Math 4301 Mathematical Analysis I

Lecture 1

Topic: Ordered Fields

- **Natural Numbers**
- The natural numbers (or positive integers) is the set

$$\mathbb{N} = \{1, 2, ...\}.$$

- Non-negative integers is the set

$$\mathcal{N} = \{0, 1, 2, ...\}.$$

- The main property of the non-negative integers is:

  Principle of Mathematical Induction

  If $S \subseteq \mathcal{N}$ and $0 \in S$ and $(k+1) \in \mathcal{S}$ whenever $k \in S$, then $S = \mathcal{N}$.

- An ordering relation $\leq$ on a set S is called *well-order* if every non-empty subset A of S has a smallest element.

  Proposition $\mathcal{N}$ is well-ordered by the relation $\leq$.

  That is, $\mathcal{N}$ has the *well-ordering property*:

  If S is nonempty subset of $\mathcal{N}$, then there exist a smallest element in S;

  i.e. there is an $s_0 \in S$, such that, for all $x \in S$,

$$s_0 \leq x.$$

Proof. We prove the statement by reductio ad impossibile.

- Suppose that $S \subseteq \mathcal{N}$ has **no smallest element**.
- Define $T = \mathcal{N} \backslash S$
- Since $0 \in \mathcal{N}$ is the smallest element of S and $S \subset \mathcal{N}$, it follows

$$0 \notin S$$

- Let

$$T_0 = \{n \in \mathcal{N} \ : \ \{0, 1, 2, ..., n\} \subseteq T\}.$$

- Since $0 \notin S$, $0 \in T$, so

$$\{0\} \subseteq T.$$

- Hence $0 \in T_0$.
- Suppose that $k \in T_0$, then

$$\{0, 1, 2, ..., k\} \subseteq T.$$

- If $(k+1) \notin T$, then

$$(k+1) \in S.$$

- Since

$$\begin{aligned} \{0,1,2,...,k\} &\subset \mathcal{N} \backslash S = T, \text{ thus} \\ S &\subset \mathcal{N} \backslash \{0,1,2,...,k\} = \{k+1, k+2, ...\}. \end{aligned}$$

- It follows

  $$(k+1) = \min S,$$

  a contradiction.

- Therefore,

  $$(k+1) \in T$$

- Since

  $$\{0,1,2,...,k\} \subseteq T,$$

  it follows that

  $$\{0,1,2,...,k,k+1\} \subseteq T,$$

- Thus, by the definition of T_0

  $$(k+1) \in T_0.$$

- Consequently, T_0 satisfies **PMI**.

- Hence

  $$T_0 = \mathcal{N},$$

  so $T = \mathcal{N}$, and

- Therefore,

  $$T = \mathcal{N} \backslash S = \mathcal{N},$$

  so

  $$S = \emptyset.$$

A contradiction. This finishes our proof. ■

- **Remark** One shows that the *Well-Ordering Property* of $\mathcal{N}$ implies the *Principle of Mathematical Induction* (**PMI**).

  Example We show that, for all $n \in \mathbb{N}$,

  $$\sum_{j=1}^{n} j^2 = \frac{1}{6} n(n+1)(2n+1).$$

  applying the **PMI**

  Proof Let

  $$S = \left\{ n \in \mathbb{N} : \sum_{j=1}^{n} j^2 = \frac{1}{6} n(n+1)(2n+1) \right\}.$$

- Since

  $$1 = \sum_{j=1}^{1} j^2 = \frac{1}{6} 1 \cdot (1+1)(2 \cdot 1 + 1) = 1$$

  is true,

  $$1 \in S.$$

- Moreover, if $k \in S$, then
$$\sum_{j=1}^{k} j^2 = \frac{1}{6}k\left(k+1\right)\left(2k+1\right)$$

- Therefore,
$$\begin{aligned}\sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^{k} j^2 + \left(k+1\right)^2 = \frac{1}{6}k\left(k+1\right)\left(2k+1\right) + \left(k+1\right)^2 \\ &= \frac{1}{6}\left(k\left(2k+1\right) + 6\left(k+1\right)\right)\left(k+1\right) \\ &= \frac{1}{6}\left(2k^2 + 7k + 6\right)\left(k+1\right) \\ &= \frac{1}{6}\left(k+1\right)\left(k+2\right)\left(2k+3\right),\end{aligned}$$

  thus
$$\left(k+1\right) \in S,$$
  so by **PMI**,
$$S = \mathbb{N}.$$

- That is, for all $n \in \mathbb{N}$,
$$\sum_{j=1}^{n} j^2 = \frac{1}{6}n\left(n+1\right)\left(2n+1\right).$$

Integers

- The set of integers is defined by
$$\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$$

Proposition Let $S \subseteq \mathbb{Z}$ and assume that $0 \in S$ and
$$\left(k+1\right), \ \left(k-1\right) \in S$$
whenever $k \in S$, then
$$S = \mathbb{Z}.$$

Ordered Fields

- A set $\mathbb{F}$ with two binary operations
$$\begin{aligned} + &: \mathbb{F} \times \mathbb{F} \to \mathbb{F}, \\ \cdot &: \mathbb{F} \times \mathbb{F} \to \mathbb{F}\end{aligned}$$
  (called *addition* and *multiplication*) and relation $\leq$ (called order) is called an *ordered field*, if the following properties are satisfied:

- **Axioms of a commutative field**

a) *Addition axioms*: Addition $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ (we write $+\left(a,b\right) = a+b$) satisfies properties:

a1) For all $x, y, z \in \mathbb{F}$,
$$x + \left(y+z\right) = \left(x+y\right) + z$$

a2) For all $x, y \in \mathbb{F}$,

$$x + y = y + x$$

a3) There is $0 \in \mathbb{F}$, such that, for all $x \in \mathbb{F}$,

$$x + 0 = x$$

a4) For each $x \in \mathbb{F}$, there is $-x \in \mathbb{F}$, such that

$$x + (-x) = 0$$

b) *Multiplication axioms*: Multiplication $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ (we write $\cdot(a, b) = a \cdot b$) satisfies properties:

b1) For all $x, y, z \in \mathbb{F}$,

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

b2) For all $x, y \in \mathbb{F}$,

$$x \cdot y = y \cdot x$$

b3) There is $1 \in \mathbb{F}$, $1 \neq 0$, such that, for all $x \in \mathbb{F}$,

$$x \cdot 1 = x$$

b4) For each $x \in \mathbb{F}$, $x \neq 0$, there is $x^{-1} \in \mathbb{F}$, such that

$$x \cdot x^{-1} = 1$$

c) *Distributivity Law*

c1) For all $x, y, z \in \mathbb{F}$,

$$(x + y) \cdot z = x \cdot z + y \cdot z$$

- **Order Axioms**: The order $\leq$ on $\mathbb{F}$ satisfies properties:

d1) For all $x \in \mathbb{F}$,

$$x \leq x$$

d2) For all $x, y \in \mathbb{F}$, if $x \leq y$ and $y \leq x$ then

$$x = y$$

d3) For all $x, y, z \in \mathbb{F}$, if $x \leq y$ and $y \leq z$ then

$$x \leq z$$

d4) For all $x, y \in \mathbb{F}$,

$$x \leq y \text{ or } y \leq x$$

d5) For all $x, y, z \in \mathbb{F}$, if $x \leq y$ then

$$x + z \leq y + z$$

d6) For all $x, y \in \mathbb{F}$, if $0 \leq x$ and $0 \leq y$, then

$$0 \leq xy$$

- Define the relation $<$ on $\mathbb{F}$, by setting

$$x < y \text{ iff } x \leq y \text{ and } x \neq y.$$

- Analogously, define
$$x \geq y \text{ iff } y \leq x$$
and
$$x > y \text{ iff } y < x.$$

Proposition For all $x, y \in \mathbb{F}$, $x < y$ or $x = y$ or $x > y$.

Proof Exercise.

Proposition For all $x, y, z \in \mathbb{F}$, the following properties hold:

1. **i**) If for all $x \in \mathbb{F}$,
$$x + y = x \text{ then } y = 0;$$
ii) If for all $x \in \mathbb{F}$,
$$x \cdot y = x \text{ then } y = 1.$$

- **Proof** We prove **i**) and **ii**) is left as an exercise.
- By $a3$)
$$y = 0 + y$$
- By our assumption,
$$0 + y = 0,$$
so by $a2$)
$$\begin{aligned} y &= 0 + y \\ &= y + 0 = 0 \end{aligned}$$
- Therefore,
$$y = 0.$$

Remark In particular, we showed that 0 is unique neutral element of $\mathbb{F}$ for the addition.

2. **i**) If $x + y = 0$, then $y = -x$;
ii) If $x \cdot y = 1$ then $y = x^{-1}$.

- **Proof** We prove **i**) and **ii**) is left as an exercise.
- We see that by properties $a3$) and $a4$) that
$$\begin{aligned} y &= 0 + y = (-x + x) + y \\ &= -x + (x + y) \end{aligned}$$
- By assumption $x + y = 0$ and by $a3$), so
$$\begin{aligned} y &= 0 + y = (-x + x) + y \\ &= -x + (x + y) = -x + 0 \\ &= -x \end{aligned}$$
so $y = -x$.

3. **i**) If $x + y = x + z$ then $y = z$;
ii) If $x + z \leq y + z$, then $x \leq y$.

- **Proof** We prove **i**) and **ii**) is left as an exercise.
- Indeed,

$$\begin{aligned} y &= 0 + y = (-x + x) + y \\ &= -x + (x + y) = -x + (x + z) \\ &= (-x + x) + z = 0 + z = z, \end{aligned}$$

so $y = z$.

4. **i**) If $xy = xz$ and $x \neq 0$, then $y = z$;
   ii) If $xy \leq xz$ and $x > 0$ then $y \leq z$.
5. $0 \cdot x = 0$

- **Proof** Indeed, we see that

$$\begin{aligned} 0 \cdot x &= (0 + 0) \cdot x \\ &= 0 \cdot x + 0 \cdot x \end{aligned}$$

- Since

$$0 \cdot x + 0 = 0 \cdot x + 0 \cdot x$$

by the previous property

$$0 \cdot x = 0.$$

6. If $x \cdot y = 0$ then $x = 0$ or $y = 0$
7. $-(-x) = x$
8. $-x = (-1) \cdot x$
9. If $x \neq 0$, then $x^{-1} \neq 0$ and $\left(x^{-1}\right)^{-1} = x$.
10. If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$ and $(xy)^{-1} = x^{-1}y^{-1}$
11. If $x \leq 0$ then $0 \leq -x$
12. $0 < 1$
13. If $x \leq y$ then $-y \leq -x$
14. $-xy = (-x) \cdot y = x \cdot (-y)$
15. **i**) If $x \leq y$ and $0 \leq z$, then $xz \leq yz$;
    ii) If $x \leq y$ and $z \leq 0$, then $yz \leq xz$
16. **i**) If $x \leq 0$ and $y \leq 0$ then $xy \geq 0$;
    ii) If $x \leq 0$ and $y \geq 0$, then $xy \leq 0$
17. For all $x \in \mathbb{F}$, $x^2 \geq 0$

- **Exercise** Let $\mathbb{F}$ be an ordered field.
  Show that, for all x, $y \in \mathbb{F}$

1. $xy \leq \frac{1}{2}\left(x^2 + y^2\right)$

2. $x^2 - y^2 = (x-y)(x+y)$

3. If $0 \leq x < y$ then $x^2 < y^2$

- For $x \in \mathbb{F}$, define
$$|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$
and we call it the *absolute value of x.*

Proposition Let $x, y \in \mathbb{F}$, then

1. $|x| \geq 0$
2. $|x| = 0$ iff $x = 0$
3. $|xy| = |x| \cdot |y|$
4. $x \leq |x|$
5. $|x+y| \leq |x| + |y|$
6. $||x| - |y|| \leq |x-y|$

Proof. For 1):

- Let $x \in \mathbb{F}$, then
$$x < 0 \text{ or } x = 0 \text{ or } x > 0.$$
- If $x \geq 0$, then
$$|x| = x \geq 0.$$
- If $x < 0$, then $-x > 0$, so
$$|x| = -x > 0.$$
- Therefore, for all $x \in \mathbb{F}$, $|x| \geq 0$.

This finishes our proof. ■

Proof. For 2):

- If $x = 0$, then $x \geq 0$, so
$$|0| = 0.$$
- Conversely, suppose that $x \neq 0$.
- Then either
$$x < 0 \text{ or } x > 0.$$
- Therefore, if $x < 0$,
$$|x| = -x > 0$$
and if $x > 0$,
$$|x| = x > 0$$
- Hence,
$$|x| > 0.$$

- We showed that:

  if $x \neq 0$ then $|x| \neq 0$,

  so we showed the *contrapositive*, i.e.

  if $|x| = 0$ then $x = 0$.

This finishes our proof. ■

Proof. For 3):

- If $x, y \geq 0$, then
$$xy \geq 0,$$
  so
$$|xy| = xy = |x|\,|y|\,.$$

- If $x > 0$ and $y < 0$, then
$$xy < 0$$
  and
$$\begin{aligned} |xy| &= -xy \\ &= (-1)\,xy \\ &= x\,((-1)\,y) \\ &= x\,(-y) = |x|\,|y|\,. \end{aligned}$$

- Analogously, if $x < 0$ and $y > 0$.

- If $x = 0$ or $y = 0$, then
$$xy = 0$$
  so $|xy| = 0$.

- Therefore,
$$|x|\,|y| = 0,$$
  so
$$|xy| = |x|\,|y|\,.$$

- Finally, if $x < 0$ and $y < 0$, then
$$xy > 0,$$
  so
$$|xy| = xy = (-x)\,(-y) = |x|\,|y|$$

This finishes our proof. ■

Proof. For 4):

- Indeed, if $x \geq 0$, then
$$|x| = x, \text{ so } x \leq |x|\,.$$

- If $x < 0$, then
$$\begin{aligned} |x| &= -x > 0, \text{ so } x < 0 < -x = |x|\,, \text{ thus} \\ x &< |x|\,. \end{aligned}$$

- Therefore, for all x,
$$x \leq |x|.$$

This finishes our proof. ■

Proof. For 5):

- We first show that, for all $x, y \in \mathbb{F}$,
$$|x| \leq y \text{ iff } -y \leq x \leq y.$$
- We assume that $|x| \leq y$.
- Since $x \in \mathbb{F}$,
$$x < 0 \text{ or } x = 0 \text{ or } 0 \leq x.$$
- Assume that $x < 0$, then
$$-x = |x| \leq y,$$
so $-y \leq x$, thus
$$-y \leq x \leq |x| \leq y,$$
- Thus
$$-y \leq x \leq y.$$
- If $x \geq 0$, then
$$0 \leq x = |x| \leq y,$$
so $0 \leq y$.
- Hence $-y \leq 0$ and
$$-y \leq 0 \leq x = |x| \leq y,$$
so
$$-y \leq x \leq y.$$
- Assume that
$$-y \leq x \leq y.$$
- Since $x \in \mathbb{F}$
$$x < 0 \text{ or } x = 0 \text{ or } x > 0.$$
- If $x \geq 0$, then
$$|x| = x \leq y,$$
so
$$|x| \leq y.$$
- If $x < 0$, then
$$-x > 0.$$
- Since $-y \leq x$,
$$-x \leq y,$$
and
$$|x| = -x \leq y.$$

- This shows that
$$|x| \leq y \text{ iff } -y \leq x \leq y.$$

- We now observe that
$$\begin{array}{ccc} \underbrace{|x+y|}_{|z|} & \leq & \underbrace{|x|+|y|}_{a} \\ & & \text{iff} \\ -\left(\underbrace{|x|+|y|}_{a}\right) & \leq & \underbrace{x+y}_{z} \leq \left(\underbrace{|x|+|y|}_{a}\right). \end{array}$$

- Since, for all $x, y \in \mathbb{F}$,
$$\begin{array}{rcl} -|x| & \leq & x \leq |x| \text{ and} \\ -|y| & \leq & y \leq |y| \end{array}$$
since $-|x| \leq x$ then
$$-|x| - |y| \leq x - |y| \leq x + y \leq |x| + |y|$$
so
$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

This finishes our proof. ■

Proof. For 6):

- Since
$$\begin{array}{rcl} |x| & = & |x - y + y| \\ & \leq & |x - y| + |y|, \end{array}$$
it follows that
$$|x| - |y| \leq |x - y|$$

- Since
$$\begin{array}{rcl} |y| & = & |y - x + x| \\ & \leq & |x - y| + |x|, \end{array}$$
it follows that
$$\begin{array}{rcl} -(|x| - |y|) & \leq & |x - y|, \text{ so} \\ -|x - y| & \leq & |x| - |y| \end{array}$$

- Therefore,
$$-|x - y| \leq |x| - |y| \leq |x - y|$$

- Hence
$$||x| - |y|| \leq |x - y|.$$

This finishes our proof. ■

- **Field of Rational Numbers**

- The set of all *rational numbers* is defined as
$$\mathbb{Q} = \left\{ \frac{m}{n} : n, m \in \mathbb{Z},\ n \neq 0 \right\}.$$
- One checks that $\mathbb{Q}$ with $+$ and $\cdot$ defined in a familiar way satisfies all properties of an ordered field.
- One of the main property of $\mathbb{Q}$ is:
  Proposition $\mathbb{Q}$ is *dense in itself.* That is,
  for all $x, y \in \mathbb{Q}$, if $x < y$, then there is $z \in \mathbb{Q}$, such that
$$x < z < y.$$

Proof. Let $x, y \in \mathbb{Q}$, and assume that $x < y$.

- Define
$$z = \frac{1}{2}(x + y).$$
- Since $x, y \in \mathbb{Q}$,
$$x + y \in \mathbb{Q}$$
  and since $\frac{1}{2} \in \mathbb{Q}$,
$$z = \frac{1}{2}(x + y) \in \mathbb{Q}.$$
- We show that
$$x < z < y.$$
- We see that
$$\begin{array}{rcl} x & < & y, \text{ so } x + x < y + x \\ x & < & y, \text{ so } x + y < y + y \\ y + x & = & x + y, \\ 2x & = & x + x < x + y < y + y = 2y, \text{ so} \\ 2x & < & x + y < 2y. \end{array}$$
- Since $\frac{1}{2} > 0$,
$$\frac{2x}{2} < \frac{1}{2}(x + y) < \frac{2y}{2}$$
- Consequently,
$$x < z < y.$$

This finishes our proof. ■

- **Proposition** $\mathbb{Q}$ is countable.

Proof. Let $n \in \mathbb{N}$ and define
$$A_n = \left\{ \frac{m}{n} : m \in \mathbb{Z} \right\} \subset \mathbb{Q}.$$

- We see that
$$\mathbb{Q} = \bigcup_{n=1}^{\infty} A_n.$$

- Since each A_n is countable,
  by theorem $\mathbb{Q}$ is countable as a countable union of countable sets.

This finishes our proof. ■

- **Proposition** (*Archimedean Property*) If $x \in \mathbb{Q}$, then there is $n \in \mathbb{Z}$, such that
$$x < n.$$

Proof. Let $x \in \mathbb{Q}$.

- If $x \leq 0$, then take $n = 1 \in \mathbb{Z}$ and
$$x \leq 0 < 1 = n,$$
- Hence, there is $n \in \mathbb{Z}$, such that
$$x < n.$$
- Assume that $x > 0$.
- If $x \in \mathbb{Z}$, then
$$n = x + 1 \in \mathbb{Z}$$
  and
$$x = x + 0 < x + 1 = n.$$
- Therefore, assume that $x \notin \mathbb{Z}$.
- Since $x \in \mathbb{Q}$, $x = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $p \geq 1$ and $q > 1$.
- Let $n = 2p$.
- Since $1 < 2$,
$$p < 2p$$
  and since
$$1 < q,$$
  multiplying by $2p$ gives
$$2p < 2pq = nq.$$
- Therefore,
$$\begin{aligned} p &< 2p < nq, \text{ so} \\ p &< nq. \end{aligned}$$
- Since $q > 1 > 0$, consequently
$$\frac{p}{q} < \frac{nq}{q} = n.$$
- We showed that, for
$$x = \frac{p}{q},$$
  there is an integer n, such that
$$x < n.$$

This finishes our proof. ■

- **Proposition** Let $\mathbb{F}$ be an order field.
  The following conditions are equivalent:

1. If $x \in \mathbb{F}$, then there is $n \in \mathbb{Z}$, such that
$$x < n.$$
2. If $x, y \in \mathbb{F}$ and $0 < x < y$ then there is an integer n, such that
$$y < nx$$
3. If $x \in \mathbb{F}$ and $x > 0$, then there is an integer $n > 0$, such that
$$0 < \frac{1}{n} < x.$$

Proof. We **show that** $1) \to 2)$.

- Let
$$0 < x < y$$
- Since $x > 0$, in particular $x \neq 0$,
$$\frac{1}{x} \in \mathbb{F}.$$
- Consider $\frac{y}{x} \in \mathbb{F}$.
- By 1), there is $n \in \mathbb{Z}$, such that
$$\frac{y}{x} < n$$
- Since $x > 0$,
$$y < nx.$$
- We **show that** $2) \to 3)$.
  Assume that $x > 0$.
- If $0 < x < 1$, then by 2), there is $n \in \mathbb{N}$, such that
$$1 < nx,$$
  so
$$0 < \frac{1}{n} < x.$$
- If $x \geq 1$, take $n > 1$.
- Then $\frac{1}{n} < 1 \leq x$, so
$$0 < \frac{1}{n} < x.$$
- Finally, we **show that** $3) \to 1)$**.**
  Let $x \in \mathbb{F}$ and assume that $x \leq 0$.
- Since $0 < 1$ in $\mathbb{F}$,
$$x \leq 0 < 1,$$
  we take $n = 1$.

- Assume that $x > 0$.
- Thus
$$0 < \frac{1}{x},$$
so there is $n \in \mathbb{N}$, such that
$$0 < \frac{1}{n} < \frac{1}{x},$$
- Hence
$$x < n.$$

This finishes our proof. ■

- **Completeness**
Definition Let $\mathbb{F}$ be an ordered field and $S \subseteq \mathbb{F}$.
A number $M \in \mathbb{F}$ is called *an upper bound for* S if for all $x \in S$,
$$x \leq M.$$
A number $\beta \in \mathbb{F}$ is called *the least upper bound* (or *supremum*) for S if
i) β is an upper bound of S, and
ii) if β' is an upper bound for S, then
$$\beta \leq \beta'.$$
- The least upper bound for S (if exists) is denoted by $\sup S$, i.e.
$$\beta = \sup S.$$
- If S is not bounded above, then we say that $\sup S$ is infinite and we write
$$\sup S = +\infty.$$
- We also note that if $S = \emptyset$, then it makes sense to define
$$\sup S = -\infty.$$
- Analogously, we define a *lower bound of* S and *the greatest lower bound* denoted by $\inf S$ provided it exists.
- By conventions
$$\inf S = \begin{cases} -\infty & \text{if} \quad S \text{ is not bounded below} \\ +\infty & \text{if} \quad S = \emptyset. \end{cases}$$
Proposition Let $S \subseteq \mathbb{F}$, then
$$\beta = \sup S$$
iff
i) β is an upper bound of S, i.e. for all $x \in S$,
$$x \leq \beta,$$
and
ii) $\beta - \epsilon$ is not an upper bound of S, for any $\epsilon > 0$
(that is, no number smaller than β is an upper bound of S), i.e.
For all $\epsilon > 0$, there is $x \in S$, such that,
$$\beta - \epsilon < x.$$

Proof. We show that

$$\beta = \sup S$$

iff β satisfies both **i**) and **ii**).

- Assume that $\beta = \sup S$.
- Since for all $x \in S$,

  $$x \leq \beta,$$

  β is an upper bound for S, so **i**) holds.
- Let $\epsilon > 0$.
- Since β is the least upper bound,

  $$\beta - \epsilon < \beta$$

  is not an upper bound for S, so there is

  $$x \in S,$$

  such that

  $$\beta - \epsilon$$

  so **ii**) also holds.
- Assume that β satisfies both **i**) and **ii**).
- Since β satisfies **i**), for all $x \in S$,

  $$x \leq \beta.$$

- Therefore, β is an upper bound of S.
- We need to show that β is the least upper bound of S.
- Suppose that β' is an upper bound of S and assume that

  $$\beta' < \beta.$$

- Let $\epsilon = \left(\beta - \beta'\right) > 0$.
- By **ii**) there is $x \in S$, such that

  $$\beta - \epsilon < x.$$

- Therefore,

  $$\begin{aligned} \beta' &= \beta - \left(\beta - \beta'\right) \\ &= \beta - \epsilon \\ &< x \end{aligned}$$

  so β' is not an upper bound of S. Contradiction.
- It follows that

  $$\beta \leq \beta'.$$

- Therefore,

  $$\beta = \sup S.$$

This finishes our proof. ∎

- **The least upper bound property (LUB)**
  Every *nonempty and bounded above subset* $S \subseteq \mathbb{F}$ has the least upper bound, that is,
  There is $\beta \in \mathbb{F}$, such that
$$\beta = \sup S.$$

Definition An ordered field $\mathbb{F}$ is called *complete* it satisfies the least upper bound property.

Proposition Every complete ordered field $\mathbb{F}$ is Archimedean.

Proof. We prove this statement by reductio ad impossibile.

- Assume that, there is
$$\alpha \in \mathbb{F},$$
  such that for all $n \in \mathbb{N}$,
$$n \leq \alpha.$$
- Let $S = \mathbb{N}$.
- Since $1 \in \mathbb{N}$, $S \neq \emptyset$.
- Since, for all $n \in \mathbb{N}$,
$$n \leq \alpha,$$
  S is bounded.
- Since $\mathbb{F}$ is complete, S has the least upper bound and let
$$\beta = \sup S \in \mathbb{F}.$$
- Take $\epsilon = 1$.
- By Proposition, there is
$$n \in S,$$
  such that,
$$\beta - \epsilon = \beta - 1 < n.$$
- Since $\mathbb{F}$ is an ordered field,
$$\begin{aligned} \beta &= (\beta - 1) + 1 \\ &< n + 1 \end{aligned}$$
- Since $n \in \mathbb{N}$ and $\mathbb{N}$ satisfies **PMI**,
$$(n+1) \in \mathbb{N}.$$
- We see that, there is
$$(n+1) \in S,$$
  such that
$$\beta < n + 1$$
- Therefore, β is not an upper bound of S.
- Contradiction since we assumed that
$$\beta = \sup S.$$

- It follows that:

  For every $x \in \mathbb{F}$, there is $n \in \mathbb{N}$, such that $x \leq n$.

- Therefore, $\mathbb{F}$ has Archimedean property.

This finishes our proof. ∎

- **Theorem** There exists a unique (up to an isomorphism of ordered fields) a complete ordered field called the *field of real numbers* and we denote it by $\mathbb{R}$.

Proof. See any textbook with a construction of $\mathbb{R}$. ∎

- **Exercise** Let
$$S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \subset \mathbb{R}.$$
  Show that $\inf S = 0 = \epsilon$.

- We use the following result

  Let $S \subseteq F$ be nonempty and bounded below.

  Then $\alpha = \inf S$ iff

  i) $\forall x \in S$, $\alpha \leq x$, and

  ii) $\forall \epsilon > 0$, $\exists x \in S \ni x < \alpha + \epsilon$.

- Since $n > 0$, for all $n \in \mathbb{N}$
$$\frac{1}{n} > 0.$$

- It follows that
$$0 \leq x, \text{ for all } x \in S, \text{ so } i) \text{ is true.}$$

- Let $\epsilon > 0$ be given.

- Since $\mathbb{R}$ is complete, as we showed, $\mathbb{R}$ is Archimedean.

- Since $\epsilon \in \mathbb{R}$ and $\epsilon > 0$, by the Archimedean property of $\mathbb{R}$,

  there is $n \in \mathbb{N}$, such that
$$\underbrace{x = \frac{1}{n}}_{\in S} < \epsilon = 0 + \epsilon$$
  so $ii)$ holds.

- It follows from the proposition that
$$\begin{aligned} 0 &= \inf S \\ &= \inf \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}. \end{aligned}$$

  Exercise Let $a < b$ and
$$S = (a, b) = \{x \in \mathbb{R} : a < x < b\}.$$
  Show that
$$b = \sup S.$$

- Indeed, for every $x \in S$,
$$x \leq b,$$
so b is an upper bound of S.
- We show that b is the least upper bound for S, i.e.
- $b - \epsilon$ is not an upper bound of S, for any $\epsilon > 0$.
- Let $x = \max\left\{b - \frac{\epsilon}{2}, \frac{a+b}{2}\right\}$.
- Since $a < b$
$$\begin{aligned} a &= \frac{a+a}{2} \\ &< \frac{a+b}{2} \\ &< \frac{b+b}{2} \\ &= b, \end{aligned}$$
it follows that
$$\frac{a+b}{2} \in S.$$
- Therefore,
$$\begin{aligned} a &< \frac{a+b}{2} \\ &\leq \max\left\{b - \frac{\epsilon}{2}, \frac{a+b}{2}\right\} \\ &= x \end{aligned}$$
and since
$$b - \frac{\epsilon}{2} < b$$
and
$$\frac{a+b}{2} < b,$$
- We see that
$$\begin{aligned} x &= \max\left\{b - \frac{\epsilon}{2}, \frac{a+b}{2}\right\} \\ &< b. \end{aligned}$$
- It follows that
$$\begin{aligned} a &< x < b, \text{ so} \\ x &\in S. \end{aligned}$$
- Since
$$\begin{aligned} b - \epsilon &< b - \frac{\epsilon}{2} \\ &\leq \max\left\{b - \frac{\epsilon}{2}, \frac{a+b}{2}\right\} \\ &= x, \end{aligned}$$

- it follows that

$$b - \epsilon < x$$

and since $x \in S$,

$$b - \epsilon$$

is not an upper bound of S.

- It follows that,
- b is the least upper bound of S.
- Hence

$$b = \sup S.$$

Exercise Let $a < b$ and $S = (a, b) = \{x \in \mathbb{R} : a < x < b\}$.
Show that

$$a = \inf S.$$

Exercise Suppose that $A \subseteq B \subseteq \mathbb{R}$, $A \neq \emptyset$ and B is bounded.
Show that

$$\inf B \leq \inf A \leq \sup A \leq \sup B$$