4. Closeness

Math 4341 (Topology)

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- Y is open iff $Y = \operatorname{Int} Y$ and Y is closed iff $Y = \overline{Y}$. Moreover, $\operatorname{Int} Y$ is the largest open subset contained in Y, and $\overline{Y}$ is the smallest closed subset containing Y.
- Note that $\operatorname{Int}(X \setminus Y) = X \setminus \overline{Y}$ and $\overline{X \setminus Y} = X \setminus \operatorname{Int}(Y)$.

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  - Since $\overline{Y} \cup \overline{Z}$ is a closed subset containing $Y \cup Z$, we have $\overline{Y \cup Z} \subset \overline{Y} \cup \overline{Z}$.
  - Since $Y \subset \overline{Y \cup Z}$ and the latter set is closed, we have $\overline{Y} \subset \overline{Y \cup Z}$. For the same reason $\overline{Z} \subset \overline{Y \cup Z}$. Hence $\overline{Y} \cup \overline{Z} \subset \overline{Y \cup Z}$.

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  - The *boundary* of Y, denoted ∂Y, is the set

    $$\partial Y = \{x \in X \mid U \cap Y \neq \emptyset \text{ and } U \cap Y^c \neq \emptyset \text{ for all nbhds } U \text{ of } x\}.$$

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- **Example.** Let $Y = [0,1) \cup \{2\} \subset \mathbb{R}$. Then $\text{Int}\, Y = (0,1)$, $\bar{Y} = [0,1] \cup \{2\}$, $\partial Y = \{0,1,2\}$, and $Y' = [0,1]$.

- **Theorem 4.2**. Let $Y \subset X$. Then
  - (i) $\partial Y = X \setminus (\operatorname{Int} Y \cup \operatorname{Int}(X \setminus Y)) = \overline{Y} \cap \overline{X \setminus Y}$,
  - (ii) $\overline{Y} = Y \cup \partial Y$, and
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- *Proof*. (i) is equivalent to $X \setminus \partial Y = \operatorname{Int} Y \cup \operatorname{Int}(X \setminus Y)$.
  - Let $x \in X \setminus \partial Y$. Then there is a nbhd U of x so that $U \subset Y$ or $U \subset X \setminus Y$. Hence $x \in \operatorname{Int} Y$ or $x \in \operatorname{Int}(X \setminus Y)$.
  - Suppose $x \in \operatorname{Int} Y$. Then there is an open set U such that $x \in U \subset Y$. Since $U \cap Y^c = \emptyset$, we have $x \notin \partial Y$. Similarly, if $x \in \operatorname{Int}(X \setminus Y)$, there is an open nbhd U of x with $U \cap Y = \emptyset$, so once again $x \notin \partial Y$. This shows (i).

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- (ii) follows from (i):

$$Y \cup \partial Y = Y \cup (\overline{Y} \cap \overline{X \setminus Y}) = \overline{Y} \cap X = \overline{Y}.$$

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  - Let $x \in \partial Y \setminus Y$. Then any nbhd U of x intersects Y, and since $x \notin Y$, this necessarily means that U intersects Y in a point which is not x itself. Hence $x \in Y' \setminus Y$.
  - Let $x \in Y' \setminus Y$. Then any nbhd U of x will intersect Y; it will also intersect $X \setminus Y$, since x belongs to that set. Hence $x \in \partial Y \setminus Y$. This completes the proof.

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- Finally, we use (ii) and show that $Y \cup \partial Y = Y \cup Y'$. To see this, it suffices to show that $\partial Y \setminus Y = Y' \setminus Y$.
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  - Let $x \in Y' \setminus Y$. Then any nbhd U of x will intersect Y; it will also intersect $X \setminus Y$, since x belongs to that set. Hence $x \in \partial Y \setminus Y$. This completes the proof.
- **Remark.** The above theorem provides us with the following useful characterization of the closure: we see that $x \in \overline{Y}$ if and only if every nbhd of x intersects Y.

- **Example**. We claim that $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e. $\overline{\mathbb{Q}} = \mathbb{R}$. To see this, it suffices to show that $\partial\mathbb{Q} = \mathbb{R}$.

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- **Example**. We claim that $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e. $\overline{\mathbb{Q}} = \mathbb{R}$. To see this, it suffices to show that $\partial\mathbb{Q} = \mathbb{R}$.
  - Let $x \in \mathbb{R}$ and U a nbhd of x. Since U contains some open interval and any interval contains both rational and irrational numbers, we have $U \cap \mathbb{Q} \neq \emptyset$ and $U \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$. This is exactly the condition that $x \in \partial\mathbb{Q}$.

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  - T_2 (or *Hausdorff*) if for every pair $x \neq y$ in X, there exists nbhds U_x and U_y of x and y respectively s.t. $U_x \cap U_y = \emptyset$.

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- **Proposition 4.4.** X is T_1 iff $\{x\}$ is closed for all $x \in X$.
- *Proof.* $(\Leftarrow)$ Let $x \neq y$ in X. Then $X \setminus \{x\}$ is a nbhd of y not containing x, and $X \setminus \{y\}$ is a nbhd of x not containing y.
  $(\Rightarrow)$ Every $y \in X$ has a nbhd U_y not containing x. Since $X \setminus \{x\} = \bigcup_{y \neq x} U_y$ is open, $\{x\}$ is closed.

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- **Example.** All metric spaces (with the metric topology) are Hausdorff.

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- **Example.** The constant sequence is convergent, regardless of the topology on the space.

- **Proposition 4.6.** Let (X, d) be a metric space with the metric topology. Then a sequence $\{x_n\}$ in X converges to $x \in X$ if and only if

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  - Let U be a nbhd of x. Then there exists an $\epsilon > 0$ s.t. $B_d(x, \epsilon) \subset U$. Now by assumption there is an $N > 0$ s.t. $x_n \in B_d(x, \epsilon) \subset U$ for all $n > N$.

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- *Proof.* Assume $x \neq y$. Choose U and V disjoint nbhds of x and y respectively. By definition of convergence, we get $N_U, N_V > 0$ such that $x_n \in U$ for all $n > N_U$ and $x_n \in V$ for all $n > N_V$. For $n > \max(N_U, N_V)$ we therefore have $x_n \in U \cap V = \emptyset$, which is a contradiction.

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- **Definition.** Let X be a topological space.

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- **Example.** All metric spaces are first-countable.

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  - Choose $x_n \in U_n \cap A$ for every n. We claim that $x_n \to x$. To see this, let U be any nbhd of x. Since X is first-countable, there is an $N \in \mathbb{N}$ s.t. $B_N \subset U$. For all $n > N$, we have $x_n \in U_n \subset U_N \subset B_N \subset U$ which means that $x_n \to x$.

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      $$B = (-a_1, a_1) \times (-a_2, a_2) \times \cdots \times (-a_n, a_n) \times \cdots$$
      where $a_i > 0$.
    - The sequence $(\frac{1}{2}a_1, \frac{1}{2}a_2, \cdots, \frac{1}{2}a_n, \cdots)$ belongs to $A \cap B$, so B intersects A. Hence U also intersects A.

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  - Then B contains $\mathbf{0}$, but it contains no member of the sequence $(\mathbf{a}_n)$ since the nth coordinate x_{nn} of a_n does not belong to the interval $(-x_{nn}, x_{nn})$.

- **Theorem 4.9.** Let X and Y be topological spaces. If $f : X \to Y$ be continuous, then $x_n \to x$ in X implies that $f(x_n) \to f(x)$ in Y. The converse holds if X is first-countable; that is, if $x_n \to x$ implies that $f(x_n) \to f(x)$ for all convergent sequences $\{x_n\}$, then f is continuous.

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- *Proof.* Suppose f is continuous and $\{x_n\}$ is a sequence with $x_n \to x$. Let us show that $f(x_n) \to f(x)$. Let $U \subset Y$ be a nbhd of $f(x)$. Then $f^{-1}(U)$ is a nbhd of x. Since $x_n \to x$, we can choose an $N > 0$ such that $x_n \in f^{-1}(U)$ for all $n > N$. Thus $f(x_n) \in U$ for all $n > N$, so $f(x_n) \to f(x)$.

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  - Let $x \in \overline{A}$ be arbitrary. Then by Lemma 4.8 (the sequence lemma), there is a sequence $\{x_n\}$ with $x_n \in A$ such that $x_n \to x$. Since $f(x_n) \in B$ and $f(x_n) \to f(x)$, we have $f(x) \in \overline{B} = B$. Hence $x \in f^{-1}(B) = A$.