Green's Functions and Applications

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Introduction

Green's functions are a method of solving linear, non-homogenous differential equations. They are helping functions that, when operated on by the same differential operator as the original problem, are forced by the Dirac delta function instead of a complicated forcing function. Once a Green's function has been found, it can be multiplied by the original forcing function and then integrated to solve the original problem. Green's functions are useful for solving IVPs and BVPs for ODEs and PDEs with difficult forcing functions. They have applications in quantum mechanics, electro dynamics, signal processing, heat transfer, and other fields of engineering and physics.

Context

Linear Operators

Recall that an operator is a mapping or function that acts on elements of a space to produce elements of another space. A linear operator L is a mapping or function that is usually defined to satisfy conditions of additivity, $L(a+b) = L(a) + L(b)$, and multiplicativity, $L(c \cdot z) = c \cdot L(z)$, where c is some constant. Differential operators are linear operators. For example, in the differential equation $y'' - y' = 1$, L is defined as $L = \left(\frac{d^2}{dx^2} - \frac{d}{dx}\right)$, so that $L[y] = \left(\frac{d^2}{dx^2} - \frac{d}{dx}\right) y = y'' - y'$. We can now write $L[y] = 1$. Generally, any linear, non-homogenous differential equation can be written as $L[y] = f(x)$ or $Ly = f(x)$. Separating out the differential operator is necessary for defining Green's functions.

Heaviside and Dirac Delta Functions

The Heaviside (unit step) function $\theta(x-s)$ is defined as:

$$\theta(x-s) = \begin{cases} 0, & x < s \\ 1, & x \geq s \end{cases}$$

and the Dirac delta function $\delta(x-s)$ is defined as:

$$\delta(x-s) = \begin{cases} 0, & x \neq s \\ \infty, & x = s \end{cases}$$

The delta function is the derivative of the Heaviside function centered at a point s, yielding the property $\frac{d}{dx}[\theta(x-s)] = \delta(x-s)$.

The delta function has two important properties that are used for deriving Green's functions:

1. $\int_a^b \delta(x-s)\,ds = 1$ for $a < s < b$
2. $\int_a^b f(x)\delta(x-s)\,dx = f(s)$ for $a < s < b$

Definition of Green's Functions

Consider the 2nd order, linear, non-homogenous differential equation $L[u] = f(x)$ where $u = u(x)$, with boundary conditions $u(a) = \alpha$ and $u(b) = \beta$. The Green's function for the linear operator L is the function $G(x,s)$ such that

$$LG(x,s) = \delta(x-s).$$

Green's functions can be found for higher order differential equations as well, as long as relevant boundary conditions are provided.

The above definition is used to solve the general case by exploiting the properties of the delta function:

$$L[G(x,s)\cdot f(s)] = \delta(x-s)\cdot f(s)$$

$$\int_a^b L[G(x,s)f(s)]\,ds = \int_a^b \delta(x-s)f(s)\,ds = f(x)$$

$$L\int_a^b G(x,s)f(s)\,ds = f(x)$$

$$u(x) = \int_a^b G(x,s)f(s)\,ds$$

Properties of Green's Functions

Green's functions all have the following properties:

1. For each x, $s \to G(x,s)$ satisfies $G'' = 0$ with respect to s, except when $x = s$.
2. G satisfies the boundary conditions $G(0,s) = 0 = G(L,s)$.
3. $G(s+0,s) - G(s-0,s) = 0$.
4. $\frac{\partial G}{\partial x}(s+0,s) - \frac{\partial G}{\partial x}(s-0,s) = -1$.
5. $G(x,s) = G(s,x)$.

$G(x,s)$ is uniquely determined by conditions 1 through 4. Condition 1 ensures that $G(x,s)$ satisfies the homogenous equation. Condition 2 ensures it satisfies the boundary conditions. Condition 3 requires continuity at $x = s$, and condition 4 requires a precise jump discontinuity at $x = s$.

Green's for ODE, IVP Problems

General Case

For the general case, consider the initial value problem with homogeneous initial conditions:

$$A(x)y'' + B(x)y' + C(x)y = f(x)$$

with $y(0) = y_0$ and $y'(0) = v_0$. The Green's Function for this IVP is

$$G(x,s) = \frac{y_1(s)y_2(x) - y_1(x)y_2(s)}{A(s) \cdot W(s)}$$

where $y_1(s)$ and $y_2(s)$ are fundamental solutions of the differential equation (they solve the homogeneous version of the differential equation y_h), and

$$W(s) = \begin{vmatrix} y_1(s) & y_2(s) \\ y_1'(s) & y_2'(s) \end{vmatrix}$$

is the Wronskian.

Then the solution to the IVP is

$$y(x) = y_h(x) + \int_0^s G(x,s)f(s)\,ds$$

BVPs

To find solutions to boundary value problems with Green's functions, let

$$G(x) = \begin{cases} G_1(x), & x < t \\ G_2(x), & x \geq t \end{cases}$$

where $a < t < b$, a and b are boundaries.

Then G must satisfy the properties of Green's functions.

Example Problems

Spring-Mass-Damper System

Consider the spring-mass-damper system described by the differential equation:

$$y'' + y = 2\cos x,$$

where $y(x)$ represents the displacement, and $2\cos x$ is an external forcing function. Our objective is to find the displacement $y(x)$.

Impulse Response Concept

To understand the system's response to any forcing function, we examine its response to an impulse. This impulse is represented by the Dirac delta function $\delta(x - s)$, which is defined by its sifting property:

$$\int_{-\infty}^{\infty} f(x)\delta(x - s)\,dx = f(s)$$

for any test function $f(x)$. The Dirac delta function is not a function in the traditional sense but rather a distribution that captures the idea of an infinitely high, infinitesimally narrow spike at $x = s$.

The system's response to this impulse is captured by the Green's function $G(x, s)$. The Green's function $G(x, s)$ for a differential operator L is defined as the solution to the equation:

$$L[G(x, s)] = \delta(x - s),$$

where L is the differential operator associated with the system. For our spring-mass-damper system, the differential operator is given by:

$$L[y] = y'' + y.$$

Differential Operator and Green's Function

The differential equation governing the system's dynamics is:

$$L[y] = y'' + y = f(x),$$

where $f(x)$ is a given forcing function. To find the Green's function $G(x, s)$, we solve:

$$y'' + y = \delta(x - s).$$

The Green's function $G(x, s)$ thus satisfies:

$$G''(x, s) + G(x, s) = \delta(x - s).$$

Constructing the Green's Function

To construct the Green's function, we assume a piecewise form for $G(x, s)$:

$$G(x, s) = \begin{cases} A(s)\sin(x) + B(s)\cos(x), & \text{for } x < s, \\ C(s)\sin(x) + D(s)\cos(x), & \text{for } x \geq s. \end{cases}$$

Here, $A(s)$, $B(s)$, $C(s)$, and $D(s)$ are coefficients that we determine using specific boundary conditions at $x = s$.

Boundary Conditions

Continuity at $x = s$ The Green's function $G(x, s)$ must be continuous at $x = s$. This condition ensures that there is no sudden jump in the displacement y at the point of application of the impulse. Mathematically, this is expressed as:

$$\lim_{x \to s^-} G(x, s) = \lim_{x \to s^+} G(x, s).$$

Given the piecewise form, this continuity condition translates to:

$$A(s)\sin(s) + B(s)\cos(s) = C(s)\sin(s) + D(s)\cos(s).$$

Discontinuity in the Derivative The derivative of the Green's function $G(x, s)$ must have a discontinuity of 1 at $x = s$. This condition arises from the presence of the Dirac delta function $\delta(x - s)$ on the right-hand side of the differential equation:

$$G''(x, s) + G(x, s) = \delta(x - s).$$

To understand why, integrate both sides of the differential equation over an infinitesimally small interval around s:

$$\int_{s-\epsilon}^{s+\epsilon} \left[G''(x,s) + G(x,s)\right] \, dx = \int_{s-\epsilon}^{s+\epsilon} \delta(x-s) \, dx.$$

Since $G(x,s)$ is continuous, the integral of $G(x,s)$ over this small interval contributes negligibly as $\epsilon \to 0$. Therefore,

$$\left. \frac{\partial G(x,s)}{\partial x} \right|_{x=s^+} - \left. \frac{\partial G(x,s)}{\partial x} \right|_{x=s^-} = \int_{s-\epsilon}^{s+\epsilon} \delta(x-s) \, dx = 1.$$

This condition ensures that the Green's function correctly models the effect of an impulse applied at $x = s$.

By solving these boundary conditions, we determine the coefficients and thus the explicit form of the Green's function.

Solution for $G(x,s)$

After solving the conditions, we find the Green's function for our system:

$$G(x,s) = \begin{cases} \sin(x-s), & \text{for } x \geq s, \\ \sin(s-x), & \text{for } x < s. \end{cases}$$

Using the Green's Function

Once we have the Green's function, we can use it to construct the particular solution to the original nonhomogeneous differential equation. For a given forcing function $f(x)$, the particular solution $y_p(x)$ is given by:

$$y_p(x) = \int_{-\infty}^{\infty} G(x,s) f(s) \, ds.$$

In our specific problem with $f(x) = 2\cos x$, the integral becomes:

$$y_p(x) = \int_0^x G(x,s) \cdot 2\cos(s) \, ds.$$

Substituting the form of $G(x,s)$, we evaluate the integral and find the particular solution, which combined with the homogeneous solution, gives us the complete solution to the original differential equation.

Evaluating the Integral

Substitute the form of $G(x,s)$:

$$y_p(x) = \int_0^x 2\sin(x-s)\cos(s)\,ds.$$

Using the trigonometric identity $\sin(x-s) = \sin x\cos s - \cos x\sin s$, we rewrite the integral:

$$y_p(x) = 2\sin x\int_0^x \cos^2(s)\,ds - 2\cos x\int_0^x \sin(s)\cos(s)\,ds.$$

Evaluating these integrals, we find:

$$\int_0^x \cos^2(s)\,ds = \frac{x}{2} + \frac{\sin(2x)}{4},$$

$$\int_0^x \sin(s)\cos(s)\,ds = \frac{1}{4}\left(1-\cos(2x)\right),$$

Thus, the particular solution is:

$$y_p(x) = x\sin x.$$

General Solution

The general solution to the original differential equation is the sum of the particular and homogeneous solutions:

$$y(x) = y_p(x) + y_h(x).$$

Applying initial conditions $y(0) = 4$ and $y'(0) = 0$, we solve for the constants in $y_h(x) = C_1\sin x + C_2\cos x$:

$$y(x) = x\sin x + 4\cos x.$$

Three-Dimensional Heat Equation

Consider the nonhomogeneous heat equation in three dimensions:

$$u_t - K\nabla^2 u = h,$$

in the region $\mathbb{R}^3 \times [0,T]$ with the initial condition $u(P,0) = 0$. Our goal is to find the temperature distribution $u(P,t)$ over time.

Green's Function for the Heat Equation

To solve this equation, we first need to find the Green's function for the three-dimensional heat equation. The Green's function $G(P,Q;t)$ represents the temperature response at a point P due to an instantaneous point heat source at another point Q. It satisfies the homogeneous heat equation with an initial condition corresponding to the Dirac delta function:

$$G_t - K\nabla^2 G = 0, \quad G(P,Q;0) = \delta(P-Q).$$

Fourier Transform Approach

We use the Fourier transform to convert the heat equation into the frequency domain. Let $\hat{G}(\xi,t)$ be the Fourier transform of $G(P,Q;t)$:

$$\hat{G}(\xi,t) = \int_{\mathbb{R}^3} G(P,Q;t) e^{-i\xi\cdot(P-Q)}\, d(P-Q).$$

The heat equation in the frequency domain becomes:

$$\hat{G}_t + K|\xi|^2\hat{G} = 0.$$

This is an ordinary differential equation in t. Solving this ODE, we get:

$$\hat{G}(\xi,t) = \hat{G}(\xi,0)e^{-K|\xi|^2 t}.$$

Given the initial condition $G(P,Q;0) = \delta(P-Q)$, we have $\hat{G}(\xi,0) = 1$. Therefore,

$$\hat{G}(\xi,t) = e^{-K|\xi|^2 t}.$$

Next, we use the inverse Fourier transform to convert back to the spatial domain:

$$G(P,Q;t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-K|\xi|^2 t} e^{i\xi\cdot(P-Q)}\, d\xi.$$

Evaluating the Inverse Fourier Transform

The inverse Fourier transform of a Gaussian is another Gaussian. The integral:

$$G(P,Q;t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-K|\xi|^2 t} e^{i\xi\cdot(P-Q)}\, d\xi,$$

can be evaluated using the known result for the Gaussian integral. This gives us:

$$G(P,Q;t) = (4\pi K t)^{-3/2} \exp\left(-\frac{|P-Q|^2}{4Kt}\right).$$

Constructing the Solution

To solve our original heat equation, we use the Green's function to construct the solution. We integrate the effects of the heat source over all points in space and over all past times. The solution is:

$$u(P,t) = \int_0^t \iiint_{\mathbb{R}^3} G(P,Q;t-s)h(Q,s)\,dQ\,ds.$$

Substituting the expression for Green's function:

$$G(P,Q;t-s) = (4\pi K(t-s))^{-3/2} \exp\left(-\frac{|P-Q|^2}{4K(t-s)}\right),$$

we get:

$$u(P,t) = \int_0^t \iiint_{\mathbb{R}^3} \frac{1}{[4\pi K(t-s)]^{3/2}} \exp\left(-\frac{|P-Q|^2}{4K(t-s)}\right) h(Q,s)\,dQ\,ds.$$

This integral sums the contributions of the heat source $h(Q,s)$ from all spatial points Q and times s, accounting for the diffusion of heat from these sources over time. The exponential term reflects how the heat spreads out from the source point Q to the point P over time $t-s$.

Fourier Representation of Green's Functions

Green's functions may be represented by Fourier sine series. The nth Fourier sine coefficient of G is

$$B_n(s) = \frac{2}{L}\int_0^L G(x,s)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

This looks similar to how we define the solution to an ODE with Green's functions,

$$u(x) = \int_a^b G(x,s)f(s)\,ds,$$

with

$$\frac{2}{L}\sin\left(\frac{n\pi x}{L}\right)f(s).$$

We can find that

$$B(s) = \frac{2}{L}\left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi s}{L}\right)$$

is the solution to

$$B''(s) = -\frac{2}{L}\sin\left(\frac{n\pi s}{L}\right),$$

with boundary conditions $B(0) = 0 = B(L)$. Thus, the Fourier representation of Green's function takes the form:

$$G(x,s) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi s}{L}\right).$$

Green's Functions for PDEs – Solving the Poisson Equation – Example

Consider the 3-Dimensional Poisson equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u(x,y,z) = f(x,y,z)$$

with boundary condition (Dirichlet): $u = h(x,y,z)$ on S, with boundary of domain D. We can use Green's functions to represent the solution to the PDE.

First, we know that for the 3D Laplacian operator $L = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) = \nabla^2$, the Green's function satisfies $LG = \delta(p-x, q-y, s-z)$ on D and $G = 0$ on S, where p, q, and s are dummy variables for integration.

Green's Identities

We must use Green's identities, which are derived from the divergence theorem:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_D \nabla\cdot\mathbf{F}\, dV,$$

meaning the flux of a vector field $\mathbf{F}$ through a closed surface or boundary S equals the divergence of $\mathbf{F}$ integrated over the entire volume D that the boundary S encloses.

We let $\mathbf{F} = v(\nabla u)$ where ∇u is the gradient of u. Then plugging this into the divergence theorem gives

$$\iint_S v\frac{\partial u}{\partial n}\, dS = \iiint_D (\nabla v\cdot\nabla u)\, dV + \iiint_D (v\nabla^2 u)\, dV$$

This is Green's first identity. Note that u and v are arbitrary, thus, they can be easily interchanged. Swapping them yields

$$\iint_S u\frac{\partial v}{\partial n}\, dS = \iiint_D (\nabla u\cdot\nabla v)\, dV + \iiint_D (u\nabla^2 v)\, dV$$

Subtracting the second identity from the first identity yields Green's second identity:

$$\iint_S \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) dS = \iiint_D \left(v\nabla^2 u - u\nabla^2 v \right) dV$$

Poisson Equation

Replace v with G in Green's identities:

$$\iint_S G\frac{\partial u}{\partial n}\, dS = \iiint_D (\nabla G \cdot \nabla u)\, dV + \iiint_D (G\nabla^2 u)\, dV$$

$$\iint_S \left(G\frac{\partial v}{\partial n} - u\frac{\partial G}{\partial n} \right) dS = \iiint_D \left(G\nabla^2 u - u\nabla^2 G \right) dV$$

Then we find a formula for the solution u to the Poisson PDE. Plug in the definition into Green's second identity:

$$\iint_S \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n} \right) dS = \iiint_D \left(Gf - u\delta(p-x, q-y) \right) dp\, dq\, ds$$

$$\iint_S \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n} \right) dS = \iiint_D Gf\, dV - u(x,y)$$

Solving for $u(x,y)$ and applying the known boundary conditions, we arrive at:

$$u(x,y) = \iint_S \left(h\frac{\partial G}{\partial n} \right) dS + \iiint_D Gf\, dV$$

Now, in a specific case, plug f, h, and G into the equation and it gives a solution to the PDE.

Important Green's Functions

The solutions to the 2D and 3D Poisson equation problems and their Green's functions are useful in theoretical physics. For example, the solution represents the potential field caused by a given electric charge or mass density distribution. This allows physicists to calculate an electrostatic or gravitational force field.

- 2D Poisson Equation: $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(x,y) = f(x,y)$
- 2D Laplacian operator: $L = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$

- $G = \frac{1}{2\pi} \ln r$, $r = \sqrt{x^2 + y^2}$
- 3D Poisson Equation: $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u(x, y, z) = f(x, y, z)$
- 3D Laplacian operator: $L = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$
- $G = -\frac{1}{4\pi r}$, $r = \sqrt{x^2 + y^2 + z^2}$

Sources

Text

- "Partial Differential Equations and Boundary-Value Problems with Applications (Third Edition)," Chapter 8: Green's Functions, Mark A. Pinsky
- "Partial Differential Equations," Schaum's Outlines, Paul DuChateau, David W. Zachmann
- "Boundary Value Problems and Partial Differential Equations," Mayer Humi, William B. Biller

Video

- "Introduction to Greens Functions from a simple example," Daniel An, `https://youtu.be/3pcokxvGD1k?si=NVFW2gv7hkKHjyOO`
- "Green's Functions — Chapter 3 Differential Equations," Sabetta Talks Math, `https://youtu.be/EOhelmznQAs?si=YMxte3QlMkTwzkKg`
- "Green's function for Sturm-Liouville problems," Nathan Kutz, `https://youtu.be/8l4VsNSX_kw?si=irzuUuN7p8r7sPRS`
- "Section 4.8 – Green's Functions – Part 1," Professor Yenerall's Math Help, `https://youtu.be/gc1eoe-C5-Q?si=lgZRHLSzvukcOWOY`
- "Introducing Green's Functions for Partial Differential Equations (PDEs)," Khan Academy, `https://youtu.be/xNqLZnM-PPY?si=Mn-vGT43PiMmVSO_`

Summary

Overall, Green's functions constitute a unique way of solving linear, non-homogenous differential equations using the properties of the Dirac delta function. They are constructed out of independent solutions to the homogenous problem $Ly = 0$, and are uniquely determined by 5 properties: proportionality to the homogenous solution (1), boundary conditions (2), continuity (3 and 4), and symmetry/reciprocity (5). Green's functions work by breaking a forcing function f up into a bunch of impulses and creating sub-problems around those. Solving for G in response to those impulse inputs and integrating the results over the whole domain gives the complete solution. Green's functions can be represented by Fourier sine series and thus are useful in studying real-world phenomena that can be discretized, with applications in particle physics and digital signal processing.