*This guide is a review of content from multiple courses at the **precalculus** level. Its purpose is to provide a comprehensive resource of basic concepts that are helpful to remember in higher level math courses, but are too often forgotten.*

Contents

1 Arithmetic

1.1 Addition

Axiom 1.1 (Identity Property of Addition) $0 + a = a + 0 = a$.

Axiom 1.2 (Additive Inverse Property of Addition) $a + (-a) = 0$. In other words, $-a + a = 0$

Axiom 1.3 (Commutativity) $a + b = b + a$.

Axiom 1.4 (Associativity) $(a + b) + c = a + (b + c)$.

Theorem 1.5 If $a + b = 0$, then $b = -a$ and $a = -b$.

Theorem 1.6 $a = -(-a)$.

Theorem 1.7 $-(a + b) = -a - b$

Theorem 1.8 If $a + b = a + c$, then $b = c$.

1.2 Multiplication

Axiom 1.9 (Commutativity) $ab = ba$.

Axiom 1.10 (Associativity) $(ab)c = a(bc)$.

Axiom 1.11 (Identity) $1a = a$.

Axiom 1.12 (0 Property) $0a = 0$.

Axiom 1.13 (Distributive Property) $a(b + c) = ab + ac$.

Axiom 1.14 $(b + c)a = ba + ca$.

Definition 1.15 (Exponents) $a^n = \underbrace{aa \cdots a}_{n}$

Axiom 1.16 $a^{m+n} = a^m a^n$.

Axiom 1.17 $(a^m)^n = a^{mn}$.

Axiom 1.18 $(ab)^n = a^n b^n$.

Theorem 1.19 $(-1)a = -a$.

Theorem 1.20 $-(ab) = (-a)b$.

Theorem 1.21 $-(ab) = a(-b)$.

Theorem 1.22 $(-a)(-b) = ab$.

Theorem 1.23 $(a + b)^2 = a^2 + 2ab + b^2$.

Theorem 1.24 $(a - b)^2 = a^2 - 2ab + b^2$

Theorem 1.25 $(a + b)(a - b) = a^2 - b^2$.

1.3 Even and Odd Integers; Divisibility

Definition 1.26 An *even* number is any integer k of the form $k = 2n$, where n is an integer.

Definition 1.27 An *odd* number is any integer k of the form $k = 2n + 1$, where n is an integer.

Definition 1.28 (Divisibility) An integer n is divisible by an integer k if and only if there is an integer d called the *divisor* such that $n = dk$.

Theorem 1.29 Let a, b be positive integers.

- If a is even and b is even, then $a + b$ is even.
- If a is even and b is odd, then $a + b$ is odd.
- If a is odd and b is even, and $a + b$ is odd.
- If a is odd and b is odd, then $a + b$ is even.

Theorem 1.30 Let a be a positive integer. If a is even, then a^2 is even. If a is odd, then a^2 is odd.

1.4 Rational Numbers

Definition 1.31 A *rational number* is a number $\frac{m}{n}$, where m and n are integers, and $n \neq 0$.

Definition 1.32 The *lowest form* of a rational number a, is the number $\frac{s}{r}$ where the common divisor of s and r is 1.

Axiom 1.33

$$\frac{a}{d} + \frac{b}{d} = \frac{a+b}{d}.$$

Axiom 1.34

$$\frac{m}{n} + \frac{r}{s} = \frac{ms+rn}{ns}$$

Axiom 1.35

$$\frac{m}{n} \cdot \frac{r}{s} = \frac{mr}{ns}.$$

Axiom 1.36

$$a^k = \left(\frac{m}{n}\right)^k = \frac{m^k}{n^k}$$

Theorem 1.33

$$\frac{m}{n} = \frac{r}{s} \text{ if and only if } ms = rn.$$

Theorem 1.34 (Cancellation rule for fractions) Let a be a non-zero integer. Let m, n be integers, $n \neq 0$. Then

$$\frac{am}{an} = \frac{m}{n}.$$

Theorem 1.35

$$-\frac{m}{n} = \frac{m}{-n}.$$

Theorem 1.36 Any positive rational number has an expression as a fraction in lowest form.

Theorem 1.37

$$\frac{0}{n} = \frac{0}{1} = 0.$$

Theorem 1.38

$$-\frac{m}{n} = \frac{m}{-n}.$$

Theorem 1.39 There is no positive rational number whose square is 2.

Definition 1.40 (Multiplicative Inverse) If a is a rational number $\neq 0$, then there exists a rational number, denoted by a^{-1}, such that

$$a^{-1}a = aa^{-1} = 1.$$

a^{-1} is called the *multiplicative inverse* of a.

Definition 1.41 If a is a rational number $\frac{m}{n}$, then $a^{-1} = \frac{n}{m}$.

$$\frac{m}{n} \cdot \frac{n}{m} = \frac{mn}{mn} = 1.$$

Theorem 1.42 If a, b are rational numbers, and $ab = 1$, then $a = b^{-1}$.

Theorem 1.42 If $ab = 0$, then $a = 0$ or $b = 0$.

Theorem 1.43 (Cross-multiplication) Let a, b, c, d be rational numbers, and assume that $b \neq 0$ and $d \neq 0$.

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc.$$

Furthermore,

$$\text{If } ad = bc, \text{ then } \frac{a}{b} = \frac{c}{d}.$$

Theorem 1.44 (Cancellation law for multiplication) Let a be a rational number $\neq 0$.

$$\text{If } ab = ac, \text{ then } b = c.$$

2 Real Numbers

2.1 Positivity

Axiom 2.1 If a, b are positive, so are the product ab and the sum $a + b$.

Axiom 2.2 If a is a real number, then either a is positive, or $a = 0$, or $-a$ is positive, and these possibilities are mutually exclusive.

Definition 2.3 (Absolute Value) $|x| = \sqrt{x^2}$.

Theorem 2.4 If $a > 0$, then there exists a number b such that $b^2 = a$.

Theorem 2.5 If x, y are numbers such that $x^2 = y^2$, then $x = y$ or $x = -y$.

2.2 Powers and Roots

Definition 2.6 Let r be the nth root of a. Then $r^n = a$ and $r = a^{\frac{1}{n}}$.

Theorem 2.6 Let a, b be positive real numbers. Then

$$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}.$$

Theorem 2.7 $a^0 = 1$.

Theorem 2.8 $a^{-x} = \frac{1}{a^x}$.

Theorem 2.9 $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = (a^{\frac{1}{n}})^m$.

2.3 Inequalities

Theorem 2.10 If $a > b$ and $b > c$, then $a > c$.

Theorem 2.11 If $a > b$ and $c > 0$, then $ac > bc$.

Theorem 2.12 If $a > b$ and $c < 0$, then $ac < bc$.

3 Quadratic Equations

Theorem 3.1 Let a, b, c be real numbers and $a \neq 0$. The solutions of the quadratic equation

$$ax^2 + bc + c = 0$$

are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

4 Distance and Angles

4.1 Distance

Axiom 4.1 For any points P, Q, we have $d(P, Q) \geq 0$.

Theorem 4.2 $d(P, Q) = 0$ if and only if $P = Q$.

Theorem 4.3 $d(P, Q) = d(Q, P)$.

Theorem 4.4 (Triangle Inequality) $d(P, M) \leq d(P, Q) + d(Q, M)$.

Axiom 4.5 $d(P, M) = d(P, Q) + d(Q, M)$ if and only if Q lies on the segment between P and M.

Axiom 4.6 Let P, M be points in the plane, and let $d = d(P, M)$. If c is a number such that $0 \leq c \leq d$, then there exists a unique point Q on the segment $\overline{PM}$such that $d(P, Q) = c$.

4.2 Angles

Axiom 4.7 Two lines which are not parallel meet in exactly one point.

Definition 4.8 An *angle* is the region of the plane determined by two rays with a common vertex.

4.3 Right Triangles and the Pythagorean Theorem

Axiom 4.9 If two right triangles $\triangle PQM$ and $\triangle P'Q'M'$ have legs $\overline{PQ}, \overline{PM}$ and $\overline{P'Q'}, \overline{P', M'}$, respectively of equal lengths, then: (a) the corresponding angles of the triangles have equal measure, (b) their areas are equal, and (c) the length of $\overline{QM}$ is equal to the length of $\overline{Q'M'}$.

Theorem 4.10 If A, B are the angles of a right triangle other than the right angles, then

$$m(A) + m(B) = 90°.$$

Theorem 4.11 The area of a right triangle whose legs have lengths a, b is equal to

$$\frac{ab}{2}.$$

Theorem 4.12 (Pythagorean Theorem) Let a, b be the lengths of the two legs of a right triangle, and let c be the length of the hypotenuse. Then

$$a^2 + b^2 = c^2.$$

Corollary 4.13 Let P, Q be distinct points in the plane. Let M be also a point in the plane. We have

$$d(P, M) = d(Q, M)$$

if and only if M lies on the perpendicular bisector of $\overline{PQ}$.

5 Area and Applications

Definition 5.1 If S is an arbitrary region of the plane whose area can be approximated by the area of a finite number of rectangles, then if S is dilated by a scalar r,

$$\text{the area of } rS = r^2(\text{area of } S).$$

Definition 5.2 Let C_1 be the circle of radius 1 and C_r the circle of radius r. Then the circumference of of C_1 is equal to 2π, and the circumference of C_r is equal to $2\pi r$.

6 Coordinates and Geometry

6.1 Coordinate Systems

Definition 6.1 The *Cartesian coordinate system* is a representation of the $\mathbb{R}^2$ plane with a horizontal and vertical axis, and horizontal and vertical delimiters at every individual unit.

Definition 6.2 A *point* in Cartesian coordinates is an ordered pair of real numbers of the form $P = (x, y)$. A point can be represented as a physical point on the written representation of the coordinate system, where x and y indicate the location of the point relative to the origin.

Definition 6.3 The *origin* of the Cartesian coordinate system is defined as the point $O = (0, 0)$.

6.2 Distance Between Points

Theorem 6.4 Let x_1, x_2 be points on a line. Then the distance between x_1 and x_2 is equal to

$$\sqrt{(x_1 - x_2)}.$$

Theorem 6.5 Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be points in $\mathbb{R}^2$. Then the distance between x and y is given by the equation

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

7 Operations on Points

7.1 Dilations and Reflections

Definition 7.1 A dilation of a point A is defined as $cA = (ca_1, ca_2)$, where c is a scalar.

Definition 7.2 A reflection of a point A is defined as $R(A) = -A = (-a_1, -a_2)$.

Theorem 7.3 Let r be a positive number. If A, B are points, then

$$d(rA, rB) = r \cdot d(A, B).$$

Theorem 7.4 Let c be a number. Then

$$d(cA, cB) = |c| \cdot d(A, B).$$

7.2 Addition, Subtraction, and the Parallelogram Law

Definition 7.5 $A + B = (a_1 + b_1, a_2 + b_2)$.

Axiom 7.6 (Commutativity) $A + B = B + A$.

Axiom 7.7 (Associativity) $A + (B + C) = (A + B) + C$.

Axiom 7.8 (Zero Element) Let $O = (0, 0)$. Then $A + O = O + A = A$.

Axiom 7.9 (Additive Inverse) If $A = (a_1, a_2)$ then the point $-A = (-a_1, -a_2)$ is such that $A + (-A) = O$.

Definition 7.10 The *norm* of a point A is defined as $|A| = \sqrt{a_1^2 + a_2^2}$. This value represents the distance between A and the origin.

Axiom 7.11 (Point-Scalar Associativity) If $b.c$ are numbers, then $b(cA) = (bc)A$.

Axiom 7.12 (Point-Scalar Distributivity) If b, c are numbers, and A, B are points, then

$$(b + c)A = bA + cA \quad \text{and} \quad c(A + B) = cA + cB.$$

Axiom 7.13 (Identity) $1A = A$.

Axiom 7.14 (Zero Property) $0A = O$.

Theorem 7.15 $d(A, B) = |A - B| = |B - A|$.

Theorem 7.16 $|cA| = |c||A|$.

Theorem 7.17 The circle of radius r and center A is the translation by A of the circle of radius r and center O.

8 Segments, Rays, and Lines

8.1 Segments

Definition 8.1 A *line segment* is the set of points satisfying the equation $LS = P + t(Q - P)$ where $0 \leq t \leq 1$. An alternate formulation is $LS = sP + (1 - s)Q$ where $0 \leq s \leq 1$.

8.2 Rays

Definition 8.2 A *ray* is the set of points satisfying the equation $R = P + tA$ where $t \geq 0$. An alternate formulation is $R = P + t(Q - P)$ where $t \geq 0$.

8.3 Lines

Definition 8.3 A *line* is the set of points satisfying the equation $L = P + tA$ for all t.

Axiom 8.4 Euclid's postulates for parallel lines are as follows:

- Two lines which are not parallel meet in exactly one point.
- Given a line L and a point P, there exists a unique line through P parallel to L.
- If L_1, L_2, and L_3 are lines, if L_1 is parallel to L_2 and L_2 is parallel to L_3, then L_1 is parallel to L_3.

Axiom 8.5 Euclid's postulates for perpendicular lines are as follows:

- Given a line L and a point P, there exists a unique line through P perpendicular to L.
- If L_1 is perpendicular to L_2 and L_2 is parallel to L_3, then L_1 is perpendicular to L_3.
- If L_1 is perpendicular to L_2 and L_2 is perpendicular to L_3, then L_1 is parallel to L_3.

Definition 8.6 The *standard form* of the line is $ax + by = c$ where a, b, c are constants and x, y correspond to the set of all points $P = (x, y)$ that satisfy the equation.

Definition 8.7 Suppose points P and Q lie on line L_1 and points M and N lie on line L_2. Then lines L_1 and L_2 are parallel if and only if

$$Q - P = c(M - N).$$

9 Trigonometry

9.1 Radians

Definition 9.1 *Radians* are a measure of what percent of a circle's circumference an angle corresponds to. Radians are represented by $\frac{x}{2\pi}$.

Definition 9.2 Suppose x is an angle measured in degrees. Then

$$x \text{ degrees} = \frac{\pi}{180} x \text{ radians.}$$

Definition 9.3 Any angle expressed in terms of radians has an infinite number of equivalent forms, given by the equation

$$x = 2\pi n + w, \quad 0 \leq w < 2\pi.$$

9.2 Trigonometric Functions

Definition 9.4

$$\text{sine} x = \frac{b}{r} = \frac{b}{\sqrt{a^2 + b^2}}$$

Definition 9.5

$$\text{cosine} x = \frac{a}{r} = \frac{b}{\sqrt{a^2 + b^2}}$$

Definition 9.14

$$\tan x = \frac{\sin x}{\cos x}$$

whenever x is a number which is not of the form

$$\frac{\pi}{2} + n\pi$$

and n is an integer.

9.3 Reciprocal identities

Theorem 9.10 $\csc x = \frac{1}{\sin x}$.

Theorem 9.11 $\sec x = \frac{1}{cos x}$.

Theorem 9.12 $\cot x = \frac{1}{\tan x}$.

9.4 Periodicity identities

Definition 9.6 Given any angle x, we have

- $\sin x = \sin(x \pm 2n\pi)$
- $\cos x = \cos(x \pm 2n\pi)$
- $\tan x = \tan(x \pm \pi)$
- $\cot x = \cot(x \pm \pi)$
- $\sec x = \sec(x \pm 2\pi)$
- $\csc x = \csc(x \pm 2\pi)$.

9.5 Co-function identities

Theorem 9.8 For any number x, we have

- $\cos x = \sin(\frac{\pi}{2} - x)$
- $\sin x = \cos(\frac{\pi}{2} - x)$
- $\cot x = \tan(\frac{\pi}{2} - x)$
- $\tan x = \cot(\frac{\pi}{2} - x)$
- $\sec x = \csc(\frac{\pi}{2} - x)$
- $\csc x = \csc(\frac{\pi}{2} - x)$.

9.6 Pythagorean identities

Theorem 9.7 $\sin^2 x + \cos^2 x = 1$.

Theorem 9.8 $\tan^2\theta + 1 = \sec^2\theta$.

Theorem 9.9 $1 + \cot^2\theta = \csc^2\theta$.

Theorem 9.17 $\cos^2 x = \frac{1+\cos 2x}{2}$.

Theorem 9.18 $\sin^2 x = \frac{1-\cos 2x}{2}$.

9.7 Even-odd identities

Theorem 9.9 For all numbers x, we have

$$\sin(-x) = -\sin x \quad \cos(-x) = \cos x \quad \tan(-x) = -\tan x.$$

9.8 Double Angle Formulas

Theorem 9.15 $\sin 2x = 2\sin x\cos x$.

Theorem 9.16 $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$.

Theorem 9.17 $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

9.9 Sum and Difference Formulas

Theorem 9.19 For any angles x, y we have

- $\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$
- $\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$
- $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$.

9.10 Product to sum formulas

Theorem 9.20 For any angles x, y we have

- $\sin x\sin y = \frac{1}{2}\left[\cos(x-y) - (x+y)\right]$
- $\cos x\cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$
- $\sin x\cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$
- $\cos x\sin y = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right]$

9.11 Half-angle Formulas

Theorem 9.21 For any angles x, y we have

- $\sin(\frac{x}{2}) = \pm\sqrt{\frac{1-\cos x}{2}}$
- $\cos(\frac{x}{2}) = \pm\sqrt{\frac{1+\cos x}{2}}$
- $\tan(\frac{x}{2}) = \frac{1-\cos x}{\sin x}$

9.12 Sum to Product

Theorem 9.22 For any angles x, y we have

- $\sin x \pm \sin y = 2\sin\left(\frac{x\pm y}{2}\right)\cos\left(\frac{x\mp y}{2}\right)$
- $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
- $\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

9.13 Law of Sines

Theorem 9.23 Given a triangle with angles A, B, C and sides of length a, b, c we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

9.14 Law of Cosines

Theorem 9.24 Given a triangle with angles A, B, C and sides of length a, b, c we have

$$a^2 = b^2 + c^2 - 2bc\cos A$$

9.15 Polar Coordinates

Definition 9.10 The *Polar Coordinate System* uses points of the form (r, θ) where r is a distance from the origin and θ is the angle from the positive x-axis.

Definition 9.11 The conversion equations between Polar and Cartesian coordinates are as follows:

- $r = \sqrt{x^2 + y^2}$ where x and y are the coordinates of a point in $\mathbb{R}^2$.
- $x = r\cos\theta$ and $y = r\sin\theta$.

Definition 9.12 The length of an arc of a circle is given by the equation $s = r\theta$.

Definition 9.13 The area of a sector of a circle is given by the equation $A = \frac{1}{2}r^2\theta$.

10 Analytic Geometry

10.1 Lines

Definition 10.1 The *slope-intercept form* of a line is $y = mx + b$, where y is the second member of a point, m indicates the slope of the line, x is the first member of a point, and b is the y-intercept.

Definition 10.2 The slope of a line can be determined by the equation

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

where x and y are two points on the line and $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Definition 10.3 The *point-slope* form of a line is given by the equation $y - y_1 = m(x - x_1)$, where y is the second member of any point on the line, y_1 is the second member of a particular point on the line, m is the slope of the line, x is the first member corresponding the same point as y, and x_1 is the first member corresponding to the same point as y_1.

10.2 Circle, Parabola, Ellipse, Hyperbola

Definition 10.4 The equation for a *circle* is $(x - a)^2 + (y - b)^2 = r^2$, where a, b are constants, r is the radius of the circle, and x, y correspond to the first and second members of points (respectively) on the circle.
Definition 10.5 The equation for a *parabola* is $(y - b) = c(x - a)^2$, where a, b, c are constants, and x, y correspond to the first and second members of points (respectively) on the parabola.

Definition 10.6 The equation for an *ellipse* is $\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$ where $u = ax$ and $v = by$, and a, b are constants and x, y correspond to the first and second members of points (respectively) on the ellipse.

Definition 10.7 The equation for a *hyperbola* is $(y - a) = \frac{1}{x-b}$ where a, b are constants and x, y correspond to the first and second members of points (respectively) on the hyperbola).

11 Complex Numbers

Definition 11.1 A *complex number* is an ordered pair of real numbers (a, b) that can be written in the form $a + bi$, where i is such that $i^2 = -1$.

Axiom 11.2 The properties of complex numbers are as follows:

- Addition is commutative and associative.
- Multiplication is commutative, associative, and distributive with respect to addition.
- Every real number is a complex number, if a, b are real numbers, then their sum and product as complex numbers are the same as their sum and product as real numbers, respectively.
- Similarly, $0z = 0$.
- Each complex number z has an additive inverse, namely $(-1)z$, so that $z + (-1)z = 0$.

Definition 11.3 Complex coordinates have the form $z = (x, y)$ where x, y are real numbers.

Definition 11.4 If z, w are complex coordinate points and $z = (x, y), w = (u, v)$, then $z + w = (x + u, y + v)$.

Definition 11.5 If $z = (x, y)$ is a complex coordinate and c is a scalar, then $cz = (cx, cy)$.

Definition 11.6 If $z = (x, y), w = (u, v)$ are complex coordinates, then $zw = (xu - yv, xv + yu)$.

Definition 11.7 The complex conjugate of a complex number $z = a+bi$ is denoted $\bar{z}$ and is defined $\bar{z} = a - bi$.

Definition 11.8 $z^{-1} = \frac{\bar{z}}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$.

Definition 11.9 $|z| = \sqrt{x^2 + y^2}$.

Theorem 11.10 $z\bar{z} = a^2 + b^2$.

Theorem 11.11 $zz^{-1} = z^{-1}z = 1$.

Theorem 11.12 $z + \bar{z} = 2x = 2Re(z)$.

Theorem 11.13 $z - \bar{z} = 2y = 2Im(z)$.

Theorem 11.14 Let z, w be complex numbers. Then

$$\overline{zw} = \bar{z}\bar{w}, \quad \overline{z + w} = \bar{z} + \bar{w}, \quad \bar{\bar{z}} = z.$$

Theorem 11.15 $z\bar{z} = |z|^2$.

Theorem 11.16 The absolute value of complex numbers satisfies the following properties. If z, w are complex numbers, then $|zw| = |z||w|$ and $|z + w| \leq |z| + |w|$.

12 Functions

12.1 Definition of a Function

Definition 12.1 Let f, g, h be functions, then:

- $(f + g)(x) = f(x) + g(x)$
- $f + g = g + f$
- $f + 0 = 0 + f = f$
- $f + (-f) = 0$
- $(fg)(x) = f(x)g(x)$
- $(f + g)h = fh + gh$

Definition 12.2 A function f is called *even* if and only if $f(x) = f(-x)$.

Definition 12.3 A function f is called *odd* if and only if $f(x) = -f(-x)$.

12.2 Polynomial Functions

Definition 12.4 A polynomial function is a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.

Theorem 12.5 Let f be a polynomial of degree $\leq n$ and let c be a root. Then there exists a polynomial g of degree $\leq n-1$ such that for all numbers x we have

$$f(x) = (x-c)g(x).$$

Theorem 12.6 Let f be a polynomial. Let $a_0, \ldots, a_n$ be numbers such that $a_n \neq 0$, and such that we have

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

for all x. Then f has at most n roots.

Theorem 12.7 Let f be a polynomial, which can be written in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

and also in the form

$$f(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0.$$

Then

$$a_i = b_i \quad \text{for every} i = 0, \ldots, n.$$

Theorem 12.8 (Euclidean algorithm) Let f and g be non-zero polynomials. Then there exist polynomials q, r such that deg$r <$deg$g$ and such that

$$f(x) = q(x)g(x) + r.$$

12.3 Logarithms

Definition 12.9 Let a be a number > 1. If $y = a^x$, then we shall say that x is the log of y to the base a, and write $x = \log_a y$. Given a number $y > 0$,there exists a number x such that $a^x = y$.

Theorem 12.9 For any numbers x, y, we have

$$\log_a(xy) = \log_a x + \log_a y.$$

Theorem 12.10 We have $\log_a 1 = 0$.

Theorem 12.11 If $x < y$, then $\log_a x < \log_a y$.

Theorem 12.12 $\log a^x = x \log a$.

12.4 Natural Logarithm

Definition 12.13 The *natural logarithm* is the logarithm with base $e : \log_e x$.

Theorem 12.13 The following are the properties of natural logarithms:

- $\ln x_1 x_2 = \ln x_1 + \ln x_2$
- $\ln \frac{x_1}{x_2} = \ln x_1 - \ln x_2$.
- $\ln x^b = b \ln x$
- $e^{\ln x} = x$ and $\ln e^x = x$
- $\lim_{x \to 0+} \ln x = -\infty$ and $\lim_{x \to \infty} \ln x = \infty$.

13 Inversion

Definition 13.1 Let f be a function with domain A and range B. For each x in A, there is exactly one y in such that $y = f(x)$. Now, for each y in B, suppose that there is exactly one x in A such that $f(x) = y$. Then we can define a new function on B, called the inverse of f, such that $g(y) = x$ means $y = f(x)$. So the value of g at each point y in B is that unique x in A such that $f(x) = y$.

Definition 13.2 If f is a function, and f^{-1} is the inverse of f, the

- $f^{-1}(f(x)) = x$
- $f^{-1} \circ f(x) = x$
- $f(f^{-1}(x)) = x$
- $f \circ f^{-1}(x) = x$

14 Sequences and Series

Definition 14.1 An arithmetic sequence is a sequence of numbers that grows or decreases linearly.

Definition 14.2 The general term equation for arithmetic sequences is

$$a_n = a_1 + (n-1) * d$$

Definition 14.3 An arithmetic series is a series that increases or decreases linearly.

Definition 14.4 The partial sum equation for an arithmetic series is

$$S_n = (a_1 + a_n) * \left(\frac{n}{2}\right)$$

Definition 14.5 A geometric sequence is a sequence of numbers that grows or decreases multiplicatively.

Definition 14.6 The general term equation for geometric sequences is

$$a_n = a_1 * (r)^{n-1}$$

Definition 14.7 A geometric series is a series that grows or decreases multiplicatively.

Definition 14.8 The partial sum equation for for geometric series is

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

15 Graphing Functions

15.1 Right-Left Translation

Definition 15.1 Let $c > 0$. Start with the graph of some function $f(x)$. Keep the x-axis and y-axis fixed, but move the graph c units to the right, or c units to the left. You get the graph of two new functions:

1. Moving the $f(x)$ graph c units to the right gives the graph of $f(x - c)$.
2. Moving the $f(x)$ graph c units to the left gives the graph of $f(x + c)$.

Note 15.2 To see the reason for (15.1), suppose the graph of $f(x)$ is moved c units to the right: it becomes then the graph of a new function $g(x)$ whose relation to $f(x)$ is described by:

$$\text{value of } g(x) \text{ at } x_0 = \text{value of } f(x) \text{ at } x_0 - c = f(x_0 - c).$$

This shows that $g(x) = f(x - c)$.

15.2 Up-Down Translation

Definition 15.3 If $c > 0$,

1. Moving the $f(x)$ graph c units up gives the graph of $f(x) + c$.
2. Moving the $f(x)$ graph c units down gives the graph of $f(x) - c$.

This is because moving the graph by, for example, c units up has the effect of adding c units to each function value, and therefore gives us the graph of the function $f(x) + c$.

15.3 Changing Scale: Stretching and Shrinking

Definition 15.4 Let $c > 1$. To stretch the x-axis by the factor c means to move the point 1 to the position formerly occupied by c, and in general, the point x_0 to the position formerly occupied by cx_0. Similarly, to shrink the x-axis by the factor c means to move x_0 to the position previously taken by $\frac{x_0}{c}$.

1. Stretching the x-axis by c changes the graph of $f(x)$ into that of $f(\frac{x}{c})$.
2. Shrinking the x-axis by c changes the graph of $f(x)$ into that of $f(cx)$.

When we horizontally shrink a function, the new function has the same value at x_0 that $f(x)$ has at $\frac{x_0}{c}$, so that it is given by the rule $x_0 \to f(\frac{x_0}{c})$, which means that it is the function $f(\frac{x}{c})$.

Definition 15.5 If the y-axis is stretched by the factor $c > 1$, each y-value is multiplied by c, so the new graph is that of the function $cf(x)$:

1. Stretching the y-axis by c changes the graph of $f(x)$ into that of $cf(x)$.
2. Shrinking the y-axis by c changes the graph of $f(x)$ into that of $\frac{f(x)}{c}$.

15.4 Reflecting in the x- and y-axes: Even and Odd Functions

Definition 15.6 To reflect the graph of $f(x)$ in the y-axis, just flip the plane over the y-axis. This carries the point (x, y) into the point $(-x, y)$, and the graph of $f(x)$ into the graph of $f(-x)$. Namely, the new function has the same y-value at x_0 as $f(x)$ has at $-x_0$, so it is given by the rule $x_0 \to f(-x_0)$ and is the function $f(-x)$.

Definition 15.7 Reflecting the xy-plane in the x-axis carries (x, y) to the point $(x, -y)$ and the graph of $f(x)$ gets carried into that of $-f(x)$.

Definition 15.8 Reflecting first in the y-axis and then in the x-axis carries the point (x, y) into the point $(-x, -y)$. This is called a reflection through the origin. The graph of $f(x)$ gets carried into the graph of $-f(-x)$, by combining the results from (15.6) and (15.7).

Theorem 15.9 The follwoing rules predict the odd- or even-ness of the product or quotient of two odd or even functions:

- even $\cdot$ even $=$ odd
- even/even $=$ even
- odd $\cdot$ odd $=$ even
- odd/odd $=$ even
- odd $\cdot$ even $=$ odd
- odd/even $=$ odd

15.5 Periodic Functions

Definition 15.10 In general, let $c > 0$; we say that $f(x)$ is *periodic*, with period c, if

1. $f(x + c) = f(x)$ for all x, and
2. c is the smallest positive number for which 1. is true.

15.6 General Sinusoidal Wave

Definition 15.11 The general sinusoidal wave is defined as

$$A \sin k(x - \phi), \quad A, k > 0, \phi \geq 0$$

which has

- period $\frac{2\pi}{k}$ (the wave repeats itself every $\frac{2\pi}{k}$ units)
- angular frequency k (has k complete cycles as x goes from 0 to 2π)
- amplitude A (the wave oscillates between A and $-A$
- phase angle ϕ (the midpoint of the wave is at $x = \phi$)

15.7 Reflection in the Diagonal Line

Definition 15.12 Reflection across the diagonal $y = x$ can be carried out by keeping each point of the function on the diagonal fixed, and the x- and y-axes are interchanged: each (x, y) becomes (y, x).

Note 15.13 Note that we have to keep the domain restricted when reflecting a function across the diagonal to ensure that the function remains a function (i.e. so that there is no value x such that there exists two distinct points $(x, y) \neq (x, y')$).

A The Decimal Numeral System

A **numeral system** is an organized way to write and manipulate this type of symbol, for example the Hindu–Arabic numeral system allows combinations of numerical digits like "5" and "0" to represent larger numbers like 50. In addition to their use in counting and measuring, numerals are often used for labels (as with telephone numbers), for ordering (as with serial numbers), and for codes (as with ISBNs).

The **decimal numeral system** is the standard system for denoting integer and non-integer numbers.

For writing numbers, the decimal system uses ten decimal digits, a decimal mark, and, for negative numbers, a minus sign "–". The decimal digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; the decimal separator is the dot "."

For representing a non-negative number, a decimal consists of:
i. either a (finite) sequence of digits such as 2017, or, in full generality,

$$a_m a_{m-1} \ldots a_0 \tag{1}$$

ii. or two sequences of digits separated by a decimal mark such as 3.14159, 15.00, or, in full generality,

$$a_m a_{m-1} \ldots a_0 . b_1 b_2 \ldots b_n \tag{2}$$

The numeral $a_m a_{m-1} \ldots a_0 . b_1 b_2 \ldots b_n$ represents the number

$$a_m 10^m + a_{m-1} 10^{m-1} + \cdots + a_0 10^0 + \frac{b_1}{10^1} + \frac{b_2}{10^2} + \cdots + \frac{b_n}{10^n} \tag{3}$$

Therefore, the contribution of each digit to the value of a number depends on its position in the numeral. That is, the decimal system is a **positional numeral system**. For representing a negative number, a minus sign is placed before a_m.

The **integer part** of a decimal is the integer written to the left of the decimal separator. For a non-negative decimal, it is the largest integer that is not greater than the decimal. The part from the decimal separator to the right is the **fractional part**, which equals the difference between the numeral and its integer part.

When the integral part of a numeral is zero, it may occur, typically in computing, that the integer part is not written (for example .1234, instead of 0.1234). In normal writing, this is generally avoided because of the risk of confusion between the decimal mark and other punctuation.

It is generally assumed that, if $m > 0$, the first digit a_m is not zero, but, in some circumstances, it may be useful to have one or more 0's on the left. This does not change the value represented by the decimal. For example, $3.14 = 03.14 = 003.14$. Similarly, if $b_n = 0$, it may be removed, and conversely, trailing zeros may be added without changing the represented number: for example, 15

$= 15.0 = 15.00$ and $5.2 = 5.20 = 5.200$. Sometimes the extra zeros are used for indicating the accuracy of a measurement. For example, 15.00 meters may indicate that the measurement error is less than one centimeter (0.01 meters), while 15 meters may mean that the length is roughly fifteen meters, and that the error may exceed 10 centimeters.

The numbers that may be represented in the decimal system are the **decimal fractions**, that is the fractions of the form $\frac{a}{10^n}$, where a is an integer, and n is a non-negative integer. For example, the numerals 0.8, 14.89, 0.00024 represent the fractions $\frac{8}{10}$, $\frac{1489}{100}$, and $\frac{24}{100000}$. More generally, a decimal with n digits after the separator represents the fraction with denominator 10^n, whose numerator is the integer obtained by removing the separator.

Decimal numerals do not allow an exact representation for all real numbers, e.g. for the real number π. Nevertheless, they allow approximating every real number with any desired accuracy, e.g., the decimal 3.14159 approximates the real π, being less than 10^{-5} off; and so decimals are widely used in science, engineering and everyday life.

Long division allows computing the infinite decimal expansion of a rational number. If the rational number is a decimal fraction, the division stops eventually, producing a decimal numeral, which may be prolongated into an infinite expansion by adding infinitely many zeros. If the rational number is not a decimal fraction, the division may continue indefinitely. However, as all successive remainders are less than the divisor, there are only a finite number of possible remainders, and after some place, the same sequence of digits must be repeated indefinitely in the quotient. That is, one has a **repeating decimal**.

The decimal system has been extended to infinite decimals, for representing any real number, by using an infinite sequence of digits after the decimal separator. In this context, the decimal numerals with a finite number of non–zero places after the decimal separator are sometimes called terminating decimals. A repeating decimal is an infinite decimal that after some place repeats indefinitely the same sequence of digits (for example $5.123144144144144... = 5.123144$). An infinite decimal represents a rational number if and only if it is a repeating decimal or has a finite number of nonzero digits.

A.1 Fractions

A **fraction** (from Latin fractus, "broken") represents a part of a whole or, more generally, any number of equal parts.

In a fraction, the number of equal parts being described is the **numerator** (from Latin numerātor, "counter" or "numberer"), and the type or variety of the parts is the **denominator** (from Latin dēnōminātor, "thing that names or designates"). As an example, the fraction 8/5 amounts to eight parts, each of which is of the type named "fifth". In terms of division, the numerator corresponds to the **dividend**, and the denominator corresponds to the **divisor**.

Uses for fractions include representing **ratios** and **division**. Thus the fraction 3/4 is also used to represent the ratio 3 : 4 (the ratio of the part to the whole) and the division $3 \div 4$ (three divided by four). Fractions can be used to represent division because division is an operation that can turn a number represented by a fraction into a decimal numeral; the fraction and decimal numeral are exactly equivalent, just represented in different ways. The non-zero denominator in the case using a fraction to represent division is an example of the rule that division by zero is undefined.

We can also write negative fractions, which represent the opposite of a positive fraction. For example if 1/2 represents a half dollar profit, then –1/2 represents a half dollar loss. Because of the rules of division of signed numbers, which require that, for example, negative divided by positive is negative, –(1/2), –1/2 and 1/–2, all represent the same fraction, negative one-half. Because a negative divided by another negative produces a positive, –1/–2 represents positive one-half.

A.1.1 Common Fractions

A **simple fraction** (also known as a common fraction or vulgar fraction) is a rational number written as a/b where a and b are both integers. As with other fractions, the denominator (b) cannot be zero. Examples include 1/2, –8/5, 8/–5, – (8/5), and 3/17. Simple fractions can be positive or negative, and they can be proper or improper (see below). Mixed numerals and decimals (see below) are not simple fractions, though, unless irrational, they can be evaluated to a simple fraction.

A.1.2 Proper and Improper Fractions

Common fractions can be classified as either **proper** or **improper**. When the numerator and the denominator are both positive, the fraction is called proper if the numerator is less than the denominator, and improper otherwise. In general, a common fraction is said to be a proper fraction if the absolute value of the fraction is strictly less than one—that is, if the fraction is greater than –1 and less than 1. It is said to be an improper fraction, or sometimes top-heavy fraction, if the absolute value of the fraction is greater than or equal to 1. Examples of proper fractions are 2/3, –3/4, and 4/9; examples of improper fractions are 9/4, –4/3, and 3/3.

A.1.3 Reciprocals

The **reciprocal** of a fraction is another fraction with the numerator and denominator exchanged. The reciprocal of 3/7, for instance, is 7/3. The product of a fraction and its reciprocal is 1, hence the reciprocal is the **multiplicative inverse** of a fraction. The reciprocal of a proper fraction is improper, and the reciprocal of an improper fraction not equal to 1, that is, numerator and denominator are not equal, is a proper fraction.

When the numerator and denominator of a fraction are equal (7/7, for example), its value is 1, and the fraction therefore is improper. Its reciprocal also has the value 1, and is improper, too.

Any integer can be written as a fraction with the number one as denominator. For example, 17 can be written as 17/1, where 1 is sometimes referred to as the **invisible denominator**. Therefore, every fraction or integer, except for zero, has a reciprocal.

A.1.4 Ratios

A **ratio** is a relationship between two or more numbers that can be sometimes expressed as a fraction. Typically, a number of items are grouped and compared in a ratio, specifying numerically the relationship between each group. Ratios are expressed as "group 1 to group 2 ... to group n". For example, if a car lot had 12 vehicles, of which 2 are white, 6 are red, and 4 are yellow, then the ratio of red to white to yellow cars is 6 to 2 to 4. The ratio of yellow cars to white cars is 4 to 2 and may be expressed as 4:2 or 2:1.

A ratio is often converted to a fraction when it is expressed as a ratio to the whole. In the above example, the ratio of yellow cars to all the cars on the lot is 4:12 or 1:3. We can convert these ratios to a fraction and say that 4/12 of the cars or 1/3 of the cars in the lot are yellow. Therefore, if a person randomly chose one car on the lot, then there is a one in three chance or probability that it would be yellow.

In general, a comparison of the quantities of a two-entity ratio can be expressed as a fraction derived from the ratio. For example, in a ratio of 2:3, the amount, size, volume, or quantity of the first entity is 2/3 that of the second entity.

If there are 2 oranges and 3 apples, the ratio of oranges to apples is 2:3, and the ratio of oranges to the total number of pieces of fruit is 2:5. These ratios can also be expressed in fraction form: there are 2/3 as many oranges as apples, and 2/5 of the pieces of fruit are oranges.

A.1.5 Decimals, fractions, and percentages

A **decimal fraction** is a fraction whose denominator is not given explicitly, but is understood to be an integer power of ten. Decimal fractions are commonly expressed using decimal notation in which the implied denominator is determined by the number of digits to the right of a decimal separator. Thus for 0.75, the numerator is 75 and the implied denominator is 10 to the second power, *viz.* 100, because there are two digits to the right of the decimal separator. In decimal numbers greater than 1 (such as 3.75), the fractional part of the number is expressed by the digits to the right of the decimal (with a value of 0.75 in this case). 3.75 can be written either as an improper fraction, 375/100, or as a mixed number, $3\frac{75}{100}$.

Decimal fractions can also be expressed using **scientific notation** with negative exponents, such as 6.023×10^{-7}, which represents 0.0000006023. The 10^{-7} represents a denominator of 10^7. Dividing by 10^7 moves the decimal point 7 places to the left.

Decimal fractions with infinitely many digits to the right of the decimal separator represent an **infinite series**. For example, $1/3 = 0.333...$ represents the infinite series 3/10 + 3/100 + 3/1000 + ...

Another kind of fraction is the **percentage** (Latin per centum meaning "per hundred", represented by the symbol %), in which the implied denominator is always 100. Thus, 51% means 51/100. Percentages greater than 100 or less than zero are treated in the same way, e.g. 311% equals 311/100, and –27% equals –27/100.

Whether common fractions or decimal fractions are used is often a matter of taste and context. Common fractions are used most often when the denominator is relatively small. By mental calculation, it is easier to multiply 16 by 3/16 than to do the same calculation using the fraction's decimal equivalent (0.1875). And it is more accurate to multiply 15 by 1/3, for example, than it is to multiply 15 by any decimal approximation of one third. Monetary values are commonly expressed as decimal fractions with denominator 100, i.e., with two decimals, for example $3.75.

A.1.6 Mixed Numerals

A **mixed numeral** (also called a mixed fraction or mixed number) is a traditional denotation of the sum of a non-zero integer and a proper fraction (having the same sign). It is used primarily in measurement: $2\frac{3}{16}$ inches, for example. The sum is implied without the use of a visible operator such as the appropriate "+". For example, in referring to two entire cakes and three quarters of another cake, the numerals denoting the integer part and the fractional part of the cakes are written next to each other as $2\frac{3}{4}$ instead of the unambiguous notation $2 + 3/4$. Negative mixed numerals, as in $-2\frac{3}{4}$, are treated like $-(2 + 3/4)$ Any such sum of a whole plus a part can be converted to an improper fraction by applying the rules of adding unlike quantities.

An improper fraction can be converted to a mixed number as follows:

1. Divide the numerator by the denominator. In the example, 11/4, divide 11 by 4. $11 \div 4 = 2$ with remainder 3.
2. The quotient (without the remainder) becomes the whole number part of the mixed number. The remainder becomes the numerator of the fractional part. In the example, 2 is the whole number part and 3 is the numerator of the fractional part.

3. The new denominator is the same as the denominator of the improper fraction. In the example, they are both 4. Thus $11/4 = 2\frac{3}{4}$.

A.1.7 Converting Repeating Decimals to Fractions

Given a repeating decimal, it is possible to calculate the fraction that produced it. For example:

$$\begin{aligned}
x &= 0.333333\ldots \\
10x &= 3.33333\ldots && \text{(multiplying each side of the above line by 10)} \\
9x &= 3 && \text{(subtracting the 1st line from the second)} \\
x &= 3/9 = 1/3 && \text{(reducing to lowest terms)}
\end{aligned}$$

Another example:

$$\begin{aligned}
x &= 0.836363636\ldots \\
10x &= 8.3636363636\ldots && \text{(multiplying by 10 to move decimal to start of repetition)} \\
1000x &= 836.36363636\ldots && \text{(multiplying by 100 to move decimal to end of first repeating decimal)} \\
990x &= 828 && \text{(subtracting to clear decimals)} \\
x &= \frac{828}{990}
\end{aligned}$$

A.2 Rounding

Rounding a number means replacing it with a different number that is approximately equal to the original, but has a shorter, simpler, or more explicit representation; for example, replacing \$23.4476 with \$23.45, or the fraction 312/937 with 1/3, or the expression $\sqrt{2}$ with 1.414.

Rounding is often done to obtain a value that is easier to report and communicate than the original. Rounding can also be important to avoid misleadingly precise reporting of a computed number, measurement or estimate; for example, a quantity that was computed as 123,456 but is known to be accurate only to within a few hundred units is usually better stated as "about 123,500".

On the other hand, rounding of exact numbers will introduce some round-off error in the reported result. Rounding is almost unavoidable when reporting many computations – especially when dividing two numbers in integer or fixed-point arithmetic; when computing mathematical functions such as square roots, logarithms, and sines; or when using a floating-point representation with a fixed number of significant digits. In a sequence of calculations, these rounding errors generally accumulate, and in certain ill-conditioned cases they may make the result meaningless.

A **wavy equals sign** ($\approx$: approximately equal to) is sometimes used to indicate rounding of exact numbers, e.g., $9.98 \approx 10$. This sign was introduced by ***Alfred George Greenhill*** in 1892.

Ideal characteristics of rounding methods include:

1. Rounding should be done by a function. This way, when the same input is rounded in different instances, the output is unchanged.

2. Calculations done with rounding should be close to those done without rounding.

    - As a result of (1) and (2), the output from rounding should be close to its input, often as close as possible by some metric.

3. To be considered rounding, the range will be a subset of the domain. A classical range is the integers, **Z**.
4. Rounding should preserve symmetries that already exist between the domain and range. With finite precision (or a discrete domain), this translates to removing bias.
5. A rounding method should have utility in computer science or human arithmetic where finite precision is used, and speed is a consideration.

But, because it is not usually possible for a method to satisfy all ideal characteristics, many methods exist.

A.3 Significant Figures

The **significant figures** (also known as the significant digits) of a number are digits that carry meaning contributing to its measurement resolution. This includes all digits except:

- All leading zeros. For example "013" has 2 significant figures: 1 and 3;
- Trailing zeros when they are merely placeholders to indicate the scale of the number (exact rules are explained at identifying significant figures); and
- Spurious digits introduced, for example, by calculations carried out to greater precision than that of the original data, or measurements reported to a greater precision than the equipment supports.

A.3.1 Rules For Significant Figures

- All non-zero digits are significant: 1, 2, 3, 4, 5, 6, 7, 8, 9.
- Zeros between non-zero digits are significant: 102, 2005, 50009.
- Leading zeros are never significant: 0.02, 001.887, 0.000515. (0.02 has one significant figure)
- In a number with or without a decimal point, trailing zeros (those to the right of the last non-zero digit) are significant provided they are justified by the precision of their derivation: 389,000; 2.02000; 5.400; 57.5400. More information through additional graphical symbols or explicit information on errors is needed to clarify the significance of trailing zeros.

A.3.2 Scientific Notation

In most cases, the same rules apply to numbers expressed in scientific notation. However, in the normalized form of that notation, placeholder leading and trailing digits do not occur, so all digits are significant. For example, 0.00012 (two significant figures) becomes 1.2×10^{-4}, and 0.00122300 (six significant figures) becomes 1.22300×10^{-3}. In particular, the potential ambiguity about the significance of trailing zeros is eliminated. For example, 1300 to four significant figures is written as 1.300×10^{-3}, while 1300 to two significant figures is written as 1.3×10^{-3}.

The part of the representation that contains the significant figures (as opposed to the base or the exponent) is known as the **significand** or **mantissa**.

A.3.3 Rounding and Decimal Places

The basic concept of significant figures is often used in connection with rounding. Rounding to significant figures is a more general-purpose technique than rounding to n decimal places, since it handles numbers of different scales in a uniform way. For example, the population of a city might only be known to the nearest thousand and be stated as 52,000, while the population of a country might only be known to the nearest million and be stated as 52,000,000. The former might be in error by hundreds, and the latter might be in error by hundreds of thousands, but both have two significant figures (5 and 2). This reflects the fact that the significance of the error is the same in both cases, relative to the size of the quantity being measured.

To round to n significant figures:

- Identify the significant figures before rounding. These are the n consecutive digits beginning with the first non-zero digit.
- If the digit immediately to the right of the last significant figure is greater than 5 or is a 5 followed by other non-zero digits, add 1 to the last significant figure. For example, 1.2459 as the result of a calculation or measurement that only allows for 3 significant figures should be written 1.25.
- If the digit immediately to the right of the last significant figure is a 5 not followed by any other digits or followed only by zeros, rounding requires a tie-breaking rule. For example, to round 1.25 to 2 significant figures:
  - Round **half away from zero** (also known as "5/4") rounds up to 1.3. This is the default rounding method implied in many disciplines if not specified.
  - Round **half to even**, which rounds to the nearest even number, rounds down to 1.2 in this case. The same strategy applied to 1.35 would instead round up to 1.4.
- Replace non-significant figures in front of the decimal point by zeros.
- Drop all the digits after the decimal point to the right of the significant figures (do not replace them with zeros).

A.3.4 Arithmetic

As there are rules for determining the number of significant figures in directly *measured quantities*, there are rules for determining the number of significant figures in quantities *calculated* from these measured quantities.

Only measured quantities figure into the determination of the number of significant figures in calculated quantities. Exact mathematical quantities like the π in the formula for the area of a circle with radius r, πr^2 has no effect on the number of significant figures in the final calculated area. Similarly the $1/2$ in the formula for the kinetic energy of a mass m with velocity v, $1/2mv^2$, has no bearing on the number of significant figures in the final calculated kinetic energy. The constants π and $1/2$ are considered for this purpose to have an infinite number of significant figures.

For quantities created from measured quantities by multiplication and division, the calculated result should have as many significant figures as the measured number with the least number of significant figures. For example,

$$1.234 \times 2.0 = 2.468\ldots \approx 2.5, \tag{4}$$

with only two significant figures. The first factor has four significant figures and the second has two significant figures. The factor with the least number of significant figures is the second one with only two, so the final calculated result should also have a total of two significant figures. However see below regarding intermediate results.

For quantities created from measured quantities by addition and subtraction, the last significant decimal place (hundreds, tens, ones, tenths, and so forth) in the calculated result should be the same as the leftmost or largest decimal place of the last significant figure out of all the measured quantities in the terms of the sum. For example,

$$100.0 + 1.234 = 101.234 \ldots \approx 101.2, \tag{5}$$

with the last significant figure in the tenths place. The first term has its last significant figure in the tenths place and the second term has its last significant figure in the thousandths place. The leftmost of the decimal places of the last significant figure out of all the terms of the sum is the tenths place from the first term, so the calculated result should also have its last significant figure in the tenths place.

A.3.5 Logs

Given a measurement inside of a logarithm (e.g $\log_{10}(7.310 \times 10^3)$), there are rules for extracting the significant figures from the logarithm.

1. Factor the number inside of the logarithm by the base of the logarithm.
2. The number of significant figures is determined by the factor that is not the base of the logarithm.

For example, $\log_{10}(7.310 \times 10^3)$ evaluates to 3.8639 since there are four significant figures in the mantissa inside of the logarithm. Note that $\log_{10}(7.310 \times 10^3) = \log_{10}(7.310) + \log_{10}(10^3) = 0.8639 + 3 = 3.8639$.

A.3.6 Exponents

Given a measurement inside an exponent, we can extract significant figures from the exponent (e.g. $10^{0.389}$).

1. Separate the whole number in the exponent from the fractional part of the exponent (e.g. $10^{12.389} = 10^{12} \cdot 10^{0.389}$).
2. The number of significant figures in the answer is determined by the number of significant figures in the fractional part of the exponent. (e.g. $10^{12} \cdot 10^{0.389} = 2.45 \times 10^{12}$ (3 sig figs).

B Number Systems

A **number** is a mathematical object (an object that can be formally defined, from which one can do mathematical proofs and use deductive reasoning) used to count, measure, and label. A written symbol like "5" that represents a number is called a **numeral**. Numbers can be classified into sets, called **number systems**, such as the natural numbers and the real numbers. The main categories are Natural, Integer, Rational, Real, and Complex.

B.1 Natural Numbers

The **natural numbers** are those used for counting (as in "there are six coins on the table") and ordering (as in "this is the third largest city in the country"). The natural numbers usually include 0. In common mathematical terminology, words colloquially used for counting are "**cardinal numbers**" and words connected to ordering represent "**ordinal numbers**."

The natural numbers are a basis from which many other number sets may be built by extension: the integers, by including the neutral element 0 and an additive inverse ($-n$) for each nonzero natural number n; the rational numbers, by including a multiplicative inverse ($1/n$) for each nonzero integer n (and also the product of these inverses by integers); the real numbers; the complex numbers, by including with the real numbers the unresolved square root of minus one (and also the sums and products thereof); and so on. These chains of extensions make the natural numbers canonically embedded (identified) in the other number systems.

B.1.1 Prime Numbers

A **prime number** is a natural number greater than 1 that cannot be formed by multiplying two smaller natural numbers. A natural number greater than 1 that is not prime is called a composite number. For example, 5 is prime because the only ways of writing it as a product, 1×5 or 5×1, involve 5 itself. However, 6 is composite because it is the product of two numbers (2×3) that are both smaller than 6. Primes are central in number theory because of the fundamental theorem of arithmetic: every natural number greater than 1 is either a prime itself or can be factorized as a product of primes that is unique up to their order.

B.2 Integers

An **integer** (from the Latin integer meaning "whole") is a number that can be written without a fractional component. For example, 21, 4, 0, and –2048 are integers, while 9.75, $5\frac{1}{2}$, and $\sqrt{2}$ are not. The set of integers consists of zero (0), the positive natural numbers (1, 2, 3, . . .), also called whole numbers or counting numbers, and their additive inverses (the negative integers, i.e., –1, –2, –3, . . .).

B.2.1 Negatives and Their Meaning

In mathematics, a **negative number** is a real number that is less than zero. Negative numbers represent opposites. If positive represents a movement to the right, negative represents a movement to the left. If positive represents above sea level, then negative represents below sea level. If positive represents a deposit, negative represents a withdrawal. They are often used to represent the magnitude of a loss or deficiency. A debt that is owed may be thought of as a negative asset, a decrease in some quantity may be thought of as a negative increase. If a quantity may have either of two opposite senses, then one may choose to distinguish between those senses—perhaps arbitrarily—as positive and negative. In the medical context of fighting a tumor, an expansion could be thought of as a negative shrinkage. Negative numbers are used to describe values on a scale that goes below zero, such as the Celsius and Fahrenheit scales for temperature. The laws of arithmetic for negative numbers ensure that the common sense idea of an opposite is reflected in arithmetic. For example, $-(-3) = 3$ because the opposite of an opposite is the original value. Zero is usually thought of as neither positive nor negative.

B.2.2 Absolute Value

In mathematics, the **absolute value** or **modulus** $|x|$ of a real number x is the non-negative value of x without regard to its sign. Namely, $|x| = x$ for a positive x, $|x| = -x$ for a negative x (in which case $-x$ is positive), and $|0| = 0$. For example, the absolute value of 3 is 3, and the absolute value of –3 is also 3. The absolute value of a number may be thought of as its distance from zero.

B.2.3 Parity

In mathematics, **parity** is the property of an integer's inclusion in one of two categories: even or odd. An integer is **even** if it is divisible by two and **odd** if it is not even. For example, 6 is even because there is no remainder when dividing it by 2. By contrast, 3, 5, 7, 21 leave a remainder of 1 when divided by 2. Examples of even numbers include –4, 0, 82 and 178. In particular, zero is an even number. Some examples of odd numbers are –5, 3, 29, and 73.

A formal definition of an even number is that it is an integer of the form $n = 2k$, where k is an integer; it can then be shown that an odd number is an integer of the form $n = 2k + 1$. It is important to realize that the above definition of parity applies only to integer numbers, hence it cannot be applied to numbers like 1/2 or 4.201.

A number (i.e., integer) expressed in the decimal numeral system is even or odd according to whether its last digit is even or odd. That is, if the last digit is 1, 3, 5, 7, or 9, then it is odd; otherwise it is even. The same idea will work using any even base.

B.2.4 Composite Numbers

A **composite number** is a positive integer that can be formed by multiplying two smaller positive integers. Equivalently, it is a positive integer that has at least one divisor other than 1 and itself. Every positive integer is composite, prime, or the unit 1, so the composite numbers are exactly the numbers that are not prime and not a unit.

For example, the integer 14 is a composite number because it is the product of the two smaller integers 2×7. Likewise, the integers 2 and 3 are not composite numbers because each of them can only be divided by one and itself.

Every composite number can be written as the product of two or more (not necessarily distinct) primes. For example, the composite number 299 can be written as 13×23, and the composite number 360 can be written as $2^3 \times 3^2 \times 5$; furthermore, this representation is unique up to the order of the factors. This fact is called the **fundamental theorem of arithmetic**.

B.3 Rational Numbers

In mathematics, a **rational number** is any number that can be expressed as the quotient or fraction (or ratio) p/q of two integers, a numerator p and a non-zero denominator q. Since q may be equal to 1, every integer is a rational number.

The decimal expansion of a rational number always either terminates after a finite number of digits or begins to repeat the same finite sequence of digits over and over. Moreover, any repeating or terminating decimal represents a rational number. These statements hold true not just for base 10, but also for any other integer base (e.g. binary, hexadecimal).

B.4 Irrational Numbers

In mathematics, the **irrational numbers** are all the real numbers which are not rational numbers, the latter being the numbers constructed from ratios (or fractions) of integers. Among irrational numbers are the ratio π of a circle's circumference to its diameter, Euler's number e, the golden ratio ϕ, and the square root of two; in fact all square roots of natural numbers, other than of perfect squares, are irrational.

It can be shown that irrational numbers, when expressed in a positional numeral system (e.g. as decimal numbers, or with any other natural basis), do not terminate, nor do they repeat, i.e., do not contain a subsequence of digits, the repetition of which makes up the tail of the representation. For example, the decimal representation of the number π starts with 3.14159, but no finite number of digits can represent π exactly, nor does it repeat.

B.5 Real Numbers

In mathematics, a **real number** is a value of a continuous quantity that can represent a distance along a line. The real numbers include all the rational numbers, such as the integer –5 and the fraction $\frac{4}{3}$, and all the irrational numbers, such as $\sqrt{2}$ (1.41421356..., the square root of 2, an irrational algebraic number). In addition to measuring distance, real numbers can be used to measure quantities such as time, mass, energy, velocity, and many more.

Real numbers can be thought of as points on an infinitely long line called the **real line** (also known as the number line), where the points corresponding to integers are equally spaced. Any real number can be determined by a possibly infinite decimal representation, such as that of 8.632, where each consecutive digit is measured in units one tenth the size of the previous one.

B.6 Complex Numbers

A **complex number** is a number that can be expressed in the form $a + bi$, where a and b are real numbers, and i is a solution of the equation $x^2 = -1$. Because no real number satisfies this equation, i is called an **imaginary number**. For the complex number $a + bi$, a is called the **real part**, and b is called the **imaginary part**. Despite the historical nomenclature "imaginary," complex numbers are regarded in the mathematical sciences as just as "real" as the real numbers, and are fundamental in many aspects of the scientific description of the natural world.

Formally, the **complex number system** can be defined as the algebraic extension of the ordinary real numbers by an imaginary number i. This means that complex numbers can be added, subtracted, and multiplied, as polynomials in the variable i, with the rule $i^2 = -1$ imposed. Furthermore, complex numbers can also be divided by nonzero complex numbers.

B.7 Infinity

It is easy to see that there is no largest natural number. Suppose there was one, call it L. Now add one to it, forming $M = L + 1$. We know that $L + 1 = M > L$, contradicting our assertion that L was the largest. This lack of a largest object, lack of a boundary, lack of termination in series, is of enormous importance in mathematics and physics. If there is no largest number, if there is no "edge" to space or time, then it in some sense they run on forever, without termination.

In spite of the fact that there is no actual largest natural number, we have learned that it is highly advantageous in many context to invent a pretend one. This pretend number doesn't actually exist as a number, but rather stands for a certain reasoning process. In fact, there are a number of properties of numbers that we can only understand or evaluate if we imagine a very large number used as a boundary or limit in some computation, and then let that number mentally increase without bound. Note well that this is a mental trick, no more, encoding the observation that there is no largest number and so we can increase a number parameter without bound. We call this unboundedness **infinity** – the lack of a finite boundary – and give it the symbol ∞ in mathematics.

C Arithmetic

Arithmetic is a branch of mathematics that consists of the study of numbers, especially the properties of the traditional operations on them—addition, subtraction, multiplication and division. Arithmetic is an elementary part of **number theory**, and number theory is considered to be one of the top-level divisions of modern mathematics, along with algebra, geometry, and analysis.

C.1 Addition

The **addition** of two whole numbers is the total amount of those values combined. For example, a combination of three apples and two apples together make a total of five apples. This observation is equivalent to the mathematical expression "3 + 2 = 5" i.e., "3 add 2 is equal to 5."

The numbers or the objects to be added in general addition are collectively referred to as the **terms** or the **addends**; this terminology carries over to the summation of multiple terms.

Besides counting items, addition can also be defined on other types of numbers, such as integers, real numbers and complex numbers. This is part of arithmetic, a branch of mathematics. In algebra, another area of mathematics, addition can be performed on abstract objects such as vectors and matrices.

C.1.1 Interpretations

There are two main interpretations of addition:

1. Combining Sets: Possibly the most fundamental interpretation of addition lies in combining sets: *When two or more disjoint collections are combined into a single collection, the number of objects in the single collection is the sum of the numbers of objects in the original collections.* This interpretation is easy to visualize, with little danger of ambiguity.

2. Extending a Length: A second interpretation of addition comes from extending an initial length by a given length: *When an original length is extended by a given amount, the final length is the sum of the original length and the length of the extension.* The sum $a+b$ can be interpreted as a binary operation that combines a and b, in an algebraic sense, or it can be interpreted as the addition of b more units to a. Under the latter interpretation, the parts of a sum $a+b$ play asymmetric roles, and the operation $a+b$ is viewed as applying the unary operation $+b$ to a.The unary view is also useful when discussing subtraction, because each unary addition operation has an inverse unary subtraction operation, and vice versa.

C.1.2 Properties

1. Commutativity: Addition is **commutative**: one can change the order of the terms in a sum, and the result is the same. Symbolically, if a and b are any two numbers, then

$$a+b=b+a \tag{6}$$

The fact that addition is commutative is known as the **commutative law of addition.**

2. Associativity: Addition is **associative**: when adding three or more numbers, the order of operations does not matter. For any three numbers a, b, and c, it is true that $(a + b) + c = a + (b + c)$. For example, $(1 + 2) + 3 = 3 + 3 = 6 = 1 + 5 = 1 + (2 + 3)$.

3. Identity Element: When adding zero to any number, the quantity does not change; zero is the **identity element** for addition, also known as the **additive identity**. In symbols, for any a,
$$a + 0 = 0 + a = a. \tag{7}$$

4. Successor: Within the context of integers, addition of one also plays a special role: for any integer a, the integer $(a+1)$ is the least integer greater than a, also known as the **successor** of a. For instance, 3 is the successor of 2 and 7 is the successor of 6. Because of this succession, the value of $a+b$ can also be seen as the b^{th} successor of a, making addition iterated succession. For example, 6 + 2 is 8, because 8 is the successor of 7, which is the successor of 6, making 8 the 2nd successor of 6.

5. Units: To numerically add physical quantities with units, they must be expressed with **common units**. For example, adding 50 milliliters to 150 milliliters gives 200 milliliters. However, if a measure of 5 feet is extended by 2 inches, the sum is 62 inches, since 60 inches is synonymous with 5 feet. On the other hand, it is usually meaningless to try to add 3 meters and 4 square meters, since those units are incomparable; this sort of consideration is fundamental in **dimensional analysis**.

C.1.3 Operations

Integer addition is performed by lining up the ones place value of two or more numbers and adding each column of numbers from right to left, producing a sum. This method is usually improved by the carry method. In elementary arithmetic, a **carry** is a digit that is transferred from one column of digits to another column of more significant digits. For example, when 6 and 7 are added to make 13, the "3" is written to the same column and the "1" is carried to the left. When used in subtraction the operation is called a **borrow**. This method works because each column is representative of a decimal place in base 10. When one carries, they make explicit the existence of another, more significant number resultant from earlier addition that must be added to the following decimal place.

Decimal addition is completed by lining up the decimals of two numbers vertically and completing addition regularly.

C.2 Subtraction

Subtraction is an arithmetic operation that represents the operation of removing objects from a collection. The result of a subtraction is called a **difference**. Subtraction is signified by the minus sign (–). For example, say we have 5 apples, and take away 2. This is represented by 5 – 2 apples—meaning 5 apples with 2 taken away, which is a total of 3 apples. Therefore, the difference of 5 and 2 is 3, that is, 5 – 2 = 3. Subtraction represents removing or decreasing physical and abstract quantities using different kinds of objects including negative numbers, fractions, irrational numbers, vectors, decimals, functions, and matrices.

Formally, the number being subtracted is known as the **subtrahend**, while the number it is subtracted from is the **minuend**. The result is the difference.

C.2.1 Properties

1. Anticommutativity: Subtraction is **anti-commutative**, meaning that if one reverses the terms in a difference left-to-right, the result is the negative of the original result. Symbolically, if a and b are any two numbers, then

$$a - b = -(b - a) \tag{8}$$

2. Non-associativity: Subtraction is **non-associative**, which comes up when one tries to define repeated subtraction. Should the expression "$a - b - c$" be defined to mean $(a - b) - c$ or $a - (b - c)$? These two possibilities give different answers. To resolve this issue, one must establish an order of operations, with different orders giving different results.

3. Predecessor: In the context of integers, subtraction of one also plays a special role: for any integer a, the integer $(a - 1)$ is the largest integer less than a, also known as the predecessor of a.

C.2.2 Operations

The borrow method is used for subtraction.

C.3 Multiplication

The **multiplication** of whole numbers may be thought as a repeated addition; that is, the multiplication of two numbers is equivalent to adding as many copies of one of them, the **multiplicand**, as the value of the other one, the **multiplier**. The multiplier can be written first and multiplicand second; both can be called **factors**. The result of multiplication is called the **product**.

$$a \times b = \underbrace{b + \cdots + b}_{a} \tag{9}$$

For example, 4 multiplied by 3 (often written as 3×4 and spoken as "3 times 4") can be calculated by adding 3 copies of 4 together:

$$3 \times 4 = 4 + 4 + 4 = 12$$

Multiplication can also be visualized as counting objects arranged in a rectangle (for whole numbers) or as finding the area of a rectangle whose sides have given lengths. The area of a rectangle does not depend on which side is measured first, which illustrates the commutative property. The product of two measurements is a new type of measurement, for instance multiplying the lengths of the two sides of a rectangle gives its area, this is the subject of dimensional analysis.

C.3.1 Properties

1. Units: One can only meaningfully add or subtract quantities of the same type but can multiply or divide quantities of different types. Four bags with three marbles each can be thought of as: 4 bags $\times$ 3 marbles per bag $=$ 12 marbles. When two measurements are multiplied together the product is of a type depending on the types of the measurements. The general theory is given by dimensional analysis.

2. Commutative Property: The order in which two numbers are multiplied does not matter:

$$x \cdot y = y \cdot x \tag{10}$$

3. Associative Property: Expressions solely involving multiplication or addition are invariant with respect to order of operations:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \tag{11}$$

4. Distributive Property: Holds with respect to multiplication over addition. This identity is of prime importance in simplifying algebraic expressions:

$$x \cdot (y + z) = x \cdot y + x \cdot z \tag{12}$$

5. Identity Element: The multiplicative identity is 1; anything multiplied by 1 is itself. This feature of 1 is known as the **identity property**:

$$x \cdot 1 = x \tag{13}$$

6. Property of 0: Any number multiplied by 0 is 0. This is known as the **zero property of multiplication**:

$$x \cdot 0 = 0 \tag{14}$$

7. Negation: –1 times any number is equal to the additive inverse of that number.

$$(-1) \cdot x = (-x) \text{ where } x + (-x) = 0 \tag{15}$$

   Similarly, $(-1) \times (-1) = 1$

8. Inverse Element: Every number x, except 0, has a **multiplicative inverse** $\frac{1}{x}$ such that $x \cdot \frac{1}{x} = 1$

9. Order Preservation: Multiplication by a positive number preserves order: For $a > 0$, if $b > c$ then $ab > ac$. Multiplication by a negative number reverses order: For $a < 0$, if $b > c$ then $ab < ac$.

C.3.2 Operations

Multiplication also uses the carry method.

This example uses long multiplication to multiply 23,958,233 (multiplicand) by 5,830 (multiplier) and arrives at 139,676,498,390 for the result (product).

$$\begin{array}{cccccccccccc}
 & & & & 2 & 3 & 6 & 5 & 8 & 2 & 3 & 3 \\
 & & & \times & & & & & 5 & 8 & 3 & 0 \\
\cline{4-12}
 & & & 7 & 0 & 9 & 7 & 4 & 6 & 9 & 9 & \cdot \\
 & 1 & 8 & 9 & 2 & 6 & 5 & 8 & 6 & 4 & & \\
1 & 1 & 8 & 2 & 9 & 1 & 1 & 6 & 5 & & & \\
\hline
1 & 3 & 7 & 9 & 2 & 7 & 4 & 9 & 8 & 3 & 9 & 0
\end{array}$$

This method relies on the base ten system: Multiplication starts with the multiplier's ones place multiplied by multiplicand. Then, the tens place of the multiplier is multiplied by the multiplicand and so on.

C.3.3 Exponentiation

Exponentiation is a mathematical operation, written as b^n, involving two numbers, the base b and the exponent or power n. When n is a positive integer, exponentiation corresponds to repeated multiplication of the base: that is, b^n is the product of multiplying n bases:

$$b^n = \underbrace{b \times \cdots \times b}_{n} \tag{16}$$

The **exponent** is usually shown as a superscript to the right of the base. In that case, b^n is called "b raised to the n^{th} power," "b raised to the power of n," "the n^{th} power of b," "b to the n^{th}," or most briefly as "b to the n." For any positive integers m and n, one has

$$b^n \times b^m = b^{n+m} \tag{17}$$

To extend this property to non-positive integer exponents, b^0 is defined to be 1, and b^{-n} with n as a positive integer and b not zero is defined as $\frac{1}{b^n}$. In particular, b^{-1} is equal to $\frac{1}{b}$, the reciprocal of b.

C.3.4 Identities and Properties

The following identities hold for all integer exponents, provided that the base is non-zero:

- Exponentiation is not commutative. For example, $2^3 = 8^4 \neq 3^2 = 9$.
- Exponentiation is not associative. For example, $(2^3)^4 = 8^4 = 4096$, whereas $2^{(3^4)} = 2^{81} = 2417851639229258349412352$. Without parentheses, the conventional order of operations in superscript notation is top-down (or right-associative), not bottom-up (or left-associative).

C.4 Division

In its simplest form, division can be viewed either as a **quotition** or a **partition**. In terms of quotition, 20 5 means the number of 5s that must be added to get 20. In terms of partition, 20 5 means the size of each of 5 parts into which a set of size 20 is divided. For example, 20 apples divide into four groups of five apples, meaning that twenty divided by five is equal to four. This is denoted as 20 / 5 = 4, or 20 5 = 4. Notationally, the **dividend** is divided by the **divisor** to get a **quotient**. In the example, 20 is the dividend, 5 is the divisor, and 4 is the quotient.

Unlike the other basic operations, when dividing natural numbers there is sometimes a remainder that will not go evenly into the dividend; for example, 10 3 leaves a remainder of 1, as 10 is not a multiple of 3. Sometimes this remainder is added to the quotient as a fractional part, so 10 3 is equal to 3 1/3 or 3.33..., but in the context of integer division, where numbers have no fractional part, the remainder is kept separately or discarded. When the remainder is kept as a fraction, it leads to a rational number. The set of all rational numbers is created by every possible division using integers. In modern mathematical terms, this is known as extending the system.