HOMEWORK 7 SOLUTIONS – MATH 4341

Problem 1. (a) Construct an explicit homeomorphism from any open interval $(a, b) \subset \mathbb{R}$ to the open interval $(-1, 1)$.

(b) Construct an explicit homeomorphism from any open interval $(a, b) \subset \mathbb{R}$ to $\mathbb{R}$.

Proof. (a) Define $f : (a, b) \to \mathbb{R}$ by $f(x) = 2\frac{x-a}{b-a} - 1$ for all $x \in (a, b)$. Since $a < x < b$, we have $0 < \frac{x-a}{b-a} < 1$ which implies that $f(x) \in (-1, 1)$. So $f : (a, b) \to (-1, 1)$ is a continuous function.

For every $y \in (-1, 1)$, the equation $y = f(x) = 2\frac{x-a}{b-a} - 1$ has a unique solution $x = a + \frac{1}{2}(b - a)(y + 1)$. Since $-1 < y < 1$, we have $a < x < b$, i.e. $x \in (a, b)$.

Hence $f : (a, b) \to (-1, 1)$ is a continuous bijective function, and $f^{-1}(y) = a + \frac{1}{2}(b - a)(y + 1)$. Since $f^{-1} : (-1, 1) \to (a, b)$ is continuous, f is a homeomorphism.

(b) We already know that $g : (-1, 1) \to \mathbb{R}$ given by $g(x) = \tan\left(\frac{\pi}{2}x\right)$ is a homeomorphism. Hence $g \circ f : (a, b) \to \mathbb{R}$ given by $(g \circ f)(x) = \tan\left(\pi\left(\frac{x-a}{b-a} - \frac{1}{2}\right)\right)$ is a homeomorphism from (a, b) to $\mathbb{R}$. □

Problem 2. Let $C \subset \mathbb{R}^2$ be the unit circle $x^2 + y^2 = 1$. Construct an explicit homeomorphism from $C \setminus \{(1, 0)\}$ to $\mathbb{R}$.

Proof. Let $f : C \setminus \{(1, 0)\} \to \mathbb{R}$ be defined by $f((x, y)) = \frac{y}{1-x}$. Note that $(0, \frac{y}{1-x})$ is the point of intersection of the y-axis and the line passing through the points $(1, 0)$ and (x, y). [Explanation: The equation of the line passing through the points $(1, 0)$ and (x, y) is $(1 - t)(1, 0) + t(x, y) = (1 - t + tx, ty)$. At the point of intersection with the y-axis we have $1 - t + tx = 0 \Longrightarrow t = \frac{1}{1-x}$, so the y-coordinate is $ty = \frac{y}{1-x}$.]

f is bijective, since its inverse is given by $f^{-1}(z) = (\frac{z^2-1}{z^2+1}, \frac{2z}{z^2+1})$ for $z \in \mathbb{R}$. Note that $(\frac{z^2-1}{z^2+1}, \frac{2z}{z^2+1})$ is the point of intersection of $C \setminus \{(1, 0)\}$ and the line passing through the points $(1, 0)$ and $(0, z)$. It is obtained by solving x, y from the equations $x^2 + y^2 = 1$ and $\frac{y}{1-x} = z$. [Explanation: $y = z(1 - x) \Longrightarrow 1 - x^2 = y^2 = z^2(1 - x)^2 \Longrightarrow 1 + x = z^2(1 - x) \Longrightarrow x = \frac{z^2-1}{z^2+1} \Longrightarrow y = z(1 - x) = \frac{2z}{z^2+1}$.]

Both f and f^{-1} are continuous, hence f is a homeomorphism. □

Problem 3. Show that the set of all irrational numbers $\mathcal{I} \subset \mathbb{R}$ is not connected.

Proof. Let $U = \mathcal{I} \cap (-\infty, 0)$ and $V = \mathcal{I} \cap (0, \infty)$. Then $\mathcal{I} = U \cup V$ is a separation (why?). □

Problem 4. Suppose that A is a connected subset of $\mathbb{R}$. Show that if $a < b$ are real numbers in A, then the closed interval $[a, b]$ is contained in A.

Proof. Assume there exists $r \in (a, b)$ such that $r \notin A$. Let $U = A \cap (-\infty, r)$ and $V = A \cap (r, \infty)$. Then $A = U \cup V$ is a separation (why?), which contradicts the connectedness of A. Hence $[a, b] \subset A$. □