HOMEWORK 11 SOLUTIONS – MATH 4341

Problem 1. Given a path $f : [0,1] \to X$ with $f(0) = p$ and $f(1) = q$. Let e_r be the constant path at $r \in X$, i.e. $e_r(x) = r$ for $x \in [0,1]$.

(a) Find explicit formulas for $f \star e_q$ and $e_p \star f$.

(b) Find an explicit formula for a path homotopy from f to $f \star e_q$.

(c) Find an explicit formula for a path homotopy from f to $e_p \star f$.

Proof. (a)

$$\begin{aligned} f \star e_q &= \begin{cases} f(2x) & x \in [0,1/2] \\ e_q(2x-1) & x \in [1/2,1] \end{cases} \\ &= \begin{cases} f(2x) & x \in [0,1/2] \\ q & x \in [1/2,1] \end{cases}. \end{aligned}$$

$$\begin{aligned} e_p \star f &= \begin{cases} e_p(2x) & x \in [0,1/2] \\ f(2x-1) & x \in [1/2,1] \end{cases} \\ &= \begin{cases} p & x \in [0,1/2] \\ f(2x-1) & x \in [1/2,1] \end{cases}. \end{aligned}$$

(b) A path homotopy from f to $f \star e_q$:

$$\begin{aligned} H(x,t) &= \begin{cases} f(\frac{x-0}{1-\frac{t}{2}}) & x \in [0, 1-\frac{t}{2}] \\ e_q(\frac{x-(1-\frac{t}{2})}{\frac{t}{2}}) & x \in [1-\frac{t}{2}, 1] \end{cases} \\ &= \begin{cases} f(\frac{2x}{2-t}) & x \in [0, 1-\frac{t}{2}] \\ q & x \in [1-\frac{t}{2}, 1] \end{cases}. \end{aligned}$$

(c) A path homotopy from f to $e_p \star f$:

$$\begin{aligned} K(x,t) &= \begin{cases} e_p(\frac{x-0}{\frac{t}{2}}) & x \in [0, \frac{t}{2}] \\ f(\frac{x-\frac{t}{2}}{1-\frac{t}{2}}) & x \in [\frac{t}{2}, 1] \end{cases} \\ &= \begin{cases} p & x \in [0, \frac{t}{2}] \\ f(\frac{2x-t}{2-t}) & x \in [\frac{t}{2}, 1] \end{cases}. \end{aligned}$$

□

Problem 2. Given a path $f : [0,1] \to X$ with $f(0) = p$ and $f(1) = q$. Let $\overline{f} : [0,1] \to X$ be the reverse path of f, i.e. $\overline{f}(x) = f(1-x)$ for $x \in [0,1]$.

(a) Find explicit formulas for $f \star \overline{f}$ and $\overline{f} \star f$.

(b) Find an explicit formula for a path homotopy from e_p to $f \star \overline{f}$.

(c) Find an explicit formula for a path homotopy from e_q to $\overline{f} \star f$.

Proof. (a)

$$
\begin{aligned}
f \star \overline{f} &= \begin{cases} f(2x) & x \in [0,1/2] \\ \overline{f}(2x-1) & x \in [1/2,1] \end{cases} \\
&= \begin{cases} f(2x) & x \in [0,1/2] \\ f(2-2x) & x \in [1/2,1] \end{cases}.
\end{aligned}
$$

$$
\begin{aligned}
\overline{f} \star f &= \begin{cases} \overline{f}(2x) & x \in [0,1/2] \\ f(2x-1) & x \in [1/2,1] \end{cases} \\
&= \begin{cases} f(1-2x) & x \in [0,1/2] \\ f(2x-1) & x \in [1/2,1] \end{cases}.
\end{aligned}
$$

(b) A path homotopy from e_p to $f \star \overline{f}$:

$$
H(x,t) = \begin{cases} f(2tx) & x \in [0,1/2] \\ f(t(2-2x)) & x \in [1/2,1] \end{cases}.
$$

(c) A path homotopy from e_q to $\overline{f} \star f$:

$$
K(x,t) = \begin{cases} f(t(1-2x)) & x \in [0,1/2] \\ f(t(2x-1)) & x \in [1/2,1] \end{cases}.
$$

□

Problem 3. Given paths $f, g, h : [0,1] \to X$ with $f(1) = g(0)$ and $g(1) = h(0)$.
(a) Find explicit formulas for $(f \star g) \star h$ and $f \star (g \star h)$.
(b) Find an explicit formula for a path homotopy from $(f \star g) \star h$ to $f \star (g \star h)$.

Proof. (a)

$$
\begin{aligned}
(f \star g) \star h &= \begin{cases} (f \star g)(2x) & x \in [0,1/2] \\ h(2x-1) & x \in [1/2,1] \end{cases} \\
&= \begin{cases} f(4x) & x \in [0,1/4] \\ g(4x-1) & x \in [1/4,1/2] \\ h(2x-1) & x \in [1/2,1] \end{cases}.
\end{aligned}
$$

$$
\begin{aligned}
f \star (g \star h) &= \begin{cases} f(2x) & x \in [0,1/2] \\ (g \star h)(2x-1) & x \in [1/2,1] \end{cases} \\
&= \begin{cases} f(2x) & x \in [0,1/2] \\ g(4x-2) & x \in [1/2,3/4] \\ h(4x-3) & x \in [3/4,1] \end{cases}.
\end{aligned}
$$

(b) A path homotopy from $(f \star g) \star h$ to $f \star (g \star h)$:

$$F(x,t) = \begin{cases} f\left(\frac{x-0}{\frac{1+t}{4}}\right) & x \in [0, \frac{1+t}{4}] \\ g\left(\frac{x-\frac{1+t}{4}}{\frac{1}{4}}\right) & x \in [\frac{1+t}{4}, \frac{2+t}{4}] \\ h\left(\frac{x-\frac{2+t}{4}}{1-\frac{2+t}{4}}\right) & x \in [\frac{2+t}{4}, 1] \end{cases}$$

$$= \begin{cases} f\left(\frac{4x}{1+t}\right) & x \in [0, \frac{1+t}{4}] \\ g(4x-(1+t)) & x \in [\frac{1+t}{4}, \frac{2+t}{4}] \\ h\left(\frac{4x-(2+t)}{2-t}\right) & x \in [\frac{2+t}{4}, 1] \end{cases}.$$

□

Problem 4. Let $h : X \to Y$ be a continuous function between two topological spaces. Given paths $f, g : [0,1] \to X$ with $f(1) = g(0)$. Show that

$$h \circ (f \star g) = (h \circ f) \star (h \circ g).$$

Proof. Since

$$(f \star g)(x) = \begin{cases} f(2x) & x \in [0, 1/2] \\ g(2x-1) & x \in [1/2, 1] \end{cases}$$

we have

$$\begin{aligned} (h \circ (f \star g))(x) &= h((f \star g)(x)) \\ &= \begin{cases} h(f(2x)) & x \in [0,1/2] \\ h(g(2x-1)) & x \in [1/2,1] \end{cases} \\ &= \begin{cases} (h \circ f)(2x) & x \in [0,1/2] \\ (h \circ g)(2x-1) & x \in [1/2,1] \end{cases} \\ &= ((h \circ f) \star (h \circ g))(x). \end{aligned}$$

Hence $h \circ (f \star g) = (h \circ f) \star (h \circ g)$. □

Problem 5. Find an explicit formula for the map $r(x)$ in the proof of Theorem 7.11 (Brouwer fixed point theorem for D^2) in the lecture notes.

Proof. We have $r(x) - x = t(x - h(x))$ for some $t > 0$. This implies that $r(x) = x + t(x - h(x))$. From $\langle r(x), r(x) \rangle = ||r(x)||^2 = 1$ we get

$$||x||^2 + t^2||x-h(x)||^2 + 2t\langle x, x-h(x) \rangle = 1.$$

Let $a = ||x - h(x)||^2$, $b = \langle x, x - h(x) \rangle$ and $c = ||x||^2 - 1$. Then $at^2 + 2bt + c = 0$. Since $a > 0$ and $c < 0$, this quadratic equation has exactly one positive solution t_0. Explicitly, we have

$$t_0 = \frac{\langle x, x-h(x) \rangle + \sqrt{\langle x, x-h(x) \rangle^2 + (1 - ||x||^2)||x-h(x)||^2}}{||x-h(x)||^2}.$$

Then $r(x) = x + t_0(x - h(x))$. □