Math 4301 Mathematical Analysis I

Lecture 10

Topic: Continuous functions

- **Continuous Functions**

  **Definition** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $c \in A$ we say that $f$ is *continuous at* $c$

i) if $c$ is an isolated point of $A$

ii) if $c$ is an *accumulation point* of $A$ and $\lim_{x\to c} f(x) = f(c)$.

  We say that $f$ *is continuous on* $A$ if $f$ is continuous at each point $c \in A$.

  **Example**: Let

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$$

  and $y_n \to y_0$ and $n \to \infty$.

  Define $f : A \to \mathbb{R}$ by

$$f(x) = \begin{cases} y_n & if \quad x = \frac{1}{n} \\ y_0 & if \quad x = 0 \end{cases}$$

  We show that $f$ is continuous on $A$.

- Each point $\frac{1}{n} \in A$ is an isolated point of $A$,
  so $f$ is continuous by the definition.
- We show that $f$ is also continuous at $c = 0$.
- Notice that $c = 0$ is an accumulation point of $A$.
- It suffices to show that

$$\lim_{x\to 0} f(x) = y_0.$$

- Let $\epsilon > 0$ **be given**.
  Since $y_n \to y_0$ as $n \to \infty$,
  there is $N \in \mathbb{N}$, such that, for $n > N$,

$$|y_n - y_0| < \epsilon.$$

- **Let** $\delta = \frac{1}{N} > 0$**.**
- Notice that, **for every** $x \in A$,
  **if** $0 < |x - 0| < \delta$, **then**

$$x = \frac{1}{n} < \delta = \frac{1}{N}.$$

- Therefore, $n > N$, so

$$|f(x) - f(0)| = \left| f\left(\frac{1}{n}\right) - y_0 \right| = |y_n - y_0| < \boldsymbol{\epsilon}.$$

- It follows that $\lim_{x\to 0} f(x) = y_0$.

- Therefore, $f$ is continuous on $A$.

  **Theorem** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $c \in A$.

  The function $f$ is continuous at $c$ iff

  for every $\epsilon > 0$, there is $\delta > 0$, such that,

  for all $x \in A$, if $|x - c| < \delta$ then

$$|f(x) - f(c)| < \epsilon.$$

**Proof.** We show that both conditions are equivalent.

- **Assume that $f$ is continuous at $c \in A$ and let $\epsilon > 0$ be given.**
- *If $c$ is an isolated point of $A$,*

  then there is $\delta > 0$, such that

$$D(c, \delta) \cap A = \{c\}.$$

  **Example**: $A = (1, 2) \cup \{3\}$ then $x = 3$ is an isolated point.

- Therefore, for all $x \in A$,

  if $|x - c| < \delta$, then $x = c$, so

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon.$$

- *Assume that $x \in A'$ (i.e. $c$ is an accumulation point of $A$).*
- Since $f$ is continuous at $c$,

$$\lim_{x \to c} f(x) = f(c),$$

  there is $\delta > 0$, such that, for all $x \in A$, if

$$0 < |x - c| < \delta,$$

  then

$$|f(x) - f(c)| < \epsilon.$$

- Assume that $x \in A$ and $|x - c| < \delta$.
- If $x \neq c$, then $0 < |x - c| < \delta$, so

$$|f(x) - f(c)| < \epsilon.$$

- If $x = c$, then clearly,

$$|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon.$$

- We showed, that if $f$ is continuous at $c$,

  then for all $\epsilon > 0$, we can find $\delta > 0$, such that, for every $x \in A$, if

$$|x - c| < \delta,$$

  then

$$|f(x) - f(c)| < \epsilon.$$

- **Conversely, assume that the $(\epsilon - \delta)$ condition holds.**
- If $c$ is an isolated point of $A$,

  then by the definition $f$ is continuous at $c$.

- Assume that $c \in A' \cap A$ (i.e. $c$ is an accumulation point of $A$ that is also in $A$).
- We show that
$$\lim_{x \to c} f(x) = f(c).$$
- Let $\epsilon > 0$ **be given**.
Since $(\epsilon - \delta)$ condition holds,
there is $\delta > 0$, such that, for all $x \in A$, if
$$|x - c| < \delta$$
then
$$|f(x) - f(c)| < \epsilon.$$
- In particular, for all $x \in A$,
if $0 < |x - c| < \delta$ then
$$|f(x) - f(c)| < \epsilon.$$
- It follows that
$$\lim_{x \to c} f(x) = f(c).$$

This finishes our proof. ■

- Here is another topological reformulation of continuity.\
**Theorem** A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at $c \in A$ iff
for every neighborhood $V$ of $f(c)$,
there is a neighborhood $U$ of $c$, such that, for all $x \in A \cap U$,
$$f(x) \in V.$$

**Proof.** We show that both conditions are equivalent.

- **Assume that** $f : A \subseteq \mathbb{R} \to \mathbb{R}$ **is continuous at** $c \in A$ **and let** $V$ **be a neighborhood of** $f(c)$.
- Since $V$ is a neighborhood and $f(c) \in V$, there is
$$D(f(c), \epsilon) = (f(c) - \epsilon, f(c) + \epsilon) \subseteq V.$$
- Since $f$ is continuous at $c$,
there is $\delta > 0$, such that, for all $x \in A$,
if $|x - c| < \delta$ then
$$|f(x) - f(c)| < \epsilon.$$
- Let
$$U = (c - \delta, c + \delta) = D(c, \delta).$$
- Then, for all, $x \in A \cap U$,
$$|x - c| < \delta.$$

- Hence, for all $x \in A \cap U$,
$$|f(x) - f(c)| < \epsilon.$$

- It follows that for all $x \in A \cap U$,
$$f(x) \in (f(c) - \epsilon, f(c) + \epsilon) \subseteq V.$$

- Since $(f(c) - \epsilon, f(c) + \epsilon) \subseteq V$,
it follows that, for all $x \in A \cap U$,
$$f(x) \in V.$$

- **Conversely, assume that for every neighborhood $V$ of $f(c)$,**
**there is a neighborhood $U$ of $c$, such that,**
**for all $x \in A \cap U$,**
$$f(x) \in V.$$

- We show that $f$ is continuous at $c$.

- Since
$$V = (f(c) - \epsilon, f(c) + \epsilon)$$
a neighborhood of $f(c)$,
then there is a neighborhood $U$ of $c$, such that,
for all $x \in A \cap U$, $f(x) \in V$.

- Since $U$ is a neighborhood of $c$,
there is $\delta > 0$, such that
$$(c - \delta, c + \delta) \subseteq U.$$

- Notice that: if $x \in A$ and $x \in (c - \delta, c + \delta)$ i.e. $|x - c| < \delta$, then
$$x \in (c - \delta, c + \delta) \cap A \subseteq U \cap A.$$

- Therefore,
$$f(x) \in (f(c) - \epsilon, f(c) + \epsilon) = V,$$
so for all $x \in A$,
if $|x - c| < \delta$, then
$$|f(x) - f(c)| < \epsilon.$$

This finishes our proof. ■

- **Example**: Let $f : \mathbb{R} \to \mathbb{R}$,
$$f(x) = x^2 - 2x + 3.$$
We show that $f$ is continuous at each $c \in \mathbb{R}$.

- Let $\epsilon > 0$ be given.

- If $|x - c| < \delta$, then
$$\begin{aligned} |f(x) - f(c)| &= \left|\left(x^2 - 2x + 3\right) - \left(c^2 - 2c + 3\right)\right| = \left|x^2 - c^2 - 2x + 2c\right| \\ &= |(x - c)(x + c) - 2(x - c)| = |x - c|\,|x + c - 2|. \end{aligned}$$

- If $|x-c|<\delta$, then
$$\begin{aligned} |x+c-2| &= |x-c+2c-2| = |(x-c)+2(c-1)| \\ &\leq |x-c|+2|c-1| \leq \delta+2|c-1|. \end{aligned}$$
- Therefore, if $\delta<1$,
$$|x+c-2| \leq \delta+2|c-1| < 1+2|c-1|.$$
- It follows that, if $\delta<1$, and $|x-c|<\delta$, then
$$|f(x)-f(c)| = |x-c||x+c-2| < \delta(1+2|c-1|).$$
- If we take
$$\delta = \frac{1}{2}\min\left\{1, \frac{\epsilon}{1+2|c-1|}\right\},$$
then for $x\in\mathbb{R}$, if $|x-c|<\delta$, then
$$\begin{aligned} |f(x)-f(c)| &= |x-c||x+c-2| < \delta(1+2|c-1|) \\ &\leq \frac{\epsilon}{1+2|c-1|}(1+2|c-1|) = \epsilon. \end{aligned}$$
- Hence $f$ is continuous at $c$.
**Exercise**: Show that $f:\mathbb{R}\to\mathbb{R}$,
$$f(x)=\sin(x)$$
is continuous at $c\in\mathbb{R}$.
**Example**: We show that
$$\begin{aligned} f &: \mathbb{R}\setminus\{0\}\to\mathbb{R}, \\ f(x) &= \frac{1}{x} \end{aligned}$$
is continuous at $c\in\mathbb{R}\setminus\{0\}$.
- We notice that, since $c\in\mathbb{R}\setminus\{0\}$,
there is $\delta=|c|>0$, such that,
for all $x\in(c-\delta,c+\delta)$, $x\neq 0$.
- Assume that $|x-c|<\delta$, then
$$|f(x)-f(c)| = \left|\frac{1}{x}-\frac{1}{c}\right| = \frac{|x-c|}{|x||c|}$$
- If $\delta<\frac{|c|}{2}$, then, for $x\in(c-\delta,c+\delta)$,
$$|x-c|<\delta<\frac{|c|}{2}$$
and
$$|x| = |(x-c)+c| \geq |c|-|x-c| > |c|-\delta > |c|-\frac{|c|}{2} = \frac{|c|}{2}.$$

- It follows that, if $x \in \mathbb{R}\setminus\{0\}$, $|x-c|<\delta$ and $\delta < \frac{|c|}{2}$, then

$$|f(x)-f(c)| = \left|\frac{1}{x}-\frac{1}{c}\right| = \frac{|x-c|}{|x||c|} \le \frac{2|x-c|}{|c|^2} < \frac{2}{|c|^2}\delta.$$

- Therefore, if we take

$$\delta = \min\left\{\frac{|c|}{2}, \frac{|c|^2\epsilon}{2}\right\} > 0,$$

- we see that, for $x \in \mathbb{R}\setminus\{0\}$,
  if $|x-c|<\delta$ then

$$|f(x)-f(c)| \le \frac{2|x-c|}{|c|^2} < \frac{2}{|c|^2}\delta \le \frac{2}{|c|^2}\frac{|c|^2\epsilon}{2} = \epsilon.$$

**Exercise**: Show that $f:[0,\infty)\to\mathbb{R}$,

$$f(x)=\sqrt{x}$$

is continuous at $c\in[0,\infty)$.

- Let us remark on yet another equivalent conditions for continuity.
- Let $f:\mathbb{R}\to\mathbb{R}$, $c\in\mathbb{R}$ and $\delta>0$ be given.
- Consider an open disk $D(c,\delta)$ and define the *oscillation* of $f$ on $D(c,\delta)$ as follows

$$\omega_f(D(c,\delta)) = \sup\{|f(x)-f(y)| \mid x,y\in D(c,\delta)\}$$

and the *oscillation of $f$ at $c$* by

$$\omega_f(c) = \inf\{\omega_f(D(c,\delta)) \mid \delta>0\}.$$

**Example**: Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1 & \text{if} \quad x<0 \\ 2 & \text{if} \quad x\ge 0 \end{cases}$$

- Take $D(0,\delta)=(-\delta,\delta)$, then

$$x=-\frac{\delta}{2}\in(-\delta,\delta) \text{ and } y=\frac{\delta}{2}\in(-\delta,\delta)$$

- Since $x<0$ and $y>0$,

$$f(x)=-1 \text{ and } f(y)=2,$$

so

$$|f(x)-f(y)| = |-1-2| = 3$$

- Therefore,

$$\begin{aligned} 3 &\le |f(x)-f(y)| \le \sup\{|f(x)-f(y)| \mid x,y\in D(0,\delta)\} \\ &= \omega_f(D(0,\delta)). \end{aligned}$$

- One needs to show that
$$\omega_f\left(D\left(0,\delta\right)\right) \leq 3 \text{ – Please think about it.}$$
- Consider cases:
- **a**) $x, y < 0$,
- **b**) $x, y \geq 0$ **c**) $x < 0 \leq y$ and compute $\left|f(x) - f(y)\right|$...
- Therefore,

$$\begin{aligned} \omega_f\left(D\left(0,\delta\right)\right) &= \sup\left\{\left|f(x) - f(y)\right| \ \mid x, y \in D\left(0,\delta\right)\right\} \\ &= 3 \end{aligned}$$

so

$$\begin{aligned} \omega_f(0) &= \inf\left\{\omega_f\left(D\left(0,\delta\right)\right) \ \mid \delta > 0\right\} = \inf\left\{3\right\} \\ &= 3. \end{aligned}$$

**Proposition** Function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $c \in \mathbb{R}$ iff $\omega_f(c) = 0$.

**Proof.** We show that both conditions are equivalent.

- **Assume that $f$ is continuous at $c$.**
- It is sufficient to show that for all $\epsilon > 0$,
$$0 \leq \omega_f(c) < \epsilon.$$
- Since
$$\omega_f(c) = \inf\left\{\omega_f\left(D\left(c,\delta\right)\right) \ \mid \delta > 0\right\},$$
- it is sufficient to show that, for $\epsilon > 0$,
there is $\delta > 0$, such that
$$\omega_f\left(D\left(c,\delta\right)\right) < \epsilon.$$
- Since $f$ is continuous at $c$, there is $\delta > 0$, such that,
for all $x \in \mathbb{R}$, if $|x - c| < \delta$, then
$$\left|f(x) - f(c)\right| < \frac{\epsilon}{4}.$$
- That is, for all $x \in D\left(c,\delta\right)$,
$$\left|f(x) - f(c)\right| < \frac{\epsilon}{4}.$$
- Since, for all $x, y \in D\left(c,\delta\right)$,
$$\begin{aligned} \left|f(x) - f(y)\right| &= \left|f(x) - f(c) + f(c) - f(y)\right| \\ &\leq \left|f(x) - f(c)\right| + \left|f(y) - f(c)\right| \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2} \end{aligned}$$
- it follows that, for all $x, y \in D\left(c,\delta\right)$,
$$\left|f(x) - f(y)\right| < \frac{\epsilon}{2}.$$

- Therefore,
$$\omega_f\left(D\left(c,\delta\right)\right)=\sup\left\{\left|f\left(x\right)-f\left(y\right)\right| \ \mid x,y\in D\left(c,\delta\right)\right\}\leq\frac{\epsilon}{2}<\epsilon,$$
so $\omega_f\left(D\left(c,\delta\right)\right)<\epsilon$.
- Hence, we showed that
$$\omega_f\left(c\right)=\inf\left\{\omega_f\left(D\left(c,\delta\right)\right) \ \mid \delta>0\right\}<\epsilon$$
for all $\epsilon>0$.
- It follows that $\omega_f\left(c\right)=0$.
- **Conversely, assume that $\omega_f\left(c\right)=0$ and $\epsilon>0$ be given.**
- We show that $f$ is continuous at $x=c$.
- Since
$$\omega_f\left(c\right)=\inf\left\{\omega_f\left(D\left(c,\delta\right)\right) \ \mid \delta>0\right\}=0,$$
- there is $\delta>0$, such that
$$\omega_f\left(D\left(c,\delta\right)\right)<\omega_f\left(c\right)+\epsilon=\epsilon,$$
that is, there is $\delta>0$, such that
$$\omega_f\left(D\left(c,\delta\right)\right)<\epsilon.$$
- Therefore, for all $x,y\in D\left(c,\delta\right)$,
$$\left|f\left(x\right)-f\left(y\right)\right|<\epsilon,$$
- In particular, for all
$$x\in D\left(c,\delta\right), \ \left|f\left(x\right)-f\left(c\right)\right|<\epsilon.$$
- That is, we showed, that for any $\epsilon>0$,
there is $\delta>0$, such that for all $x\in\mathbb{R}$,
if $\left|x-c\right|<\delta$, then
$$\left|f\left(x\right)-f\left(c\right)\right|<\epsilon.$$
- This shows that $f$ is continuous at $c$.

This finishes our proof. ■

- **Example**: Let $f:\mathbb{R}\to\mathbb{R}$ be defined by
$$f\left(x\right)=\begin{cases}1 & if \quad x\in\mathbb{Q}\\ 0 & if \quad x\in\mathbb{R}\backslash\mathbb{Q}\end{cases}.$$
We show that $f$ is not continuous at each $c\in\mathbb{R}$.
- Indeed, if $c\in\mathbb{R}$, for every $\delta>0$,
$$D\left(c,\delta\right)=\left(c-\delta,c+\delta\right)$$
contains both rational and the irrational numbers, that is,
$$D\left(c,\delta\right)\cap\mathbb{Q}\neq\emptyset \text{ and } D\left(c,\delta\right)\cap\left(\mathbb{R}\backslash\mathbb{Q}\right)\neq\emptyset.$$

- Let $x \in D(c,\delta) \cap \mathbb{Q}$ and $y \in D(c,\delta) \cap (\mathbb{R}\backslash\mathbb{Q})$, then

$$1 = |f(x) - f(y)| \leq \omega_f(D(c,\delta)).$$

- It follows that,

$$\omega_f(c) = \inf\{\omega_f(D(c,\delta)) \mid \delta > 0\} \geq 1,$$

so $f$ is not continuous at $c \in \mathbb{R}$.

**Exercise**: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \left\{ \begin{array}{lll} \frac{1}{q} & if & x = \frac{p}{q},\ \gcd(p,q) = 1,\ p \in \mathbb{Z},\ q \in \mathbb{N} \\ 0 & if & x \in \mathbb{R}\backslash\mathbb{Q} \cup \{0\} \end{array} \right. .$$

Show that $f$ is continuous at $c$ iff $c \in \mathbb{R}\backslash\mathbb{Q}$.

We start by showing that $f$ is **not continuous at each** $c \in \mathbb{Q}\backslash\{0\}$.

- Let $\delta > 0$, then for each $\frac{p}{q}$,

$$\gcd(p,q) = 1,\ p,q \in \mathbb{Z},\ q \neq 0,$$

disk $D\left(\frac{p}{q},\delta\right)$ and $\mathbb{R}\backslash\mathbb{Q}$ intersect non-empty, i.e.

$$D\left(\frac{p}{q},\delta\right) \cap \mathbb{R}\backslash\mathbb{Q} = \left(\frac{p}{q} - \delta, \frac{p}{q} + \delta\right) \cap \mathbb{R}\backslash\mathbb{Q} \neq \emptyset.$$

- Let $x \in D\left(\frac{p}{q},\delta\right) \cap \mathbb{R}\backslash\mathbb{Q}$, then

$$\frac{1}{q} = \left|0 - \frac{1}{q}\right| = \left|f(x) - f\left(\frac{p}{q}\right)\right| \leq \omega_f\left(D\left(\frac{p}{q},\delta\right)\right).$$

- It follows that

$$\omega_f\left(\frac{p}{q}\right) = \inf\left\{\omega_f\left(D\left(\frac{p}{q},\delta\right)\right) \mid \delta > 0\right\} \geq \frac{1}{q} > 0,$$

so $f$ is not continuous at $\frac{p}{q}$.

- **We show that $f$ is continuous at $c \in \mathbb{R}\backslash\mathbb{Q}$.**
- It suffices to show that, for $\epsilon > 0$,

$$\omega_f(c) < \epsilon.$$

- Since by the definition

$$\omega_f(c) = \inf\{\omega_f(D(c,\delta)) \mid \delta > 0\},$$

it is sufficient to show that, there is, $\delta > 0$, such that,

$$\omega_f(D(c,\delta)) < \frac{\epsilon}{2}.$$

- Since $\frac{\epsilon}{2} > 0$, there is $n \in \mathbb{N}$, such that, $0 < \frac{1}{n} < \frac{\epsilon}{2}$.
- For each $1 \leq q \leq n$, let

$$S_q = \left\{\frac{p}{q} : p \in \mathbb{Z} \text{ and } \gcd(p,q) = 1\right\} \cap (c-1, c+1).$$

- We see that $S_q$ must be **finite**, otherwise

$$c - 1 < \frac{p}{q} < c + 1,$$

for infinitely many $p \in \mathbb{Z}$, such that $\gcd(p, q) = 1$.
That is,

$$(c - 1)\, q < p < (c + 1)\, q,$$

for infinitely many $p \in \mathbb{Z}$, $\gcd(p, q) = 1$ which is impossible.

- It follows that

$$S = \bigcup_{q=1}^{n} S_q$$

is also **finite**.

- Since $c \in \mathbb{R} \backslash \mathbb{Q}$, for all $x \in S$,

$$|x - c| > 0.$$

- Therefore,

$$\delta = \min\{|x - c| : x \in S\} > 0.$$

and since $S \subset (c - 1, c + 1)$,

$$\delta < 1$$

- Consider

$$D(c, \delta) = (c - \delta, c + \delta).$$

- If $x \in \mathbb{R} \backslash \mathbb{Q} \cap D(c, \delta)$, then

$$f(x) = 0 < \epsilon.$$

- Now, if $x = \frac{p}{q}$ then either $1 \leq q \leq n$ or $q > n$.
- If $1 \leq q \leq n$, then

$$x \notin D(c, \delta)$$

by the definition of $\delta > 0$.

- Therefore, $q > n$, so

$$f(x) = \frac{1}{q} < \frac{1}{n} < \frac{\epsilon}{2}.$$

- It follows that,

$$\text{for all } x, y \in D(c, \delta),\ \ |f(x) - f(y)| < \frac{\epsilon}{2}$$

and hence

$$\omega_f(D(c, \delta)) < \frac{\epsilon}{2}.$$

- Therefore,

$$0 \leq \omega_f(c) = \inf\{\omega_f(D(c, \delta))\ \mid \delta > 0\} < \frac{\epsilon}{2} < \epsilon.$$

- We showed that, for every $\epsilon > 0$,

$$\omega_f(c) < \epsilon,$$

so $\omega_f(c) = 0$.

- It follows that $f$ is continuous at $c \in \mathbb{R} \backslash \mathbb{Q}$.
- Finally, if $c = 0$, since
$$0 \leq f(x) \leq |x|$$
we have
$$0 \leq \lim_{x \to 0} f(x) \leq \lim_{x \to 0} |x| = 0$$
then
$$\lim_{x \to 0} f(x) = 0 = f(0),$$
so $f$ is continuous at $c = 0$.

**Proposition** Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be continuous at $c \in A$. Then

i) $\alpha f + \beta g$ is continuous at $c$;

ii) $f \cdot g$ is continuous at $c$;

iii) $\frac{f}{g}$ is continuous at $c$ provided that $g(c) \neq 0$.

**Proof.** The statement follows from theorem about limits ■

- **Proposition** If $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $g : B \subseteq \mathbb{R} \to \mathbb{R}$ are continuous and $f(A) \subseteq B$, then $g \circ f : A \to \mathbb{R}$ is continuous.

**Proof.** We show that, for every open subset $W \subseteq \mathbb{R}$, such that $(g \circ f)(c) \in W$, $(g \circ f)^{-1}(W)$ is open in $A$.

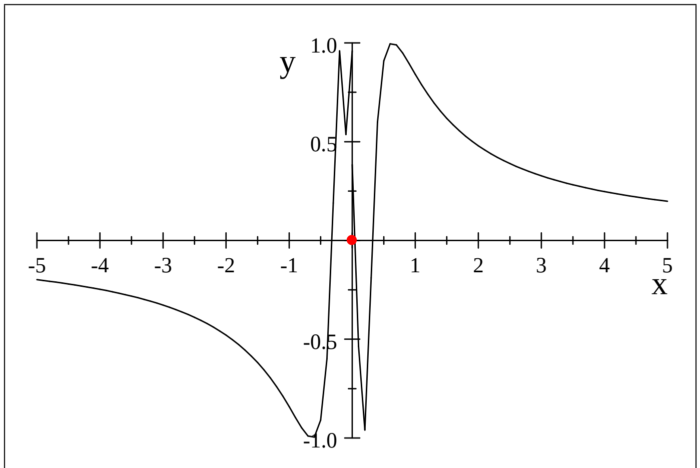
- Let $c \in A$.
- We see that if $W \subseteq \mathbb{R}$ is a neighborhood of $(g \circ f)(c)$,
  then since $g$ is continuous,
  there is a neighborhood $V \subseteq \mathbb{R}$,
  such that $f(c) \in V$ and for every $y \in V \cap B$,
$$g(y) \in W.$$
- Since $f(A) \subseteq B$ and $f(c) \in B$,
  then $V$ is a neighborhood of $f(c)$.
- Since $f$ is continuous,
  there is a neighborhood $U$ of $c$, such that, for all $x \in U \cap A$,
$$f(x) \in V \cap B.$$
  Therefore, $g(f(x)) \in W$.
- We showed that, for all $x \in U \cap A$,
$$(g \circ f)(x) \in W.$$
- It follows that $g \circ f$ is continuous at each $c \in A$, so $g \circ f$ is continuous on $A$.

This finishes our proof. ■

- **Exercise**: Show that $f:\mathbb{R}\to\mathbb{R}$ given by

$$f(x)=\begin{cases}\sin\left(\frac{1}{x}\right) & if \quad x\neq 0\\ 0 & if \quad x=0\end{cases}$$

is continuous on $\mathbb{R}\setminus\{0\}$ and it is not continuous at $c=0$.




$y=\sin\left(\frac{1}{x}\right),\ x\neq 0$

Let $\delta>0$ and consider $D(0,\delta)$.

- We would like to find

$$\omega_f(D(0,\delta))$$

**Idea**: Notice that

$$\begin{aligned}\sup\left\{\sin\left(\frac{1}{x}\right):x\in D(0,\delta)\right\} &= 1 \text{ and}\\ \inf\left\{\sin\left(\frac{1}{x}\right):x\in D(0,\delta)\right\} &= -1\end{aligned}$$

- Therefore,

$$\omega_f(D(0,\delta))\leq|1-(-1)|=2$$

- Notice that,

$$\begin{aligned}x_n &= \frac{1}{\frac{\pi}{2}+2n\pi}\to 0 \text{ as } n\to\infty \text{ and}\\ y_n &= \frac{1}{-\frac{\pi}{2}+2n\pi}\to 0 \text{ as } n\to\infty\end{aligned}$$

- Therefore, there is $N\in\mathbb{N}$, such that $x_N,y_N\in(-\delta,\delta)$.
- Therefore,

$$\begin{aligned}2 &= |1-(-1)|=\left|\sin\left(\frac{\pi}{2}+2N\pi\right)-\sin\left(-\frac{\pi}{2}+2N\pi\right)\right|\\ &= \left|\sin\left(\frac{1}{x_N}\right)-\sin\left(\frac{1}{y_N}\right)\right|=|f(x_N)-f(y_N)|\leq\omega_f(D(0,\delta))\end{aligned}$$

We showed that

$$\omega_f(D(0,\delta))=2,\ \text{for every } \delta>0.$$

- Consequently, we see that

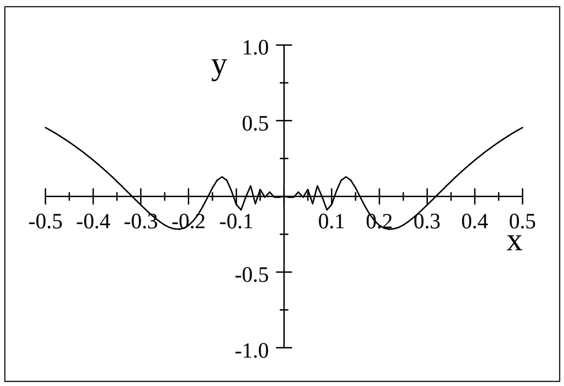
$$\begin{aligned}\omega_f(0) &= \inf\{\omega_f(D(0,\delta)) \mid \delta > 0\} \\ &= \inf\{2\} = 2 > 0.\end{aligned}$$

- By theorem, $f$ is not continuous at $x = 0$.
- To finish our proof, one shows that if $x \neq 0$, then $\omega_f(x) = 0$.
- We leave this part as an exercise.

  **Exercise**: Show that $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

  is continuous.




  **Remark** We notice that using the oscillation $\omega_f$ of $f : \mathbb{R} \to \mathbb{R}$ we can define the set of its discontinuities as

$$D_f = \{x \in \mathbb{R} : \omega_f(x) > 0\}.$$

  **Definition** Let $f : \mathbb{R} \to \mathbb{R}$ then, for every $\epsilon > 0$ the set

$$G_\epsilon(f) = \{x \in \mathbb{R} : \omega_f(x) < \epsilon\}$$

  is open in $\mathbb{R}$.

**Proof.** We show that, for every $y \in G_\epsilon(f)$, there is $\delta > 0$, such that

$$D(y,\delta) \subseteq G_\epsilon(f).$$

- Let $y \in G_\epsilon(f)$.
- Then

$$\omega_f(y) = \inf\{\omega_f(D(y,\delta)) \mid \delta > 0\} < \epsilon.$$

- It follows that, there is $\delta > 0$, such that $\omega_f(D(y,\delta)) < \epsilon$.
- Let $z \in D(y,\delta)$, $z \neq y$ and

$$\eta = \delta - |y - z| > 0.$$

- Then

$$\begin{aligned} D(z,\eta) &\subseteq D(y,\delta), \text{ hence} \\ \omega_f(D(z,\eta)) &\leq \omega_f(D(y,\delta)) < \epsilon. \end{aligned}$$

- It follows that

$$\omega_f(z) = \inf\{\omega_f(D(z,\alpha)) \mid \alpha > 0\} \leq \omega_f(D(y,\delta)),$$

  hence

$$\omega_f(z) \leq \omega_f(y) < \epsilon,$$

  and therefore, $z \in G_\epsilon(f)$, for all $z \in D(y,\delta)$.

- Consequently, $D(y,\delta) \subseteq G_\epsilon(f)$, so $G_\epsilon(f)$ is open in $\mathbb{R}$.

This finishes our proof. ■

- **Remark** We note that if $A \subseteq \mathbb{R}$ is bounded then also its closure $\overline{A}$ is bounded.
- Since $A$ is bounded, there is $R > 0$, such that $A \subset R(0,R)$, so by the property of closure, we see that

$$\overline{A} \subseteq \overline{D(0,R)} = \{x \in \mathbb{R}^n : |x| \leq R\}$$

- Since $\overline{D(0,R)}$ is bounded, then $\overline{A}$ is also bounded.

  **Corollary** Let $A \subseteq \mathbb{R}$ be bounded and $f : A \to \mathbb{R}$.

  Define $\widetilde{f} : \mathbb{R} \to \mathbb{R}$, by

$$\widetilde{f}(x) = \begin{cases} f(x) & if \quad x \in A \\ 0 & if \quad x \in \mathbb{R}\backslash A \end{cases}$$

  Then the set

$$D_\epsilon\left(\widetilde{f}\right) = \left\{x \in \mathbb{R} : \omega_{\widetilde{f}}(x) \geq \epsilon\right\}$$

  is compact in $\mathbb{R}$.