HOMEWORK 4 SOLUTIONS – MATH 4341

Problem 1. (a) Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are two different topologies on a set X. When is the identity map $id : X \to X$ given by $id(x) = x$ a continuous map from $(X, \mathcal{T}_1)$ to $(X, \mathcal{T}_2)$?

(b) Show that the subspace topology $\mathcal{T}_Y$ is the smallest topology on $Y \subset X$ for which the inclusion $\iota : Y \to X$ is a continuous map.

Proof. (a) $id : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous $\Leftrightarrow id^{-1}(U) \in \mathcal{T}_1$ for all $U \in \mathcal{T}_2 \Leftrightarrow U \in \mathcal{T}_1$ for all $U \in \mathcal{T}_2 \Leftrightarrow \mathcal{T}_2 \subset \mathcal{T}_1$.

(b) We first prove that $\iota : (Y, \mathcal{T}_Y) \to X$ is continuous. Let $V \subset X$ be an open set. Then $\iota^{-1}(V) = Y \cap V \in \mathcal{T}_Y$. Hence $\iota : (Y, \mathcal{T}_Y) \to X$ is continuous.

Suppose $\iota : (Y, \mathcal{T}) \to X$ is continuous. We will show that $\mathcal{T}_Y \subset \mathcal{T}$. Let $U \in \mathcal{T}_Y$. Then $U = Y \cap V$ for some open set $V \subset X$. Since $\iota^{-1}(V) = Y \cap V = U$ and $\iota : (Y, \mathcal{T}) \to X$ is continuous, we have $U \in \mathcal{T}$. Hence $\mathcal{T}_Y \subset \mathcal{T}$. $\square$

Problem 2. (a) Let $Y \subset X$ be an open subset of a topological space X. Show that a set $U \subset Y$ is open in the subspace topology on Y if and only if U is open in X.

(b) Let $Y \subset X$ be a closed subset of a topological space X. Show that a set $U \subset Y$ is closed in the subspace topology on Y if and only if U is closed in X.

Proof. (a) Suppose $U \subset Y$ is open in X. Since $U = Y \cap U$, we have U is open in Y.

Suppose $U \subset Y$ is open. Then $U = Y \cap V$ for some open set $V \subset X$. Since both Y and V are open in X, $U = Y \cap V$ is open in X.

(b) Suppose $U \subset Y$ is closed in X. Then $X \setminus U$ is open in X. Since $Y \setminus U = Y \cap (X \setminus U)$, we have $Y \setminus U$ is open in Y. This means that U is closed in Y.

Suppose $U \subset Y$ is closed. Then $Y \setminus U$ is open in Y, and hence $Y \setminus U = Y \cap V$ for some open set $V \subset X$. We have

$$U = Y \setminus (Y \setminus U) = Y \setminus (Y \cap V) = (Y \setminus Y) \cup (Y \setminus V) = Y \setminus V = Y \cap (X \setminus V).$$

Since both Y and $X \setminus V$ are closed in X, $U = Y \cap (X \setminus V)$ is closed in X. $\square$

Problem 3. Let (X_1, d_1) and (X_2, d_2) be metric spaces. Define a function on $X_1 \times X_2$ by

$$d((x_1, x_2), (y_1, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2)).$$

(a) Show that d is a metric on $X_1 \times X_2$.

(b) Show that the metric topology on $X_1 \times X_2$ induced by d is the product topology, where X_1 and X_2 have the metric topologies from d_1 and d_2 respectively.

Proof. (a) (positivity) $d((x_1, x_2), (y_1, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2)) \geq 0$ and equality holds iff $d_1(x_1, y_1) = 0$ and $d_2(x_2, y_2) = 0$, i.e. $x_1 = y_1$ and $x_2 = y_2$, so $(x_1, x_2) = (y_1, y_2)$.

(symmetry) $d((x_1, x_2), (y_1, y_2)) = d((y_1, y_2), (x_1, x_2))$, since $d_1(x_1, y_1) = d_1(y_1, x_1)$ and $d_2(x_2, y_2) = d_2(y_2, x_2)$.

(triangle inequality) We have $d((x_1, x_2), (y_1, y_2)) + d((y_1, y_2), (z_1, z_2)) \geq d_1(x_1, y_1) + d_1(y_1, z_1) \geq d_1(x_1, z_1)$. Similarly, $d((x_1, x_2), (y_1, y_2)) + d((y_1, y_2), (z_1, z_2)) \geq d_2(x_2, y_2) + d_2(y_2, z_2) \geq d_2(x_2, z_2)$. Hence

$$d((x_1, x_2), (y_1, y_2)) + d((y_1, y_2), (z_1, z_2)) \geq \max(d_1(x_1, z_1), d_2(x_2, z_2)) = d((x_1, x_2), (z_1, z_2)).$$

(b) We will apply Lemma 2.4 in the lecture notes. Let $B_d((y_1, y_2), r)$ be an open ball in the topology $\mathcal{T}_d$ and $(x_1, x_2) \in B_d((y_1, y_2), r)$. Then

$$r > d((x_1, x_2), (y_1, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2))$$

and $(x_1, x_2) \in B_{d_1}(y_1, r) \times B_{d_2}(y_2, r) \subset B_d((y_1, y_2), r)$. Indeed, if $(z_1, z_2) \in B_{d_1}(y_1, r) \times B_{d_2}(y_2, r)$, then $d_1(y_1, z_1) < r$ and $d_2(y_2, z_2) < r$. This implies that $d((y_1, y_2), (z_1, z_2)) = \max(d_1(y_1, z_1), d_2(y_2, z_2)) < r$. Hence $(z_1, z_2) \in B_d((y_1, y_2), r)$.

Conversely, let $U_1 \times U_2$ be a basis element for the product topology on $X_1 \times X_2$, and $(x_1, x_2) \in U_1 \times U_2$. Since $U_1 \subset X_1$ is open and $x_1 \in U_1$, there exists an open ball $B_{d_1}(y_1, r_1)$ such that $x_1 \in B_{d_1}(y_1, r_1) \subset U_1$. Similarly, there exists an open ball $B_{d_2}(y_2, r_2)$ such that $x_2 \in B_{d_2}(y_2, r_2) \subset U_2$. Let

$$r = \min(r_1 - d(x_1, y_1), r_2 - d(x_2, y_2)) > 0.$$

Then $(x_1, x_2) \in B_d((x_1, x_2), r) \subset B_{d_1}(y_1, r_1) \times B_{d_2}(y_2, r_2) \subset U_1 \times U_2$. Indeed, if $(z_1, z_2) \in B_d((x_1, x_2), r)$ then $d((x_1, x_2), (z_1, z_2)) < r$, i.e.

$$\max(d_1(x_1, z_1), d_2(x_2, z_2)) < \min(r_1 - d(x_1, y_1), r_2 - d(x_2, y_2)).$$

This implies that $d_1(x_1, z_1) < r_1 - d(x_1, y_1)$ and $d_1(x_2, z_2) < r_2 - d(x_2, y_2)$, i.e. $d_1(x_1, z_1) + d(x_1, y_1) < r_1$ and $d_1(x_2, z_2) + d(x_2, y_2) < r_2$. Together with the triangle inequality, we obtain $d(y_1, z_1) < r_1$ and $d(y_2, z_2) < r_2$. Hence $z_1 \in B_{d_1}(y_1, r_1)$ and $z_2 \in B_{d_2}(y_2, r_2)$, so $(z_1, z_2) \in B_{d_1}(y_1, r_1) \times B_{d_2}(y_2, r_2)$. $\square$