Math 4301 Mathematical Analysis I
Lecture 3
Topic: Monotone Sequence Property

- **Summary of Previous Lecture**
- **Archimedean Property**
  We say that an ordered field $\mathbb{F}$ satisfies the *archimedean* property if
  for every $x \in \mathbb{F}$, there is $n \in \mathbb{N}$, such that,
  $$x < n.$$
  We say that $\mathbb{F}$ is an *archimedean field* if $\mathbb{F}$ satisfies the archimedean property.
- **Definition** Let $\mathbb{F}$ be an ordered field and $S \subseteq \mathbb{F}$.
  A number $M \in \mathbb{F}$ is called an *upper bound* for S if for all $x \in S$,
  $$x \leq M.$$
- **Definition** A number $\beta \in \mathbb{F}$ is called *the least upper bound* (or *supremum*) for S if

i) β is an upper bound of S, and

ii) if β' is an upper bound for S, then $\beta \leq \beta'$.

- The least upper bound for S is denoted by $\sup S$, i.e.
  $$\beta = \sup S.$$
- If S is not bounded above, then
  $$\sup S = +\infty.$$
- If $S = \emptyset$, then
  $$\sup S = -\infty.$$
- **The least upper bound property**
  Every *nonempty and bounded above* subset $S \subseteq \mathbb{F}$ has the least upper bound, that is,
  there is $\beta \in \mathbb{F}$, such that
  $$\beta = \sup S.$$
- **Definition** An ordered field $\mathbb{F}$ is called *complete* if $\mathbb{F}$ satisfies the least upper bound property.
- **Theorem** Every complete ordered field $\mathbb{F}$ is Archimedean.
- **Theorem** There exists a *unique* (up to an isomorphism of ordered fields) *complete ordered field* that we call the *field of real numbers* $\mathbb{R}$.
- **Proposition** $\mathbb{Q} \subset \mathbb{R}$ is dense in $\mathbb{R}$. That is,

i) If $x, y \in \mathbb{R}$ and $x < y$, then there is $r \in \mathbb{Q}$, such that,
$$x < r < y.$$

ii) If $x \in \mathbb{R}$, $\epsilon > 0$, then there is $r \in \mathbb{Q}$ with

$$|x - r| < \epsilon.$$

- **Proposition** Equation

$$x^2 = 2$$

  has no solutions in $\mathbb{Q}$.

- **Proposition** There is $\alpha \in \mathbb{R}$, $\alpha > 0$ such that

$$\alpha^2 = 2.$$

- **Proposition** $\mathbb{Q}$ is not a complete ordered field.

- **Definition** Let $\{x_n\}$ be a sequence in $\mathbb{F}$ and $x \in \mathbb{F}$.
  We say that $\{x_n\}$ converges to x if
  for every $\epsilon > 0$, there is $N \in \mathbb{N}$, such that,
  for all $n \in \mathbb{N}$, if $n \geq N$, then

$$|x_n - x| < \epsilon.$$

- We write

$$\lim_{n\to\infty} x_n = x$$

  or $x_n \to x$ as $n \to \infty$.

- **Proposition** In an ordered field $\mathbb{F}$, if $x_n \to x$ and $x_n \to y$ as $n \to \infty$, then

$$x = y.$$

- **Proposition** In any ordered field $\mathbb{F}$ a convergent sequence is bounded.
  Monotone Sequence Property

- Recall, a sequence $\{x_n\}$ is said to be *increasing* (*nondecreasing*) if, for all $n \in \mathbb{N}$,

$$x_n < x_{n+1} \ \left(x_n \leq x_{n+1}\right).$$

- Analogously, we define a *decreasing* (*nonincerasing*) sequence.
  Monotone Sequence Property (MSP)
  Let $\mathbb{F}$ be an ordered field. We say that $\mathbb{F}$ has the *monotone sequence property* if every *nondecreasing and bounded above sequence* in $\mathbb{F}$ converges to a point in $\mathbb{F}$.

- We show that the *monotone sequence property* (MSP) is **equivalent** to the *least upper bound property* (LUB).

- In particular, we can define a complete ordered field as follows.
  Completeness Property (CP)
  An ordered field is said to be *complete* if it satisfies the *monotone sequence property.*
  Example Assume that $\mathbb{F}$ is a complete ordered field.
  Let $x_n = \left(1 - \frac{1}{n}\right)$. Since

$$\frac{1}{n+1} < \frac{1}{n},$$

it follows that

$$-\frac{1}{n} < -\frac{1}{n+1},$$

so

$$x_n = 1 - \frac{1}{n} < 1 - \frac{1}{n+1} = x_{n+1}.$$

- Therefore, $\{x_n\}$ increases.
- Furthermore, since $\frac{1}{n} > 0$,

$$-\frac{1}{n} < 0.$$

- Thus,

$$\underbrace{\left(1 - \frac{1}{n}\right)}_{x_n} < 1,$$

so for all $n \in \mathbb{N}$,

$$|x_n| < 1,$$

i.e. $\{x_n\}$ is bounded.

- Therefore, $\{x_n\}$ converges in $\mathbb{F}$.

Remark There are two formulations for the **Completeness Property**

LUB Every nonempty and bounded above subset $S \subseteq \mathbb{F}$ has a least upper bound in $\mathbb{F}$.

MSP Every nondecreasing sequence that is bounded above converges.

- **Question**: Is **LUB** equivalent to **MSP?**

Proposition If $\mathbb{F}$ satisfies the *monotone sequence property* then $\mathbb{F}$ is archimedean.

Proof. Suppose that $\mathbb{F}$ is not archimedean.

- Thus, there is $x \in \mathbb{F}$, such that, for all $n \in \mathbb{N}$,

$$n \leq x.$$

- Therefore, the sequence

$$x_n = n$$

is bounded above by x.

- Since

$$x_n = n < (n+1) = x_{n+1},$$

the sequence $\{x_n\}$ increases.

- Since $\mathbb{F}$ satisfies (**MSP**), there is $a \in \mathbb{F}$, such that,

$$a = \lim_{n \to \infty} x_n.$$

- Thus, for $\epsilon = \frac{1}{2}$, there is $N \in \mathbb{N}$, such that, for every $n > N$,

$$|x_n - a| < \frac{1}{2}.$$

- Let $n > N$, then $(n+1) > n > N$, and

$$x_{n+1} - x_n = (n+1) - n = 1.$$

- Consequently,

$$\begin{aligned} 1 &= |x_{n+1} - x_n| = |(x_{n+1} - a) + (a - x_n)| \\ &\leq |x_{n+1} - a| + |x_n - a| \\ &< \frac{1}{2} + \frac{1}{2} = 1, \text{ so} \end{aligned}$$

- $1 < 1$, a contradiction.

This completes our proof. ■

- **Theorem** If $\mathbb{F}$ has the *least upper bound property* then $\mathbb{F}$ satisfies the *monotone sequence property.*

Proof. Let $\{x_n\}$ be a nondecreasing and bounded sequence in $\mathbb{F}$.

- Consider
$$S = \{x_n : n \in \mathbb{N}\}.$$

- Notice that $x_1 \in S$, so
$$S \neq \emptyset.$$

- Moreover, since $\{x_n\}$ is bounded, there is $K \in \mathbb{F}$, such that, for all $n \in \mathbb{N}$,
$$|x_n| \leq K.$$

- Therefore, for all $n \in \mathbb{N}$,
$$-K \leq x_n \leq K,$$
i.e. we showed that, for every $x \in S$,
$$x \leq K.$$

- Therefore S is nonempty and bounded subset of $\mathbb{F}$.

- Since $\mathbb{F}$ satisfies (LUB), it follows that
$$\sup S \in \mathbb{F}.$$

- Let $\alpha = \sup S \in \mathbb{F}$.

- We show that,
$$\lim_{n \to \infty} x_n = \alpha,$$

- Let $\epsilon > 0$ be given.

- Since $\alpha = \sup S$ is the least upper bound for S,
$$\alpha - \epsilon < \alpha$$
is not an upper bound of S.

- Therefore, there is, $x \in S$, such that
$$\alpha - \epsilon < x.$$
- Since $x \in S$,
$$x = x_N,$$
for some $N \in \mathbb{N}$.
- Therefore,
$$\alpha - \epsilon < x_N$$
- Let $n > N$, then $x_N \leq x_n$, so
$$\alpha - \epsilon < x_N \leq x_n$$
- Since α is an upper bound for S,
$$x_n \leq \alpha,$$
for all $n \in \mathbb{N}$.
- Therefore,
$$\alpha - \epsilon < x_N \leq x_n \leq \alpha < \alpha + \epsilon.$$
- So, if $n > N$, then
$$\begin{aligned} \alpha - \epsilon &< x_n < \alpha + \epsilon, \text{ so} \\ |x_n - \alpha| &< \epsilon. \end{aligned}$$
- We showed that:
For every $\epsilon > 0$, there is $N \in \mathbb{N}$, such that for all $n > N$,
$$|x_n - \alpha| < \epsilon.$$
- Hence, by the definition
$$\lim_{n\to\infty} x_n = \alpha$$
as claimed.

This finishes our proof. ■

- **Exercise** Show that in a complete ordered field $\mathbb{F}$ the sequence $\left\{\frac{1}{n}\right\}$ converges to 0.
- Indeed, as we showed before, in an ordered field with the least upper bound property,
$$\inf S = 0,$$
where $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.
- Since $\left\{\frac{1}{n}\right\}$ is decreasing and bounded ($0 < \frac{1}{n} \leq 1$, for all $n \in \mathbb{N}$),
one shows that
$$0 = \inf S = \lim_{n\to\infty} \frac{1}{n}.$$
Exercise Show that in a complete ordered field $\mathbb{F}$ the sequence $\left\{\frac{1}{2^n}\right\}$ converges to 0.
- The **converse of the above theorem is true**, namely
Theorem In an ordered field $\mathbb{F}$ with the *monotone sequence property* the following properties hold:

i) (*Least upper bound property*) Every nonempty and bounded above subset $S \subseteq \mathbb{F}$ has the least upper bound in $\mathbb{F}$.

ii) (*Greatest lower bound property*) Every nonempty and bounded below subset $S \subseteq \mathbb{F}$ has the greatest lower bound in $\mathbb{F}$.

Proof. For **i**), let M be an upper bound for S (i.e. for all $x \in S$, $x \leq M$) and fix $n \in \mathbb{N}$.

- Consider sequence
$$M - \frac{1}{2^n},\ M - \frac{2}{2^n},\ M - \frac{3}{2^n},\ \ldots.$$
- Let k_n be the least positive integer k such that
$$M - \frac{k}{2^n}$$
is not an upper bound of S, i.e.
$$k_n = \min\left\{k \in \mathbb{N} : \exists x \in S \ni M - \frac{k}{2^n} < x\right\}$$
- Notice that such k_n exists:
Since $S \neq \emptyset$ we let $x \in S$.
Since $\mathbb{F}$ is Archimedean, there is $k \in \mathbb{N}$, such that
$$M - \frac{k}{2^n} < x$$
(show this).
- Let
$$b_n = M - \frac{k_n}{2^n}$$
and notice that
b_n **is not an upper bound for** S, for all $n \in \mathbb{N}$.
- Since k_n the least positive integer k such that
$$M - \frac{k}{2^n}$$
is not an upper bound of S,
$$\begin{aligned} M - \frac{k_n - 1}{2^n} &= \left(M - \frac{k_n}{2^n}\right) + \frac{1}{2^n} \\ &= b_n + \frac{1}{2^n} \end{aligned}$$
is an upper bound of S.
- Thus,
$$b_n + \frac{1}{2^n}$$
is an upper bound for S**.**

- Furthermore, we see that
$$b_1 \leq b_2 \leq ...$$
and
$$b_n = M - \frac{k_n}{2^n} \leq M,$$
- It follows that $\{b_n\}$ **is monotonically increasing and bounded above.**
- Since $\mathbb{F}$ satisfies the *monotone sequence property*,
$\{b_n\}$ converges in $\mathbb{F}$.
- Let $b_n \to b \in \mathbb{F}$ as $n \to \infty$.
- We show that:
$$\sup S = b.$$
- Notice that
$$b_n \leq b,$$
for all $n \in \mathbb{N}$.
Indeed, suppose that
$$b_N > b,$$
for some $N \in \mathbb{N}$.
Since $\{b_n\}$ increases
$$b_N \leq b_n$$
for all $n > N$.
Therefore,
$$b_N \leq \lim_{n\to\infty} b_n \text{ (show this)}$$
Hence,
$$\begin{aligned} b &< b_N \leq \lim_{n\to\infty} b_n = b \\ b &< b \end{aligned}$$
a contradiction.
- **We show that b is an upper bound for S.**
Suppose b is not an upper bound for S.
Then there is $x \in S$ such that
$$b < x$$
(i.e. b is not an upper bound of S).
Let
$$\epsilon = (x - b) > 0.$$
Since $\frac{1}{2^n} \to 0$ as $n \to \infty$, there is $n \in \mathbb{N}$, such that
$$\frac{1}{2^n} < \epsilon.$$

Now, as we showed, $b_n \leq b$, hence,

$$\begin{aligned} x &= b + (x - b) \\ &= b + \epsilon \\ &\geq b_n + \epsilon \\ &> b_n + \frac{1}{2^n}. \end{aligned}$$

A contradiction, since by our construction

$$b_n + \frac{1}{2^n}$$

is an upper bound for S.

- It follows that, for all $x \in S$,
$$x \leq b.$$
- Since $b_n \to b$, for $\epsilon > 0$, there is $N \in \mathbb{N}$, such that
$$b - b_N = |b - b_N| < \epsilon.$$
- Since b_N is not an upper bound for S, there is $x \in S$, such that
$$b_N \leq x.$$
- It follows that
$$-x \leq -b_N.$$
- Hence,
$$b - x \leq b - b_N < \epsilon$$
and, in particular
$$b - x < \epsilon$$
- It follows that
$$b - \epsilon < x.$$
- Therefore, the number
$$b - \epsilon$$
is not an upper bound of S.
- In summary, we showed that:
$i)$ The number b is an upper bound for S;
$ii)$ For any $\epsilon > 0$, there is $x \in S$, such that
$$b - \epsilon < x.$$
- Hence,
$$b = \sup S$$

as claimed. ∎

- **Remark** As we showed above:
  (MSP) is **equivalent** to (LUB)
  In particular, since $\mathbb{R}$ is complete,
  Every monotone and bounded sequence $\{x_n\} \subseteq \mathbb{R}$ converges in $\mathbb{R}$.
  Example In a complete ordered field $\mathbb{F}$ let

$$x_0 = 0$$

  and

$$x_{n+1} = \sqrt{x_n + 2}, n \geq 0.$$

  Show that $\{x_n\}$ converges.

- Indeed, we show that

$$0 \leq x_n < 2$$

  and $\{x_n\}$ is monotonically increasing.

- Clearly,

$$0 \leq x_1 = \sqrt{2} < 2$$

  and assume that

$$0 \leq x_n < 2.$$

- Therefore,

$$x_n + 2 < 4,$$

  so

$$\sqrt{x_n + 2} < \sqrt{4},$$

  thus

$$x_{n+1} = \sqrt{x_n + 2} < 2,$$

  hence

$$0 \leq x_{n+1} < 2.$$

- By PMI, for all $n \in \mathbb{N}$,

$$0 \leq x_n < 2.$$

- Since

$$\begin{aligned}
x_{n+1} - x_n &= \sqrt{x_n + 2} - x_n \\
&= \frac{\left(\sqrt{x_n + 2} - x_n\right)\left(\sqrt{x_n + 2} + x_n\right)}{\left(\sqrt{x_n + 2} + x_n\right)} \\
&= \frac{x_n + 2 - x_n^2}{\sqrt{x_n + 2} + x_n} \\
&= \frac{(x_n + 1)(2 - x_n)}{\sqrt{x_n + 2} + x_n} \\
&> 0,
\end{aligned}$$

  it follows that

$$x_{n+1} > x_n,$$

  for all $n \in \mathbb{N}$.

- Hence $\{x_n\}$ is monotone and bounded,
  since *the field is complete*,
  MSP holds in $\mathbb{F}$, so $\{x_n\}$ converges.
- We find its limit.
- Let
$$a = \lim_{n\to\infty} x_n \,.$$
- Since
$$x_{n+1} = \sqrt{x_n + 2},$$
$$x_{n+1}^2 = x_n + 2.$$
- Since
$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = a,$$
  it follows that
$$\begin{aligned} a^2 &= \lim_{n\to\infty} x_{n+1} \cdot \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_{n+1}^2 \\ &= \lim_{n\to\infty} (x_n + 2) \\ &= \lim_{n\to\infty} x_n + 2 \\ &= a + 2. \end{aligned}$$
- Thus,
$$a^2 = a + 2$$
- Hence $a = 2$ or $a = -1$.
- However, $x_n \geq 0$, for all $n \in \mathbb{N}$, so
$$a = \lim_{n\to\infty} x_n \geq 0, \text{ so } a = 2.$$
- It follows that
$$a = \lim_{n\to\infty} x_n = 2.$$

Proposition Let $\{x_n\}$ be the sequence defined by
$$x_n = \sum_{k=1}^{n} \frac{1}{k}.$$
Then $\{x_n\}$ monotonically increasing, unbounded, and it does not converge.

Proof. Clearly,
$$\begin{aligned} x_{n+1} - x_n &= \sum_{k=1}^{n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} \\ &= \frac{1}{n+1} \\ &> 0 \end{aligned}$$

- Therefore, $\{x_n\}$ is **monotonically increasing**.
- We show that $\{x_n\}$ is **not bounded above**.
- Let $M \in \mathbb{R}$, $M > 0$.
- We show that there is $k \in \mathbb{N}$, such that
$$x_k > M.$$
- Since $\mathbb{R}$ is archimedean, there is $n \in \mathbb{N}$, such that
$$2M < n.$$
- Let $k = 2^n$, then
$$\begin{aligned} x_k &= x_{2^n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{2} + ... + \underbrace{\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + ... + \frac{1}{2^n}}_{2^{n-1}} \\ &\geq \left(1 + \frac{1}{2}\right) + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + ... + 2^{n-1} \cdot \frac{1}{2^n} \\ &= 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + ... + \frac{1}{2}}_{n} \\ &= 1 + \frac{n}{2} \\ &> 1 + M > M. \end{aligned}$$
- It follows that
$$x_k > M.$$
- Therefore, $\{x_n\}$ is unbounded.
- Since convergent sequence must be bounded, it follows that $\{x_n\}$ does not converge.

This finishes our proof. ■

- **Remark** If a sequence $\{x_n\}$ is *monotonically increasing and unbounded above*, then we say that $\{x_n\}$ diverges to ∞ and we write
$$x_n \to \infty \text{ as } n \to \infty.$$

  Definition A sequence $x_n \to \infty$ as $n \to \infty$, if for any $M \in \mathbb{F}$, there is $N \in \mathbb{N}$, such that, for all $n \geq N$,
$$x_n \geq M.$$

  Analogously, $x_n \to -\infty$ as $n \to \infty$, if for any $M \in \mathbb{F}$, there is $N \in \mathbb{N}$, such that, for all $n \geq N$,
$$x_n \leq M.$$

Proposition The sequence $\{x_n\}$ defined by

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

is strictly monotone increasing and converges to a limit e, where

$$2 < e < 3.$$

Proof. Using the binomial theorem

$$\begin{aligned}
\left(1 + \frac{1}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \\
&= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \ldots + \frac{n(n-1) \cdot \ldots \cdot 2 \cdot 1}{n!} \cdot \frac{1}{n^n} \\
&= 2 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots + \frac{1}{n!} \cdot \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdot \ldots \cdot \frac{2}{n} \cdot \frac{1}{n}
\end{aligned}$$

- Each term in the sum is positive and increases as n increases and the number of terms increases with n.
- Therefore, $\{x_n\}$ is strictly increasing and

  $$x_n > 2,$$

  for all $n \geq 2$.
- Since

  $$0 \leq \left(1 - \frac{k}{n}\right) < 1, \text{ for } 1 \leq k \leq n,$$

  it follows that

  $$\left(1 + \frac{1}{n}\right)^n < 2 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}.$$

- Since for $n \geq 1$,

  $$n! \geq 2^{n-1},$$

  it follows that

  $$\begin{aligned}
  \left(1 + \frac{1}{n}\right)^n &< 2 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} \\
  &< 2 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} \\
  &= 2 + \frac{1}{2} \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} \\
  &< 2 + 1 = 3.
  \end{aligned}$$

- So the sequence $\{x_n\}$ is bounded above by a number strictly less that 3.
- Since $\{x_n\}$ is increasing and bounded, and $\mathbb{R}$ is complete,
- it follows that $\{x_n\}$ converges.
  We let

  $$e = \lim_{n \to \infty} x_n,$$

  then $2 < e < 3$.

This finishes our proof. ■

- **Cauchy Sequences**

  Definition A sequence $\{x_n\}$ of real numbers is called a Cauchy sequence if for every $\epsilon > 0$ there is $N \in \mathbb{N}$, such that, for $m, n \geq N$,

$$|x_m - x_n| < \epsilon.$$

  Example Let $x_n = \frac{1}{n}$, $n = 1, 2, ...$
  We show that $\{x_n\}$ is a Cauchy sequence.

- Let $\epsilon > 0$ be given.
- Since $\mathbb{R}$ is archimedean, there is $N \in \mathbb{N}$, such that

$$\frac{1}{N} < \frac{\epsilon}{2}$$

- Let $n, m > N$, then

$$\begin{aligned} |x_n - x_m| &\leq |x_n| + |x_m| \\ &= \frac{1}{n} + \frac{1}{m} \\ &< \frac{1}{N} + \frac{1}{N} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

  Proposition Every convergent sequence is Cauchy.

Proof. Let $x_n \to x$ as $n \to \infty$ and take $\epsilon > 0$.

- By the definition, there is $N \in \mathbb{N}$, such that,
  for $n > N$,

$$|x_n - x| < \frac{\epsilon}{2}.$$

- Therefore, if $m, n > N$,

$$\begin{aligned} |x_n - x_m| &= |(x_n - x) + (x - x_m)| \\ &\leq |x_n - x| + |x_m - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

- So, $\{x_n\}$ is Cauchy.

This finishes our argument. ■

- **Proposition** Every Cauchy sequence is bounded.

Proof. Let $\epsilon = 1$.

- Since $\{x_n\}$ is Cauchy, there is $N \in \mathbb{N}$, such that,
  for all $n, m > N$

$$|x_m - x_n| < 1.$$

- In particular,
$$N+1>N,$$
so for all $n>N$,
$$|x_{N+1}-x_n|<1.$$

- Hence, for $n>N$,
$$\begin{aligned} |x_n| &= |(x_n-x_{N+1})+x_{N+1}| \\ &\le |x_n-x_{N+1}|+|x_{N+1}| \\ &< 1+|x_{N+1}|. \end{aligned}$$

- Let
$$K=\max\{|x_1|,\ldots|x_N|\}.$$

- Then
$$|x_n|\le K, \text{ for } n=1,2,\ldots,N.$$

- Therefore, for all $n\in\mathbb{N}$,
$$|x_n|\le\max\{K,1+|x_{N+1}|\}.$$

Therefore, $\{x_n\}$ is bounded. ■

- **Definition** Let $\{x_n\}$ be a sequence and
$$n(1)<n(2)<\ \ldots,$$
be an increasing sequence of positive integers, then the sequence
$$\begin{aligned} &x_{n(1)},\ x_{n(2)},\ \ldots,\ \text{that is} \\ &x_{n_1},\ x_{n_1},\ \ldots \end{aligned}$$
is called a subsequence of the sequence $\{x_n\}$.
Example Let $x_n=(-1)^n$, then
$$x_{2k}=(-1)^{2k}=1$$
and
$$x_{2k+1}=(-1)^{2k+1}=-1$$
are subsequences
Our goal is to prove the following result:
Theorem Every Cauchy sequence in $\mathbb{R}$ converges.
Theorem (*Bolzano-Weierstrass*) Every bounded sequence $\{x_n\}$ in $\mathbb{R}$ has a convergent subsequence $\{x_{n_k}\}$.

Proof. Since $\{x_n\}$ is bounded,

- there is $M\ge 0$, such that, for all $n\in\mathbb{N}$,
$$-M\le x_n\le M.$$

- Consider the interval
$$I=[-M,M]$$
and its subintervals
$$[-M,0] \text{ and } [0,M].$$

- At least one of then must contain x_n for infinite number of $n \in \mathbb{N}$.
- Call this subinterval I_0 and select $n_0 \in \mathbb{N}$ with

$$x_{n_0} \in I_0.$$

- Split I_0 into half and
  let I_1 be a subinterval for which

$$x_n \in I_1$$

  for infinitely many $n \in \mathbb{N}$.
- Since there are infinitely many $n \in \mathbb{N}$, for which $x_n \in I_1$,
  there is $n_1 > n_0$, such that

$$x_{n_1} \in I_1.$$

- We continue in such a manner to obtain sequence of subintervals I_k,
  indices

$$n_k > n_{k-1} > ... > n_1 > n_0$$

  and points

$$x_{n_k} \in I_k.$$

  We observe that:
- $I_0 \supseteq I_1 \supseteq I_2 \supseteq ...$
- $I_k = [a_k, b_k]$ with

$$b_k - a_k = \frac{M}{2^k}.$$

- $n_0 < n_1 < n_2 < ... < n_k < ...$
- $x_{n_k} \in I_k$.
- **We show that $\{x_{n_k}\}$ converges.**
- Consider the sequence $\{a_k\}$ of the left ends of intervals I_k.
- Since $I_{k+1} \subset I_k$, for all $k \in \mathbb{N}$, we see that

$$a_0 \leq a_1 \leq ...,$$

  so the sequence $\{a_k\}$ is **monotonically increasing.**
- The sequence $\{a_k\}$ is also **bounded** since

$$a_k \in I_k \subset I,$$

  so

$$-M \leq a_k \leq M.$$

- By **completeness property** of $\mathbb{R}$:
- $\{a_k\}$ converges to some real number $x \in I$ (why $x \in I$?)
- Now, we ready to **show that subsequence $\{x_{n_k}\}$ also converges to** x.

- Indeed, for all $k \in \mathbb{N}$:

$$\begin{aligned} |x_{n_k} - x| &= |(x_{n_k} - a_k) + (a_k - x)| \\ &\leq |x_{n_k} - a_k| + |a_k - x| \end{aligned}$$

- Since $x_{n_k} \in I_k$, i.e.

$$a_k \leq x_{n_k} \leq b_k$$

it follows that

$$\begin{aligned} \max\{|x_{n_k} - a_k|, |x_{n_k} - b_k|\} &\leq |I_k| \\ &= b_k - a_k \\ &\leq \frac{M}{2^k}. \end{aligned}$$

- Let $\epsilon > 0$ be given.

- Since

$$\frac{M}{2^k} \to 0,$$

there is $N_1 \in \mathbb{N}$, such that, for $k > N_1$,

$$\frac{M}{2^k} < \frac{\epsilon}{2}.$$

- Since $\{a_k\}$ converges to x,

there is $N_2 \in \mathbb{N}$, such that, for $k > N_2$,

$$|a_k - x| < \frac{\epsilon}{2}.$$

- Thus, if $k > \max\{N_1, N_2\}$,

$$\begin{aligned} |x_{n_k} - x| &\leq |x_{n_k} - a_k| + |a_k - x| \\ &< \frac{M}{2^k} + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

- It follows that

$$\lim_{k \to \infty} x_{n_k} = x,$$

so $\{x_{n_k}\}$ is a convergent subsequence of $\{x_n\}$.

This finishes our proof. ■

- **Example** Is $x_n = \sin(n)$, $n = 1, 2, ...$ a bounded sequence?

- Since

$$-1 \leq \sin(n) \leq 1,$$

it follows that

$$-1 \leq x_n \leq 1$$

- By the **B-W** Theorem, $\{x_n\}$ has a convergent subsequence

$$x_{n_k} = \sin(n_k),$$

$k = 1, 2, ...$ with the limit $x \in [-1, 1]$.

Exercise Show that $\left\{e^{\sin(3n)}\right\}$ has convergent subsequence.

Lemma Let $\{x_n\}$ be a Cauchy sequence and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. If $\{x_{n_k}\}$ converges to x then $\{x_n\}$ converges to x.

Proof. Let $\epsilon > 0$ be given.

- Since $\{x_n\}$ is Cauchy,
  there is $N \in \mathbb{N}$, such that, for all $m, n > N$,

$$|x_m - x_n| < \frac{\epsilon}{2}.$$

- Since $\{x_{n_k}\}$ converges to x,
  there is $K > N$, such that

$$|x_{n_K} - x| < \frac{\epsilon}{2}.$$

- Now, if $m > \max\{N, K\}$, then $m > n_K$ and

$$\begin{aligned} |x_m - x| &= |(x_m - x_{n_K}) + (x_{n_K} - x)| \\ &\leq |x_m - x_{k_K}| + |x_{n_K} - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

- Therefore, $x_n \to x$ as $n \to \infty$.

This finishes our proof. ■

- **Remark** The sequence $x_n = \frac{1}{n}$ is Cauchy in $\mathbb{R}\backslash\{0\}$,
  but it does not converge in $\mathbb{R}\backslash\{0\}$ since $0 \notin \mathbb{R}\backslash\{0\}$.

  Theorem Every Cauchy sequence $\{x_n\}$ in $\mathbb{R}$ converges to some point in $\mathbb{R}$.

Proof. Since Cauchy sequence $\{x_n\}$ is bounded, by $(B - W)$,

- $\{x_n\}$ has convergent subsequence $\{x_{n_k}\}$.
- If $x_{n_k} \to x$, then, by Lemma,

$$x_n \to x, \text{ as } n \to \infty$$

so $\{x_n\}$ is convergent. ■

- **Example** Assume that $\{x_n\}$ is a sequence of real numbers such that

$$|x_n - x_{n+1}| < \frac{1}{2^n},$$

  for all $n \in \mathbb{N}$. Show that $\{x_n\}$ converges.

- We show that $\{x_n\}$ is a Cauchy sequence.

- Let $m = n + k$, where $k > 0$.
- Then

$$\begin{aligned}
|x_n - x_{n+k}| &= |(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + ... + (x_{n+k-1} - x_{n+k})| \\
&\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + ... + |x_{n+k-1} - x_{n+k}| \\
&\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + ... + \frac{1}{2^{n+k-1}} = \frac{1}{2^n}\left(1 + \frac{1}{2} + ... + \frac{1}{2^{k-1}}\right) \\
&= \frac{1}{2^n} \cdot \frac{1 - \left(\frac{1}{2}\right)^k}{1 - \frac{1}{2}} \\
&< \frac{1}{2^n} \cdot \frac{1}{1 - \frac{1}{2}} \\
&= \frac{1}{2^{n-1}} \leq \frac{1}{n}.
\end{aligned}$$

- Let $\epsilon > 0$ be given.
- There is $N \in \mathbb{N}$, such that
$$\frac{1}{N} < \epsilon.$$
- Therefore, for all $m, n > N$, if $m = n + k$, $k > 0$, then
$$|x_n - x_m| < \frac{1}{n} < \frac{1}{N} < \epsilon.$$
- We showed that $\{x_n\}$ is a Cauchy sequence.
- From the theorem above, it follows that $\{x_n\}$ converges.

Review Susequences

- It important to remember that a sequence $\{x_n\}$ is a function
$$\begin{aligned} X &: \mathbb{N} \to \mathbb{R}, \\ X(n) &= x_n, \end{aligned}$$
- hence, if
$$\begin{aligned} n &: \mathbb{N} \to \mathbb{N}, \\ n(k) &= n_k, \end{aligned}$$
is *strictly increasing* (so $n(k) < n(k+1)$, for all k) then
$$X \circ n : \mathbb{N} \to \mathbb{R},$$
is defined by
$$(X \circ n)(k) = X(n(k)) = X(n_k) = x_{n_k}$$
is a subsequence of the sequence $\{x_n\}$.
- Thus each strictly increasing function
$$n : \mathbb{N} \to \mathbb{N}$$
gives a subsequence of $\{x_n\}$.

- In our example
$$x_n = X(n) = (-1)^{n+1}$$
and let
$$n : \mathbb{N} \to \mathbb{N}$$
be defined by,
$$n(k) = n_k = 2k,$$
then clearly n is strictly increasing and
$$\begin{aligned} x_{2k} &= x_{n_k} = (X \circ n)(k) \\ &= X(n(k)) = X(2k) \\ &= (-1)^{(2k)+1} \\ &= -1 \end{aligned}$$

- if we take $n : \mathbb{N} \to \mathbb{N}$ defined by,
$$n(k) = n_k = 2k - 1,$$
then clearly n is strictly increasing and
$$\begin{aligned} x_{2k-1} &= x_{n_k} = (X \circ n)(k) \\ &= X(n(k)) = X(2k-1) \\ &= (-1)^{(2k-1)+1} \\ &= 1. \end{aligned}$$

- In such a way we obtained two different subsequences of $\{x_n\}$.
- The first is defined as $x_{2k} = -1$, for all k and the second is $x_{2k-1} = 1$, for all k.
- We see though that different strictly increasing function $n : \mathbb{N} \to \mathbb{N}$ may still yield same subsequences.
- For instance, if
$$x_n = X(n) = (-1)^{n+1}$$
and we take
$$n : \mathbb{N} \to \mathbb{N}$$
defined by
$$n(k) = 2k,$$
then clearly n is strictly increasing and
$$\begin{aligned} x_{2k} &= (X \circ n)(k) \\ &= X(n(k)) = X(2k) \\ &= (-1)^{2k+1} \\ &= -1. \end{aligned}$$

- However, the function,
$$m : \mathbb{N} \to \mathbb{N}$$
defined by,
$$m(k) = 4k$$

is also strictly increasing and

$$\begin{aligned}(X \circ m)(k) &= X(m(k)) \\ &= X(4k) = (-1)^{4k+1} \\ &= -1\end{aligned}$$

is also a subsequence of $\{x_n\}$.

- We see however that
$$(X \circ n)(k) = (X \circ m)(k), \text{ for all } k \in \mathbb{N}.$$
- Therefore, for a given subsequence $\{x_{n_k}\}$ of $\{x_n\}$
there might be several strictly increasing functions
$$n : \mathbb{N} \to \mathbb{N},$$
such that
$$x_{n_k} = (X \circ n)(k).$$
- Finally, we notice that every strictly increasing function
$$n : \mathbb{N} \to \mathbb{N}$$
determines an infinite naturally ordered set which is the image of $\mathbb{N}$ via n, that is
$$\begin{aligned}S &= n(\mathbb{N}) = \{n(k) : k = 1, 2, ...\} \\ &= \{n_1, n_2, ...\}\end{aligned}$$
and each infinite naturally ordered subset
$$\{n_1, n_2, ...\} \subseteq \mathbb{N}$$
determines a unique strictly increasing function
$$\begin{aligned}n &: \mathbb{N} \to \mathbb{N}, \text{ defined by setting} \\ n(k) &= n_k\end{aligned}$$
- Therefore, each naturally ordered subset
$$\{n_1, n_2, ...\} \subseteq \mathbb{N}$$
determines a unique subsequence
$$x_{n_k} = (X \circ n)(k)$$
- Furthermore, since
$$x_{n_k} = (X \circ n)(k) = X(n_k), \text{ for all } k = 1, 2, ..$$
we may also view each subsequence $\{x_{n_k}\}$ of $\{x_n\}$ as a restriction of X to the subset $\{n_1, n_2, ...\} \subseteq \mathbb{N}$.
- Therefore, there is a surjective map from the set of all strictly increasing functions
$$n : \mathbb{N} \to \mathbb{N}$$
(equivalently naturally ordered subsets of $\mathbb{N}$) to the set of all subsequences of $\{x_n\}$.

Exercise Find a bounded sequence with three subsequences converging to three different numbers.

Solution For $n \in \mathbb{N}$, let
$$x_n = \begin{cases} \frac{1}{n} & if \quad n = 3k \\ 1 - \frac{1}{n} & if \quad n = 3k+1 \\ 2 + \frac{1}{n} & if \quad n = 3k+2 \end{cases}.$$

- We see that

$$\begin{aligned} x_{3k} &= \frac{1}{3k}, \\ x_{3k+1} &= 1 - \frac{1}{3k+1}, \text{ and} \\ x_{3k+2} &= 2 + \frac{1}{3k+2}. \end{aligned}$$

- One shows that

$$\begin{aligned} \lim_{k\to\infty} x_{3k} &= 0, \\ \lim_{k\to\infty} x_{3k+1} &= 1, \text{ and} \\ \lim_{k\to\infty} x_{3k+2} &= 2. \end{aligned}$$