Math 4301 Mathematical Analysis I

Lecture 14

Topic: Number Series

- **Number Series**

  Definition Let $(a_n)_{n\geq 1}$ be a sequence of real numbers and define sequence $(S_n)_{n\geq 1}$ by

$$S_n = \sum_{k=1}^{n} a_n = a_1 + a_2 + ... + a_n.$$

  then

$$a_1 + a_2 + ... = \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_n$$

  is called an *infinite number series.*

- We say that the series $\sum_{n=1}^{\infty} a_n$ is *convergent* to $S \in \mathbb{R}$ if $S_n \to S$ as $n \to \infty$ and we write

$$S = \sum_{n=1}^{\infty} a_n.$$

  and we call S the sum of the series $\sum_{n=1}^{\infty} a_n$.

- If $\sum_{n=1}^{\infty} a_n$ is *not convergent*, we say that the series is *divergent.* Moreover, if

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_n = \pm\infty$$

  then we say that the *series is diverges to* $\pm\infty$ and we write

$$\sum_{n=1}^{\infty} a_n = \begin{cases} +\infty & if \quad \lim_{n\to\infty} \sum_{k=1}^{n} a_n = +\infty \\ -\infty & if \quad \lim_{n\to\infty} \sum_{k=1}^{n} a_n = -\infty \end{cases}$$

  Definition Let $\{a_n\}$ be a sequence of real numbers. The series

$$\sum_{n=1}^{\infty} \left(\underbrace{a_n - a_{n+1}}_{b_n} \right)$$

  is called a *telescoping series.*

  Exercise Given a telescoping series

$$\sum_{n=1}^{\infty} (a_n - a_{n+1}),$$

  show that such a series converges iff

$$\lim_{n\to\infty} a_n = a.$$

  Example: We show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

- Since
$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$
it follows that
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} (a_n - a_{n+1})$$
is a telescoping series.

- Moreover,
$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - \lim_{n\to\infty} a_{n+1} \\ &= \frac{1}{1} - \lim_{n\to\infty} \frac{1}{n+1} = 1 - 0 = 1. \end{aligned}$$

Example The series $\sum_{n=1}^{\infty} 1$ diverges to ∞.

- Indeed, we see that
$$S_n = \sum_{k=1}^{n} 1 = n.$$
- Since for all $M \geq 0$, there is $n > M$.
- Thus, for every $M \geq 0$, there is $n \in \mathbb{N}$, such that $S_n > M$.
- It follows that
$$\lim_{n\to\infty} S_n = \infty$$
and therefore
$$\sum_{n=1}^{\infty} 1 = \infty.$$

Example: The series $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges.

- We see that
$$S_n = \sum_{k=1}^{n} (-1)^{n+1} = \begin{cases} 1 & if \quad n \text{ is odd} \\ 0 & if \quad n \text{ is even} \end{cases}$$
hence $S_n = \frac{1-(-1)^{n+1}}{2}$.
- We see that $S_{2n} = 0$, for all $n \in \mathbb{N}$ and $S_{2n-1} = 1$ for all $n \in \mathbb{N}$.
- Since $\{S_{2n}\}$ and $\{S_{2n-1}\}$ are subsequences of $\{S_n\}$ and
$$\lim_{n\to\infty} S_{2n} = 0 \neq 1 = \lim_{n\to\infty} S_{2n-1}$$
it follows that $\{S_n\}$ diverges.

- Therefore, $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges.

  Remark

- We defined the infinite sum

$$\sum_{n=1}^{\infty} a_n = a_1 + a_1 + ...$$

as $\lim_{n\to\infty} S_n$, where

$$\begin{aligned} S_1 &= a_1 \\ S_n &= S_{n-1} + a_n, \ n \geq 2 \end{aligned}$$

Remark: For a series $\sum_{n=1}^{\infty} a_n$ its sums can be defined in a different way, for example its *Cesàro sum* C

is defined by

$$C = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} S_k$$

Example: Find *Cesàro sum* for $\sum_{n=1}^{\infty} (-1)^{n+1}$.

- In particular, for

$$a_n = (-1)^{n+1},$$

since

$$S_1 = 1, \ S_2 = 0, \ S_3 = 1, \ ..., \ S_n = \frac{1 + (-1)^{n+1}}{2}, \ ...$$

it follows that

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} S_k &= \frac{1}{n} \sum_{k=1}^{n} \frac{1 + (-1)^{k+1}}{2} = \frac{1}{2n} \sum_{k=1}^{n} \left(1 + (-1)^{k+1}\right) \\ &= \frac{1}{2n} \left(n + \sum_{k=1}^{n} (-1)^{k+1}\right) \\ &= \frac{1}{2} + \frac{1}{2n} \sum_{k=1}^{n} (-1)^{k+1} = \begin{cases} \frac{1}{2} + \frac{1}{2n} & if \quad n \text{ is odd} \\ \frac{1}{2} & if \quad n \text{ is even} \end{cases} \end{aligned}$$

and

$$C = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} S_k = \frac{1}{2}.$$

- **Cesàro summation** is important in the theory of *Fourier series*.
- There are also other notions of summation for series.

  Remark: Algebraic rules for finite sums cannot be blindly apply to infinite sums, For instance

$$\begin{aligned} S &= \sum_{n=1}^{\infty} (-1)^n = -1+1-1+... = (-1+1)+(-1+1)+... \\ &= 0+0+...=0 \end{aligned}$$

but

$$\begin{aligned} S &= \sum_{n=1}^{\infty} (-1)^n = -1+1-1+... = -1+(1-1)+(1-1)+... \\ &= -1+0+0+...=-1. \end{aligned}$$

Example: Assume that $|q|<1$ then the series

$$\sum_{n=0}^{\infty} q^n = \sum_{n=1}^{\infty} q^{n-1}$$

converges to $\frac{1}{1-q}$.

- Notice that, by the definition, the series $\sum_{n=1}^{\infty} q^{n-1}$ if the sequence of its partial sums converges.
- We will start from determining the sequence of partial sums.
- We know that

  $$S_n = 1+q+q^2+...+q^n$$

  so

  $$qS_n = q+q^2+q^3+...+q^{n+1}$$

- hence

  $$\begin{aligned} S_n(q-1) &= qS_n - S_n = \left(q+q^2+q^3+...+q^{n+1}\right) - \left(1+q+q^2+...+q^n\right) \\ &= q^{n+1}-1, \end{aligned}$$

  so

  $$S_n(q-1) = q^{n+1}-1$$

- Since $|q|<1$, then $q \neq 1$, so $q-1 \neq 0$ and we can solve for S_n :

  $$S_n = \frac{q^{n+1}-1}{q-1} = \frac{1-q^{n+1}}{1-q}$$

- We see that

  $$S_n = \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$$

  so

  $$|S_n - S| = \left| \frac{1-q^{n+1}}{1-q} - \frac{1}{1-q} \right| = \frac{|q|^{n+1}}{|1-q|}$$

- Since $|q|^{n+1} \to 0$ as $n \to \infty$.
- Let $\epsilon > 0$ be given.

- Since $|q| < 1$, then $|q|^{n+1} \to 0$ as $n \to \infty$, there is $N \in \mathbb{N}$, such that for $n > N$,

$$|q|^{n+1} < \epsilon\,|1-q|\,,$$

- Therefore, for $n > N$,

$$|S_n - S| = \frac{|q|^{n+1}}{|1-q|} < \frac{\epsilon\,|1-q|}{|1-q|} = \epsilon.$$

- Therefore,

$$\lim_{n\to\infty} S_n = \frac{1}{1-q},$$

hence

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q} \text{ if } |q| < 1.$$

Proposition Let $a_n \geq 0$ for all $n \in \mathbb{N}$.

Series $\sum_{n=1}^{\infty} a_n$ converges iff the sequence $\{S_n\}$ of its partial sums is bounded.

Proof. Let $\{S_n\}$ be a sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$.

- If the series converges then $\{S_n\}$ is convergent, so $\{S_n\}$ is bounded.
- **Conversely**, assume that $\{S_n\}$ is bounded, since $a_n \geq 0$, then

$$0 \leq S_1 \leq S_2 \leq \ldots$$

- Hence $\{S_n\}$ is non-decreasing.
- Since $\{S_n\}$ is bounded, by theorem, $\{S_n\}$ converges, so by the definition, $\sum_{n=1}^{\infty} a_n$ converges.

This finishes our proof. ■

Theorem 0.1 (Cauchy Condition) *The series* $\sum_{n=1}^{\infty} a_n$ *converges* if and only if *for every* $\epsilon > 0$ *there is* $N \in \mathbb{N}$, *such that, for all* $n > m > N$

$$\left| \sum_{k=m+1}^{n} a_n \right| < \epsilon$$

Proof. Exercise. ■

- **Proposition** If $\sum_{n=1}^{\infty} a_n$ converges then

$$\lim_{n\to\infty} a_n = 0.$$

Proof. Exercise. ■

- **Remark**: Observe that we can use the above to show that $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges.

- Indeed, since $a_n = (-1)^n$ has no limit, $\sum_{n=1}^{\infty} (-1)^{n+1}$ cannot converge.

Example: Show that

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

diverges.

- We see that

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0,$$

so by the proposition, $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

Example: We show that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

- We show that the sequence $\{S_n\}$ is not bounded above.
- We see that

$$\begin{aligned}
S_{2^0} &= S_1 = 1 \\
S_{2^1} &= S_2 = 1 + \frac{1}{2} \\
S_{2^2} &= S_4 = \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) > \left(1 + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + 2 \cdot \frac{1}{2} \\
&\vdots \\
S_{2^n} &= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + ... + \left(\frac{1}{2^{n-1}} + \frac{1}{2^{n-1}+1} + ... + \frac{1}{2^n}\right) \\
&> \left(1 + \frac{1}{2}\right) + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{2} + ... + \underbrace{\left(\frac{1}{2^n} + ... + \frac{1}{2^n}\right)}_{2^{n-1}} = 1 + \underbrace{\frac{1}{2} + ... + \frac{1}{2}}_{n} \\
&= 1 + \frac{n}{2}
\end{aligned}$$

- It follows that $\{S_n\}$ is not bounded, so $\{S_n\}$ diverges to ∞.
- Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{n\to\infty} S_n = \infty.$$

Proposition The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

Proof. If $p = 1$ then, as we showed, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

- If $0 \leq p < 1$, then

$$1 \leq n^p \leq n,$$

so

$$\frac{1}{n} \leq \frac{1}{n^p}$$

- Hence, for all n
$$\sum_{k=1}^{n} \frac{1}{k} \leq \sum_{k=1}^{n} \frac{1}{k^p}.$$
- Since the sequence $\left\{ \sum_{k=1}^{n} \frac{1}{k} \right\}_n$ is not bounded above, $\left\{ \sum_{k=1}^{n} \frac{1}{k^p} \right\}$ is also not bounded above.
- Thus,
$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$
diverges, for all $0 \leq p \leq 1$.
- If $p < 0$, then
$$\lim_{n\to\infty} \frac{1}{n^p} = \lim_{n\to\infty} n^{-p} = \infty, \text{ since } -p > 0.$$
- Hence $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.
- Assume that $p > 1$ and let
$$S_n = \sum_{k=1}^{n} \frac{1}{k^p}$$
- It is clear that
$$0 \leq S_n < S_{n+1}, \text{ for all } n \in \mathbb{N}.$$
- Therefore, the sequence $\{S_n\}$ is increasing.
- It suffices to show that $\{S_n\}$ is bounded above.
- Indeed,
$$\begin{aligned}
S_{2^n-1} &= \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + ... \\
&\quad + \left(\frac{1}{(2^{n-1})^p} + \frac{1}{(2^{n-1}+1)^p} + ... + \frac{1}{(2^{n-1}-1)^p}\right) \\
&\leq \frac{1}{1^p} + \frac{2}{2^p} + \frac{4}{4^p} + ... + \frac{2^{n-1}}{(2^{n-1})^p} = \frac{1}{1^{p-1}} + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + ... + \frac{1}{(2^{n-1})^{p-1}} \\
&= 1 + \left(\frac{1}{2^{p-1}}\right)^1 + \left(\frac{1}{2^{p-1}}\right)^2 + ... + \left(\frac{1}{2^{p-1}}\right)^{n-1} \\
&= \frac{1 - \left(\frac{1}{2^{p-1}}\right)^n}{1 - \frac{1}{2^{p-1}}} < \frac{1}{1 - \frac{1}{2^{p-1}}} = \frac{2^{p-1}}{2^{p-1}-1}
\end{aligned}$$
- Therefore, for all $n \in \mathbb{N}$,
$$S_{2^n-1} \leq \frac{2^{p-1}}{2^{p-1}-1}.$$

- If $m \in \mathbb{N}$ is given, then there is $n \in \mathbb{N}$, such that $2^n - 1 \geq m$.
- Therefore, for all $m \in \mathbb{N}$.
$$S_m \leq S_{2^n-1} \leq \frac{2^{p-1}}{2^{p-1}-1}.$$
- Thus, $\{S_n\}$ is bounded.
- Since $\{S_n\}$ is increasing and bounded above, $\{S_n\}$ is converges.
- This shows that, if $p > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.
- In summary we showed that
$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$
- converges iff $p > 1$.

This finishes our proof. ■

- **Definition** A series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ converges and

  it is conditionally convergent if $\sum_{n=1}^{\infty} a_n$ but $\sum_{n=1}^{\infty} |a_n|$ is divergent.

  Example: Let us consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.
- We see that
$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}.$$
- Thus, the series is not absolutely convergent.
- One can show that
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$
  converges.
- Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

  Theorem Let $\{a_n\} \subset \mathbb{R}$ and assume that

i) $0 \leq a_{n+1} \leq a_n$, for all $n \in \mathbb{N}$

ii) $\lim_{n\to\infty} a_n = 0$.

  Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof. Let
$$S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k.$$

- Since $0 \leq a_{n+1} \leq a_n$,
$$\begin{aligned} S_1 &= a_1 \geq 0, \\ S_2 &= a_1 - a_2 \geq 0, \\ S_3 &= (a_1 - a_2) + a_3 \geq 0, \\ &... \\ S_n &= \sum_{k=1}^{n} (-1)^{k+1} a_k \geq 0. \end{aligned}$$
- Therefore, $\{S_n\}$ is bounded below by 0.
- Consider $\{S_{2n}\}$ and $\{S_{2n-1}\}$.
- For $\{S_{2n}\}$ since
$$S_{2n+2} = S_{2n} + (a_{2n+1} - a_{2n+2}) \geq S_{2n} \text{ as } a_{2n+1} - a_{2n+2} \geq 0,$$
it follows that
$$S_{2n} \leq S_{2n+2}$$
- Therefore, $\{S_{2n}\}$ is non-decreasing.
- Furthermore, since
$$\begin{aligned} S_2 &= a_1 - a_2 \leq a_1 \\ S_4 &= a_1 - a_2 + a_3 - a_4 \\ &= a_1 - (a_2 - a_3) - a_4 \leq a_1 \\ &... \\ S_{2n} &= \sum_{k=1}^{n} (-1)^{k+1} a_k \\ &= a_1 - (a_2 - a_3) - ... - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1. \end{aligned}$$
it follows that $\{S_{2n}\}$ is non-decreasing and bounded above by a_1.
- By the Monotone Sequence Property of $\mathbb{R}$,
$\{S_{2n}\}$ is convergent.
- Let
$$S_{2n} \to L_1 \text{ as } n \to \infty.$$
- Analogously, since $a_{2n} - a_{2n+1} \geq 0$,
$$\begin{aligned} S_{2n+1} &= a_1 - a_2 + a_3 - ... - a_{2n} + a_{2n+1} \\ &= S_{2n-1} - (a_{2n} - a_{2n+1}) \leq S_{2n-1}. \end{aligned}$$

- Thus,

$$S_{2n-1} \geq S_{2n+1}$$

  i.e. $\{S_{2n-1}\}$ is non-increasing.

- Since $\{S_n\}$ is bounded below by 0,
  $\{S_{2n-1}\}$ is also bounded below by 0.

- It follows that $\{S_{2n-1}\}$ is non-increasing and bounded below.

- By the Monotone Sequence Property,
  $\{S_{2n-1}\}$ converges in $\mathbb{R}$.

- Let

$$S_{2n-1} \to L_2 \text{ as } n \to \infty.$$

- Since for all $n \in \mathbb{N}$:

$$S_{2n} = S_{2n-1} - a_{2n}$$

  and sequences $\{a_n\}$, $\{S_{2n}\}$ and $\{S_{2n-1}\}$ converge,
  $\{a_{2n}\}$ converges (as a subsequence of a convergent sequence) and

$$\begin{aligned} L_1 &= \lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} (S_{2n-1} - a_{2n}) = \lim_{n\to\infty} S_{2n-1} - \lim_{n\to\infty} a_{2n} \\ &= L_2 - 0 = L_2. \end{aligned}$$

- Therefore,

$$L_1 = L_2 = L.$$

- We show that $\{S_n\}$ is convergent.

- Let $\epsilon > 0$ be given.

- Since $S_{2n} \to L$ and $S_{2n-1} \to L$,
  there are $N_1, N_2 \in \mathbb{N}$, such that,
  if $n > N = \max\{N_1, N_2\}$, then

$$|S_{2n} - L| < \epsilon \text{ and } |S_{2n-1} - L| < \epsilon.$$

- Therefore, if $n > N$,

$$|S_n - L| < \epsilon.$$

This finishes our argument. ■

- **Definition** Let $x \in \mathbb{R}$, denote by

$$x^+ = \max\{0, x\}$$

  and call it the positive part of x.
  Analogously,

$$x^- = \max\{0, -x\}$$

  is called the negative part of x.

- We observe that, for any $x \in \mathbb{R}$,

$$\begin{aligned} 0 &\leq x^+ \leq |x| \text{ and } 0 \leq x^- \leq |x|, \\ x &= x^+ - x^- \text{ and} \\ |x| &= x^+ + x^- \end{aligned}$$

Example: Let us consider the series $\sum_{n=1}^{\infty} a_n$, where $a_n = \frac{(-1)^{n+1}}{n}$.

- Notice that

$$\begin{aligned} a_n^+ &= \begin{cases} 0 & if \quad n \text{ is even} \\ \frac{1}{n} & if \quad n \text{ is odd} \end{cases} \quad \text{and} \\ a_n^- &= \begin{cases} 0 & if \quad n \text{ is odd} \\ \frac{1}{n} & if \quad n \text{ is even} \end{cases} \end{aligned}$$

- Then

$$\begin{aligned} \sum_{n=1}^{\infty} a_n^+ &= 1 + 0 + \frac{1}{3} + ... = \sum_{n=1}^{\infty} \frac{1}{2n-1} \text{ and} \\ \sum_{n=1}^{\infty} a_n^- &= 0 + \frac{1}{2} + 0 + \frac{1}{4} + ... = \sum_{n=1}^{\infty} \frac{1}{2n} \\ &= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + ...\right) = \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{1}{n}\right) \end{aligned}$$

so

$$\sum_{n=1}^{\infty} a_n^- \text{ diverges.}$$

- Since $\sum_{n=1}^{\infty} a_n^+ \geq \sum_{n=1}^{\infty} a_n^-$, $\sum_{n=1}^{\infty} a_n^+$ also diverges.

Theorem If $\sum_{n=1}^{\infty} a_n$ absolutely converges then it converges.

Moreover, $\sum_{n=1}^{\infty} a_n$ converges absolutely iff

$\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ converge and

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^- \text{ and} \\ \sum_{n=1}^{\infty} |a_n| &= \sum_{n=1}^{\infty} a_n^+ + \sum_{n=1}^{\infty} a_n^-. \end{aligned}$$

Proof. Let $\{\overline{S_n}\}$ denotes the sequence of partial sums of $\sum_{n=1}^{\infty} |a_n|$.

- **Assume that** $\sum_{n=1}^{\infty} a_n$ **absolutely converges.**
- Since $\sum_{n=1}^{\infty} a_n$ absolutely converges,

  $\sum_{n=1}^{\infty} |a_n|$ converges.
- Therefore, for $\epsilon > 0$,
  there is $N \in \mathbb{N}$, such that, for all $n > m > N$

$$\left|\overline{S_n} - \overline{S_m}\right| = \sum_{k=m+1}^{n} |a_n| < \epsilon$$

- Since

$$|S_n - S_m| = \left|\sum_{k=m+1}^{n} a_n\right| \leq \sum_{k=m+1}^{n} |a_n| < \epsilon$$

- It follows that the sequence of partial sums $\{S_n\}$ of $\sum_{n=1}^{\infty} a_n$ satisfies Cauchy condition.
- Therefore, $\{S_n\}$ converges, so $\sum_{n=1}^{\infty} a_n$ converges.
- Let $\{S_n^+\}$ and $\{S_n^-\}$ be partial sums of $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ respectively.
- Since

$$\begin{aligned} 0 &\leq \left|\overline{S_n} - \overline{S_m}\right| = \sum_{k=m+1}^{n} |a_n| = \sum_{k=m+1}^{n} \left(a_n^+ + a_n^-\right) = \sum_{k=m+1}^{n} a_n^+ + \sum_{k=m+1}^{n} a_n^- \\ &= \left|S_n^+ - S_m^+\right| + \left|S_n^- - S_m^-\right| \end{aligned}$$

  and

$$\begin{aligned} 0 &\leq \left|S_n^+ - S_m^+\right| = \sum_{k=m+1}^{n} a_n^+ \leq \sum_{k=m+1}^{n} |a_n| = \left|\overline{S_n} - \overline{S_m}\right| \\ 0 &\leq \left|S_n^- - S_m^-\right| = \sum_{k=m+1}^{n} a_n^- \leq \sum_{k=m+1}^{n} |a_n| = \left|\overline{S_n} - \overline{S_m}\right| \end{aligned}$$

- It follows that $\{\overline{S_n}\}$ **is Cauchy iff both sequences** $\{S_n^+\}$ **and** $\{S_n^-\}$ **are Cauchy sequences.**
- Therefore, $\sum_{n=1}^{\infty} a_n$ converges absolutely iff both $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ converge.

- Finally, if $\sum_{n=1}^{\infty} a_n$ converges absolutely,

  then the sequences $\{S_n\}$, $\{\overline{S_n}\}$, $\{S_n^+\}$ and $\{S_n^-\}$ converge and

$$\begin{aligned}\sum_{n=1}^{\infty} |a_n| &= \lim_{n\to\infty} \overline{S_n} = \lim_{n\to\infty} \left(S_n^+ + S_n^-\right) \\ &= \lim_{n\to\infty} S_n^+ + \lim_{n\to\infty} S_n^- = \sum_{n=1}^{\infty} a_n^+ + \sum_{n=1}^{\infty} a_n^-.\end{aligned}$$

- Analogously,

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(S_n^+ - S_n^-\right) \\ &= \lim_{n\to\infty} S_n^+ - \lim_{n\to\infty} S_n^- = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-.\end{aligned}$$

This finishes our proof. ■

- **Theorem** (*Comparison Test*) Assume that $b_n \geq 0$ and $|a_n| \leq b_n$, for all $n \in \mathbb{N}$.

  If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

  If the series $\sum_{n=1}^{\infty} |a_n|$ diverges then the series $\sum_{n=1}^{\infty} b_n$ diverges.

Proof. Exercise ■

- **Example**: Determine if the series

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

  converges?

- Since $\frac{1}{n} \in [0, \pi]$,

$$0 \leq \sin\left(\frac{1}{n}\right),$$

  for all $n = 1, 2, ...$

- We will show that

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

  diverges.

- Since

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

  and $\frac{1}{n} \to 0$,

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$$

- Therefore, for $\epsilon = \frac{1}{2}$,
  there is $N \in \mathbb{N}$, such that, for $n > N$,

$$\begin{aligned}
\left| \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} - 1 \right| &< \frac{1}{2}, \text{ so} \\
-\frac{1}{2} &< \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} - 1 < \frac{1}{2}, \text{ hence} \\
\frac{1}{2} &= 1 - \frac{1}{2} < \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} < 1 + \frac{1}{2} = \frac{3}{2}.
\end{aligned}$$

- It follows that

$$\frac{1}{2} < \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}, \text{ for } n > N.$$

- In particular,

$$\frac{1}{2n} < \sin\left(\frac{1}{n}\right), \text{ for } n > N.$$

- Since $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges, by comparison test,
  $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ diverges.

  Theorem (*Ratio Test*) Let $\{a_n\} \subset \mathbb{R}$ and $a_n \neq 0$, for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

  where $L \in [0, \infty)$ or $L = \infty$.

  If $0 \leq L < 1$ then $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

  If $L > 1$ or $L = \infty$ then $\sum_{n=1}^{\infty} a_n$ diverges.

  If $L = 1$, then the ratio test is inconclusive.

Proof. Suppose that $0 \leq L < 1$, then

- there is K, such that

$$L < K < 1.$$

- Let $\epsilon = (K - L) > 0$.
- Since

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L,$$

  there is $N \in \mathbb{N}$, such that, for $n > N$

$$\begin{aligned}
\left| \frac{a_{n+1}}{a_n} \right| - L &< (K - L), \text{ so} \\
\left| \frac{a_{n+1}}{a_n} \right| &< K, \text{ for } n > N.
\end{aligned}$$

- It follows that
$$|a_{n+1}| < K\,|a_n|\,,\ \text{for } n > N.$$
- In particular,
$$|a_{N+2}| < K\,|a_{N+1}|$$
- Therefore,
$$|a_{N+3}| < K\,|a_{N+2}| < K^2\,|a_{N+1}|$$
and one shows that
$$|a_{N+m}| < K^{m-1}\,|a_{N+1}|\,,\ \text{for } m \geq 2.$$
- It follows that,
if $n > N+1$, then $n - N \geq 2$ and
$$|a_n| < K^{(n-N)-1}\,|a_{N+1}|\,.$$
- Therefore,
$$\begin{aligned}
\sum_{n=1}^{\infty}|a_n| &= \sum_{n=1}^{N}|a_n| + \sum_{n=N+1}^{\infty}|a_n| \\
&\leq \sum_{n=1}^{N}|a_n| + |a_{N+1}| + \sum_{n=N+2}^{\infty}|a_n| \\
&= \sum_{n=1}^{N}|a_n| + |a_{N+1}| + \sum_{n=N+2}^{\infty}K^{n-N-1}\,|a_{N+1}| \\
&= \sum_{n=1}^{N}|a_n| + |a_{N+1}|\left(1 + \sum_{n=N+2}^{\infty}K^{n-N-1}\right) \\
&= \sum_{n=1}^{N}|a_n| + |a_{N+1}|\left(1 + \sum_{n=1}^{\infty}K^{n}\right) \\
&= \sum_{n=1}^{N}|a_n| + |a_{N+1}|\sum_{n=0}^{\infty}K^{n} \\
&\leq \underbrace{\sum_{n=1}^{N}|a_n|} + \underbrace{\frac{|a_{N+1}|}{1-K}} < \infty
\end{aligned}$$
- It follows that, $\sum_{n=1}^{\infty}|a_n|$ converges, so
$\sum_{n=1}^{\infty}a_n$ also converges.
- If $L > 1$, then there is K, such that $L > K > 1$.
- Let $\epsilon = (L-K) > 0$.

- Since
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L,$$
there is $N \in \mathbb{N}$, such that, for $n > N$
$$\begin{aligned} -(L-K) &< \left|\frac{a_{n+1}}{a_n}\right| - L, \text{ so} \\ K &< \left|\frac{a_{n+1}}{a_n}\right|, \text{ for } n > N. \end{aligned}$$

- Therefore,
$$K|a_n| < |a_{n+1}|,$$
for $n > N$.

- In particular,
$$\begin{aligned} K|a_{N+1}| &< |a_{N+2}| \\ K^2|a_{N+1}| &< K|a_{N+2}| < |a_{N+3}|, \end{aligned}$$
so by induction
$$K^{m-1}|a_{N+1}| < |a_{N+m}|, \text{ for } m \geq 2.$$

- Consequently,
$$K^{n-N-1}|a_{N+1}| < |a_n|, \text{ for } n > N+1$$

- Since $K^{n-N-1}|a_{N+1}| \to \infty$ as $n \to \infty$.

- It follows that $\lim_{n\to\infty}|a_n| = \infty$.

- In particular, $\lim_{n\to\infty} a_n$ cannot be 0.

- Hence $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

This finishes our proof. ■

- **Example**: Consider series
$$\sum_{n=1}^{\infty}\frac{1}{n}.$$

- As we know this series diverges and
$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+1} = 1.$$

- However, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, and also
$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1$$

- Thus in both cases

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1.$$

Theorem (*Root Test*) Let $\{a_n\}$ be sequence of real numbers and

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = L$$

where $L \in [0, \infty)$ or $L = \infty$.

If $0 \leq L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

If $L > 1$ or $L = \infty$ then $\sum_{n=1}^{\infty} a_n$ diverges.

If $L = 1$ then the root test is inconclusive.

Proof. Exercise ■