These notes are taken from James Munkres' *Topology* and are a useful reference when constructing proofs.

## Contents



# 1 Fundamental Concepts

## 1.1 Basic Notation

**Definition 1.1** *Sets* are collections of *objects* or *elements*.

**Notation 1.2** We shall use capital letters $A, B, \ldots$ to denote sets, and lowercase letters $a, b, \ldots$ to denote the objects or elements belonging to these sets.

**Notation 1.3** If an object $a$ belongs to a set $A$, we express this fact by the notation $a \in A$. If $a$ does not belong to $A$, we express this fact by writing $a \notin A$.

**Notation 1.4** The equality symbol $=$ is used to mean *logical identity.* Thus, when we write $a = b$, we mean that "$a$" and "$b$" are symbols for the same object. Similarly, the equation $A = B$ states that "$A$" and "$B$" are symbols for the same set; that is, $A$ and $B$ consist of precisely the same objects. If $a$ and $b$ are different objects, we write $a \neq b$; and if $A$ and $B$ are different sets, we write $A \neq B$.

**Notation 1.5** We say that $A$ is a *subset* of $B$ if every element of $A$ is also an element of $B$; and we express this fact by writing $A \subset B$.

**Definition 1.6** If $A = B$, then $A \subset B$ and $B \subset A$.

**Definition 1.7** If $A \subset B$ and $A$ is different from $B$, we say that $A$ is a *proper subset* of $B$, and we write $A \subsetneq B$.

**Notation 1.8** The relations $\subset$ and $\subsetneq$ are called *inclusion* and *proper inclusion*, respectively.

**Notation 1.9** If $A \subset B$, we also write $B \supset A$, which is read "$B$ *contains* $A$."

**Notation 1.10** If the set has only a few elements, we can specify the set by listing the elements, e.g. $A = \{a, b, c\}$. The usual way to specify a set, however, is to take some set $A$ of objects and some *property* that elements of $A$ may or may not possess, not to form the set consisting of all elements $A$ having that property. E.g. one might take the set of real numbers and form the subset $B$ consisting of all even integers: $B = \{x \mid x \text{ is an even integer}\}$. Here the braces stand for the words "the set of," and the vertical bar stands for the words "such that."

## 1.2 The Union of Sets

**Definition 1.11** Given two sets $A$ and $B$, one can form a set from them that consists of all the elements of $A$ together with all the elements of $B$. This set is called the *union* of $A$ and $B$ and is denoted by $A \cup B$. Formally, we define

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

## 1.3 The Intersection of Sets and the Empty Set

**Definition 1.12** Given sets $A$ and $B$, another way one can form a set to take the common part of $A$ and $B$. This set is called the *intersection* of $A$ and $B$ and is denoted by $A \cap B$. Formally, we define

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

**Definition 1.13** The *empty set*, denoted by $\emptyset$, which we think of as "the set having no elements." This means that for each object $x$, the relation $x \in \emptyset$ does not hold; in other words, for every $x, x \notin \emptyset$.

**Notation 1.14** We express the statement that $A$ and $B$ have no elements in common by the equation

$$A \cap B = \emptyset.$$

We also express this fact by saying that $A$ and $B$ are *disjoint.*

**Note 1.15** The empty set is a convention which is used because without it, one would have to prove that two sets $A$ and $B$ do have elements in common before one could use the notation $A \cap B$. Similarly, the notation $C = \{x \mid x \in A \text{ and } x \text{ has a certain property}\}$ could not be used if it happened that no element $x$ of $A$ had the given property.

**Definition 1.16** For every set $A$ we have the equation

$$A \cup \emptyset = A \quad \text{and} \quad A \cap \emptyset = \emptyset.$$

**Definition 1.17** For every set $A$, we have that $\emptyset \subset A$ because it is a statement of the form "For every object $x$, if $x$ belongs to the empty set, then $x$ also belongs to the set $A$." It is never the case that $x$ belongs to $\emptyset$, so the implication is always true.

**Definition 1.18** In an implication $P \to Q$, $P$ is called the *hypothesis* and $Q$ is called the *conclusion.*

**Definition 1.19** In an implication $P(x) \to Q(x)$, if the hypothesis $P(x)$ is false for all $x$, then the implication is said to be *vacuously true.*

## 1.4 Contrapositive and Converse

**Definition 1.20** Given a statement of the form "$P \to Q$," its *contrapositive* is defined to be the statement "If $Q$ is not true, then $P$ is not true."

**Note 1.21** An implication and its contrapositive are *logically equivalent.* We can see this by looking at the truth-table of both.

**Definition 1.22** Given an implication "$P \to Q$," the statement $Q \to P$, which is called the *converse* of $P \to Q$.

## 1.5 The Difference of Two Sets

**Definition 1.23** The *difference* of two sets denoted by $A - B$, and defined as the set consisting of those elements of $A$ that are not in $B$. Formally,

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$

It is also called the *complement* of $B$ relative to $A$, or the complement of $B$ in $A$.

## 1.6 Rules of Set Theory

**Theorem 1.24** (The Distributive Laws) For any sets $A, B, C$, the equations $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ hold.

**Theorem 1.25** (DeMorgan's laws) For any sets $A, B, C$, the equations $A - (B \cup C) = (A - B) \cap (A - C)$ and $A - (B \cap C) = (A - B) \cup (A - C)$.

## 1.7 Collections of Sets

**Definition 1.26** Given a set $A$, we can consider sets whose elements are subsets of $A$. When we have a set whose elements are sets, we shall often refer to it as a *collection* of sets and denote it by a script letter such as $\mathscr{A}$ or $\mathscr{B}$.

**Definition 1.27** Given a set $A$, the set of all subsets of $A$ is called the *power set* of $A$ and is denoted $\mathcal{P}(A)$.

**Note 1.28** We make a distinction between the object $a$, which is an *element* of the set $A$, and the one-element set $\{a\}$, which is a *subset* of $A$. E.g. if $A$ is the set $\{a, b, c\}$, then the statements

$$a \in A, \quad \{a\} \subset A, \quad \text{and} \quad \{a\} \in \mathcal{P}(A)$$

are all correct, but the statements $\{a\} \in A$ and $a \subset A$ are not.

### 1.8 Arbitrary Unions and Intersections

**Definition 1.29** Given a collection $\mathscr{A}$ of sets, the *union* of the elements of $\mathscr{A}$ is defined by the equation

$$\bigcup_{A \in \mathscr{A}} A = \{x \mid x \in A \text{ for at least one } A \in \mathscr{A}\}.$$

**Definition 1.30** Given a collection $\mathscr{A}$ of sets, the *intersection* of the elements of $\mathscr{A}$ is defined by the equation

$$\bigcap_{A \in \mathscr{A}} A = \{x \mid x \in A \text{ for every } A \in \mathscr{A}\}.$$

**Note 1.31** In (1.29), if one of the sets $A$ happens to be the empty set, then that set simply contributes no elements to the union. In (1.30), if one of the sets happens to be the empty set, then the entire intersection results in the empty set.

**Note 1.32** In (1.29), if $\mathscr{A}$ is the empty collection, then the union evaluates to the empty set. In (1.30), if $\mathscr{A}$ is the empty collection, we don't define the intersection and determine its value depending on the specific context.

### 1.9 Cartesian Products

**Definition 1.33** Given sets $A$ and $B$, we define their Cartesian product $A \times B$ to be the set of all ordered pairs $(a, b)$ for which $a$ is an element of $A$ and $b$ is an element of $B$. Formally

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

**Note 1.34** (1.33) assumes that the concept of "ordered pair" is already given. It can be taken as a primitive concept (as was the notion of "set") or it can be given a definition in terms of the set operations already introduced. One definition in terms of set operations is $(a, b) = \{\{a\}, \{a, b\}\}$ that defines the ordered pair $(a, b)$ as a collection of sets. If $a \neq b$, this definition says that $(a, b)$ is a collection of two sets, one of which is a one-element set and the other is a two-element set. The first coordinate of the ordered pair is defined to be the element belonging to both sets, and the second coordinate is the element belonging to only one of the sets. If $a = b$, then $(a, b)$ is a collection containing only one set $\{a\}$.

**Notation 1.35** An ordered pair $(a, b)$ can also be denoted by $a \times b$.

## 2 Functions

**Definition 2.1** (Rule of Assignment) A *rule of assignment* is a subset $r$ of the Cartesian product $C \times D$ of two sets, having the property that each element of $C$ appears as the first coordinate of *at most one* ordered pair belonging to $r$. Formally, a subset $r$ of $C \times D$ is a rule of assignment if

$$[(c, d) \in r \text{ and } (c, d') \in r] \rightarrow [d = d'].$$

**Definition 2.2** (Domain and Range) Given a rule of assignment $r$, the *domain* of $r$ is defined to be the subset of $C$ consisting of all first coordinates of elements of $r$, and the *image set* of $r$ is defined as the subset of $D$ consisting of all second coordinates of elements of $r$. Formally,

$$\text{domain } r = \{c \mid \text{there exists } d \in D \text{ such that } (c, d) \in r\},$$

$$\text{range } r = \{d \mid \text{there exists } c \in C \text{ such that } (c, d) \in r\}.$$

Note that given a rule of assignment $r$, its domain and image are entirely determined.

**Definition 2.3** (Function) A *function* $f$ is a rule of assignment $r$, together with a set $B$ that contains the image set of $r$. The domain $A$ of the rule $r$ is also called the *domain* of the function $f$; the image set of $r$ is also called the *image set* of $f$; and the set $B$ is called the *range* of $f$.

**Notation 2.4** If $f$ is a function having domain $A$ and range $B$, we express this fact by writing $f : A \to B$, which is read "$f$ is a function from $A$ to $B$," or " $f$ is a mapping from $A$ into $B$."

**Notation 2.5** If $f : A \to B$ and if $a$ is an element of $A$, we denote by $f(a)$ the unique element of $B$ that the rule determining $f$ assigns to $a$; it is called the *value* of $f$ at $a$, or sometimes the *image* of $a$ under $f$. Formally, if $r$ is the rule of the function $f$, then $f(a)$ denotes the unique element of $B$such that $(a, f(a)) \in r$.

**Note 2.6** We can specify functions by their rule of assignment and range, or by their domain, range and equation.

**Definition 2.7** (Restriction) If $f : A \to B$ and if $A_0$ is a subset of $A$, we define the *restriction* of $f$ to $A_0$ to be the function mapping $A_0$ into $B$ whose rule is $\{(a, f(a)) \mid a \in A_0\}$. It is denoted by $f|A_0$, which is read "$f$ restricted to $A_0$."

**Definition 2.8** (Composite Functions) Given function $f : A \to B$ and $g : B \to C$, we define the *composite* $g \circ f$ of $f$ and $g$ as the functions $g \circ f : A \to C$ defined by the equation $(g \circ f)(a) = g(f(a))$. Formally, $g \circ f : A \to C$ is the function whose rule is

$$\{(a, c) \mid \text{For some } b \in B, f(a) = b \text{ and } g(b) = c\}.$$

Note that $g \circ f$ is defined only when the range of $f$ equals the domain of $g$.

**Definition 2.9** (Injective, Surjective, Bijective Functions) A function $f : A \to B$ is said to be *injective* (or *on-to-one*) if for each pair of distinct points of $A$, their images under $f$ are distinct. It is said to be *surjective* (or $f$ is said to map $A$ *onto* $B$) if every element of $B$ is the image of some element of $A$ under the function $f$. If $f$ is both injective and surjective, it is said to be *bijective* (or is called a *one-to-one correspondence*). Formally, $f$ is injective if

$$[f(a) = f(a')] \to [a = a'],$$

and $f$ is surjective if

$$[b \in B] \to [b = f(a) \text{ for at least one } a \in A].$$

**Definition 2.10** (Inverse Functions) If $f$ is bijective, there exists a function from $B$ to $A$ called the *inverse* of $f$. It is denoted by $f^{-1}$ and is defined by letting $f^{-1}(b)$ be that unique element $a$ of $A$ for which $f(a) = b$. Given $b \in B$, the fact that $f$ is surjective implies that there exists such an element $a \in A$; the fact that $f$ is injective implies that there is only one such element $a$.

**Theorem 2.11** The composite of two injective functions is injective; the composite of two surjective functions is surjective; the composite of two bijective functions is bijective; if $f$ is bijective, $f^{-1}$ is bijective.

**Theorem 2.12** Let $f : A \to B$. If there are functions $g : B \to A$ and $h : B \to A$ such that $g(f(a)) = a$ for every $a$ in $A$ and $f(h(b)) = b$ for every $b$ in $B$ then $f$ is bijective and $g = h = f^{-1}$.

**Definition 2.13** Let $f : A \to B$. If $A_0$ is a subset of $A$, we denote by $f(A_0)$ the set of all images of points of $A_0$ under the function $f$; this set is called the *image* of $A_0$ under $f$. Formally,

$$f(A_0) = \{b \mid b = f(a) \text{ for at least one } a \in A_0\}.$$

On the other hand, if $B_0$ is a subset of $B$, we denote by $f^{-1}(B_0)$ the set of all elements of $A$ whose images under $f$ lie in $B_0$; it is called the *preimage* of $B_0$ under $f$ (or the counterimage or the inverse image of $B_0$).Formally,

$$f^{-1}(B_0) = \{a \mid f(a) \in B_0\}.$$

Of course, there may be no points of $a$ of $A$ whose images lie in $B_0$; in that case, $f^{-1}(B_0)$ is empty.

**Theorem 2.14** Given a function $f : A \to B$, the operation $f^{-1}$ when applied to subsets of $B$ preserves inclusions, unions, intersections, and differences of sets. The operation $f$, when applied to subsets of $A$, preserves only inclusions and unions.

**Theorem 2.15** If $f : A \to B$ and if $A_0 \subset A$ and $B_0 \subset B$, then

$$A_0 \subset f^{-1}(f(A_0)) \quad \text{and} \quad f((f^{-1}(B_0)) \subset B_0.$$

The first inclusion is an equality if $f$ is injective, and the second inclusion is an equality if $f$ is surjective.

# 3 Relations

**Definition 3.1** (Relation) A *relation* on a set $A$ is a subset $C$ of the Cartesian product $A \times A$.

**Notation 3.2** If $C$ is a relation on $A$, we use the notation $xCy$ to mean the same thing as $(x, y) \in C$. This is read as "$x$ is in the relation $C$ to $y$."

**Note 3.3** A rule of assignment $r$ for a function $f : A \to A$ is also a subset of $A \times A$. But it is a subset of a special kind, namely, one such that each element of $A$ appears as the first coordinate of an element of $r$ exactly once. *Any* subset of $A \times A$ is a relation on $A$.

## 3.1 Equivalence Relations and Partitions

**Definition 3.4** (Equivalence Relation) An *equivalence relation* on a set $A$ is a relation $C$ on $A$ having the following three properties:

1. (Reflexivity) $xCx$ for every $x$ in $A$.
2. (Symmetry) If $xCy$,then $yCx$.
3. (Transitivity) If $xCy$ and $yCz$, then $xCz$.

**Notation 3.5** We can use any symbol for an equivalence relation, e.g. $\sim$ .

**Definition 3.6** (Equivalence Class) Given an equivalence relation $\sim$ on a set $A$ and an element $x$ of $A$, we define a certain subset $E$ of $A$, called the *equivalence class* determined by $x$, by the equation

$$E = \{y \mid y \sim x\}.$$

Note that the equivalence class $E$ determined by $x$ contains $x$, since $x \sim x$.

**Theorem 3.7** Two equivalence classes $E$ and $E'$ are either disjoint or equal.

**Notation 3.8** Given an equivalence relation on a set $A$, let us denote by $\mathscr{E}$ the collection of all equivalence classes determined by this relation. (3.7) shows that distinct elements of $\mathscr{E}$ are disjoint. Furthermore, the union of the elements of $\mathscr{E}$ equals all of $A$ because every element of $A$ belongs to an equivalence class.

**Definition 3.9** A *partition* of a set $A$ is a collection of disjoint nonempty subsets of $A$ whose union is all of $A$.

**Note 3.10** An equivalence class is a particular example of a partition.

**Theorem 3.11** Given a partition $\mathscr{D}$ of $A$, there is exactly one equivalence relation on $A$ from which it is derived.

## 3.2 Order Relations

**Definition 3.12** (Order Relations) A relation $C$ on a set $A$ is called an *order relation* (or simple order or near order) if it has the following properties:

1. (Comparability) For every $x$ and $y$ in $A$ for which $x \neq y$, either $xCy$ or $yCx$.
2. (Nonreflexivity) For no $x$ in $A$ does the relation $xCx$ hold.
3. (Transitivity) If $xCy$ and $yCz$, then $xCz$.

**Notation 3.13** The ‘$<$’ symbol is commonly used to denote an order relation. Stated in this notation, the properties of an order relation become”

1. If $x \neq y$, then either $x < y$ or $y < x$.
2. If $x < y$, then $x \neq y$.
3. If $x < y$ and $y < z$, then $x < z$.

**Notation 3.14** We shall use the notation $x \leq y$ to stand for the statement “either $x < y$ or $x = y$”; and we shall use the notation $y > x$ to stand for the statement “$x < y$.” We write $x < y < z$ to mean “$x < y$ and $y < z$. ”

**Definition 3.15** (Intervals) If $X$ is a set and ¡ is an order relation on $X$, and if $a < b$, we use the notation $(a, b)$ to denote the set

$$\{x \mid a < x < b\};$$

it is called an *open interval* in $X$. If this set is empty, we call $a$ the *immediate predecessor* of $b$, and we call $b$ the *immediate successor* of $a$.

**Definition 3.16** (Order Type) Suppose that $A$ and $B$ are two sets with order relations $<_A$ and $<_B$ respectively. We say that $A$ and $B$ have the same *order type* if there is a bijective correspondence between them that preserves order; that is, if there exists a bijective function $f : A \to B$ such that

$$a_1 <_A a_2 \to f(a_1) <_B f(a_2).$$

**Definition 3.17** (Dictionary Order) Suppose that $A$ and $B$ are two sets with order relations $<_A$ and $<_B$ respectively. Define an order relation $<$ on $A \times B$ by defining

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$, or if $a_1 = a_2$ and $b_1 <_B b_2$. It is called the *dictionary order relation* on $A \times B$.

**Definition 3.18** (Largest/Smallest Elements) Suppose that $A$ is a set ordered by the relation $<$. Let $A_0$ be a subset of $A$. We say that the element $b$ is the *largest element* of $A_0$ if $b \in A_0$ and if $x \leq b$ for every $x \in A_0$. Similarly, we say that $a$ is the *smallest element* of $A_0$ if $a \in A_0$ and if $a \leq x$ for every $x \in A_0$.

**Definition 3.19** (Bounded Above) We say that the subset $A_0$ of $A$ is *bounded above* if there is an element $b$ of $A$ such that $x \leq b$ for every $x \in A_0$; the element $b$ is called an *upper bound* for $A_0$. If the set of all upper bounds for $A_0$ has a smallest element, that element is called the *least upper bounds*, or the *supremum*, of $A_0$. It is denoted by $\sup A_0$; it may or may not belong to $A_0$. If it does, it is the largest element of $A_0$.

**Definition 3.20** (Bounded Below) $A_0$ is *bounded below* if there is an element $a$ of $A$ such that $a \leq x$ for every $x \in A_0$; the element $a$ is called a *lower bound* for $A_0$. If the set of all lower bounds for $A_0$ has a largest element, that element is called the *greatest lower bound,* or the *infimum,* of $A_0$. It is denoted by $\inf A_0$; it may or may not belong to $A_0$. If it does, it is the smallest element of $A_0$

**Definition 3.21** (LUB and GLB) An ordered set $A$ is said to have the *least upper bound property* if every nonempty subset $A_0$ of $A$ that is bounded above has a least upper bound. Analogously, the set $A$ is said to have the *greatest lower bound property* if every nonempty subset $A_0$ of $A$ that is bounded below has a greatest lower bound.

**Theorem 3.22** $A$ has the least upper bound property if and only if it has the greatest lower bound property.

**Definition 3.23** A *binary operation* on a set $A$ is a function $f$ mapping $A \times A$ into $A$.

**Notation 3.24** We usually write the symbol for the binary opreation between the two coordinates of hte point in question, writing the value of the function at $(a, a')$ as $afa'$. It is more common to use some symbol than a letter to denote a binary operation, e.g. $+, \cdot, \circ$.

## 4 Cartesian Products

**Definition 4.1** Let $\mathscr{A}$ be nonempty collection of sets. An *indexing function* for $\mathscr{A}$is a surjective function $f$ from some set $J$, called the *index set,* to $\mathscr{A}$. The collection $\mathscr{A}$, together with the indexing function $f$, is called an *indexed family of sets.* Given $\alpha \in J$, we shall denote the set $f(\alpha)$ by the symbol $A_\alpha$. And we shall denote the indexed family itself by the symbol $\{A_\alpha\}_{\alpha \in J}$, which is read "the family of all $A_\alpha$, as $\alpha$ ranges over $J$," Sometimes we write only $\{A_\alpha\}$, if it is clear what the index set is.

**Notation 4.2** Although an indexing function is required to be surjective, it is not required to be injective. It is entirely possible for $A_\alpha$ and $A_\beta$ to be the same set of $\mathscr{A}$ even though $\alpha \neq \beta$.

**Notation 4.3** Indexing functions give a new notation for arbitrary unions and intersections of sets. Suppose that $f : J \to \mathscr{A}$ is an indexing function for $\mathscr{A}$; Let $A_\alpha$ denote $f(\alpha)$. Then we define

$$\bigcup_{\alpha \in J} A_\alpha = \{x \mid \text{for at least one } \alpha \in J, x \in A_\alpha\},$$

and

$$\bigcap_{\alpha \in J} A_\alpha = \{x \mid \text{for every } \alpha \in J, x \in A_\alpha\}.$$

**Definition 4.4** Let $m$ be a positive integer. Given a set $X$, we define an *m-tuple* of elements of $X$ to be a function

$$\mathbf{x} : \{1, \ldots, m\} \to X.$$

If $\mathbf{x}$ is an $m$-tuple, we often denote the value of $\mathbf{x}$ at $i$ by the symbol $x_i$ instead of $\mathbf{x}$(i) and call it the $i$th *coordinate* of $x$. And we often denote the function $\mathbf{x}$ itself by the symbol

$$(x_1, \ldots, x_m).$$

**Definition 4.5** Let $\{A_1, \ldots, A_m\}$ be a family of sets indexed with the set $\{1, \ldots, m\}$. Let $X = A_1 \cup \cdots \cup A_m$. We define the *Cartesian product* of this indexed family, denoted by

$$\prod_{i=1}^{m} A_i \quad \text{or} \quad A_1 \times \cdots \times A_m,$$

to be the set of all $m$-tuples $(x_1, \ldots, x_m)$ of elements of $X$ such that $x_i \in A_i$ for each $i$.

**Definition 4.6** Given a set $X$, we define an $\omega$-tuple of elements of $X$ to be a function

$$\mathbf{x} : \mathbb{Z}^+ \to X;$$

we also call such a function a *sequence*, or an *infinite sequence*, of elements of $X$. If $\mathbf{x}$ is an $\omega$-tuple, we often denote the value of $\mathbf{x}$ at $i$ by $x_i$ rather than $\mathbf{x}(i)$ and call it the $i$th *coordinate* of $\mathbf{x}$. We denote $\mathbf{x}$ itself by the symbol

$$(x_1, x_2, \ldots) \quad \text{or} \quad (x_n)_{n \in \mathbb{Z}_+}.$$

**Definition 4.7** Let $\{A_1, A_2, \ldots\}$ be a family of sets, indexed with the positive integers; let $X$ be the union of the sets in this family. The *Cartesian product* of this indexed family of sets, denoted by

$$\prod_{i \in \mathbb{Z}_+} A_i \quad \text{or} \quad A_1 \times A_2 \times \ldots,$$

is defined to be the set of all $\omega$-tuples $(x_1, x_2, \ldots)$ of elements of $X$ such that $x_i \in A_i$ for each $i$.

**Note 4.8** There is nothing in definitions (4.4)-(4.8) that requires the sets $A_i$ to be different from one another. Indeed, they may all equal the same set $X$. In that case, the Cartesian product $A_1 \times \cdots \times A_m$ is just the set of all $m$-tuples of elements of $X$, which we denote by $X^m$.Similarly, the product $A_1 \times A_2 \times \ldots$ is just the set of all $\omega$-tuples of elements of $X$, which we denote by $X^\omega$.

## 5 Finite Sets

**Notation 5.1** If $n$ is a positive integer, we use $S_n$ to denote the set of positive integers less than $n$; it is called a *section* of the positive integers.

**Definition 5.2** A set is said to be *finite* if there is a bijective correspondence of $A$ with some section of the positive integers. That is, $A$ is finite if it is empty or if there is a bijection

$$f : A \to \{1, \ldots, n\}$$

for some positive integer $n$. In the former case, we say that $A$ has *cardinality 0*; in the latter case, we say that $A$ has *cardinality n*.

**Lemma 5.3** Let $n$ be a positive integer. Let $A$ be a set; let $a_0$ be an element of $A$. Then there exists a bijective correspondence $f$ of the set $A$ with the set $\{1, \dots, n+1\}$ if and only if there exists a bijective correspondence $g$ of the set $A - \{a_0\}$ with the set $\{1, \dots, n\}$.

**Theorem 5.4** Let $A$ be a set; suppose that there exists a bijection $f : A \to \{1, \dots, n\}$ for some $n \in \mathbb{Z}_+$. Let $B$ be a proper subset of $A$. Then there exists no bijection $g : B \to \{1, \dots, n\}$; but (provided $B \neq \emptyset$) there does exist a bijection $h : B \to \{1, \dots, m\}$ for some $m < n$.

**Corollary 5.5** If $A$ is finite, there is no bijection of $A$ with proper subset of itself.

**Corollary 5.6** $\mathbb{Z}_+$ is note finite.

**Corollary 5.7** The cardinality of a finite set $A$ is uniquely determined by $A$.

**Corollary 5.8** If $B$ is a subset of the finite set $A$, then $B$ is finite. If $B$ is a proper subset of $A$, then the cardinality of $B$ is less than the cardinality of $A$.

**Corollary 5.9** Let $B$ be a nonempty set. Then the following are equivalent:

1. $B$ is finite.
2. There is a surjective function from a section of the positive integers onto $B$.
3. There is an injective function from $B$ into a section of the positive integers.

**Corollary 5.10** Finite unions and finite Cartesian products of finite sets are finite.

# 6 Countable and Uncountable Sets

**Definition 6.1** A set $A$ is said to be *infinite* if it is not finite. It is said to be *countably infinite* if there is a bijective correspondence

$$f : A \to \mathbb{Z}_+.$$

**Definition 6.2** A set is said to be *countable* if it is either finite or countably infinite. A set that is not countable is said to be *uncountable.*

**Theorem 6.3** Let $B$ be a nonempty set. Then the following are equivalent:

1. $B$ is countable.
2. There is a surjective function $f : \mathbb{Z}_+ \to B$.
3. There is an injective function $g : B \to \mathbb{Z}_+$.

**Theorem 6.4** If $C$ is an infinite subset of $\mathbb{Z}_+$, then $C$ is countably infinite.

**Definition 6.5** A *recursion formula* defines a function in terms of itself. A definition given by such a formula is called a *recursive definition.*

**Theorem 6.6** (Principle of Recursive Definition) Let $A$ be a set. Given a formula that defines $h(1)$ as a unique element of $A$, and for $i > 1$ defines $h(i)$ uniquely as an element of $A$ in terms of the values of $h$ for positive integers less than $i$, this formula determines a unique function $h : \mathbb{Z}_+ \to A$.

**Theorem 6.7** A subset of a countable set is countable.

**Theorem 6.8** The set $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable infinite.

**Theorem 6.9** A countable union of countable sets is countable.

**Theorem 6.10** A finite product of countable sets is countable.

**Theorem 6.11** Let $X$ denote the two element set $\{0, 1\}$. Then the set $X^\omega$ is uncountable.

**Theorem 6.12** Let $A$ be a set. There is no injective map $f : \mathcal{P}(A) \to A$, and there is no surjective map $g : A \to \mathcal{P}(A)$.

# 7 The Principle of Recursive Definition

**Theorem 7.1** (General Principle of Recursive Definition) Let $A$ be a set; let $a_0$ be an element of $A$. Suppose $\rho$ is a function that assigns, to each function $f$ mapping a nonempty section of the positive integers into $A$, an element of $A$. Then there exists a unique function

$$h : \mathbb{Z}_+ \to A$$

such that

$$h(1) = a_0$$

$$h(i) = \rho(h|\{1, \dots, i-1\}) \quad \text{for } i > 1.$$

The above formula is called a *recursion formula* for $h$. It specifies $h(1)$, and it expresses the value of $h$ at $i > 1$ in terms of $h$ for positive integers less than $i$.

# 8 Infinite Sets and the Axiom of Choice

**Theorem 8.1** Let $A$ be a set. The following statements about $A$ are equivalent:

1. There exists an injective function $f : \mathbb{Z}_+ \to A$.
2. There exists a bijection of $A$ with a proper subset of itself.
3. $A$ is infinite.

**Definition 8.2** (Axiom of Choice) Given a collection $\mathscr{A}$ of disjoint nonempty sets, there exists a set $C$ consisting of exactly one element from each element of $\mathscr{A}$; that is, a set $C$ such that $C$ is contained in the union of the elements of $\mathscr{A}$, and for each $A \in \mathscr{A}$, the set $C \cap A$ contains a single element.

**Theorem 8.3** (Existence of a Choice Function) Given a collection $\mathscr{B}$ of nonempty sets (not necessarily disjoint), there exists a function

$$c : \mathscr{B} \to \bigcup_{B \in \mathscr{B}} B$$

such that $c(B)$ is an element of $B$, for each $B \in \mathscr{B}$. The function $c$ is called a *choice function* for the collection $\mathscr{B}$.

**Note 8.4** The difference between (8.3) and (8.2) is that in (8.3) the sets of the collection $\mathscr{B}$ are not required to be disjoint. For example, one can allow $\mathscr{B}$ to be the collection of all nonempty subsets of a given set.

**Note 8.5** There are two forms of the axiom of choice:

1. The *finite axiom of choice* which asserts that given a finite collection $\mathscr{A}$ of disjoint nonempty sets, there exists a set $C$ consisting of exactly one element from each element of $\mathscr{A}$.
2. The *strong axiom of choice* which asserts that applies to an *arbitrary* collection $\mathscr{A}$ of nonempty sets.

# 9 The Maximum Principle

**Definition 9.1** Given a set $A$, a relation $\prec$ on $A$ is called a *strict partial order* on $A$ if it has the following two properties:

1. (Nonreflexivity) The relation $a \prec a$ never holds.
2. (Transitivity) If $a \prec b$ and $b \prec c$, then $a \prec c$.

**Theorem 9.2** (The Maximum Principle) Let $A$ be a set; let $\prec$ be a strict partial order on $A$. Then there exists a maximal simply ordered subset $B$ of $A$. In other words, there exists a subset $B$ of $A$ such that $B$ is simply ordered by $\prec$ and such that no subset of $A$ that properly contains $B$ is simply ordered by $\prec$ .

**Definition 9.3** Let $A$ be a set and let $\prec$ be a strict partial order on $A$. If $B$ is a subset of $A$, an *upper bound* on $B$ is an element $c$ of $A$ such that for every $b$ in $B$, either $b = c$ or $b \prec c$. A *maximal element* of $A$ is an element $m$ of $A$ such that for no element $a$ of $A$ does the relation $m \prec a$ hold.

**Lemma 9.4** (Zorn's Lemma) Let $A$ be a set that is strictly partially ordered. If every simply ordered subset of $A$ has an upper bound in $A$, then $A$ has a maximal element.

**Definition 9.5** A *partial order* $a \preceq b$ is defined as meaning either $a \prec b$ or $a = b$