1. List all the quadratic residues and quadratic nonresidues mod 17.

The quadratic residues of 17 are $1, 4, 9, 16, 8, 2, 15, 13$, because
$(\pm1)^2 \equiv 1 \bmod 17$, $(\pm2)^2 \equiv 4 \bmod 17$, $(\pm3)^2 \equiv 9 \bmod 17$, $(\pm4)^2 \equiv 16 \bmod 17$,
$(\pm5)^2 \equiv 8 \bmod 17$, $(\pm6)^2 \equiv 2 \bmod 17$, $(\pm7)^2 \equiv 15 \bmod 17$, $(\pm8)^2 \equiv 13 \bmod 17$.

That leaves the quadratic non-residues of 17: $3, 5, 6, 7, 10, 11, 12, 14$.

2. (a) Use Euler's criterion to show that 5 is a quadratic residue of 29.

(b) Find all the solutions mod 29 to the congruence $x^2 \equiv 5 \bmod 29$.

(a) $5^{\frac{29-1}{2}} \equiv 1 \bmod 29$, because....

$5^2 \equiv 25 \equiv -4 \bmod 29$, $5^4 \equiv (-4)^2 \equiv 16 \equiv -13 \bmod 29$, $5^8 \equiv (-13)^2 \equiv 169 \equiv 24 \equiv -5 \bmod 29$,

so $5^{14} \equiv 5^{2+4+8} \equiv 5^2 5^4 5^8 \equiv (-4)(-13)(-5) \equiv -260 \equiv -28 \equiv 1 \bmod 29$.

(b) $x^2 \equiv 5 \equiv 34 \equiv 63 \equiv 3^2 \cdot 7 \equiv 3^2 \cdot 36 \equiv 3^2 \cdot 6^2 \bmod 29$.

So $x \equiv \pm18 \bmod 29$.

3. Use properties of the Legendre symbol to calculate $(\frac{51}{31})$ with minimal effort.

$(\frac{51}{31}) = (\frac{20}{31}) = (\frac{4\cdot5}{31}) = (\frac{4}{31})(\frac{5}{31}) = (\frac{2^2}{31})(\frac{5}{31}) = (\frac{5}{31}) = (\frac{36}{31}) = (\frac{6^2}{31}) = 1$.

4. Let p be an odd prime. Let a be an integer relatively prime with p. Let a^{-1} denote the multiplicative inverse of a mod p.

Show that a is a quadratic residue mod p if and only if a^{-1} is a quadratic residue mod p.

$$1 = \left(\frac{1}{p}\right) = \left(\frac{aa^{-1}}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{a^{-1}}{p}\right).$$

Thus either both $(\frac{a}{p})$ and $(\frac{a^{-1}}{p})$ equal 1, or both equal -1, since their product equals 1. So a and a^{-1} are either both quadratic residues, or both quadratic non-residues.

5. Let p be an odd prime. Prove that

$$\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) = 0.$$

If a is a quadratic residue mod p then $(\frac{a}{p}) = 1$ and if a is a quadratic nonresidue mod p then $(\frac{a}{p}) = -1$. There are an equal number of quadratic residues and quadratic nonresidues mod p. So the 1's and -1's will cancel out.