MATH 4334.001/CS 4334.001
Fall 2023
Paper Homework 2 Solutions

1. In the Taylor series for the function $f(x) = \sin x + \cos x$ centered at $a = \pi/4$, find the third nonzero term.

We have

$$
\begin{aligned}
f(x) = \sin x + \cos x &\implies f(\pi/4) = \sin(\pi/4) + \cos(\pi/4) = \sqrt{2} \\
f'(x) = \cos x - \sin x &\implies f'(\pi/4) = \cos(\pi/4) - \sin(\pi/4) = 0 \\
f''(x) = -\sin x - \cos x &\implies f''(\pi/4) = -\sin(\pi/4) - \cos(\pi/4) = -\sqrt{2} \\
f^{(3)}(x) = -\cos x + \sin x &\implies f^{(3)}(\pi/4) = 0 \\
f^{(4)}(x) = \sin x + \cos x &\implies f^{(4)}(\pi/4) = \sqrt{2}
\end{aligned}
$$

so that, in general,

$$f^{(2k)}(\pi/4) = (-1)^k \sqrt{2}$$

Hence, the third nonzero term in the Taylor series expansion is

$$\frac{f^{(4)}(\pi/4)}{4!} \left(x - \frac{\pi}{4}\right)^4 = \frac{\sqrt{2}}{24} \left(x - \frac{\pi}{4}\right)^4$$

2. Find $T_5(x)$, the Taylor polynomial of degree 5, about the center $a = 0$ for the following functions:

   (a) $f(x) = e^{x^2}$
   (b) $f(x) = \cos 2x$
   (c) $f(x) = \ln(1 + x)$
   (d) $f(x) = \sin^2 x$

(a)

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

and hence,

$$\boxed{T_5(x) = 1 + x^2 + \frac{x^4}{2}}$$

(b)

$$\cos 2x = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}$$

and hence,

$$\boxed{T_5(x) = 1 - 2x^2 + \frac{2}{3}x^4}$$

(c)

$$\ln(1 + x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n + 1}$$

and hence,

$$\boxed{T_5(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}}$$

(d) The derivatives of $f(x) = \sin^2 x$ at 0 are calculated to be

$$f(0) = 0,\ f'(0) = 0,\ f''(0) = 2,\ f^{(3)}(0) = 0,\ f^{(4)}(0) = -8,\ f^{(5)}(0) = 0$$

and hence,

$$\begin{aligned} T_5(x) &= \frac{f''(0)}{2!}x^2 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= \boxed{x^2 - \frac{x^4}{3}} \end{aligned}$$

3. (a) Determine $T_4(x)$ for $f(x) = x^{-2}$ about the center $a = 1$.
   (b) Use this result to approximate $f(0.9)$ and $f(1.1)$.
   (c) Use the Taylor remainder to find an error bound for each of the two approximations in part(b).

(a) The derivatives of $f(x) = x^{-2}$ at $x = 1$ are

$$f(1) = 1,\ f'(1) = -2,\ f''(1) = 3!,\ f^{(3)}(1) = -4!,\ f^{(4)}(1) = 5!$$

so that

$$T_4(x) = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4$$

(b) We have

$$\begin{aligned} f(0.9) &\approx T_4(0.9) \\ &= 1 - 2(0.9-1) + 3(0.9-1)^2 - 4(0.9-1)^3 + 5(0.9-1)^4 \\ &= 1 + 2(0.1) + 3(0.1)^2 + 4(0.1)^3 + 5(0.1)^4 \\ &= 1 + 0.2 + 0.03 + 0.004 + 0.0005 \\ &= 1.2345 \end{aligned}$$

and

$$\begin{aligned} f(1.1) &\approx T_4(1.1) \\ &= 1 - 0.2 + 0.03 - 0.004 + 0.0005 \\ &= 0.8265 \end{aligned}$$

(c) The Taylor remainder is $6(x-1)^5/c^7$, where c lies between x and 1.
For $\boxed{x = 0.9}$, the error is

$$\boxed{6(0.1)^5/c^7 \leq 6(0.1)^5/(0.9)^7 \approx 0.00012544509}$$

where the upper bound results from evaluating c at the worst case $c = 0.9$.
For $\boxed{x = 1.1}$, the error is

$$\boxed{6(0.1)^5/c^7 \leq 6(0.1)^5/(1.0)^7 \approx 0.00006}$$

Note that the actual error at $x = 0.9$ is $|f(0.9) - T_4(0.9)| = 0.00006790123$, and the actual error at $x = 1.1$ is $|f(1.1) - T_4(1.1)| = 0.00005371901$.

4. Carry out Exercise 3 (a)-(c) for $f(x) = \ln x$.

(a) $T_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$

(b) $f(0.9) \approx T_4(0.9) = -0.105358\overline{3}$
$f(1.1) \approx T_4(1.1) = 0.095308\overline{3}$

(c) The Taylor remainder is $(x-1)^5/(5c^5)$, where c lies between x and 1. At $x = 0.9$, the maximum error is ≈ 0.000003387. At $x = 1.1$, the maximum error is ≈ 0.000002.

5. Convert the following decimal numbers to octal numbers.

   (a) 27.1
   (b) 12.34
   (c) 3.14

(a) $(27.1)_{10} = (33.0\overline{6314})_8$
(b) $(12.34)_{10} = (14.256050\ldots)_8$
(c) $(3.14)_{10} = (3.107534\ldots)_8$

6. Convert the following numbers:

   (a) $(100\ 101\ 101)_2 \ = \ (\qquad)_8 \ = \ (\qquad)_{10}$

   (b) $(0.694)_{10} \ = \ (\qquad)_8 \ = \ (\qquad)_2$

   (c) $(361.4)_8 \ = \ (\qquad)_2 \ = \ (\qquad)_{10}$

(a) $(100\ 101\ 101)_2 = (455)_8 = 5 + (5 \times 8) + (4 \times 8^2) = (301)_{10}$

(b)

$$\begin{aligned}(0.694)_{10} &= (0.54324\ 77371\ 6\ldots)_8 \\ &= (0.101\ 100\ 011\ 010\ 100\ 111\ 111\ 011\ 111\ 001\ 110\ldots)_2\end{aligned}$$

(c) $(361.4)_8 = (011\ 110\ 001.100)_2 = (241.5)_{10}$