Math 4301 Mathematical Analysis I

Lecture 7

Topic: Topology of Real Numbers

- **Open and Closed Sets – Review**

  Open Sets

- Open disk

  In the real line $\mathbb{R}$, an open disk is just an open interval

$$\begin{aligned} D(x_0, \epsilon) &= \{x \in \mathbb{R} : |x - x_0| < \epsilon\} \\ &= (x_0 - \epsilon, x_0 + \epsilon). \end{aligned}$$

- We define open set as follows:

  We say that $U \subseteq \mathbb{R}$ is called open if,

- for every $x_0 \in U$, there is $\epsilon > 0$, such that

$$D(x_0, \epsilon) \subseteq U.$$

  Examples: Open disk = open interval in $\mathbb{R}$

  Theorem: If U_α is open for all $\alpha \in \Gamma$,

  then $\bigcup_{\alpha \in \Gamma} U_\alpha$ is also open

- **Theorem**: If $U_1, U_2, ..., U_n$ are open,

  then $\bigcap_{i=1}^{n} U_i$ is also open.

- Notice that $\emptyset$, $\mathbb{R}$ are open.

  Closed Sets

- We say that $A \subseteq \mathbb{R}$ is closed if $\mathbb{R} \backslash A = A^c$ is open

  Examples: Closed interval $[a, b]$ is closed.

  Examples: Finite sets are closed,

$$A = \{x_1, x_2, x_3, ..., x_n\} \subseteq \mathbb{R}$$

  Examples: $\mathbb{Z} \subseteq \mathbb{R}$ is closed but

$$\mathbb{Q} \subseteq \mathbb{R}$$

  is not closed.

  Theorem: If A_α is closed for all $\alpha \in \Gamma$,

  then $\bigcap_{\alpha \in \Gamma} A_\alpha$ is also closed.

  Theorem: If $A_1, A_2, ..., A_n$ are closed

  then $\bigcup_{i=1}^{n} A_i$ is also closed.

  Topology of Real Numbers

  Definition Let $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$.

  Then x is:

- An *interior* point of A if
  there exists $\delta > 0$ such that
$$D(x, \delta) \subseteq A$$
- An *isolated* point of A if $x \in A$ and
  there exists $\delta > 0$ such that,
  x is the only point in A that belongs to $D(x, \delta)$.
- A *boundary* point of A if
  for every $\delta > 0$ the
$$D(x, \delta) \cap A \neq \emptyset \text{ and } D(x, \delta) \cap A^c \neq \emptyset.$$
- An *accumulation* point of A if
  for every $\delta > 0$ the
$$A \setminus \{x\} \cap D(x, \delta) \neq \emptyset.$$

Remark: We define

$$\begin{aligned}
\operatorname{Int}(A) &= \{x \in A : x \text{ is an interior point of } A\} \\
\partial A &= \{x \in \mathbb{R} : x \text{ is a boundary point of } A\} \\
A' &= \{x \in \mathbb{R} : x \text{ is an accumulation point of } A\}
\end{aligned}$$

Closure of $A \subseteq \mathbb{R}$

- Using the properties of closed sets,
  we can define closure of $A \subseteq \mathbb{R}$ as follows.
- Let
$$\mathcal{C}(A) = \{C \mid A \subseteq C \text{ and } C \text{ is closed}\}$$
- and then the *closure of* A in X is defined as follows:
$$\overline{A} = \bigcap_{C \in \mathcal{C}(A)} C.$$

From the definition of $\overline{A}$ it follows immediately that

1. $\overline{A}$ is closed in X and $A \subseteq \overline{A}$.

2. If C is closed and $A \subseteq C$,

then $\overline{A} \subseteq C$ (therefore, $\overline{A}$ is the smallest closed subset of $\mathbb{R}$ that contains A).

3. If C is closed,

then $\overline{C} = C$, in particular, we have
$$\overline{\overline{A}} = \overline{A}.$$

- **Example** Let
$$A = (a,\ b) \subset \mathbb{R}.$$
Find $\overline{A}$ using the definition of the closure.
Notice that A is not closed, because
$$\mathbb{R} \backslash A = A^c = (-\infty, a] \cup [b, \infty)$$
is not open, since
$$D\,(a, \epsilon) \cap A \neq \emptyset,$$
for all $\epsilon > 0$.
- Since in analogous way we can show that
$$[a, b)\,, \quad (a, b]$$
are not closed,
- the minimal closed subset of $\mathbb{R}$ that contains A is $[a, b]$.
- Therefore, we see that
$$\overline{A} = [a, b]\,.$$

Proposition Let A, $B \subseteq \mathbb{R}$.

Then the closure satisfies the following properties.

i) If $A \subseteq B$ then
$$\overline{A} \subseteq \overline{B}.$$

ii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

iii) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

Proof. We prove for instance **i**).

- Notice that $B \subseteq \overline{B}$.
- Since
$$A \subseteq B, \text{ then } A \subseteq \overline{B}$$
- Since $\overline{B}$ is closed and it contains A,
$$\overline{A} \subseteq \overline{B}.$$
- We prove for instance **ii**)
- We see that, by i)
$$A \subseteq \overline{A} \text{ and } B \subseteq \overline{B},$$
- thus
$$A \cup B \subseteq \overline{A} \cup \overline{B}$$
- Since $\overline{A} \cup \overline{B}$ is closed and
$$A \cup B \subseteq \overline{A} \cup \overline{B},$$
then
$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$$

- since $\overline{A \cup B}$ is the smallest closed set that contains $A \cup B$.

  Conversely

- We see that
$$A \subseteq A \cup B, \text{ so by } i)\ \overline{A} \subseteq \overline{A \cup B}$$
  and
$$B \subseteq A \cup B, \text{ so by } i)\ \overline{B} \subseteq \overline{A \cup B}.$$

- Therefore,
$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$
  which finishes our proof.

Exercise show **iii**). ∎

- **Remark** The converse inclusion given in property
  (**iii**) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ is not true.

- For instance,
  let $A = (0, 1) \subset \mathbb{R}$ and $B = (1, 2) \subset \mathbb{R}$.

- Then
$$\overline{A} = [0, 1] \text{ and } \overline{B} = [1, 2]$$
  therefore
$$A \cap B = \emptyset, \text{ so } \overline{A \cap B} = \emptyset$$
  but
$$\overline{A} \cap \overline{B} = \{1\}.$$

- Therefore, the inclusion
$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$$

- can be proper, i.e.
$$\overline{A \cap B} \subset \overline{A} \cap \overline{B}.$$

  Proposition Let $A \subseteq \mathbb{R}$.
  Then
$$x \in \overline{A} \Leftrightarrow \forall \epsilon > 0,\ D(x, \epsilon) \cap A \neq \emptyset.$$

Proof. We will prove an equivalent statement,

- that is
$$x \notin \overline{A} \Leftrightarrow \exists \epsilon > 0,\ D(x, \epsilon) \cap A = \emptyset.$$

- Suppose that $x \notin \overline{A}$, then $x \in \mathbb{R} \backslash \overline{A}$.

- But $\overline{A}$ is closed, so $\mathbb{R} \backslash \overline{A}$ is open,

- therefore, there is $\epsilon > 0$, such that
$$D(x, \epsilon) \subseteq \mathbb{R} \backslash \overline{A},$$

- thus
$$D(x, \epsilon) \cap \overline{A} = \emptyset.$$

- However, $A \subseteq \overline{A}$, so

$$\begin{aligned} D(x,\epsilon) \cap A &\subseteq D(x,\epsilon) \cap \overline{A} = \emptyset, \text{ so} \\ D(x,\epsilon) \cap A &= \emptyset. \end{aligned}$$

Conversely

- Assume that, there is $\epsilon > 0$, such that

$$D(x,\epsilon) \cap A = \emptyset.$$

- Therefore, we see that

$$\begin{aligned} A &\subseteq \mathbb{R} \backslash D(x,\epsilon), \text{ so} \\ \overline{A} &\subseteq \overline{\mathbb{R} \backslash D(x,\epsilon)}, \end{aligned}$$

but $\mathbb{R} \backslash D(x,\epsilon)$ is closed
since it is a complement of an open set.

- Therefore,

$$\begin{aligned} \overline{A} &\subseteq \overline{\mathbb{R} \backslash D(x,\epsilon)} = \mathbb{R} \backslash D(x,\epsilon), \text{ so} \\ \overline{A} \cap D(x,\epsilon) &= \emptyset \end{aligned}$$

- Since $x \in D(x,\epsilon)$, then

$$x \notin \overline{A}.$$

This finishes our proof. ■

- **Example** We find the closures for the following subsets of $\mathbb{R}$:

a. $A = \mathbb{Z}$.
Notice that

$$A^c = \bigcup_{i \in \mathbb{Z}} (i, i+1)$$

is open as a union of open sets,
so A is closed by the definition.
It follows that

$$\overline{A} = \overline{\mathbb{Z}} = \mathbb{Z}.$$

b. $A = \mathbb{Q}$.
We see that $\mathbb{Q}$ is not closed and
the minimal closed subset of $\mathbb{R}$ that contains $\mathbb{Q}$ is $\mathbb{R}$, so

$$\overline{\mathbb{Q}} = \mathbb{R}.$$

Another way to see this,
we notice that $\mathbb{Q} \subseteq \overline{\mathbb{Q}}$ and if $x \in \mathbb{R}$,
then

$$D(x,\epsilon) \cap \mathbb{Q} \neq \emptyset,$$

for any $\epsilon > 0$, b/c any open interval in $\mathbb{R}$ contains a rational number.

- Therefore, by previous theorem

$$x \in \overline{\mathbb{Q}}, \text{ so } \mathbb{R} \subseteq \overline{\mathbb{Q}}, \text{ so } \overline{\mathbb{Q}} = \mathbb{R}.$$

Proposition $A \subseteq \mathbb{R}$.
Then $x \in \overline{A}$ if and only if
there is a sequence $\{x_n\}$ in A such that

$$x_n \to x.$$

Proof. Assume $x \in \overline{A}$, then

- for every $\epsilon > 0$,

$$D(x, \epsilon) \cap A \neq \emptyset.$$

- We take $\epsilon = 1, \frac{1}{2}, \ldots$, so

$$D\left(x, \frac{1}{k}\right) \cap A \neq \emptyset$$

- so there is

$$x_k \in D\left(x, \frac{1}{k}\right) \cap A,$$

for $k = 1, 2, \ldots.$

- Since

$$x_k \in D\left(x, \frac{1}{k}\right) \cap A,$$

so $x_k \in A$.

- Consider sequence $\{x_k\}$, as we see

$$\{x_k\} \subset A.$$

- We show that this sequence converges.
- Take $\epsilon > 0$, there is $K \in \mathbb{N}$, such that

$$\frac{1}{K} < \epsilon.$$

- If $k > K$, then

$$\frac{1}{k} < \frac{1}{K}.$$

- Therefore, since

$$x_k \in D\left(x, \frac{1}{k}\right) \subseteq D\left(x, \frac{1}{K}\right),$$

thus

$$|x - x_k| < \frac{1}{k} < \frac{1}{K} < \epsilon, \text{ for all } k > K.$$

- This give us that $x_n \to x$.
- Suppose that there is a sequence $\{x_n\}$ in A, such that

$$x_n \to x.$$

- Take $\epsilon > 0$, then there is $K \in \mathbb{N}$, such that, for $k > K$,

$$\begin{aligned} |x_k - x| &< \epsilon, \\ \text{i.e. } x_k &\in D(x, \epsilon), \text{ for } k > K. \end{aligned}$$

- Since $x_k \in A$,

$$x_k \in D(x, \epsilon) \cap A.$$

- In particular, for any $\epsilon > 0$,

$$D(x, \epsilon) \cap A \neq \emptyset,$$

  so $x \in \overline{A}$

This finishes our proof ■

- As we see from the examples above,
  for any $A \subseteq X$, there are two types of points in $\overline{A}$:
- 1) $x \in A$ and
- 2) $x \in \overline{A}$ and $x \notin A$.
- It is desirable, especially when we want to define
  limit of a function at a point $x_0 \in \overline{A}$,
  to describe the later points as
  accumulation points or *limit points* of A.

  Accumulation Points of $A \subseteq \mathbb{R}$

  Definition Let $A \subseteq \mathbb{R}$.
  We say that $x \in \mathbb{R}$ is a an *accumulation point* of A, if

$$\forall \epsilon > 0, \; D(x, \epsilon) \cap (A \backslash \{x\}) \neq \emptyset.$$

  The set of all *accumulation points* of A is denoted by A'.

  Example Let $A = (a, b) \subseteq \mathbb{R}$.
  Find all accumulation points of A.

- If $x < a$ or $x > b$, then for

$$\begin{gathered} \epsilon = \min\{a - x, x - b\} > 0, \\ D(x, \epsilon) \cap (A \backslash \{x\}) = \emptyset. \end{gathered}$$

- Therefore, $x \in \mathbb{R} \backslash [a, b]$ is not an accumulation point of A.
- If $x \in [a, b]$, then
  for all $\epsilon > 0$,

$$D(x, \epsilon) \cap (A \backslash \{x\}) \neq \emptyset.$$

- Indeed, if $x \in (a, b)$, then
  there is $\epsilon > 0$, such that

$$D(x, \epsilon) \subseteq A$$

  we simply take

$$\epsilon = \min\{x - a, b - x\} > 0.$$

- Since $D(x,\epsilon)$ is infinite
$$D(x,\epsilon) \cap (A \setminus \{x\}) \neq \emptyset.$$
- If $\delta \geq \epsilon$, then
$$\emptyset \neq D(x,\epsilon) \cap (A \setminus \{x\}) \subseteq D(x,\delta) \cap (A \setminus \{x\})$$
and therefore
$$D(x,\delta) \cap (A \setminus \{x\}) \neq \emptyset.$$
- If $0 < \delta < \epsilon$, then
$$D(x,\delta) \subseteq D(x,\epsilon) \subseteq (a,b) = A$$
and $D(x,\delta)$ is infinite, so
$$D(x,\delta) \cap (A \setminus \{x\}) \neq \emptyset.$$
- It follows that if $x \in A$
then $x \in A'$.
- Furthermore, if $x = a$,
then for $\epsilon > 0$,
$$y = \min\left\{\frac{a+b}{2}, a + \frac{\epsilon}{2}\right\} \in D(x,\epsilon) \cap (A \setminus \{x\}),$$
so
$$D(x,\epsilon) \cap (A \setminus \{x\}) \neq \emptyset.$$
- If $x = b$, then
$$y = \max\left\{\frac{a+b}{2}, b - \frac{\epsilon}{2}\right\} \in D(x,\epsilon) \cap (A \setminus \{x\}),$$
so
$$D(x,\epsilon) \cap (A \setminus \{x\}) \neq \emptyset.$$
- It follows that $a, b \in A'$.
- Therefore, we prove that
$$A' = [a,b].$$
- The following connection holds between the closure and
the set of accumulation points.
Theorem Let $A \subseteq \mathbb{R}$. Then
$$\overline{A} = A \cup A'$$

Proof. We show that $A \cup A' \subseteq \overline{A}$.

- If $x \in A'$, then
$$\forall \epsilon > 0,\ D(x,\epsilon) \cap A \setminus \{x\} \neq \emptyset.$$
- Since
$$D(x,\epsilon) \cap A \setminus \{x\} \subseteq D(x,\epsilon) \cap A,$$
$$\forall \epsilon > 0,\ D(x,\epsilon) \cap A \neq \emptyset,$$
hence $x \in \overline{A}$.

- Since $A \subseteq \overline{A}$,

$$A \cup A' \subseteq \overline{A}.$$

Conversely

- if $x \in \overline{A}$, then either $x \in A$ or

$$\begin{aligned} x &\notin A \text{ and} \\ \forall \epsilon &> 0,\ D(x,\epsilon) \cap A \neq \emptyset. \end{aligned}$$

- If $x \in A$, then since $A \subset A \cup A'$,

$$x \in A \cup A'.$$

- If $x \notin A$ and

$$\forall \epsilon > 0, D(x,\epsilon) \cap A \neq \emptyset$$

then, since $x \notin A$,

$$\begin{aligned} D(x,\epsilon) \cap A &= D(x,\epsilon) \cap A \backslash \{x\},\ \text{so} \\ \forall \epsilon &> 0,\ D(x,\epsilon) \cap A \backslash \{x\} \neq \emptyset, \end{aligned}$$

- Consequently,

$$x \in A' \subseteq A \cup A',$$

so again

$$x \in A \cup A',$$

We showed that

$$\overline{A} = A \cup A'$$

which finish our argument here. ■

- **Corollary** Let $A \subseteq \mathbb{R}$.
  A is closed in $\mathbb{R}$ $\left(A = \overline{A}\right)$ if and only if $A' \subseteq A$.

Proof. Since

$$\overline{A} = A \cup A'$$

and $A \subseteq \overline{A}$, then

- if $A' \subseteq A$,

$$\begin{aligned} \overline{A} &= A \cup A' \subseteq A \cup A = A,\ \text{so} \\ \overline{A} &\subseteq A. \end{aligned}$$

- Therefore, since $A \subseteq \overline{A}$,

$$\overline{A} = A,$$

so A is closed.

This finishes our proof. ■

- **Example** Let us consider

$$A = [a, b] \subseteq \mathbb{R}.$$

We observe that

$$\begin{aligned} A' &= [a, b], \text{ so} \\ A' &\subseteq A, \text{ consequently} \\ A &= \overline{A}. \end{aligned}$$

Therefore, we have that A is closed.

Boundary of $A \subseteq \mathbb{R}$

Definition Let $A \subseteq \mathbb{R}$.

- We say that $x \in \mathbb{R}$ is a boundary point of A if

$$\forall \epsilon > 0, \ \ (D(x, \epsilon) \cap A \neq \emptyset) \wedge (D(x, \epsilon) \cap \mathbb{R} \backslash A \neq \emptyset).$$

- The collection of all boundary points of A in $\mathbb{R}$ is denoted by ∂A and we call it boundary of A.

Remark It is clear from the definition that,
$x \in X$ is a boundary point of A iff $x \in \overline{A}$ and $x \in \overline{\mathbb{R} \backslash A}$.
Therefore, the following connection holds between boundary and closure operations:

$$\partial A = \overline{A} \cap \overline{\mathbb{R} \backslash A}.$$

Example Find ∂A for the following subsets A of $\mathbb{R}$.

a. $A = \{x_1, \ x_2, \ ..., \ x_n\}, n \in \mathbb{N}$

We will use the formula

$$\partial A = \overline{A} \cap \overline{\mathbb{R} \backslash A}$$

- Since A is finite, A is closed.
- Therefore,

$$\overline{A} = A.$$

- Assume that

$$x_1 < x_2 < ... < x_n.$$

- Therefore,

$$\mathbb{R} \backslash A = (-\infty, x_1) \cup (x_1, x_2) \cup ... \cup (x_n, \infty).$$

- One can check that each x_i, $i = 1, 2, ..., n$ is an accumulation point of $\mathbb{R} \backslash A$.
- It follows that

$$\mathbb{R} \backslash A \cup (\mathbb{R} \backslash A)' \supseteq \mathbb{R} \backslash A \cup A = \mathbb{R}.$$

- Therefore, we see that

$$\overline{\mathbb{R} \backslash A} = \mathbb{R}.$$

- It follows that

$$\begin{aligned}\partial A &= \overline{A} \cap \overline{\mathbb{R}\backslash A}\\ &= A \cap \mathbb{R}\\ &= A.\end{aligned}$$

b. $A = \mathbb{Q}$

- Since $\overline{\mathbb{Q}} = \mathbb{R}$ (as we proved today) and
  as we can show $\overline{\mathbb{R}\backslash\mathbb{Q}} = \mathbb{R}$.
- Then

$$\begin{aligned}\partial A &= \overline{A} \cap \overline{\mathbb{R}\backslash A}\\ &= \overline{\mathbb{Q}} \cap \overline{\mathbb{R}\backslash\mathbb{Q}}\\ &= \mathbb{R} \cap \mathbb{R}\\ &= \mathbb{R}.\end{aligned}$$

  Notice that $\mathbb{Q} \subsetneq \partial\mathbb{Q}$

  Proposition Let $A \subseteq \mathbb{R}$.

  Then A is closed in $\mathbb{R}$ if and only if

$$\partial A \subseteq A.$$

Proof. We show that the statements are equivalent.

- If A is closed then $\overline{A} = A$.
- Since $\partial A \subseteq \overline{A}$ ($\partial A = \overline{A} \cap \overline{\mathbb{R}\backslash A}$),
- it follows that

$$\partial A \subseteq \overline{A} = A,$$

  so $\partial A \subseteq A$.

- **Conversely**
  we show that $\overline{A} = A$.
- Clearly $A \subseteq \overline{A}$, so
  it suffices to show that $\overline{A} \subseteq A$.
- If $x \in \overline{A}$ and $x \notin A$, then

$$x \in \mathbb{R}\backslash A$$

  and

- For all $\epsilon > 0$,

$$D(x, \epsilon) \cap \mathbb{R}\backslash A \neq \emptyset.$$

- Otherwise, there is $\epsilon > 0$, such that

$$D(x, \epsilon) \cap \mathbb{R}\backslash A = \emptyset,$$

  so

$$D(x, \epsilon) \cap A \neq \emptyset,$$

  a contradiction.

- Thus,
$$x \in \overline{A} \cap \overline{\mathbb{R} \backslash A} = \partial A \subseteq A,$$
so
- Hence, $x \in A$.
- It follows that $\overline{A} \subseteq A$.
So A is closed.

This finishes our proof. ■

- **Example** Let $A = [a, b]$.
Then $A' = [a, b]$ but
$$\partial A = \{a, b\}$$
so $\partial A \subset A'$.
Interior of $A \subseteq \mathbb{R}$
- Let
$$\mathcal{U}(A) = \{U \subseteq \mathbb{R} \mid U \subseteq A \text{ and } U \text{ is open}\}.$$
- Define interior of A as follows
$$\operatorname{Int}(A) = \bigcup_{U \in \mathcal{U}(A)} U.$$
- It is clear that $\operatorname{Int}(A)$ satisfies the following properties:

1. $\operatorname{Int}(A)$ is open in $\mathbb{R}$ and
$$\operatorname{Int}(A) \subseteq A.$$
2. If U is open and $U \subseteq A$, then
$$U \subseteq \operatorname{Int}(A).$$
3. If U is open in $\mathbb{R}$, then
$$\operatorname{Int}(U) = U$$
and, in particular,
for any $A \subseteq X$,
$$\operatorname{Int}(\operatorname{Int}(A)) = \operatorname{Int}(A).$$

- Before, we show some examples of computations related to $\operatorname{Int}(A)$, it is useful to discuss characterization of the points from the interior.

Proposition 0.1 *Let $A \subseteq \mathbb{R}$. Then*
$$x \in \operatorname{Int}(A) \Leftrightarrow \exists \epsilon > 0 \ni D(x, \epsilon) \subseteq A.$$

Proof. We show that if
$$x \in \operatorname{Int}(A) \Rightarrow \exists \epsilon > 0 \ni D(x, \epsilon) \subseteq A.$$

- If $x \in \text{Int}(A)$, then since
$$\text{Int}(A) = \bigcup_{U \in \mathcal{U}(A)} U,$$
there is $U \subseteq A$, U - open, such that
$$x \in U.$$

- Since U is open, then
$$\exists \epsilon > 0 \ni D(x, \epsilon) \subseteq U.$$

- Since $U \subseteq A$,
$$\exists \epsilon > 0 \ni D(x, \epsilon) \subseteq A.$$

Conversely

we show that, if $\exists \epsilon > 0 \ni D(x, \epsilon) \subseteq A$ then $x \in \text{Int}(A)$

- If $\exists \epsilon > 0 \ni D(x, \epsilon) \subseteq A$, then
$$D(x, \epsilon) \subseteq \text{Int}(A)$$
since $\text{Int}(A)$ is the largest open subset of A and
$D(x, \epsilon)$ is an open subset of A.

- Since $x \in D(x, \epsilon)$,
$$x \in \text{Int}(A).$$

This finishes our argument. ■

- **Example** Let $A \subseteq \mathbb{R}$.
We find $\text{Int}(A)$ for the following subsets A of $\mathbb{R}$.

1. $A = \{x\}$

- We notice that
$$\text{Int}(A) \subseteq A, \text{ so}$$
if $y \in \text{Int}(A)$ then
$$y \in A = \{x\},$$
so $y = x$.

- Take $\epsilon > 0$ and notice that
$$\begin{aligned} D(x, \epsilon) &= (x - \epsilon, x + \epsilon) \\ &\nsubseteq \{x\} \\ &= A. \end{aligned}$$

- Therefore, $x \notin \text{Int}(A)$.

- Therefore, we see that
$$\text{Int}(A) = \emptyset.$$
We can also see it as follows:

- $\text{Int}(A)$ is the largest open subset of A.

- Since A is finite,
  it cannot contain any non-trivial open set
  (b/c open set must contain an open disk and disk is infinite).
- It follows that largest open subset of A is the empty set $\emptyset$.
- Therefore,
$$\operatorname{Int}(A) = \emptyset.$$

2. $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$

- Since $\operatorname{Int}(A) \subseteq A$,
  if $x \in \operatorname{Int}(A)$, then
$$x = \frac{1}{n},$$
  for some $n \in \mathbb{N}$.
- However, we see that
$$D\left(\frac{1}{n}, \epsilon\right) \nsubseteq A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$$
  since $D\left(\frac{1}{n}, \epsilon\right)$ is uncountable while A is countable.
- Therefore,
$$\frac{1}{n} \notin \operatorname{Int}(A).$$
- It follows that
$$\operatorname{Int}(A) = \emptyset.$$

c. $A = \mathbb{Q}$.

- If $x \in \operatorname{Int}(A)$, the $x \in \mathbb{Q}$.
- However,
$$D(x, \epsilon) \nsubseteq \mathbb{Q}$$
  since $D(x, \epsilon)$ contains irrational number
  ($D(x, \epsilon)$ is not countable and $\mathbb{Q}$ is countable).
- It follows that
$$\operatorname{Int}(A) = \emptyset.$$

d. $A = (a, b]$

- Since the largest open subset of $(a, b]$ is (a, b), then
$$\operatorname{Int}((a, b]) = (a, b).$$

Proposition Let $A, B \subseteq \mathbb{R}$.
Then the following is true

i) If $A \subseteq B$ then
$$\operatorname{Int}(A) \subseteq \operatorname{Int}(B)$$

ii) $\mathrm{Int}\,(A \cup B) \supseteq \mathrm{Int}\,(A) \cup \mathrm{Int}\,(B)$

iii) $\mathrm{Int}\,(A \cap B) = \mathrm{Int}\,(A) \cap \mathrm{Int}\,(B)$

Proof. We prove $i)$ and we leave $ii)$ and $iii)$ as an exercise.

- For instance, for $i)$:
- Since

$$\mathrm{Int}\,(A) \subseteq A$$

  and $A \subseteq B$, then

$$\mathrm{Int}\,(A) \subseteq B$$

  and $\mathrm{Int}\,(A)$ is open.
- Since $\mathrm{Int}\,(B)$ is the largest subset of B then

$$\mathrm{Int}\,(A) \subseteq \mathrm{Int}\,(B).$$

This finishes our proof. ■

- **Exercise** Find $\mathrm{Int}\,([1,\ 3])$
  Since, as we can show,
  $(1,3)$ is the largest open subset of $[1,\ 3]$, then

$$\mathrm{Int}\,([1,\ 3]) = (1,3).$$

  Proposition Let $A \subseteq \mathbb{R}$. Then

$$\overline{A} = \mathrm{Int}\,(A) \cup \partial A,$$

  in particular,

$$\begin{aligned} \mathrm{Int}\,(A) &= \overline{A} \backslash \partial A \\ &= \overline{A} \backslash \left(\overline{\mathbb{R} \backslash A} \right). \end{aligned}$$

Proof. We show that

$$\overline{A} \supseteq \mathrm{Int}\,(A) \cup \partial A.$$

- Since

$$\overline{A} \supseteq \mathrm{Int}\,(A)$$

  and $\overline{A} \supseteq \partial A$,

$$\overline{A} \supseteq \mathrm{Int}\,(A) \cup \partial A$$

  Conversely, we show that

$$\mathrm{Int}\,(A) \cup \partial A \supseteq \overline{A}$$

- If $x \in \overline{A}$, then

$$\forall \epsilon > 0,\ D\,(x,\ \epsilon) \cap A \neq \emptyset.$$

- We consider two cases

$$\mathbf{1})\ \exists \epsilon > 0 \ni D\,(x,\ \epsilon) \subseteq A$$

  or

$$\mathbf{2})\ \forall \epsilon > 0, D\,(x,\ \epsilon) \cap (\mathbb{R} \backslash A) \neq \emptyset.$$

- In the first case, clearly
$$x \in \mathrm{Int}\,(A)\,.$$
- In the second case,
$$\forall \epsilon > 0,\ \ (D\,(x,\ \epsilon) \cap A \neq \emptyset) \wedge (D\,(x,\ \epsilon) \cap (\mathbb{R} \backslash A) \neq \emptyset)\,,$$
so $x \in \partial A$.
- Therefore,
$$x \in \mathrm{Int}\,(A) \ \text{ or } x \in \partial A.$$
- It follows that
$$\overline{A} \subseteq \mathrm{Int}\,(A) \cup \partial A.$$
- Hence
$$\overline{A} = \mathrm{Int}\,(A) \cup \partial A.$$
- It is easy to see that
- if $x \in \mathrm{Int}\,(A)$ then $x \notin \partial A$ thus
$$\mathrm{Int}\,(A) \cap \partial A = \emptyset.$$
- Therefore,
$$\begin{aligned} \mathrm{Int}\,(A) &= \overline{A} \backslash \partial A \\ &= \overline{A} \backslash \left(\overline{A} \cap \overline{\mathbb{R} \backslash A} \right) \\ &= \overline{A} \backslash \overline{\mathbb{R} \backslash A}, \ \text{hence} \\ \mathrm{Int}\,(A) &= \overline{A} \backslash \overline{\mathbb{R} \backslash A}. \end{aligned}$$

The last identity completes our proof. ■