5. Connectedness

Math 4341 (Topology)

Homeomorphisms

- **Definition**. A bijection $f : X \to Y$ between two topological spaces is called a *homeomorphism* if both f and its inverse f^{-1} are continuous. In this case, we say that X and Y are *homeomorphic* and we write $X \simeq Y$.

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- Note that $\simeq$ satisfies the property of an equivalence relation.
- **Example**. Let $f : (-1, 1) \to \mathbb{R}$ be the bijective map

$$f(x) = \tan\left(\frac{\pi x}{2}\right)$$

whose inverse is $f^{-1}(x) = \frac{2}{\pi} \arctan x$. Then both f and f^{-1} are continuous so $(-1, 1)$ and $\mathbb{R}$ are homeomorphic.

- **Example**. Let $B^n := B(0,1)$ be the unit ball in $\mathbb{R}^n$. Then $B^n \simeq \mathbb{R}^n$. This is because the map $f : B^n \to \mathbb{R}^n$ given by

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- **Definition**. Let X and Y be topological spaces. A function $f : X \to Y$ is called an *embedding* if $f : X \to f(X)$ is a homeomorphism; where $f(X) \subset Y$ has the subspace topology.
- **Example**. If X is a top. space and $Y \subset X$ a subspace, then the inclusion $\iota : Y \to X$ given by $\iota(x) = x$ is an embedding.

The n-dimensional sphere

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  - Define the *stereographic projection* of x by
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  - Geometrically, if one draws a straight line through x and p, then its intersection with $\mathbb{R}^n \times \{0\}$ is the point $(f(x), 0)$.
  - f is continuous because each of its components are.
  - f has a continuous inverse $g : \mathbb{R}^n \to S^n \setminus \{p\}$ given by

$$g(y_1, \ldots, y_n) = (t(y)y_1, \ldots, t(y)y_n, 1 - t(y)),$$

  where $t(y) = 2/(1+ \| y \|^2)$.

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- **Lemma 5.2**. A topological space X is connected iff $\emptyset$ and X are the only subsets of X that are both open and closed.
- **Example**. The rational numbers $\mathbb{Q} \subset \mathbb{R}$ are not connected.
  - Choose any irrational number $a \in \mathbb{R}$. Then

    $$\mathbb{Q} = ((-\infty, a) \cap \mathbb{Q}) \cup (\mathbb{Q} \cap (a, \infty)),$$

    which is a separation.

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- **Lemma 5.3**. Let $X = U \cup V$ for disjoint open sets U and V, and let $Y \subset X$. If Y is connected, then $Y \subset U$ or $Y \subset V$.
- **Proof**. We will show the contrapositive of the statement, so assume that $Y \cap U \neq \emptyset$ and $Y \cap V \neq \emptyset$. Then

$$Y = Y \cap X = Y \cap (U \cup V) = (Y \cap U) \cup (Y \cap V)$$

is a separation of Y (since $Y \cap U$ and $Y \cap V$ are disjoint, non-empty and open in the subspace topology).

- **Theorem 5.4**. Let $\{A_i\}_{i \in I}$ be a collection of connected subspaces of a topological space X with a common point $x \in X$; i.e. $x \in A_i$ for all $i \in I$. Then $\bigcup_{i \in I} A_i$ is connected.

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- **Proof**. Suppose $\bigcup_{i \in I} A_i = U \cup V$ is a separation.
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  - Assume without loss of generality that $x \in U$. By Lemma 5.3 we have for each i that either $A_i \subset U$ or $A_i \subset V$.
  - Since $x \in A_i$ we must have $A_i \subset U$ for all $i \in I$. This implies that $\bigcup_{i \in I} A_i \subset U$, so V must be empty.

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  - By definition of the subspace topology, there are open sets U' and V' in X so that $U = B \cap U'$, $V = B \cap V'$, and

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  - Hence $B \subset \bar{U} \subset X \setminus V$ which means $B \cap V = \emptyset$, so $V = \emptyset$.

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- **Corollary 5.7**. [Intermediate value theorem] Let $f : X \to \mathbb{R}$ be continuous and assume that X is connected. If there is an $r \in \mathbb{R}$ and $x, y \in X$ so that $f(x) < r < f(y)$, then there is a $z \in X$ with $f(z) = r$.

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- *Proof*. By Theorem 5.6 $f(X) \subset \mathbb{R}$ is connected. This implies that $r \in [f(x), f(y)] \subset f(X)$ (why ???).

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  - Fix $x_0 \in A$ and let $A_{x_0} = \{x_0\} \times Y$. Then A_{x_0} is the image of the continuous map $Y \to X \times Y$ given by $y \mapsto (x_0, y)$, so A_{x_0} is connected by Theorem 5.6.

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  - Similarly $B_y = X \times \{y\}$ is connected for all $y \in Y$. By Theorem 5.4, $A_{x_0} \cup B_y$ is connected for all $y \in Y$ since (x_0, y) is contained in both A_{x_0} and B_y.

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  - Similarly $B_y = X \times \{y\}$ is connected for all $y \in Y$. By Theorem 5.4, $A_{x_0} \cup B_y$ is connected for all $y \in Y$ since (x_0, y) is contained in both A_{x_0} and B_y.
  - Now clearly,
  $$X \times Y = \bigcup_{y \in Y} A_{x_0} \cup B_y,$$
  and all the sets on the RHS have the common point $(x_0, *)$, where $* \in Y$. Therefore, $X \times Y$ is connected by Theorem 5.4.

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- *Proof*. Recall from Proposition 5.1 that $S^n \setminus \{p\} \simeq \mathbb{R}^n$, where p is the north pole. It follows that $S^n \setminus \{p\}$ is connected. Since $\overline{S^n \setminus \{p\}} = S^n$, the result follows from Theorem 5.5.

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- **Prop. 5.11**. For any $n \in \mathbb{N}$, we have $S^n \not\simeq \mathbb{R}$.
- *Proof*. Suppose $f : S^n \to \mathbb{R}$ is a homeomorphism. Then $S^n \setminus \{p\} \simeq \mathbb{R} \setminus \{f(p)\}$. We obtain a contradiction, since $S^n \setminus \{p\} \simeq \mathbb{R}^n$ is connected while $\mathbb{R} \setminus \{f(p)\}$ is not connected.

Paths and path-connectedness

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  - This means that it is not possible to find paths from points in U to points in V. Hence X is not path-connected.

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  - Some examples of convex subsets are the upper half-plane

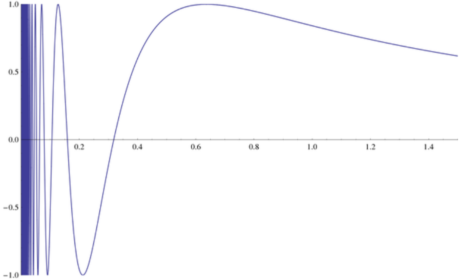
$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

    and any ball $B(x, r)$, $x \in \mathbb{R}^n$, $r > 0$.

- A connected space does not need to be path-connected. A counter-example is the so-called topologist's sine curve

$$S = \{(x, y) \in \mathbb{R}^2 \mid y = \sin(1/x), x > 0\} \cup \{(0, y) \mid -1 \leq y \leq 1\},$$

that is, the closure of the graph of $x \mapsto \sin(1/x)$ for $x > 0$. For details, see Section 24 of Munkres' textbook.



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  - If $x \sim y$ and $y \sim z$ we get connected sets A and B such that $x, y \in A$ and $y, z \in B$. Let $C = A \cup B$. Then $x, z \in C$, and C is connected by Theorem 5.4, so $x \sim z$.

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  - (iv) We will show that $C_i = \overline{C_i}$. Write $C_i = [x]$ for any $x \in C_i$. Let $y \in \overline{C_i}$. Then $\overline{C_i}$ is a subset containing both x and y, and $\overline{C_i}$ is connected by Theorem 5.5, so $y \in [x] = C_i$.

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  - Since the topology on $\mathbb{Q}$ is not the discrete one, the connected component $\{x\}$ is not open for any x.

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- Note that $[y, c + \varepsilon/2] \subset (c - \varepsilon, c + \varepsilon) \subset U$. Hence $[a, c + \varepsilon/2] = [a, y] \cup [y, c + \varepsilon/2] \subset U$. Then $c + \varepsilon/2 \in S$, which contradicts $c = \sup S$.