Math 4301 Mathematical Analysis I
Lecture 6
Topic: Topology of Real Numbers

- **Topology of Real Numbers**
- Recall, the properties of an absolute value:

$$|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

i) $\forall x, y \in \mathbb{R}$, $|y - x| \geq 0$ and
$\forall x, y \in \mathbb{R}$, $((|y - x| = 0) \Longleftrightarrow (x = y))$

ii) $\forall x, y \in \mathbb{R}$, $|y - x| = |x - y|$

iii) $\forall x, y, z \in \mathbb{R}$, $|y - x| \leq |y - z| + |z - y|$

- Define a function

$$\begin{aligned} d &: \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \text{ by} \\ d(x, y) &= |x - y|. \end{aligned}$$

- Since the absolute value satisfies $i) - iii)$,
function d satisfies the following properties

1. $d(x, y) \geq 0$, for all $x, y \in \mathbb{R}$ and $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$, for all $x, y \in \mathbb{R}$.
3. $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in \mathbb{R}$

- **Remark**: A function

$$d : X \times X \to \mathbb{R}$$

that satisfies $1) - 3)$ is called a **distance or a metric** on X.
Definition Let $x_0 \in \mathbb{R}$ and $\epsilon > 0$.
An open disk centered at x_0 (or an $\epsilon-$**neighborhood of** x_0) is the set

$$D(x_0, \epsilon) = \{x \in \mathbb{R} \mid d(x,\ x_0) < \epsilon\}$$

where $d(x,\ y) = |x - y|$, $x, y \in \mathbb{R}$.
The number $\epsilon > 0$ is referred to as the **radius** of $D(x_0, \epsilon)$.

- Observe that an $\epsilon-$**disk centered at** x_0 is simply an open interval with endpoints $x_0 - \epsilon$, $x_0 + \epsilon$.
- Indeed, we see that

$$\begin{aligned} D(x_0,\ \epsilon) &= \{x \in \mathbb{R} \mid d(x,\ x_0) < \epsilon\} \\ &= \{x \in \mathbb{R} \mid\ |x - x_0| < \epsilon\} \\ &= \{x \in \mathbb{R} \mid\ -\epsilon < x - x_0 < \epsilon\} \\ &= \{x \in \mathbb{R} \mid x_0 - \epsilon < x < x_0 + \epsilon\} \\ &= (x_0 - \epsilon,\ x_0 + \epsilon) \end{aligned}$$

an open interval with the endpoints $x_0 - \epsilon$ and $x_0 + \epsilon$, whose center is at x_0.

- Note that endpoints $x_0 - \epsilon$ and $x_0 + \epsilon$ are of the distance ϵ from the center x_0 of the interval.
- The notion of an $\epsilon-$**disk centered at** x_0 is used in the definition of an open set in $\mathbb{R}$ as follows.
  Definition Let $U \subseteq \mathbb{R}$.
  We say that U is open in $\mathbb{R}$, if
$$\forall x_0 \in U,\ \exists \epsilon > 0 \ni D\left(x_0,\ \epsilon\right) \subseteq U.$$
  Example Show that an open interval
$$(a,\ b) \subseteq \mathbb{R},$$
  where $a < b$ is open.
  Solution: We need to show that
$$\forall x_0 \in (a,\ b)\,,\ \exists \epsilon > 0 \ni D\left(x_0,\ \epsilon\right) \subseteq (a,\ b)\,.$$
- Let $x_0 \in (a,\ b)$, and
  we need to find $\epsilon > 0$, such that
$$D\left(x_0,\ \epsilon\right) \subseteq (a,\ b)\,.$$
- Since $x_0 \in (a,\ b)$,
$$a < x_0 < b.$$
- Therefore, both numbers:
$$x_0 - a > 0 \text{ and } b - x_0 > 0.$$
- If we take
$$\epsilon = \min\left\{x_0 - a,\ b - x_0\right\},$$
  we see that $\epsilon > 0$ (since $x_0 - a > 0$ and $b - x_0 > 0$, so
  the minimum of two positive numbers is also a positive number).
- Moreover,
$$\left(x_0 - \epsilon,\ x_0 + \epsilon\right) \subseteq (a,\ b)\,.$$
  Indeed, since for all $x \in \left(x_0 - \epsilon,\ x_0 + \epsilon\right)$,
$$x_0 - \epsilon < x < x_0 + \epsilon.$$
- Since
$$x_0 - a \geq \min\left\{x_0 - a,\ b - x_0\right\},$$
  it follows that
$$\begin{aligned} a &= x_0 - \left(x_0 - a\right) \\ &\leq x_0 - \min\left\{x_0 - a,\ b - x_0\right\} \\ &= x_0 - \epsilon < x \end{aligned}$$
  and $b - x_0 \geq \min\left\{x_0 - a,\ b - x_0\right\}$
$$\begin{aligned} x &< x_0 + \epsilon \\ &= x_0 + \min\left\{x_0 - a,\ b - x_0\right\} \\ &\leq x_0 + b - x_0 \\ &= b. \end{aligned}$$

- It follows that, for all $x \in (x_0 - \epsilon,\ x_0 + \epsilon)$,

$$a < x < b,$$

- Thus,

$$D(x_0,\ \epsilon) = (x_0 - \epsilon,\ x_0 + \epsilon) \subseteq (a,\ b).$$

- By the definition of an open set, an open interval $(a,\ b)$ is open in $\mathbb{R}$.

  Example $S = [1, 2)$ is not open.

- Take $x_0 = 1$ and let $\epsilon > 0$ be given.

- Consider

$$D(1, \epsilon) = (1 - \epsilon, 1 + \epsilon).$$

- Look at $[1, 2)$ and

$$(1 - \epsilon, 1 + \epsilon) = D(1, \epsilon)$$

- We see that $\left(1 - \frac{\epsilon}{2}\right) < 1$, so

$$\left(1 - \frac{\epsilon}{2}\right) \notin S = [1, 2)$$

  however

$$1 - \epsilon < \left(1 - \frac{\epsilon}{2}\right) < 1 + \epsilon,$$

  so

$$\left(1 - \frac{\epsilon}{2}\right) \in D(1, \epsilon) = (1 - \epsilon, 1 + \epsilon).$$

- We showed that,

  for every $\epsilon > 0$,

$$D(1, \epsilon) \nsubseteq S.$$

- Therefore, S is not open.

  Proposition Let $x \in \mathbb{R}$ and $\epsilon > 0$.

  The open disk $D(x, \epsilon) = (x - \epsilon, x + \epsilon)$ is open.

Proof. Let $\epsilon > 0$ be given, and consider $D(x, \epsilon) \subseteq \mathbb{R}$.

- We need to show that

$$\forall y \in D(x, \epsilon),\ \exists \eta > 0 \ni D(y, \eta) \subseteq D(x, \epsilon).$$

- Let $y \in D(x, \epsilon)$, thus,

  by the definition of $D(x, \epsilon)$,

$$|x - y| < \epsilon,$$

  hence

$$\eta = \epsilon - |x - y| > 0.$$

- We show that

$$D(y, \eta) \subseteq D(x, \epsilon).$$

- Let $z \in D(y, \eta)$, thus

$$|y - z| < \eta.$$

- By the triangle inequality,

$$\begin{aligned} |x-z| &= |(x-y)+(y-z)| \\ &\leq |x-y|+|y-z| \\ &< |x-y|+\eta \\ &= |x-y|+\underbrace{\epsilon-|x-y|}_{\eta} \\ &= \epsilon. \end{aligned}$$

- We showed that,
  for every $z \in \mathbb{R}$,
  if $z \in D(y,\eta)$, then $z \in D(x,\epsilon)$.

- Therefore,

$$D(y,\eta) \subseteq D(x,\epsilon),$$

  and hence $D(x,\epsilon)$ is open.

This finishes our proof. ■

- Consider the following examples.

- $A=(1,2)$ and $B=\left(\frac{3}{2},3\right)$, then

$$A \cup B = (1,3)$$

  so we see that union of open sets is open.

- $\mathcal{A}=\{(x-1,x+1) : x \in \mathbb{R}\}$ – uncountable family of open intervals.

- Is $\bigcup \mathcal{A}$ an open set? – it is open by the result below.
  Proposition Arbitrary union of open sets is open and
  finite intersection of open sets is open.

Proof. Let $\mathcal{B}$ be a family of open sets and

- let

$$x \in \bigcup \mathcal{B} = \bigcup_{B \in \mathcal{B}} B.$$

- Therefore, there is $B \in \mathcal{B}$, such that $x \in B$.

- Since B is open, there is $\epsilon > 0$, such that

$$D(x,\epsilon) \subseteq B.$$

- Since $B \subseteq \bigcup \mathcal{B} = \bigcup_{B \in \mathcal{B}} B$,

$$D(x,\epsilon) \subseteq \bigcup_{B \in \mathcal{B}} B.$$

- Therefore, $\bigcup_{B \in \mathcal{B}} B$ is open.

- Suppose that
$$\mathcal{B} = \{B_1, B_2, ..., B_n\}$$
is a finite family of open sets and let $x \in \bigcap\limits_{B \in \mathcal{B}} B$.
- Therefore, $x \in B_i$, $i = 1, 2, ..., n$.
- Since each B_i is open,
there is $\epsilon_i > 0$, such that $D(x, \epsilon_i) \subseteq B_i$, $i = 1, 2, ..., n$.
- Define
$$\epsilon = \min \{\epsilon_i : i = 1, 2, ..., n\} > 0.$$
- Since
$$D(x, \epsilon) \subseteq D(x, \epsilon_i)$$
and
$$D(x, \epsilon_i) \subseteq B_i,$$
for all $i = 1, 2, ..., n$.
- It follows that $D(x, \epsilon) \subseteq B_i$, for all $i = 1, 2, ..., n$.
- Consequently,
$$D(x, \epsilon) \subseteq \bigcap_{B \in \mathcal{B}} B.$$
Therefore, $\bigcap\limits_{B \in \mathcal{B}} B$ is open.

This finishes our proof. ■

- **Remark:** It is worth to mention that if
$$\mathcal{B} = \{B_1, B_2, ...\}$$
is a infinite family of open sets,
then $\bigcap\limits_{B \in \mathcal{B}} B$ might no be open.
- For example, let
$$\mathcal{B} = \{B_n : n \in \mathbb{N}\},$$
where $B_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$.
- Then each
$$B_n = D\left(0, \frac{1}{n}\right)$$
is clearly open,

- but the intersection

$$\begin{aligned} \bigcap_{B\in\mathcal{B}} B &= \bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right) \\ &= \{0\} \end{aligned}$$

  is not open since

$$(-\epsilon,\epsilon) = D(0,\epsilon) \nsubseteq \{0\},$$

  for any $\epsilon > 0$.

- This is because, for instance,

$$D(0,\epsilon) = (-\epsilon,\epsilon)$$

  is infinite and $\{0\}$ is finite.

  Exercise Show that any open subset of $\mathbb{R}$ is a countable union of disjoint intervals.

  Definition Let $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$.

  We say that A is a *neighborhood* of x

  if $x \in A$ and there is $\delta > 0$, such that

$$D(x,\delta) \subseteq A.$$

  Remark: We observe that a neighborhood A of $x \in A$ does not need to be open.

- For instance, if $x \in (a,b)$,

  then $A = [a,b]$ is a neighborhood of x.

- Indeed,

$$x \in (a,b) \subset [a,b],$$

  so $x \in [a,b]$.

- Let

$$\delta = \min\{|x-a|,|x-b|\} > 0.$$

- Then

$$\begin{aligned} D(x,\delta) &\subseteq (a,b) \\ &\subset A \\ &= [a,b], \end{aligned}$$

  hence A is a neighborhood of x.

- Using the notion of a neighborhood,

  we can also define an open set as follows.

  Definition $U \subseteq \mathbb{R}$ is *open* if

  for every $x \in U$ there is a *neighborhood* A of x such that

$$A \subseteq U.$$

- Using the notion of a neighborhood,

  we can define convergence of sequences.

  Proposition Let $\{x_n\}$ be a sequence of real numbers.

  Then

  $$x_n \to x \text{ as } n \to \infty$$

  iff for every neighborhood A of x

  there is $N \in \mathbb{N}$, such that

  $$x_n \in A,$$

  for $n > N$.

- Recall, $x_n \to x$ as $n \to \infty$ if

  for every $\epsilon > 0$, there is $N \in \mathbb{N}$, such that,

  for every $n > N$,

  $$|x_n - x| < \epsilon,$$

  i.e. $x_n \in D(x, \epsilon)$, for $n > N$.

Proof. Assume that $x_n \to x$ and $n \to \infty$ and

- let $A \subseteq \mathbb{R}$ be a neighborhood of x.
- There is $\delta > 0$, such that

  $$D(x, \delta) \subseteq A.$$

- Since

  $$x_n \to x,$$

  there is $N \in \mathbb{N}$, such that, for $n > N$,

  $$|x_n - x| < \delta.$$

- But $|x_n - x| < \delta$ iff

  $$x - \delta < x_n < x + \delta.$$

- Therefore, $|x_n - x| < \delta$ iff

  $$x_n \in D(x, \delta).$$

- It follows that, $x_n \in D(x, \delta)$, for all $n > N$.
- Since

  $$D(x, \delta) \subseteq A,$$

  we see that, for all $n > N$, $x_n \in A$.

  Conversely

- Let $\delta > 0$ be given.
- Since for any neighborhood A of x

  there is $N \in \mathbb{N}$, such that

  $$x_n \in A,$$

  for all $n > N$.

- If we take
$$A = D(x, \delta),$$
then $x \in A$ and A is a neighborhood of x,
- Therefore, there is $N \in \mathbb{N}$, such that
$$x_n \in A,$$
for all $n > N$.
- Since $x_n \in D(x, \delta)$, $n > N$ iff
$$|x_n - x| < \delta,$$
for all $n > N$.
- This shows that $x_n \to x$ and $n \to \infty$.

This finishes our proof. ■

- **Relatively open sets**
  Definition Let $A \subseteq \mathbb{R}$ and $B \subseteq A$.
  We say that B is relatively open in A (or open in A) if
  there is an open subset $U \subseteq \mathbb{R}$, such that
$$B = A \cap U.$$
  Example: Let $A = [1, 2]$ and
$$B = \left[1, \frac{3}{2}\right) \subset A = [1, 2].$$
- Then B is open in A, since
$$\begin{aligned} \underbrace{\left[1, \frac{3}{2}\right)}_{B} &= B = A \cap \left(0, \frac{3}{2}\right) \\ &= \underbrace{[1, 2]}_{A} \cap \underbrace{\left(0, \frac{3}{2}\right)}_{U} \end{aligned}$$
and the set $U = \left(0, \frac{3}{2}\right)$ is open.
- Therefore, B is open in A.
  Remark: If U is open in $\mathbb{R}$,
  then $U \subseteq A$ is open in $A \subseteq \mathbb{R}$.
- The converse is not always true, i.e. if B is open in A
  then B might be no longer open in $\mathbb{R}$.
  Definition Let $A \subseteq \mathbb{R}$ and $x \in A$.
  A *relative neighborhood* in A of x is a set
$$V = A \cap U,$$

where $U \subseteq \mathbb{R}$ is a neighborhood of x in $\mathbb{R}$.

Proposition A set $B \subseteq A$ is *relatively open* in A if and only if for every $x \in B$ there is a relative neighborhood V in A such that

$$V \subseteq B.$$

Proof. Let $x \in B$, since B is open in A,

- there is an open set $U \subseteq \mathbb{R}$, such that
$$B = A \cap U.$$
- Since $x \in B$,
$$x \in A \cap U,$$
so $x \in U$.
- Since U is open in $\mathbb{R}$,
there is a neighborhood G of x such that
$$G \subseteq U.$$
- Therefore,
$$V = G \cap B \subseteq B$$
is a relative neighborhood of x.
Conversely
- **Assume that**: For each $x \in B$ there is
$$V_x = A \cap U_x$$
such that
$$V_x \subseteq B$$
and U_x is a neighborhood of x in $\mathbb{R}$.
- Since U_x is a neighborhood of x in $\mathbb{R}$,
there is $\delta_x > 0$, such that
$$D(x, \delta_x) \subseteq U_x.$$
- Let $G = \bigcup_{x \in B} D(x, \delta_x)$.
- Clearly G is open in $\mathbb{R}$ as a union of open sets.
- It is sufficient to show that
$$B = G \cap A.$$
We show that
$$B \subseteq G \cap A$$

- Let $x \in B$.

  Since $B \subseteq A$ and
  $$x \in D(x, \delta_x)$$
  then $x \in A$ and
  $$x \in \bigcup_{x \in B} D(x, \delta_x) = G,$$
  so
  $$x \in G \cap A,$$
  hence $B \subseteq G \cap A$.

  We show that
  $$G \cap A \subseteq B$$

- Since $D(x, \delta_x) \subseteq U_x$,
  $$\begin{aligned} D(x, \delta_x) \cap A &\subseteq U_x \cap A \\ &= V_x \subseteq B. \end{aligned}$$

- Therefore,
  $$D(x, \delta_x) \cap A \subseteq B,$$
  for every $x \in B$.

- Hence,
  $$\bigcup_{x \in B} (D(x, \delta_x) \cap A) \subseteq B$$
  and
  $$\begin{aligned} G \cap A &= \left(\bigcup_{x \in B} D(x, \delta_x) \right) \cap A \\ &= \bigcup_{x \in B} (D(x, \delta_x) \cap A) \\ &\subseteq B. \end{aligned}$$

- It follows that
  $$G \cap A \subseteq B.$$

- Therefore,
  $$G \cap A = B$$
  and G is open in $\mathbb{R}$.

- Consequently, B is open in A.

This finishes our proof. ■

- **Closed sets**

  Definition $F \subseteq \mathbb{R}$ is *closed* if
$$F^c = \mathbb{R} \backslash F$$
  is open in $\mathbb{R}$.

  Example: Let
$$F = [0,1] \subseteq \mathbb{R},$$
  then $F^c = (-\infty, 0) \cup (1, \infty)$.

- Since both $(-\infty, 0)$ and $(1, \infty)$ are open,
  thus $(-\infty, 0) \cup (1, \infty)$ is open as a union of open sets.

- Therefore, F^c is open, so
  F is closed.

  Example: $A = [0, 1)$ is neither open nor closed.

- Indeed, as we showed before,
  A is not open.

- Since
$$A^c = (-\infty, 0) \cup [1, \infty)$$
  is not open since for $1 \in A^c$,
$$D(1, \epsilon) \nsubseteq A^c, \ \epsilon > 0.$$

- Hence, A^c is not open.

- Therefore, A^c is not open.

- Hence A is not closed.

  Example: $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subseteq \mathbb{R}$ is neither open nor closed.

- Indeed, if $D \subseteq \mathbb{R}$ is a neighborhood of $\frac{1}{n}$,
  then
$$D\left(\frac{1}{n}, \delta\right) \subseteq D$$
  for some $\delta > 0$.

- Since $D\left(\frac{1}{n}, \delta\right)$ is not countable and A is countable,
$$D \nsubseteq A.$$

- Furthermore, $0 \in A^c$ and if U is a neighborhood of 0,
  then there is $\delta > 0$, such that
$$D(0, \delta) \subseteq U.$$

- Since $\delta > 0$, there is $n \in \mathbb{N}$, such that
$$0 < \frac{1}{n} < \delta$$
  and since $\frac{1}{n} \in A$,

- then

$$D\left(0,\delta\right) \cap A \neq \emptyset,$$

  so

$$U \cap A \neq \emptyset.$$

- Therefore, $U \nsubseteq A^c$.
- It follows that A is not closed.

  Proposition A set $F \subseteq \mathbb{R}$ is closed *if and only if*

  the limit of every convergent sequence in F belongs to F.

Proof. Let $\{x_n\} \subseteq F$ be a sequence in F and

- assume that $x_n \to x$ as $n \to \infty$.
- **Suppose** that $x \notin F$, then

$$x \in F^c.$$

- Since F^c is open,

  there is a neighborhood D of x, such that

$$D \subseteq F^c$$

- Since $x_n \to x$, there is $N \in \mathbb{N}$, such that,

  for all $n > N$,

$$x_n \in D.$$

- Therefore, $x_n \notin F$, for all $n > N$,

  however $x_n \in F$, for all $n \in \mathbb{N}$,

  we arrive at **contradiction**.

  Conversely

- **Suppose that** the limit of every convergent sequence in F belongs to F.
- Let $x \in F^c$.
- Suppose that F^c is not open, so for all $n \in \mathbb{N}$,

$$D\left(x, \frac{1}{n}\right) \nsubseteq F^c.$$

- Then, for all $n \in \mathbb{N}$,

$$D\left(x, \frac{1}{n}\right) \cap F \neq \emptyset.$$

- For each n, let

$$x_n \in D\left(x, \frac{1}{n}\right) \cap F.$$

- Consider the sequence

$$\{x_n\} \subseteq F.$$

- If $\epsilon > 0$ is given,
  then there is $N \in \mathbb{N}$ such that
$$0 < \frac{1}{N} < \epsilon.$$
- If $n > N$, then
$$\frac{1}{n} < \frac{1}{N}$$
  and thus
$$D\left(x, \frac{1}{n}\right) \subset D\left(x, \frac{1}{N}\right).$$
- Since $x_n \in D\left(x, \frac{1}{n}\right)$, hence for $n > N$,
$$x_n \in D\left(x, \frac{1}{N}\right).$$
- Therefore,
$$|x_n - x| < \frac{1}{N} < \epsilon,$$
  for all $n > N$.
- It follows that $x_n \to x$ so $x \in F$.
  A contradiction since we assumed that
  limits of all convergent sequences in F are in F.
- Therefore, there is $n \in \mathbb{N}$, such that
$$D\left(x, \frac{1}{n}\right) \subseteq F^c.$$
- Since $x \in F^c$,
  it follows that F^c is **open**, so F is **closed**.

This finishes our proof. ■

- **Proposition** An arbitrary intersection of closed sets is closed, and
  a finite union of closed sets is closed.

Proof. If $\mathcal{C}$ is a family of closed sets, then
$$\mathbb{R}\backslash \bigcap_{C\in\mathcal{C}} C = \bigcup_{C\in\mathcal{C}} (\mathbb{R}\backslash C).$$
- Since $C \in \mathcal{C}$,
- $\mathbb{R}\backslash C$ is open and, as we showed it before,
$$\bigcup_{C\in\mathcal{C}} \mathbb{R}\backslash C$$
  is open, so
$$\mathbb{R}\backslash \bigcap_{C\in\mathcal{C}} C$$
  is open.

- Therefore, $\bigcap_{C\in\mathcal{C}} C$ is closed.
- Now, if

$$\mathcal{C} = \{C_j : 1 \le j \le n\},$$

then

$$\begin{aligned} \mathbb{R}\backslash \bigcup_{C\in\mathcal{C}} C &= \bigcap_{C\in\mathcal{C}} (\mathbb{R}\backslash C) \\ &= \bigcap_{j=1}^{n} \mathbb{R}\backslash C_j. \end{aligned}$$

- Since $C \in \mathcal{C}$, $\mathbb{R}\backslash C$ is open and, as we have showed this before,

$$\bigcap_{j=1}^{n} \mathbb{R}\backslash C_j$$

is open.

- Therefore, $\mathbb{R}\backslash \bigcup_{C\in\mathcal{C}} C$ is open.
- Hence $\bigcup_{C\in\mathcal{C}} C$ is closed.

This finishes our proof. ■

- **Remark:** If $\mathcal{C}$ is infinite then $\bigcup_{C\in\mathcal{C}} C$ may not be closed.
- Indeed, let

$$\mathcal{C} = \left\{ \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right] : n \in \mathbb{N} \right\}.$$

- We see that if $C \in \mathcal{C}$ then C is closed,
- however

$$\bigcup_{C\in\mathcal{C}} C = (-1, 1)$$

is not closed.

Closure of $A \subseteq \mathbb{R}$

- Using the properties of closed sets,
- we can define closure of $A \subseteq \mathbb{R}$ as follows.
- Let

$$\mathcal{C}(A) = \{C \mid A \subseteq C \text{ and } C \text{ is closed}\}$$

and then the *closure of* A in X is defined as follows:

$$\overline{A} = \bigcap_{C\in\mathcal{C}(A)} C.$$

- From the definition of $\overline{A}$ it follows immediately that

1. $\overline{A}$ is closed in X and $A \subseteq \overline{A}$.

2. If C is closed and $A \subseteq C$,

then $\overline{A} \subseteq C$ (therefore, $\overline{A}$ is the smallest closed subset of $\mathbb{R}$ that contains A).

3. If C is closed,

then $\overline{C} = C$, in particular,

$$\overline{\overline{A}} = \overline{A}.$$

- **Example** Let

$$A = (a,\ b) \subset \mathbb{R}.$$

  Find $\overline{A}$ using the definition of the closure.

- Notice that A is not closed.
- Moreover, since

$$\begin{aligned} \mathbb{R} \backslash A &= A^c \\ &= (-\infty, a] \cup [b, \infty) \end{aligned}$$

  and $D(a, \epsilon) \cap A \neq \emptyset$, for all $\epsilon > 0$.

- It follows that A is not open.
- Since in analogous way we can show that
  $[a, b)$, $(a, b]$ are not closed,
  the minimal closed subset of $\mathbb{R}$ that contains A is $[a, b]$.
- Therefore, we see that

$$\overline{A} = [a, b] \, .$$