HOMEWORK 10 SOLUTIONS – MATH 4341

Problem 1. Let $S^n \subset \mathbb{R}^{n+1}$ be the standard unit n-sphere, i.e.

$$S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}.$$

Suppose $\{A_k\}_{k=1}^{\infty}$ is a sequence of non-empty closed sets in S^n such that $A_1 \supset A_2 \supset \cdots \supset A_k \supset A_{k+1} \supset \dots$. Show that $\cap_{k=1}^{\infty} A_k$ is non-empty.

Proof. Assume that $\cap_{k=1}^{\infty} A_k = \emptyset$. Then

$$S^n = S^n \setminus (\cap_{k=1}^{\infty} A_k) = \cup_{k=1}^{\infty} (S^n \setminus A_k).$$

Since $A_k \subset S^n$ is closed, $S^n \setminus A_k$ is open in S^n. This implies that $\{S^n \setminus A_k\}_k$ is open cover of S^n. Since S^n is compact, there exist $k_1 < k_2 < \cdots < k_r$ such that $\{S^n \setminus A_{k_1}, S^n \setminus A_{k_2}, \cdots, S^n \setminus A_{k_r}\}$ is also an open cover of S^n. This means that

$$S^n = \cup_{i=1}^{r} (S^n \setminus A_{k_i}) = S^n \setminus (\cap_{i=1}^{r} A_{k_i}).$$

Hence $\cap_{i=1}^{r} A_{k_i} = \emptyset$. Since $A_{k_1} \supset A_{k_2} \supset \cdots \supset A_{k_r}$ we have $\emptyset = \cap_{i=1}^{r} A_{k_i} = A_{k_r}$. This contradicts the fact that A_{k_r} is an non-empty set. □

Problem 2. Show that every compact subspace of a metric space is bounded and closed.

Proof. Suppose K is a compact subspace of a metric space X. Since X is Hausdorff, Theorem 6.2 in the lecture notes implies that K is closed. To show that K is bounded, we fix $x_0 \in X$ and note that $\{B_d(x_0, n)\}_{n=1}^{\infty}$ is an open cover of X. Consequently, $\{K \cap B_d(x_0, n)\}_{n=1}^{\infty}$ is an open cover of K. Since K is compact, there exists $n_1, \dots, n_r$ such that

$$K = (K \cap B_d(x_0, n_1)) \cup \cdots \cup (K \cap B_d(x_0, n_r)).$$

This means that $K \subset B_d(x_0, n_1) \cup \cdots \cup B_d(x_0, n_r)$ and hence K is bounded. □

Problem 3. Show that a bounded and closed subset of a metric space is not always compact.

Proof. Consider any infinite set X with the discrete metric, i.e. $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$. Then every subset of X is bounded and closed. However X is not compact, since the open cover $\{\{x\}\}_{x \in X}$ does not have a finite subcover. □

Problem 4. Suppose A is a compact subspace of the Hausdorff space X and $x \in X \setminus A$. Show that there exist disjoint open sets U and V of X containing x and A respectively.

Proof. For $y \in A$, we can find disjoint neighbourhoods U_y and V_y of x and y respectively, since X is Hausdorff. Now the collection $\{A \cap V_y\}_{y \in A}$ is an open cover of A, and since A is compact, we can choose finitely many $y_1, \dots, y_n$ such that $\{A \cap V_{y_i}\}_{i=1,\dots,n}$ is a finite subcover. In particular, $A \subset V_{y_1} \cup \cdots \cup V_{y_n}$.

Let $U = U_{y_1} \cap \cdots \cap U_{y_n}$ and $V = V_{y_1} \cup \cdots \cup V_{y_n}$. Then U and V are open subsets of X containing x and A respectively. Note that U and V are disjoint, since $(U \cap V) \subset \cup_{i=1}^{n} (U_i \cap V_i) = \emptyset$. □

Problem 5. Let A and B be disjoint compact subspaces of a Hausdorff space X. Show that there exist disjoint open sets U and V containing A and B respectively.

Proof. For every $x \in A$, by problem 4, there exist disjoint open sets U_x and V_x in X containing x and B respectively (since $x \notin B$ and B is a compact subspace of a Hausdorff space X). The collection $\{A \cap U_x\}_{x \in A}$ is an open cover of the compact space A, hence there exist $U_{x_1}, \dots, U_{x_r}$ such that

$$A = (A \cap U_{x_1}) \cup \cdots \cup (A \cap U_{x_r}). \tag{0.1}$$

Let $U = U_{x_1} \cup \cdots \cup U_{x_r}$. Then (0.1) implies that U is an open subset of X containing A.

Let $V = V_{x_1} \cap \cdots \cap V_{x_r}$. Then V is an open subset of X containing B. Moreover we have $U \cap V = \emptyset$, since $(U \cap V) \subset \cup_{i=1}^{r} (U_{x_i} \cap V_{x_i}) = \emptyset$. $\square$