HOMEWORK 3 – MATH 4341
DUE DATE: MONDAY 09/18/2023

Problem 1. Suppose $\mathcal{B}$ is a basis for a topology on a set X. Let $\mathcal{T}$ be the intersection of all topologies on X that contain $\mathcal{B}$. Show that
(a) $\mathcal{T}$ is a topology on X,
(b) $\mathcal{T}$ is equal to $\mathcal{T}_{\mathcal{B}}$, the topology generated by $\mathcal{B}$.

Problem 2. Let (X, d) be a metric space and

$$\mathcal{B} = \{B_d(x, 2^{-n}) \mid x \in X, n \in \mathbb{N}\}.$$

Show that
(a) $\mathcal{B}$ is a basis for a topology on X,
(b) the topology generated by $\mathcal{B}$ is equal to the metric topology on X.

Problem 3. If $f : X \to Y$ is a function between two sets and $A \subset Y$ is a subset, we define the *preimage* of A to be

$$f^{-1}(A) = \{x \in X \mid f(x) \in A\}.$$

Show that:
(1) If $f : X \to Y$ and $\{A_i\}_{i \in I}$ is a family of subsets of Y, then

$$f^{-1}\Big(\bigcup_{i \in I} A_i\Big) = \bigcup_{i \in I} f^{-1}(A_i) \qquad \text{and} \qquad f^{-1}\Big(\bigcap_{i \in I} A_i\Big) = \bigcap_{i \in I} f^{-1}(A_i).$$

(2) If $A \subset Y$, then $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$.
(3) If $g : Y \to Z$ is another map and $B \subset Z$, then

$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B)).$$