Math 4301 Mathematical Analysis I

Lecture 8

Topic: Compact and Connected Sets

- **Compact subsets of $\mathbb{R}$**

  Definition A subset $K \subseteq \mathbb{R}$ is called *sequentially compact (*or *compact)* if every sequence in K has a convergent subsequence whose limit belongs to K.

  Example: $A = (0,1) \subseteq \mathbb{R}$ is not sequentially compact.

- Consider $\{x_n\} \subset A$,
$$x_n = \frac{1}{n+1} \in (0,1)\,.$$

- We see that $x_n \in A$ and
  since $\{x_n\}$ is convergent in $\mathbb{R}$ to 0, i.e. $x_n \to 0$ as $n \to \infty$,
  every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has limit 0.

- Since $0 \notin A$, it follows that
  A is not seqentially compact.

  Remark: We observe that
$$A = (0,1)$$
  is not closed subset of $\mathbb{R}$.

- In particular, the sequence $\{x_n\}$ is a sequence in A
  which converges to a point that is not in A.

  Remark: We observe that $A = \mathbb{N}$ is closed but it is not bounded.

  Theorem (*Bolzano-Weierstrass*) A subset $A \subset \mathbb{R}$ is sequentially compact if and only if it is closed and bounded.

Proof. Assume that $A \subset \mathbb{R}$ is sequentially compact.

- We show that A is closed and bounded.
- **We show that A is closed.**
- Let $\{x_n\} \subseteq A$ be a sequence in A and and
  assume that $x_n \to x$, where $x \in \mathbb{R}$.
- **We want to show that $x \in A$.**
- Since $\{x_n\}$ converges,
  every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ also converges to x,
  i.e. $x_{n_k} \to x$ as $k \to \infty$.
- Since A is sequentially compact,
$$x \in A.$$
- It follows that every convergent sequence in A has limit that belongs to A.
- Therefore, by previous theorem,
  A is **closed**.

- **We show that A is bounded.**
- Suppose that A is unbounded above,
  so for each $n \in \mathbb{N}$, there is
  $$x_n \in A,$$
  such that $x_n > n$
- Clearly, $x_n \to \infty$ as $n \to \infty$,
  so every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is unbounded and
  $$x_{n_k} \to \infty.$$
- It follows that $\{x_n\}$ has no convergent subsequence.
- This contradicts the assumption that A is sequentially compact, so
  A must be bounded.
- **We show that if A is closed and bounded then it is sequentially compact.**
- Assume that $A \subseteq \mathbb{R}$ is closed and bounded.
- We want to show that if $\{x_n\}$ is a sequence in A,
  then $\{x_n\}$ has a convergent subsequence with its limit in A.
- Consider $\{x_n\} \subseteq A$.
- Since A is bounded,
  $\{x_{n_k}\}$, by Bolzano-Weierstrass theorem,
  $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ and
  let $x_{n_k} \to x$, $x \in \mathbb{R}$.
- We want to show that $x \in A$.
- Since $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$,
  then $\{x_{n_k}\}$ is also a sequence in A.
- Since A is closed and $x_{n_k} \to x$ is a convergent sequence in A,
  it follows that
  $$x \in A.$$
- Therefore, A is *sequentially compact.*

This finishes our proof. ■

- **Example:** $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ is sequentially compact.
- A is closed because $A' = \{0\}$, so
  $$\overline{A} = A \cup A' = A.$$

- A is also bounded because,
  if $n \in \mathbb{N}$, then
  $$0 < \frac{1}{n} \leq 1,$$
  thus for every $x \in A$,
  $$0 \leq x \leq 1.$$
- Therefore, by the Bolzano–Weierstrass theorem,
  A is *sequentially compact.*
  Proposition If $K \subseteq \mathbb{R}$, $K \neq \emptyset$ is *sequentially compact*,
  then both $\min K$ and $\max K$ exist.

Proof. Exercise. ■

- Let $K \subset \mathbb{R}$ be bounded.
- Define
  $$\operatorname{diam}(K) = \sup\{|x-y| : x, y \in K\}$$
  and we call it the diameter of K.
  Example: Let $A = \{1,\ 2,\ 3\}$, then
  $$\begin{aligned} \operatorname{diam}(K) &= \sup\{|x-y| : x, y \in A\} \\ &= \sup\{|1-1|, |1-2|, |2-2|, |1-3|, |2-3|, |3-3|\} \\ &= \sup\{0, 1, 2\} = 2. \end{aligned}$$
  Theorem Let $K_n \subseteq \mathbb{R}$ be *nonempty and sequentially compact*, for all $n \in \mathbb{N}$.
  Assume that $K_{n+1} \subseteq K_n$, for all $n \in \mathbb{N}$.
  Then
  $$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$
  Moreover, if $\operatorname{diam}(K_n) \to 0$ as $n \to \infty$
  then $\bigcap_{n=1}^{\infty} K_n$ consists of a *single point.*

Proof. Since $K_n \neq \emptyset$, let $x_n \in K_n$, $n \in \mathbb{N}$.

- Since $K_{n+1} \subseteq K_n$, for all $n \in \mathbb{N}$,
  then $x_n \in K_1$, for all $n \in \mathbb{N}$.
- Since K is compact,
  it follows that $\{x_n\}$ has a convergent subsequence.
- Let $x_{n_k} \to x$.
- Since $n_k \geq k$,
  then $x_{n_k} \in K_k$ and $K_{k+1} \subseteq K_k$.

- Therefore,
$$\left\{x_{n_j} : j \geq k\right\} \subseteq K_k,$$
and since K_k is closed $x \in K_k$ for all $k \in \mathbb{N}$,
it follows that
$$x \in \bigcap_{n=1}^{\infty} K_n, \text{ so } \bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$
- Now, let us assume that
$$\operatorname{diam}(K_n) \to 0.$$
- Let $x, y \in \bigcap\limits_{n=1}^{\infty} K_n$, so $x, y \in K_n$, for all n.
- Let $\epsilon > 0$ be given.
- Since $\operatorname{diam}(K_n) \to 0$,
there is $n \in \mathbb{N}$, such that
$$\operatorname{diam}(K_n) < \epsilon.$$
- Therefore,
$$|x - y| \leq \operatorname{diam}(K_n) < \epsilon.$$
- Since $\epsilon > 0$ is arbitrary,
it follows that
$$|x - y| = 0,$$
so $x = y$.
- We showed that $\bigcap\limits_{n=1}^{\infty} K_n$ consists of a single point.

This finishes our proof. ■

- **Example**: Let $\mathcal{B} = \left\{\left[2 - \frac{1}{n}, 4 + \frac{1}{n}\right] : n \in \mathbb{N}\right\}$.
- Notice that
$$A_n = \left[2 - \frac{1}{n}, 4 + \frac{1}{n}\right]$$
is both closed and bounded, so
A_n is sequentially compact.
- Moreover,
$$2 - \frac{1}{n} < 2 - \frac{1}{n+1}$$
and
$$4 + \frac{1}{n+1} < 4 + \frac{1}{n},$$
so
$$A_{n+1} \subset A_n, \text{ for all } n \in \mathbb{N}.$$

- Therefore, by theorem
$$\bigcap \mathcal{B} \neq \emptyset.$$

Example: Let $\mathcal{B} = \left\{ \left(1, 1 + \frac{1}{n}\right] : n \in \mathbb{N} \right\}$.

- Notice that
$$\bigcap \mathcal{B} = \bigcap_{n=1}^{\infty} \left(1, 1 + \frac{1}{n}\right] = \emptyset.$$

- Since $1 \notin \left(1, 1 + \frac{1}{n}\right]$,
  then $1 \notin \bigcap \mathcal{B}$.
- If $x < 1$,
  then
$$x \notin \bigcap \mathcal{B}$$
  and analogously
  if $x > 2$, then
$$x \notin \bigcap \mathcal{B}.$$

- Moreover, if $1 < x \leq 2$,
  then $x - 1 > 0$, so
  there is $n \in \mathbb{N}$, such that
$$\begin{aligned} x - 1 &> \frac{1}{n}, \text{ so} \\ x &> 1 + \frac{1}{n}. \end{aligned}$$

- It follows that
$$x \notin \left(1, 1 + \frac{1}{n}\right],$$

- so $x \notin \bigcap \mathcal{B}$.
- In summary,
  there is no $x \in \mathbb{R}$, such that
$$x \in \bigcap \mathcal{B}.$$

- It follows that
$$\bigcap \mathcal{B} = \emptyset.$$

- Notice that although
$$\left(1, 1 + \frac{1}{n+1}\right] \subset \left(1, 1 + \frac{1}{n}\right],$$

  for all $n \in \mathbb{N}$,
  however each $\left(1, 1 + \frac{1}{n}\right]$ is not sequentially compact (because it is not closed).

  Exercise: Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that
$$a_n \leq b_n,$$

for all $n \in \mathbb{N}$ and

$$\begin{aligned} a_n &\leq a_{n+1}, \\ b_{n+1} &\leq b_n. \end{aligned}$$

Define

$$\mathcal{B} = \{[a_n, b_n] : n \in \mathbb{N}\}.$$

Show that $\bigcap \mathcal{B} \neq \emptyset$.

Exercise: Let

$$A_k = \left\{\frac{1}{n} : n \geq k\right\} \cup \{0\}, \ k \in \mathbb{N}.$$

Show that $\bigcap_{k=1}^{\infty} A_k = \{0\}$.

- Compactness can also be defined in terms of open set.

  Definition Let $A \subseteq \mathbb{R}$ and $\mathcal{B}$ be a family of subsets of $\mathbb{R}$.

  We say that $\mathcal{B}$ *covers* A is

$$A \subseteq \bigcup_{B \in \mathcal{B}} B = \bigcup \mathcal{B}.$$

  Example: Let

$$A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subseteq \mathbb{R}$$

  and

$$\mathcal{B} = \{A_k : k \in \mathbb{N}\},$$

  where

$$A_k = \left[\frac{1}{k}, 1\right].$$

- **We show that $\mathcal{B}$ covers A.**
- Notice that $\frac{1}{k} \in A_k$, so $\left\{\frac{1}{k}\right\} \subset A_k$.
- Therefore,

$$\begin{aligned} A &= \left\{\frac{1}{k} : k \in \mathbb{N}\right\} \subseteq \bigcup_{k=1}^{\infty} \left\{\frac{1}{k}\right\} \\ &\subset \bigcup_{k=1}^{\infty} \left[\frac{1}{k}, 1\right] = \bigcup_{B \in \mathcal{B}} B = \bigcup \mathcal{B}. \end{aligned}$$

- Hence, $\mathcal{B}$ covers A.

  Definition Let $\mathcal{B}$ be a *covering* of A.

  A subfamily $\mathcal{C} \subseteq \mathcal{B}$ is called a *subcovering* of A if

  $\mathcal{C}$ is a *covering* of A, i.e.

$$A \subseteq \bigcup_{B \in \mathcal{C}} B = \bigcup \mathcal{C}.$$

  If $\mathcal{C} \subseteq \mathcal{B}$ and $\mathcal{C}$ is finite,

  we call it a *finite subcovering* of A, i.e.

Definition 0.1 *i)* $\mathcal{C} = \{C_1, C_2, ..., C_n\} \subseteq \mathcal{B}$ *and*

ii) $A \subset \bigcup_{i=1}^{n} C_i$

- **Example:** Let $A = (0,1]$ then
$$\mathcal{B} = \left\{ \left(\frac{1}{n}, 2 \right] : n \in \mathbb{N} \right\}$$
is a covering of A that has no finite subcovering.

- We show that
$$A \subseteq \bigcup \mathcal{B},$$
i.e. $\mathcal{B}$ covers A.

- If $x \in A = (0,1]$,
then there is $N \in \mathbb{N}$, such that
$$\frac{1}{N} < x,$$
so $x \in \left(\frac{1}{N}, 2\right]$.

- Therefore,
$$A \subseteq \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 2 \right] = \bigcup \mathcal{B},$$
so $\mathcal{B}$ **covers** A.

- Notice that if
$$\mathcal{C} = \left\{ \left(\frac{1}{n_1}, 2 \right], \ \left(\frac{1}{n_2}, 2 \right], \ ..., \ \left(\frac{1}{n_k}, 2 \right] \right\} \subset \mathcal{B},$$
then $\mathcal{C}$ is not a subcovering of A.

- Let $N = \max\{n_i : i = 1, 2, ..., k\}$.

- Then
$$\bigcup_{C \in \mathcal{C}} C = \bigcup_{j=1}^{k} \left(\frac{1}{n_j}, 2 \right] = \left(\frac{1}{N}, 2 \right].$$

- Since
$$0 < \frac{1}{2N} < \frac{1}{N},$$
then $\frac{1}{2N} \in (0,1] = A$ but
$$\frac{1}{2N} \notin \bigcup_{C \in \mathcal{C}} C = \left(\frac{1}{N}, 2 \right].$$

- Therefore, $A \nsubseteq \bigcup_{C \in \mathcal{C}} C$.

- It follows that $\mathcal{C}$ is not a subcovering of A.

- Therefore, one **cannot find a finite family $C \subseteq B$ that covers A.**

  Exercise: Let $A = (0,1]$ and $\epsilon > 0$.

  Define $\mathcal{D} = \mathcal{B} \cup \{(-\epsilon, \epsilon)\}$,
  $$\mathcal{B} = \left\{ \left(\frac{1}{n}, 2 \right] : n \in \mathbb{N} \right\}.$$
  Show that $\mathcal{D}$ is a covering of A that has a finite subcovering.

  Definition We say that a covering $\mathcal{B}$ of A is an open covering if each $B \in \mathcal{B}$ is an open subset of $\mathbb{R}$.

  Definition Let $K \subseteq \mathbb{R}$. We say that K is *compact* if every open covering $\mathcal{B}$ of K has a finite subcovering $\mathcal{C} \subseteq \mathcal{B}$.

  Example: We show that
  $$A = [a, b] \subset \mathbb{R}$$
  is compact.

- Suppose that $\mathcal{B}$ is an open covering A that has no finite subcovering.
- Define
  $$X = \{x \in A : [a, x] \text{ is covered by a finite number number of } B's \text{ form } \mathcal{B}\}.$$
- **We show that X is nonempty and bounded so**
  $$\alpha = \sup X$$
  exists.
- Since $\mathcal{B}$ covers A,
  there is $B \in \mathcal{B}$, such that
  $$a \in B.$$
- It follows that $X \neq \emptyset$.
- Since $X \subseteq A$ and A is bounded,
  then X is bounded.
- By completeness
  $$\alpha = \sup X \in \mathbb{R}.$$
- Since, for all $x \in X$,
  $$a \leq x \leq b,$$
  it follows that $a \leq \alpha \leq b$.
- **Suppose that $a \leq \alpha < b$ and $\alpha \in X$.**
- Then there are
  $$B_1, B_2, ..., B_k \in \mathcal{B},$$
  such that
  $$[a, \alpha] \subseteq \bigcup_{i=1}^{k} B_i$$

- There is a finite collection
$$\mathcal{C} = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{B}$$
that covers $[a, \alpha]$.
- Since $\mathcal{C}$ covers $[a, \alpha]$,
there is $B_j \in \mathcal{C}$, such that
$$\alpha \in B_j.$$
- However, B_j is open, so
there is $\epsilon > 0$ and $\epsilon < b - \alpha$, such that,
$$(\alpha - \epsilon, \alpha + \epsilon) \subseteq B_j.$$
- Therefore,
$$\left[a, \alpha + \frac{\epsilon}{2}\right] \subseteq \bigcup_{i=1}^{k} B_i$$
so
$$\alpha + \frac{\epsilon}{2} \in X \text{ and } \alpha < \alpha + \frac{\epsilon}{2} < b,$$
a **contradiction** since
$$\alpha = \sup X.$$
- **Assume that**
$$a \leq \alpha < b$$
and $\alpha \notin X$.
- Notice that $\alpha \in A$ since $a \leq \alpha < b$.
- Since $\mathcal{B}$ is an open covering A,
there is $B \in \mathcal{B}$ such that
$$\alpha \in B.$$
- Since B is open,
there is $\epsilon > 0$, such that
$$(\alpha - \epsilon, \alpha + \epsilon) \subseteq B.$$
- Since $\alpha < b$,
$$(\alpha - \epsilon, \alpha + \epsilon) \cap [a, \alpha] \neq \emptyset.$$
- Since $\alpha = \sup X$,
there is $x \in X$, such that
$$\alpha - \epsilon < x \leq \alpha.$$
- Since $x \in X$,
there are
$$B_1, B_2, ..., B_k \in \mathcal{B},$$
such that
$$[a, x] \subseteq \bigcup_{i=1}^{k} B_i.$$

- Therefore,
$$\mathcal{C} = \{B_1, B_2, ..., B_k, B\} \subseteq \mathcal{B}$$
and
$$[a, \alpha] \subseteq \bigcup_{C \in \mathcal{C}} C.$$

- It follows that $\alpha \in X$.
- **Contradiction since we assumed that $\alpha \notin X$.**
- **Assume that $\alpha = b$ and $\alpha \notin X$.**
- Since $\mathcal{B}$ is an open covering A,
there is $B \in \mathcal{B}$ such that
$$\alpha \in B.$$

- Since B is open
there is $\epsilon > 0$, such that
$$(\alpha - \epsilon, \alpha + \epsilon) \subseteq B$$
and
$$(\alpha - \epsilon, \alpha + \epsilon) \cap [a, \alpha] \neq \emptyset.$$

- Since $\alpha = \sup X$,
there is $x \in X$, such that
$$\alpha - \epsilon < x \leq \alpha = b.$$

- Since $x \in X$,
there are
$$B_1, B_2, ..., B_k \in \mathcal{B},$$
such that
$$[a, x] \subseteq \bigcup_{i=1}^{k} B_i.$$

- Therefore,
$$\mathcal{C} = \{B_1, B_2, ..., B_k, B\} \subseteq \mathcal{B}$$
and
$$[a, \alpha] \subseteq \bigcup_{C \in \mathcal{C}} C.$$

- Therefore, $\alpha \in X$.
Contradiction because we assumed that $\alpha \notin X$.
- It follows that $\alpha = b$ and $\alpha \in X$.

- Since
$$[a,\alpha] = [a,b] \subseteq \bigcup_{i=1}^{k} B_i,$$
for some
$$B_1, B_2, ..., B_k \in \mathcal{B},$$
so $[a,b]$ is covered by a finite family
$$\mathcal{C} = \{B_1, B_2, ..., B_k\} \subseteq \mathcal{B}.$$

- It follows that every open covering $\mathcal{B}$ of A has a finite subcovering
$$\mathcal{C} \subseteq \mathcal{B}.$$

- Therefore, $A = [a,b]$ is compact.
Exercise: Let $A = (0,1]$ and
$$\mathcal{B} = \left\{ \left(\frac{1}{n}, 2 \right) : n \in \mathbb{N} \right\}.$$
Show that $\mathcal{B}$ is an open covering of A that has no finite subcovering.
Exercise: Let $A = \mathbb{N}$ and
$$\mathcal{B} = \{(n-1, n+1) \subset \mathbb{R} : n \in \mathbb{N}\}.$$
Show that $\mathcal{B}$ is an open covering of $\mathbb{N}$ that has no finite subcovering.
Theorem (*Heine-Borel*) A subset K of $\mathbb{R}$ is *compact*
if and only if
K is *closed and bounded.*

Proof. We show that if K is closed and bounded then K is compact.

- **Assume that K is closed and bounded**.
- Let $\mathcal{B}$ be an open covering of K.
- Since K is bounded,
there are $a, b \in \mathbb{R}$, such that
$$K \subseteq [a,b].$$
- Since K is closed,
$\mathbb{R} \backslash K$ is open and
$$\mathcal{D} = \mathcal{B} \cup \{\mathbb{R} \backslash K\}$$
is an *open covering* of $[a,b]$.
- **Indeed**, if $x \in K$ then
since $\mathcal{B}$ be an open covering of K,
there is $B \in \mathcal{B}$, such that
$$x \in B.$$

- Therefore,
$$x \in \bigcup_{B \in \mathcal{B}} B \subset \bigcup_{D \in \mathcal{D}} D.$$
- If $x \in [a,b] \setminus K$,
then $x \notin K$, so
$$x \in \mathbb{R} \setminus K.$$
- Since $\mathbb{R} \setminus K \in \mathcal{D}$,
$$x \in \bigcup_{D \in \mathcal{D}} D.$$
- **As we proved**, $[a,b]$ is compact, so
there is a finite subcovering
$$\mathcal{C} \subseteq \mathcal{D}.$$
- Since $K \subset [a,b]$, $\mathcal{C}$ is also a finite subcovering of K.
- Let
$$\mathcal{C}_K = \{C \in \mathcal{C} : C \cap K \neq \emptyset\}.$$
- We observe that $\mathcal{C}_K$ is finite and
$$\mathbb{R} \setminus K \notin \mathcal{C}_K,$$
so $\mathcal{C}_K \subseteq \mathcal{B}$.
- Furthermore,
$$K \subseteq \bigcup_{C \in \mathcal{C}_K} C.$$
- **Indeed,** if $x \in K$,
since $\mathcal{C} \subseteq \mathcal{D}$ covers K and $x \notin \mathbb{R} \setminus K$,
there is $C \in \mathcal{C}$ such that
$$C \neq \mathbb{R} \setminus K \text{ and } x \in C.$$
- Therefore,
$$C \cap K \neq \emptyset$$
and $C \in \mathcal{C}$, so
$$C \in \mathcal{C}_K$$
- Therefore, $\mathcal{B}$ has a finite subcollection
$$\mathcal{C}_K \subseteq \mathcal{B}$$
that covers K.
- **Consequently, K is compact.**
- **Conversely, assume that $K \subseteq \mathbb{R}$ is compact.**
- Let $\mathcal{B} = \{(-n,n) \subseteq \mathbb{R} : n \in \mathbb{N}\}$.
- Observe that
$$\mathbb{R} = \bigcup_{n=1}^{\infty} (-n,n) = \bigcup_{B \in \mathcal{B}} B.$$

- Since $K \subseteq \mathbb{R}$,
  $\mathcal{B}$ is also an open covering of K.
- Since K is compact,
  there is $\mathcal{C} \subset \mathcal{B}$ such that $\mathcal{C}$ covers K and $\mathcal{C}$ is finite.
- Let $\mathcal{C} = \{(-n_i, n_i) : i = 1, 2, ..., k\}$.
- Define
$$N = \max\{n_1, n_2, ..., n_k\}.$$
- Therefore,
$$\bigcup_{C \in \mathcal{C}} C = (-N, N).$$
- Consequently,
$$\begin{aligned} K &\subseteq \bigcup_{C \in \mathcal{C}} C = (-N, N), \text{ so} \\ K &\subset [-N, N]. \end{aligned}$$
- **Therefore, K is bounded.**
- **To finish our proof we show that K is closed**.
- It suffices to show that $\mathbb{R} \backslash K$ is open.
- Let $x \in \mathbb{R} \backslash K$ and let
$$\begin{aligned} \mathcal{B} &= \left\{ \mathbb{R} \backslash \left[x - \frac{1}{n}, x + \frac{1}{n} \right] : n \in \mathbb{N} \right\} \\ &= \left\{ \left(-\infty, x - \frac{1}{n} \right) \cup \left(x + \frac{1}{n}, \infty \right) : n \in \mathbb{N} \right\}. \end{aligned}$$
- Since
$$\bigcap_{n=1}^{\infty} \left[x - \frac{1}{n}, x + \frac{1}{n} \right] = \{x\},$$
  it follows
$$\begin{aligned} \bigcup_{B \in \mathcal{B}} B &= \bigcup_{n=1}^{\infty} \mathbb{R} \backslash \left[x - \frac{1}{n}, x + \frac{1}{n} \right] \\ &= \mathbb{R} \backslash \bigcap_{n=1}^{\infty} \left[x - \frac{1}{n}, x + \frac{1}{n} \right] = \mathbb{R} \backslash \{x\}. \end{aligned}$$
- Furthermore, because $x \notin K$,
$$K \subseteq \bigcup_{B \in \mathcal{B}} B.$$
- Moreover,
$$\left(-\infty, x - \frac{1}{n} \right) \cup \left(x + \frac{1}{n}, \infty \right) \subseteq \mathbb{R}$$
  is open for all $n \in \mathbb{N}$, so
  $\mathcal{B}$ is an open covering of K.

- Since K is compact,
  there is a finite subcollection
  $$\mathcal{C} \subseteq \mathcal{B}$$
  that covers K.
- Let
  $$\mathcal{C} = \left\{ \left(-\infty, x - \frac{1}{n_i}\right) \cup \left(x + \frac{1}{n_i}, \infty\right) : i = 1, 2, ..., k \right\}$$
  and
  $$N = \max\left\{n_1, n_2, ..., n_k\right\}.$$
- Because
  $$K \subseteq \bigcup_{C \in \mathcal{C}} C = \mathbb{R} \backslash \left[x - \frac{1}{N}, x + \frac{1}{N}\right],$$
  we see that
  $$\mathbb{R} \backslash K \supseteq \left[x - \frac{1}{N}, x + \frac{1}{N}\right] \supset \left(x - \frac{1}{N}, x + \frac{1}{N}\right),$$
- Hence
  $$x \in \left(x - \frac{1}{N}, x + \frac{1}{N}\right) \subset \mathbb{R} \backslash K.$$
- So, $\mathbb{R} \backslash K$ is open, so
  K is closed.

This finishes our proof. ■

- **Corollary** A subset K of $\mathbb{R}$ is compact iff K is sequentially compact.

Proof. We apply Heine-Borel theorem and Bolzano-Weierstrass theorem.

- By Heine-Borel theorem $K \subseteq \mathbb{R}$ is compact
  iff
  K is closed and bounded.
- By Bolzano-Weierstrass theorem K is closed and bounded
  iff
  K is sequentially compact.
- Therefore, K is compact
  iff
  K is sequentially compact.

This finishes our proof. ■