HOMEWORK 5 SOLUTIONS – MATH 4341

Problem 1. Let X be a topological space and $Y \subset X$. Show that $\mathrm{Int}Y$ is equal to the set of all points $x \in X$ such that some neighborhood of x is a subset of Y.

Proof. Let $x \in \mathrm{Int}Y$. Since

$$\mathrm{Int}Y = \bigcup_{\substack{U \subset Y \\ U \text{ open in } X}} U,$$

there exists $U \subset Y$ such that U is open in X and $x \in U$. Then U is a neighborhood of x which is also a subset of Y.

Let $x \in X$ such that some neighborhood U of x is a subset of Y. Since $U \subset Y$ is open in X, we have $x \in U \subset \mathrm{Int}Y$. □

Problem 2. Let X be a topological space and $Y \subset X$. Show that:

(1) $\mathrm{Int}Y = X \setminus \overline{(X \setminus Y)}$.
(2) $\overline{Y} = X \setminus \mathrm{Int}(X \setminus Y)$.

Proof. (1) is equivalent to $\overline{X \setminus Y} = X \setminus \mathrm{Int}Y$. By definition we have

$$\overline{X \setminus Y} = \bigcap_{\substack{X \setminus Y \subset V \\ V \text{ closed in } X}} V$$

Let $U = X \setminus V$. Then $X \setminus Y \subset V$ is equivalent to $X \setminus Y \subset X \setminus U$, which means that $U \subset Y$. Moreover, V is closed in X is equivalent to U is open in X. Hence

$$\begin{aligned} \overline{X \setminus Y} &= \bigcap_{\substack{U \subset Y \\ U \text{ open in } X}} (X \setminus U) \\ &= X \setminus \bigcup_{\substack{U \subset Y \\ U \text{ open in } X}} U \\ &= X \setminus \mathrm{Int}Y. \end{aligned}$$

(2) is obtained from (1) by replacing Y by $X \setminus Y$. □

Problem 3. Let X be a topological space and $Y, Z \subset X$. Show that:

(1) $\overline{Y \cup Z} = \overline{Y} \cup \overline{Z}$.
(2) $\overline{Y \cap Z} \subset \overline{Y} \cap \overline{Z}$. Find an example where $\overline{Y \cap Z} \neq \overline{Y} \cap \overline{Z}$.
(3) $\mathrm{Int}Y \cup \mathrm{Int}Z \subset \mathrm{Int}(Y \cup Z)$. Find an example where $\mathrm{Int}Y \cup \mathrm{Int}Z \neq \mathrm{Int}(Y \cup Z)$.
(4) $\mathrm{Int}Y \cap \mathrm{Int}Z = \mathrm{Int}(Y \cap Z)$.

Proof. (1) Since $\overline{Y} \cup \overline{Z}$ is a closed subset containing $Y \cup Z$, it contains $\overline{Y \cup Z}$. Hence $\overline{Y \cup Z} \subset \overline{Y} \cup \overline{Z}$.

Since $Y \subset \overline{Y \cup Z}$ and the latter set is closed, we have $\overline{Y} \subset \overline{Y \cup Z}$. For the same reason $\overline{Z} \subset \overline{Y \cup Z}$. Hence $\overline{Y} \cup \overline{Z} \subset \overline{Y \cup Z}$.

(2) Since $Y \cap Z \subset Y \subset \overline{Y}$ and the latter set is closed, we have $\overline{Y \cap Z} \subset \overline{Y}$. For the same reason $\overline{Y \cap Z} \subset \overline{Z}$. Hence $\overline{Y \cap Z} \subset \overline{Y} \cap \overline{Z}$.

Take $Y = (-\infty, 0)$ and $Z = (0, \infty)$ in $X = \mathbb{R}$. Then $\overline{Y \cap Z} = \overline{\emptyset} = \emptyset$ and $\overline{Y} \cap \overline{Z} = (-\infty, 0] \cap [0, \infty) = \{0\}$. Hence $\overline{Y \cap Z} \neq \overline{Y} \cap \overline{Z}$.

(3) Since $\mathrm{Int} Y \cup \mathrm{Int} Z \subset Y \cup Z$ and $\mathrm{Int} Y \cup \mathrm{Int} Z$ is open in X, we have $\mathrm{Int} Y \cup \mathrm{Int} Z \subset \mathrm{Int}(Y \cup Z)$.

Take $Y = (-\infty, 0]$ and $Z = [0, \infty)$ in $X = \mathbb{R}$. Then $\mathrm{Int} Y \cup \mathrm{Int} Z = (-\infty, 0) \cup (0, \infty) = \mathbb{R} \setminus \{0\}$, and $\mathrm{Int}(Y \cup Z) = \mathrm{Int}\mathbb{R} = \mathbb{R}$. Hence $\mathrm{Int} Y \cup \mathrm{Int} Z \neq \mathrm{Int}(Y \cup Z)$.

(4) Since $\mathrm{Int} Y \cap \mathrm{Int} Z \subset Y \cap Z$ and $\mathrm{Int} Y \cap \mathrm{Int} Z$ is open in X, we have $\mathrm{Int} Y \cap \mathrm{Int} Z \subset \mathrm{Int}(Y \cap Z)$.

Since $\mathrm{Int}(Y \cap Z) \subset Y \cap Z \subset Y$ and $\mathrm{Int}(Y \cap Z)$ is open in X, we have $\mathrm{Int}(Y \cap Z) \subset \mathrm{Int} Y$. Similarly, $\mathrm{Int}(Y \cap Z) \subset \mathrm{Int} Z$. Hence $\mathrm{Int}(Y \cap Z) \subset \mathrm{Int} Y \cap \mathrm{Int} Z$. □

Problem 4. Let $\mathbb{R}_\ell$ be the set of all real numbers $\mathbb{R}$ with the lower limit topology, and $\mathcal{I}$ be the set of all irrational real numbers. Prove that

(1) $\partial \mathcal{I} = \mathbb{R}_\ell$.

(2) $\mathcal{I}$ is dense in $\mathbb{R}_\ell$.

Proof. (1) Let x be any real number. For any neighborhood $U \subset \mathbb{R}_\ell$ of x, there exist real numbers $a < b$ such that $x \in [a, b) \subset U$. The interval $[a, b)$ contains infinitely many irrational numbers and infinitely many rational numbers, so it intersects both $\mathcal{I}$ and $\mathbb{Q} = \mathbb{R}_\ell \setminus \mathcal{I}$. Hence U intersects both $\mathcal{I}$ and $\mathbb{Q} = \mathbb{R}_\ell \setminus \mathcal{I}$. This implies that $x \in \partial \mathcal{I}$. Hence $\partial \mathcal{I} = \mathbb{R}_\ell$.

(2) Since $\mathbb{R}_\ell = \partial \mathcal{I} \subset \overline{\mathcal{I}}$, we must have $\overline{\mathcal{I}} = \mathbb{R}_\ell$. Hence $\mathcal{I}$ is dense in $\mathbb{R}_\ell$. □