HW 2 solutions.

① $z = -5\sqrt{3} + 5i, \quad \omega = -3\sqrt{3} - 3i$

$$\tan\theta_1 = \frac{5}{-5\sqrt{3}} = -\frac{1}{\sqrt{3}} \Rightarrow \theta_1 = \frac{5\pi}{6}$$

$$\tan\theta_2 = \frac{-3}{-3\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta_2 = \frac{7\pi}{6}$$

$$\arg(z\omega) = (\theta_1 + \theta_2) + 2n\pi = \left(\frac{5\pi}{6} + \frac{7\pi}{6}\right) + 2n\pi = 2\pi + 2n\pi = 2n\pi, \quad n = 0, \pm 1, \ldots$$

$$\arg\left(\frac{z}{\omega}\right) = (\theta_1 - \theta_2) + 2n\pi = \left(\frac{5\pi}{6} - \frac{7\pi}{6}\right) + 2n\pi = -\frac{2\pi}{6} + 2n\pi = -\frac{\pi}{3} + 2n\pi; n = 0, \pm 1, \ldots$$

so,

$$\boxed{\begin{aligned} &\arg(z\omega) = 2n\pi, \; n = 0, \pm 1, \pm 2, \ldots \\ &\arg\left(\frac{z}{\omega}\right) = -\frac{\pi}{3} + 2n\pi, \; n = 0, \pm 1, \pm 2, \ldots \\ &\operatorname{Arg}(z\omega) = 0 \\ &\operatorname{Arg}\left(\frac{z}{\omega}\right) = -\frac{\pi}{3} \end{aligned}}$$

② $z_0 = -16 - 16\sqrt{3}\,i$, $r_0 = \sqrt{(-16)^2 + (-16\sqrt{3})^2} = \sqrt{256 + 256\cdot 3}$

$= \sqrt{256\cdot 4} = 32$

$\tan\theta_0 = \frac{-16\sqrt{3}}{-16} = \sqrt{3} \Rightarrow \theta_0 = \frac{4\pi}{3}$, $n = 5$.

$$C_k = \sqrt[n]{r_0}\; e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)}, \quad k = 0, 1, \dots, (n-1)$$

$$= \sqrt[5]{32}\; e^{i\left(\frac{4\pi}{15} + \frac{2k\pi}{5}\right)}, \quad k = 0, 1, 2, 3, 4.$$

$$= 2e^{i\left(\frac{4\pi}{15} + \frac{2k\pi}{5}\right)}, \quad k = 0, 1, 2, 3, 4.$$

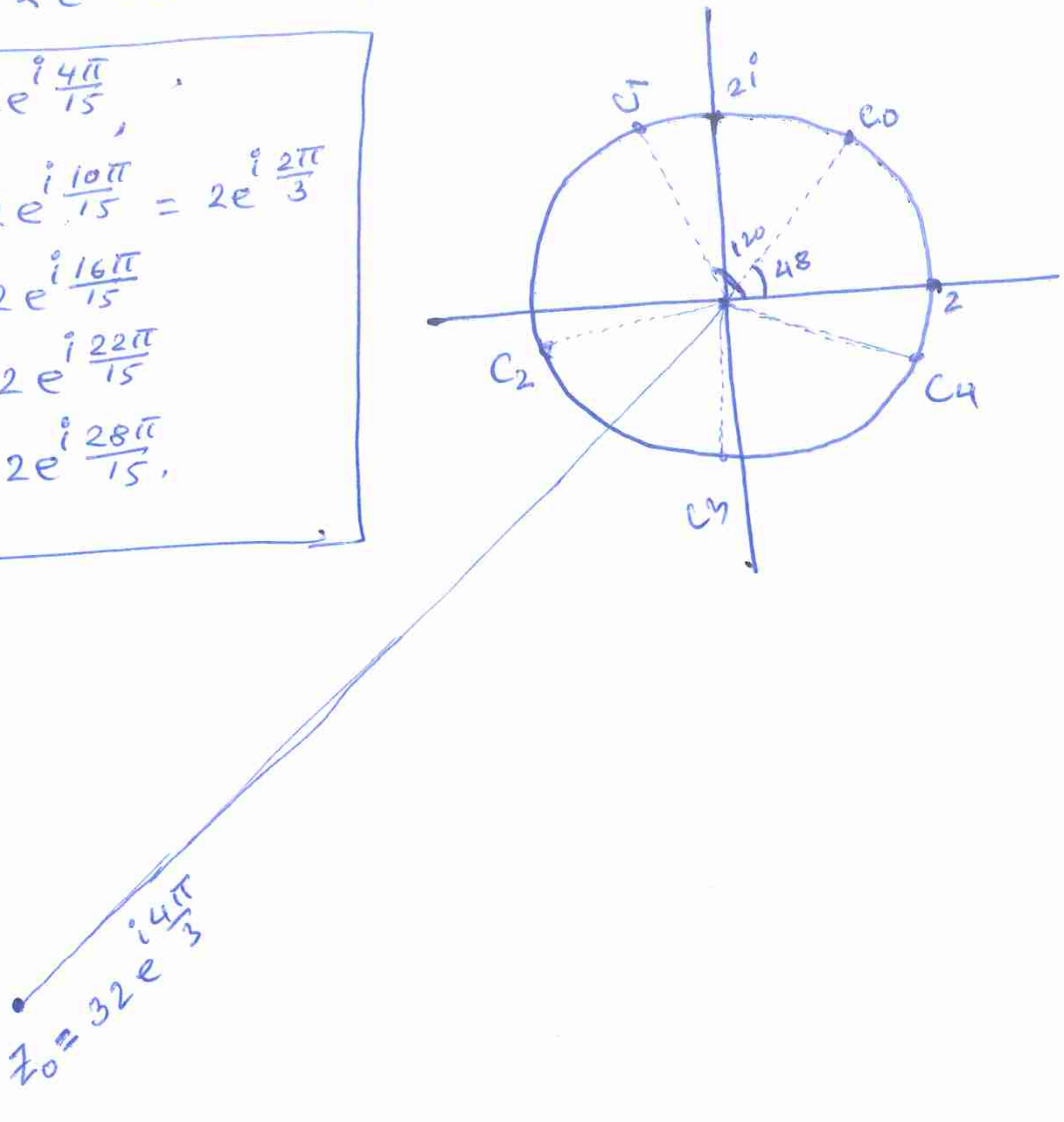
$C_0 = 2e^{i\frac{4\pi}{15}}$,

$C_1 = 2e^{i\frac{10\pi}{15}} = 2e^{i\frac{2\pi}{3}}$

$C_2 = 2e^{i\frac{16\pi}{15}}$

$C_3 = 2e^{i\frac{22\pi}{15}}$

$C_4 = 2e^{i\frac{28\pi}{15}}$.



③ $z^4 + 81 = 0$

$\Rightarrow z^4 = -81$; $r_0 = 81$, $\tan\theta_0 = \frac{0}{81} = 0 \Rightarrow \theta_0 = \pi$.

$\Rightarrow z = \sqrt[4]{-81} = \sqrt[4]{r_0}\, e^{i\left(\frac{\theta_0}{4} + \frac{2k\pi}{4}\right)}$; $k = 0, 1, 2, 3$.

$= \sqrt[4]{81}\, e^{i\left(\frac{\pi}{4} + \frac{2k\pi}{4}\right)}$; $k = 0, 1, 2, 3$;

$= 3e^{i\left(\frac{\pi}{4} + \frac{2k\pi}{4}\right)}$; $k = 0, 1, 2, 3$.

$= 3e^{i\frac{\pi}{4}}, \quad 3e^{i\frac{3\pi}{4}}, \quad 3e^{i\frac{5\pi}{4}}, \quad 3e^{i\frac{7\pi}{4}}$

$$\boxed{z = 3\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right), \ 3\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right), \ 3\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right), \ 3\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)}.$$

④ Given $\varepsilon > 0$, we find a $\delta > 0$ s.t.

$$\left|\left(\frac{3z}{5}+2\right)-\left(\frac{16+3i}{5}\right)\right| < \varepsilon \text{ whenever } 0<|z-(2+i)|<\delta.$$

Now, $\left|\left(\frac{3z}{5}+2\right)-\left(\frac{16+3i}{5}\right)\right| < \varepsilon$

$$\Rightarrow \left|\frac{3z}{5}+2-\frac{16}{5}-\frac{3i}{5}\right| < \varepsilon$$

$$\Rightarrow \left|\frac{3z+10-16-3i}{5}\right| < \varepsilon$$

$$\Rightarrow \left|\frac{3z-6-3i}{5}\right| < \varepsilon$$

$$\Rightarrow \left|\frac{3}{5}\left(z-(2+i)\right)\right| < \varepsilon$$

$$\Rightarrow |z-(2+i)| < \frac{5\varepsilon}{3}.$$

Choose $\delta = \frac{5\varepsilon}{3}$.

Verification: when $\delta = \frac{5\varepsilon}{3}$ and $0<|z-(2+i)|<\delta$;

$$\left|\left(\frac{3z}{5}+2\right)-\left(\frac{16+3i}{5}\right)\right| = \left|\frac{3z+10-16-3i}{5}\right| = \left|\frac{3}{5}\left[z-(2+i)\right]\right|$$

$$= \frac{3}{5}|z-(2+i)|$$

$$< \frac{3}{5}\delta$$

$$= \frac{3}{5}\left(\frac{5\varepsilon}{3}\right)$$

$$= \varepsilon.$$

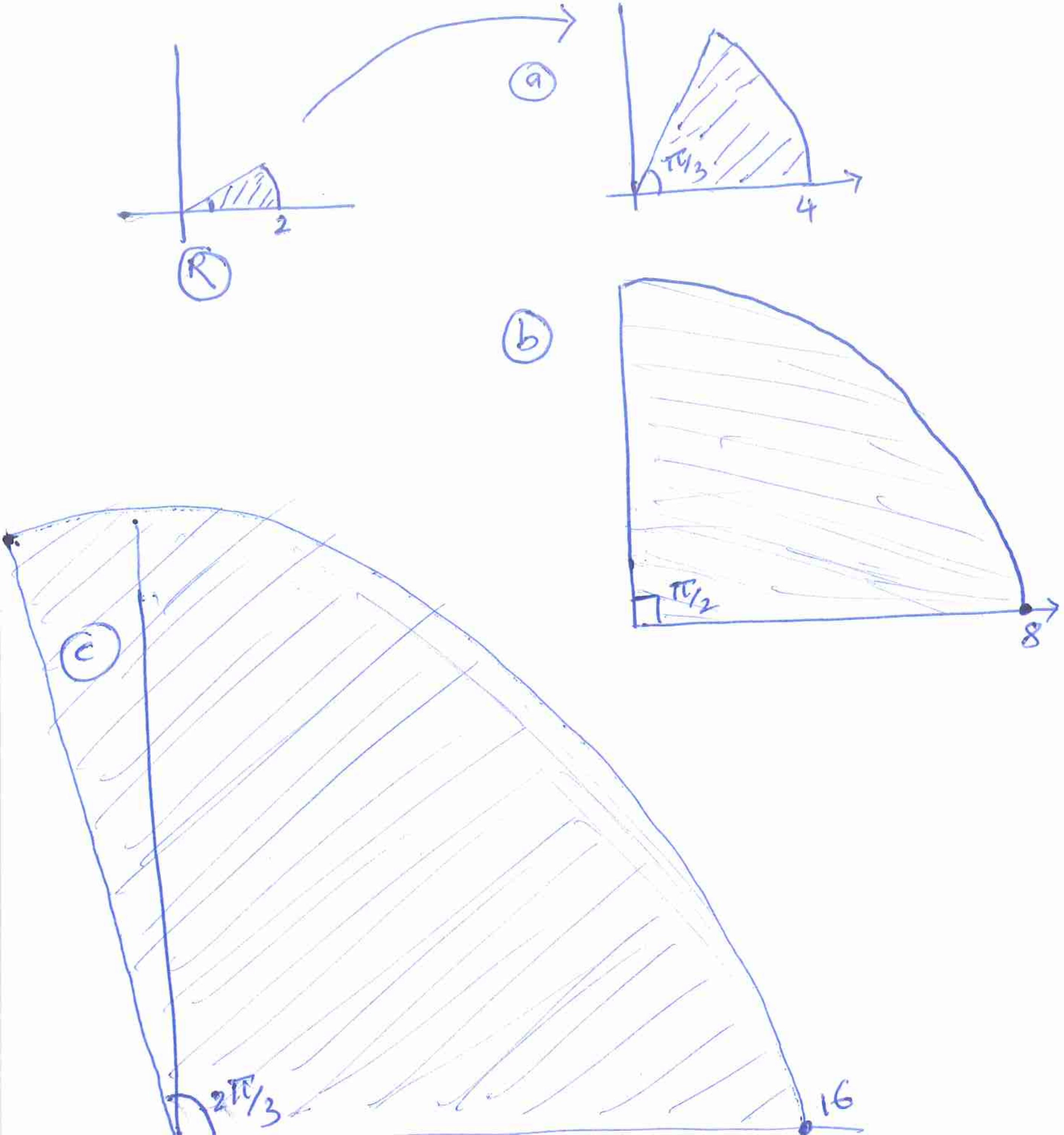
we verified $\boxed{\lim_{z\to 2+i} \frac{3z}{5}+2 = \frac{16+3i}{5}}$

⑤ If $z = re^{i\theta}$, $0 \le r \le 2$, $0 \le \theta \le \frac{\pi}{6}$

Then ⓐ $\omega = z^2 = r^2 e^{i2\theta} = Re^{i\phi}$, $0 \le R \le 4$, $0 \le \phi \le \pi/3$.

ⓑ $\omega = z^3 = r^3 e^{i3\theta} = Re^{i\phi}$, $0 \le R \le 8$, $0 \le \phi \le \pi/2$

ⓒ $\omega = z^4 = r^4 e^{i4\theta} = Re^{i\phi}$, $0 \le R \le 16$, $0 \le \phi \le 2\pi/3$.



⑥ $\lim\limits_{z\to i} \dfrac{z^2+1}{z^6+1} \left(\dfrac{0}{0}\right)$

$= \lim\limits_{z\to i} \dfrac{(z-i)(z+i)}{(z-i)(z^5+iz^4-z^3-iz^2+z+i)}$

$= \lim\limits_{z\to i} \dfrac{z+i}{z^5+iz^4-z^3-iz^2+z+i}$

$= \dfrac{i+i}{i^5+i^5-i^3-i^3+i+i}$

$= \dfrac{2i}{2i^5-2i^3+2i}$

$= \dfrac{2i}{2i+2i+2i}$

$= \dfrac{2i}{6i}$

$= \dfrac{1}{3}.$

$z-i \,\big)\, z^6+1 \,\big(\, z^5+iz^4-z^3-iz^2+z+i$

$\underset{-}{z^6}\underset{+}{-}iz^5$

iz^5+1

$\underset{-}{iz^5}\underset{-}{+}z^4$

$-z^4+1$

$\underset{+}{-}z^4 \pm iz^3$

$-iz^3+1$

$\underset{-}{iz^3}\underset{+}{-}z^2$

$+z^2+1$

$\underset{-}{z^2}\underset{+}{-}iz$

$-iz+1$

$\underset{+}{-}iz\underset{-}{+}1$

0

⑦ $\lim\limits_{z\to 1+i} \dfrac{z-1-i}{z^2-2z+2}$ $\quad \frac{0}{0}$

$\left\{ z = \dfrac{2 \pm \sqrt{4-8}}{2} = \dfrac{2 \pm 2i}{2} = 1 \pm i \right.$

$z = 1+i, \; z = 1-i$

↳ are zeros.

$$= \lim_{z\to 1+i} \frac{\cancel{z-(1+i)}}{\cancel{[z-(1+i)]}[z-(1-i)]}$$

$$= \lim_{z\to 1+i} \frac{1}{z-(1-i)}$$

$$= \frac{1}{(1+i)-(1-i)}$$

$$= \frac{1}{2i}$$

$$= \frac{i}{2i^2}$$

$$= -\frac{i}{2}.$$

Therefore $\lim\limits_{z\to 1+i} \left[\dfrac{z-1-i}{z^2-2z+2}\right]^2 = \left[\lim\limits_{z\to 1+i} \dfrac{z-1-i}{z^2-2z+2}\right]^2 = \left(\dfrac{-i}{2}\right)^2$

$$= \frac{i^2}{4}$$

$$= -\frac{1}{4}.$$

⑧ $\lim\limits_{z\to 1+i} \dfrac{z^4+4}{z^2-2z+2} \quad \left(\dfrac{0}{0}\right)$

$= \lim\limits_{z\to 1+i} \dfrac{\cancel{[z-(1+i)]}[z-(-1+i)][z-(-1-i)][z-(1-i)]}{\cancel{[z-(1+i)]}[z-(1-i)]}$

$= \lim\limits_{z\to 1+i} \dfrac{[z-(-1+i)][z-(-1-i)]\cancel{[z-(1-i)]}}{\cancel{[z-(1-i)]}}$

$= [(1+i)-(-1+i)][1+i-(-1-i)]$

$= (1+\cancel{i}+1-\cancel{i})(1+i+1+i)$

$= 2(2+2i)$

$= 4+4i$

Here: $z = 1+i = \sqrt{2}e^{i\pi/4}$

$\Rightarrow z^4 = (\sqrt{2})^4 e^{i\pi}$

$= 4(-1)$

$= -4$

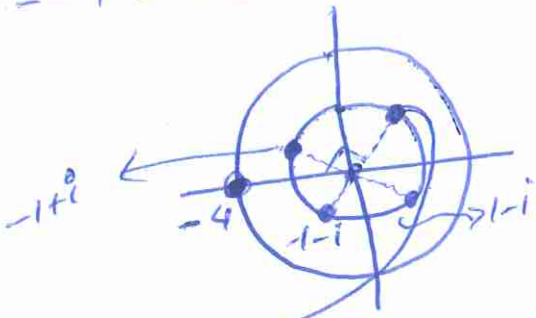
So $z^4+4=0$ when $z=1+i$

see Q#7. solutions to $z^2-2z+2=0$ are

$z=1+i$ & $z=1-i$

$$\cancel{\begin{array}{r l} & z^3+(1+i)z^2 \\ z-1-i \,\big) & z^4+4 \\ & z^4 - z^3 - iz^3 \\ & - \quad + \quad + \\ \hline & z^3+iz^3+4 \\ & z^3 \end{array}}$$

$z^4 = -4 = 4e^{i\pi}$



$\to (4)^{1/4} e^{i\pi/4}$

$= \sqrt{2}(\cos\pi/4 + i\sin\pi/4)$

$= 1+i$.