2\}, \infty) = (a_1, \infty) \cap (a_2, \infty)$.

(b) Let $x < y$ and U be any nbhd of x. We will show that U contains y. (This will imply that $\mathbb{R}$ is not Hausdorff in this topology.) Indeed, there exists $a \in \mathbb{R}$ such that $x \in (a, \infty) \subset U$. Since $a < x < y$, we have $y \in (a, \infty) \subset U$. □

Problem 3. Let $\mathbb{R}^\omega$ be the countably infinite product of $\mathbb{R}$, i.e., $\mathbb{R}^\omega = \prod_{i=1}^\infty X_i$ where each $X_i = \mathbb{R}$. We equip $\mathbb{R}^\omega$ with the product topology. Let A be the subset of $\mathbb{R}^\omega$ consisting of all points whose coordinates are positive, i.e. $A = \{(x_1, x_2, \cdots) \mid x_i > 0 \ \forall i = 1, 2, \cdots\}$.

(a) Show that the origin $\mathbf{0} = (0, 0, \cdots)$ is a limit point of A.

(b) Construct an explicit sequence in A converging to $\mathbf{0}$ in $\mathbb{R}^\omega$.

Proof. (a) It is equivalent to show that any neighborhood $U \subset \mathbb{R}^\omega$ of $\mathbf{0}$ intersects A. Let U be a neighborhood of $\mathbf{0}$. Then there exists a basis element $B = \prod_{i=1}^\infty U_i$ such that $\mathbf{0} \in B \subset U$, where each $U_i \subset X_i = \mathbb{R}$ is open. Since $\mathbf{0} \in B$, we have $0 \in U_i$. Since $U_i \subset \mathbb{R}$ is open, there exists an open interval (a_i, b_i) such that $0 \in (a_i, b_i) \subset U_i$.

Let $x = (x_1, x_2, \cdots)$ where $x_i = b_i/2 > 0$. Then $x \in A$. Since $x_i = b_i/2 \in (a_i, b_i) \subset U_i$, we have $x \in B \subset U$. Hence $x \in U \cap A$.

(b) For every $n \in \mathbb{N}$, let x_n be the constant sequence $(1/n, 1/n, \cdots)$. Then $x_n \in A$. We claim that $x_n \to \mathbf{0}$ in the product topology. Indeed, let U be a neighborhood of $\mathbf{0}$. Then there exists a basis element $B = \prod_{i=1}^\infty U_i$ such that $\mathbf{0} \in B \subset U$, where each $U_i \subset X_i$ is open and $U_i = X_i$ for $i > i_0$. Since $\mathbf{0} \in B$, we have $0 \in U_i$. Since $U_i \subset \mathbb{R}$ is open, there exists an open interval (a_i, b_i) such that $0 \in (a_i, b_i) \subset U_i$.

Choose $n_0 \in \mathbb{N}$ such that $1/n_0 < \min\{b_1, b_2, \cdots, b_{i_0}\}$. For $n \geq n_0$ we have $1/n \leq 1/n_0 < \min\{b_1, b_2, \cdots, b_{i_0}\}$, so $1/n \in (a_i, b_i) \subset U_i$ for $1 \leq i \leq i_0$. Since $1/n \in U_i = \mathbb{R}$ for $i > i_0$, we conclude that $1/n \in U_i$ for all $i \in \mathbb{N}$. This implies that $x_n = (1/n, 1/n, \cdots) \in \prod_{i=1}^\infty U_i = B \subset U$ for all $n \geq n_0$. Hence $x_n \to \mathbf{0}$ in the product topology. □