HOMEWORK 1 SOLUTIONS – MATH 4341

Problem 1. Let A, B, C be three sets. Use definition to show that

$$\begin{aligned} A \setminus (B \cup C) &= (A \setminus B) \cap (A \setminus C), \\ A \setminus (B \cap C) &= (A \setminus B) \cup (A \setminus C). \end{aligned}$$

Proof. (1) Take $x \in A \setminus (B \cup C)$. Then $x \in A$ and $x \notin B \cup C$. Since $x \notin B \cup C$, we have $x \notin B$ and $x \notin C$. Combining with $x \in A$, we obtain $x \in A \setminus B$ and $x \in A \setminus C$. Hence $x \in (A \setminus B) \cap (A \setminus C)$.

Take $x \in (A \setminus B) \cap (A \setminus C)$. Then $x \in A \setminus B$ and $x \in A \setminus C$. This implies that $x \in A$, $x \notin B$ and $x \notin C$. Since $x \notin B$ and $x \notin C$, we have $x \notin B \cup C$. Combining with $x \in A$, we obtain $x \in A \setminus (B \cup C)$.

(2) Take $x \in A \setminus (B \cap C)$. Then $x \in A$ and $x \notin B \cap C$. Since $x \notin B \cap C$, we have $x \notin B$ or $x \notin C$. Combining with $x \in A$, we obtain $x \in A \setminus B$ or $x \in A \setminus C$. Hence $x \in (A \setminus B) \cup (A \setminus C)$.

Take $x \in (A \setminus B) \cup (A \setminus C)$. Then $x \in A \setminus B$ or $x \in A \setminus C$. This implies that $x \in A$, $x \notin B$ or $x \notin C$. Since $x \notin B$ or $x \notin C$, we have $x \notin B \cap C$. Combining with $x \in A$, we obtain $x \in A \setminus (B \cap C)$. $\square$

Problem 2. Let $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ be two collections of sets. Show that

$$\begin{aligned} \left(\bigcup_{i \in I} A_i \right) \cap \left(\bigcup_{j \in J} B_j \right) &= \bigcup_{i \in I,\, j \in J} (A_i \cap B_j), \\ \left(\bigcap_{i \in I} A_i \right) \cup \left(\bigcap_{j \in J} B_j \right) &= \bigcap_{i \in I,\, j \in J} (A_i \cup B_j). \end{aligned}$$

Proof. (1) Take any $x \in \left(\bigcup_{i \in I} A_i\right) \cap \left(\bigcup_{j \in J} B_j\right)$. Then $x \in \bigcup_{i \in I} A_i$ and $x \in \bigcup_{j \in J} B_j$. Since $x \in \bigcup_{i \in I} A_i$, there exists $i_0 \in I$ such that $x \in A_{i_0}$. Similarly, since $x \in \bigcup_{j \in J} B_j$, there exists $j_0 \in J$ such that $x \in B_{j_0}$. Now $x \in A_{i_0}$ and $x \in B_{j_0}$ imply that $x \in A_{i_0} \cap B_{j_0}$. Hence $x \in \bigcup_{i \in I,\, j \in J} (A_i \cap B_j)$.

Take any $x \in \bigcup_{i \in I,\, j \in J} (A_i \cap B_j)$. Then there exists $i_0 \in I$ and $j_0 \in J$ such that $x \in A_{i_0} \cap B_{j_0}$. This implies that $x \in A_{i_0}$ and $x \in B_{j_0}$. Since $x \in A_{i_0}$, we have $x \in \bigcup_{i \in I} A_i$. Similarly, since $x \in B_{j_0}$ we have $x \in \bigcup_{j \in J} B_j$. Hence $x \in \left(\bigcup_{i \in I} A_i\right) \cap \left(\bigcup_{j \in J} B_j\right)$.

(2) Take any $x \in \left(\bigcap_{i \in I} A_i\right) \cup \left(\bigcup_{j \in J} B_j\right)$. Then either $x \in \bigcap_{i \in I} A_i$ or $x \in \bigcap_{j \in J} B_j$. Without loss of generality, we may assume that $x \in \bigcap_{i \in I} A_i$. Then $x \in A_i$ for all $i \in I$. This implies that $x \in A_i \cup B_j$ for all $i \in I$ and $j \in J$. Hence $x \in \bigcap_{i \in I,\, j \in J} (A_i \cup B_j)$.

Take any $x \in \bigcap_{i \in I,\, j \in J} (A_i \cup B_j)$. Assume that $x \notin \left(\bigcap_{i \in I} A_i\right) \cup \left(\bigcap_{j \in J} B_j\right)$. Then $x \notin \bigcap_{i \in I} A_i$ and $x \notin \bigcap_{j \in J} B_j$. Since $x \notin \bigcap_{i \in I} A_i$, there exists $i_0 \in I$ such that $x \notin A_{i_0}$. Similarly, since $x \notin \bigcap_{j \in J} B_j$, there exists $j_0 \in J$ such that $x \notin B_{j_0}$. Since $x \notin A_{i_0} \cup B_{j_0}$, we have $x \notin \bigcap_{i\in I,\, j\in J}(A_i \cup B_j)$. This contradicts $x \in \bigcap_{i\in I,\, j\in J}(A_i \cup B_j)$. Hence $x \in \left(\bigcap_{i\in I} A_i\right) \cup \left(\bigcap_{j\in J} B_j\right)$. $\square$

Problem 3. (a) Suppose C is a subset of $\mathbb{R} \times \mathbb{R}$ such that C is equal to the Cartesian product of two subsets of $\mathbb{R}$. Show that if two points (a_1, b_1) and (a_2, b_2) are elements in C then two points (a_1, b_2) and (a_2, b_1) are also elements in C.

(b) Determine whether the subset $C = \{(x, y) \mid x^2 + y^3 > 7\}$ of $\mathbb{R} \times \mathbb{R}$ is equal to the Cartesian product of two subsets of $\mathbb{R}$.

Proof. (a) Suppose $C = A \times B$, where A and B are subsets of $\mathbb{R}$. Since $(a_1, b_1) \in C = A \times B$, we have $a_1 \in A$ and $b_1 \in B$. Similarly, since $(a_2, b_2) \in A \times B$, we have $a_2 \in A$ and $b_2 \in B$.

Since $a_1 \in A$ and $b_2 \in B$, we have $(a_1, b_2) \in A \times B = C$. Similarly, since $a_2 \in A$ and $b_1 \in B$ we have $(a_2, b_1) \in A \times B = C$.

(b) Assume $C = \{(x, y) \mid x^2 + y^3 > 7\}$ is equal to the Cartesian product of two subsets of $\mathbb{R}$. Note that $(3, 0)$ and $(0, 2)$ are elements in C. By (a), $(0, 0)$ and $(3, 2)$ must also be elements in C. This contradicts the fact that $(0, 0)$ is not an element in C (since $0^2 + 0^3 < 7$). Hence C is not equal to the Cartesian product of two subsets of $\mathbb{R}$. $\square$

Problem 4. Let $f : A \to B$ be a function. We define a relation C on A by setting xCy if $f(x) = f(y)$. Show that C is an equivalence relation.

Proof. C is reflexive: $(x, x) \in C$ since $f(x) = f(x)$.

C is symmetric: if $(x, y) \in C$ then $f(x) = f(y)$. This implies that $f(y) = f(x)$, so $(y, x) \in C$.

C is transitive: if $(x, y) \in C$ and $(y, z) \in C$, then $f(x) = f(y)$ and $f(y) = f(z)$. This implies that $f(x) = f(y) = f(z)$, so $(x, z) \in C$. $\square$

Problem 5. Define a relation on $\mathbb{Q}$ by

$$C = \{(x, y) \mid x - y \text{ is an even integer}\}.$$

(a) Show that C is an equivalence relation.

(b) Describe the set of equivalence classes of C.

Proof. (a) C is reflexive: $(x, x) \in C$ since $x - x = 0$ is an even integer.

C is symmetric: if $(x, y) \in C$ then $x - y$ is an even integer. This implies that $y - x = -(x - y)$ is also an even integer, so $(y, x) \in C$.

C is transitive: if $(x, y) \in C$ and $(y, z) \in C$, then $x - y$ and $y - z$ are even integers. This implies that $x - z = (x - y) + (y - z)$ is an even integer, so $(x, z) \in C$.

(b) For every $x \in \mathbb{Q}$, there exists a unique $k \in \mathbb{Z}$ such that $x \in [k, k + 1)$. If k is even, then x is equivalent to an element $x - k \in \mathbb{Q} \cap [0, 1)$. If k is odd, then x is equivalent to an element $x - (k - 1) \in \mathbb{Q} \cap [1, 2)$. Since no elements in $\mathbb{Q} \cap [0, 2)$ are equivalent to each other, the set of equivalence classes is in bijection with $\mathbb{Q} \cap [0, 2)$. $\square$