MATH 4301
Exam 3
Wednesday, August 9, 2023 by 3:00pm, FO 3.704G

Each Problem is 20 points. You may choose any 4 out of the 5 to complete for full 80 points of credit. If a fifth problem is completed, it will be counted as a bonus[1]. Please on your own and do not discuss the problems with your classmates or anyone else. Show all work and/or reasoning. A separate component posted to `eLearning` will be worth an additional 20 points. If you have any questions please let me know.

1a. (10 pts) Determine whether series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}+1}$$

converges or diverges. **Justify your answer:** Please show all details; If you are using theorems from class, please state which ones and then carefully check their assumptions.

[1] I will count half of the lowest scored problem as a bonus.

1b. (10 pts) Show that if the series $\sum_{n=1}^{\infty} |a_n|$ converges and a sequence $\{b_n\}$ is bounded then $\sum_{n=1}^{\infty} |a_n b_n|$ converges.

Hint: Let $S_n = \sum_{k=1}^{n} |a_k|$ is nth partial sum of $\sum_{n=1}^{\infty} |a_n|$ and $\overline{S}_n = \sum_{k=1}^{n} |a_k b_k|$ be partial sum of $\sum_{n=1}^{\infty} |a_n b_n|$. Start by showing that there is a constant $K \geq 0$, such that

$$\overline{S}_n \leq K S_n.$$

2. (20 pts) Using the definition of Riemann integrability to show that $f : [0,1] \to \mathbb{R}$ given by $f(x) = \begin{cases} 1 & \text{if} \quad x = \frac{1}{2} \\ 0 & \text{if} \quad x \neq \frac{1}{2} \end{cases}$ is Riemann integrable and find $\int_0^1 f(x)\,dx$.

- Let $\epsilon > 0$ be given and $P = \{0 = x_0 < x_1 < ... < x_n = 1\}$ be a partition of $[0,1]$. For each of the following cases compute $L(f,P)$, $U(f,P)$ and $U(f,P) - L(f,P)$

i) (5pts) $x_i = \frac{1}{2}$ for some $i \in \{0, 1, ..., n\}$

ii) (5pts) $x_i \neq \frac{1}{2}$ for all $i \in \{0, 1, ..., n\}$.

- (5pts) Using $i)$ and $ii)$, find $\delta > 0$ such that if $\Delta x_j = x_j - x_{j-1} < \delta$, then $U(f,P) - L(f,P) < \epsilon$. Now, assume that $\Delta x_j < \delta$, for $j = 1, 2, ..., n$. Show that $U(f,P) - L(f,P) < \epsilon$ and argue that f is Riemann integrable over $[0,1]$.

- (5pts) Find lower Darboux integral $\underline{\int_0^1} f$ and use it to find $\int_0^1 f(x)\,dx$. **Justify your answer.**

3. (20 pts) Let $f:[a,b]\to\mathbb{R}$ be bounded and assume that there is a partition $P=\{x_0,x_1,...,x_n\}$ of $[a,b]$, such that $U(f,P)=L(f,P)$ (the upper and the lower Darboux sums are equal). Show that f is constant, i.e., for all $x\in[a,b]$, $f(x)=c$, for some $c\in\mathbb{R}$.

- *Hint*: Notice that by the assumption $U(f,P)-L(f,P)=\sum_{i=1}^{n}(M_i(f)-m_i(f))\Delta x_i=0$. **i**) Show that, for $i=1,2,...,n$, $f(x)=f(x_{i-1})=f(x_i)$, for $x\in[x_{i-1},x_i]$; **ii**) Show that $f(x)=f(a)$ for all $x\in[a,b]$.

4a. (10 pts) Show that the following sequence of functions

$$f_n(x) = \frac{nx^2 + 2x}{n}, \ x \in [0,1]$$

is uniformly convergent on $[0,1]$ and find

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx.$$

Justify your answer.

4b. (10 pts) Find the radius of convergence R and the interval of convergence I for the following power series

i) $\sum_{n=0}^{\infty} \frac{n+2}{n^4+1} x^n$ **Justify your answer.**

ii) $\sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{n5^n}$ **Justify your answer.**

5a. (10 pts) For the following power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \left(\frac{x^n}{n!}\right)^2$$

i) (4pts) Find its radius of convergence R and the interval of convergence I.

ii) (6pts) Show that $f : I \to \mathbb{R}$ given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \left(\frac{x^n}{n!}\right)^2$$

is continuous on I.

5b) (10 pts) Use the following theorem from **Lecture 19**

- **Theorem** *Let* $f_n : (a,b) \to \mathbb{R}$ *be differentiable* , $f : (a,b) \to \mathbb{R}$ *and* $f_n(x) \to f(x)$ *(pointwise). Suppose that* $f_n' : (a,b) \to \mathbb{R}$,

$$f_n'(x) = \frac{d}{dx} f_n(x)$$

is continuous and $f_n' \to g$ *(uniformly), where* $g : (a,b) \to \mathbb{R}$, $g(x) = \lim_{n\to\infty} f_n'(x)$. *Then* f *is differentiable on* (a,b) *and* $f' = g$, *that is,*

$$g(x) = \lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} f_n'(x) = f'(x), \text{ for all } x \in (a,b).$$

to prove the following:

Corollary Let $f_n : (a,b) \to \mathbb{R}$ be differentiable and $f_n' : (a,b) \to \mathbb{R}$ be continuous for each $n \in \mathbb{N}$. Assume that

i) $\sum_{n=1}^{\infty} f_n \to f$ (pointwise) on (a,b), i.e., $f : (a,b) \to \mathbb{R}$, $f(x) = \sum_{n=1}^{\infty} f_n(x)$ and

ii) $\sum_{n=1}^{\infty} f_n' \to g$ (uniformly) on (a,b), where $g : (a,b) \to \mathbb{R}$, $g(x) = \sum_{n=1}^{\infty} f_n'(x)$.

Then $f : (a,b) \to \mathbb{R}$ is differentiable and

$$f'(x) = \left(\sum_{n=1}^{\infty} f_n(x)\right)' = \sum_{n=1}^{\infty} f_n'(x) = g(x), \text{ for all } x \in (a,b).$$