Math 4301 Mathematical Analysis I

Lecture 13

Topic: Differentiability

- **Differentiability**

  Corollary Let $f:(a,b)\to\mathbb{R}$ be *differentiable* for all $x\in(a,b)$ and $f'(x)=0$.

  Then f is constant.

Proof. It is sufficient to show that for all $x\in(a,b)$,

$$f(x)=c,$$

where $c\in\mathbb{R}$.

- Let $y\in(a,b)$ and $c=f(y)$.
- Let $x\neq y$ and assume that $x<y$.
- Then f is continuous on $[x,y]\subset(a,b)$ and differentiable on (x,y).
- Therefore, by the **MVT**, there is $z\in(x,y)$, such that

$$f'(z)=\frac{f(y)-f(x)}{y-x}.$$

- Since, $f'(z)=0$, we have

$$f(y)-f(x)=0,\ \text{so}$$
$$c=f(y)=f(x).$$

- Analogously we show that,

  if $y<x$ then

$$f(x)=f(y)=c.$$

- Therefore, for all $x\in(a,b)$,

$$f(x)=c.$$

This finishes our proof. ■

- **Corollary** If $f,g:(a,b)\to\mathbb{R}$ are differentiable on (a,b) and $f'(x)=g'(x)$, for all $x\in(a,b)$, then

$$f(x)=g(x)+C,$$

  for some constant $C\in\mathbb{R}$.

Proof. Exercise ■

- **Proposition** Let $f:(a,b)\to\mathbb{R}$ be differentiable on (a,b).

i) $f'(x)\geq 0$ for all $x\in(a,b)$ iff f is non-decreasing on (a,b)
(if $f'(x)>0$, for all $x\in(a,b)$, then f is strictly increasing).

ii) $f'(x)\leq 0$ for all $x\in(a,b)$ iff f is non-increasing on (a,b)
(if $f'(x)<0$, for all $x\in(a,b)$, then f is strictly decreasing).

Proof. We only prove $a)$ since a proof for $b)$ is completely analogous.

- Assume that $a < x < y < b$.
- Since f is continuous on $[x, y]$ and differentiable on $(x, y) \subset (a, b)$,
- it follows from the **Mean Value Theorem** that,
  there is $z \in (x, y)$, such that
$$0 \leq f'(z) = \frac{f(y) - f(x)}{y - x}.$$
- Hence, $f(y) \geq f(x)$, so f is non-decreasing on (a, b).
- Notice that if $f'(x) > 0$, for all $x \in (a, b)$, then
$$0 < f'(z) = \frac{f(y) - f(x)}{y - x}$$
  and $f(y) > f(x)$, so f is strictly increasing.
- **Conversely, suppose that f is non-decreasing on (a, b).**
- Let $z \in (a, b)$.
- Since f is non-decreasing on (a, b), for $x > z$, $x \in (a, b)$
$$f(x) - f(z) \geq 0,$$
  and $x < z$
$$f(x) - f(z) \leq 0$$
- Therefore,
$$\frac{f(x) - f(z)}{x - z} \geq 0$$
  for all $x \neq z$.
- It follows that
$$f'(z) = \lim_{x \to z} \frac{f(x) - f(z)}{x - z} \geq 0.$$

This finishes our proof. ■

- **Example**: Let $f : \mathbb{R} \to \mathbb{R}$, be given by $f(x) = x^3$,
  then $f'(x) = 3x^2 \geq 0$ so f is non-decreasing.
  Moreover, we see that $f'(0) = 0$ however, f is strictly increasing.
  Example: Let $f : \mathbb{R} \to \mathbb{R}$, be given by
$$f(x) = \begin{cases} \frac{1}{2}x + x^2 \sin\left(\frac{1}{x}\right) & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}.$$
  We show that $f'(0) > 0$, but there is no open neighborhood $D(0, \delta) = (-\delta, \delta)$, $\delta > 0$ of $x = 0$, such that f is either increasing or decreasing on $D(0, \delta)$.

- We see that
$$f'(x)=\begin{cases}\frac{1}{2}+2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right) & if \quad x\neq 0\\ \frac{1}{2} & if \quad x=0\end{cases}$$

- Therefore,
$$f'\left(\frac{1}{2n\pi}\right)=\frac{1}{2}+\frac{2}{2n\pi}\sin(2n\pi)-\cos(2n\pi)=\frac{1}{2}-1=-\frac{1}{2}<0$$

  and
$$\begin{aligned} f'\left(\frac{1}{\frac{\pi}{2}+2n\pi}\right) &= \frac{1}{2}+\frac{2}{\frac{\pi}{2}+2n\pi}\sin\left(\frac{\pi}{2}+2n\pi\right)-\cos\left(\frac{\pi}{2}+2n\pi\right)\\ &= \frac{1}{2}+\frac{4}{\pi+4\pi n}>0. \end{aligned}$$

- In particular, $f'(0)>0$ does not imply that there is an open neighborhood of 0 on which f is non-increasing or non-decreasing.

  Theorem (*Taylor's Theorem*) Suppose that f and its first n derivatives are continuous on $[a,b]$, differentiable on (a,b) and $x_0\in[a,b]$.

  Then for each $x\in[a,b]$, $x\neq x_0$ there is c in the interval with the endpoints x_0 and x, such that
$$f(x)=\underbrace{f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+...+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{n\text{th Taylor polynomial } p_n}+\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

Proof. We show that for given $x_0\in[a,b]$ and $x\neq x_0$, there is c in

- Let $x\in[a,b]$, then there is $M\in\mathbb{R}$, such that
$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+...+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+M(x-x_0)^{n+1}.$$

- We show that $M=\frac{f^{(n+1)}(c)}{(n+1)!}$.

- Define $F:[a,b]\to\mathbb{R}$ by
$$F(t)=f(t)+\frac{f'(t)}{1!}(x-t)+\frac{f''(t)}{2!}(x-t)^2+...+\frac{f^{(n)}(t)}{n!}(x-t)^n+M(x-t)^{n+1}.$$

- We see that, $F(x)=f(x)$ and
$$\begin{aligned} F(x_0) &= f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+...+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+M(x-x_0)^{n+1}\\ &= f(x). \end{aligned}$$

- Now, F is continuous on the closed interval I with the endpoints x and x_0 and it is differentiable on the interior of I.

- By Rolle's theorem, there is c in the interior of I, such that,
$$F'(c)=0.$$

- However,

$$\begin{aligned}
\frac{d}{dt}F(t) &= f'(t) \\
&\quad + \frac{f''(t)}{1!}(x-t) - \frac{f'(t)}{1!} \\
&\quad + \frac{f'''(t)}{2!}(x-t)^2 - \frac{f''(t)}{1!}(x-t) \\
&\quad + \frac{f^{(4)}(t)}{3!}(x-t)^3 - \frac{f^{(3)}(t)}{2!}(x-t)^2 \\
&\quad \vdots \\
&\quad + \frac{f^{(n+1)}(t)}{n!}(x-t)^n - \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} \\
&\quad - M(n+1)(x-t)^n
\end{aligned}$$

- Hence, we have

$$\frac{d}{dt}F(t) = \frac{f^{(n+1)}(t)}{n!}(x-t)^n - M(n+1)(x-t)^n.$$

  and for $t = c$,

$$0 = F'(c) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n - M(n+1)(x-c)^n.$$

- Since $c \neq x$,

$$\frac{f^{(n+1)}(c)}{n!} - M(n+1) = 0, \text{ so } M = \frac{f^{(n+1)}(c)}{(n+1)!}.$$

This finishes our proof. ■

- Let

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,$$

  $x \in [a,b]$.

- We call this polynomial, the nth Taylor's polynomial and

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

  will be called the nth reminder.

- We observe that, if

$$p_n(x) = a_0 + a_1(x-x_0) + \ldots + a_n(x-x_0)^n$$

  is nth degree polynomial such that

$$p_n^{(j)}(x_0) = f^{(j)}(x_0) \text{ for all } j = 0,1,2,\ldots,n$$

  then

$$a_j = \frac{f^{(j)}(x_0)}{j!}, \text{ for } j = 0,1,2,\ldots,n.$$

- Hence $p_n(x) = P_n(x)$, for all $x \in [a,b]$.
- Indeed, since
$$p_n^{(j)}(x_0) = f^{(j)}(x_0) \text{ for all } j = 0,1,2,...,n,$$
in particular,
$$\begin{aligned} a_0 &= p_n^{(0)}(x_0) = f^{(0)}(x_0) = f(x_0), \text{ so } a_0 = f(x_0). \\ 1!a_1 &= p_n^{(1)}(x_0) = f^{(1)}(x_0), \text{ so } a_1 = \frac{f^{(1)}(x_0)}{1!} \\ 2!a_2 &= p_n^{(2)}(x_0) = f^{(2)}(x_0), \text{ so } a_2 = \frac{f^{(2)}(x_0)}{2!} \\ &\vdots \\ n!a_n &= p_n^{(n)}(x_0) = f^{(n)}(x_0), \text{ so } a_n = \frac{f^{(n)}(x_0)}{n!}. \end{aligned}$$
- Therefore, the statement follows.

Example: We find $P_n(x)$ for $f(x) = e^x$ and $x_0 = 0$.
Since $f^{(n)}(x) = e^x$, $f^{(n)}(0) = 1$, so
$$P_n(x) = \sum_{k=0}^{n} \frac{x^n}{n!}.$$

Example: We find $P_n(x)$ for $f(x) = \sin(x)$ and $x_0 = 0$.

- Since
$$\begin{aligned} f(x) &= \sin(x), \text{ then } f(0) = 0 \\ f'(x) &= \cos(x) = \sin\left(x + 1\cdot\frac{\pi}{2}\right), \text{ then } f'(0) = 1 \\ f''(x) &= -\sin(x) = \sin\left(x + 2\cdot\frac{\pi}{2}\right), \text{ then } f''(0) = 0 \\ f^{(3)}(x) &= -\cos(x) = \sin\left(x + 3\cdot\frac{\pi}{2}\right), \text{ then } f^{(3)}(0) = -1 \\ &\vdots \\ f^{(n)}(x) &= \sin\left(x + n\cdot\frac{\pi}{2}\right), \text{ then} \\ f^{(n)}(0) &= \sin\left(n\cdot\frac{\pi}{2}\right) = \begin{cases} (-1)^k & if \quad n = 2k+1 \\ 0 & if \quad n = 2k \end{cases}, \ k = 0,1,... \end{aligned}$$
- Therefore,
$$P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

Example: We find $P_n(x)$ for $f(x) = \cos(x)$ and $x_0 = 0$.

- Since

$$
\begin{aligned}
f(x) &= \cos(x), \text{ then } f(0)=1 \\
f'(x) &= -\sin(x) = \cos\left(x + 1\cdot\frac{\pi}{2}\right), \text{ then } f'(0)=0 \\
f''(x) &= -\cos(x) = \cos\left(x + 2\cdot\frac{\pi}{2}\right), \text{ then } f''(0)=-1 \\
f^{(3)}(x) &= \sin(x) = \cos\left(x + 3\cdot\frac{\pi}{2}\right), \text{ then } f^{(3)}(0)=0 \\
&\vdots \\
f^{(n)}(x) &= \cos\left(x + n\cdot\frac{\pi}{2}\right), \text{ then} \\
f^{(n)}(0) &= \cos\left(n\cdot\frac{\pi}{2}\right) = \begin{cases} (-1)^k & if \quad n=2k \\ 0 & if \quad n=2k+1 \end{cases}, \ k=0,1,...
\end{aligned}
$$

- Therefore,

$$P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}.$$

- **Exercise**: Show that

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$$

Solution: Let $x \neq 0$ and $x_0 = 0$.

- By Taylor's Theorem, there is c_x in the interval with the endpoints x and 0, such that

$$
\begin{aligned}
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{\cos(c_x)}{4!}x^4, \text{ so} \\
\frac{1-\cos(x)}{x^2} &= \frac{\frac{x^2}{2!} - \frac{\cos(c_x)}{4!}x^4}{x^2} = \frac{1}{2!} - \frac{\cos(c_x)}{4!}x^2.
\end{aligned}
$$

- Now, let $\epsilon > 0$ be given and assume that $0 < |x| < \delta$.
- Then

$$
\begin{aligned}
\left| \frac{1-\cos(x)}{x^2} - \frac{1}{2} \right| &= \left| \left(\frac{1}{2!} - \frac{\cos(c_x)}{4!}x^2 \right) - \frac{1}{2} \right| \\
&= \frac{|\cos(c_x)|}{4!} |x|^2 \leq \frac{|x|^2}{24} < \frac{\delta^2}{24}.
\end{aligned}
$$

- If $0 < \delta < 2\sqrt{6\epsilon}$, then for $0 < |x| < \delta$,

$$\left| \frac{1-\cos(x)}{x^2} - \frac{1}{2} \right| < \frac{\delta^2}{24} < \frac{\left(2\sqrt{6\epsilon}\right)^2}{24} = \epsilon.$$

- We showed that

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}.$$

Exercise: Let $f(x) = \sqrt{x}$. Find p_2 for f at $x_0 = 9$ to estimate $\sqrt{8.8}$. What is the error?

- **Solution:** For $f(x) = \sqrt{x}$,

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x}} \\ f''(x) &= -\frac{1}{4\sqrt{x^3}} \\ f'''(x) &= \frac{3}{8\sqrt{x^5}} \end{aligned}$$

- Therefore,

$$\begin{aligned} f'(9) &= \frac{1}{2\sqrt{9}} = \frac{1}{6} \\ f''(9) &= -\frac{1}{4\sqrt{9^3}} = -\frac{1}{108}. \end{aligned}$$

and by Taylor's theorem, for $x \neq 9$,
there is $c_x \in I_x$, where

$$I_x = \begin{cases} [9, x] & if \quad x > 9 \\ [x, 9] & if \quad x < 9 \end{cases}$$

such that

$$f(x) = p_2(x) + R_2(c_x),$$

where

$$\begin{aligned} p_2(x) &= \sum_{i=0}^{2} \frac{f^{(i)}(9)}{(i)!}(x-9)^i = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 \text{ and} \\ R_2(c_x) &= \frac{f^{(3)}(c_x)}{3!}(x-9)^3 = \frac{\frac{3}{8\sqrt{c_x^5}}}{6}(x-9)^3 = \frac{1}{16\sqrt{c_x^5}}(x-9)^3 \end{aligned}$$

- If $x = 8.8$, then

$$p_2(8.8) = 3 + \frac{1}{6}(8.8-9) - \frac{1}{216}(8.8-9)^2 \approx 2.966\,5.$$

and

$$\begin{aligned} \left|\sqrt{8.8} - p_2(8.8)\right| &= |R_2(c_{8.8})| = \left|\frac{f^{(3)}(c_{8.8})}{3!}\right| |8.8-9|^3 \\ &= \frac{0.008}{16\sqrt{c_x^5}} = \frac{1}{2000\sqrt{c_{8.8}^5}}. \end{aligned}$$

- Since $c_{8.8} \in (8.8, 9)$,

$$c_{8.8} > 8.8,$$

so $\frac{1}{c_{8.8}} < \frac{1}{8.8}$.

- Therefore,

$$\frac{1}{2000\sqrt{c_{8.8}^5}} < \frac{1}{2000\sqrt{(8.8)^5}} \approx 2.176\,5 \times 10^{-6}.$$

- Hence,
$$\left|\sqrt{8.8} - p_2(8.8)\right| \leq 2.176\,5 \times 10^{-6}.$$

- So the error of the approximation of $\sqrt{8.8}$ that we obtain using Taylor's theorem is $\leq 2.176\,5 \times 10^{-6}$.

  Theorem (*L'Hospital's Rule*) Let f, g be continuous on $[a, b]$ and differentiable on (a, b).
  Suppose that $c \in [a, b]$ and $f(c) = g(c) = 0$, and there is $\delta > 0$, such that
$$g'(x) \neq 0$$
  for $x \in D^*(c, \delta) \cap (a, b)$, where $D^*(c, \delta) = D(c, \delta) \setminus \{c\}$.
  If $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$ then
$$\lim_{x \to c} \frac{f(x)}{g(x)} = L.$$

Proof. Let $\{x_n\} \subseteq D^*(c, \delta) \cap (a, b)$ and assume that $x_n \to c$ as $n \to \infty$.

- We show that
$$\lim_{n \to \infty} \frac{f(x_n)}{g(x_n)} = L.$$

- Consider the closed interval I_n with the endpoints c and x_n

- Observe that f and g are continuous on I_n and differentiable in its interior.

- By Cauchy's Mean Value Theorem,
  there is c_n in the interior of I_n, such that
$$f'(c_n)(g(x_n) - g(c)) = g'(c_n)(f(x_n) - f(c)).$$

- Since, $f(c) = g(c) = 0$,
$$f'(c_n)\, g(x_n) = g'(c_n)\, f(x_n).$$

- Since $c_n \in \operatorname{Int}(I_n) \subset D^*(c, \delta) \cap (a, b)$,
$$g'(c_n) \neq 0.$$

- If $g(x_n) = 0$ then since
$$g(c) = 0,$$
  g is continuous on the closed interval with the endpoints x_n and c and differentiable in its interior, by Rolle's theorem, there is c_n' in the interior of the interval with the endpoints c and x_n, such that
$$g'(c_n') = 0.$$

- However, as we see,
$$c_n' \in I_n \subset D^*(c, \delta) \cap (a, b),$$
  so
$$g'(c_n') \neq 0,$$
  a contradiction.

- Therefore, we must be
$$g'(x_n) \neq 0$$
and hence
$$\frac{f'(c_n)}{g'(c_n)} = \frac{f(x_n)}{g(x_n)}$$
- Observe that since $x_n \to c$,
then $c_n \to c$ as c_n is a point in the interior of I_n.
- Therefore
$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \lim_{n\to\infty} \frac{f'(c_n)}{g'(c_n)} = L$$
since
$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = L.$$
- It follows that
$$\lim_{x\to c} \frac{f(x)}{g(x)} = L.$$

This finishes our proof. ■

- **Remark:** Let $a > 0$ and
$$f, g : (a, \infty) \to \mathbb{R}$$
be differentiable and
$$g'(x) \neq 0,$$
for all $x \in (a, \infty)$.
- If
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0 \text{ and } \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L,$$
then
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$
- Indeed, we define
$$F(x) = f\left(\frac{1}{x}\right)$$
and
$$G(x) = g\left(\frac{1}{x}\right),$$
for $x \in \left(0, \frac{1}{a}\right)$.
- We see that
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \text{ iff } \lim_{x\to 0^+} \frac{F(x)}{G(x)} = L.$$

- Since $F'(x) = -\frac{f'\left(\frac{1}{x}\right)}{x^2}$ and $G'(x) = -\frac{g'\left(\frac{1}{x}\right)}{x^2}$,
  then for $x \in \left(0, \frac{1}{a}\right)$,
$$\frac{F'(x)}{G'(x)} = \frac{f'\left(\frac{1}{x}\right)}{g'\left(\frac{1}{x}\right)}.$$
- Since
$$\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} G(x) = 0 \text{ and } G'(x) \neq 0$$
- for all $x \in \left(0, \frac{1}{a}\right)$, by L'Hospital's Rule
$$\lim_{x\to 0^+} \frac{F(x)}{G(x)} = \lim_{x\to 0^+} \frac{F'(x)}{G'(x)}.$$
- Therefore,
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{F(x)}{G(x)} = \lim_{x\to 0^+} \frac{F'(x)}{G'(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L.$$
- Analogous argument applies for two differentiable functions
- $f, g : (-\infty, a) \to \mathbb{R}$, where $a < 0$, such that
$$g'(x) \neq 0,$$
  for all $x \in (-\infty, a)$, and
- $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} g(x) = 0$ and $\lim_{x\to-\infty} \frac{f'(x)}{g'(x)} = L$.
  Theorem (*Inverse Function Theorem*) Suppose that $f : (a, b) \to \mathbb{R}$ and
$$f'(x) > 0$$
  for all $x \in (a, b)$ or $f'(x) < 0$ for all $x \in (a, b)$.
  Then f is bijective onto $f((a, b))$, f^{-1} is differentiable on its domain, and
$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)}$$
  where $y = f(x)$.
  Example: Let $f(x) = \sin(x)$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- Then $f'(x) = \cos(x) > 0$, for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- By the Inverse Function Theorem, f has the inverse
$$f^{-1} : (-1, 1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
  which we call $\arcsin = f^{-1}$.
- Now,
$$\begin{aligned}\frac{d}{dy}(\arcsin(y)) &= \left(f^{-1}\right)'(y) = \frac{1}{f'(x)} = \frac{1}{\cos x} \\ &= \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}.\end{aligned}$$

Proof. Since f is monotone (strictly increasing if $f'(x) > 0$ or strictly decreasing if $f'(x) < 0$),

- f is injective, so
$$f^{-1} : f((a,b)) \to (a,b)$$
is defined and $f((a,b))$ is an interval since (a,b) is an interval and f is continuous.
- Suppose that $f'(x) > 0$, for all $x \in (a,b)$,
so f is strictly increasing.
- We want to show that $f^{-1} : f((a,b)) \to (a,b)$ is continuous.
- Let $U \subset (a,b)$ is open, it is sufficient to show that
$$\left(f^{-1}\right)^{-1}(U) = f(U) \subseteq f((a,b)) - \text{ Why?}$$
is open.
- Let $y \in f(U)$, so there is $x \in U$, such that
$$y = f(x).$$
- Since U is open, there is an open interval $(x_1, x_2) \subseteq U$, such that $x \in (x_1, x_2)$.
- Since f increases
$$f(x_1) < f(x) < f(x_2)$$
and
$$f((x_1, x_2)) \subseteq (f(x_1), f(x_2)).$$
- If $c \in (f(x_1), f(x_2))$,
then by the Intermediate Value Theorem,
there is $z \in (x_1, x_2)$, such that
$$f(z) = c,$$
so
$$(f(x_1), f(x_2)) \subseteq f((x_1, x_2)).$$
- It follows that
$$y \in (f(x_1), f(x_2)) = f((x_1, x_2)) \subseteq f(U) = \left(f^{-1}\right)^{-1}(U),$$
so $\left(f^{-1}\right)^{-1}(U) = f(U)$ is open.
- It follows that f^{-1} is **continuous.**
- Now let $y = f(x)$,
then $x = f^{-1}(y)$ and $y_0 = f(x_0)$,
so $x_0 = f^{-1}(y_0)$.
- Since f^{-1} is continuous,
$$\lim_{y \to y_0} f^{-1}(y) = \lim_{y \to y_0} x = x_0.$$
Then
$$\begin{aligned} \left(f^{-1}\right)'(y_0) &= \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{x - x_0}{f(x) - f(x_0)} \\ &= \lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}. \end{aligned}$$

This finishes our proof. ■