```
Math 4301 Mathematical Analysis I
Lecture 20
Topic: Power Series
```

- **Definition** Let $\{a_n\}$ be a sequence of real numbers.

  The power series centered at $c \in \mathbb{R}$ and coefficients a_n is the series

$$\sum_{n=0}^{\infty} a_n (x-c)^n .$$

Remark We see that

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 (x-c)^0 + a_1 (x-c)^1 + ...$$

thus for $x = c$,

$$\begin{aligned} \sum_{n=0}^{\infty} a_n (c-c)^n &= a_0 (c-c)^0 + a_1 (c-c)^1 + ... \\ &= a_0 (c-c)^0 + 0 + ... = a_0 \end{aligned}$$

since, in the context of power series, our convention is $0^0 = 1$.

Remark Notice that if $f_n(x) = a_n (x-c)^n$, $x \in \mathbb{R}$, then

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + \sum_{n=1}^{\infty} a_n (x-c)^n = \sum_{n=0}^{\infty} f_n(x)$$

so power series are special case of series of functions.

Example The following are examples of power series

a. $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + ...$, $|x| < 1$ – series centered at $c = 0$ and coefficients $a_n = 1$, for all $n \in \mathbb{N} \cup \{0\}$

b. $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ...$ – series centered at $c = 0$ and coefficients $a_n = \frac{1}{n!}$, for all $n \in \mathbb{N} \cup \{0\}$

c. $\sum_{n=0}^{\infty} (n!) x^n$ – series centered at $c = 0$ and coefficients $a_n = n!$, for all $n \in \mathbb{N} \cup \{0\}$

d. $f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$, $|x| < 1$ – series centered at $c = 0$ and coefficients $a_n = (-1)^n$, for all $n \in \mathbb{N} \cup \{0\}$

e. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n} (x-1)^n$ – series centered at $c = 1$ and

coefficients $a_n = \frac{(-1)^n}{n}$, for all $n \in \mathbb{N}$.

Remark We observer that every power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

converges at $x = c$.

One of natural questions is to determine the domain (the largest subset of $\mathbb{R}$) for f defined by

$$f(x) = \sum_{n=0}^{\infty} f_n(x).$$

Theorem Let $\{a_n\}$ be a real sequence, $n \in \mathbb{N} \cup \{0\}$ and

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

be the power series centered at x_0. There is R, $0 \leq R \leq +\infty$, such that,

ii) $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges absolutely for $|x - x_0| < R$,

ii) $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ diverges for $|x - x_0| > R$

iii) If $0 \leq \rho < R$ then power series converges uniformly on the interval $[c-\rho, c+\rho]$.

Proof. Assume without the loss of generality that that $x_0 = 0$.

- Otherwise we take
$$y = x - x_0.$$
- Therefore, we consider a power series
$$\sum_{n=0}^{\infty} a_n x^n.$$
- Let
$$S = \left\{ |x| : \lim_{n\to\infty} a_n x^n = 0 \right\} \subseteq \mathbb{R}.$$
- Since
$$\lim_{n\to\infty} a_n 0^n = 0,$$
it follows that $0 \in S$.
- Thus $S \neq \emptyset$ and if, in addition, S is bounded,
then by the **least upper bound property** of $\mathbb{R}$,
$\sup S$ is a real number.
In this case define
$$R = \sup S.$$

- If S is not bounded above,
$$R = +\infty.$$
- If $R = 0$, i.e.
$$\sup S = 0$$
then
$$S = \{0\}$$
so $x = 0$ and $\sum_{n=0}^{\infty} a_n x^n$ converges only for $x = 0$.
- Assume that $0 < R \leq +\infty$
- Let $x \in \mathbb{R}$ and $|x| < R$.
- If $x = 0$, then
$$\sum_{n=0}^{\infty} a_n x^n = a_0 0^0 + a_1 0^1 + ... = a_0$$
Here, by the convention, $0^0 = 1$.
- Therefore, we assume that $x \neq 0$.
- If $R < \infty$, then
$$R = \sup S.$$
- By the definition, for
$$\epsilon = \frac{1}{2}(R - |x|) > 0,$$
there is $\rho \in S$, such that
$$R - \epsilon < \rho.$$
- Since $0 < |x| < R$,
$$\begin{aligned} R - \epsilon &= R - \frac{1}{2}(R - |x|) \\ &= \frac{1}{2}(R + |x|) > \frac{1}{2}(|x| + |x|) \\ &= |x| > 0 \end{aligned}$$
it follows that
$$0 < |x| < R - \epsilon < \rho.$$
- Therefore,
$$0 < |x| < \rho.$$
- Since
$$\rho \in S,$$
$$\lim_{n\to\infty} a_n \rho^n = 0,$$
thus by the theorem (convergent sequences are bounded),
sequence $\{a_n \rho^n\}$ is bounded.

- Therefore, there is $M > 0$, such that, for all $n \in \mathbb{N}$,

$$|a_n \rho^n| \leq M.$$

- We see that

$$|a_n x^n| = \left| a_n \rho^n \left(\frac{x}{\rho} \right)^n \right| = |a_n \rho^n| \left(\frac{|x|}{\rho} \right)^n \leq M \left(\frac{|x|}{\rho} \right)^n .$$

- Let $q_x = \frac{|x|}{\rho}$.
- Since $0 < |x| < \rho$,

$$0 < \frac{|x|}{\rho} < 1,$$

  that is

$$q_x \in (0, 1) .$$

- Therefore, the number series $\sum_{n=0}^{\infty} q_x^n$ converges so the series

$$\sum_{n=0}^{\infty} M \left(\frac{|x|}{\rho} \right)^n = \sum_{n=0}^{\infty} M q_x^n = M \sum_{n=0}^{\infty} q_x^n = \frac{M}{1 - q_x}$$

  also converges.
- Since

$$0 \leq |a_n x^n| \leq M q_x^n,$$

  series

$$\sum_{n=0}^{\infty} |a_n x^n|$$

  converges.
- Hence

$$\sum_{n=0}^{\infty} a_n x^n$$

  converges absolutely.
- If $R = \infty$, then S is not bounded above.
- Let $x \in \mathbb{R}$ (and $x \neq 0$).
- Since S is not bounded above, by the definition, there is $\rho \in S$, such that

$$0 < |x| < \rho.$$

- As before, since $\rho \in S$, the sequence $\{a_n \rho^n\}$ is bounded.
- Therefore, there is $M > 0$, such that

$$|a_n \rho^n| < M, \text{ for all } n = 0, 1, ...$$

- Since
$$0 < \frac{|x|}{\rho} < 1$$
the series $\sum_{n=0}^{\infty} M \left(\frac{|x|}{\rho}\right)^n$ converges.
- Since for all $n = 0, 1, 2, ...$
$$0 \le |a_n x^n| \le M \left(\frac{|x|}{\rho}\right)^n,$$
by the comparison test the series
$$\sum_{n=0}^{\infty} |a_n x^n|$$
converges, so the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent.
- If $|x| > R = \sup S$ then
$$|x| \notin S = \{|x| : a_n x^n \to 0 \text{ as } n \to \infty\}$$
otherwise R is not an upper bound of S.
- Since $|x| \notin S$,
$$a_n x^n \nrightarrow 0,$$
thus the series
$$\sum_{n=0}^{\infty} a_n x^n$$
diverges.

iii) Let $R > 0$, $0 \le \rho < R$ and $x \in [-\rho, \rho]$.

- Since $\rho < R$, there is $\delta \in \mathbb{R}$, such that $\rho < \delta < R$.
- Since $0 < \delta < R$, the number series
$$\sum_{n=0}^{\infty} a_n \delta^n$$
converges absolutely,

  i.e., the series $\sum_{n=0}^{\infty} |a_n \delta^n|$ is convergent.
- In particular, the sequence $\{|a_n \delta^n|\}$ is bounded,
so there is $M \ge 0$, such that,
$$|a_n \delta^n| \le M,$$
for all $n = 0, 1, ...$

- Therefore, for all $x \in [-\rho, \rho]$,

$$\begin{aligned} |a_n x^n| &= |a_n| |x|^n \leq |a_n| \rho^n \\ &= |a_n \delta^n| \left(\frac{\rho}{\delta}\right)^n \leq M r^n, \end{aligned}$$

where $r = \frac{\rho}{\delta} < 1$.

- Since the series $\sum_{n=0}^{\infty} M r^n$ converges, by the Weierstrass M-test,

the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-\rho, \rho]$.

This finishes our proof. ■

- **Definition** Let a_n, $n \in \mathbb{N} \cup \{0\}$ be sequence and $c \in \mathbb{R}$. The number

$$R = \sup \left\{ |x| : \lim_{n \to \infty} a_n x^n = 0 \right\}$$

is called *the radius of convergence* of $\sum_{n=0}^{\infty} a_n (x - c)^n$.

Exercise Let $R > 0$ be the radious of convergence of $\sum_{n=0}^{\infty} a_n (x - c)^n$. Show that, for all $[a, b] \subseteq (c - R, c + R)$ the $\sum_{n=0}^{\infty} a_n (x - c)^n$ converges uniformly and absolutely on $[a, b]$.

Exercise Let $R > 0$ be the radious of convergence of $\sum_{n=0}^{\infty} a_n (x - c)^n$. Define

$$\begin{aligned} f &: (c - R, c + R) \to \mathbb{R}, \text{ by} \\ f(x) &= \sum_{n=0}^{\infty} a_n (x - c)^n. \end{aligned}$$

Show that f is continuous on $(c - R, c + R)$.

Remark From theorem above we see that for $\sum_{n=0}^{\infty} a_n (x - c)^n$, the set

$$S = \left\{ x \in \mathbb{R} : \lim_{n \to \infty} a_n x^n = 0 \right\} \pm c$$

is an interval with endpoints $c \pm R$ and for each $x \in S$, the power series converges. Howerer, using the theorem, one cannot determine if

$$c \pm R \in S.$$

Therefore, for a power series centered at c and radius of convergence $R > 0$, the set S is one of the intervals

$$(c - R, c + R), \ [c - R, c + R), \ (c - R, c + R], \ [c - R, c + R].$$

- The interval I in the form above that equals S is called *the interval of convergence* for the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$.

  Remark If I is the interval of convergence for $\sum_{n=0}^{\infty} a_n (x-c)^n$, then I is the domain of

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n .$$

- The following theorem allows us to find radious of convergence effectively.

  Theorem Let a_n, $n \in \mathbb{N} \cup \{0\}$, $c \in \mathbb{R}$ and assume that there is $N \in \mathbb{N}$, such that, for all $n \geq N$, $a_n \neq 0$.
  If $\left|\frac{a_{n+1}}{a_n}\right| \to L$ as $n \to \infty$ or diverges to ∞ then

$$R = \begin{cases} \frac{1}{L} & if \quad 0 < L < \infty \\ \infty & if \quad L = 0 \\ 0 & if \quad L = \infty \end{cases}$$

  is the radious of convergence for $\sum_{n=0}^{\infty} a_n (x-c)^n$.

Proof. Assume that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \geq 0$$

- If $0 < L < \infty$ and $|x-c| < \frac{1}{L}$, then

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}(x-c)^{n+1}}{a_n (x-c)^n}\right| &= |x-c| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ &= |x-c| L < 1. \end{aligned}$$

- Therefore, by the ratio test the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

  converges at x.

- Hence, we showed that

$$\frac{1}{L} \leq R.$$

- If $|x-c| > \frac{1}{L}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}(x-c)^{n+1}}{a_n (x-c)^n}\right| = |x-c| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-c| L > 1$$

  so the power series diverges at x.

- It follows that
$$R = \frac{1}{L}.$$

- If $L = 0$, then for all $x \in \mathbb{R}$,
$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1} (x-c)^{n+1}}{a_n (x-c)^n} \right| &= |x-c| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= |x-c| \cdot 0 = 0 < 1 \end{aligned}$$

  so the power series converges at x.

- Hence,
$$R = \infty.$$

- Finally, assume that
$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty,$$

  then
$$\lim_{n\to\infty} \left| \frac{a_{n+1} (x-c)^{n+1}}{a_n (x-c)^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x-c| < 1$$

  iff $|x-c| = 0$, so $x = c$.

- Therefore, the power series
$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

  converges if and only if
$$x = c.$$

- It follows that $R = 0$.

This finishes our proof. ■

- **Theorem** (*Hadamard*) Let a_n, $n \in \mathbb{N} \cup \{0\}$, $c \in \mathbb{R}$, and
$$L = \limsup |a_n|^{\frac{1}{n}}.$$

  Then
$$R = \begin{cases} \frac{1}{L} & if \quad 0 < L < \infty \\ \infty & if \qquad L = 0 \\ 0 & if \qquad L = \infty \end{cases}$$

  is the radious of convergence of $\displaystyle\sum_{n=0}^{\infty} a_n (x-c)^n$.

Proof. Assume that
$$\limsup \sqrt[n]{|a_n|} = L \geq 0$$

- If $0 < L < \infty$ and $|x-c| < \frac{1}{L}$, then

$$\begin{aligned}\limsup \sqrt[n]{|a_n (x-c)^n|} &= |x-c| \limsup \sqrt[n]{|a_n|} \\ &= |x-c| L < 1\end{aligned}$$

  hence, by the root test the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

  converges at x.

- Hence, we showed that

$$\frac{1}{L} \leq R.$$

- If $|x-c| > \frac{1}{L}$, then

$$\begin{aligned}\limsup \sqrt[n]{|a_n (x-c)^n|} &= |x-c| \limsup \sqrt[n]{|a_n|} \\ &= |x-c| L > 1\end{aligned}$$

  so the power series diverges at x.

- It follows that

$$R = \frac{1}{L}.$$

- If $L = 0$, then for all $x \in \mathbb{R}$,

$$\begin{aligned}\limsup \sqrt[n]{|a_n (x-c)^n|} &= |x-c| \limsup \sqrt[n]{|a_n|} \\ &= |x-c| \cdot 0 = 0 < 1\end{aligned}$$

  so the power series converges at x.

- Therefore,

$$R = \infty.$$

- Finally, assume that

$$\limsup \sqrt[n]{|a_n|} = \infty,$$

  then

$$\limsup \sqrt[n]{|a_n (x-c)^n|} = \limsup \sqrt[n]{|a_n|} \, |x-c| < 1$$

  iff $|x-c| = 0$, so $x = c$.

- Therefore, the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

  converges if and only if

$$x = c.$$

- It follows that $R = 0$.

This finishes our proof. ■

- **Example** Consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

  We find its radius and the interval of convergence.

- Since
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+1} = 1,$$
  then the radius of convergence $R = \frac{1}{1} = 1$ for this series.

- If $x = 0 + R = 1$, then
$$\sum_{n=1}^{\infty}\frac{1}{n} = \infty$$

- if $x = 0 - R = -1$, then
$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$
  converges.

- Therefore, the series converges for each
$$x \in [0 - R, 0 + R) = [-1, 1)\,,$$
  so $R = 1$ is the radius of convergence for the series
  and $I = [-1, 1)$ is its interval of convergence.

- We see that
$$f(x) = \sum_{n=1}^{\infty}\frac{x^n}{n}$$
  is well defined for $x \in [-1, 1)$.

  Example Consider the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

  We find its radius and the interval of convergence.

- We see that
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty}\frac{n!}{(n+1)!} = 0$$

- Therefore, $R = \infty$ is the radius of convergence for the series and consequently, its interval of convergence is
$$I = (-\infty, \infty)\,.$$

- Therefore, the domain of the function given by
$$f(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$
  is the set $\mathbb{R}$ of all real numbers.

Example Consider power series $\sum_{n=0}^{\infty}(n!)\,x^n$.

We find its radius and the interval of convergence.

- We see that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1) = \infty$$

- Therefore, $R = 0$ is the radius of convergence for the series and consequently, $\sum_{n=0}^{\infty}(n!)\,x^n$ converges only for $x = 0$.

- Thus, the domain D of the function given by

$$f(x) = \sum_{n=0}^{\infty}(n!)\,x^n$$

is $D = \{0\}$.

Example Consider the power series $\sum_{n=0}^{\infty}(-1)^n x^{2^n}$.

We find its radius and the interval of convergence.

- We see that

$$a_n = \begin{cases} (-1)^k & if \quad n = 2^k \\ 0 & if \quad n \neq 2^k \end{cases}$$

- Moreover,

$$|a_n x^n| = \begin{cases} |x|^n & if \quad n = 2^k \\ 0 & if \quad n \neq 2^k \end{cases}$$

- Therefore, if $|x| < 1$, by the comparison test the series $\sum_{n=0}^{\infty}(-1)^n x^{2^n}$ converges at x.

- Since

$$\limsup \sqrt[n]{|a_n|} = 1$$

it follows that $R = 1$.

- Moreover, since

$$\sum_{n=0}^{\infty}(-1)^n (1)^{2^n} \text{ diverges and } \sum_{n=0}^{\infty}(-1)^n (-1)^{2^n} \text{ diverges,}$$

it follows that

$$I = (-1, 1)$$

is its interval of convergence.

Proposition Let

$$\begin{aligned} f(x) &= \sum_{n\geq 0} a_n x^n, \ |x| < R_1, \\ g(x) &= \sum_{n\geq 0} b_n x^n, \ |x| < R_2. \end{aligned}$$

and $R = \min\{R_1, R_2\}$. Then for $|x| < R$,

i) $(f+g)(x) = \sum_{n\geq 0} (a_n + b_n) x^n$ and

ii) $(fg)(x) = \sum_{n\geq 0} c_n x^n$, where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$.

Proof. Exercise ■

- **Remark** We note that radious of convergence for $f+g$ and fg might larger than

$$R = \min\{R_1, R_2\}.$$

Proposition Let

$$f(x) = \sum_{n\geq 0} a_n x^n, \ \ |x| < R, \ \ R > 0.$$

If $a_0 \neq 0$, then there is $R' > 0$, such that

$$\frac{1}{f(x)} = \sum_{n\geq 0} b_n x^n, \ \ |x| < R',$$

where $b_0 = \frac{1}{a_0}$ and for $n \geq 1$,

$$b_n = -\frac{1}{a_0} \sum_{j=0}^{n-1} a_{n-j} b_j.$$

Proof. Since

$$f(x) \frac{1}{f(x)} = 1$$

and

$$f(x) \frac{1}{f(x)} = \sum_{n\geq 0} c_n x^n, \ \text{where } c_n = \sum_{j=0}^{n} a_{n-j} b_j,$$

it follows that

$$c_0 = 1 \text{ and } c_n = 0, \ \text{for all } n \geq 1.$$

- Therefore,

$$a_0 b_0 = 1,$$

and for $n \geq 1$

$$c_n = \sum_{j=0}^{n} a_{n-j} b_j = \sum_{j=0}^{n-1} a_{n-j} b_j + a_0 b_n = 0.$$

- Hence,

$$b_0 = \frac{1}{a_0}, \ b_n = -\frac{1}{a_0} \sum_{j=0}^{n-1} a_{n-j} b_j, \ n \geq 1.$$

- We show that there is $R' > 0$, such that

$$\frac{1}{f(x)} = \sum_{n\geq 0} b_n x^n, \ \ |x| < R'.$$

- We may assume WLOG that $a_0 = 1$ otherwise,
  we take $\frac{1}{a_0} f$.

- Since

$$g\left(x\right)=\sum_{n\geq 1}\left|a_n\right|\left|x\right|^n$$

  is continous at $x_0=0$ and $g\left(0\right)=0$,
  there is $\delta>0$, such that, for all $|x|<\delta$,

$$\left|g\left(x\right)\right|<1.$$

- Since

$$\begin{aligned}\left|f\left(x\right)\right| &= |1+\sum_{n\geq 1}a_nx^n|\geq 1-|\sum_{n\geq 1}a_nx^n|=1-|\lim_{n\to\infty}\sum_{k=1}^{n}a_kx^k| \\ &= 1-\lim_{n\to\infty}|\sum_{k=1}^{n}a_kx^k|\geq 1-\lim_{n\to\infty}\sum_{k=1}^{n}|a_k|\left|x\right|^k \\ &= 1-\sum_{n\geq 1}\left|a_n\right|\left|x\right|^n>0.\end{aligned}$$

- Therefore, $\frac{1}{f(x)}$ is defined for $|x|<\delta$.
- We show that $|b_n|\leq\frac{1}{\delta^n}$.
  Indeed, $|b_0|=1\leq\frac{1}{\delta^0}$, and assume that

$$|b_{n-1}|\leq\frac{1}{\delta^{n-1}}.$$

  Thus, for $n\geq 1$

$$\begin{aligned}|b_n| &= |-\frac{1}{a_0}\sum_{j=0}^{n-1}a_{n-j}b_j|\leq\sum_{j=0}^{n-1}|a_{n-j}||b_j|\leq\sum_{j=0}^{n-1}\frac{|a_{n-j}|}{\delta^j} \\ &= \frac{1}{\delta^n}\sum_{j=0}^{n-1}\frac{|a_{n-j}|}{\delta^{j-n}}=\frac{1}{\delta^n}\sum_{j=0}^{n-1}|a_{n-j}|\delta^{n-j} \\ &= \frac{1}{\delta^n}\sum_{j=1}^{n}|a_j|\delta^j\leq\frac{1}{\delta^n}\sum_{n=1}^{\infty}|a_n|\delta^n<\frac{1}{\delta^n}\end{aligned}$$

- It follows from Hadamard theorem that

$$\frac{1}{R'}=\limsup|b_n|^{\frac{1}{n}}\leq\frac{1}{\delta}$$

  hence

$$R'\geq\delta>0.$$

This completes our argument. ■

- **Corollary** Let

$$f\left(x\right)=\sum_{n\geq 0}a_nx^n,\ \ |x|<R_1,\ \ R_1>0$$

  and

$$g\left(x\right)=\sum_{n\geq 0}b_nx^n,\ \ |x|<R_2,\ \ R_2>0.$$

  If $b_0\neq 0$, then there are c_n and $R>0$, such that

$$\frac{f\left(x\right)}{g\left(x\right)}=\sum_{n\geq 0}c_nx^n,\ \ |x|<R.$$

Proof. Exercise ■

- **Example** Let
$$f(x)=\sum_{n\geq 0}\frac{x^n}{(n+1)!},\ x\in\mathbb{R}$$
Since $f(0)=1$, by theorem
$$\frac{1}{f(x)}=\sum_{n\geq 0}b_nx^n,$$
where $b_0=1$ and for $n\geq 1$,
$$b_n=-\frac{1}{a_0}\sum_{j=0}^{n-1}a_{n-j}b_j=-\sum_{j=0}^{n-1}\frac{1}{(n-j+1)!}b_j$$
is defined.
- Since
$$f(x)=\sum_{n\geq 0}\frac{x^n}{(n+1)!}=\frac{e^x-1}{x},\ x\neq 0$$
one shows that
$$\frac{1}{f(x)}=\frac{x}{e^x-1}=1-\frac{1}{2}x+\sum_{n\geq 0}\frac{B_{2n}}{(2n)!}x^{2n}.$$
- Since
$$\sum_{j=0}^{n-1}\frac{1}{(n-j+1)!}b_j+b_n=\sum_{j=0}^{n}\frac{1}{(n-j+1)!}b_j=0$$
and $B_n=n!b_n$,
$$\begin{aligned}\sum_{j=0}^{n}\frac{1}{(n-j+1)!}b_j &= \sum_{j=0}^{n}\frac{1}{(n-j+1)!}\frac{B_j}{j!}=\frac{1}{(n+1)!}\sum_{j=0}^{n}\frac{(n+1)!}{(n-j+1)!j!}B_j\\ &= \frac{1}{(n+1)!}\sum_{j=0}^{n}\binom{n+1}{j}B_j=0\end{aligned}$$
so
$$\sum_{j=0}^{n}\binom{n+1}{j}B_j=0$$
and thus for $n\geq 1$,
$$B_n=-\frac{1}{n+1}\sum_{j=0}^{n-1}\binom{n+1}{j}B_j.$$
- It follows that coefficients of
$$\frac{1}{f(x)}=1-\frac{1}{2}x+\sum_{n\geq 0}\frac{B_{2n}}{(2n)!}x^{2n}$$
can be determined recursively.

- Moreover, one proves that

$$\frac{1}{R} = \limsup \left| \frac{B_n}{n!} \right|^{\frac{1}{n}} = \frac{1}{2\pi}$$

so

$$R = 2\pi$$

is the radious of convergence.

Theorem Let $R > 0$ be the radious of convergence of

$$f(x) = \sum_{n \geq 0} a_n x^n, \ |x| < R.$$

Then f is differentiable and

$$f'(x) = \sum_{n \geq 1} n a_n x^{n-1}, \ |x| < R.$$

Proof. We first show that the series

$$f'(x) = \sum_{n \geq 1} n a_n x^{n-1}$$

- has radious of convergence R.
- Indeed, let $x \in (-R, R)$.
- There is $\delta > 0$, such that

$$|x| < \delta < R.$$

- Thus,

$$r = \frac{|x|}{\delta} < 1$$

and

$$|n a_n x^{n-1}| = n |a_n| |x|^{n-1} = \frac{n}{\delta} \left(\frac{|x|}{\delta} \right)^{n-1} |a_n| \delta^n = \frac{n r^{n-1}}{\delta} |a_n| \delta^n$$

- Since

$$\lim_{n \to \infty} \frac{(n+1) r^n}{n r^{n-1}} = r < 1,$$

it follows that

$$\lim_{n \to \infty} n r^{n-1} = 0.$$

- Therefore, the sequence $\left\{ n r^{n-1} \right\}$ is bounded, so there is $M \geq 0$, such that,

$$0 \leq n r^{n-1} \leq M$$

as convergent sequences are bounded.

- Hence

$$|n a_n x^{n-1}| \leq \frac{M}{\delta} |a_n| \delta^n.$$

- Since $|x| < \delta < R$,

$$\sum_{n \geq 1} |a_n| \delta^n$$

converges.

- Therefore, $\sum_{n\geq 1} na_n x^{n-1}$ converges absolutely at x.
- Consequently,

$$f'(x) = \sum_{n\geq 1} na_n x^{n-1}$$

  is absolutely convergent for each $x \in \mathbb{R}$, such that $|x| < R$.
- Assume that $|x| > R$, then

$$|na_n x^{n-1}| = \frac{n}{|x|} |a_n| |x|^n \geq \frac{1}{|x|} |a_n| |x|^n .$$

  Since the series $\sum_{n\geq 1} |a_n| |x|^n$ diverges,

  it follows that the series $\sum_{n\geq 1} na_n x^{n-1}$ also diverges.
- Consequently, $R > 0$ is the radious of convergence for

$$f'(x) = \sum_{n\geq 1} na_n x^{n-1}.$$

- Showing that f is differentiable for all $x \in (-R, R)$ and

$$f'(x) = \sum_{n\geq 1} na_n x^{n-1}$$

  is left as an exercise.

This finishes our proof. ■