1. What is wrong with the following proof?

STATEMENT: In any set of n flowers, the flowers have the same color.

Proof.
Base case: If $n = 1$ then there is only one flower in the set so clearly all flowers in the set have the same color.

Inductive step:
Suppose the claim is true for a set of n flowers.
Now consider a set of $n + 1$ flowers. Remove one flower from the set. Then the remaining n flowers have the same color by the hypothesis above. It remains to show that the flower we removed is also of this color. Put that flower back in the set and remove a different flower. Applying the inductive hypothesis again shows that the remaining n flowers have the same color. Therefore all $n + 1$ flowers have the same color and the claim is true for $n + 1$.

By induction, the claim is true for all natural numbers n. □

The issue is with the inductive step. If one flower is removed from the set of $n + 1$ flowers, then the remaining n flowers do have the same color by the inductive hypothesis. When the flower is put back and a different flower removed, the remaining n flowers do have the same color by the inductive hypothesis. But there's no guarantee that the two colors are the same! So it is not proven that the $n + 1$ flowers altogether have the same color.

2. Use induction to prove that
$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \ldots + n \cdot 2^n = 2 + (n-1)2^{n+1}.$$

Base case. $1 \cdot 2^1 = 2 + (1-1)2^{1+1}$.

Inductive step. Suppose $1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \ldots + n \cdot 2^n = 2 + (n-1)2^{n+1}$. (Inductive hypothesis).

Then
$$\begin{aligned}
&1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \ldots + n \cdot 2^n + (n+1) \cdot 2^{n+1} \\
&= 2 + (n-1)2^{n+1} + (n+1) \cdot 2^{n+1} \\
&= 2 + 2(n2^{n+1}) \\
&= 2 + n \cdot 2^{n+2} \\
&= 2 + ((n+1) - 1) \cdot 2^{(n+1)+1}.
\end{aligned}$$

3. The Fibonacci numbers F_n are defined as follows:
$$F_1 = 1,\ F_2 = 1,$$
and
$$F_k = F_{k-1} + F_{k-2}$$

for $k > 2$. Use induction to prove that

$$F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$$

for all $n \in \mathbb{N}$.

Base case. $F_1 = F_{1+2} - 1$ is true because $F_1 = 1$ and $F_3 = 2$.

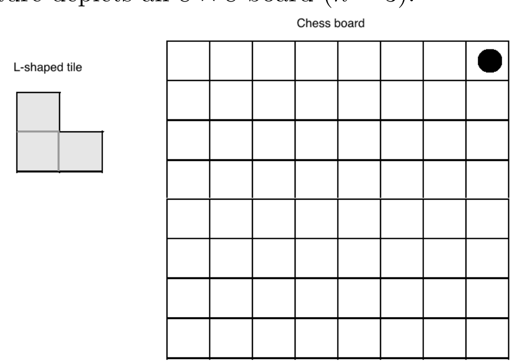
Inductive step. Suppose $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$. (Inductive hypothesis).

Then

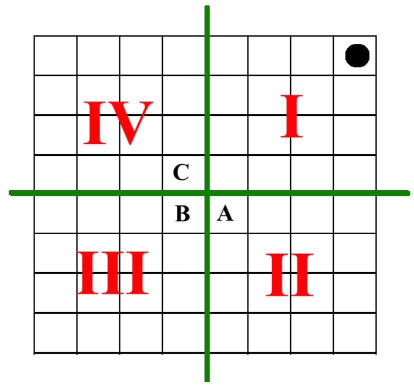
$$\begin{aligned}
&F_1 + F_2 + F_3 + \ldots + F_n + F_{n+1} \\
&= F_{n+2} - 1 + F_{n+1} \\
&= F_{n+3} - 1 \qquad \text{(by the recursive relation)} \\
&= F_{(n+1)+2} - 1.
\end{aligned}$$

4. A $2^n \times 2^n$ chess board has a stone placed on one of its a corner squares. The rest of the board needs to be completely covered by non-overlapping L-shaped tiles (an L-shaped tile consists of 3 squares arranged in the shape of an L). Prove by induction that this can be done for any $n \in \mathbb{N}$.

The following picture depicts an 8×8 board ($n = 3$).



Hint: For the inductive step, divide the board into quarters (as pictured below for the case $n = 3$). What are the dimensions of each quarter? The quarter labeled I can be covered with L-shaped tiles apart from the square with the stone – Why? The quarter labeled II can be covered with L-shaped tiles apart from the square labeled A – Why? The quarter labeled III can be covered with L-shaped tiles apart from the square labeled B – Why? The quarter labeled IV can be covered with L-shaped tiles apart from the square labeled C – Why? Finally it remains to cover the squares A, B, C – can you do it?



Base case. A $2^1 \times 2^1$ board with a stone on a corner square can be covered with an L-shaped tile placed on the remaining three squares.

Inductive step. Suppose that a $2^n \times 2^n$ board with a stone on a corner square can have the rest of its squares covered by L-shaped tiles. (Inductive hypothesis).

Consider a $2^{n+1} \times 2^{n+1}$ board. Divide the board into quarters as pictured in the hint. Each may be viewed as a separate $2^n \times 2^n$ board.

Quarter I can be covered, apart from the square with the stone, by L-shaped tiles using the inductive hypothesis.

Quarter II can be covered, apart from square A on which we pretend is a stone, by L-shaped tiles using the inductive hypothesis.

Quarter III can be covered, apart from square B on which we pretend is a stone, by L-shaped tiles using the inductive hypothesis.

Quarter IV can be covered, apart from square C on which we pretend is a stone, by L-shaped tiles using the inductive hypothesis.

Finally, an L-shaped tile can be placed so as to cover squares A, B, C. Then all squares on the $2^{n+1} \times 2^{n+1}$ board are covered apart from the one with the stone.

5. Use induction to prove that $3^{2n} - 1$ is divisible by 8 for every natural number n.

Base case. $3^{2\cdot 1} - 1 = 8$ is divisible by 8.

Inductive step. Suppose that $3^{2n} - 1$ is divisible by 8. (Inductive hypothesis). This implies that $3^{2n} - 1 = 8k$ for some $k \in \mathbb{Z}$. So $3^{2n} = 8k + 1$.

Then

$$\begin{aligned}
&3^{2(n+1)} - 1 \\
&= 3^{2n} \cdot 3^2 - 1 \\
&= (8k+1) \cdot 9 - 1 \\
&= 72k + 8 \\
&= 8(9k+1).
\end{aligned}$$

This shows that $3^{2(n+1)} - 1$ is divisible by 8.