HOMEWORK 3 SOLUTIONS – MATH 4341

Problem 1. Suppose $\mathcal{B}$ is a basis for a topology on a set X. Let $\mathcal{T}$ be the intersection of all topologies on X that contain $\mathcal{B}$. Show that
(a) $\mathcal{T}$ is a topology on X,
(b) $\mathcal{T}$ is equal to $\mathcal{T}_{\mathcal{B}}$, the topology generated by $\mathcal{B}$.

Proof. (a) Let $\mathbf{S}$ be the set of all topologies on X that contain $\mathcal{B}$. Then

$$\mathcal{T} = \bigcap_{\mathcal{S} \in \mathbf{S}} \mathcal{S}.$$

We will check 3 conditions for $\mathcal{T}$ to be a topology.

– (T1): Any topology on X contains both $\emptyset$ and X. Since $\mathcal{T}$ is the intersection of a family of topologies on X, it also contains both $\emptyset$ and X.

– (T2): Suppose $\{U_i\}_{i \in I}$ is an indexed family of elements of $\mathcal{T}$. Since $U_i \in \mathcal{T}$, for any $\mathcal{S} \in \mathbf{S}$ we have $U_i \in \mathcal{S}$. Since $\mathcal{S}$ is a topology, $(\bigcup_{i \in I} U_i) \in \mathcal{S}$. This holds true for all $\mathcal{S} \in \mathbf{S}$. Hence $(\bigcup_{i \in I} U_i) \in (\bigcap_{\mathcal{S} \in \mathbf{S}} \mathcal{S}) = \mathcal{T}$.

– (T3): Suppose $U_1, \cdots, U_n$ are elements of $\mathcal{T}$. Since $U_1, \cdots, U_n \in \mathcal{T}$, for any $\mathcal{S} \in \mathbf{S}$ we have $U_1, \cdots, U_n \in \mathcal{S}$. Since $\mathcal{S}$ is a topology, $(U_1 \cap \cdots \cap U_n) \in \mathcal{S}$. This holds true for all $\mathcal{S} \in \mathbf{S}$. Hence $(U_1 \cap \cdots \cap U_n) \in (\bigcap_{\mathcal{S} \in \mathbf{S}} \mathcal{S}) = \mathcal{T}$.

(b) Since $\mathcal{T}$ be the intersection of all topologies on X that contain $\mathcal{B}$ and $\mathcal{T}_{\mathcal{B}}$ is a topology on X containing $\mathcal{B}$, we have $\mathcal{T} \subset \mathcal{T}_{\mathcal{B}}$.

Take any $U \in \mathcal{T}_{\mathcal{B}}$. Since $\mathcal{B}$ is a basis for $\mathcal{T}_{\mathcal{B}}$ we can write $U = \bigcup_{i \in I} B_i$ for some $B_i \in \mathcal{B}$. Since $B_i \in \mathcal{B} \subset \mathcal{T}$ and $\mathcal{T}$ is closed under taking unions ($\mathcal{T}$ is a topology on X by part (a)), we have $U = \bigcup_{i \in I} B_i \in \mathcal{T}$. This implies that $\mathcal{T}_{\mathcal{B}} \subset \mathcal{T}$. Hence $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$ as desired. $\square$

Problem 2. Let (X, d) be a metric space and

$$\mathcal{B} = \{B_d(x, 2^{-n}) \mid x \in X, n \in \mathbb{N}\}.$$

Show that
(a) $\mathcal{B}$ is a basis for a topology on X,
(b) the topology generated by $\mathcal{B}$ is equal to the metric topology on X.

Proof. We will apply Lemma 2.4 in the lecture notes to prove both (a) and (b). Recall that the metric topology $\mathcal{T}_d$ is generated by the basis consisting of all open balls $B_d(x, r)$ where $x \in X$ and $r > 0$. Clearly, we have $\mathcal{B} \subset \mathcal{T}_d$.

For any $U \in \mathcal{T}_d$ and $x \in U$, there exists $r > 0$ such that $B_d(x, r) \subset U$. Since $\lim_{n \to \infty} 2^{-n} = 0 < r$, we can choose $n_0 \in \mathbb{N}$ such that $2^{-n_0} < r$. Then $B_d(x, 2^{-n_0}) \subset B_d(x, r) \subset U$. Hence, by Lemma 2.4 in the lecture notes, $\mathcal{B}$ is a basis for $\mathcal{T}_d$. This means that both (a) and (b) hold true. $\square$

Problem 3. If $f : X \to Y$ is a function between two sets and $A \subset Y$ is a subset, we define the *preimage* of A to be

$$f^{-1}(A) = \{x \in X \mid f(x) \in A\}.$$

Show that:

(1) If $f : X \to Y$ and $\{A_i\}_{i\in I}$ is a family of subsets of Y, then
$$f^{-1}\Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I} f^{-1}(A_i) \qquad \text{and} \qquad f^{-1}\Big(\bigcap_{i\in I} A_i\Big) = \bigcap_{i\in I} f^{-1}(A_i).$$
(2) If $A \subset Y$, then $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$.
(3) If $g : Y \to Z$ is another map and $B \subset Z$, then
$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B)).$$

Proof. (1) We have $x \in f^{-1}\big(\bigcup_{i\in I} A_i\big) \Leftrightarrow f(x) \in \bigcup_{i\in I} A_i \Leftrightarrow \big(\exists i \in I : f(x) \in A_i\big) \Leftrightarrow \big(\exists i \in I : x \in f^{-1}(A_i)\big) \Leftrightarrow x \in \bigcup_{i\in I} f^{-1}(A_i)$. Hence $f^{-1}\big(\bigcup_{i\in I} A_i\big) = \bigcup_{i\in I} f^{-1}(A_i)$.

We have $x \in f^{-1}\big(\bigcap_{i\in I} A_i\big) \Leftrightarrow f(x) \in \bigcap_{i\in I} A_i \Leftrightarrow \big(\forall i \in I : f(x) \in A_i\big) \Leftrightarrow \big(\forall i \in I : x \in f^{-1}(A_i)\big) \Leftrightarrow x \in \bigcap_{i\in I} f^{-1}(A_i)$. Hence $f^{-1}\big(\bigcap_{i\in I} A_i\big) = \bigcap_{i\in I} f^{-1}(A_i)$.

(2) We have $x \in f^{-1}(Y \setminus A) \Leftrightarrow f(x) \in Y \setminus A \Leftrightarrow f(x) \notin A \Leftrightarrow x \notin f^{-1}(A) \Leftrightarrow x \in X \setminus f^{-1}(A)$. Hence $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$.

(3) We have $x \in (g \circ f)^{-1}(B) \Leftrightarrow (g \circ f)(x) \in B \Leftrightarrow g(f(x)) \in B \Leftrightarrow f(x) \in g^{-1}(B) \Leftrightarrow x \in f^{-1}(g^{-1}(B))$. Hence $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$. $\square$