Math 4301 Mathematical Analysis I

Lecture 11

Topic: Properties of Continuous functions

- **Uniformly Continuous Functions**
- Recall, $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at $a \in A$ if
  for every $\epsilon > 0$ there is $\delta > 0$, such that, $x \in A$, $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$
  Definition Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$. We say that f is *uniformly continuous* on A if
  for every $\epsilon > 0$, there is $\delta > 0$, such that, for all $x, y \in A$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.
  Example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = 2x + 3$$

We show that f is uniformly continuous on $\mathbb{R}$.

- Let $\epsilon > 0$ be given.
- Take $\delta = \frac{\epsilon}{2} > 0$ and
  assume that $x, y \in \mathbb{R}$ and $|x - y| < \delta$.
- Then

$$\begin{aligned} |f(x) - f(y)| &= |2x + 3 - (2y + 3)| = |2(x - y)| \\ &= 2|x - y| < 2\delta = 2 \cdot \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Remark Notice that is $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ are uniformly continuous then

- $\alpha f + \beta g$ is uniformly continuous, $\alpha, \beta \in \mathbb{R}$.
  Remark A product of two uniformly continuous functions is not necessarily uniformly continuous.
  As we see $f, g : \mathbb{R} \to \mathbb{R}$, $f(x) = g(x) = x$ then f and g are both uniformly continuous.
  However, $f \cdot g$ is not uniformly continuous.
  Example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^2$$

We show that f is not uniformly continuous on $\mathbb{R}$.

- We show that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$ if we can show that
  there is $\epsilon > 0$, **such that,**
  for every $\delta > 0$,
  there are $x, y \in \mathbb{R}$ **such that**

$$|x - y| < \delta \textbf{ and } \left|x^2 - y^2\right| \geq \epsilon.$$

- Let $\epsilon > 0$ and $\delta > 0$.
- **Idea**: We want

$$\left|x^2 - y^2\right| = |x - y|\,|x + y| \geq \epsilon$$

for $x, y \in \mathbb{R}$ such that $|x - y| < \delta$.

- First, we see that, there are $x, y \in \mathbb{R}$, such that,

$$x - y = \frac{\delta}{2}$$

and

$$x + y = \frac{2}{\delta}\epsilon.$$

- We can solve the system of linear equations

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{\delta}{2} \\ \frac{4\epsilon}{\delta} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\delta + \frac{2}{\delta}\epsilon \\ \frac{2}{\delta}\epsilon - \frac{1}{4}\delta \end{pmatrix}$$

- Indeed, we see that if

$$x = \frac{1}{4}\delta + \frac{2}{\delta}\epsilon \text{ and } y = \frac{2}{\delta}\epsilon - \frac{1}{4}\delta$$

- Then $|x - y| = \frac{\delta}{2} < \delta$ and $|x + y| = \frac{4\epsilon}{\delta}$, so

$$|f(x) - f(y)| = |x - y|\,|x + y| = 2\epsilon > \epsilon.$$

- Therefore, there is $\epsilon > 1$, such that,
  for every $\delta > 0$, there are $x, y \in \mathbb{R}$, such that $|x - y| < \delta$ and $|f(x) - f(y)| > \epsilon$.

- It follows that f is not uniformly continuous on $\mathbb{R}$.

Remark In the proof above we showed that, for an arbitrary $\epsilon > 0$ and $\delta > 0$
there is a pair $x, y \in \mathbb{R}$, such that $|x - y| < \delta$ and $|f(x) - f(y)| > \epsilon$.

Remark If $f, g : A \subseteq \mathbb{R} \to \not{\mathbb{R}}$ are uniformly continuous and bounded
then $f \cdot g$ is uniformly continuous.

Example: Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ then f is continuous and bounded on $\mathbb{R}$,
therefore

$$(f \cdot f)(x) = f(x) f(x) = \sin^2(x)$$

is also uniformly continuous.

Definition We say that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is Lipschitz function if
there is $L \geq 0$, such that, for all $x, y \in A$,

$$|f(x) - f(y)| \leq L\,|x - y|.$$

Example Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 5x + 6$.
Notice that, if $x, y \in \mathbb{R}$, then

$$\begin{aligned} |f(x) - f(y)| &= |(5x + 6) - (5x + 6)| = |5(x - y)| \\ &= 5\,|x - y| \end{aligned}$$

If we take $L = 6$, then for all, $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| = 5\,|x - y| \leq 6\,|x - y| = L\,|x - y|$$

We see that f is Lipshitz.

Example Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$, then one can show that, for all $x, y \in \mathbb{R}$

$$|\sin(x) - \sin(y)| \leq |x - y|$$

Therefore, if we take $L = 1$, we see that, for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq L|x - y|.$$

Example Notice that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable for all x and there is $M \geq 0$, such that, for all $x \in \mathbb{R}$,

$$|f'(x)| \leq M,$$

then one shows using MVT that f is Lipshitz.
In fact, we can take $L = M$ and one can show that

$$|f(x) - f(y)| \leq L|x - y|, \text{ for all } x, y \in \mathbb{R}.$$

Proposition If $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is Lipschitz function then f is uniformly continuous.

Proof. Exercise. ■

- **Remark** A *Lipshitz function* f with $0 \leq L < 1$ is called a *contraction.*
- How can we show that a function $f : A \to \mathbb{R}$ is *not uniformly continuous*
- We need to show that
  there is $\epsilon > 0$, **such that, for every** $\delta > 0$,
  there are $x, y \in A$ **such that**

  $$|x - y| < \delta \textbf{ and } |f(x) - f(y)| \geq \epsilon.$$

- Therefore, in particular,
  there is $\epsilon > 0$ such that for all $n \in \mathbb{N}$, there are

  $$x_n,\ y_n \in A,$$

  such that

  $$|x_n - y_n| < \frac{1}{n}$$

  and

  $$|f(x_n) - f(y_n)| \geq \epsilon.$$

Remark A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is **not uniformly continuous** on A iff there is $\epsilon > 0$ and sequences $\{x_n\}$, $\{y_n\}$ of points in A,

$$\{x_n\},\ \{y_n\} \subseteq A,$$

such that, $|x_n - y_n| \to 0$ as $n \to \infty$ and

$$|f(x_n) - f(y_n)| \geq \epsilon,$$

for all $n \in \mathbb{N}$.

Exercise: Show that

$$\begin{aligned} f &: (0,a] \to \mathbb{R}, \\ f(x) &= \frac{1}{x} \end{aligned}$$

is *not uniformly continuous* on $(0,a]$, $a > 0$.

- Let $x_n = \frac{2}{n}$ and $y_n = \frac{1}{n}$, and see that

$$|x_n - y_n| = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

- We see that

$$\begin{aligned} |f(x_n) - f(y_n)| &= \left| \frac{1}{\frac{2}{n}} - \frac{1}{\frac{1}{n}} \right| = \frac{1}{2}n \\ &\geq \frac{1}{2}, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

- We can take $\epsilon = \frac{1}{2} > 0$ to see that

there are sequences $\{x_n\}$, $\{y_n\}$ of points in $A = (0,a]$,

$$\{x_n\}, \ \{y_n\} \subseteq A,$$

such that, $|x_n - y_n| \to 0$ as $n \to \infty$ and

$$|f(x_n) - f(y_n)| \geq \epsilon = \frac{1}{2},$$

for all $n \in \mathbb{N}$.

Exercise: Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is **NOT uniformly continuous**

Which sequences x_n and y_n will work?

Try $x_n = n + \frac{1}{n}$ and $y_n = n$, $n \in \mathbb{N}$.

Exercise: Show that $f : [a,b] \to \mathbb{R}$, $f(x) = x^2$ is **uniformly continuous** on $[a,b]$, $a < b$.

- Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, then for $B \subseteq A$, and $C \subseteq \mathbb{R}$, recall

$$\begin{aligned} f(B) &= \{f(x) : x \in B\} \text{ and} \\ f^{-1}(C) &= \{x \in A : f(x) \in C\} \end{aligned}$$

are called the *image* of B via f and *preimage* of C via f, respectively.

Example $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and $B = [-1,1]$, then

$$f(B) = \left\{x^2 : x \in [-1,1]\right\} = [0,1]$$

and if $C = [1,2]$, then

$$\begin{aligned} f^{-1}(C) &= \left\{x \in \mathbb{R} : x^2 \in [1,2]\right\} \\ &= \left[-\sqrt{2}, -1\right] \cup \left[1, \sqrt{2}\right]. \end{aligned}$$

Theorem Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be continuous on is an interval I.

Then $f(I)$ is an interval.

Proof. Suppose that $f(I)$ is not an interval,

- then there are $a, b \in f(I)$ and $c \in \mathbb{R}$, such that
$$a < c < b \text{ and } c \notin f(I).$$
- Let $U = (-\infty, c)$ and $V = (c, \infty)$.
- Clearly, U and V are disjoint and
$$f(I) = (f(I) \cap U) \cup (f(I) \cap V).$$
- Since $a \in f(I) \cap U$ and $b \in f(I) \cap V$, both sets $f(I) \cap U$ and $f(I) \cap V$ are nonempty.
- Therefore, U and V is a separation of $f(I)$.
- Since U, V are open and f is continuous
$$I \cap f^{-1}(U) \text{ and } I \cap f^{-1}(V)$$
are open in I.
- Since $a \in f(I) \cap U$ and $b \in f(I) \cap V$,
there are $x \in I \cap f^{-1}(U)$ and $y \in I \cap f^{-1}(V)$.
- Since
$$\emptyset = f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V),$$
so
$$\left(I \cap f^{-1}(U)\right) \cap \left(I \cap f^{-1}(V)\right) = \emptyset.$$
- Furthermore, since $f(I) \subseteq U \cup V$, then
$$I \cap f^{-1}(U) \cup I \cap f^{-1}(V) = I.$$
- Therefore, $I \cap f^{-1}(U)$ and $I \cap f^{-1}(V)$ is a separation of I.
- Contradiction since I is connected.

This finishes our proof. ■

- **Definition** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$. We say that f is an *open mapping* if
for every subset $B \subseteq A$, such that B is open in A,
$f(B)$ is an open subset of $\mathbb{R}$.
- We see that if $f : A \subseteq \mathbb{R} \to \mathbb{R}$, $f(x) = c$, for $c \in \mathbb{R}$ is a constant mapping,
then f is continuous but it is not open.
- However, the following is true.
Proposition Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be continuous and injective. Then
$$f^{-1} : f(A) \to A,$$
defined by
$$f^{-1}(y) = x$$
iff $y = f(x)$ is an open map.

Proof. Exercise. ■

- **Theorem** If K is compact and $f : K \subseteq \mathbb{R} \to \mathbb{R}$ is continuous, then $f(K) \subseteq \mathbb{R}$ is compact.

Proof. We show that every $\mathcal{B}$ be an open covering of $f(K)$ has a finite subcovering.

- Let $\mathcal{B}$ be an open covering of $f(K)$, then $\mathcal{C} = \left\{ f^{-1}(B) : B \in \mathcal{B} \right\}$ is an open covering of K.
- Since K is compact, there is $\mathcal{D} = \{C_1, ..., C_k\} \subseteq \mathcal{C}$ that covers K.
- Therefore, $K \subseteq \bigcup\limits_{i=1}^{k} C_i$ and

$$f(K) \subseteq f\left(\bigcup_{i=1}^{k} C_i \right) \subseteq \bigcup_{i=1}^{k} f(C_i) \subseteq \bigcup_{i=1}^{k} B_i,$$

  so $\{B_1, ..., B_k\} \subseteq \mathcal{B}$ is a finite covering of $f(K)$ that is a subcovering of $\mathcal{B}$.
- It follows that $f(K)$ is compact.

This finishes our proof. ■

- **Theorem** Let $f : K \subseteq \mathbb{R} \to \mathbb{R}$ be continuous and assume that K is compact. Then f is uniformly continuous on K.

Proof. We show that if K is compact,

- then $f : K \subseteq \mathbb{R} \to \mathbb{R}$ is uniformly continuous.
- **Suppose that this is not a case**.
- Then there is $\epsilon > 0$ and sequences $\{x_n\}, \{y_n\} \subseteq K$ such that

$$\lim_{n\to\infty} |x_n - y_n| = 0$$

  and

$$|f(x_n) - f(y_n)| \geq \epsilon,$$

  for all n.
- Since K is compact and $\{x_n\} \subseteq K$, it follows that $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$.
- Let $\lim_{k\to\infty} x_{n_k} = x$.
- Since $\lim_{n\to\infty} |x_n - y_n| = 0$, $\lim_{k\to\infty} (x_{n_k} - y_{n_k}) = 0$ and therefore

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} (x_{n_k} + (y_{n_k} - x_{n_k})) = \lim_{k\to\infty} x_{n_k} + \lim_{k\to\infty} (y_{n_k} - x_{n_k}) = x.$$

- Since f is continuous, it follows that

$$f\left(x_{n_k}\right),\ f\left(y_{n_k}\right) \to f\left(x\right),$$

so

$$\lim_{k\to\infty}\left|f\left(x_{n_k}\right)-f\left(y_{n_k}\right)\right| = \left|\lim_{k\to\infty} f\left(x_{n_k}\right) - \lim_{k\to\infty} f\left(y_{n_k}\right)\right| = 0.$$

- Therefore, there is $n_k \in \mathbb{N}$, such that

$$\left|f\left(x_{n_k}\right)-f\left(y_{n_k}\right)\right| < \epsilon.$$

- **Contradiction** since $\left|f\left(x_n\right)-f\left(y_n\right)\right| \geq \epsilon$, for all n.

This finishes our proof. ■

- **Example**: Let

$$A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$$

and $y_n \to y_0$ and $n \to \infty$.
Define $f : A \to \mathbb{R}$ by

$$f\left(x\right) = \begin{cases} y_n & if \quad x = \frac{1}{n} \\ y_0 & if \quad x = 0 \end{cases}.$$

We show that f is *uniformly continuous*.

- As we showed, A is closed and bounded, so it is compact.

- Since, as we showed it before, f is continuous,
it follows that f is uniformly continuous.
Theorem (*The Intermediate Value Theorem*) Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$.
Then for all $a, b \in f\left(I\right)$ and for every $c \in \mathbb{R}$, if $a < c < b$
then there is $x \in I$, such that

$$f\left(x\right) = c.$$

Proof. Exercise. ■

- **Theorem** (*Extreme Value Theorem*) Let $f : K \subseteq \mathbb{R} \to \mathbb{R}$ be continuous and assume that K is compact.
Then there are $x, y \in K$, such that

$$f\left(x\right) = \inf_K\left(f\right) \text{ and } f\left(y\right) = \sup_K\left(f\right).$$

Proof. Notice that since $f : K \subseteq \mathbb{R} \to \mathbb{R}$ is continuous and K is compact, then

- $f\left(K\right) \subset \mathbb{R}$ is compact, so $A = f\left(K\right)$ is closed and bounded by H-B theorem.

- Since A is bounded and non-empty,
then A has both $\sup A$ and $\inf A$.

- Since A is closed, then

$$\sup A,\ \inf A \in A = f\left(K\right).$$

- Since $\sup A, \inf A \in f(K)$, there are $x, y \in K$, such that
$$f(x) = \inf A = \inf_A (f)$$
and
$$f(y) = \sup A = \sup_A (f).$$

This finishes our proof. ■

- **Definition** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$. We say that f is

i) *increasing*, if for all $x, y \in A$, $f(x) < f(y)$ whenever $x < y$

ii) *decreasing*, if for all $x, y \in A$, $f(x) > f(y)$ whenever $x < y$

iii) *non-decreasing*, if for all $x, y \in A$, $f(x) \leq f(y)$ whenever $x < y$

iv) *non-incereasing*, if for all $x, y \in A$, $f(x) \geq f(y)$ whenever $x < y$
A function f is called *monotone* if f is non-increasing or non-decreasing.
Theorem If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is monotone and $c \in \text{Int}(I)$,
then $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist.

Proof. We prove that $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist.

- We show that is f is non-decreasing
then f has left and right limits for each $c \in \text{Int}(I)$.
- Assume that f is non-decreasing.
- Let
$$S = \{f(x) : x \in I \text{ and } x < c\}$$
and
$$T = \{f(x) : x \in I \text{ and } x > c\}.$$

Claim: Both S and T are nonempty.

- Indeed, since $c \in \text{Int}(I)$,
there is $\delta > 0$, such that
$$(c - \delta, c + \delta) \subseteq I.$$
- Therefore,
$$(c - \delta, c) \cap I \neq \emptyset \text{ and } (c, c + \delta) \cap I \neq \emptyset.$$
- It follows that
$$\emptyset \neq f((c - \delta, c) \cap I) \subseteq S$$
and
$$\emptyset \neq f((c, c + \delta) \cap I) \subseteq T$$
are nonempty.
- Since $f(x) \leq f(c)$, for all $x < c$,
then S is bounced above

- Since $f(c) \leq f(x)$, for all $c < x$,
  then T is bounded below.

- By completeness, there are $\alpha, \beta \in \mathbb{R}$, such that

$$\begin{aligned} \alpha &= \sup S \text{ and} \\ \beta &= \inf T. \end{aligned}$$

- We show that

$$\begin{aligned} \alpha &= \lim_{x \to c^-} f(x) \text{ and} \\ \beta &= \lim_{x \to c^+} f(x). \end{aligned}$$

- Let $\epsilon > 0$ be given.

- Since α is the least upper bound,
  there is $y \in S$, such that $\alpha - \epsilon < y$, i.e.

$$\alpha - y < \epsilon.$$

- Since $y \in S$,
  there is $a \in I$, $a < c$, such that,

$$f(a) = y.$$

- Let $\delta = c - a > 0$.

- If $0 < c - x < \delta$, then

$$0 < c - x < c - a = \delta,$$

  so $x > a$.

- Since f is non-decteasing,

$$f(a) \leq f(x),$$

  so

$$0 \leq \alpha - f(x) \leq \alpha - \underbrace{f(a)}_{y} < \epsilon.$$

- Therefore, $|f(x) - \alpha| < \epsilon$.

- It follows that $\lim_{x \to c^-} f(x) = \alpha$.

- Analogously one shows that

$$\beta = \lim_{x \to c^+} f(x).$$

This completes our proof. ∎

- **Remark**: If $c \in \partial I$, where I is an interval,
  then $c = \inf I$ or $c = \sup I$.

- If $c = \inf I$, then

$$\lim_{x \to c^+} f(x) = \inf \{f(x) : x \in I \text{ and } c < x\}.$$

- If $c = \sup I$, then
$$\lim_{x \to c^-} f(x) = \sup \{f(x) : x \in I \text{ and } x < c\}.$$
- We leave a proof of these facts as an exercise.

  Definition Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, $A_{c^-} = \{x \in A : x < c\}$ and
$$c \in A \cap A'_{c^-},$$
  where
$$A_{c^-} = \{x \in A : x < c\}.$$
  We say that f is *left continuous* at c if
$$\lim_{x \to c^-} f(x) = f(c).$$
  Analogously, if
$$c \in A \cap A'_{c^+},$$
  where
$$A_{c^+} = \{x \in A : x > c\}$$
  and
$$\lim_{x \to c^+} f(x) = f(c),$$
  then we say that f is *right continuous* at c.

  Definition Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $c \in A \cap A'_{c^-}$.or $c \in A \cap A'_{c^+}$

  We say that f has a *jump discontinuity* at c if $\lim_{x \to c^-} f(x)$ is defined and
$$\lim_{x \to c^-} f(x) \neq f(c)$$
  or $\lim_{x \to c^+} f(x)$ exist and
$$\lim_{x \to c^+} f(x) \neq f(c).$$
  Proposition Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be monotone

  then every discontinuity of f is a *jump discontinuity.*

Proof. Exercise ■