§1. Set Theory

Math 4341 (Topology)

Sets and subsets

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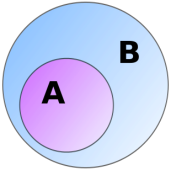
- If A is a set and a is an element of A, we write $a \in A$. If b is not an element of A, then we write $b \notin A$.
- If B is another set which contains all the elements of A (that is, if $a \in A$ implies that $a \in B$), then we say that A is a *subset* of B and write $A \subset B$.



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- If $A \subset B$ but $A \neq B$, we write $A \subsetneq B$ and say that A is a *proper subset* of B.
- Example 1.1: When a set contains only a few elements, we simply list them. For example if A contains only a, b and c, we write $A = \{a, b, c\}$. Then if $B = \{a, b, c, d\}$, we see that $A \subset B$ and since $d \in B$ but $d \notin A$, we have $A \subsetneq B$.

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- Example 1.2: Sets are often given by the properties of their elements. For example, the set consisting of all odd numbers is written as

$$\{x \mid x \text{ is an odd integer}\},$$

and is read as "x such that x is an odd integer".

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- Example 1.3: Some examples:

$$\begin{aligned} \mathcal{P}(\emptyset) &= \{\emptyset\}, \\ \mathcal{P}(\{a\}) &= \{\emptyset, \{a\}\}, \\ \mathcal{P}(\{a,b\}) &= \{\emptyset, \{a\}, \{b\}, \{a,b\}\}, \\ \mathcal{P}(\{a,b,c\}) &= \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}. \end{aligned}$$

Finite union and intersection

- Given two sets A and B, we define their *union* $A \cup B$ and *intersection* $A \cap B$ by

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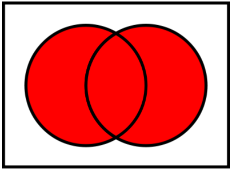

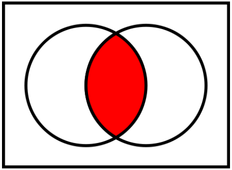

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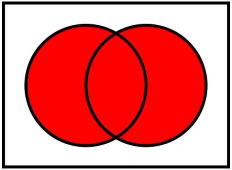

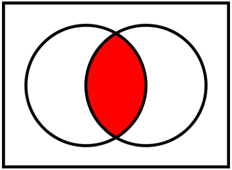

- The sets A and B are called *disjoint* if $A \cap B = \emptyset$.

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- Given two sets A and B, we define the *difference* $A \setminus B$ by

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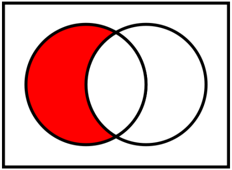

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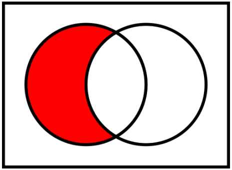
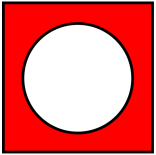


- If $A \subset X$, we define the *complement* of A in X, written A^c, as

$$A^c = X \setminus A.$$



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- Let I be any set. Then a collection $\{A_i\}_{i\in I}$ of sets A_i is called a family of sets parametrized by I.
- For such a family, define the union and the intersection by

$$\bigcup_{i\in I} A_i = \{x \mid \exists i \in I \text{ such that } x \in A_i\},$$
$$\bigcap_{i\in I} A_i = \{x \mid x \in A_i \ \forall i \in I\}.$$

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Cartesian products

- If A and B are two sets, then the *Cartesian product* $A \times B$ is the set of all pairs (a, b), where $a \in A$ and $b \in B$. That is

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- Example 1.4: $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ (n times).
- In general, if $\{A_i\}_{i \in I}$ is a family of sets then the Cartesian product $\prod_{i \in I} A_i$ is the set of all functions $a : I \to \bigcup_{i \in I} A_i$ such that $a(i) \in A_i$ for all $i \in I$.

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  - *transitive* if xCy and yCz implies that xCz for all $x, y, z \in A$,
  - *total* if either xCy or yCx when $x, y \in A$.
- Example 1.6: The relation in Example 1.5 is reflexive, anti-symmetric, transitive, and total, but it is not symmetric.

Partial order and equivalence relation

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  - When C is an equivalence relation, we use the notation $x \sim y$ for xCy and say that x is equivalent to y.
- Example 1.8: Fix $p \in \mathbb{N}$. Let $C \subset \mathbb{Z} \times \mathbb{Z}$ be the subset of pairs (m, n) such that $m - n$ is a multiple of p, i.e. $m - n = kp$ for some $k \in \mathbb{Z}$. Then C is an equivalence relation on $\mathbb{Z}$.

Equivalence classes

- Given any equivalence relation on a set A, it is possible to partition A into smaller sets consisting of elements that are equivalent to each other. More precisely, for $x \in A$ let

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- Lemma 1.3: Let $\sim$ denote an equivalence relation on a set A. Then for two elements $x, x' \in A$, the equivalence classes $[x]$ and $[x']$ are either disjoint or equal.
- Proof: It is equivalent to show that if $[x]$ and $[x']$ are not disjoint then $[x] = [x']$.

Equivalence classes (contd)

- Suppose $[x]$ and $[x']$ are not disjoint. Then there is a $z \in A$ such that $z \in [x]$ and $z \in [x']$. That is, $z \sim x$ and $z \sim x'$.

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- Showing $[x] \subset [x']$: Let $y \in [x]$. Then $y \sim x$, so $x \sim y$ (by symmetry). Since $z \sim x$, by transitivity, $z \sim y$. Thus $y \sim z$ (by symmetry). Since $z \sim x'$, by transitivity, $y \sim x'$. This means that $y \in [x']$. Hence $[x] \subset [x']$.

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- Showing $[x'] \subset [x]$: similar.

Equivalence classes (contd)

- The set of equivalence classes on a set A with respect to an equivalence relation $\sim$ is denoted $A/\sim$. That is,

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- Example 1.9: Consider the relation $\sim$ from Example 1.8. The equivalence class of an integer $n \in \mathbb{Z}$ is the set of integers

$$[n] = \{\dots, n-2p, n-p, n, n+p, n+2p, \dots\}.$$

and we can write $\mathbb{Z}$ as the union of p equivalence classes:

$$\mathbb{Z} = [0] \cup [1] \cup [2] \cup \cdots \cup [p-1].$$

Similarly,

$$\mathbb{Z}/\sim = \{[0], [1], \dots, [p-1]\}.$$