Math 4301 Mathematical Analysis I
Lecture 19
Topic: Sequences and series of functions

- **Proposition** Suppose that $f_n : [a,b] \to \mathbb{R}$ are Riemann integrable over $[a,b]$,
$$f : [a,b] \to \mathbb{R}$$
and $f_n \to f$ (uniformly) on $[a,b]$.
Then f is integrable over $[a,b]$ and
$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\ dx = \int_a^b f(x)\ dx.$$

Proof. Let us assume (we prove it latter) that f is integrable over $[a,b]$.

- We show that
$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\ dx.$$
- Let $\epsilon > 0$ be given.
- Since $f_n \to f$ (uniformly) on $[a,b]$, there is $N \in \mathbb{N}$, such that, for all $n > N$ and $x \in [a,b]$,
$$|f_n(x) - f(x)| < \frac{\epsilon}{b-a}.$$
- We see that, for $n > N$,
$$\begin{aligned} \left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\ dx\right| &= \left|\int_a^b (f_n(x) - f(x))\,dx\right| \le \int_a^b |f_n(x) - f(x)|\,dx \\ &< \int_a^b \frac{\epsilon}{b-a}dx = \frac{\epsilon}{b-a}\int_a^b\ dx \\ &= \frac{\epsilon}{(b-a)} \cdot (b-a) = \epsilon. \end{aligned}$$
since
$$\left|\int_a^b h(x)\,dx\right| \le \int_a^b |h(x)|\,dx$$
- **We show that f is Riemann integrable**.
- Recall, $f : [a,b] \to \mathbb{R}$ is **Riemann integrable**, if for all $\epsilon > 0$, there is a partition
$$P = \{x_0, x_1, ..., x_n\}$$
of $[a,b]$, such that
$$U(f,P) - L(f,P) < \epsilon.$$
- Let $\epsilon > 0$ be given and since $f_n \to f$ (uniformly) on $[a,b]$, there is $N \in \mathbb{N}$, such that, for all $n > N$ and $x \in [a,b]$,
$$|f_n(x) - f(x)| < \frac{\epsilon}{4(b-a)}.$$

- If $n = N + 1$, then since f_{N+1} is bounded,
  there is $M_{N+1} \geq 0$, such that, for all $x \in [a, b]$,

$$|f_{N+1}(x)| \leq M_{N+1}$$

- Since $n = N + 1 > N$, for all $x \in [a, b]$,

$$\begin{aligned} |f(x)| &= |(f(x) - f_{N+1}(x)) + f_{N+1}(x)| \\ &\leq |f_{N+1}(x) - f(x)| + |f_{N+1}(x)| \\ &\leq \frac{\epsilon}{4(b-a)} + M_{N+1}, \end{aligned}$$

- It follows that f is **bounded** on $[a, b]$.
- Hence, both $\overline{\int_a^b} f$ and $\underline{\int_a^b} f$ are defined for f.
- Let $n > N$.
- Since $f_n : [a, b] \to \mathbb{R}$ is integrable,
  there is a partition

$$P = \{x_0, x_1, ..., x_m\}$$

  of $[a, b]$, such that

$$U(f_n, P) - L(f_n, P) < \frac{\epsilon}{2}$$

- Since, $n > N$ and all $x \in [a, b]$,

$$\begin{aligned} |f_n(x) - f(x)| &= |f(x) - f_n(x)| < \frac{\epsilon}{4(b-a)}, \text{ i.e.} \\ -\frac{\epsilon}{4(b-a)} &< f(x) - f_n(x) < \frac{\epsilon}{4(b-a)}, \end{aligned}$$

- so

$$f_n(x) - \frac{\epsilon}{4(b-a)} < f(x) < f_n(x) + \frac{\epsilon}{4(b-a)},$$

  and for all $i = 1, 2, ..., m$ and for all $x \in [x_{i-1}, x_i]$:

$$f_n(x) - \frac{\epsilon}{4(b-a)} < f(x) < \frac{\epsilon}{4(b-a)} + f_n(x),$$

- Since

$$f_n(x) - \frac{\epsilon}{4(b-a)} < f(x),$$

  and, for all $x \in [x_{i-1}, x_i]$,

$$m_i(f_n) = \inf\{f_n(x) : x \in [x_{i-1}, x_i]\} \leq f_n(x),$$

  it follows

$$m_i(f_n) - \frac{\epsilon}{4(b-a)} < f(x)$$

- Furthermore, from the above
$$m_i\left(f_n\right)-\frac{\epsilon}{4\left(b-a\right)}$$
is a lower bound for
$$\left\{f\left(x\right):x\in\left[x_{i-1},x_i\right]\right\}$$
- Since $m_i\left(f\right)$ is the greatest lower bound
$$m_i\left(f_n\right)-\frac{\epsilon}{4\left(b-a\right)}<\inf\left\{f\left(x\right):x\in\left[x_{i-1},x_i\right]\right\}=m_i\left(f\right).$$
- Moreover, since
$$f\left(x\right)<\frac{\epsilon}{4\left(b-a\right)}+f_n\left(x\right)$$
and
$$f_n\left(x\right)\leq\sup\left\{f_n\left(x\right):x\in\left[x_{i-1},x_i\right]\right\}=M_i\left(f_n\right),$$
so
$$\begin{aligned}f\left(x\right) &< \frac{\epsilon}{4\left(b-a\right)}+f_n\left(x\right)\\ &\leq \sup\left\{f_n\left(x\right):x\in\left[x_{i-1},x_i\right]\right\}+\frac{\epsilon}{4\left(b-a\right)}\\ &= M_i\left(f_n\right)+\frac{\epsilon}{4\left(b-a\right)}\end{aligned}$$
- Therefore,
$$M_i\left(f_n\right)+\frac{\epsilon}{4\left(b-a\right)}$$
is an upper bound for
$$\left\{f\left(x\right):x\in\left[x_{i-1},x_i\right]\right\}.$$
- Since $M_i\left(f\right)$ is the least upper bound for
$$\left\{f\left(x\right):x\in\left[x_{i-1},x_i\right]\right\},$$
$$\begin{aligned}M_i\left(f\right) &= \sup\left\{f\left(x\right):x\in\left[x_{i-1},x_i\right]\right\}\\ &\leq M_i\left(f_n\right)+\frac{\epsilon}{4\left(b-a\right)}\end{aligned}$$
- Hence
$$\begin{aligned}m_i\left(f_n\right)-\frac{\epsilon}{4\left(b-a\right)} &\leq m_i\left(f\right)\ \text{ and}\\ M_i\left(f\right) &\leq M_i\left(f_n\right)+\frac{\epsilon}{4\left(b-a\right)}.\end{aligned}$$
- Therefore,
$$\sum_{i=1}^{m}\left(m_i\left(f_n\right)-\frac{\epsilon}{4\left(b-a\right)}\right)\Delta x_i\leq\sum_{i=1}^{m}m_i\left(f\right)\Delta x_i=L\left(f,P\right)$$
and
$$U\left(f,P\right)=\sum_{i=1}^{m}M_i\left(f\right)\Delta x_i\leq\sum_{i=1}^{m}\left(M_i\left(f\right)+\frac{\epsilon}{4\left(b-a\right)}\right)\Delta x_i.$$

- Since

$$\begin{aligned}\sum_{i=1}^{m}\left(m_i\left(f_n\right)-\frac{\epsilon}{4\left(b-a\right)}\right)\Delta x_i &= \sum_{i=1}^{m} m_i\left(f_n\right)\Delta x_i-\frac{\epsilon}{4\left(b-a\right)}\sum_{i=1}^{m}\Delta x_i\\ &= L\left(f_n,P\right)-\frac{\epsilon}{4\left(b-a\right)}\left(b-a\right)\\ &= L\left(f_n,P\right)-\frac{\epsilon}{4},\end{aligned}$$

  and

$$\begin{aligned}\sum_{i=1}^{m}\left(M_i\left(f\right)+\frac{\epsilon}{4\left(b-a\right)}\right)\Delta x_i &= \sum_{i=1}^{m} M_i\left(f\right)\Delta x_i+\sum_{i=1}^{m}\frac{\epsilon}{4\left(b-a\right)}\Delta x_i\\ &= \sum_{i=1}^{m} M_i\left(f\right)\Delta x_i+\frac{\epsilon}{4\left(b-a\right)}\left(b-a\right)\\ &= U\left(f_n,P\right)+\frac{\epsilon}{4},\end{aligned}$$

  so

$$\begin{aligned}L\left(f_n,P\right)-\frac{\epsilon}{4} &\leq L\left(f,P\right) \text{ and}\\ U\left(f,P\right) &\leq U\left(f_n,P\right)+\frac{\epsilon}{4}.\end{aligned}$$

- We see that

$$\begin{aligned}U\left(f,P\right)-L\left(f,P\right) &\leq \left(U\left(f_n,P\right)+\frac{\epsilon}{4}\right)-\left(L\left(f_n,P\right)-\frac{\epsilon}{4}\right)\\ &= U\left(f_n,P\right)-L\left(f_n,P\right)+\frac{\epsilon}{2}\\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon\end{aligned}$$

- Therefore, f is Riemann integrable.

This finishes our proof. ■

- **Corollary** Suppose that $f_n:[a,b]\to\mathbb{R}$ is Riemann integrable and $\sum_{n=1}^{\infty} f_n$ converges uniformly to $f:[a,b]\to\mathbb{R}$.
  Then f is Riemann integrable over $[a,b]$ and

$$\int_a^b f\left(x\right)dx=\sum_{n=1}^{\infty}\int_a^b f_n\left(x\right)dx.$$

  i.e.

$$\int_a^b\left(\sum_{n=1}^{\infty} f_n\left(x\right)\right)dx=\sum_{n=1}^{\infty}\int_a^b f_n\left(x\right)dx$$

Proof. Let

$$s_n=\sum_{i=1}^{n} f_n.$$

- Then $s_n \to f$ (uniformly) so by previous theorem

$$\begin{aligned}\sum_{n=1}^{\infty}\int_a^b f_n(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\int_a^b f_n(x)\,dx \\ &= \lim_{n\to\infty}\int_a^b s_n(x)\,dx = \int_a^b \lim_{n\to\infty} s_n(x)\,dx \\ &= \int_a^b f(x)\;\;dx = \int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx.\end{aligned}$$

This finishes our proof. ■

- **Exercise** Let $f_n : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = nx\left(1-x^2\right)^n.$$

  Is is true that

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$$

- **Solution** We compute the pointwise limit of the sequence $\{f_n\}$.
- If $x_0 = 0,\ 1$, then

$$\begin{aligned}\lim_{n\to\infty} f_n(0) &= \lim_{n\to\infty} n\cdot 0\left(1-0^2\right)^n = \lim_{n\to\infty} n\cdot 1\left(1-1^2\right)^n \\ &= \lim_{n\to\infty} f_n(1) = 0.\end{aligned}$$

- Thus, assume that $0 < x < 1$, so

$$\begin{aligned}\lim_{n\to\infty}\frac{f_{n+1}(x)}{f_n(x)} &= \lim_{n\to\infty}\frac{(n+1)\,x\left(1-x^2\right)^{n+1}}{nx\left(1-x^2\right)^n} \\ &= \lim_{n\to\infty}\frac{(n+1)\left(1-x^2\right)}{n} = \left(1-x^2\right) < 1.\end{aligned}$$

- It follows that

$$\lim_{n\to\infty} f_n(x) = 0.$$

- Let $f : [0,1] \to \mathbb{R}$, be given by,

$$f(x) = \lim_{n\to\infty} f_n(x) = 0.$$

- We see that since

$$\lim_{n\to\infty} f_n(x) = f(x),$$

  for all $x \in [0,1]$, then $\{f_n\}$ converges to f pointwise.
- Notice that

$$\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 f(x)\,dx = \int_0^1 0dx = 0$$

- Therefore,

$$\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = 0.$$

- We compute

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx=\lim_{n\to\infty}\int_0^1 nx\left(1-x^2\right)^n dx$$

- Let $u=1-x^2$, then $dx=-2xdx$, so

$$\begin{aligned}\int_0^1 f_n(x)\,dx &= \int_0^1 nx\left(1-x^2\right)^n dx=-\frac{n}{2}\int_1^0 u^n du\\ &= \frac{n}{2}\int_0^1 u^n du=\frac{n}{2(n+1)}\end{aligned}$$

so

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx=\lim_{n\to\infty}\frac{n}{2(n+1)}=\frac{1}{2}.$$

- As we see

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx\neq\int_0^1\lim_{n\to\infty}f_n(x)\,dx.$$

- This also shows that $\{f_n\}$ cannot converge uniformly to f and that the pointwise convergence of the sequence $\{f_n\}$ is usually insufficient for the statement of the above theorem to hold.

  Exercise Show that

$$\int_0^1\left(\sum_{n=1}^{\infty}\frac{x}{\left(x^2+n\right)^2}\right)dx=\frac{1}{2}.$$

- **Solution** Let

$$f_n:[0,1]\to\mathbb{R},\ f_n(x)=\frac{x}{\left(x^2+n\right)^2}.$$

- We see that

$$\begin{aligned}f_n'(x) &= \frac{d}{dx}\left(\frac{x}{\left(x^2+n\right)^2}\right)=\frac{n-3x^2}{\left(x^2+n\right)^3}=0\text{ iff}\\ x_n &= \pm\sqrt{\frac{n}{3}},\ n\in\mathbb{N}.\end{aligned}$$

- Since $x_n\notin[0,1]$, $x_n=\sqrt{\frac{n}{3}}$, $n\geq 4$.
- Moreover, since

$$f_n(0)=0,\ f_n(1)=\frac{1}{(n+1)^2}$$

  and for $n\geq 4$ and $x\in[0,1]$,

$$f_n'(x)=\frac{n-3x^2}{\left(x^2+n\right)^3}>0$$

  so f_n increases.

- Thus, for all $x\in[0,1]$,

$$\left|\frac{x}{\left(x^2+n\right)^2}\right|\leq f_n(1)=\frac{1}{(n+1)^2}.$$

- Let $M_n = \frac{1}{(n+1)^2}$.
- Since
$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} < \infty,$$
by the Weierstrass M-test, $\sum_{n=1}^{\infty} f_n$ converges uniformly.
- Consequently, by the theorem
$$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{x}{(x^2+n)^2} \right) dx = \sum_{n=1}^{\infty} \int_0^1 \frac{x}{(x^2+n)^2} dx.$$
- Since
$$\int_0^1 \frac{x}{(x^2+n)^2} dx = \frac{1}{2} \int_0^1 \frac{2x}{(x^2+n)^2} dx,$$
let $u = x^2 + n$, $du = 2xdx$,
$$\begin{aligned} \int_0^1 \frac{x}{(x^2+n)^2} dx &= \frac{1}{2} \int_n^{n+1} \frac{1}{u^2} du = -\frac{1}{2} \left. \frac{1}{u} \right|_n^{n+1} \\ &= \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+1} \right). \end{aligned}$$
- Thus,
$$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{x}{(x^2+n)^2} \right) dx = \sum_{n=1}^{\infty} \int_0^1 \frac{x}{(x^2+n)^2} dx = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right).$$
- Let
$$\begin{aligned} s_n &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{n+1}. \end{aligned}$$
- Since
$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1} \right) = 1.$$
- It follows that
$$\begin{aligned} \int_0^1 \left(\sum_{n=1}^{\infty} \frac{x}{(x^2+n)^2} \right) dx &= \sum_{n=1}^{\infty} \int_0^1 \frac{x}{(x^2+n)^2} dx \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{2} \cdot 1 = \frac{1}{2}. \end{aligned}$$

Differentiation of Sequences and Series of Functions

Theorem Let $f_n : (a,b) \to \mathbb{R}$ be differentiable, $f : (a,b) \to \mathbb{R}$ and $f_n(x) \to f(x)$ (pointwise). Suppose that $f_n' : (a,b) \to \mathbb{R}$,
$$f_n'(x) = \frac{d}{dx} f_n(x)$$

is continuous and $f_n' \to g$ (uniformly), where

$$g : (a,b) \to \mathbb{R}.$$

Then f is differentiable on (a,b) and $f' = g$, that is,

$$g(x) = \lim_{n\to\infty} f_n'(x) = f'(x),$$

for all $x \in (a,b)$.

That is,

$$\frac{d}{dx}\left(\underbrace{\lim_{n\to\infty} f_n(x)}_{f(x)}\right) = \underbrace{\lim_{n\to\infty} f_n'(x)}_{g(x)}$$

Proof. Let $x_0 \in (a,b)$.

- Using the Fundamental Theorem of Calculus,

$$\begin{aligned} f_n(x) &= f_n(x_0) + \int_{x_0}^{x} f_n'(t)\,dt, \text{ hence} \\ f(x) &= \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(x_0) + \lim_{n\to\infty} \int_{x_0}^{x} f_n'(t)\,dt \\ &= f(x_0) + \int_{x_0}^{x} \lim_{n\to\infty} f_n'(t)\,dt = f(x_0) + \int_{x_0}^{x} g(t)\,dt. \end{aligned}$$

- Since g is continuous on (a,b),
  it follows that f is differentiable and

$$f'(x) = g(x)$$

  as we claimed.

This finishes our proof. ■

- **Corollary** Let $f_n : (a,b) \to \mathbb{R}$ be differentiable and
  $\sum_{n=1}^{\infty} f_n$ converges (pointwise) on (a,b) to $f : (a,b) \to \mathbb{R}$.
  If $\sum_{n=1}^{\infty} f_n'$ converges (uniformly) to $g : (a,b) \to \mathbb{R}$,
  then $f : (a,b) \to \mathbb{R}$ is differentiable and

$$f' = g,$$

  that is, for all $x \in (a,b)$:

$$\left(\sum_{n=1}^{\infty} f_n\right)'(x) = \sum_{n=1}^{\infty} f_n'(x).$$

Proof. Proof is left as an exercise. ■

- **Example** Let $n \in \mathbb{N}$ and $f_n : [0,1] \to \mathbb{R}$, be given by

$$f_n(x) = \begin{cases} 4n^2 x & \text{if} \quad x \in \left[0, \frac{1}{2n}\right] \\ 4n(1-nx) & \text{if} \quad x \in \left[\frac{1}{2n}, \frac{1}{n}\right] \\ 0 & \text{if} \quad x \in \left[\frac{1}{n}, 1\right] \end{cases}.$$

- One show that

$$\int_0^1 f_n(x)\,dx = \frac{1}{2} \cdot \left(\frac{1}{n} \cdot 2n\right) = 1,$$

  for all $n \in \mathbb{N}$.

- Therefore,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1.$$

  Since $f_n(x) \to 0 = f(x)$ (pointwise) and

$$\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = 0.$$

- We see that

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx \neq \int_a^b \lim_{n\to\infty} f_n(x)\ dx.$$

- We conclude that $f_n(x) \to f(x)$ pointwise but not uniformly.

  Example Let $g_n : [-1,1] \to \mathbb{R}$,

$$g_n(x) = \frac{nx^2}{1+nx^2}.$$

  Is it true that $g : (-1,1) \to \mathbb{R}$, given by

$$g(x) = \lim_{n\to\infty} g_n(x)$$

  is differentiable?

  Solution: Let $x \in [-1,1]$.

  If $x \neq 0$, then

$$\begin{aligned} \lim_{n\to\infty} g_n(x) &= \lim_{n\to\infty} \frac{nx^2}{1+nx^2} = \lim_{n\to\infty} \frac{x^2}{\frac{1}{n}+x^2} = \frac{x^2}{\lim_{n\to\infty}\left(\frac{1}{n}\right)+x^2} \\ &= \frac{x^2}{x^2} = 1. \end{aligned}$$

  If $x = 0$, then

$$g_n(0) = 0,$$

  for all $n \in \mathbb{N}$.

- Thus,

$$\lim_{n\to\infty} g_n(x) = \begin{cases} 1 & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

- Therefore,

$$g : (-1,1) \to \mathbb{R},$$

  given by

$$g(x) = \lim_{n\to\infty} g_n(x)$$

  is not differentiable at $x = 0$.

- What type of convergence do we have for the sequence $\{g_n'\}$?
- We see that

$$g_n'(x) = \frac{d}{dx}\left(\frac{nx^2}{1+nx^2}\right) = \frac{2nx}{(nx^2+1)^2}, \text{ so}$$

for $x \in (-1,1)$,

$$\lim_{n\to\infty} g_n'(x) = \lim_{n\to\infty} \frac{2nx}{(nx^2+1)^2} = \lim_{n\to\infty} \frac{2x}{n\left(x^2+\frac{1}{n}\right)^2} = 0.$$

- Let $h : (-1,1) \to \mathbb{R}$, be given by

$$h(x) = 0,$$

for all $x \in (-1,1)$.

- Therefore,

$$g_n'(x) \to h(x) \text{ (pointwise).}$$

- We check if $g_n' \to h$ uniformly.
- Indeed,

$$\begin{aligned} g_n''(x) &= \frac{d}{dx}\left(\frac{2nx}{(nx^2+1)^2}\right) = \frac{-2n\left(3nx^2-1\right)}{(nx^2+1)^3} = 0, \text{ iff} \\ 3nx^2-1 &= 0, \text{ so } x_n = \pm\frac{1}{\sqrt{3n}}. \end{aligned}$$

- Therefore,

$$g_n'\left(\pm\frac{1}{\sqrt{3n}}\right) = \frac{2n\left(\pm\frac{1}{\sqrt{3n}}\right)}{\left(\frac{n}{3n}+1\right)^2} = \pm\frac{3}{8}\sqrt{3n}$$

- Since

$$\lim_{n\to\infty} g_n'\left(\pm\frac{1}{\sqrt{3n}}\right) = \pm\infty,$$

the sequence g_n' is not uniformly convergent h.

Example Verify that

$$\int_0^t e^x dx = e^t - 1$$

using

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \; t \in \mathbb{R}.$$

and the corollary above.

Solution: By the Weierstrass M-test,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges uniformly on every finite interval.

- Let $x > 0$ be given and consider interval $[0, x]$, then

$$\begin{aligned}
\int_0^x e^t dt &= \int_0^x \sum_{n=0}^{\infty} \frac{t^n}{n!} dt = \sum_{n=0}^{\infty} \int_0^x \frac{t^n}{n!} dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \\
&= \sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} - 1 = e^x - 1.
\end{aligned}$$

Example Find $\sum_{n=1}^{\infty} \frac{x^n}{n}$, if $|x| < 1$ and compute the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.
Solution: We know that, for all $\epsilon > 0$,
$\sum_{n=0}^{\infty} x^n$ converges uniformly to

$$\begin{aligned}
f &: [-1+\epsilon, 1-\epsilon] \to \mathbb{R}, \\
f(x) &= \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,
\end{aligned}$$

Moreover, the series representing f on

$$[-1+\epsilon, 1-\epsilon]$$

is convergent uniformly.

- Let $x \in (-1, 1)$.
  There is $\epsilon > 0$, such that $x \in [-1+\epsilon, 1-\epsilon]$, so

$$\begin{aligned}
-\ln(1-x) &= \int_0^x \frac{1}{1-t} dt = \int_0^x f(t) dt = \int_0^x \sum_{n=0}^{\infty} t^n dt \\
&= \sum_{n=0}^{\infty} \int_0^x t^n dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n}
\end{aligned}$$

- The formula above is valid for all $\epsilon > 0$, on the interval

$$[-1+\epsilon, 1-\epsilon].$$

- It follows that the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges pointwise to

$$\begin{aligned}
f &: (-1, 1) \to \mathbb{R}, \\
f(x) &= -\ln(1-x).
\end{aligned}$$

- For $x \neq 1$, since

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}.$$

after integrating both sides

$$\begin{aligned}
\int_0^x \left(\sum_{k=0}^{n} t^k \right) dt &= \sum_{k=0}^{n} \int_0^x t^k dt = \sum_{k=0}^{n} \frac{x^{k+1}}{k+1} \\
&= -\ln(1-x) - \int_0^x \frac{t^{n+1}}{1-t} dt, \text{ hence} \\
\left| \sum_{k=0}^{n} \int_0^x t^k dt + \ln(1-x) \right| &= \left| \int_0^x \frac{t^{n+1}}{1-t} dt \right|.
\end{aligned}$$

- Thus

$$\left|\sum_{k=0}^{n} \frac{x^{k+1}}{k+1} + \ln(1-x)\right| = \left|\int_0^x \frac{t^{n+1}}{1-t} dt\right|$$

- If $x = -1$, since

$$\frac{1}{1-t} \leq 1,$$

for $-1 \leq t < 0$,

$$\begin{aligned}\left|\sum_{k=0}^{n} \frac{(-1)^{k+1}}{k+1} + \ln 2\right| &\leq \left|\int_0^{-1} \frac{t^{n+1}}{1-t} dt\right| \leq \int_{-1}^{0} \left|\frac{t^{n+1}}{1-t}\right| dt \leq \int_{-1}^{0} \left|t^{n+1}\right| dt \\ &= \int_0^1 t^{n+1} dt = \frac{1}{n+2}.\end{aligned}$$

- Thus, we see that

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^{k+1}}{k+1} &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} = -\ln 2, \text{ so} \\ -\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} &= -\ln 2\end{aligned}$$

- It follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$$