Math 4301 Mathematical Analysis I

Lecture 4

Topic: Cluster Points

- **Cluster Points**

  Definition Let $\{x_n\}$ be a sequence of real numbers and $x \in \mathbb{R}$.

  We say that x is a *cluster point* of $\{x_n\}$, if for all $\epsilon > 0$, the set

$$\{n \in \mathbb{N} : |x_n - x| < \epsilon\}$$

  is infinite.

- Let $\{x_n\}$ be a real sequence and C be the *set of all cluster points* of $\{x_n\}$, i.e.

$$C = \{x : \ x \text{ is a cluster point of } \{x_n\}\}.$$

  Example: Find the set C of all cluster points of the sequence

$$x_n = (-1)^n + \frac{1}{n}.$$

- Clearly,

$$C = \{-1, 1\}.$$

- Notice that

$$x_n = \begin{cases} 1 + \frac{1}{2k} & if \quad n = 2k \\ -1 + \frac{1}{2k-1} & if \quad n = 2k - 1 \end{cases}, \ k \in \mathbb{N}.$$

- Therefore, there are two subsequences $\{x_{2k}\}$ and $\{x_{2k-1}\}$ of $\{x_n\}$, such that

$$x_{2k} \to 1 \text{ and } x_{2k-1} \to -1.$$

- Since for $\epsilon > 0$, there are

$$N_1, N_2 \in \mathbb{N},$$

  such that, for $k > N_1$

$$|x_{2k} - 1| < \epsilon$$

  and, for $k > N_2$

$$|x_{2k-1} - (-1)| < \epsilon$$

- It follows that sets

$$\{n \in \mathbb{N} : |x_n - 1| < \epsilon\}$$

  is infinite (since $2k \in \{n \in \mathbb{N} : |x_n - 1| < \epsilon\}$, $k > N_1$) and

$$\{n \in \mathbb{N} : |x_n + 1| < \epsilon\}$$

  is also infinite (since $(2k - 1) \in \{n \in \mathbb{N} : |x_n + 1| < \epsilon\}$, $k > N_2$).

- Therefore, we know that

$$-1, 1 \in C.$$

- We show that **if** $x \neq \pm 1$, **then** x **is not a cluster point of** $\{x_n\}$.

- Define
$$\epsilon = \frac{1}{2}\min\{|x-1|, |x+1|\} > 0$$

- Since $x_{2k} \to 1$ and $x_{2k-1} \to -1$, there are
$$N_1, N_2 \in \mathbb{N},$$
such that, if $k > N_1$, then
$$|x_{2k} - 1| < \epsilon$$
and if $k > N_2$, then
$$|x_{2k-1} - (-1)| < \epsilon.$$

- Therefore, if $n > \max\{N_1, N_2\}$, then
$$|x_n - 1| < \epsilon \text{ or } |x_n + 1| < \epsilon.$$

- Hence, if $n > \max\{N_1, N_2\}$ and n is even then
$$|x_n - x| = |(x_n - 1) + (1 - x)| \geq ||x_n - 1| - |1 - x||.$$

- Since
$$|x_n - 1| < \epsilon \leq \frac{1}{2}|x-1| < |x-1|,$$
so
$$\begin{aligned} |x_n - x| &\geq ||x_n - 1| - |1-x|| = |1-x| - |x_n - 1| \\ &> |1-x| - \epsilon \geq |1-x| - \underbrace{\frac{1}{2}|1-x|}_{\geq \epsilon} = \frac{1}{2}|1-x| \geq \epsilon. \end{aligned}$$

- Analogously, one shows that
$$n > \max\{N_1, N_2\}$$
and n is odd then
$$|x_n - x| \geq \epsilon.$$

- We showed that, if $n > \max\{N_1, N_2\}$, then $|x_n - x| \geq \epsilon$.

- Therefore, the set
$$\{n \in \mathbb{N} : |x_n - x| < \epsilon\} \text{ is finite}$$

- Consequently, x is not a cluster point of $\{x_n\}$.

- Therefore,
$$C = \{-1, 1\}.$$

Theorem Let $\{x_n\}$ be a sequence of real numbers and $x \in \mathbb{R}$.

1. x is a cluster point of $\{x_n\}$ iff
   for every $\epsilon > 0$ and for every $N \in \mathbb{N}$, there is $n \in \mathbb{N}$, such that
$$n > N \text{ and } |x_n - x| < \epsilon.$$

2. x is a cluster point of $\{x_n\}$ iff
   there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that,

$$x_{n_k} \to x \text{ as } k \to \infty.$$

3. $x_n \to x$ as $n \to \infty$ iff
   every subsequence of $\{x_n\}$ converges to x.
4. $x_n \to x$ as $n \to \infty$ iff
   the sequence is bounded and x is its only cluster point.
5. $x_n \to x$ as $n \to \infty$ iff
   every subsequence of $\{x_n\}$ has a further subsequence that converges to x.

Proof. We prove each of the statements $(1-5)$.

- **For statement** (1): x is a cluster point of $\{x_n\}$ iff
  for every $\epsilon > 0$ and for every $N \in \mathbb{N}$, there is $n \in \mathbb{N}$, such that

$$n > N \text{ and } |x_n - x| < \epsilon.$$

- **Assume** that x is a cluster point of $\{x_n\}$ and
  let $\epsilon > 0$ be given and $N \in \mathbb{N}$.
- Since x is a cluster point of $\{x_n\}$,
  by the definition the set

$$\{n \in \mathbb{N} : |x_n - x| < \epsilon\}$$

  is infinite.
- Since N is a finite number and

$$\{n \in \mathbb{N} : |x_n - x| < \epsilon\}$$

  is infinite, there is

$$n \in \{n \in \mathbb{N} : |x_n - x| < \epsilon\},$$

  such that

$$n > N, \text{ so } |x_n - x| < \epsilon.$$

- **Conversely**, let $\epsilon > 0$ be given.
- We need to show that

$$\{n \in \mathbb{N} : |x_n - x| < \epsilon\}$$

  is infinite.
- Then for $N_1 = 1$, there is $n_1 > N_1$, such that,

$$|x_{n_1} - x| < \epsilon.$$

- Take $N_2 = n_1$, then there is $n_2 > N_2 = n_1$ (i.e. $n_1 < n_2$), such that

$$|x_{n_2} - x| < \epsilon.$$

- If we repeat the above construction,
  we obtain a sequence of natural numbers

$$n_1 < n_2 < n_3 < ...,$$

  such that

$$|x_{n_k} - x| < \epsilon, \ k = 1, 2, ...$$

- It follows that

$$\begin{aligned} n_k &\in \{n \in \mathbb{N} : |x_n - x| < \epsilon\}, \text{ so} \\ \{n_k : k = 1, 2, ...\} &\subseteq \{n \in \mathbb{N} : |x_n - x| < \epsilon\} \end{aligned}$$

  so $\{n \in \mathbb{N} : |x_n - x| < \epsilon\}$ is infinite and thus,

- x is a cluster point.
  For statement (2) : x is a cluster point of $\{x_n\}$ iff
  there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that,

$$x_{n_k} \to x \text{ as } k \to \infty.$$

- **Assume** that x is a cluster point.
- By (1), for every $\epsilon > 0$ and $N \in \mathbb{N}$,
  there is $n > N$, such that

$$|x_n - x| < \epsilon.$$

- We construct a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that,

$$x_{n_k} \to x.$$

- Let

$$\epsilon_k = \frac{1}{k}, \text{ for } k = 1, 2, ...$$

- Take $\epsilon_1 = \frac{1}{1}$ and $N_1 = 1$, then
  there is $n_1 > N_1$, such that

$$|x_{n_1} - x| < \frac{1}{1}$$

- Take $\epsilon_2 = \frac{1}{2}$ and $N_2 = n_1$, then
  there is $n_2 > N_2$, such that

$$|x_{n_2} - x| < \frac{1}{2}$$

  and then by induction,
  if $\epsilon_k = \frac{1}{k}$, taking $N_k = n_{k-1}$, there is $n_k > N_k$, such that

$$|x_{n_k} - x| < \frac{1}{k}$$

- We constructed a sequence of natural numbers
$$n_1 < n_2 < ...,$$
such that
$$|x_{n_k} - x| < \frac{1}{k}.$$
- Clearly, $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.
- **We show now that $x_{n_k} \to x$.**
- Let $\epsilon > 0$ be given.
- There is $K \in \mathbb{N}$, such that
$$0 < \epsilon_K = \frac{1}{K} < \epsilon.$$
- Since, for $k > K$, $\epsilon_k < \epsilon_K$, then, for $k > K$,
$$|x_{n_k} - x| < \epsilon_k = \frac{1}{k} < \frac{1}{K} = \epsilon_K < \epsilon.$$

  It follows that $\{x_{n_k}\}$ **converges to x.**
- **Conversely**, suppose that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ that converges to x.
- **We show that x is a cluster point of $\{x_n\}$.**
- By assumption, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that
$$x_{n_k} \to x \text{ as } k \to \infty.$$
- Therefore, if $\epsilon > 0$ be given,
  there is $K \in \mathbb{N}$, such that, for all $k > K$,
$$|x_{n_k} - x| < \epsilon.$$
- Therefore, if $k > K$
$$n_k \in \{n \in \mathbb{N} : |x_n - x| < \epsilon\}$$
  Hence,
$$\{n_k : k > K\} \subseteq \{n \in \mathbb{N} : |x_n - x| < \epsilon\},$$
  so
$$\{n \in \mathbb{N} : |x_n - x| < \epsilon\}$$
  is infinite, and
- by the definition, x is a cluster point of $\{x_n\}$.
  For statement (3) : $x_n \to x$ as $n \to \infty$ iff
  every subsequence of $\{x_n\}$ converges to x.
- **Assume** that $x_n \to x$ as $n \to \infty$ and
  suppose that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.
- Let $\epsilon > 0$ be given.

- Since $x_n \to x$, then there is $N \in \mathbb{N}$, such that, for

$$n > N, \;\; |x_n - x| < \epsilon.$$

- Notice that if

$$n_1 < n_2 < n_3 < ...$$

  is a sequence of natural numbers, then
  $n_k \geq k$, for all $k = 1, 2, ...$
- Indeed, clearly $n_1 \geq 1$.
- Since $n_2 > n_1$, then

$$n_2 > n_1 \geq 1,$$

  so $n_2 \geq 2$.
- By induction, if $n_k \geq k$, then
  $n_{k+1} > n_k \geq k$, so

$$n_{k+1} \geq k + 1.$$

- Therefore, if $k > N$, then

$$n_k \geq k > N,$$

  so

$$|x_{n_k} - x| < \epsilon.$$

- It follows that

$$x_{n_k} \to x \text{ as } k \to \infty.$$

- **Conversely**, suppose that $\{x_n\}$ is not convergent to x.
- **We construct a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $\{x_{n_k}\}$ does not converge to x.**
- If $\{x_n\}$ is not convergent to x, then
  there is $\epsilon > 0$, such that,
  for every $N \in \mathbb{N}$, there is $n > N$, such that

$$|x_n - x| \geq \epsilon.$$

- Using the above condition, we take

$$N_1 = 1,$$

  the we get $n_1 > N_1$, such that

$$|x_{n_1} - x| \geq \epsilon$$

- Taking $N_2 = n_1$,
  there is $n_2 > N_2 = n_1$ (i.e. $n_1 < n_2$) such that

$$|x_{n_2} - x| \geq \epsilon$$

- Inductively, we construct a sequence of natural numbers

$$n_1 < n_2 < n_3 < ....,$$

  such that

$$|x_{n_k} - x| \geq \epsilon, \text{ for all } k = 1, 2, ...$$

- We see that such a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ cannot converge to x.
- **Contradiction** since we assumed that every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x.
- Therefore, it must be
$$x_n \to x.$$

For statement $(4) : x_n \to x$ as $n \to \infty$ iff
the sequence is bounded and x is its only cluster point.

- **Assume that** $x_n \to x$ **as** $n \to \infty$.
- By previous theorem $\{x_n\}$ is bounded.
- Since $x_n \to x$, by (3),
every $\{x_{n_k}\}$ subsequence of $\{x_n\}$, converges to x,
i.e. $x_{n_k} \to x$, as $k \to \infty$.
- In particular, **there is subsequence** $\{x_{n_k}\}$ **of** $\{x_n\}$ **that converges to** x,
so by (2), x is a cluster point of $\{x_n\}$.
- Moreover, if y is a cluster point of $\{x_n\}$, then
by (2), there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to y.
- Since $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and $x_n \to x$,
by (3), $x_{n_k} \to x$ as $k \to \infty$.
- Since limit of a convergent sequence is unique,
it follows that
$$y = x.$$
- Therefore, $\{x_n\}$ has a unique cluster point x and,
as we showed it before, $\{x_n\}$ is bounded.
- **Conversely**, assume that $\{x_n\}$ has a unique cluster point x and $\{x_n\}$ is bounded.
- **We show that every subsequence** $\{x_{n_k}\}$ **of** $\{x_n\}$ **converges to** x**, so by** (3)
$$x_n \to x.$$
- Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ and
assume that $x_{n_k} \nrightarrow x$ as $k \to \infty$.
- Since $x_{n_k} \nrightarrow x$ as $k \to \infty$,
there is $\epsilon > 0$, such that,
for every $K \in \mathbb{N}$, there is $k > K$, such that
$$|x_{n_k} - x| \geq \epsilon$$
- Let $K = 1$, then
there is $k_1 > 1$, such that
$$\left|x_{n_{k_1}} - x\right| \geq \epsilon$$

- Let $K = k_1$, then

  there is $k_2 > k_1$, such that
  $$\left|x_{n_{k_2}} - x\right| \geq \epsilon$$
  and by induction.
- Let $K = k_l$, then there is
  $$k_{l+1} > k_l > ... > k_1,$$
  such that
  $$\left|x_{n_{k_{l+1}}} - x\right| \geq \epsilon$$
- We obtained a subsequence $\left\{x_{n_{k_l}}\right\}$ of $\{x_{n_k}\}$, such that,

  for all $l = 1, 2, ...$
  $$\left|x_{n_{k_l}} - x\right| \geq \epsilon.$$
- Since $\{x_n\}$ is bounded,

  $\{x_{n_k}\}$ is also bounded, so also

  $\left\{x_{n_{k_l}}\right\}$ is bounded.
- By Bolzano-Weierstrass theorem,

  $\left\{x_{n_{k_l}}\right\}$ has a convergent subsequence $\left\{x_{n_{k_{l_j}}}\right\}$.
- Let $x_{n_{k_{l_j}}} \to y$, as $j \to \infty$.
- Since $\left\{x_{n_{k_{l_j}}}\right\}$ is a subsequence of $\left\{x_{n_{k_l}}\right\}$, for all $j = 1, 2, ...$
  $$\left|x_{n_{k_{l_j}}} - x\right| \geq \epsilon$$
- Therefore, $y \neq x$.
- Since $\left\{x_{n_{k_{l_j}}}\right\}$ is a subsequence of $\left\{x_{n_{k_l}}\right\}$,

  $\left\{x_{n_{k_l}}\right\}$ is a subsequence of $\{x_{n_k}\}$, and $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, $\left\{x_{n_{k_{l_j}}}\right\}$ is a subsequence of $\{x_n\}$.
- By (2), y is a cluster point of $\{x_n\}$ and $y \neq x$.
- **Contradiction** since we assumed that

  $\{x_n\}$ has a unique cluster point.
- Therefore, we showed that

  every subsequence of $\{x_n\}$ converges to x.
- By (3), $x_n \to x$.

  For statement (5) : $x_n \to x$ as $n \to \infty$ iff

  every subsequence of $\{x_n\}$ has a further subsequence that converges to x.
- **Assume that $x_n \to x$ as $n \to \infty$.**

- By (3) , every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x.
- In particular, by (3) each subsequence of $\{x_{n_k}\}$ converges to x.
- Thus, every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has a further subsequence that converges to x.
- **Conversely**, assume that every subsequence $\{x_{n_k}\}$ of a sequence $\{x_n\}$ has a subsequence $\left\{x_{n_{k_l}}\right\}$ that converges to x.
- **We show that $\{x_n\}$ is bounded and x is a unique cluster point of $\{x_n\}$.**
- If $\{x_n\}$ is unbounded, then $\{x_n\}$ has a subsequence $\{x_{n_k}\}$, such that
$$x_{n_k} \to \infty \text{ or } x_{n_k} \to -\infty.$$
- Such a subsequence has no further subsequence $\left\{x_{n_{k_l}}\right\}$ that converges to x.
- Therefore, $\{x_n\}$ **is bounded**.
- Suppose that $y \neq x$ is a cluster point of $\{x_n\}$, then by (2) there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that
$$x_{n_k} \to y.$$
- Since $\{x_{n_k}\}$, converges, by (3) every subsequence $\left\{x_{n_{k_l}}\right\}$ of $\{x_{n_k}\}$ also converges to y. **Contradiction.**
- Therefore $y = x$ and x is the unique cluster point of $\{x_n\}$.
- Since $\{x_n\}$ is bounded and it has a unique cluster point, by (4) , it follows that
$$x_n \to x \text{ as } n \to \infty.$$

This finishes our proof. ■

- **Example:** Find the set C of all cluster points of the sequence
$$x_n = \sin\left(\frac{n\pi}{2}\right), \; n = 1, 2, ...$$
- Notice that
$$x_n = \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{if} \quad n = 2k \\ 1 & \text{if} \quad n = 4k+1 \\ -1 & \text{if} \quad n = 4k+3 \end{cases}$$

- Therefore, we can show that

$$C = \{-1, 0, 1\}.$$

Exercise: Show (using the method as above) that

$$C = \{-1, 0, 1\}.$$

Example: Construct a sequence $\{x_n\}$ such that its set of cluster points is

$$C = \{x_1, x_2, ..., x_m\}.$$

Hint: Consider sequence

$$x_n = \begin{cases} x_1 + \frac{1}{n} & if & n = mk \\ x_2 + \frac{1}{n} & if & n = mk + 1 \\ \vdots & \vdots & \vdots \\ x_m + \frac{1}{n} & if & n = mk + (m-1) \end{cases}$$

Show that

$$C = \{x_1, x_2, ..., x_m\}.$$

Example: Construct a sequence $\{x_n\}$ such that its set of cluster points is

$$C = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}.$$

Hint: Consider set

$$\left\{\frac{1}{n} + \frac{1}{k} : n, k \in \mathbb{N}\right\}.$$

One can define a sequence $\{x_m\}$, such that

$$x_m = \frac{1}{n} + \frac{1}{k}$$

(for m there will be unique k and n).
Show that

$$C = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$$

for this sequence.

Example: Construct a sequence $\{x_n\}$ such that its set of cluster points is

$$C = [0, 1].$$

Hint: Consider sequence of all rational numbers form the interval $[0, 1]$.
Show that such a sequence has the set of all cluster points precisely $[0, 1]$.

Upper and Lower Limits

Example: As we showed, the set of its cluster points of the sequence

$$x_n = (-1)^n + \frac{1}{n}$$

and we showed that

$$C = \{-1, 1\}$$

is the set of all of its cluster points.

- Recall, for a real sequence $\{x_n\}$
  let C be the set of its all cluster points i.e.

$$C = \{x : \ x \text{ is a cluster point of } \{x_n\}\}.$$

- We define the upper and the lower limits of $\{x_n\}$ as follows.
  Definition For a real $\{x_n\}$ sequence,
  the *limit superior* of $\{x_n\}$ is defined as follows.
  Let C be the set of cluster points of $\{x_n\}$:
  If $C \neq \emptyset$ and $\{x_n\}$ is bounded from above, then

$$\limsup (x_n) = \sup (C).$$

  If $C = \emptyset$ and $\{x_n\}$ is bounded from above, then

$$\limsup (x_n) = -\infty.$$

  If $\{x_n\}$ is not bounded above, then

$$\limsup (x_n) = +\infty.$$

  Definition For a real $\{x_n\}$ sequence,
  the *limit inferior* of $\{x_n\}$ is defined as follows.
  Let C be the set of cluster points of $\{x_n\}$:
  If $C \neq \emptyset$ and $\{x_n\}$ is bounded from below, then

$$\liminf (x_n) = \inf (C).$$

  If $C = \emptyset$ and $\{x_n\}$ is bounded from below, then

$$\liminf (x_n) = +\infty$$

  If $\{x_n\}$ is not bounded below, then

$$\liminf (x_n) = -\infty.$$

- **Example**: Let $x_n = (-1)^n$.
  As we showed it before, the set of cluster points is

$$C = \{-1, 1\},$$

  and since $\{x_n\}$ is bounded,

- therefore, by the definition

$$\begin{aligned} \limsup (x_n) &= \sup \{-1, 1\} = 1 \text{ and} \\ \liminf (x_n) &= \inf \{-1, 1\} = -1. \end{aligned}$$

  Example: Let $x_n = n$.

- Clearly $\{x_n\}$ is not bounded above.

- Therefore,
$$\limsup (x_n) = +\infty.$$
- Moreover, since $\{x_n\}$ is bounded below and
  the set of cluster points of $\{x_n\}$ is empty, i.e.
$$C = \emptyset,$$
- It follows that
$$\liminf (x_n) = +\infty.$$
- Indeed $C = \emptyset$, since for $x \in \mathbb{R}$.
- If $x < 1$, then we take for
$$\epsilon = |1 - x| > 0$$
  and we see that
$$\begin{aligned} & \{n \in \mathbb{N} : |x_n - x| < \epsilon\} \\ = & \{n \in \mathbb{N} : |n - x| < \epsilon\} \\ = & \emptyset \end{aligned}$$
  so x is not a cluster point of (x_n).
- If $x \geq 1$, then by the Archimedean property of $\mathbb{R}$ and
  the principle of well-order for $\mathbb{N}$,
  there is the a unique $m \in \mathbb{N}$, such that
$$m \leq x < (m + 1).$$
- Then if
$$0 < \epsilon < \min \{|x - m|, |(m + 1) - x|\}$$
  the set
$$\begin{aligned} \{n \in \mathbb{N} : |x_n - x| < \epsilon\} &= \{n \in \mathbb{N} : |n - x| < \epsilon\} \\ &\subseteq \{m\} \end{aligned}$$
  so it is finite,
- hence x is not a cluster point of (x_n).
  Example: If $x_n = -n$,
  then (x_n) is not bounded below, so
$$\liminf (x_n) = -\infty.$$
  Since (x_n) is bounded above and
  the set of all of its cluster points C is empty, i.e.
$$C = \emptyset$$
- Therefore
$$\limsup (x_n) = -\infty.$$

- To show that the set of all cluster points of (x_n) is empty

  we can use a similar argument as we used in the proof in the previous example.

  Example: Let

$$x_n = \begin{cases} 1+\frac{1}{k} & if \quad n = 5k \\ 1-\frac{1}{k} & if \quad n = 5k+1 \\ 0 & if \quad n = 5k+2 \\ -1+\frac{1}{k} & if \quad n = 5k+3 \\ -1-\frac{1}{k} & if \quad n = 5k+4 \end{cases}$$

- One can show that

  the set C of all cluster points of $\{x_n\}$ is

$$C = \{-1, 0, 1\}.$$

- Indeed, we see that

$$\begin{aligned} \lim_{k\to\infty} x_{5k} &= \lim_{k\to\infty} x_{5k+1} = -1 \\ \lim_{k\to\infty} x_{5k+2} &= 0 \\ \lim_{k\to\infty} x_{5k+3} &= \lim_{k\to\infty} x_{5k+4} = 1. \end{aligned}$$

- Therefore, by theorem,

$$C \supseteq \{-1, 0, 1\}.$$

- If $x \notin \{-1, 0, 1\}$, then

$$\epsilon = \frac{1}{3}\min\{|x-1|, |x|, |x+1|\} > 0.$$

- One argues that the set

$$\{n \in \mathbb{N} : |x_n - x| < \epsilon\}$$

  is finite (we leave this to show as an exercise).

- Therefore, we see that

$$C = \{-1, 0, 1\}.$$

- Notice that

$$\begin{aligned} -2 &\leq x_n \leq 2, \text{ so} \\ |x_n| &\leq 2, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

- Therefore, $\{x_n\}$ is bounded above and below, so

$$\begin{aligned} \liminf\{x_n\} &= \inf C = \inf\{-1, 0, 1\} = -1 \text{ and} \\ \limsup\{x_n\} &= \sup C = \sup\{-1, 0, 1\} = 1. \end{aligned}$$