6. Compactness

Math 4341 (Topology)

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- **Example**. The real line $\mathbb{R}$ is not compact since the open cover $\mathcal{U}$ consisting of open sets $U_n = (-n, n)$, $n \in \mathbb{N}$, does not have a finite subcover.
- **Example**. The half-open interval $(0, 1] \subset \mathbb{R}$ is not compact since the open cover $\mathcal{U}$ consisting of open sets $U_n = (\frac{1}{n}, 1]$, $n \in \mathbb{N}$, does not have a finite subcover.

- **Example**. The subspace $A = \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is not compact. Note that $U_n = \{1/n\}$ is an open set in the subspace topology, so $\mathcal{U} = \{U_n \mid n \in \mathbb{N}\}$ is an open cover of A. Clearly, we can not find a finite subcover, since any finite subcover would cover only finitely many points of A.

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  - Since $\mathcal{U}$ is an open cover, we can also find open sets $U_1, \dots, U_N \in \mathcal{U}$ so that $1/k \in U_k$ for all $k = 1, \dots, N$.

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  - Since $\mathcal{U}$ is an open cover, we can also find open sets $U_1, \ldots, U_N \in \mathcal{U}$ so that $1/k \in U_k$ for all $k = 1, \ldots, N$.
  - We now see that the collection $U, U_1, \ldots, U_N$ together form a finite subcover of X.

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  - By definition, we can find for every $i \in I$ open subsets V_i of X so that $U_i = A \cap V_i$. Since the U_i cover A, it follows that the family $\mathcal{V} = \{V_i\}_{i \in I} \cup \{A^c\}$ is an open cover of X. Note that A^c is open since A is closed.

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  - Since X is compact, there is a finite subcover $V_{i_1}, \ldots, V_{i_n} \in \mathcal{V}$ of X. Going back, we see that $V_{i_1} \cap A, \ldots, V_{i_n} \cap A \in \mathcal{U}$ form a finite subcover of A, which is what we wanted to prove.

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  - Take any $x_0 \in A^c$. For every point $y \in A$, we can find disjoint nbhds U_y and V_y of x_0 and y respectively, since X is Hausdorff.
  - Now the collection $\{A \cap V_y\}_{y \in A}$ is an open cover of A and since A is compact, we can choose finitely many $y_1, \ldots, y_n$ so that $\{A \cap V_{y_i}\}_{i=1,\ldots,n}$ is a finite subcover. In particular, $A \subset V_{y_1} \cup \cdots \cup V_{y_n}$.

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    - Let $U^{x_0} = U_{y_1} \cap \cdots \cap U_{y_n}$. Then U^{x_0} is open and $U^{x_0} \subset A^c$: if $z \in U^{x_0}$, then $z \in V_{y_i}^c$, so $z \in (V_{y_1} \cup \cdots \cup V_{y_n})^c \subset A^c$.

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- Assume now that Y is Hausdorff and f is bijective. We have to show that f^{-1} is continuous.
    - Let $U \subset X$ be open. Since U^c is a closed subspace of a compact space X, it is compact by Theorem 6.1.
    - By the first part of the theorem, $f(U)^c = f(U^c)$ is also compact. By Theorem 6.2, this implies that $f(U)^c$ is closed. Hence $f(U)$ is open.

- **The Tube Lemma**. Let X and Y be topological spaces where Y is compact. If N is an open set of $X \times Y$ which contains $\{x_0\} \times Y$ for some $x_0 \in X$, then N contains a "tube" $W \times Y$, where $W \subset X$ is a neighbourhood of x_0.

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- Since $\{U_y \times V_y\}_{y \in Y}$ is an open cover of the compact set $\{x_0\} \times Y$, we can find $y_1, \ldots, y_n \in Y$ such that $\{U_{y_1} \times V_{y_1}, \ldots, U_{y_n} \times V_{y_n}\}$ covers $\{x_0\} \times Y$.

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- Let $W = \bigcap_{i=1}^n U_{y_i}$. Then W is open, and is a non-empty neighbourhood of x_0. Moreover

$$W \times Y = W \times (V_{y_1} \cup \cdots \cup V_{y_n}) \subset N.$$

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- $(\Leftarrow)$ It suffices to show that a product $X \times Y$ of two compact spaces is compact. Let $\mathcal{U}$ be an open cover of $X \times Y$. For every $x \in X$ the space $\{x\} \times Y$ is compact, so we can find a finite collection $\{U_1^x, \cdots, U_n^x\} \subset \mathcal{U}$ that covers $\{x\} \times Y$.

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- We claim that $\{U_i^{x_j} \mid i = 1, \ldots, n,\ j = 1, \ldots, m\} \subset \mathcal{U}$ covers $X \times Y$. To see this, let $(x, y) \in X \times Y$. Then there exists $j \in \{1, \ldots, m\}$ such that $x \in W_{x_j}$. Since $(x, y) \in N_{x_j}$, there exists $i \in \{1, \ldots, n\}$ so that $(x, y) \in U_i^{x_j}$.

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- ($\Rightarrow$) Suppose X is compact. Let $\mathcal{C}$ be any collection of closed subsets of X with the FIP. We want to show $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.
- Assume $\bigcap_{C \in \mathcal{C}} C = \emptyset$. Then $\bigcup_{C \in \mathcal{C}} (X \setminus C) = X$, so $\{X \setminus C\}_{C \in \mathcal{C}}$ is an open cover of X. Since X is compact, there exist $C_1, \cdots, C_n \in \mathcal{C}$ such that $\bigcup_{i=1}^{n} (X \setminus C_i) = X$. This implies that $\bigcap_{i=1}^{n} C_i = \emptyset$, which contradicts the FIP of $\mathcal{C}$.

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- **Theorem 6.6**. If X is first countable, then compactness of X implies sequential compactness.
- *Proof*. Assume that X is first countable and compact. Let $\{x_n\}$ be any sequence. We claim that $\exists x_0 \in X$ s.t. for any nbhd U of x_0 there are infinitely many n so that $x_n \in U$.

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- *Proof*. Assume that X is first countable and compact. Let $\{x_n\}$ be any sequence. We claim that $\exists x_0 \in X$ s.t. for any nbhd U of x_0 there are infinitely many n so that $x_n \in U$.
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- Let $\{B_i\}_{i=1}^{n}$ be a countable basis at x and let $U_k = \bigcap_{i=1}^{k} B_i$. Then $x_n \in U_k$ for infinitely many n, so in particular we can choose $x_{n_k} \in U_k$ for some increasing sequence n_k.

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- **Fact**. If X is a metric space with the metric topology, then compactness and sequential compactness of X are equivalent.

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- ($\Leftarrow$) If A is closed and bounded, $\exists K > 0$ s.t. $A \subset [-K, K]^n$. $[-K, K]$ is compact by Theorem 6.7, so $[-K, K]^n$ is compact by Theorem 6.4. Thus A is compact by Theorem 6.1.

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- **Corollary 6.9**. If X is compact and $f : X \to \mathbb{R}$ is continuous, there are x_1 and x_2 with $f(x_1) = \sup f(X)$, $f(x_2) = \inf f(X)$.

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- **Theorem 6.11**. (Bolzano–Weierstrass) A set $A \subset \mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded.