MATH 4334.001/CS 4334.001
Fall 2023
Paper Homework 4 Solutions

1. Calculate $f(10^{-2})$ for the function

$$f(x) = e^x - x - 1$$

using five significant digits. Then compare this by evaluating $f(10^{-2})$ directly by using $e^{0.01} \approx 1.0101$.

We have

$$\begin{aligned} f(x) &= e^x - x - 1 \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} - x - 1 \\ &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - x - 1 \\ &= \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \end{aligned}$$

so that

$$f(10^{-2}) = (1/2) \times 10^{-4} + (1/6) \times 10^{-6} + (1/24) \times 10^{-8} + (1/120) \times 10^{-10}$$

$$\begin{array}{l} 0.00005 \\ 0.00000\ 01666\ 7 \\ 0.00000\ 00004\ 1667 \\ \underline{0.00000\ 00000\ 0083} \\ 0.00005\ 01670\ 8750 \end{array}$$

Hence,

$$\boxed{f(10^{-2}) \approx 5.0167 \times 10^{-5}}$$

Direct evaluation yields

$$\begin{aligned} f(10^{-2}) &= e^{0.01} - 0.01 - 1 \\ &\approx 1.0101 - 0.01 - 1 \\ &= 0.0001 \end{aligned}$$

2. Determine the values of $x$ for which there is loss of significance in evaluating the function

$$f(x) = \frac{1-(1-x)^3}{x}$$

Then find an alternative form that avoids the problem.

The numerator of $f(x)$ subtracts nearly equal numbers for $x$ near 0. We can avoid this problem by simplifying the function as

$$\begin{aligned} f(x) &= \frac{1-(1-x)^3}{x} \\ &= \frac{1-(1-3x+3x^2-x^3)}{x} \\ &= \frac{3x-3x^2+x^3}{x} \\ &= 3-3x+x^2 \end{aligned}$$

3. Write a function that computes accurate values of

$$f(x) = \sqrt[4]{x+4} - \sqrt[4]{x}$$

for positive $x$.

We have

$$\begin{aligned}
f(x) = \sqrt[4]{x+4} - \sqrt[4]{x} &= \frac{\left(\sqrt[4]{x+4} - \sqrt[4]{x}\right)\left(\sqrt[4]{x+4} + \sqrt[4]{x}\right)}{\left(\sqrt[4]{x+4} + \sqrt[4]{x}\right)} \\
&= \frac{\left(\sqrt{x+4} - \sqrt{x}\right)}{\left(\sqrt[4]{x+4} + \sqrt[4]{x}\right)} \\
&= \frac{\left(\sqrt{x+4} - \sqrt{x}\right)\left(\sqrt{x+4} + \sqrt{x}\right)}{\left(\sqrt[4]{x+4} + \sqrt[4]{x}\right)\left(\sqrt{x+4} + \sqrt{x}\right)} \\
&= \frac{4}{\left(\sqrt[4]{x+4} + \sqrt[4]{x}\right)\left(\sqrt{x+4} + \sqrt{x}\right)}
\end{aligned}$$

$a \leftarrow \sqrt{x}$
$b \leftarrow \sqrt{x+4}$
$f \leftarrow 4/(\sqrt{a} + \sqrt{b})(a+b)$

4. Calculate both zeros of

$$3x^2 - 9^{14}x + 100 = 0$$

correct to 3 significant digits.

The quadratic formula gives the two zeros

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

where,

$$a = 3,\ b = -9^{14},\ c = 100$$

We see that the numerator in the second zero $x_2$ subtracts nearly equal numbers. Hence, $x_2$ should be computed in alternative form. Thus, the zeros are

$$\begin{aligned} x_1 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \approx \boxed{7.63 \times 10^{12}} \\ x_2 &= \frac{2c}{-b + \sqrt{b^2 - 4ac}} \approx \boxed{4.37 \times 10^{-12}} \end{aligned}$$