`Math 4301 Mathematical Analysis I`
`Lecture 9`
`Topic: Connected sets and limits`

- **Connected subsets of $\mathbb{R}$**

  Definition $A \subseteq \mathbb{R}$ is *disconnected* if there is a pair U and V of subsets of $\mathbb{R}$ such that

i) U and V are both open and nonempty;

ii) U and V are disjoint, i.e. $U \cap V = \emptyset$;

iii) $A = (A \cap U) \cup (A \cap V)$, where $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$.

A pair of subsets of $\mathbb{R}$ that satisfies **i**) $-$ **iii**) as the above is called a *separation* of A.

We say that A set is *connected* if it not disconnected (or equivalently there is *no separation* of A).

Example: Let $A = \emptyset \subset \mathbb{R}$,

then A is connected since A has no separation.

- Otherwise, there will be open and disjoint subsets U, V of $\mathbb{R}$, such that $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$, but $A = \emptyset$, so

$$\begin{aligned} A \cap U &= \emptyset \cap U = \emptyset \text{ and} \\ A \cap V &= \emptyset \cap V = \emptyset. \end{aligned}$$

So there is no separation.

Example: Let $A = \{a\}$, $a \in \mathbb{R}$,

then A has no separation, hence it is connected.

- Indeed, if U, V is a separation of A,

  then

$$A = (A \cap U) \cup (A \cap V).$$

Since $a \in A$,

then $a \in (A \cap U) \cup (A \cap V)$, so

$a \in (A \cap U)$ or $a \in (A \cap V)$.

- If $a \in (A \cap U)$,

  then $a \in U$, so $a \notin V$.

- Since $U \cap V = \emptyset$,

  $a \notin A \cap V$.

- Notice that $A = \{a\}$, so

- since $a \notin A \cap V$,

  it follows that $A \cap V = \emptyset$.

  Example: Let $A = \{1, 2\} \subset \mathbb{R}$ is *disconnected*.

- We show that A has a separation.

- Indeed, subsets

$$U = \left(0, \frac{3}{2}\right)$$

and

$$V = \left(\frac{3}{2}, 3\right)$$

are both open and nonempty, disjoint and

$$\begin{aligned} A &= (A \cap U) \cup (A \cap V) \\ &= \{1, 2\}, \end{aligned}$$

where

$$A \cap U = \{1\} \neq \emptyset$$

and

$$A \cap V = \{2\} \neq \emptyset.$$

Therefore, the pair of subsets U and V of $\mathbb{R}$ is a separation of A, so A is disconnected.

Example: Let $x \in \mathbb{R}$ and $A = \mathbb{R} \backslash \{x\}$.

- Then $U = (-\infty, x)$ and $V = (x, \infty)$ is a separation of A since both U and V are open, nonempty, disjoint and

$$A = (A \cap U) \cup (A \cap V),$$

where $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$.

- Therefore, $A = \mathbb{R} \backslash \{x\}$ is disconnected.

Example: Let $A = \mathbb{Q} \subset \mathbb{R}$ and $x \in \mathbb{R} \backslash \mathbb{Q}$.

- Define $U = (-\infty, x)$ and $V = (x, \infty)$ is a separation of A since both U and V are open, nonempty, disjoint and

$$A = (A \cap U) \cup (A \cap V),$$

where $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$.

- Therefore, $\mathbb{Q}$ is disconnected.
- For $a < b$, let

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

is called an open interval.

- In general, we define an interval as follows

Definition Let $I \subseteq \mathbb{R}$. We say that I is an *interval* if

$$\forall x, y \in I, \ \forall z \in \mathbb{R}, \ (x < z < y) \Rightarrow z \in I.$$

- It follows from the definition of the interval that if I is an interval in $\mathbb{R}$ then

$$I = \begin{cases} \emptyset \\ \{a\} \\ [a,b] \\ (a,b] \\ [a,b) \\ (a,b) \end{cases}, \text{ where } a,b \in \mathbb{R} \text{ and } a < b$$

$$\text{or } I = \begin{cases} (-\infty,a] \\ [a,\infty) \\ (-\infty,a) \\ (a,\infty) \\ (-\infty,\infty) \end{cases}, \text{ where } a \in \mathbb{R}.$$

- **Theorem** Let I be interval in $\mathbb{R}$.
  Then I is *connected*.

Proof. We show that I has no separation.

- If $I = \emptyset$ or $I = \{a\}$, for some $a \in \mathbb{R}$,
  then I is connected since there is no separation of A.
- **Assume that** $I \neq \emptyset$ **and** $I \neq \{a\}$.
- *Suppose by contradiction* that I is disconnected and
  let U and V be a separation of I.
- Since $I \cap U$ and $I \cap V$ are nonempty
  then there is $a \in I \cap U$ and $b \in I \cap V$.
- **We may assume without lose of generality that** $a < b$**.**
- Since $[a,b]$ is nonempty and bounded,
  the set $[a,b] \cap U \subseteq [a,b]$ is nonempty and bounded.
- By *completeness* of $\mathbb{R}$, there is $\alpha \in \mathbb{R}$, such that

$$\alpha = \sup\left([a,b] \cap U\right).$$

- Since, for all $x \in [a,b] \cap U$,

$$a \leq x \leq b,$$

  it follows that $\alpha \leq b$.
- Moreover, since $a \in [a,b] \cap U$, $a \leq \alpha$.
- Therefore,

$$a \leq \alpha \leq b.$$

- If $x \in [a,b) \cap U$,
  then since U is open and $x \in U$,
  there is $0 < \epsilon < (b-x)$, such that

$$[x, x+\epsilon) \subset U.$$

- Therefore, $x \neq \alpha$ so, in particular,
$$a < \alpha$$
and
if $\alpha < b$,
then $\alpha \notin U$ (otherwise $\alpha \in [a, b) \cap U$, a contradiction)
- If $y \in (a, b] \cap V$,
since V is open and $y \in V$,
there is $0 < \epsilon < (y - a)$, such that
$$(y - \epsilon, y] \subset V.$$
- Since
$$U \cap V = \emptyset,$$
therefore
$$(y - \epsilon, y] \cap U = \emptyset$$
and $y \neq \alpha$.
- Hence, in particular, $\alpha < b$ and
if $a < \alpha$,
then $\alpha \notin V$ (otherwise $\alpha \in (a, b] \cap V$, a contradiction).
- Since, as we showed, $a < \alpha < b$.
- Furthermore, because $a, b \in I$ and $a < \alpha < b$,
therefore
$$\alpha \in I = (I \cap U) \cup (I \cap V) .$$
- Hence $\alpha \in I \cap U$ or $\alpha \in I \cap V$ which is impossible since as we showed, if
$$a < \alpha < b$$
then $\alpha \notin U$ and $\alpha \notin V$.

This completes our proof. ■

- **Proposition** If $A \subseteq \mathbb{R}$ is connected then A is an interval.

Proof. We show that if A is not an interval then A has a separation.

- Suppose that $A \subseteq \mathbb{R}$ is connected and A is not an interval.
- Then there are $a, b \in A$ and $c \in \mathbb{R}$, such that
$a < c < b$ and $c \notin A$.
- Let $U = (-\infty, c)$ and $V = (c, \infty)$.
- As we see U and V are open, nonempty and disjoint.
- Moreover, $a \in A \cap U$ and $b \in A \cap V$, so
both $A \cap U$ and $A \cap V$ are nonempty.

- Since
$$A = (A \cap U) \cup (A \cap V),$$
it follows that U and V is a separation of A.
- This contradicts to our assumption that A is connected.

This completes our proof. ■

- **Corollary** $A \subseteq \mathbb{R}$ is connected iff A is an interval.
  Limits of functions
  Definition Let $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$, and
  suppose that $c \in \mathbb{R}$ is an *accumulation point* of A ($c \in A'$).
  Then
  $$\lim_{x\to c} f(x) = L$$
  if for every $\epsilon > 0$ there is $\delta > 0$ such that, for all $x \in A$, if
  $$0 < |x - c| < \delta,$$
  then
  $$|f(x) - L| < \epsilon.$$
  We write $f(x) \to L$ as $x \to c$.
  Remark: It is important to note that $c \in A'$ rather than $c \in A$.
- In particular, if c is an isolated point of A,
  then $\lim_{x\to c} f(x)$ is **not defined**.
- Also, notice that we may also write
  $|f(x) - L| \to 0$ as $x \to c$.
- For instance $f(x) = \frac{1}{x}$ is defined for $x \in \mathbb{R}\setminus\{0\}$.
- Notice that $0 \notin \mathbb{R}\setminus\{0\}$, but we can still be asked to compute
  $$\lim_{x\to 0} f(x).$$
- This is because $0 \in (\mathbb{R}\setminus\{0\})'$ (Note: $\mathbb{R}\setminus\{0\} = (-\infty, 0) \cup (0, \infty)$)
  Example: Let $A = (0, 1) \cup \{2\}$ and
  define $f : A \to \mathbb{R}$, $f(x) = 2$, $x \in A$.
- Can we define $\lim_{x\to 2} f(x)$?
- Notice that $x = 2$ is not in A', so
  $\lim_{x\to 2} f(x)$ is not defined for $x = 2$.

- This is because, if $\delta < \frac{1}{2}$,
  then no point $x \in A$ satisfies
$$0 < |x - 2| < \delta$$
  therefore, for any $L \in \mathbb{R}$, implication
$$0 < |x - 2| < \delta \Rightarrow |f(x) - L| < \epsilon$$
  is true.

- Therere will be no uniquness for the number L, so
  the notion of the limit will not be well defined.

- To avoid this, we define the limit operation for accumulation points of the domain only.
  Example: Let
$$f : \mathbb{R} \setminus \{2\} \to \mathbb{R},\ f(x) = \frac{x^2 - 4}{x - 2}.$$
  We show that
$$\lim_{x \to 2} f(x) = 4.$$

- We observe that
$$c = 2 \in A'$$
  where $A = \mathbb{R} \setminus \{2\}$, so
  we may ask if $\lim_{x \to 2} f(x)$ exists for $c = 2$.

- Let $\epsilon > 0$ be given.

- If we assume that $x \in \mathbb{R} \setminus \{2\}$ and
  $0 < |x - 2| < \delta$, then
$$|f(x) - 4| = \left| \frac{x^2 - 4}{x - 2} - 4 \right| = \left| \frac{(x - 2)(x + 2)}{x - 2} - 4 \right|$$

- Now, since $0 < |x - 2|$,
  we see that $x \neq 2$, so
$$\frac{x^2 - 4}{x - 2} = x + 2.$$

- Therefore,
$$\left| \frac{(x - 2)(x + 2)}{x - 2} - 4 \right| = |(x + 2) - 4| = |x - 2| < \delta.$$

- Consequently, if $\epsilon > 0$ is given,
  we take for $\delta = \frac{\epsilon}{2}$ (or δ and positive number such that $\delta < \epsilon$).

- Then, for all $x \in \mathbb{R} \setminus \{2\}$
  if $0 < |x - 2| < \delta$,
$$|f(x) - 4| = |x - 2| < \delta = \frac{\epsilon}{2} < \epsilon.$$
  It follows that $\lim_{x \to 2} f(x) = 4$.

Proposition Let $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$ and $c \in A'$.

If

$$\lim_{x \to c} f(x) = L_1$$

and

$$\lim_{x \to c} f(x) = L_2,$$

then

$$L_1 = L_2.$$

Therefore, if $\lim_{x \to c} f(x)$ exists then it is unique.

Proof. Suppose that $L_1 \neq L_2$, then $\epsilon = \frac{1}{3}|L_1 - L_2| > 0$.

- Since

  $$\lim_{x \to c} f(x) = L_1 \text{ and } \lim_{x \to c} f(x) = L_2,$$

  there are $\delta_1 > 0$ and $\delta_2 > 0$, such that,

  for all $x \in A$, if

  $$0 < |x - c| < \delta_1 \text{ then } |f(x) - L_1| < \epsilon$$

  and

  $$0 < |x - c| < \delta_2 \text{ then } |f(x) - L_2| < \epsilon$$

- Now, if $\delta = \min\{\delta_1, \delta_2\} > 0$,

  then for every $x \in A$,

  if $0 < |x - c| < \delta$, we

  $$|f(x) - L_1| < \epsilon \text{ and } |f(x) - L_2| < \epsilon.$$

- Therefore, for every $x \in A$,

  if $0 < |x - c| < \delta$, then

  $$\begin{aligned} 3\epsilon &= |L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \\ &\leq |f(x) - L_1| + |f(x) - L_2| < 2\epsilon, \text{ so} \\ 3\epsilon &< 2\epsilon, \text{ thus since } \epsilon > 0, \text{ we have } 3 < 2, \text{ a contradiction.} \end{aligned}$$

This finishes our proof. ■

- **Proposition** Let $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$ and $c \in A'$.

  Then $\lim_{x \to c} f(x) = L$

  if and only if

  for every sequence $\{x_n\} \subseteq A \setminus \{c\}$

  if $\lim_{n \to \infty} x_n = c$ then

  $$\lim_{n \to \infty} f(x_n) = L.$$

Proof. We show that conditions $\lim_{x \to c} f(x) = L$ and

- *for every sequence* $\{x_n\} \subseteq A \setminus \{c\}$

  if $\lim_{n \to \infty} x_n = c$

  then $\lim_{n \to \infty} f(x_n) = L$ are equivalent.

- **Assume that** $\lim_{x\to x} f(x) = L$ **and let** $\{x_n\} \subseteq A\setminus\{c\}$ **and**
$$\lim_{n\to\infty} x_n = c.$$
- Let $\epsilon > 0$ be given.
- Since $\lim_{x\to c} f(x) = L$,
there is $\delta > 0$, such that,
for every $x \in A\setminus\{c\}$,
if $0 < |x - c| < \delta$, then
$$|f(x) - L| < \epsilon.$$
- Since $\lim_{n\to\infty} x_n = c$,
there is $N \in \mathbb{N}$, such that for all $n > N$,
$$|x_n - c| < \delta.$$
- Since $\{x_n\} \subseteq A\setminus\{c\}$, for all $n > N$,
$$0 < |x_n - c| < \delta.$$
- Therefore, for $n > N$,
$$|f(x_n) - L| < \epsilon.$$
- It follows that
$$\lim_{n\to\infty} f(x_n) = L.$$
- **Conversely, assume by contradiction that**
$$\lim_{x\to c} f(x) \neq L$$
and *for every sequence* $\{x_n\} \subseteq A\setminus\{c\}$
if $\lim_{n\to\infty} x_n = c$ *then*
$$\lim_{n\to\infty} f(x_n) = L.$$
- Since $\lim_{x\to c} f(x) \neq L$,
there is $\epsilon > 0$, such that,
for every $\delta > 0$, there is $x \in A$, such that
$$0 < |x - c| < \delta$$
and
$$|f(x) - L| \geq \epsilon.$$
- We take $\delta = \frac{1}{n} > 0$, $n = 1, 2, ...$
- Since $c \in A'$,
$$D\left(c, \frac{1}{n}\right) \cap A\setminus\{c\} \neq \emptyset,$$
so let
$$x_n \in D\left(c, \frac{1}{n}\right) \cap A\setminus\{c\}.$$

- We notice that the sequence
$$\{x_n\} \subseteq A \backslash \{c\}$$
and, for all $n \in \mathbb{N}$,
$$|x_n - c| < \frac{1}{n}.$$

- Moreover, for all $n \in \mathbb{N}$,
$$|f(x_n) - L| \geq \epsilon$$

- Clearly, $x_n \to c$.
Indeed, for $\delta > 0$,
there is $N \in \mathbb{N}$, such that,
$$\frac{1}{N} < \delta.$$

- Since
$$D\left(c, \frac{1}{N}\right) \cap A \backslash \{c\} \supseteq D\left(c, \frac{1}{n}\right) \cap A \backslash \{c\},$$
for $n > N$,
then $|x_n - c| < \frac{1}{N}$, for $n > N$.

- It follows that,
for $\delta > 0$, there is $N \in \mathbb{N}$, such that,
for $n > N$,
$$|x_n - c| < \frac{1}{N} < \delta.$$

- Since
$$\{x_n\} \subseteq A \backslash \{c\}$$
and $x_n \to c$ as $n \to \infty$,
then by our assumption
$$\lim_{n \to \infty} f(x_n) = L.$$

- Hence, there is $N_1 \in \mathbb{N}$, such that,
for $n > N_1$,
$$|f(x_n) - L| < \epsilon.$$

- Let $n > N_1$,
then by construction of $\{x_n\}$,
$$|x_n - c| < \frac{1}{n} \text{ and } |f(x_n) - L| \geq \epsilon.$$

- Therefore, for $n > N_1$,
$$\epsilon \leq |f(x_n) - L| < \epsilon.$$

  Contradiction.

This finishes our proof. ■

- **Remark** Let $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$ and $c \in A'$.
- We observe that $\lim_{x \to c} f(x)$ does not exist if
- there are sequences $\{x_n\}$,

$$\{y_n\} \subseteq A \backslash \{c\},$$

such that

$$\lim_{n \to \infty} x_n = c = \lim_{n \to \infty} y_n$$

and

$$\lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n)$$

- There is a sequence

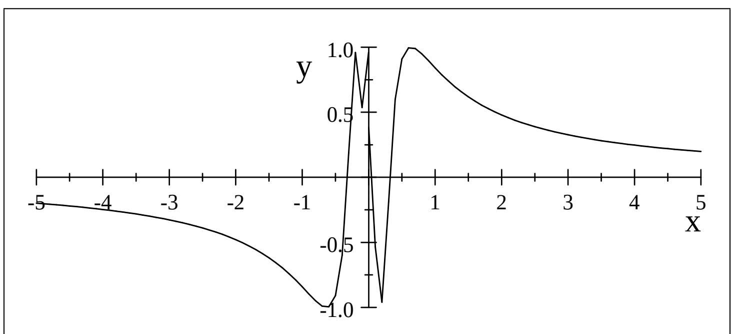
$$\{x_n\} \subseteq A \backslash \{c\} \text{ and } \lim_{n \to \infty} x_n = c$$

such that

- $\lim_{n \to \infty} f(x_n)$ does not exist or
- $\{f(x_n)\}$ is divergent to $\pm\infty$.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$



$f(x) = \sin\left(\frac{1}{x}\right)$

- We see that if $x_n = \frac{1}{n\pi}$ and $y_n = \frac{1}{\frac{\pi}{2} + 2n\pi}$,

then

$$x_n, y_n \in \mathbb{R} \backslash \{0\}$$

and

$$\lim_{n \to \infty} x_n = 0 = \lim_{n \to \infty} y_n.$$

- However,

$$\begin{aligned} \lim_{n \to \infty} f(x_n) &= \lim_{n \to \infty} \sin\left(\frac{1}{x_n}\right) = \lim_{n \to \infty} \sin(n\pi) = 0 \text{ and} \\ \lim_{n \to \infty} f(y_n) &= \lim_{n \to \infty} \sin\left(\frac{1}{y_n}\right) = \lim_{n \to \infty} \sin\left(\frac{\pi}{2} + 2n\pi\right) = 1, \end{aligned}$$

so

$$\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n).$$

Therefore, f has no limit at $c = 0$.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{x} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

We show that f has no limit at $x = 0$.

- Let $x_n = \frac{1}{n}$, $n = 1, 2, ...$
- Then $x_n \in \mathbb{R}\backslash\{0\}$, for all $n \in \mathbb{N}$, so
  $\{x_n\} \subseteq \mathbb{R}\backslash\{0\}$ and
  $x_n \to 0$ as $n \to \infty$.
- However, we see that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{1}{x_n} = \lim_{n\to\infty} n = \infty$$

diverges, so f has no limit at $c = 0$.

Theorem Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ and $c \in A'$, $\alpha, \beta \in \mathbb{R}$.

If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = K$, then

1. $\lim_{x\to c} (\alpha f(x) + \beta g(x)) = \alpha L + \beta K$
2. $\lim_{x\to c} f(x) g(x) = LK$
3. If $K \neq 0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{K}.$$

Proof. Exercise. $\blacksquare\backslash$

- **Definition** Let $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$ and define

$$\begin{aligned} \inf_A(f) &= \inf\{f(x) : x \in A\} \text{ and} \\ \sup_A(f) &= \sup\{f(x) : x \in A\}. \end{aligned}$$

We say that f is bounded if both $\inf_A(f)$ and $\sup_A(f)$ are finite.

Example: Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{x} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}.$$

- *We show that* $\inf_{[0,1]}(f) = 0$.
  Indeed, for all $x \in [0, 1]$,
- if $x \neq 0$, then

$$f(x) = \frac{1}{x} > 0$$

- if $x = 0$, then
$$f(0) = 0.$$
- Therefore, for every $x \in [0,1]$,
$$f(x) \geq 0.$$
- It follows that 0 is a lower bound for
$$\{f(x) : x \in [0,1]\}.$$
- Hence,
$$\inf_{[0,1]}(f) \geq 0.$$
- Now, let $\epsilon > 0$ be given.
- Since
$$0 \in \{f(x) : x \in [0,1]\},$$
then there is
$$y \in \{f(x) : x \in [0,1]\},$$
such that
$$y < 0 + \epsilon.$$
- It follows that,
$$\inf_{[0,1]}(f) = 0.$$
- *Now, we show that*
$$\sup_A(f) = \infty.$$
- It is sufficient to show that
$$\{f(x) : x \in [0,1]\}$$
is not bounded above.
- Let $M > 0$ be given.
- There is $n \in \mathbb{N}$, such that $n > M$.
- Since $n \geq 1$, $\frac{1}{n} \in [0,1]$ and
$$f\left(\frac{1}{n}\right) = \frac{1}{\frac{1}{n}} = n > M.$$
- Therefore, for every $M > 0$,
there is
$$y \in \{f(x) : x \in [0,1]\},$$
such that $y > M$.

- It follows that
$$\sup_A (f) = \infty.$$

  Remark: We notice that a function $f : A \to \mathbb{R}$,
  where $A \subseteq \mathbb{R}$, is bounded
  if and only if
  there is $M \geq 0$, such that,
  for all $x \in A$,
$$|f(x)| \leq M.$$

  Definition Let $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$ and $c \in A'$.
  We say that f is *locally bounded* at c
  if there is $U \subseteq \mathbb{R}, U$ - open, $c \in U$ and $M \geq 0$, such that,
  for all $x \in U \cap A$,
$$|f(x)| \leq M.$$

  Proposition Let $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$ and $c \in A'$ and $\lim_{x \to c} f(x) = L$.
  Then there is $U \subseteq \mathbb{R}, U$ - open, $c \in U$, and $M \geq 0$, such that,
  for all $x \in U \cap A$,
$$|f(x)| \leq M.$$

  That is, if f has limit at c then f is locally bounded at c.

Proof. Exercise. ■

- **Example**: Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined by
$$f(x) = \frac{1}{x}.$$

- We show that for every $U \subseteq \mathbb{R}, U$ - open, $0 \in U$, and
  for every $M \geq 0$,
  there is $x \in U \cap \mathbb{R} \setminus \{0\}$, such that
$$f(x) > M.$$

- Indeed, let $U \subseteq \mathbb{R}, U$ - open and $0 \in U$.
- Since U is open, there is $\delta > 0$, such that,
$$0 \in (-\delta, \delta) \subseteq U.$$

- Since $\delta > 0$,
  there is $n_1 > \frac{1}{\delta}$.
- Furthermore, if $M \geq 0$ then
  there is $n_2 \in \mathbb{N}$, such that,
$$M < n_2.$$

- Let $n = \max\{n_1, n_2\}$.

- Since $n \geq n_1$,
  it follows that
$$\frac{1}{n} < \delta$$
  and since $n \geq n_2$,
  $n > M$.
- Therefore,
$$\frac{1}{n} \in (-\delta, \delta) \cap \mathbb{R} \backslash \{0\} \subseteq U \cap \mathbb{R} \backslash \{0\},$$
  so
$$\frac{1}{n} \in U \cap \mathbb{R} \backslash \{0\}$$
  and
$$f\left(\frac{1}{n}\right) = n > M.$$
  As we showed, f is not locally bounded at $c = 0$.
  Example: Let $f : \mathbb{R} \backslash \{0\} \to \mathbb{R}$,
$$f(x) = \sin\left(\frac{1}{x}\right).$$
  We show that f has no limit at $c = 0$ but f is locally bounded at $c = 0$.
- Since, for every $x \in \mathbb{R} \backslash \{0\}$,
$$\left|\sin\left(\frac{1}{x}\right)\right| \leq 1,$$
  then clearly, f is locally bounded at $c = 0$.
- However, as we showed it before,
  f has no limit at $c = 0$.
  Definition Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $c \in A'_{c^+}$, where
$$A_{c^+} = \{x \in A : c < x\}.$$
  We say that
$$\lim_{x \to c^+} f(x) = L,$$
  if for every $\epsilon > 0$,
  there is $\delta > 0$, such that,
  for all $x \in A$, if
$$0 < x - c < \delta,$$
  then
$$|f(x) - L| < \epsilon.$$
  The limit $\lim_{x \to c^+} f(x)$ is called the right limit of f at c.
  Definition Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $c \in A'_{c^-}$, where
$$A_{c^-} = \{x \in A : x < c\}.$$

We say that

$$\lim_{x \to c^-} f(x) = L,$$

if for every $\epsilon > 0$,
there is $\delta > 0$, such that,
for all $x \in A$, if

$$0 < c - x < \delta,$$

then

$$|f(x) - L| < \epsilon.$$

The limit $\lim_{x \to c^-} f(x)$ is called the right limit of f at c.

Theorem Let $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$ and $c \in A'_{c^+} \cap A'_{c^-}$.
Then

$$\lim_{x \to c} f(x) = L$$

if and only if

$$\lim_{x \to c^+} f(x) = L = \lim_{x \to c^-} f(x).$$

Proof. We show that conditions

$$\lim_{x \to c} f(x) = L$$

and

$$\lim_{x \to c^+} f(x) = L = \lim_{x \to c^-} f(x)$$

are equivalent.

- *Assume that,* $c \in A'_{c^+} \cap A'_{c^-}$ *and* $\lim_{x \to c} f(x) = L$.
- *We show that*

$$\lim_{x \to c^+} f(x) = L = \lim_{x \to c^-} f(x).$$

- Let $\epsilon > 0$ be given.
- Since $\lim_{x \to c} f(x) = L$,
  there is $\delta > 0$, such that,
  for all $x \in A$, if

$$0 < |x - c| < \delta,$$

  then

$$|f(x) - L| < \epsilon.$$

- Observe that,
  if $x \in A$ and

$$0 < x - c < \delta,$$

  then

$$0 < |x - c| = x - c < \delta,$$

  hence

$$|f(x) - L| < \epsilon.$$

- Therefore, $\lim_{x \to c^+} f(x) = L$.
- Analogously, if $x \in A$ and
$$0 < c - x < \delta,$$
then
$$0 < |x - c| = c - x < \delta,$$
hence
$$|f(x) - L| < \epsilon.$$
- Therefore,
$$\lim_{x \to c^-} f(x) = L.$$
- It follows that
$$\lim_{x \to c^+} f(x) = L = \lim_{x \to c^-} f(x)$$
- *Conversely, assume that*
$$\lim_{x \to c^+} f(x) = L = \lim_{x \to c^-} f(x).$$
We show that
$$\lim_{x \to c} f(x) = L.$$
- Since
$$\lim_{x \to c^+} f(x) = L,$$
then
for $\epsilon > 0$, there is $\delta_1 > 0$, such that,
for all $x \in A$, if
$$0 < x - c < \delta_1,$$
then
$$|f(x) - L| < \epsilon.$$
- Since
$$\lim_{x \to c^-} f(x) = L,$$
then for $\epsilon > 0$,
there is $\delta_2 > 0$, such that,
for all $x \in A$, if
$$0 < c - x < \delta_2,$$
then
$$|f(x) - L| < \epsilon.$$
- Let
$$\delta = \min\{\delta_1, \delta_2\}$$
and let $x \in A$ and assume that
$$0 < |x - c| < \delta.$$

- **If** $x > c$, then $x - c > 0$ and
$$x - c = |x - c| < \delta \leq \delta_1.$$
- Therefore,
$$0 < x - c < \delta_1,$$
so
$$|f(x) - L| < \epsilon.$$
- **If** $x < c$, then $c - x > 0$ and
therefore,
$$c - x = |x - c| < \delta \leq \delta_2.$$
- Hence,
$$0 < c - x < \delta_2,$$
so
$$|f(x) - L| < \epsilon.$$
- We showed, that,
for every $x \in A$,
if
$$0 < |x - c| < \delta,$$
then
$$|f(x) - L| < \epsilon.$$
- Consequently,
$$\lim_{x \to c} f(x) = L.$$

This finishes our proof. ■

- **Definition** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and assume that A is **not bounded above**.
We say that
$$\lim_{x \to \infty} f(x) = L,$$
if for every $\epsilon > 0$,
there is $M \in \mathbb{R}$, such that,
for all $x \in A$, if $x > M$ then
$$|f(x) - L| < \epsilon.$$

  Definition Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and assume that A is **not bounded below**.
We say that
$$\lim_{x \to -\infty} f(x) = L,$$
if for every $\epsilon > 0$,
there is $M \in \mathbb{R}$, such that,
for all $x \in A$, if $x < M$ then
$$|f(x) - L| < \epsilon.$$

Example: Let

$$\begin{aligned} f &: \mathbb{R}\setminus\{-1\} \to \mathbb{R}, \\ f(x) &= \frac{x}{x+1}. \end{aligned}$$

We show that

$$\lim_{x\to\infty} f(x) = 1.$$

- Indeed, let $\epsilon > 0$ be given and assume that

$$x > M \geq 0.$$

- Then

$$|f(x) - 1| = \left|\frac{x}{x+1} - 1\right| = \left|\frac{x - x - 1}{x+1}\right| = \frac{1}{|x+1|}$$

- Since

$$x > M \geq 0, x + 1 > 1 > 0,$$

so

$$|x+1| = x + 1.$$

- Therefore,

$$\frac{1}{|x+1|} = \frac{1}{x+1} \leq \frac{1}{x}.$$

- Since $x > M$,

$$\frac{1}{x} < \frac{1}{M}.$$

- Now, if $M > \frac{1}{\epsilon}$,
  then $M \geq 0$ and for $x > M$:

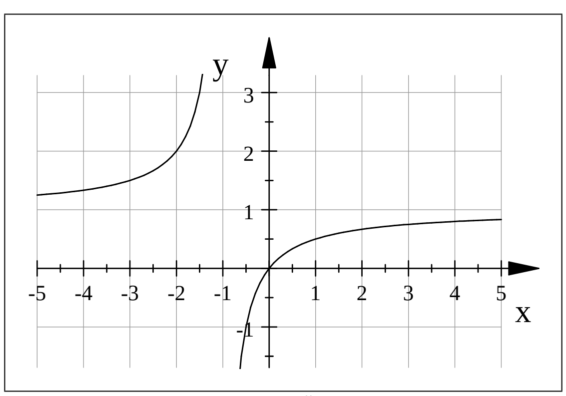
$$|f(x) - 1| = \frac{1}{x+1} \leq \frac{1}{x} < \frac{1}{M} < \epsilon.$$

- Therefore,

$$\lim_{x\to\infty} f(x) = 1.$$

- Analogously, one shows that

$$\lim_{x\to-\infty} f(x) = 1.$$



$y = \frac{x}{x+1}$

Exercise: Show that $\lim_{x\to-\infty} f(x) = 1$.

Definition Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and assume that A is **not bounded above**. We say that

$$\lim_{x\to\infty} f(x) = \infty,$$

if for every $K \in \mathbb{R}$,
there is $M \in \mathbb{R}$, such that,
for all $x \in A$,
if $x > M$ then

$$f(x) > K.$$

Definition Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and assume that A is **not bounded above**. We say that

$$\lim_{x\to\infty} f(x) = -\infty,$$

if for every $K \in \mathbb{R}$,
there is $M \in \mathbb{R}$, such that,
for all $x \in A$,
if $x > M$ then

$$f(x) < K.$$

Definition Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and assume that A is **not bounded below**. We say that

$$\lim_{x\to-\infty} f(x) = \infty,$$

if for every $K \in \mathbb{R}$,
there is $M \in \mathbb{R}$, such that,
for all $x \in A$,
if $x < M$ then

$$f(x) > K.$$

Definition Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and assume that A is **not bounded below**. We say that

$$\lim_{x\to-\infty} f(x) = -\infty,$$

if for every $K \in \mathbb{R}$,
there is $M \in \mathbb{R}$, such that,
for all $x \in A$,
if $x < M$ then

$$f(x) < K.$$

Proposition Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$, $c \in A'$ and

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = K.$$

If, for all $x \in A$,

$$f(x) \leq g(x),$$

then $L \leq K$.

Proof. We show that if or all $x \in A$,

$$f(x) \leq g(x),$$

then $L \leq K$.

- Suppose that $L > K$ and define

  $$\epsilon = \frac{1}{2}(L - K) > 0.$$

- Since $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$,
  there are δ_1, $\delta_2 > 0$, such that,
  for all $x \in A$, if

  $$0 < |x - c| < \delta_1,$$

  then

  $$|f(x) - L| < \epsilon$$

  and
  for all $x \in A$, if

  $$0 < |x - c| < \delta_2,$$

  then

  $$|g(x) - K| < \epsilon.$$

- Let $\delta = \min\{\delta_1, \delta_2\} > 0$.
- Since $c \in A'$
  there is $x \in A$, such that

  $$0 < |x - c| < \delta.$$

- Therefore,

  $$\begin{aligned} f(x) - g(x) &= (f(x) - L) + L - K + (K - g(x)) \\ &> -\epsilon + L - K - \epsilon \\ &= L - K - 2\epsilon \\ &= L - K - 2 \cdot \frac{1}{2}(L - K) = 0. \end{aligned}$$

- Consequently,
  there is $x \in A$, such that,

  $$f(x) - g(x) > 0,$$

  i.e. $f(x) > g(x)$. Contradiction.

This finishes our proof. ■

- **Proposition** Let f, g, $h : A \subseteq \mathbb{R} \to \mathbb{R}$ and $c \in A'$ and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$.
  If for all $x \in A$,

  $$f(x) \leq g(x) \leq h(x),$$

  then

  $$\lim_{x \to c} g(x) = L.$$

Proof. Proof follows from the previous result. ■