Math 4301 Mathematical Analysis I
Lecture 17
Topic: Sequences and series of functions

- **Pointwise and Uniform Convergence**

Definition Let $A \subseteq \mathbb{R}$, $x_0 \in A$ and $f_n : A \to \mathbb{R}$ be a sequence of functions. We say that the sequence $\{f_n\}$ converges at x_0 to a limit L if

$$\lim_{n\to\infty} f_n(x_0) = L, \text{ i.e.}$$

for all $\epsilon > 0$ there is $N \in \mathbb{N}$, such that, for all $n > N$,

$$|f_n(x_0) - L| < \epsilon.$$

Definition Let $A \subseteq \mathbb{R}$, $f_n : A \to \mathbb{R}$ be a sequence of functions. We say that $\{f_n\}$ is *pointwise* convergent to a function $f : A \to \mathbb{R}$ if for every $x \in A$,

$$\lim_{n\to\infty} f_n(x) = f(x), \text{ i.e.}$$

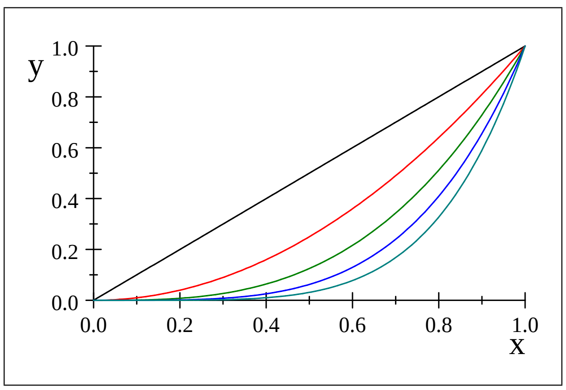
for every $x \in A$ and $\epsilon > 0$, there is $N \in \mathbb{N}$, such that, for all $n > N$,

$$|f_n(x) - f(x)| < \epsilon.$$

We write $f_n(x) \underset{n\to\infty}{\to} f(x)$ (pointwise).

Example Let $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$ and $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if} \quad x \neq 1 \\ 1 & \text{if} \quad x = 1 \end{cases}.$$



- $f_n(x) \underset{n\to\infty}{\to} f(x)$ (pointwise).
- Indeed, if $x \in (0,1)$, then for $\epsilon > 0$, there is

$$N > \log_x \epsilon, \ n \in \mathbb{N}$$

such that, if $n > N$, then

$$|f_n(x) - f(x)| = |x^n - 0| = x^n < x^N < x^{\log_x \epsilon} = \epsilon,$$

so $\lim_{n\to\infty} f_n(x) = 0 = f(x)$.

- If $x = 0$ then $f_n(0) = 0$, for all $n \in N$ and clearly $f_n(0) \to f(0) = 0$ as $n \to \infty$.
- If $x = 1$ then $f_n(1) = 1$, for all $n \in N$ and clearly $f_n(1) \to f(1) = 1$ as $n \to \infty$.
- It follows that f is a pointwise limit of the sequence $\{f_n\}$.

  Definition Let $f_n : A \subseteq \mathbb{R} \to \mathbb{R}$ be a sequence of functions.

  We say that the sequence $\{f_n\}$ *converges uniformly* to a function $f : A \to \mathbb{R}$ if, for every $\epsilon > 0$, there is $N \in \mathbb{N}$, such that, for all $n > N$ and for all $x \in A$,

$$|f_n(x) - f(x)| < \epsilon.$$

  We write $f_n \underset{n\to\infty}{\to} f$ *(*uniformly*)*.

  Remark We observe that if $f_n \underset{n\to\infty}{\to} f$ *(*uniformly*)*,

  then $f_n(x) \underset{n\to\infty}{\to} f(x)$ *(*pointwise*)*.

- Indeed, if for $\epsilon > 0$, there is $N \in \mathbb{N}$, such that, for all $n > N$ and for all $x \in A$,

$$|f_n(x) - f(x)| < \epsilon,$$

- Then, for every $x \in A$, if $n > N$, then

$$|f_n(x) - f(x)| < \epsilon.$$

- The converse is **not generally true**.
- Therefore, to find the function $f : A \to \mathbb{R}$, such that $f_n \to f$ (uniformly) we may start by computing the pointwise limit since uniform limit and pointwise limit must be the same.

  Example As we showed the sequence of functions

$$f_n : [0,1] \to \mathbb{R}, \ \ f_n(x) = x^n$$

  converges pointwise to $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if} \quad x \neq 1 \\ 1 & \text{if} \quad x = 1 \end{cases}.$$

- However, $\{f_n\}$ is does not coverage uniformly to f (so f_n is not uniformly convergent).
- Indeed, suppose that f is the uniform limit of $\{f_n\}$.
- Let $\epsilon = \frac{1}{4}$, then there is $N \in \mathbb{N}$, such that, for $n > N$ and for all $x \in [0,1]$,

$$|f_n(x) - f(x)| < \frac{1}{4}.$$

- In particular, if $n > N$, then
$$\left| f_n\left(1-\frac{1}{n}\right) - f\left(1-\frac{1}{n}\right)\right| = \left(1-\frac{1}{n}\right)^n < \frac{1}{4}.$$
Therefore,
$$\frac{1}{e} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n \leq \frac{1}{4},$$
so $e \geq 4$, a contradiction.

Exercise Show that the sequence of functions $f_n : [0,1] \to \mathbb{R}$,
$$f_n(x) = \frac{\sin(nx)}{n}$$
converges uniformly to
$$f : [0,1] \to \mathbb{R},\ f(x) = 0.$$

Definition Let $f_n : A \subseteq \mathbb{R} \to \mathbb{R}$ be a sequence of functions.
We say that series of functions $\sum_{n=1}^{\infty} f_n$ *converges pointwise* to $f : A \to \mathbb{R}$, if for every $x \in A$,
$$\lim_{n\to\infty} S_n(x) = f(x),$$
where $S_n(x) = \sum_{j=1}^{n} f_j(x)$ is the sequence of partial sums of $\sum_{n=1}^{\infty} f_n$.
If $S_n \underset{n\to\infty}{\to} f$ (uniformly), we say $\sum_{n=1}^{\infty} f_n$ *converges uniformly* to $f : A \to \mathbb{R}$.

Example We showed that if $f_n(x) = x^n$, $x \in (-1,1)$, then
$$\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$
converges pointwise to
$$f(x) = \frac{1}{1-x},\ x \in (-1,1).$$

Example Let $0 < a < 1$ and $f_n(x) = x^n$, $x \in [-a,a] \subset (-1,1)$.
We show that $\sum_{n=0}^{\infty} f_n$ converges uniformly to
$$\begin{aligned} f &: [-a,a] \to \mathbb{R}, \\ f(x) &= \frac{1}{1-x} \end{aligned}$$

- We need to show that, for $\epsilon > 0$ there is N, such that, if $n > N$ then for all $x \in [-a,a]$,
$$|s_n(x) - f(x)| < \epsilon$$
where
$$\begin{aligned} s_n(x) &= \sum_{k=0}^{n} f_k(x) = \sum_{k=0}^{n} x^k = 1 + x + \ldots + x^n \\ &= \frac{1-x^{n+1}}{1-x} \end{aligned}$$

is the nth partial sum of the series $\sum_{n=0}^{\infty} f_n$.

- We observe that, if $0 \leq x < 1$, then

$$\begin{aligned} |s_n(-x) - f(-x)| &= \left|\frac{1-(-x)^{n+1}}{1-(-x)} - \frac{1}{1-(-x)}\right| \\ &= \frac{|-x|^{n+1}}{1+x} = \frac{x^{n+1}}{1+x} \leq x^{n+1} \\ &\leq \frac{x^{n+1}}{1-x} = |s_n(x) - f(x)|. \end{aligned}$$

- We find $\sup\left\{\frac{x^{n+1}}{1-x} : x \in [0,a]\right\}$ since

$$|s_n(x) - f(x)| \leq \sup\left\{\frac{x^{n+1}}{1-x} : x \in [0,a]\right\}.$$

- Since $x \geq 0$,

$$\begin{aligned} \frac{d}{dx}\left(\frac{x^{n+1}}{1-x}\right) &= \frac{d}{dx}\left(\frac{x^{n+1}}{1-x}\right) \\ &= \frac{x^n}{(x-1)^2}\left(n(1-x)+1\right) \geq 0, \end{aligned}$$

  it follows that

$$\sup\left\{\frac{x^{n+1}}{1-x} : x \in [0,a]\right\} = \frac{a^{n+1}}{1-a}.$$

- Since, $\frac{a^{n+1}}{1-a} \to 0$ as $n \to \infty$, there is $N \in \mathbb{N}$, such that for $n > N$,

$$\frac{a^{n+1}}{1-a} < \epsilon.$$

- Let $n > N$, then for all $x \in [-a,a]$,

$$|s_n(x) - f(x)| \leq \sup\left\{\frac{x^{n+1}}{1-x} : x \in [0,a]\right\} = \frac{a^{n+1}}{1-a} < \epsilon.$$

- It follows that $s_n \to f$ (uniformly),

  so $\sum_{n=0}^{\infty} f_n$ converges uniformly to f on $[-a,a]$.

- We write

$$f = \sum_{n=0}^{\infty} f_n.$$

Example We show that $\sum_{n=0}^{\infty} f_n$, where $f_n(x) = x^n$ does not converge to

$$f(x) = \frac{1}{1-x}$$

uniformly on $(-1,1)$.

- Suppose that, there is $N \in \mathbb{N}$, such that,
  for $n > N$ and all $x \in (-1, 1)$,
$$|s_n(x) - f(x)| < 1.$$

- Thus, in particular, for $n > N$,
$$\begin{aligned}
\left| s_n\left(1 - \frac{1}{n}\right) - f\left(1 - \frac{1}{n}\right) \right| &= \left| \frac{1 - \left(1 - \frac{1}{n}\right)^{n+1}}{1 - \left(1 - \frac{1}{n}\right)} - \frac{1}{1 - \left(1 - \frac{1}{n}\right)} \right| \\
&= \left| \frac{1 - \left(1 - \frac{1}{n}\right)^{n+1}}{\frac{1}{n}} - \frac{1}{\frac{1}{n}} \right| \\
&= n\left(1 - \frac{1}{n}\right)^{n+1} < 1.
\end{aligned}$$

- It follows that
$$\lim_{n\to\infty} n\left(1 - \frac{1}{n}\right)^{n+1} \leq 1.$$

- However,
$$\lim_{n\to\infty} n\left(1 - \frac{1}{n}\right)^{n+1} = \infty$$
  since
$$\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right)\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^{n} = \frac{1}{e}.$$

- Thus $\infty \leq 1$, a contradiction.
  Example Consider series $\sum_{n=0}^{\infty} xe^{-nx}$, $x \in [0, 1]$.
  What is the pointwise limit for this series.

- We see that
$$\begin{aligned}
s_n(x) &= \sum_{k=0}^{n} xe^{-kx} = x\sum_{k=0}^{n} e^{-kx} = x\sum_{k=0}^{n}\left(\frac{1}{e^x}\right)^k \\
&= x\frac{1 - \left(\frac{1}{e^x}\right)^{n+1}}{1 - \frac{1}{e^x}} = xe^x\left(\frac{1 - e^{-(n+1)x}}{e^x - 1}\right)
\end{aligned}$$

- We see that, for $x \in [0, 1]$, $x \neq 0$,
$$\lim_{n\to\infty} e^{-(n+1)x} = \lim_{n\to\infty}\frac{1}{e^{(n+1)x}} = 0$$
  then
$$\lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} xe^x\frac{1 - e^{-(n+1)x}}{e^x - 1} = \frac{xe^x}{e^x - 1}.$$

- So we showed that
$$\lim_{n\to\infty} s_n(x) = \frac{xe^x}{e^x - 1},\ x \in (0, 1]$$
  and for $x = 0$,
$$\sum_{n=0}^{\infty} 0 \cdot e^{-n\cdot 0} = 0.$$

- Therefore,

$$\sum_{n=0}^{\infty} xe^{-nx} = \begin{cases} \frac{xe^x}{e^x-1} & if \quad x \in (0,1] \\ 0 & if \quad x = 0 \end{cases}$$

- Let $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{xe^x}{e^x-1} & if \quad x \in (0,1] \\ 0 & if \quad x = 0 \end{cases}.$$

- We showed that

$$f(x) = \sum_{n=0}^{\infty} xe^{-nx},$$

i.e. the series converges pointwise to f.

Theorem Let $f_n : A \subseteq \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions and assume that $\{f_n\}$ converges uniformly to $f : A \to \mathbb{R}$.
Then f is continuous on A.

Proof. Let $x_0 \in A$ and $\epsilon > 0$ be given.

- We need to find $\delta > 0$, such that, for all $x \in A$, if $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| < \epsilon.$$

- Since $f_n \to f$ uniformly, there is $N \in \mathbb{N}$, such that, for all $n \geq N$, and for all $x \in A$,

$$|f_n(x) - f(x)| < \epsilon/3.$$

- In particular, for all $x \in A$,

$$|f_N(x) - f(x)| < \epsilon/3.$$

- Since f_N is continuous at x_0, there is $\delta > 0$, such that, for all $x \in A$, if $|x - x_0| < \delta$, then

$$|f_N(x) - f_N(x_0)| < \epsilon/3.$$

- Finally, since for all $x \in A$,

$$|f_N(x) - f(x)| < \epsilon/3,$$

it follows that in particular, for $x_0 \in A$,

$$|f_N(x_0) - f(x_0)| < \epsilon/3.$$

- Consequently, for all $x \in A$, if $|x - x_0| < \delta$,

$$\begin{aligned} |f(x) - f(x_0)| &= |(f(x) - f_N(x)) + (f_N(x) - f_N(x_0)) + (f_N(x_0) - f(x_0))| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

- Therefore f is continuous at $x_0 \in A$.
- Since $x_0 \in A$ is arbitrary, f is continuous on A.

This finishes our proof. ■

- **Exercise** Check if $\sum_{n=0}^{\infty} xe^{-nx}$ converges to f uniformly.

  Exercise Let $f_n : [0,2] \to \mathbb{R}$, $f_n(x) = \frac{x^n}{1+x^n}$.

  Show that $\{f_n\}$ converges pointwise on $[0,2]$ but it does not converge uniformly on $[0,2]$.

  Example Let
  $$f_n(x) = xe^{-nx^2},\ x \in [0,1].$$
  Show that $\{f_n\}$ converges uniformly on $[0,1]$.

  Example Show that $f : [0,1] \to \mathbb{R}$,
  $$f(x) = \sum_{n=1}^{\infty} \frac{x^{n/2}}{n\,(n!)^2}$$
  is continuous.

  Tests for Convergence

  Proposition (*Cauchy Criterion*) Let $A \subseteq \mathbb{R}$ and $f_n : A \to \mathbb{R}$ be a sequence of functions.
  Then $\{f_n\}$ converges uniformly on A iff
  for every $\epsilon > 0$, there is $N \in \mathbb{N}$, such that,
  for all $m, n > N$, and for every $x \in A$,
  $$|f_m(x) - f_n(x)| < \epsilon.$$

Proof. Assume that $f_n \to f$ (uniformly) and let $\epsilon > 0$ be given.

- Thus, there is $N \in \mathbb{N}$, such that, for $m > N$, and for all $x \in A$,
  $$|f_n(x) - f(x)| < \frac{\epsilon}{2}.$$
- Therefore, if $m, n > N$, and $x \in A$,
  $$\begin{aligned} |f_m(x) - f_n(x)| &= |f_m(x) - f(x) + f(x) - f_n(x)| \\ &\leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$
- Conversely, assume that for $\epsilon > 0$,
  there is $N \in \mathbb{N}$, such that, for all $m, n > N$ and for all $x \in A$,
  $$|f_m(x) - f_n(x)| < \epsilon.$$
- It follows that, for all $x \in A$, if $m, n > N$, then
  $$|f_m(x) - f_n(x)| < \epsilon.$$
- Therefore, $\{f_n(x)\}$ is a Cauchy sequence, for each $x \in A$.
- Since Cauchy sequence converge in $\mathbb{R}$,
  define $f : A \to \mathbb{R}$ by
  $$f(x) = \lim_{n \to \infty} f_n(x).$$

- Since the uniform Cauchy condition holds,
  for $\epsilon > 0$, there is $N \in \mathbb{N}$, such that,
  for $m, n > N$, and for all $x \in A$,
$$|f_m(x) - f_n(x)| < \frac{\epsilon}{2}.$$
- Since $f_n(x) \to f(x)$ (pointwise), for all $x \in A$
$$|f_m(x) - f(x)| = \left| f_m(x) - \lim_{n\to\infty} f_n(x) \right| = \lim_{n\to\infty} |f_m(x) - f_n(x)| \le \frac{\epsilon}{2}$$
- Thus, for $m > N$, and for all $x \in A$,
$$|f_m(x) - f(x)| \le \frac{\epsilon}{2} < \epsilon,$$
- i.e. $f_m \to f$ (uniformly).

This finishes our proof. ■

- **Theorem** (*Weierstrass M-test*) Let $A \subseteq \mathbb{R}$ and suppose that
$$f_n : A \to \mathbb{R}$$
  satisfies the following conditions:

a. For every $n \in \mathbb{N}$, there is $M_n \ge 0$, such that, for all $x \in A$,
$$|f_n(x)| \le M_n;$$
b. The series $\sum_{n=1}^{\infty} M_n$ converges.
   Then $\sum_{n=1}^{\infty} f_n$ is uniformly and absolute convergent on A.

Proof. Since $\sum_{n=1}^{\infty} M_n$ converges,

- sequence of its partial sums $\{s_n\}$ is Cauchy.
- Thus, for $\epsilon > 0$ there is $N \in \mathbb{N}$, such that, for all $m > N$, and $k \in \mathbb{N}$,
$$|s_m - s_{m+k}| = |M_{m+1} + M_{m+2} + ... + M_{m+k}| < \epsilon.$$
- Let
$$S_n(x) = \sum_{j=1}^{\infty} f_j(x), \ x \in A,$$
  and assume that $m > N$.
- Then, for all $k \in \mathbb{N}$ and $x \in A$,
$$\begin{aligned} |S_{m+k}(x) - S_m(x)| &= |f_{m+1}(x) + f_{m+2}(x) + ... + f_{m+k}(x)| \\ &\le |f_{m+1}(x)| + |f_{m+2}(x)| + ... + |f_{m+k}(x)| \\ &\le M_{m+1} + M_{m+2} + ... + M_{m+k} \\ &= |M_{m+1} + M_{m+2} + ... + M_{m+k}| < \epsilon. \end{aligned}$$

- It follows that $\{S_n\}$ satisfies the uniform Cauchy condition.
- Thus $\{S_n\}$ converges uniformly on A.

This finishes our proof. $\blacksquare$

- **Example** Let $\{a_i\}$ be a bounded sequence of real numbers.
  Use the Weierstrass M-test to show that the series $\sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$ converges to a continuous function on all of $\mathbb{R}$. In other words, if $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$$
  then we show that f is continuous on $\mathbb{R}$.
- Let $x_0 \in \mathbb{R}$, and define $a = |x_0| + 1$, then $x_0 \in [-a, a]$.
- Let $f_n : \mathbb{R} \to \mathbb{R}$, defined by
$$f_n(x) = \frac{a_n}{n!} x^n$$
- We show that
$$f = \sum_{n=0}^{\infty} f_n$$
  i.e. the series is uniformly convergent to f on $[-a, a]$.
- Since $\{a_n\}$ is bounded, there is $M \geq 0$, such that, for all $n \in \mathbb{N}$,
$$|a_n| \leq M.$$
- Now, notice that, for all $x \in [-a, a]$:
$$|f_k(x)| = \left| \frac{a_k}{k!} x^k \right| = \frac{|a_k|}{k!} |x|^k \leq M \frac{a^k}{k!} = M_k$$
  and
$$\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} M \frac{a^k}{k!} = M \sum_{k=0}^{\infty} \frac{a^k}{k!} = Me^a < \infty$$
- so the series $\sum_{n=0}^{\infty} f_n$ is convergent uniformly on $[-a, a]$.
- Therefore, by the Weierstrass M-test,
  it follows that $\sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$ is uniformly convergent on $[-a, a]$.
- Let
$$f(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k, \; x \in [-a, a].$$

- Since each function

$$\begin{aligned} f_k &: [-a,a] \to \mathbb{R}, \\ f_k(x) &= \frac{a_k}{k!}x^k \end{aligned}$$

is continuous, each function

$$\begin{aligned} s_n &: [-a,a] \to \mathbb{R}, \\ s_n(x) &= \sum_{k=0}^{n} f_k(x) = \sum_{k=0}^{n} \frac{a_k}{k!}x^k \end{aligned}$$

is continuous, where $n \in \mathbb{N}$.

- Since the series $\sum_{k=0}^{\infty} f_k$ converges uniformly to f, the sequence (s_n) of its partial sums converges to f.
- Since the uniform limit of any sequence of continuous functions is continuous, it follows that

$$f(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!}x^k = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{a_k}{k!}x^k = \lim_{n\to\infty} s_n(x)$$

is f is continuous on $[-a,a]$.

- In particular, f is continuous at x_0.
- Since $x_0 \in \mathbb{R}$ is an arbitrary point, $\sum_{k=0}^{\infty} \frac{a_k}{k!}x^k$ converges to a continuous function on $\mathbb{R}$.

Example We show that the series

$$\sum_{n=0}^{\infty} \frac{x}{(n+x^2)^2}, \ x \in \mathbb{R}$$

converges uniformly on $\mathbb{R}$.

- Let us consider function $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \frac{x}{(n+x^2)^2}.$$

- Since

$$\begin{aligned} f_n'(x) &= \frac{d}{dx}\left(\frac{x}{(n+x^2)^2}\right) = \frac{n-3x^2}{(x^2+n)^3} = 0 \text{ iff} \\ x_n &= \pm\sqrt{\frac{n}{3}}. \end{aligned}$$

- Since $f_n'(x) > 0$ iff $x \in \left(-\sqrt{\frac{n}{3}}, \sqrt{\frac{n}{3}}\right)$ and

- $f_n'(x) < 0$ iff $x \in \left(-\infty, -\sqrt{\frac{n}{3}}\right) \cup \left(\sqrt{\frac{n}{3}}, \infty\right)$, so f_n has a **local minimum** at

$$x_n = -\sqrt{\frac{n}{3}}$$

and **local maximum** at

$$x_n = \sqrt{\frac{n}{3}}.$$

- Since

$$\begin{aligned} \lim_{x \to \pm\infty} f_n(x) &= \lim_{x \to \pm\infty} \frac{x}{(n+x^2)^2} = \lim_{x \to \pm\infty} \frac{x}{\left(x^2\left(\frac{n}{x^2}+1\right)\right)^2} \\ &= \lim_{x \to \pm\infty} \frac{x}{x^4\left(\frac{n}{x^2}+1\right)^2} = \lim_{x \to \pm\infty} \frac{1}{x^3\left(\frac{n}{x^2}+1\right)^2} = 0 \end{aligned}$$

- It follows that

$$f_n\left(-\sqrt{\frac{n}{3}}\right) = \frac{-\sqrt{\frac{n}{3}}}{\left(n+\left(-\sqrt{\frac{n}{3}}\right)^2\right)^2} = -\frac{3}{16}\frac{\sqrt{3}}{n^{\frac{3}{2}}}$$

is the **absolute minimum** of f_n and

$$f_n\left(\sqrt{\frac{n}{3}}\right) = \frac{\sqrt{\frac{n}{3}}}{\left(n+\left(\sqrt{\frac{n}{3}}\right)^2\right)^2} = \frac{3}{16}\frac{\sqrt{3}}{n^{\frac{3}{2}}}$$

is the **absolute maximum**.

- Therefore, for all $x \in \mathbb{R}$:

$$|f_n(x)| \leq \frac{3}{16}\frac{\sqrt{3}}{n^{\frac{3}{2}}}.$$

- Let $M_n = \frac{3}{16}\frac{\sqrt{3}}{n^{\frac{3}{2}}}$, $n \in \mathbb{N}$
- Since, using p-test,

$$\sum_{n=0}^{\infty} M_n = \frac{3\sqrt{3}}{16}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}} < \infty$$

- by the Weierstrass M-test, the series

$$\sum_{n=0}^{\infty}\frac{x}{(n+x^2)^2}$$

converges uniformly on $\mathbb{R}$.

Remark: We can define $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \sum_{n=0}^{\infty}\frac{x}{(n+x^2)^2}$$

since $f_n : \mathbb{R} \to \mathbb{R}$ are continuous and
the series $\sum_{n=0}^{\infty} f_n$ converges uniformly on $\mathbb{R}$,
it follows that f is continuous.

- Is the function f defined above integrable, differentiable, etc?

  Example We show that the series

$$\sum_{n=1}^{\infty} \frac{x}{n\left(1+nx^2\right)},\ x \in \mathbb{R}$$

  converges uniformly on $\mathbb{R}$.

- Let us consider function $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \frac{x}{n\left(1+nx^2\right)}.$$

- Since

$$\begin{aligned} f_n'(x) &= \frac{d}{dx}\left(\frac{x}{n\left(1+nx^2\right)}\right) = -\frac{1}{n}\frac{nx^2-1}{\left(nx^2+1\right)^2} = 0 \text{ iff} \\ x_n &= \pm\frac{1}{\sqrt{n}}. \end{aligned}$$

- Since $f_n'(x) > 0$ iff $x \in \left(-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right)$ and
- $f_n'(x) < 0$ iff $x \in \left(-\infty, -\frac{1}{\sqrt{n}}\right) \cup \left(\frac{1}{\sqrt{n}}, \infty\right)$,
- so f_n has a **local minimum** at

$$x_n = -\frac{1}{\sqrt{n}}$$

  and a **local maximum** at $x_n = \frac{1}{\sqrt{n}}$.

- Since

$$\lim_{x\to\pm\infty} f_n(x) = \lim_{x\to\pm\infty} \frac{x}{n\left(1+nx^2\right)} = \lim_{x\to\pm\infty} \frac{1}{nx\left(\frac{1}{x^2}+n\right)} = 0,$$

  it follows that

$$f_n\left(-\frac{1}{\sqrt{n}}\right) = \frac{-\frac{1}{\sqrt{n}}}{n\left(1+n\left(-\frac{1}{\sqrt{n}}\right)^2\right)} = -\frac{1}{2n^{\frac{3}{2}}}$$

- is the **absolute minimum** of f_n and

$$f_n\left(\frac{1}{\sqrt{n}}\right) = \frac{\frac{1}{\sqrt{n}}}{n\left(1+n\left(\frac{1}{\sqrt{n}}\right)^2\right)} = \frac{1}{2n^{\frac{3}{2}}}$$

- is the **absolute maximum**.
- Therefore, for all $x \in \mathbb{R}$:

$$\left|f_n(x)\right| \leq \frac{1}{2n^{\frac{3}{2}}}.$$

- Let $M_n = \frac{1}{2n^{\frac{3}{2}}}$, $n \in \mathbb{N}$

- Since, using p-test,

$$\sum_{n=0}^{\infty} M_n = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} < \infty$$

- by the **Weierstrass M-test**, the series

$$\sum_{n=1}^{\infty} \frac{x}{n\,(1+nx^2)}$$

  converges uniformly on $\mathbb{R}$.