Math 4301 Mathematical Analysis I
Lecture 12
Topic: Differentiability

- **Differentiability**

  Definition A function $f:(a,b)\to\mathbb{R}$ is *differentiable* at $x_0\in(a,b)$ if

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

  exists (and is finite).

  In such a case the value of this limit is denoted by $f'(x_0)$ and we call it the *first derivative* of f at x_0,

  that is

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}.$$

- We observe that, if $f'(x_0)$ exists, then

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h},\ \text{so} \\ \lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h}-f'(x_0)\right) &= 0,\ \text{so} \\ \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)-f'(x_0)h}{h} &= 0. \end{aligned}$$

- Since, $f'(x_0)\in\mathbb{R}$, we may also say that $f:(a,b)\to\mathbb{R}$ is differentiable at x_0

  if there is **a real number** $a\in\mathbb{R}$, such that

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)-ah}{h}=0.$$

  Example: Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=x^2$.

  Show that f is differentiable at each $x_0\in\mathbb{R}$.

- We see that

$$\begin{aligned} \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} &= \lim_{h\to 0}\frac{(x_0+h)^2-x_0^2}{h}=\lim_{h\to 0}\frac{h(h+2x_0)}{h} \\ &= \lim_{h\to 0}(h+2x_0)=2x_0 \end{aligned}$$

  thus the limit exist.

- Therefore, f is differentiable at x_0 and $f'(x_0)=2x_0$.

  Remark: Assume that $f:(a,b)\to\mathbb{R}$ and $f'(x)$ exists for all $x\in(a,b)$,

  then we can define a new function, *called the derivative of* f, $Df:(a,b)\to\mathbb{R}$,

$$Df(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x).$$

- As we can see the derivative of f at x is define as above for function with the domain that is an interval.

- It is easy to notice that we can define the first derivative for a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ for each point

  $x \in \text{Int}(A)$.

  Remark: Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in \text{Int}(A)$,

  then there is $\delta > 0$, such that $D(x_0, \delta) \subseteq A$.

- Therefore, we may define

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

  since if $0 < |h| < \delta$, then

$$x_0 + h \in D(x_0, \delta).$$

  Example: Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$.

  We show that f is differentiable for all $x \neq 0$.

- We need to check if

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

  exists.

- By theorem, $\lim_{h \to 0} \frac{|h|}{h}$ exists iff $\lim_{h \to 0^+} \frac{|h|}{h}$ and $\lim_{h \to 0^-} \frac{|h|}{h}$ both exist and they are equal.
- However, if $h < 0$, then $|h| = -h$, and therefore

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

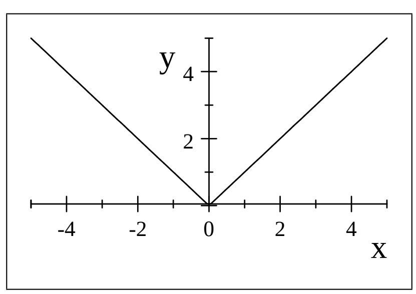
  and if $h > 0$, then $|h| = h$, so

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1.$$

- Since

$$\lim_{h \to 0^-} \frac{|h|}{h} \neq \lim_{h \to 0^+} \frac{|h|}{h}$$

  then $\lim_{h \to 0} \frac{|h|}{h}$ does not exist.



Therefore, f is not differentiable.

Example: Let $f : \mathbb{R} \to \mathbb{R}$, be given by

$$f(x) = \begin{cases} \frac{1}{x} & if \quad x \neq 0 \\ 1 & if \quad x = 0 \end{cases}$$

- We show that f is differentiable for all $x \in \mathbb{R} \setminus \{0\}$.
- Indeed, we see that, if $x \neq 0$, then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = -\lim_{h \to 0} \frac{1}{x(x+h)} = -\frac{1}{x^2}$$

so

$$f'(x) = -\frac{1}{x^2}.$$

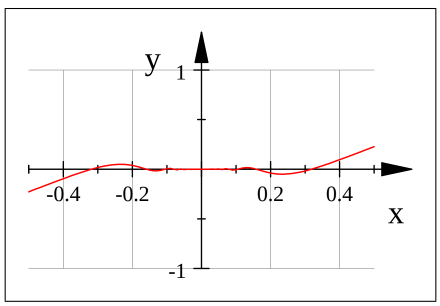
Example: Let $f : \mathbb{R} \to \mathbb{R}$, be given by $f(x) = \sqrt{|x|}$.

We show that f is differentiable at each $x \neq 0$.

Exercise: Let $f : \mathbb{R} \to \mathbb{R}$, be given by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}.$$

- Show that f is differentiable, for all $x \neq 0$.



- We see that

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 0}{h} = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right).$$

- Now, we notice that

$$\begin{aligned} 0 &\leq \left| h \sin\left(\frac{1}{h}\right) \right| = |h| \left| \sin\left(\frac{1}{h}\right) \right| \leq |h|, \text{ since } \left| \sin\left(\frac{1}{h}\right) \right| \leq 1, \text{ so} \\ 0 &\leq \left| h \sin\left(\frac{1}{h}\right) \right| \leq |h|. \end{aligned}$$

- Since $\lim_{h \to 0} |h| = 0$, the by the theorem

$$\lim_{h \to 0} \left| h \sin\left(\frac{1}{h}\right) \right| = 0, \text{ so } \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0.$$

- We showed that$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = 0$, so $f'(0) = 0$.

Exercise: Let $f : \mathbb{R} \to \mathbb{R}$, be given by

$$f(x) = |x|^3.$$

Show that f is differentiable, for all $x \in \mathbb{R}$.

Definition Let $f : [a, b] \to \mathbb{R}$ and $a \leq x_0 < b$.

Define

$$f'\left(x_0^+\right) = \lim_{h\to 0^+} \frac{f\left(x_0+h\right)-f\left(x_0\right)}{h}$$

provided that the limit exists and we call it the *right derivative* at x_0.

Analogously, if $a < x_0 \leq b$, then

$$f'\left(x_0^-\right) = \lim_{h\to 0^-} \frac{f\left(x_0+h\right)-f\left(x_0\right)}{h}$$

provided that the limit exists and we call it the left derivative of f at x_0.

Proposition Let $f : [a,b] \to \mathbb{R}$ and $a < x_0 < b$.

Then f is differentiable at x_0 iff both $f'\left(x_0^+\right)$ and $f'\left(x_0^-\right)$ exist and $f'\left(x_0^+\right) = f'\left(x_0^-\right)$.

Proof. Exercise ■

- **Exercise**: Let $f : \mathbb{R} \to \mathbb{R}$, be given by

$$f(x) = \begin{cases} x^2 & if \quad x > 0 \\ -x & if \quad x \leq 0 \end{cases}.$$

- Find $f'(0^+)$ and $f'(0^-)$.

  Is f differentiable at $x_0 = 0$?

- We see that

$$\begin{aligned} f'\left(0^+\right) &= \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{f(h)}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = 0 \text{ and} \\ f'\left(0^-\right) &= \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{f(h)}{h} = \lim_{h\to 0^+} \frac{-h}{h} = -1 \end{aligned}$$

- By the theorem, since $f'(0^+) \neq f'(0^-)$, f is not differentiable at $x = 0$.

  Theorem Let $f : (a,b) \to \mathbb{R}$ and $x_0 \in (a,b)$. If f is differentiable at x_0 then f is continuous at x_0.

Proof. We want to show that

$$\lim_{x\to x_0} f(x) = f(x_0).$$

- Let $\epsilon > 0$ be given.
- Since f is **differentiable** at x_0,

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$$

  exists.

- Therefore, there is $\delta_1 > 0$, such that, for all $x \in (a,b)$, if $0 < |x-x_0| < \delta_1$, then

$$\left|\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)\right| < \frac{\epsilon}{2}.$$

- Hence, if $0 < |x - x_0| < \min\{\delta_1, 1\}$, then

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| < \frac{\epsilon}{2}|x - x_0| < \frac{\epsilon}{2}.$$

- Therefore, when $0 < |x - x_0| < \min\{\delta_1, 1\}$

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f(x_0) - f'(x_0)(x - x_0) + f'(x_0)(x - x_0)| \\ &\leq |f(x) - f(x_0) - f'(x_0)(x - x_0)| + |f'(x_0)||x - x_0| \end{aligned}$$

- If $0 < |x - x_0| < \frac{\epsilon}{2(|f'(x_0)|+1)}$, then

$$|f'(x_0)||x - x_0| < \frac{\epsilon|f'(x_0)|}{2(|f'(x_0)| + 1)} \leq \frac{\epsilon}{2}.$$

- Let $\delta = \frac{1}{2}\min\left\{1, \delta_1, \frac{\epsilon}{2(|f'(x_0)|+1)}\right\} > 0$ and assume that for $x \in (a, b)$,
- if $0 < |x - x_0| < \delta$, then

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f(x_0) - f'(x_0)(x - x_0)| + |f'(x_0)||x - x_0| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

  If $x = x_0$, then clearly,

$$|f(x) - f(x_0)| = |f(x_0) - f(x_0)| = 0 < \epsilon.$$

- Therefore, for $\epsilon > 0$, there is $\delta > 0$, such that, for all $x \in (a, b)$, if $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| < \epsilon.$$

It follows that f is continuous at x_0. ■

- **Rules for differentiation**

  Theorem Let $f, g : (a, b) \to \mathbb{R}$ be differentiable at $x_0 \in (a, b)$. Then

i) $\alpha f + \beta g$ is differentiable at x_0 and

$$(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0).$$

ii) fg is differentiable at x_0 and

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

iii) $\frac{f}{g}$ is differentiable at x_0 and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}$$

whenever $g(x_0) \neq 0$.

Proof. Exercise. ■

- **Theorem** (*Chain Rule*) Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable at $x_0 \in \operatorname{Int}(A)$ and $g : B \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable at $f(x_0) \in \operatorname{Int}(B)$, where $f(A) \subseteq B$.
  Then $g \circ f : A \to \mathbb{R}$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(f(x_0)) f'(x_0).$$

Proof. We show that

$$(g \circ f)'(x_0) = g'(f(x_0)) f'(x_0),$$

that is,

$$\begin{aligned}(g \circ f)'(x_0) &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} \\ &= g'(f(x_0)) f'(x_0).\end{aligned}$$

- Without lose of generality we may assume that:
  $A = (a, b)$ and $B = (c, d)$,
  $x_0 \in (a, b)$, $f(x_0) \in (c, d)$ and
  $f((a, b)) \subseteq (c, d)$.
- Since g is differentiable at $f(x_0)$, it follows that

$$\lim_{y \to f(x_0)} \frac{g(y) - g(f(x_0))}{y - f(x_0)} = g'(f(x_0)).$$

- Define $h : (c, d) \to \mathbb{R}$ by

$$h(y) = \begin{cases} \frac{g(y)-g(f(x_0))}{y-f(x_0)} & if \quad y \neq f(x_0) \\ g'(f(x_0)) & if \quad y = f(x_0) \end{cases}.$$

- Since

$$\lim_{y \to f(x_0)} h(y) = \lim_{y \to f(x_0)} \frac{g(y) - g(f(x_0))}{y - f(x_0)} = g'(f(x_0))$$

  it follows that h is continuous at $f(x_0)$, i.e.

$$h(f(x_0)) = \lim_{y \to f(x_0)} h(y)$$

- Since f is differentiable at x_0, f is continuous at x_0.
- Thus,

$$h \circ f : (a, b) \to \mathbb{R}$$

  is continuous at $x_0 \in (a, b)$ as a composition of continuous functions.
- Therefore,

$$\lim_{x \to x_0} (h \circ f)(x) = h(f(x_0)) = g'(f(x_0)).$$

- Recall, since

$$h(y) = \begin{cases} \frac{g(y)-g(f(x_0))}{y-f(x_0)} & if \quad y \neq f(x_0) \\ g'(f(x_0)) & if \quad y = f(x_0) \end{cases}$$

- For all $y \in (c,d)$,
$$g(y) - g(f(x_0)) = h(y)(y - f(x_0)).$$
- Since $f((a,b)) \subseteq (c,d)$, this gives
$$g(f(x)) - g(f(x_0)) = h(f(x))(f(x) - f(x_0)), \text{ for all } x \in (a,b).$$
- Therefore, for all $x \in (a,b)$, if $x \neq x_0$, then
$$\begin{aligned} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} &= \frac{g(f(x)) - g(f(x_0))}{x - x_0} \\ &= h(f(x)) \frac{f(x) - f(x_0)}{x - x_0} \\ &= (h \circ f)(x) \frac{f(x) - f(x_0)}{x - x_0}. \end{aligned}$$
- It follows that
$$\begin{aligned} \lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} &= \lim_{x \to x_0} (h \circ f)(x) \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} (h \circ f)(x) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= g'(f(x_0)) f'(x_0). \end{aligned}$$

This finishes our proof. ■

- **Example**: Let $f : \mathbb{R} \to \mathbb{R}$, be given by
$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}.$$

  We show that f is differentiable, for all $x \in \mathbb{R}$.
- Indeed, if $x_0 \neq 0$, then
$$u(x) = \frac{1}{x}$$
  is differentiable at x_0 and
- since sin is also differentiable,
  by the chain rule $\sin \circ u$ is differentiable at x_0 and
$$(\sin \circ u)'(x_0) = \cos(u(x_0)) u'(x_0) = -\frac{1}{x_0} \cos\left(\frac{1}{x_0}\right).$$
- Therefore, by the product rule
$$f'(x_0) = 2x_0 \sin\left(\frac{1}{x_0}\right) - \cos\left(\frac{1}{x_0}\right).$$
- Now, for $x_0 = 0$,
$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

- Indeed, since
$$0 \le \left| \sin\left(\frac{1}{x}\right) \right| \le 1,$$
for all $x \neq 0$ and
$$\lim_{x\to 0} |x| = 0,$$
for all $x \neq 0$:
$$0 \le |x| \left| \sin\left(\frac{1}{x}\right) \right| \le |x|,$$
- so
$$\lim_{x\to 0} \left| x\sin\left(\frac{1}{x}\right) \right| = 0, \text{ hence } \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0.$$
- We see that
$$\begin{aligned} f' &: \mathbb{R} \to \mathbb{R}, \\ f'(x) &= \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}. \end{aligned}$$
- One shows that f' is not continuous.

  Definition A function $f : (a, b) \to \mathbb{R}$ is called a function of class $\mathcal{C}^1$ on (a, b) *(*or *continuously differentiable* on (a, b) *)* if $f' : (a, b) \to \mathbb{R}$ is continuous.

  We write, $f \in \mathcal{C}^1(a, b)$ if f is of class $\mathcal{C}^1$ on (a, b).
- We see that the function $f : \mathbb{R} \to \mathbb{R}$, be given by
$$f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}.$$
  is differentiable, but the derivative
$$\begin{aligned} f' &: \mathbb{R} \to \mathbb{R}, \text{ defined by} \\ f'(x) &= \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}. \end{aligned}$$
  is not continuous at $x = 0$, so f is not class $\mathcal{C}^1$ on $\mathbb{R}$.

  Proposition Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be injective and differentiable at $x_0 \in \text{Int}(A)$.

  If $f^{-1} : f(A) \to \mathbb{R}$ is defined by $f^{-1}(y) = x$ iff $y = f(x)$ and f^{-1} is differentiable at $f(x_0) \in \text{Int}(f(A))$

  then $f'(x_0) \neq 0$ and
$$\left(f^{-1}\right)'(f(x_0)) = \frac{1}{f'(x_0)}$$

Proof. Since $\left(f^{-1} \circ f\right)(x) = x$, for all $x \in A$ and both f and f^{-1} are differentiable at x_0 and $f(x_0)$,

- by the chain rule
$$\frac{d}{dx}\left(f^{-1} \circ f\right)(x_0) = f'(f(x_0))\, f'(x_0) = 1$$

- In particular, $f'(x_0) \neq 0$ and
$$\left(f^{-1}\right)'(f(x_0)) = \frac{1}{f'(x_0)}.$$

This finishes our proof. ■

- **Example**: Let $f : (0,\infty) \to (0,\infty)$, be given by $f(x) = x^2$.
  Then $f^{-1} : (0,\infty) \to (0,\infty)$ is defined by
$$f^{-1}(y) = \sqrt{y}$$
  and it is differentiable, for all $y \in (0,\infty)$ and
$$\left(f^{-1}\right)'(y) = \frac{1}{2\sqrt{y}}.$$

- We see that $f'(x) = 2x \neq 0$, for all $x \in (0,\infty)$, then
$$\left(f^{-1}\right)'(f(x)) = \frac{1}{2\sqrt{f(x)}} = \frac{1}{2x} = \frac{1}{f'(x)}, \text{ for all } x \in (0,\infty).$$

  Example: Let $f : \mathbb{R} \to \mathbb{R}$, be given by
$$f(x) = x^3.$$

- Then $f^{-1} : \mathbb{R} \to \mathbb{R}$ is defined by $f^{-1}(y) = \sqrt[3]{y}$ and it is differentiable, for all $y \in \mathbb{R}\setminus\{0\}$ and
$$\left(f^{-1}\right)'(y) = \frac{1}{3\sqrt[3]{y^2}}.$$

- We see that $f'(x) = 3x^2 \neq 0$, for all $x \in \mathbb{R}\setminus\{0\}$, then
$$\left(f^{-1}\right)'(f(x)) = \frac{1}{3\sqrt[3]{(f(x))^2}} = \frac{1}{3x^2} = \frac{1}{f'(x)}, \text{ for all } x \in \mathbb{R}\setminus\{0\}.$$

- We observe f^{-1} is not differentiable at $y = 0$, so the theorem does not apply and we cannot conclude that $f'(0) \neq 0$.

  Definition Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in A$. We say that

i) f has an absolute maximum at x_0 if, for all $x \in A$,
$$f(x) \leq f(x_0);$$

ii) f has a local maximum at x_0 if there exists $U \subseteq \mathbb{R}$, U-open, $x_0 \in U$ and for all $x \in A \cap U$,
$$f(x) \leq f(x_0);$$

iii) f has an absolute minimum at x_0 if, for all $x \in A$,
$$f(x_0) \leq f(x);$$

iv) f has a local minimum at x_0 if there exists $U \subseteq \mathbb{R}$, U-open, $x_0 \in U$ and for all $x \in A \cap U$,
$$f(x_0) \leq f(x);$$

- We say that $x_0 \in A$ is an extreme point of f if
  f has absolute or a local maximum or minimum at x_0.
  Proposition Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and assume that f is differentiable at $x_0 \in \operatorname{Int}(A)$ and x_0 is a local extremum point. Then
$$f'(x_0) = 0.$$

Proof. Assume *without the lose of generality* that f has a local maximum at $x_0 \in \operatorname{Int}(A)$.

- Therefore, there is $\delta > 0$, such that,
$$D(x_0, \delta) \subseteq A$$
  and for all $x \in D(x_0, \delta)$
$$f(x) \leq f(x_0).$$
- For $x \in (x_0 - \delta, x_0) \cap A$,
$$f(x) - f(x_0) \leq 0$$
  and $x - x_0 < 0$.
- Therefore,
$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0, \text{ for all } x \in (x_0 - \delta, x_0) \cap A,$$
  hence
$$f'\left(x_0^-\right) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$
- Analogously, for $x \in (x_0, x_0 + \delta) \cap A$,
$$f(x) - f(x_0) \leq 0$$
  and $x - x_0 > 0$.
- Therefore,
$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0, \text{ for all } x \in (x_0, x_0 + \delta) \cap A$$
  hence
$$f'\left(x_0^+\right) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$
- Since f is differentiable at x_0,
$$0 \geq f'\left(x_0^+\right) = f'(x_0) = f'\left(x_0^-\right) \geq 0,$$
- it follows that $f'(x_0) = 0$.

This finishes our proof. ■

- **Proposition** Let $f : [a, b] \to \mathbb{R}$ and assume that $f'(b^-)$ exists

a. If f has a local maximum at b then
$$f'\left(b^-\right) \geq 0.$$

b. If f has a local minimum at b then
$$f'\left(b^{-}\right) \leq 0.$$

Proof. Exercise ■

- **Proposition** Let $f : [a, b] \to \mathbb{R}$ and assume that $f'\left(a^{+}\right)$ exists

a. If f has a local maximum at a then $f'\left(a^{+}\right) \leq 0$.

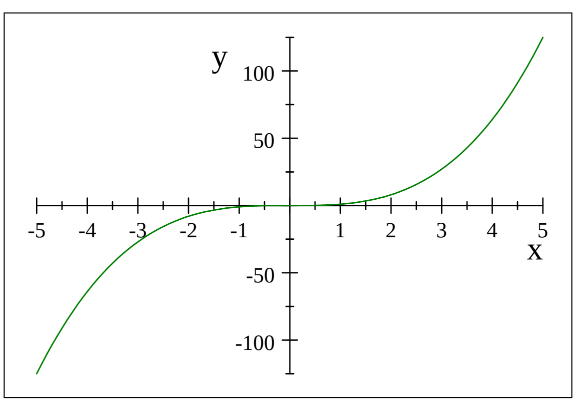
b. If f has a local minimum at a then $f'\left(a^{+}\right) \geq 0$.

Proof. Exercise ■

- **Definition** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in \operatorname{Int}(A)$. \
  We say that x_0 is a critical point of f if
$$f'\left(x_0\right) = 0$$
  or f is not differentiable at x_0.
  Example: Let $f : \mathbb{R} \to \mathbb{R}$, be given by
$$f\left(x\right) = x^3.$$

- Notice that $x_0 = 0$ is a critical point of f since
$$f'\left(0\right) = 3 \cdot 0^2 = 0.$$

- However, f has no local extremum at $x_0 = 0$.
- Indeed, if $\delta > 0$, then for
$$x \in \left(-\delta, 0\right), \ f\left(x\right) < 0$$
  and for
$$x \in \left(0, \delta\right), \ f\left(x\right) > 0,$$
  so for $x_0 = 0$ there is no open disk $D\left(0, \delta\right)$, such that for$x \in D\left(0, \delta\right)$,
$$f\left(x\right) \geq f\left(0\right)$$
  nor there is an open disk $D\left(0, \delta\right)$, such that, for all $x \in D\left(0, \delta\right)$,
$$f\left(x\right) \leq f\left(0\right).$$

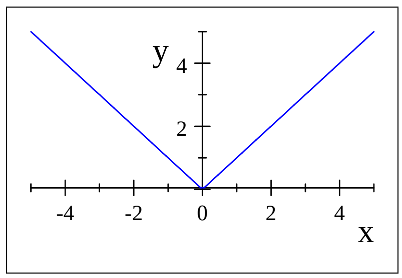


- The example above shows that critical points of f need not to be necessarily local minima or maxima.

  Example: Let $f : \mathbb{R} \to \mathbb{R}$, be given by
  $$f(x) = |x|.$$

- We observe that f is not differentiable at $x_0 = 0$, so $x_0 = 0$ is a critical point of f.

- If $\delta > 0$, then for all $x \in D(0, \delta)$,
  $$f(x) \geq f(0),$$
  so $x_0 = 0$ is a local minimum of f.



Theorem (*Rolle's Theorem*) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there is $c \in (a, b)$, such that
$$f'(c) = 0.$$

Proof. Since f is continuous on $[a, b]$ and $[a, b]$ is compact,

- by the **extreme value theorem**, there are
  $$x^*, y^* \in [a, b],$$
  such that,
  $$f(x^*) \leq f(x) \leq f(y^*).$$

- If $x^* = a$ and $y^* = b$ (or $x^* = b$ and $y^* = a$) then
  $$f(a) = f(x^*) \leq f(x) \leq f(y^*) = f(b)$$
  and since $f(a) = f(b)$,
  $$f(x) = f(a) = f(b),$$
  for all $x \in [a, b]$.

- Since f is constant,
  $$f'(x) = 0,$$
  for all $x \in (a, b)$.

- Therefore, we may assume without lose of the generality that
  $$x^* \in (a, b).$$

- From the previous theorem it must be $f(x^*) = 0$.

This finishes our proof. $\blacksquare$

- **Theorem** (*Mean Value Theorem*) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b).

  Then there is $c \in (a, b)$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Let $g : [a, b] \to \mathbb{R}$ be given by

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

- From our assumptions it follows that g is continuous on $[a, b]$ and differentiable on (a, b).
- Since $g(a) = 0$ and $g(b) = 0$, so
$$g(a) = g(b).$$
- Therefore, g satisfies assumptions of Rolle's Theorem,
  hence there is $c \in (a, b)$, such that
$$g'(c) = 0.$$
- However,
$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$
  so
$$\begin{aligned} 0 &= f'(c) - \frac{f(b) - f(a)}{b - a}, \text{ and thus} \\ f'(c) &= \frac{f(b) - f(a)}{b - a}. \end{aligned}$$

This finishes our proof. $\blacksquare$

- **Theorem** (*Cauchy Mean Value Theorem*) Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there is $c \in (a, b)$, such that

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)).$$

Proof. Let $h : [a, b] \to \mathbb{R}$ be defined by

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)).$$

- We see that

$$\begin{aligned} h(a) &= f(a)(g(b) - g(a)) - g(a)(f(b) - f(a)) \\ &= f(a)g(b) - f(a)g(a) - f(b)g(a) + f(a)g(a) \\ &= f(a)g(b) - f(b)g(a) \end{aligned}$$

  and

$$\begin{aligned} h(b) &= f(b)(g(b) - g(a)) - g(b)(f(b) - f(a)) \\ &= f(b)g(b) - f(b)g(a) - f(b)g(b) + f(a)g(b) \\ &= f(a)g(b) - f(b)g(a) \end{aligned}$$

- Since h is clearly continuous on $[a, b]$ and differentiable on (a, b),
- by Rolle's theorem, there is $c \in (a, b)$, such that

$$h'(c) = 0.$$

- Since

$$h'(x) = f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a)),$$

$$\begin{aligned} f'(c)(g(b) - g(a)) - g'(c)(f(b) - f(a)) &= 0 \text{ and thus} \\ f'(c)(g(b) - g(a)) &= g'(c)(f(b) - f(a)). \end{aligned}$$

This finishes our proof. ■