MIDTERM 1 REVIEW – MATH 4341

1. Proofs of Theorems

(1) (De Morgan's laws)

$$\begin{aligned} A \cup (\cap_{i\in I} B_i) &= \cap_{i\in I}(A \cup B_i), \\ A \cap (\cup_{i\in I} B_i) &= \cup_{i\in I}(A \cap B_i), \\ A \setminus (\cup_{i\in I} B_i) &= \cap_{i\in I}(A \setminus B_i), \\ A \setminus (\cap_{i\in I} B_i) &= \cup_{i\in I}(A \setminus B_i). \end{aligned}$$

(2) In a topological space X, we have
   (a) $\emptyset$ and X are closed,
   (b) If C_i is closed for all $i \in I$, then $\cap_{i\in I} C_i$ is also closed,
   (c) If $C_1, \dots, C_n$ are closed, then $C_1 \cup C_2 \cup \cdots \cup C_n$ is also closed.

(3) If $\mathcal{B}$ is a basis for a topology on X, then $\mathcal{T}_{\mathcal{B}} \subset \mathcal{P}(X)$ is a topology.

(4) If $\mathcal{B}$ be a basis for a topology on X, then $\mathcal{T}_{\mathcal{B}}$ is equal to the set of all unions of elements from $\mathcal{B}$.

(5) Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{C} \subset \mathcal{T}$ be a collection of open sets on X with the following property: for each set $U \in \mathcal{T}$ and each $x \in U$ there is a $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for $\mathcal{T}$.

(6) Let X be a set, and let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively; both on X. Then the followings are equivalent:
   (a) The topology $\mathcal{T}'$ is finer than $\mathcal{T}$.
   (b) For every $x \in X$ and each basis element $B \in \mathcal{B}$ satisfying $x \in B$, there is a basis element $B' \in \mathcal{B}'$ so that $x \in B' \subset B$.

(7) The topologies $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are both strictly finer than the standard topology on $\mathbb{R}$ but are not comparable with each other.

(8) If (X, d) is a metric space, then the collection

$$\mathcal{B} = \{B_d(x, r) \mid x \in X, r > 0\}$$

is a basis for a topology. (The topology generated by this basis is called the metric topology.)

(9) A set U is open in the metric topology if and only if for every point $x \in U$ there is an $r > 0$ so that $B_d(x, r) \subset U$.

(10) The preimage behaves nicely with respect to various operations of sets.
   (a) If $f : X \to Y$ and $\{A_i\}_{i\in I}$ is a family of subsets of Y, then

$$f^{-1}(\cup_{i\in I} A_i) = \cup_{i\in I} f^{-1}(A_i), \quad f^{-1}(\cap_{i\in I} A_i) = \cap_{i\in I} f^{-1}(A_i).$$

   (b) If $A \subset Y$, then $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$.
   (c) If $g : Y \to Z$ is another map and $B \subset Z$, then

$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B)).$$

(11) The following properties hold:
   (a) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is $g \circ f : X \to Z$.

   (b) A function $f : X \to Y$ is continuous if and only if the preimage of any closed set is closed.
   (c) A function $f : X \to Y$ is continuous if and only if it is continuous at x for all $x \in X$.

(12) Let (X, d_X) and (Y, d_Y) be metric spaces with their induced metric topologies. Then a function $f : X \to Y$ is continuous if and only if

$$\forall x \in X, \forall \epsilon > 0, \exists \delta > 0 : d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon.$$

(13) Let $(X, \mathcal{T})$ be a topological space, and let $Y \subset X$ be any subset of X. Then the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

defines a topology on Y. (This topology is called the subspace topology.)

(14) Let $(X, \mathcal{T})$ be a topological space, and let $(Y, \mathcal{T}_Y)$ be a subspace. Then
   (a) the inclusion map $\iota : Y \to X$ given by $\iota(y) = y$ is continuous,
   (b) if Z is a topological space and $f : X \to Z$ is a continuous map, then the restriction map $f|_Y : Y \to Z$ is also continuous,
   (c) a set $F \subset Y$ is closed in Y if and only if there is a set $G \subset X$ which is closed in X so that $F = Y \cap G$.

(15) (The pasting lemma) Let X be a topological space, and let $U, V \subset X$ be two open subsets such that $X = U \cup V$. Let $f : U \to Y$ and $g : V \to Y$ be two functions so that $f|_{U\cap V} = g|_{U\cap V}$. Then f and g are continuous with respect to the subspace topologies on U and V if and only if the function $h : X \to Y$ given by

$$h(x) = \begin{cases} f(x) & \text{if } x \in U, \\ g(x) & \text{if } x \in V, \end{cases}$$

is continuous.

(16) Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Then
   (a) The box topology on $\prod X_i$ has as basis all sets of the form $\prod U_i$, where U_i is open in X_i for each i.
   (b) The product topology on $\prod X_i$ has as basis all sets of the form $\prod U_i$, where U_i is open in X_i for each i and U_i equals X_i except for finitely many values of i.

(17) Let X be a topological space, and let $\{Y_i\}_{i\in I}$ be a family of topological spaces. A function $f : X \to \prod_{i\in I} Y_i$ consists of a family of functions $\{f_i\}_{i\in I}$ where $f_i : X \to Y_i$ for all $i \in I$. Then f is continuous if and only if f_i is continuous for every i.

2. Problems

(1) Examples in lecture notes.
(2) Homework 1, 2, 3, 4.