§3. Constructing Topologies

Math 4341 (Topology)

Subspace topology

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- **Definition**. Let $(X, \mathcal{T})$ be a topological space, and let $Y \subset X$ be any subset of X. Then the collection

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- **Lemma 3.1**. The collection $\mathcal{T}_Y$ defines a topology on Y.
- *Proof*. (T1) is obvious. (T2) and (T3) follow from De Morgan's Laws:

$$\bigcup_{i \in I}(Y \cap U_i) = Y \cap \bigcup_{i \in I} U_i, \qquad \bigcap_{i \in I}(Y \cap U_i) = Y \cap \bigcap_{i \in I} U_i.$$

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- **Example**. The subspace topology on $\mathbb{Z} \subset \mathbb{R}$ is the discrete topology on $\mathbb{Z}$: the set $\{n\}$ is open in $\mathbb{Z}$ for any integer n.

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- **Example**. The subspace topology on $\mathbb{Z} \subset \mathbb{R}$ is the discrete topology on $\mathbb{Z}$: the set $\{n\}$ is open in $\mathbb{Z}$ for any integer n.
    - On the other hand, the subspace topology on $\mathbb{Q} \subset \mathbb{R}$ is *not* the discrete topology, essentially because any non-empty open interval in $\mathbb{R}$ contains infinitely many rational numbers.

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- *Proof.* (i) follows from $\iota^{-1}(U) = U \cap Y$.
  - (ii) follows from $f|_Y^{-1}(U) = f^{-1}(U) \cap Y$.
  - (iii): Suppose $F \subset Y$ is closed in Y. Then there is an open set U in X so that $Y \setminus F = Y \cap U$. Note that

$$F = Y \setminus (Y \setminus F) = Y \setminus (Y \cap U) = Y \setminus U = Y \cap (X \setminus U).$$

Suppose $F = Y \cap G$ for some closed set $G \subset X$. Then

$$Y \setminus F = Y \setminus (Y \cap G) = Y \cap (X \setminus G),$$

which is open in Y.

Subspace topology: The pasting lemma

- **Lemma 3.5**. Let X be a topological space, and let $U, V \subset X$ be two open subsets such that $X = U \cup V$. Let $f : U \to Y$ and $g : V \to Y$ be two functions so that $f|_{U \cap V} = g|_{U \cap V}$. Then f and g are continuous w.r.t. the subspace topologies on U and V if and only if the function $h : X \to Y$ given by

$$h(x) = \begin{cases} f(x) & \text{if } x \in U, \\ g(x) & \text{if } x \in V, \end{cases}$$

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- *Proof*. $(\Rightarrow)$ follows from

$$\begin{aligned} h^{-1}(W) &= \{x \in X \mid h(x) \in W\} \\ &= \{x \in U \mid h(x) \in W\} \cup \{x \in V \mid h(x) \in W\} \\ &= \{x \in U \mid f(x) \in W\} \cup \{x \in V \mid g(x) \in W\} \\ &= f^{-1}(W) \cup g^{-1}(W). \end{aligned}$$

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- Note that if $Y \subset X$ is open (resp. closed) in X and $U \subset Y$ is open (resp. closed) in Y, then U is open (resp. closed) in X.

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- **Definition**. Let $\{X_i\}_{i \in I}$ be an indexed family of sets; let $X = \bigcup_{i \in I} X_i$. The *Cartesian product* of this indexed family, denoted by

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  - That is, it is the set of all functions

$$x : I \to \bigcup_{i \in I} X_i$$

such that $x(i) \in X_i$ for each $i \in I$.

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- **Definition**. Let $\{X_i\}_{i\in I}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space
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the collection of all sets of the form
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  - The topology generated by this basis is called the *box topology*.

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  - The topology $\mathcal{T}_{\mathcal{C}}$ generated by $\mathcal{C}$ consists of all unions of all finite intersections of elements in $\mathcal{C}$.
- **Remark**. $\mathcal{T}_{\mathcal{C}}$ is the coarsest topology containing $\mathcal{C}$, meaning that it has as few open sets as possible while still including the elements in $\mathcal{C}$ as open sets.

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  - For every $i \in I$, there is a natural map $\pi_i : X \to X_i$, called the projection onto X_i, which maps $\pi_i(x) = x(i)$, where we think of $x \in X$ as a map $I \to \bigcup_{i \in I} X_i$.

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  - We define a topology on X, called the *product topology*, to be the coarsest topology such that π_i is continuous for every i.
- **Remark**. The product topology on $X = \prod_{i \in I} X_i$ is generated by the subbasis $\mathcal{C}$ which consists of all sets of the form $\pi_i^{-1}(U)$, where U is an open set in X_i.

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  - Let U and V be open sets in X_1 and X_2 respectively. Then $\pi_1^{-1}(U) = U \times X_2$ and $\pi_2^{-1}(V) = X_1 \times V$ are examples of open sets in $X_1 \times X_2$. Their intersection is the set $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V$, and the topology on $X_1 \times X_2$ consists of all unions of this form.

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  - In symbols, if we let

$$\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_{X_1}, V \in \mathcal{T}_{X_2}\},$$

then $\mathcal{B}$ is a basis for the product topology on $X_1 \times X_2$.

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  - In symbols, if we let

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    then $\mathcal{B}$ is a basis for the product topology on $X_1 \times X_2$.
- Similarly, open sets in $X = \prod_{i \in I} X_i$ are unions of sets of the form $\prod_{i \in I} U_i$, where U_i is open in X_i for each $i \in I$, and $U_i = X_i$ for all but finitely many i.

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- **Theorem 3.8**. The box topology on $\prod X_i$ has as basis all sets of the form $\prod U_i$, where U_i is open in X_i for each i. The product topology on $\prod X_i$ has as basis all sets of the form $\prod U_i$, where U_i is open in X_i for each i and U_i equals X_i except for finitely many values of i.

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- **Remark**. Two things are clear. First, for finite products $\prod_{i=1}^{n} X_i$ the two topologies are precisely the same. Second, the box topology is in general finer than the product topology.
- What is not so clear is why we prefer the product topology to the box topology. We will find that a number of important theorems about finite prducts will also hold for arbitrary products if we use the product topology, but not if we use the box topology.

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- **Theorem 3.9**. Let X be a topological space, and let $\{Y_i\}_{i \in I}$ be a family of topological spaces. A function $f : X \to \prod_{i \in I} Y_i$ consists of a family of functions $\{f_i\}_{i \in I}$ where $f_i : X \to Y_i$ for all $i \in I$. Then f is continuous iff f_i is continuous for every i.

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- *Proof*. There are two things to prove.
  - Suppose f is continuous. Since each π_i is continuous, so is every $f_i = \pi_i \circ f$.
  - Suppose all the f_i are continuous. We will show that the preimages of elements of the subbasis are open. That is, let U be an open set in $\prod_{i\in I} Y_i$ of the form $U = \pi_j^{-1}(V)$ where V is open in Y_j. Then $f^{-1}(U) = f^{-1}(\pi_i^{-1}(V)) = f_i^{-1}(V)$, which is open by assumption.

- **Example**. Consider $\mathbb{R}^\omega$, the countably infinite product of $\mathbb{R}$ with itself. That is
$$\mathbb{R}^\omega = \prod_{i=1}^{\infty} X_i$$
where $X_i = \mathbb{R}$ for each i. Let us define a function $f : \mathbb{R} \to \mathbb{R}^\omega$ by the equation
$$f(t) = (t, t, t, \cdots);$$
the ith coordinate function of f is the function $f_i(t) = t$.

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the ith coordinate function of f is the function $f_i(t) = t$.
  - Each of the coordinate functions $f_i : \mathbb{R} \to \mathbb{R}$ is continuous; therefore, the function f is continuous if $\mathbb{R}^\omega$ is given the product topology.

The product topology

- But f is not continuous if $\mathbb{R}^\omega$ is given the box topology. Consider, for example, the basis element

$$B = (-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots$$

for the box topology. We assert that $f^{-1}(B)$ is not open in $\mathbb{R}$. If $f^{-1}(B)$ were open in $\mathbb{R}$, it would contain some interval $(-\delta, \delta)$ about the point 0. This would mean that $f((-\delta, \delta)) \subset B$. Applying π_i to both sides of the inclusion we obtain

$$f_i((-\delta, \delta)) = (-\delta, \delta) \subset (-1/i, 1/i)$$

fo all i, a contradiction.