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Math 4301 Mathematical Analysis I
Lecture 5
Topic: Upper and lower limits
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- Recall, for a real sequence $\{x_n\}$,
  let C be the set of its all cluster points i.e.

$$C = \{x : \ x \text{ is a cluster point of } \{x_n\}\}.$$

- We define the upper and the lower limits of $\{x_n\}$ as follows.
  Definition For a real $\{x_n\}$ sequence,
  the *limit superior* of $\{x_n\}$ is defined as follows.
  Let C be the set of cluster points of $\{x_n\}$:
  If $C \neq \emptyset$ and $\{x_n\}$ is bounded from above, then

$$\limsup (x_n) = \sup (C).$$

If $C = \emptyset$ and $\{x_n\}$ is bounded from above, then

$$\limsup (x_n) = -\infty.$$

If $\{x_n\}$ is not bounded above, then

$$\limsup (x_n) = +\infty.$$

Definition For a real $\{x_n\}$ sequence,
the *limit inferior* of $\{x_n\}$ is defined as follows.
Let C be the set of cluster points of $\{x_n\}$:
If $C \neq \emptyset$ and $\{x_n\}$ is bounded from below, then

$$\liminf (x_n) = \inf (C).$$

If $C = \emptyset$ and $\{x_n\}$ is bounded from below, then

$$\liminf (x_n) = +\infty$$

If $\{x_n\}$ is not bounded below, then

$$\liminf (x_n) = -\infty.$$

Example: Let

$$x_n = \left(1 + \frac{(-1)^n}{n}\right)^n.$$

Find $\liminf (x_n)$ and $\limsup (x_n)$.

- We observe that

$$x_n = \begin{cases} \left(1 + \frac{1}{2k}\right)^{2k} & if \quad n = 2k \\ \left(1 - \frac{1}{2k-1}\right)^{2k-1} & if \quad n = 2k-1 \end{cases}$$

- It follows that

$$e = \lim_{k\to\infty}\left(1+\frac{1}{2k}\right)^{2k} \text{ and } \frac{1}{e} = \lim_{k\to\infty}\left(1-\frac{1}{2k-1}\right)^{2k-1}$$

  are cluster points of $\{x_n\}$.
- If $x \neq e, \frac{1}{e}$, then there is

$$\epsilon = \frac{1}{2}\min\left\{\left|x-\frac{1}{e}\right|, |x-e|\right\} > 0,$$

  such that the set

$$\{n \in \mathbb{N} : |x_n - x| < \epsilon\} \text{ is finite.}$$

- Indeed, we see that,
  since

$$e = \lim_{k\to\infty}\left(1+\frac{1}{2k}\right)^{2k},$$

  there is $N_1 \in \mathbb{N}$, such that,
  for all $k > N_1$

$$|x_{2k} - e| < \epsilon$$

  and there is $N_2 \in \mathbb{N}$, such that,
  for all $k > N_2$:

$$\left|x_{2k-1} - \frac{1}{e}\right| < \epsilon$$

- Therefore, for

$$n > \max\{N_1, N_2\},$$

$$|x_{2k} - e| < \epsilon \text{ or } \left|x_{2k-1} - \frac{1}{e}\right| < \epsilon.$$

- So if $n > \max\{N_1, N_2\}$ and n is even,

$$\begin{aligned} |x_n - x| &= |(x_n - e) + (e - x)| \geq ||x_n - e| - |e - x|| = |e - x| - |x_n - e| \\ &\geq |e - x| - \epsilon \geq |e - x| - \frac{1}{2}|e - x| \geq \epsilon. \end{aligned}$$

- Analogously, if

$$n > \max\{N_1, N_2\}$$

  and n is odd,

$$\begin{aligned} |x_n - x| &= \left|\left(x_n - \frac{1}{e}\right) + \left(\frac{1}{e} - x\right)\right| \\ &\geq \left|\left|x_n - \frac{1}{e}\right| - \left|\frac{1}{e} - x\right|\right| \\ &= \left|\frac{1}{e} - x\right| - \left|x_n - \frac{1}{e}\right| \\ &\geq \left|\frac{1}{e} - x\right| - \epsilon \\ &\geq \left|\frac{1}{e} - x\right| - \frac{1}{2}\left|\frac{1}{e} - x\right| \\ &\geq \epsilon. \end{aligned}$$

- Thus, we showed that,

  if $n > \max\{N_1, N_2\}$, then

$$|x_n - x| \geq \epsilon,$$

  so the set

$$\{n \in \mathbb{N} : |x_n - x| < \epsilon\} \text{ is finite.}$$

- We showed that, if $x \neq e, \frac{1}{e}$, then x is not a cluster point.
- Therefore,

$$C = \left\{e,\ \frac{1}{e}\right\}$$

  is the set of all cluster points of $\{x_n\}$.

- Hence,

$$\liminf\{x_n\} = \frac{1}{e} \text{ and } \limsup\{x_n\} = e.$$

Example: Let $x_n = (-1)^n n$.

Find $\liminf(x_n)$ and $\limsup(x_n)$.

- Notice that since

$$\lim_{k\to\infty} x_{2k} = \infty$$

  and

$$\lim_{k\to\infty} x_{2k-1} = -\infty$$

  the sequence $\{x_n\}$ is not bounded above and below.

- Therefore,

$$\liminf(x_n) = -\infty \text{ and } \limsup(x_n) = \infty.$$

Example: Let

$$x_n = 3^{n\sin\left(\frac{2\pi n}{3}\right)}.$$

Find $\liminf(x_n)$ and $\limsup(x_n)$.

- We observe that

$$\sin\left(\frac{2\pi}{3}n\right) = \begin{cases} \frac{\sqrt{3}}{2} & if \quad n = 3k-2 \\ -\frac{\sqrt{3}}{2} & if \quad n = 3k-1 \\ 0 & if \quad n = 3k \end{cases}, \ k = 1, 2, 3, ...$$

Proposition Let $\{x_n\}$ be a sequence of real numbers and $x \in \mathbb{R}$. Then

1. If $\{x_n\}$ is bounded below,

$$x = \liminf(x_n)$$

   if and only if

a. For all $\epsilon > 0$ there is an $N \in \mathbb{N}$, such that

$$x - \epsilon < x_n$$

   whenever $n > N$, and

b. For all $\epsilon > 0$ and all M,
there is $n > M$ with
$$x_n < x + \epsilon$$

2. If $\{x_n\}$ is bounded above,
$$x = \limsup(x_n)$$
if and only if

a. For all $\epsilon > 0$ there is an $N \in \mathbb{N}$, such that
$$x_n < x + \epsilon$$
whenever $n \geq N$, and

b. For all $\epsilon > 0$ and all M,
there is $n > M$ with
$$x - \epsilon < x_n$$

Proof. We show that if $\{x_n\}$ is bounded below,
$$x = \liminf(x_n)$$
if and only if **1a**) and **1b**) hold.

- We start by showing **1b**).
- Let
$$C = \{y : y \text{ is a cluster point of } \{x_n\}\}.$$
Suppose that $x = \liminf(x_n)$, i.e. $x = \inf C$ (since $C \neq \emptyset$).
- Given $\epsilon > 0$, since $x + \frac{\epsilon}{2}$ is not a lower bound of C,
there is $y \in C$, such that
$$y < x + \frac{\epsilon}{2};$$
- Since, $y \in C$, the set
$$\left\{n \in \mathbb{N} : |x_n - y| < \frac{\epsilon}{2}\right\}$$
is infinite.
- Therefore, given M,
there is $n > M$, such that
$$|x_n - y| < \frac{\epsilon}{2}, \text{ i.e. } -\frac{\epsilon}{2} < x_n - y < \frac{\epsilon}{2}$$
- Hence
$$x_n < y + \frac{\epsilon}{2} < \left(x + \frac{\epsilon}{2}\right) + \frac{\epsilon}{2} = x + \epsilon.$$
for some $n > M$ and condition **1b**) follows.
- Now, we show **1a**).
Suppose that, there is $\epsilon > 0$, such that, for all $N \in \mathbb{N}$,
there is $n > N$, such that
$$x_n \leq x - \epsilon.$$

- Take $N = 1$, then
  there is $n_1 > N$, such that
  $$x_{n_1} < x - \epsilon.$$
- Take $N_2 = n_1$, then
  there is $n_2 > N_2 = n_1$, such that
  $$x_{n_2} < x - \epsilon.$$
- Using induction,
  we construct $\{x_{n_k}\}$ a subsequence of $\{x_n\}$, such that
  $$x_{n_k} < x - \epsilon,$$
  for all $k = 1, 2, ...$
- Since $\{x_n\}$ is bounded below,
  its subsequence $\{x_{n_k}\}$ is **bounded below**.
- Moreover, since
  $$x_{n_k} < x - \epsilon,$$
  for all $k = 1, 2, ...,$
  then also $\{x_{n_k}\}$ is bounded.
- Therefore, by Bolzano-Weierstrass thm.,
  $\{x_{n_k}\}$ has a convergent subsequence $\left\{x_{n_{k_l}}\right\}$ and
  let
  $$a = \lim_{l \to \infty} x_{n_{k_l}}.$$
  Since,
  $$x_{n_{k_l}} < x - \epsilon,$$
  for all $l = 1, 2, ...,$ then $a \leq x - \epsilon$.
- Therefore, a is a cluster point of $\{x_n\}$ and
  $$a < x = \inf C.$$

  Contradiction.
- **Conversely**, suppose that $x \in \mathbb{R}$ satisfies conditions **1a**) and **1b**).
- If y is a cluster point of (x_n) and
  $$y < x,$$
  we let
  $$\epsilon = \frac{1}{2}(x - y) > 0,$$
  then the set
  $$\{n \in \mathbb{N} : |x_n - y| < \epsilon\}$$
  is infinite, that is
  $$y - \epsilon < x_n < y + \epsilon$$
  for infinitely many $n \in \mathbb{N}$.

- Since

$$\begin{aligned} x_n &< y+\epsilon \\ &= y+\frac{1}{2}(x-y) \\ &= \frac{1}{2}(x+y) \\ &= x-\frac{1}{2}(x-y) \\ &= x-\epsilon \end{aligned}$$

  it follows that

$$x_n < x-\epsilon,$$

  for infinitely many $n \in \mathbb{N}$.

- This **contradicts** to condition **1a**).

- It follows that x is a lower bound for the set of all cluster points of (x_n).

- **We show that x is the greatest lower bound of the set of all cluster points** of (x_n), *that is*, if $\epsilon > 0$, then $x+\epsilon$ is not a lower bound.

- By **1b**), taking $N_1 = 1$,
  there is $n_1 > N_1$, such that

$$x_{n_1} < x+\epsilon$$

- Taking $N_2 = n_1$,
  there is $n_2 > N_2 = n_1$, such that

$$x_{n_2} < x+\epsilon.$$

- By induction, taking $N_k = n_{k-1}$,
  there is $n_k > N_k = n_{k-1}$, such that

$$x_{n_k} < x+\epsilon.$$

- Thus, we construct a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$, such that,
  for all $k \in \mathbb{N}$,

$$x_{n_k} < x+\epsilon.$$

- By **1a**), there is $N \in \mathbb{N}$, such that,
  for all $n > N$,

$$x-\epsilon < x_n.$$

- Therefore, if $k > N$,
  then $n_k \geq k > N$, so for $k > N$

$$x-\epsilon < x_{n_k}.$$

- Hence, for $k > N$,

$$\begin{aligned} x-\epsilon &< x_{n_k} < x+\epsilon, \text{ i.e.} \\ |x_{n_k}-x| &< \epsilon. \end{aligned}$$

- It follows that
$$x_{n_k} \to x,$$
so x is a cluster point of (x_n).
- Since $x < x + \epsilon$, so we showed that
$$x + \epsilon$$
is not a lower bound, so
$$x = \liminf (x_n).$$

Proof for $x = \limsup (x_n)$ is analogous. ■

- **Corollary** Let $\{x_n\}$ be a bounded sequence of real numbers and C denotes the set of its cluster points. Then
$$\liminf (x_n) \in C$$
and
$$\limsup (x_n) \in C.$$
- Let $\{x_n\}$ be a real sequence and
$$S_n = \{x_k : k \geq n + 1\}.$$
- Define sequence
$$a_n = \inf S_n \text{ and } b_n = \sup S_n.$$
- Clearly,
$$S_1 \supseteq S_2 \supseteq ...$$
therefore, for $k \geq n$,
$$S_k \subseteq S_n,$$
thus
$$a_n \leq a_k \leq b_k.$$
- If $k \leq n$, then
$$S_n \subseteq S_k,$$
so
$$a_k \leq a_n \leq b_n.$$

Example Consider sequence
$$x_n = (-1)^n + \frac{1}{n}, n \in \mathbb{N}.$$

Then
$$\begin{aligned}
a_1 &= \inf S_1 = \inf \left\{ (-1)^n + \frac{1}{n} : n \geq 1 \right\} = -1 \\
a_2 &= \inf S_2 = \inf \left\{ (-1)^n + \frac{1}{n} : n \geq 2 \right\} = -1 \\
a_3 &= \inf S_3 = \inf \left\{ (-1)^n + \frac{1}{n} : n \geq 3 \right\} = -1 \\
&\vdots \\
a_n &= -1
\end{aligned}$$

Observe that

$$a_1 \leq a_2 \leq a_3 \leq ....$$

We also see that

$$\begin{aligned}
b_1 &= \sup S_1 = \sup\left\{(-1)^n + \frac{1}{n} : n \geq 1\right\} \\
&= (-1)^2 + \frac{1}{2} = \frac{3}{2} \\
b_2 &= \sup S_2 = \sup\left\{(-1)^n + \frac{1}{n} : n \geq 2\right\} \\
&= (-1)^2 + \frac{1}{2} = \frac{3}{2} \\
b_3 &= \sup S_3 = \sup\left\{(-1)^n + \frac{1}{n} : n \geq 3\right\} \\
&= (-1)^4 + \frac{1}{4} = \frac{5}{4} \\
&\vdots \\
b_n &= \begin{cases} 1 + \frac{1}{2k} & if \quad n = 2k-1 \\ 1 + \frac{1}{2k} & if \quad n = 2k \end{cases}
\end{aligned}$$

so

$$b_1 \geq b_2 \geq b_3 \geq ...$$

- We observe that

$$\begin{aligned}
\limsup (x_n) &= \lim_{n\to\infty} b_n = 1 \\
&= \inf \{b_n : n \geq 1\} \\
\liminf (x_n) &= \lim_{n\to\infty} a_n = -1 \\
&= \sup \{a_n : n \geq 1\}
\end{aligned}$$

- Note that if sequence $\{x_n\}$ is not bounded then
the sequences $\{a_n\}$ and $\{b_n\}$ might not take their values in $\mathbb{R}$.
For instance, if $x_n = n$, then

$$\begin{aligned}
a_n &= \inf S_n = n, \ n \in \mathbb{N}, \text{ but} \\
b_n &= \sup S_n = +\infty, \text{ for all } n \in \mathbb{N}.
\end{aligned}$$

We see that still

$$\begin{aligned}
a_1 &\leq a_2 \leq ... \text{ and} \\
b_1 &\geq b_2 \geq ...
\end{aligned}$$

so

$$\begin{aligned}
\limsup (x_n) &= \lim_{n\to\infty} b_n \\
&= +\infty \\
\liminf (x_n) &= \lim_{n\to\infty} a_n \\
&= \lim_{n\to\infty} n \\
&= +\infty
\end{aligned}$$

as we have seen it before.

- In general, the following result holds.

  Proposition If $\{x_n\}$ is a sequence of real numbers, then

$$\begin{aligned}\limsup(x_n) &= \inf\{\sup S_n : n \in \mathbb{N}\} \\ &= \inf\{b_n : n \geq 1\} \\ &= \lim_{n\to\infty} b_n \text{ and} \\ \liminf(x_n) &= \sup\{\inf S_n : n \in \mathbb{N}\} \\ &= \sup\{a_n : n \geq 1\} \\ &= \lim_{n\to\infty} a_n.\end{aligned}$$

Proof. We show that

$$\limsup(x_n) = \inf\{\sup S_n : n \in \mathbb{N}\}.$$

- **If** $\{x_n\}$ **is not bounded above**,

  then by the definition

$$\limsup(x_n) = \infty$$

  and since,

$$b_n = \sup S_n = +\infty$$

  for all $n = 1, 2, ...$

$$\lim_{n\to\infty} b_n = +\infty,$$

  (with the understanding that $\inf\{\infty\} = \infty$), so

$$\limsup(x_n) = \inf\{b_n : n \in \mathbb{N}\}.$$

- **Assume** that $\{x_n\}$ is **bounded above**:

- **If** $\{x_n\}$ **has no cluster points**,

  then by the definition

$$\limsup(x_n) = -\infty.$$

- Since $\{x_n\}$ has no cluster points,

  $\{x_n\}$ cannot be bounded below and

  for each $M \in \mathbb{R}$, the set

$$\{n \in \mathbb{N} : M \leq x_n\}$$

  must be finite.

- Therefore, for each $M \in \mathbb{R}$,

  there is $N \in \mathbb{N}$, such that, for $n > N$,

$$x_n < M.$$

- It follows that

$$b_N = \sup\{x_n : n > N\} \leq M.$$

- Since $b_1 \geq b_2 \geq ...$, we see that,

  for all $n > N$, $b_n \leq M$.

- Consequently,

$$\lim_{n\to\infty} b_n = -\infty.$$

- We showed that
$$\limsup (x_n) = \inf \{b_n : n \in \mathbb{N}\}.$$
- **Assume that $\{x_n\}$ is bounded below**.
- **Therefore, $\{x_n\}$ is bounded** (since $\{x_n\}$ is also bounded above), so
by the Bolzano-Weierstrass thm.,
the set of cluster points C of $\{x_n\}$ is not empty and bounded.
- Let
$$\limsup (x_n) = \sup C.$$
- Since $b_1 \geq b_2 \geq ...$ and $\{b_n\}$ is bounded below,
then $\{b_n\}$ converges and let
$$b = \lim_{n\to\infty} b_n = \inf \{b_n : n \in \mathbb{N}\}.$$
- Now we show that
$$b = \limsup (x_n) = \sup C.$$
- **Our strategy is to use the following:**
If $\{x_n\}$ is bounded above,
$$x = \limsup (x_n)$$
if and only if

a. For all $\epsilon > 0$ there is an $N \in \mathbb{N}$, such that
$$x_n < x + \epsilon$$
whenever $n \geq N$, and

b. For all $\epsilon > 0$ and all M,
there is $n > M$ with
$$x - \epsilon < x_n$$

- We first show **a**):
Let $\epsilon > 0$ be given,
since $b + \epsilon$ is not a lower bound for
$$\{b_n : n \in \mathbb{N}\},$$
there is $N \in \mathbb{N}$, such that,
$$b_N < b + \epsilon.$$
- Since
$$b_n \leq b_N,$$
for all $n \geq N$, we see that,
for all $n \geq N$
$$b_n < b + \epsilon,$$
- Since
$$x_{n+1} \leq b_n,$$
for all n (by the definition of b_n).

- Therefore, if $n > N$,

$$x_n < b + \epsilon.$$

- **This gives us condition a).**
- Now, we show **b**):
  Let $\epsilon > 0$ and $N \in \mathbb{N}$ be given.
- Since $b \leq b_N$:

$$b - \epsilon \leq b_N - \epsilon.$$

- Since

$$b_N = \sup \{x_k : k > N\},$$

- there is $n > N$, such that

$$b_N - \epsilon < x_n.$$

- Therefore, there is $n > N$, such that

$$\begin{aligned} b - \epsilon &\leq b_N - \epsilon < x_n, \text{ so} \\ b - \epsilon &< x_n. \end{aligned}$$

- By previous result,
  we showed that indeed

$$b = \limsup (x_n).$$

- Therefore, we showed that

$$\limsup (x_n) = \inf \{b_n : n \in \mathbb{N}\}.$$

- Analogous argument shows that

$$\liminf (x_n) = \sup \{\inf S_n : n \in \mathbb{N}\}.$$

This finishes our proof. ■

- **Proposition** Let $\{x_n\}$ be a sequence of real numbers. Then

1. $\liminf (x_n) \leq \limsup (x_n)$
2. If $x_n \leq M$, for all $n \in \mathbb{N}$, then
$$\limsup (x_n) \leq M$$
3. If $M \leq x_n$, for all $n \in \mathbb{N}$, then
$$M \leq \liminf (x_n)$$
4. $\limsup (x_n) = \infty$ if and only if $\{x_n\}$ is not bounded above.
5. $\liminf (x_n) = -\infty$ if and only if $\{x_n\}$ is not bounded below.
6. If x is a cluster point of $\{x_n\}$, then
$$\liminf (x_n) \leq x \leq \limsup (x_n)$$
7. If $x = \liminf (x_n)$ is finite then x is a cluster point of $\{x_n\}$

8. If $x = \limsup (x_n)$ is finite than x is a cluster point of $\{x_n\}$

9. $x_n \to x \in \mathbb{R}$ as $n \to \infty$ iff
$$\liminf (x_n) = \limsup (x_n) = x \in \mathbb{R}.$$

Proof. We show each statement $1) - 9)$.

1. **For statement** (1) : Let

$$\begin{aligned} S_n &= \{x_k : k > n\}, \\ a_n &= \inf S_n \text{ and} \\ b_n &= \sup S_n \end{aligned}$$

($a_n = -\infty$ if $\{x_n\}$ is not bounded below and $b_n = \infty$ if $\{x_n\}$ is not bounded above).

- Since $S_n \subseteq S_k$, for $n \geq k$, therefore

$$\inf S_k \leq \inf S_n \leq \sup S_n \leq \sup S_k,$$

- Hence, for all $n \geq k$,
$$a_n \leq b_k.$$

- Therefore, each b_k is an upper bound for $\{a_1, a_2, ...\}$.

- It follows that, for all k :

$$\begin{aligned} \liminf (x_n) &= \sup \{a_n : n \in \mathbb{N}\} \\ &\leq b_k. \end{aligned}$$

- Hence, $\liminf (x_n)$ is a lower bound for $\{b_1, b_2, ...\}$, so

$$\begin{aligned} \liminf (x_n) &\leq \inf \{b_n : n \in \mathbb{N}\} \\ &= \limsup (x_n). \end{aligned}$$

2. **For statement** (2) :
Since $x_n \leq M$ for all $n \in \mathbb{N}$,

$$\begin{aligned} b_n &= \sup \{x_k : k > n\} \\ &\leq M, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Consequently,

$$\begin{aligned} \limsup (x_n) &= \inf \{b_n : n \in \mathbb{N}\} \\ &\leq M. \end{aligned}$$

3. Analogous argument shows that,
if $M \leq x_n$, then
$$M \leq \liminf (x_n).$$

4. **For statement** (4) :

   Suppose that
$$\limsup (x_n) = +\infty$$
   and $\{x_n\}$ is bounded above,

   i.e. there is $M \in \mathbb{R}$, such that
$$x_n \leq M,$$
   for all $n \in \mathbb{N}$.

- Then by (2),
$$\limsup (x_n) \leq M, \text{ a contradiction.}$$

- Conversely, if $\{x_n\}$ is not bounded above,

  then by the definition,
$$\limsup (x_n) = \infty.$$

5. Analogous arguments can be used to prove (5).

6. **For statement** (6) :

   Let C be the set of all cluster points of $\{x_n\}$.

- If $x \in C$ then clearly
$$\begin{aligned} \liminf (x_n) &\leq \inf C \leq x \leq \sup C \leq \limsup (x_n), \text{ so} \\ \liminf (x_n) &\leq x \leq \limsup (x_n) \end{aligned}$$

7. **For statement** (7) :

   Let C be the set of all cluster points of $\{x_n\}$ and

   assume that
$$x = \liminf (x_n).$$

- By previous theorem 1**a**)

- Given $\epsilon > 0$,

  there is $N \in \mathbb{N}$, such that, for all $n > N$
$$x - \epsilon < x_n.$$

- Moreover, by previous theorem 1**b**),

  there is $n_1 > N$, such that
$$x_{n_1} < x + \epsilon.$$

- Therefore,
$$x - \epsilon < x_{n_1} < x + \epsilon$$

- Taking $N = n_1$,

  there is
$$n_2 > N = n_1,$$
  such that,
$$x - \epsilon < x_{n_2} < x + \epsilon$$

- Using induction,
  we construct a sequence of natural numbers
  $$n_1 < n_2 < ...,$$
  such that
  $$x - \epsilon < x_{n_k} < x + \epsilon, \text{ for all } k = 1, 2, ...$$
- Therefore, for every $\epsilon > 0$, the set
  $$\{n \in \mathbb{N} : |x_n - x| < \epsilon\}$$
  is infinite.
- By the definition,
  $$x = \liminf (x_n)$$
  is a cluster point of $\{x_n\}$, i.e. $x \in C$.

8. Similar argument can be used to prove (8).
9. **For statement** (9) :
   Assume that $x_n \to x$,
   then $\{x_n\}$ is bounded and
   the set of cluster points of $\{x_n\}$ is $C = \{x\}$,
   hence
   $$\begin{aligned} \liminf (x_n) &= \inf C = x = \sup C = \limsup (x_n), \text{ that is} \\ \liminf (x_n) &= \limsup (x_n) = x \end{aligned}$$

- **Conversely**, if
  $$\liminf (x_n) = \limsup (x_n) = x \in \mathbb{R},$$
  then by theorem,
  there are N_1 and N_2, such that,
  for $n > N_1$,
  $$x_n < x + \epsilon$$
  and for $n > N_2$,
  $$x - \epsilon < x_n.$$
- Therefore, if
  $$n > \max \{N_1, N_2\},$$
  then
  $$\begin{aligned} x - \epsilon &< x_n < x + \epsilon, \text{ so} \\ |x_n - x| &< \epsilon. \end{aligned}$$
- It follows that $x_n \to x$ as $n \to \infty$.

This finishes our proof. ■

- **Example**: Find $\liminf (x_n)$ and $\limsup (x_n)$, for

$$x_n = 4 + (-1)^n \left(1 - \frac{1}{n}\right).$$

Justify your answer.

- Since

$$x_n = \begin{cases} 4 + \left(1 - \frac{1}{2k}\right) & if \quad n = 2k \\ 4 - \left(1 - \frac{1}{2k-1}\right) & if \quad n = 2k-1 \end{cases}$$

we see that

$$x_{2k} \to 5 \text{ and } x_{2k-1} \to 3$$

- Show that $C = \{3, 5\}$ is the set of cluster points and that

$$|x_n| \leq 5.$$

- Then by the definition

$$\begin{aligned} \liminf (x_n) &= \inf C = 3 \text{ and} \\ \limsup (x_n) &= \sup C = 5. \end{aligned}$$

Example: Find a sequence x_n with

$$\limsup (x_n) = 3$$

and

$$\liminf (x_n) = -2.$$

- Let

$$x_n = \begin{cases} 3 - \frac{1}{2k} & if \quad n = 2k \\ -2 + \frac{1}{2k-1} & if \quad n = 2k-1 \end{cases}$$

and then show that

$$\begin{aligned} \limsup (x_n) &= 3 \text{ and} \\ \liminf (x_n) &= -2 \end{aligned}$$

Example: Let $\{x_n\}$ be a sequence with

$$\liminf (x_n) = x$$

and

$$\limsup (x_n) = y,$$

where $x, y \in \mathbb{R}$.

Show that $\{x_n\}$ has subsequences $\{a_n\}$ and $\{b_n\}$, such that

$$a_n \to x \text{ and } b_n \to y.$$

Hint: Use proof from the last proposition (see 7) and 8)).

Example: Is it true that if

$$\limsup (x_n) = 2,$$

then there is $n \in \mathbb{N}$, such that

$$1.99 < x_n.$$

Justify your answer.

Hint: Use the following result:

Proposition If $\{x_n\}$ is bounded above,

$$x = \limsup (x_n)$$

if and only if

a. For all $\epsilon > 0$ there is an $N \in \mathbb{N}$, such that

$$x_n < x + \epsilon$$

whenever $n \geq N$, and

b. For all $\epsilon > 0$ and all M,
there is $n > M$ with

$$x - \epsilon < x_n$$

Example: Is it true that if

$$\limsup (x_n) = x,$$

then there is $n \in \mathbb{N}$, such that,

$$x_n \leq x.$$

Hint: Use the following result

Proposition If $\{x_n\}$ is bounded above,

$$x = \limsup (x_n)$$

if and only if

a. For all $\epsilon > 0$ there is an $N \in \mathbb{N}$, such that

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whenever $n \geq N$, and

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there is $n > M$ with

$$x - \epsilon < x_n$$

Topology of Real Numbers

- Let us start with a simple motivation from the Mathematical Analysis in which the notion of an open set arises in natural context of defining limit, continuity, and differentiation.
- Recall, the notions of *open* and *closed* are built on the notion of a distance of points in $\mathbb{R}$.

- For any $x, y \in \mathbb{R}$, to find distance between x and y $(x \leq y)$ we need to find the length of the interval $[x,\ y]$.
- The length of this interval is calculated using the absolute value function $|\quad|$ defined for all real numbers in a following way

$$d(x, y) = |y - x|$$

- For instance,

$$d(-1, 2) = |2 - (-1)| = 3.$$

- As we showed it before, the absolute value $|\quad|$ satisfies the following properties:

i) $\forall x, y \in \mathbb{R},\ |y - x| \geq 0$ and

$\forall x, y \in \mathbb{R},\ ((|y - x| = 0) \Longleftrightarrow (x = y))$

ii) $\forall x, y \in \mathbb{R},\ |y - x| = |x - y|$

iii) $\forall x, y, z \in \mathbb{R},\ |y - x| \leq |y - z| + |z - y|$

- **Remark** The property (ii) is called the *symmetry* and the property (iii) is called the *triangle inequality.*
- The notions such as *limit* and *continuity* that we are familiar with from calculus classes are based on the notion of *"being close"* for points $x, y \in \mathbb{R}$.
- For instance, we say that a function

$$f : A \to \mathbb{R},$$

where $A \subseteq \mathbb{R}$ is continuous at point $x_0 \in A$, if for any point x that is *"close"* to the point x_0 the image $f(x)$ of x, is *"close"* to $f(x_0)$.
- To make the notion of *"being close"* precise, we may say that x is *close* to x_0 if the distance of x from x_0 is *"small enough"*,
- That is, if we fix a *"small"* positive number $\epsilon > 0$, then x is *close* to x_0 if

$$|x - x_0| < \epsilon,$$

i.e. distance between x and x_0 is smaller than ϵ :

$$d(x,\ x_0) = |x - x_0| < \epsilon.$$

- Therefore, we regard each point x as *close* to x_0 if

$$d(x,\ x_0) < \epsilon$$

- For a given $\epsilon > 0$, there are, of course,
  infinitely many points $x \in \mathbb{R}$ that are *"close"* to x_0.
- That is, all points x that are *close* to x_0 form set

$$\begin{aligned} D(x_0, \epsilon) &= \{x \in \mathbb{R} \mid |x - x_0| < \epsilon\} \\ &= \{x \in \mathbb{R} \mid d(x,\ x_0) < \epsilon\}. \end{aligned}$$

  that we call $\epsilon-$**disk centered at** x_0.

  Definition Let $x_0 \in \mathbb{R}$ and $\epsilon > 0$.

  An open disk centered at x_0 (or an $\epsilon-$**neighborhood of** x_0) is the set

$$D(x_0, \epsilon) = \{x \in \mathbb{R} \mid d(x,\ x_0) < \epsilon\},$$

  where $d(x,\ y) = |x - y|$, $x, y \in \mathbb{R}$.

  The number $\epsilon > 0$ is referred to as the **radius** of $D(x_0, \epsilon)$.
- Observe that an $\epsilon-$**disk centered at** x_0 is simply
  an open interval with endpoints $x_0 - \epsilon$, $x_0 + \epsilon$.
- Indeed, we see that

$$\begin{aligned} D(x_0,\ \epsilon) &= \{x \in \mathbb{R} \mid d(x,\ x_0) < \epsilon\} \\ &= \{x \in \mathbb{R} \mid |x - x_0| < \epsilon\} \\ &= \{x \in \mathbb{R} \mid -\epsilon < x - x_0 < \epsilon\} \\ &= \{x \in \mathbb{R} \mid x_0 - \epsilon < x < x_0 + \epsilon\} \\ &= (x_0 - \epsilon,\ x_0 + \epsilon) \end{aligned}$$

  an open interval with the endpoints $x_0 - \epsilon$ and $x_0 + \epsilon$, whose center is at x_0.
- Note that endpoints $x_0 - \epsilon$ and $x_0 + \epsilon$ are of the distance ϵ
  from the center x_0 of the interval.
- The notion of an $\epsilon-$**disk centered at** x_0 is used in
  the definition of an open set in $\mathbb{R}$ as follows.

  Definition Let $U \subseteq \mathbb{R}$. We say that U is open in $\mathbb{R}$, if

$$\forall x_0 \in U,\ \exists \epsilon > 0 \ni D(x_0,\ \epsilon) \subseteq U.$$

  Example Show that an open interval $(a,\ b) \subseteq \mathbb{R}$, where $a < b$ is open.

  Solution: We need to show that

$$\forall x_0 \in (a,\ b),\ \exists \epsilon > 0 \ni D(x_0,\ \epsilon) \subseteq (a,\ b).$$

- To show the above statement,
  let $x_0 \in (a,\ b)$, and we need to find $\epsilon > 0$, such that

$$D(x_0,\ \epsilon) \subseteq (a,\ b).$$

- Since $x_0 \in (a,\ b)$, $a < x_0 < b$.

- Therefore, both numbers:

$$x_0 - a > 0 \text{ and } b - x_0 > 0.$$

- If we take

$$\epsilon = \min\{x_0 - a,\ b - x_0\},$$

  we see that $\epsilon > 0$

  (since $x_0 - a > 0$ and $b - x_0 > 0$, so the minimum of two positive numbers is also a positive number).

- Moreover,

$$(x_0 - \epsilon,\ x_0 + \epsilon) \subseteq (a,\ b).$$

- This is clear, since for all $x \in (x_0 - \epsilon,\ x_0 + \epsilon)$,

$$x_0 - \epsilon < x < x_0 + \epsilon$$

- Since $x_0 - a \geq \min\{x_0 - a,\ b - x_0\}$,

$$\begin{aligned} a &= x_0 - (x_0 - a) \leq x_0 - \min\{x_0 - a,\ b - x_0\} \\ &= x_0 - \epsilon \\ &< x \end{aligned}$$

  and $b - x_0 \geq \min\{x_0 - a,\ b - x_0\}$,

$$\begin{aligned} x &< x_0 + \epsilon = x_0 + \min\{x_0 - a,\ b - x_0\} \\ &\leq x_0 + b - x_0 \\ &= b. \end{aligned}$$

- It follows that, for all $x \in (x_0 - \epsilon,\ x_0 + \epsilon)$,

$$a < x < b,$$

- Thus, we showed that

$$D(x_0,\ \epsilon) = (x_0 - \epsilon,\ x_0 + \epsilon) \subseteq (a,\ b).$$

- By the definition of an open set,

  we conclude that an open interval $(a,\ b)$ is open in $\mathbb{R}$.