Math 4301 Mathematical Analysis I
Lecture 21
Topic: Taylor Series, Generating Functions

- **Theorem** Let
$$f(x) = \sum_{n\geq 0} a_n (x-c)^n, \ |x-c| < R, \ R > 0.$$
then
$$a_n = \frac{f^{(n)}(c)}{n!}, \ n = 0, 1, 2, ...$$

Proof. Exercise. ■

- **Corollary** Let
$$f(x) = \sum_{n\geq 0} a_n (x-c)^n, \ |x-c| < R, \ R > 0$$
and
$$g(x) = \sum_{n\geq 0} b_n (x-c)^n, \ |x-c| < R', \ R' > 0.$$
If there is $\delta > 0$, such that
$$f(x) = g(x), \ |x-c| < \delta,$$
then, for all $n = 0, 1, 2, ...$
$$a_n = b_n$$

Proof. Exercise. ■

- **Definition** Let n be a non-negative integer and $f : (a,b) \to \mathbb{R}$.
We say that f is *class* C^n on (a,b), if for all $0 \leq k \leq n$,
$$f^{(k)} : (a,b) \to \mathbb{R}$$
is continuous (here $f^{(0)} = f$) and we write $f \in C^n(a,b)$.
We say that f is *smooth* (or *class* C^∞) on (a,b) if
$$f^{(n)} : (a,b) \to \mathbb{R}$$
is continuous for all $n = 0, 1, 2, ...$ We will write $f \in C^\infty(a,b)$.
Definition We say that $f : (a,b) \to \mathbb{R}$ is *analytic* on (a,b) if for every $x_0 \in (a,b)$, there is $\delta_{x_0} > 0$, such that,
$$f(x) = \sum_{n\geq 0} a_n (x-x_0)^n, \ |x-x_0| < \delta_{x_0}, \ x \in (a,b).$$
Remark If
$$f(x) = \sum_{n\geq 0} a_n (x-x_0)^n, \ |x-x_0| < R, \ R > 0$$
then f is clearly analytic on $(x_0 - R, x_0 + R)$ and also class C^∞.
However, a class C^∞ function $f : (a,b) \to \mathbb{R}$ might not be analytic on (a,b).

Exercise Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & if \quad x > 0 \\ 0 & if \quad x \leq 0 \end{cases}.$$

Show that, $f^{(n)}(0) = 0$, for all $n = 0, 1, 2, ...$ and f is class C^{∞}.

Exercise Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & if \quad x > 0 \\ 0 & if \quad x \leq 0 \end{cases}.$$

Show that f is smooth but not analytic.

Proposition Let $f, g : (a, b) \to \mathbb{R}$ be analytic on (a, b). If there is $c \in (a, b)$, such that

$$f^{(n)}(c) = g^{(n)}(c),$$

for all $n = 0, 1, 2, ...$

Then, for all $x \in (a, b)$,

$$f(x) = g(x)$$

Proof. Let

$$E = \left\{x \in (a, b) : f^{(n)}(x) = g^{(n)}(x),\ n = 0, 1, ...\right\}.$$

- Since each

$$E_n = \left\{x \in (a, b) : f^{(n)}(x) = g^{(n)}(x)\right\}$$

is closed and

$$E = \bigcap_{n=0}^{\infty} E_n,$$

it follows that E is closed in (a, b).

- We show that $E \subseteq (a, b)$ is open. Indeed, let $x_0 \in E$ and
- since f and g are analytic on (a, b), there are $R, R' > 0$, such that

$$f(x) = \sum_{n \geq 0} a_n (x - x_0)^n,\ \ |x - x_0| < R,\ x \in (a, b)$$

and

$$g(x) = \sum_{n \geq 0} b_n (x - x_0)^n,\ \ |x - x_0| < R',\ x \in (a, b).$$

- Since $f^{(n)}(x_0) = g^{(n)}(x_0)$, $n = 0, 1, ...,$

$$a_n = \frac{f^{(n)}(x_0)}{n!} = \frac{g^{(n)}(x_0)}{n!} = b_n.$$

- Therefore, if

$$\delta = \min\{x_0 - a, b - x_0, R, R'\} > 0$$

then for all $x \in (x_0 - \delta, x_0 + \delta) \subseteq (a, b)$,

$$f(x) = \sum_{n \geq 0} a_n (x - x_0)^n = \sum_{n \geq 0} b_n (x - x_0)^n = g(x).$$

- It follows that
$$(x_0-\delta, x_0+\delta) \subseteq E_0.$$
- Therefore, for all $x \in (x_0-\delta, x_0+\delta)$,
$$f^{(n)}(x) = g^{(n)}(x),$$
so for all $n = 0, 1, 2, \ldots$
$$(x_0-\delta, x_0+\delta) \subseteq E_n.$$
- It follows that
$$(x_0-\delta, x_0+\delta) \subseteq E.$$
- Therefore, E is open in (a,b).
- By assumption, $E \neq \emptyset$ as $c \in E$ and since (a,b) is connected,
- it must be
$$E = (a,b).$$

This finishes our argument. ■

- **Generating Functions**
- Recall, the **Binomial Theorem** states that, for an integer $n \geq 0$ and $x, y \in \mathbb{R}$
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$
- In particular, when $y = 1$,
$$\underbrace{(1+x)^n}_{\substack{\text{polynomial}\\ \text{whose coefficients}\\ \text{record number of}\\ k\text{-sub. of } n\text{-set.}}} = \sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{\substack{\text{number of } k\text{-subsets}\\ \text{of an } n\text{-set}}} x^k,$$
$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$
- This is an idea for the generating function.
- Let a_n = number of ways to put a structure on n-set, $a_n \geq 0$
- Define
$$f(x) = \sum_{n\geq 0} a_n x^n$$
and call it *an ordinary generating function* (OGF) and let
$$g(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n$$
and call it *an exponential generating function* (EGF).

  Example Let a_n = number of subsets of n-set
- As we showed $a_n = \underbrace{2^n}$, $n \geq 0$.

- Then
$$f(x) = \sum_{n\geq 0} (2^n)\, x^n = \sum_{n\geq 0} (2x)^n = \frac{1}{1-2x}$$
is an *ordinary generating function* and
$$g(x) = \sum_{n\geq 0} \frac{2^n}{n!} x^n = \sum_{n\geq 0} \frac{(2x)^n}{n!} = e^{2x}$$
is *exponential generating function* for the sequence (a_n).

- In general, **ordinary and exponential generating functions** are **formal power series** (elements of $\mathbb{C}[[x]]$).

- We can of course add them and multiply as follows:

- Let
$$f(x) = \sum_{n\geq 0} a_n x^n,\ g(x) = \sum_{n\geq 0} b_n x^n,$$
then
$$(f+g)(x) = \sum_{n\geq 0} (a_n + b_n) x^n$$
and
$$f(x)g(x) = \sum_{n\geq 0} c_n x^n,$$
where
$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

- Let
$$f(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n \text{ and } g(x) = \sum_{n\geq 0} \frac{b_n}{n!} x^n$$
then
$$(f+g)(x) = \sum_{n\geq 0} \frac{(a_n + b_n)}{n!} x^n$$
and
$$(f\cdot g)(x) = \sum_{n\geq 0} \frac{c_n}{n!} x^n,$$
where
$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

- Indeed,
$$\frac{c_n}{n!} = \sum_{k=0}^{n} \frac{a_k}{k!} \cdot \frac{b_{n-k}}{(n-k)!} = \sum_{k=0}^{n} \frac{1}{n!} \binom{n}{k} a_n b_{n-k}$$
so $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$.

Example Let $a_0 = 2$, $a_1 = 1$, and

$$a_n = a_{n-2} + 2n,\ n \geq 2.$$

Find a closed form formula for a_n.

- We use the method of the generating functions to find a_n, for all $n \geq 0$.
- Let
$$A(z) = \sum_{n\geq 0} a_n z^n.$$
- Notice that if
$$f(z) = \sum_{n\geq 0} z^n = \frac{1}{1-z}$$
then
$$\begin{aligned} f'(z) &= \frac{1}{(1-z)^2} = \frac{d}{dz}\left(\frac{1}{1-z}\right) \\ &= \frac{d}{dz}\left(\sum_{n\geq 0} z^n\right) = \sum_{n\geq 1} nz^{n-1}. \end{aligned}$$
- Moreover, since
$$a_n = a_{n-2} + 2n,\ n \geq 2$$
and $a_0 = 2$, $a_1 = 1$,
$$\begin{aligned} A(z) &= \sum_{n\geq 0} a_n z^n = a_0 + a_1 z + \sum_{n\geq 2} a_n z^n \\ &= 2 + z + \sum_{n\geq 2} (a_{n-2} + 2n)\, z^n = 2 + z + z^2 \sum_{n\geq 2} a_{n-2} z^{n-2} + 2z \sum_{n\geq 2} nz^{n-1} \\ &= 2 + z + z^2 \sum_{n\geq 0} a_n z^n + 2z\left(\sum_{n\geq 1} nz^{n-1} - 1\cdot z^0\right) \\ &= 2 + z + z^2 A(z) + 2z\left(\sum_{n\geq 1} nz^{n-1} - 1\right) \\ &= 2 + z + z^2 A(z) + 2z\left(\frac{1}{(z-1)^2} - 1\right) \end{aligned}$$
- It follows that
$$\begin{aligned} A(z) &= 2 + z + z^2 A(z) + 2z\left(\frac{1}{(z-1)^2} - 1\right), \text{ so} \\ \left(1 - z^2\right) A(z) &= 2 + z + \frac{2z}{(z-1)^2} - 2z, \text{ thus} \\ \left(1 - z^2\right) A(z) &= 2 - z + \frac{2z}{(z-1)^2}. \end{aligned}$$

- This give
$$A(z)=\frac{2-z}{1-z^2}+\frac{2z}{(z-1)^2(1-z^2)}=\frac{z^3-4z^2+3z-2}{(z-1)^3(z+1)}=\sum_{n\geq 0}a_nx^n.$$

- Since
$$\frac{z^3-4z^2+3z-2}{(z-1)^3(z+1)}=\frac{5}{4(1+z)}+\frac{1}{4(1-z)}-\frac{1}{2(1-z)^2}+\frac{1}{(1-z)^3}$$

  and as we know
$$\begin{aligned}\frac{1}{1+z} &= \sum_{n\geq 0}(-1)^n z^n,\\ \frac{1}{1-z} &= \sum_{n\geq 0} z^n\\ \frac{1}{(1-z)^2} &= \frac{d}{dz}\left(\frac{1}{1-z}\right)=\sum_{n\geq 1}nz^{n-1},\ \text{and}\\ \sum_{n\geq 2}n(n-1)z^{n-2} &= \frac{d^2}{dz^2}\left(\frac{1}{1-z}\right)=\frac{2}{(1-z)^3},\ \text{so}\\ \frac{1}{(1-z)^3} &= \frac{1}{2}\sum_{n\geq 2}n(n-1)z^{n-2}\end{aligned}$$

- We obtain
$$\begin{aligned}A(z) &= \frac{5}{4(1+z)}+\frac{1}{4(1-z)}-\frac{1}{2(1-z)^2}+\frac{1}{(1-z)^3}\\ &= \frac{5}{4}\sum_{n\geq 0}(-1)^n z^n+\frac{1}{4}\sum_{n\geq 0}z^n-\frac{1}{2}\sum_{n\geq 1}nz^{n-1}+\frac{1}{2}\sum_{n\geq 2}n(n-1)z^{n-2}\\ &= \frac{5}{4}\sum_{n\geq 0}(-1)^n z^n+\frac{1}{4}\sum_{n\geq 0}z^n-\frac{1}{2}\sum_{n\geq 0}(n+1)z^n+\frac{1}{2}\sum_{n\geq 0}(n+2)(n+1)z^n\\ &= \sum_{n\geq 0}\left(\frac{5}{4}(-1)^n+\frac{1}{4}-\frac{1}{2}(n+1)+\frac{1}{2}(n+2)(n+1)\right)z^n\\ &= \sum_{n\geq 0}\left(\underbrace{\frac{5}{4}(-1)^n+\frac{1}{2}n^2+n+\frac{3}{4}}_{a_n}\right)z^n.\end{aligned}$$

- Since
$$A(z)=\sum_{n\geq 0}a_nz^n$$

- Therefore, we see that
$$\begin{aligned}a_n &= \frac{5}{4}(-1)^n+\frac{1}{2}n^2+n+\frac{3}{4}\\ &= \frac{1}{4}\left(5(-1)^n+2n^2+4n+3\right).\end{aligned}$$

- We see that

$$\begin{aligned}
a_0 &= \frac{1}{4}\left(5\,(-1)^0 + 2\cdot 0^2 + 4\cdot 0 + 3\right) = 2\\
a_1 &= \frac{1}{4}\left(5\,(-1)^1 + 2\cdot 1^2 + 4\cdot 1 + 3\right) = 1\\
&\vdots\\
a_{10} &= \frac{1}{4}\left(5\,(-1)^{10} + 2\cdot (10)^2 + 4\cdot 10 + 3\right) = 62\\
&\vdots
\end{aligned}$$

Falling Factorial

Let $\lambda \in \mathbb{C}$ (i.e. λ is a complex number).

- Define

$$\begin{aligned}
[\lambda]_0 &= 1,\\
[\lambda]_n &= \lambda(\lambda-1)\cdot\ldots\cdot(\lambda-(n-1)),\ n \geq 1.
\end{aligned}$$

- We call $[\lambda]_n$ the falling factorial.
- It allows us to define

$$\binom{\lambda}{k} = \frac{[\lambda]_k}{k!},\ k \geq 0.$$

Definition For $\lambda \in \mathbb{C}$ *define the binomial series* as follows

$$(1+x)^\lambda = \sum_{n\geq 0}\binom{\lambda}{n}x^n.$$

Proposition For all $\lambda, \mu \in \mathbb{C}$,

$$(1+x)^\lambda \cdot (1+x)^\mu = (1+x)^{\lambda+\mu}$$

Example One shows that, if $m > 0$, then for all $z \in \mathbb{C}$,

$$\frac{1}{(1-z)^m} = (1-z)^{-m} = \sum_{n\geq 0}\binom{-m}{n}(-z)^n = \sum_{n\geq 0}(-1)^n\,\frac{[-m]_n}{n!}z^n$$

Since

$$\begin{aligned}
[-m]_n &= (-m)\,(-m-1)\cdot\ldots\cdot(-m-(n-1))\\
&= (-1)^n\cdot m\cdot(m+1)\cdot\ldots\cdot(m+(n-1))\\
&= (-1)^n\,\frac{1\cdot 2\cdot\ldots\cdot(m-1)\cdot m\cdot\ldots\cdot(m+n-2)\cdot(m+n-1)}{(m-1)!}\\
&= (-1)^n\,\frac{(m+n-1)!}{(m-1)!},
\end{aligned}$$

we see that

$$\begin{aligned}
\frac{1}{(1-z)^m} &= \sum_{n\geq 0}(-1)^n\frac{[-m]_n}{n!}z^n = \sum_{n\geq 0}\frac{(-1)^{2n}(m+n-1)!}{n!\,(m-1)!}z^n \\
&= \sum_{n\geq 0}\binom{m+n-1}{m-1}z^n. \\
\binom{m+n-1}{m-1} &= \frac{(m+n-1)!}{(m-1)!n!}
\end{aligned}$$

- We obtained the following, for $m>0$, $z\in\mathbb{C}$

$$\frac{1}{(1-z)^m} = \sum_{n\geq 0}\binom{m+n-1}{m-1}z^n, \text{ for } |z|<1$$

and in particular, for $b\in\mathbb{C}\setminus\{0\}$,

$$\begin{aligned}
\frac{1}{(1-bz)^m} &= \sum_{n\geq 0}\binom{m+n-1}{m-1}b^nz^n, \quad |z|<\frac{1}{|b|}. \\
\frac{1}{(1+bz)^m} &= \sum_{n\geq 0}(-1)^n\binom{m+n-1}{m-1}b^nz^n, \quad |z|<\frac{1}{|b|}
\end{aligned}$$

Example: We find

$$(1+bx)^{\frac{1}{2}} = \sum_{n\geq 0}\binom{\frac{1}{2}}{n}(bx)^n$$

- We see that for $n\geq 1$

$$\begin{aligned}
\binom{\frac{1}{2}}{n} &= \frac{\frac{1}{2}\cdot(\frac{1}{2}-1)\cdot(\frac{1}{2}-2)\cdot\cdots\cdot(\frac{1}{2}-(n-1))}{n!} \\
&= \frac{\frac{1}{2}\cdot(-\frac{1}{2})\cdot(-\frac{3}{2})\cdot\cdots\cdot(-\frac{2n-3}{2})}{n!} \\
&= \frac{(\frac{1}{2})^n\cdot(-1)^{n-1}\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{n!} \\
&= \frac{(-1)^{n-1}}{2^n}\cdot\frac{1\cdot 2\cdot 3\cdot\cdots\cdot(2n-3)(2n-2)}{n!\cdot 2\cdot 4\cdot\cdots\cdot(2n-2)} \\
&= \frac{(-1)^{n-1}\cdot(2n-2)!}{2^n\cdot 2^{n-1}n!(n-1)!} = \frac{2\cdot(-1)^{n-1}\cdot(2n-2)!}{4^n\cdot n!(n-1)!}
\end{aligned}$$

and

$$\binom{\frac{1}{2}}{0} = \frac{[\frac{1}{2}]_0}{0!} = 1$$

- Hence

$$
\begin{aligned}
(1+bx)^{\frac{1}{2}} &= \sum_{n\geq 0}\binom{\frac{1}{2}}{n}(bx)^n \\
&= 1+\sum_{n\geq 1}\frac{2\cdot(-1)^{n-1}\cdot(2n-2)!}{n!(n-1)!4^n}\cdot b^n x^n \\
&= 1+2\sum_{n\geq 1}\frac{(-1)^{n-1}\,b^n(2n-2)!}{4^n n!(n-1)!}x^n \\
&= 1+2\sum_{n\geq 1}\frac{(-1)^{n-1}\,b^n}{n\cdot 4^n}\binom{2n-2}{n-1}x^n
\end{aligned}
$$

Fibonacci Numbers

- **Example** Let $F_0=F_1=1$ and $F_n=F_{n-1}+F_{n-2}$, for $n\geq 2$.
- The sequence $(F_n)_{n\geq 0}$ is called the Fibonacci sequence.
- For instance

$$
\begin{aligned}
F_2 &= F_0+F_1=2, \\
F_3 &= F_2+F_1=3 \\
&\vdots
\end{aligned}
$$

- **Interpretation :** $F_n=$ number of ways to climb n steps by taking 1 or 2 steps at a time
- $F_n = \underbrace{F_{n-1}}_{\substack{\text{number of ways} \\ \text{to climb } n\text{steps} \\ \text{starting with the first steps} \\ \text{to be 1}}} + \underbrace{F_{n-2}}_{\substack{\text{number of ways} \\ \text{to climb } n \text{ steps by making} \\ \text{2 steps at the beginning}}}$
- Let $F(x)=\sum_{n\geq 0}F_n x^n$ to be an ordinary generating function for the sequence F_n.
- We want to find the closed formula for F_n.

$$
\begin{aligned}
F(x)=\sum_{n\geq 0}F_n x^n &= F_0+F_1x+\sum_{n\geq 2}F_n x^n \\
&= 1+x+\sum_{n\geq 2}(F_{n-1}+F_{n-2})x^n = 1+x+x\sum_{n\geq 2}F_{n-1}x^{n-1}+x^2\sum_{n\geq 2}F_{n-2}x^{n-2} \\
&= 1+x+x\sum_{n\geq 1}F_n x^n+x^2\sum_{n\geq 0}F_n x^n = 1+x+x(F(x)-1)+x^2F(x) \\
&= 1+x+xF(x)-x+x^2F(x) = 1+(x+x^2)F(x)
\end{aligned}
$$

- Thus,

$$(1-x-x^2)F(x)=1,$$

so

$$F(x)=\frac{1}{1-x-x^2}=-\frac{1}{x^2+x-1}=-[\frac{A}{x+\alpha}+\frac{B}{x+\beta}],$$

where α,β are roots of the equation

$$x^2+x-1=0,$$

- Since
$$x^2+x-1=(x+\frac{1}{2})^2-\frac{5}{4}=0$$
if and only if
$$x=-\frac{1}{2}\mp\frac{\sqrt{5}}{2}.$$
- Therefore,
$$\alpha=\frac{1}{2}+\frac{\sqrt{5}}{2} \text{ and } \beta=\frac{1}{2}-\frac{\sqrt{5}}{2}.$$
- Now, we see that
$$\begin{aligned} A(x+\beta)+B(x+\alpha)=1, &\text{ so} \\ (A+B)x+(A\beta+B\alpha)=1 & \end{aligned}$$
- It follows that
$$\begin{aligned} A&=-B, \\ A\beta-A\alpha=1 & \\ A&=\frac{1}{\beta-\alpha}=\frac{1}{\frac{1}{2}-\frac{\sqrt{5}}{2}-\frac{1}{2}-\frac{\sqrt{5}}{2}}=-\frac{1}{\sqrt{5}} \end{aligned}$$
- Therefore,
$$A=-\frac{1}{\sqrt{5}} \text{ and } B=\frac{1}{\sqrt{5}}.$$
- Notice that
$$\frac{1}{\alpha}=\frac{1}{\frac{1}{2}+\frac{\sqrt{5}}{2}}=\frac{\frac{1}{2}-\frac{\sqrt{5}}{2}}{\frac{1}{4}-\frac{5}{4}}=-(\frac{1}{2}-\frac{\sqrt{5}}{2})=-\beta$$
and $\frac{1}{\beta}=-\alpha$.
- However,
$$\begin{aligned} \frac{1}{x+\alpha}&=\frac{1}{\alpha}\frac{1}{\frac{x}{\alpha}+1}=-\beta\cdot\frac{1}{-\beta x+1}=\beta\cdot\frac{1}{\beta x-1} \\ &=\beta\sum_{n\geq 0}\beta^n x^n=\sum_{n\geq 0}\beta^{n+1}x^n \end{aligned}$$
- Analogously,
$$\frac{1}{x+\beta}=\alpha\frac{1}{\alpha x-1}=\sum_{n\geq 0}\alpha^{n+1}x^n.$$
- Therefore,
$$\begin{aligned} -\frac{1}{\sqrt{5}}\cdot\frac{1}{x+\alpha}+\frac{1}{\sqrt{5}}\frac{1}{x+\beta} &= \frac{1}{\sqrt{5}}\left(\frac{1}{x+\beta}-\frac{1}{x+\alpha}\right) \\ &= \frac{1}{\sqrt{5}}\sum_{n\geq 0}(\alpha^{n+1}-\beta^{n+1})x^n \end{aligned}$$

- Hence

$$F_n = \frac{1}{\sqrt{5}}\left[\alpha^{n+1}-\beta^{n+1}\right] = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

$$F_{10} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{10+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{10+1}\right) = 89$$

Evaluating Sums Using Generating Functions

- $1^2+2^2+...+n^2 = \frac{1}{6}n\,(n+1)\,(2n+1)$ — How can we get this formula?
- Suppose we are given $(a_n)_{n\geq 0}$, and we would like to find

$$s_n = \sum_{k=0}^{n} a_k.$$

- Define

$$F(z) = \sum_{n=0}^{\infty} a_n z^n$$

and

$$G(z) = \sum_{n=0}^{\infty} \underbrace{1}_{b_n} \cdot z^n = \frac{1}{1-z}.$$

- Then

$$F(z)G(z) = \frac{F(z)}{1-z} = \sum_{n=0}^{\infty}\left(\underbrace{\sum_{k=0}^{n} a_k \underbrace{b_{n-k}}_{1}}_{c_n}\right) z^n$$

and since $b_n = 1$ for all n, thus

$$\frac{F(z)}{1-z} = \sum_{n=0}^{\infty}\left(\underbrace{\sum_{k=0}^{n} a_k}_{s_n}\right) z^n = \sum_{n=0}^{\infty} s_n z^n$$

- So s_n is the coefficients of z^n in $\frac{F(z)}{1-z}$.

  Example Find $a_n = 3^n$, $n = 0, 1, ...$

$$s_n = \sum_{k=0}^{n} a_n = \sum_{k=0}^{n} 3^k = 1+3+...+3^n.$$

- As we know

$$s_n = \sum_{k=0}^{n} 3^k = \frac{3^{n+1}-1}{3-1} = \frac{3^{n+1}-1}{2}.$$

- Let us find it using the method of generating functions, that is, we want to find s_n, where

$$\frac{F(z)}{1-z} = \sum s_n z^n,$$

where

$$F(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} 3^n z^n = \sum_{n=0}^{\infty} (3z)^n = \frac{1}{1-3z}$$

- Since

$$\begin{aligned}
\sum_{n\geq 0} s_n z^n &= \frac{F(z)}{1-z} = \frac{1}{1-3z}\cdot\frac{1}{1-z} = \frac{1}{2\,(z-1)} - \frac{3}{2\,(3z-1)} \\
&= \frac{3}{2}\frac{1}{1-3z} - \frac{1}{2}\frac{1}{1-z} = \frac{1}{2}\left(3\sum_{n=0}^{\infty} 3^n z^n - \sum_{n=0}^{\infty} z^n\right) \\
&= \sum_{n=0}^{\infty}\left(\frac{3^{n+1}-1}{2}\right) z^n
\end{aligned}$$

- It follows that

$$s_n = \frac{3^{n+1}-1}{2}$$

Example Find

$$s_n = \sum_{k=0}^{n} k^3.$$

- It is known that

$$s_n = \left(\frac{n(n+1)}{2}\right)^2.$$

- Let us find it using the method of generating functions, that is, we want to find s_n, where

$$\frac{F(z)}{1-z} = \sum_{n\geq 0} s_n z^n$$

and $a_n = n^3$, so

$$F(z) = \sum_{n\geq 0} a_n z^n = \sum_{n\geq 0} n^3 \cdot z^n$$

- Therefore, we need to find $F(z)$
- Since

$$\begin{aligned}
\frac{1}{(1-z)^2} &= \frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{d}{dz}\left(\sum_{n\geq 0} z^n\right) = \sum_{n\geq 1} n\cdot z^{n-1}, \\
\frac{z}{(z-1)^2} &= z\cdot\frac{d}{dz}\left(\frac{1}{1-z}\right) = \sum_{n\geq 1} n\cdot z^n = \sum_{n\geq 0} n\cdot z^n
\end{aligned}$$

- Furthermore,

$$\begin{aligned}\frac{1+z}{(1-z)^3} &= \frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)=\sum_{n\geq 1} n^2\cdot z^{n-1}\\ \frac{z(1+z)}{(1-z)^3} &= z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)=z\sum_{n\geq 1} n^2\cdot z^{n-1}=\sum_{n\geq 0} n^2\cdot z^{n}\end{aligned}$$

and therefore

$$\begin{aligned}\frac{z^2+4z+1}{(1-z)^4} &= \frac{d}{dz}\left(\frac{z(1+z)}{(1-z)^3}\right)=\sum_{n\geq 1} n^3\cdot z^{n-1},\ \text{so}\\ \frac{z\left(z^2+4z+1\right)}{(1-z)^4} &= z\frac{d}{dz}\left(\frac{z(1+z)}{(1-z)^3}\right)=z\sum_{n\geq 1} n^3\cdot z^{n-1}=\sum_{n\geq 0} n^3\cdot z^{n}=F(z)\end{aligned}$$

- Hence

$$\frac{z\left(z^2+4z+1\right)}{(1-z)^4}=\sum_{n\geq 0} n^3\cdot z^n.$$

- Since

$$\begin{aligned}\sum_{n\geq 0} s_n z^n &= \frac{F(z)}{1-z}=\frac{\frac{z\left(z^2+4z+1\right)}{(1-z)^4}}{1-z}=\frac{z\left(z^2+4z+1\right)}{(1-z)^5}\\ &= -\frac{1}{(1-z)^2}+\frac{7}{(1-z)^3}-\frac{12}{(1-z)^4}+\frac{6}{(1-z)^5}\end{aligned}$$

- Since

$$\frac{1}{(1-z)^m}=\sum_{n\geq 0}\binom{m+n-1}{m-1}z^n$$

we obtain

$$\begin{aligned}\sum_{n\geq 0} s_n z^n &= -\sum_{n\geq 0}\binom{2+n-1}{2-1}z^n+\sum_{n\geq 0}7\binom{3+n-1}{3-1}z^n-12\sum_{n\geq 0}\binom{4+n-1}{4-1}z^n+6\sum_{n\geq 0}\binom{5+n-1}{5-1}z^n\\ &= \sum_{n\geq 0}\left(-\binom{2+n-1}{2-1}+7\binom{3+n-1}{3-1}-12\binom{4+n-1}{4-1}+6\binom{5+n-1}{5-1}\right)z^n\\ &= \sum_{n\geq 0}\left(-\binom{n+1}{1}+7\binom{n+2}{2}-12\binom{n+3}{3}+6\binom{n+4}{4}\right)z^n\end{aligned}$$

- Thus

$$\begin{aligned}s_n &= -\binom{n+1}{1}+7\binom{n+2}{2}-12\binom{n+3}{3}+6\binom{n+4}{4}\\ &= \frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2\\ &= \frac{1}{4}n^2(n+1)^2\end{aligned}$$

Exercise Find the (GF) for the harmonic sequence

$$s_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n},\ n=1,2,...$$

- **Solution:** Let $a_n = \frac{1}{n}$, then
$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=1}^{\infty} \frac{1}{n} z^n,$$
so (GF) for the sequence
$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$
is
$$\frac{f(z)}{1-z} = \sum_{n=0}^{\infty} s_n z^n$$

- We see that
$$f'(z) = \sum_{n=0}^{\infty} a_n n z^{n-1} = \sum_{n=1}^{\infty} \frac{1}{n} n z^{n-1} = \sum_{n=1}^{\infty} z^{n-1} = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$
Since $f(0) = s_0 = 0$, we see that
$$f(z) = \int_0^z \frac{1}{1-s} ds = -\ln(1-z).$$

- Hence
$$\sum_{n=0}^{\infty} s_n z^n = \frac{f(z)}{1-z} = \frac{-\ln(1-z)}{1-z} = \frac{\ln(1-z)}{z-1}$$
is the generating function.

Theorem (*Basic Operations on Generating Functions (GF)*) Let $f(z)$ be the GF of $(a_n)_{n\geq 0}$ i.e.
$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then

i) If $h > 0$ is a positive integer, then the GF of $(a_{n+h})_{n\geq 0}$ is
$$\frac{f(z) - \left(a_0 + a_1 z + \ldots + a_h z^{h-1}\right)}{z^h}.$$

Thus, the GF of $(a_{n+2})_{n\geq 0}$ is
$$\sum_{n\geq 2} a_n z^{n-2} = \frac{f(z) - a_0 - a_1 z}{z^2}.$$

ii) The GF of $(P(n)\, a_n)_{n\geq 0}$, where P is a polynomial, is
$$P(z\frac{d}{dz})f(z).$$

Thus, $\left((n^2+n+1)a_n\right)_{n\geq 0}$ has the GF
$$\left(z\frac{d}{dz}\right)^2 f(z) + \left(z\frac{d}{dz}\right) f(z) + f(z),$$

where

$$\left(z\frac{d}{dz}\right)f\left(z\right)=zf'(z)$$

and

$$\left(z\frac{d}{dz}\right)^2 f\left(z\right)=z\frac{d}{dz}(zf'\left(z\right))=z(f'\left(z\right)+zf''\left(z\right))=zf'\left(z\right)+z^2f''\left(z\right)$$

So the GF of $\left((n^2+n+1)a_n\right)_{n\geq 0}$ is

$$zf'\left(z\right)+z^2f''\left(z\right)+zf'(z)+f\left(z\right)=z^2f''\left(z\right)+2zf'(z)+f\left(z\right).$$

iii) If

$$g\left(z\right)=\sum_{n=0}^{\infty}b_nz^n,$$

then

$$f(z)g(z)=\sum_{n=0}^{\infty}c_nz^n,$$

where

$$c_n=\sum_{k=0}^{n}a_kb_{n-k}$$

a. (Power Shifting Rule) *Special case*

$$z^kf(z)=\sum_{n=k}^{\infty}a_{n-k}z^n$$

since z^k is the GF of $\{b_n\}$, where

$$b_n=\begin{cases}1 & \text{if } n=k\\ 0 & \text{else}\end{cases},$$

b. Special case

$$\frac{f(z)}{1-z}=\sum_{n=0}^{\infty}s_nz^n$$

where

$$s_n=a_0+a_1+...+a_n$$

c. Similarly if f,g,h are the GF of $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, then the $f(z)g(z)h(z)$ is the GF of

$$\left(\sum_{i+j+k=n}a_ib_jc_k\right)_{n\geq 0}$$

In particular, if $f^k(z)=\underbrace{f(z)\cdot f(z)\cdot ...\cdot f(z)}_{k\text{ factors}}$, then

$$\begin{aligned}f^k\left(z\right) &= \sum_{n=0}^{\infty}c_nz^n,\text{ where}\\ c_n &= \sum_{n_1+\cdots+n_k=n}a_{n_1}a_{n_2}...a_{n_k}.\end{aligned}$$