Math 4301
Exam 2

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Question 1

(10 pts) Let $A = (0,1) \cap \mathbb{Q} \cup \{2 + \frac{1}{n} : n \in \mathbb{N}\}$. Find

1. (2.5 pts) $A' =$
2. (2.5 pts) $\bar{A} =$
3. (2.5 pts) $\text{Int}(A) =$
4. (2.5 pts) $\partial A =$

Question 2

Answer the following questions (**T**/**F**)

1. (2 pts) ______ $A = \{(-1)^n + \frac{1}{n} : n \in \mathbb{N}\}$ is closed.
2. (2 pts) ______ $A =$ is connected.
3. (2 pts) ______ $\partial(\mathbb{R} \backslash \mathbb{Q}) = \mathbb{R}$.
4. (2 pts) ______ $\text{Int}(\{\frac{1}{n} : n \in \mathbb{N}\}) = (0,1)$.
5. (2 pts) ______ $A = [1,2)$ is sequentially compact.

Question 3

(6 pts) Complete the following definition

Definition Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in A'$(A' is the set of accumulation points of A). We say that

$$\lim_{x \to x_0} f(x) = L$$

if for every $\epsilon > 0$.

Solution: ☺

Question 4

(14 pts) Using $(\epsilon - \delta)$ definition of limit, show that

$$\lim_{x \to 1} (2x^2 - x + 1) = 2$$

Proof: ☺

Question 5

Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^2 + 2 & \text{if} \quad x > 0 \\ x + 1 & \text{if} \quad x \leqslant 0 \end{cases}$$

and $\delta > 0$.

i) (8 pts) Find

$$\omega_f(D(0,\delta)) = \sup\{|f(x) - f(y)| : x, y \in D(0,\delta)\} =$$

Explain your answer using graph $y = f(x)$

ii) (6 pts) Find

$$\omega_f(0) = \inf\{\omega_f(D(0,\delta)) : \delta > 0\} =$$

Explain briefly your answer.

iii) For $x \neq 0$, find

$$\omega_f(x) = \inf\{\omega_f(D(x,\delta)) : \delta > 0\}$$

1. (3 pts) If $x < 0$ then $f(x) =$ ______, so $\omega_f(x) =$
2. (3 pts) if $x > 0$ then $f(x) =$ ______, so $\omega_f(x) =$

Solution: 😄

Question 6

(20 pts) Show that $f : (\frac{1}{2}, \infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{2x-1}$ is not uniformly continuous.

Proof: 😄

Question 7

(10 pts) (Extra Credit) Show that if A is compact and $A \neq 0$ then there are $x_0, y_0 \in A$ such that

$$x_0 = \inf A \text{ and } y_0 = \sup A$$

Proof: ☺

Question 8

(10 pts) (Extra Credit) Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ and assume that there is $M \geqslant 0$, such that, for all $x \in A$,

$$|f(x)| \leqslant M \text{ and } |g(x)| \leqslant M$$

Show that if both f and g are uniformly continuous on A, then the function $h : A \to \mathbb{R}$, given by $h(x) = f(x)g(x)$ is also uniformly continuous on A.

Proof: ☺