24.241 First-Order Logic Notes

Contents

1 Introduction to Deductive Logic

Definition 1.1 An ***argument*** is a set of two or more sentences, one of which is designated as the conclusion and the others as the premises.

Definition 1.2 An argument is ***logically valid*** if and only if it is not possible for the premises to be true and the conclusion false. An argument is ***logically invalid*** if and only if it is not logically valid.

Definition 1.3 An argument is ***logically sound*** if and only if it is logically valid and all its premises are true. An argument is ***logically unsound*** if and only if it is not logically sound.

Definition 1.4 A sentence is ***logically true*** if and only if it is not possible for the sentence to be false.

Definition 1.5 A sentence is ***logically false*** if and only if it is not possible for the sentence to be true.

Definition 1.6 A sentence is ***logically indeterminate*** if and only if it is neither logically true nor logically false.

Definition 1.7 Sentences **p** and **q** are ***logically equivalent*** if and only if it is not possible for one of these sentences to be true while the other sentence is false.

Definition 1.8 A set of sentences if ***logically consistent*** if and only if it is possible for all the members of that set to be true. A set of sentences is ***logically inconsistent*** if and only if it is not logically consistent.

Definition 1.9 A set of sentences ***logically entails*** a sentence if and only if it is impossible for all the members of the set to be true and that sentence false.

2 Sentential Logic: Syntax and Symbolization

Definition 2.1 The ***syntax*** of a language specifies the basic expressions of a language and the rules that determine which combinations of those expressions count as sentences of the language.

Definition 2.2 The ***vocabulary of SL*** consists of **Sentence Letters** and **Sentential Connectives**:

- ***Sentence Letters*** are the capital Roman letters 'A' through 'Z', with or without positive integer subscripts.
- ***Sentential Connectives*** consist of the following:
  - $\sim$ (called the 'tilde')
  - & (called the 'ampersand')
  - $\vee$ (called the 'wedge')
  - $\supset$ (called the 'horseshoe')
  - $\equiv$ (called the 'triple bar')

Definition 2.3 The ***punctuation marks*** of SL are '(' and ')'.

Definition 2.4 We define '***sentence of SL*** as follows:

1. Every sentence letter of SL is a sentence of SL.
2. If $\mathbf{P}$ is a sentence of SL, then $\sim \mathbf{P}$ is a sentence of SL.
3. If $\mathbf{P}$ and $\mathbf{Q}$ are sentences of SL, then $(\mathbf{P} \,\&\, \mathbf{Q})$ is a sentence of SL.
4. If $\mathbf{P}$ and $\mathbf{Q}$ are sentences of SL, then $(\mathbf{P} \vee \mathbf{Q})$ is a sentence of SL.
5. If $\mathbf{P}$ and $\mathbf{Q}$ are sentences of SL, then $(\mathbf{P} \supset \mathbf{Q})$ is a sentence of SL.
6. If $\mathbf{P}$ and $\mathbf{Q}$ are sentences of SL, then $(\mathbf{P} \equiv \mathbf{Q})$ is a sentence of SL.
7. Nothing is a sentence of SL unless it can be formed by repeated application of clauses 1-6.

Definition 2.5 The sentences of SL are of two sorts: ***atomic sentences*** and ***compound sentences***:

- ***Atomic Sentences*** are the sentence letters of SL. They are not formed or compounded from other sentences.
- ***Compound Sentences*** are all non-atomic sentences. They are formed or compounded from other sentences of SL. All compound sentences contain at least one sentential connectives.

Definition 2.6 There are five types of compound sentences and each type has a **main connective** and an **immediate component** or **components**:

- **Negations** are sentences of the form $\sim \mathbf{P}$.
- **Conjunctions** are sentences of the form $(\mathbf{P}\&\mathbf{Q})$.
- **Disjunctions** are sentences of the form $(\mathbf{P} \vee \mathbf{Q})$.
- **Material Conditionals** are sentences of the form $(\mathbf{P} \supset \mathbf{Q})$.
- **Material Biconditionals** are sentences of the form $(\mathbf{P} \equiv \mathbf{Q})$.

Definition 2.7 The ***components*** of a sentence $\mathbf{P}$ of SL are

- $\mathbf{P}$ itself
- The immediate components (if any) of $\mathbf{P}$
- The components of $\mathbf{P}$'s immediate components.

Definition 2.8 A sentential connective of a formal or natural language is used ***truth-functionally*** if and only if it is used to generate a compound sentence from one or more sentences in such a way that the truth-value of the generated compound is wholly determine by the truth-values of those one or more sentences from which the compound is generated no matter what those truth-values may be.

Definition 2.9 The truth table for the sentential connectives are as follows:

Negation

P	**~ P**
T	**F**
F	**T**

Conjunction

P	**Q**	**(P & Q)**
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

Disjunction

P	**Q**	**(P ∨ Q)**
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

Material Conditional

P	**Q**	**(P ⊃ Q)**
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Material Biconditional

P	**Q**	**(P ≡ Q)**
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

SUMMARY OF SOME COMMON CONNECTIVES

English Connective	*Paraphrase*	*Symbolization in SL*
not **p**	it is not the case that **p**	~ **P**
p and **q** **p** but **q** **p** however **q** **p** although **q** **p** nevertheless **q** **p** nonetheless **q** **p** moreover **q**	**p** and **q**	**P** & **Q**
p or **q** **p** unless **q**	**p** or **q**	**P** ∨ **Q**
p or **q** (exclusive sense)	**p** or **q** and it is not the case that (**p** and **q**)	(**P** ∨ **Q**) & ~ (**P** & **Q**)
if **p** then **q** **p** only if **q** **q** if **p** **q** provided that **p** **q** given **p**	if **p** then **q**	**P** ⊃ **Q**
p if and only if **q** **p** if but only if **q** **p** just in case **q**	**p** if and only if **q**	**P** ≡ **Q**

3 Sentential Logic: Semantics

Definition 3.1 ***Semantics*** is the study of truth-functionality. It consists of **truth-functional truth, falsity, and indeterminacy; truth-functional equivalence; truth-functional consistency; truth-functional entailment and validity**.

Definition 3.2 A ***truth-value assignment*** is an assignment of truth-values (**T**s and **F**s) to the atomic sentences of SL.

Definition 3.3 A sentence is ***true on a truth-value assignment*** if and only if it has the truth-value **T** on that truth-value assignment. A sentence is ***false on a truth-value assignment*** if and only if it has the truth-value **F** on that truth-value assignment.

Definition 3.4 A sentence **P** is ***truth-functionally true*** if and only if **P** is true on every truth-value assignment. A sentence **P** is ***truth-functionally false*** if and only if **P** is false on every truth-value assignment.

Definition 3.5 A sentence **P** of SL is ***truth-functionally indeterminate*** if and only if **P** is neither truth-functionally true nor truth-functionally false.

Definition 3.6 Sentences **P** and **Q** are ***truth-functionally equivalent*** if and only if there is no truth-value assignment on which **P** and **Q** have different truth-values.

Definition 3.7 A set of sentences of SL is ***truth-functionally consistent*** if and only if there is at least one truth-value assignment on which all the members of the set are true. A set of sentences of SL is ***truth-functionally inconsistent*** if and only if it is not truth-functionally consistent.

Definition 3.8 A set Γ of sentences of SL ***truth-functionally entails*** a sentence **P** of SL if and only if there is no truth-value assignment on which every member of Γ is true and **P** is false.

Notation 3.9 Γ truth-functionally entails **P** is notated '$\Gamma \vDash \mathbf{P}$'. Γ does no truth-functionally entail **P** is notated '$\Gamma \nvDash \mathbf{P}$'.

An argument of SL is ***truth-functionally valid*** if and only if there is no truth-value assignment on which all the premises are true and the conclusion is false. An argument of SL is ***truth-functionally invalid*** if and only if it is not truth-functionally valid.

Definition 3.9 The ***iterated conjunction*** of a sentence **P** is just **P**, while the iterated conjunction of sentences $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ is $\ldots(\mathbf{P}_1 \& \mathbf{P}_2) \& \ldots \& \mathbf{P}_n)$.

Definition 3.10 The ***corresponding material conditional*** for an argument of SL with a finite number of premises is the material conditional whose antecedent is the iterated conjunction of the argument's premises and whose consequent is the conclusion of the argument.

Theorem 3.11 An argument is truth-functionally valid if and only if its corresponding material conditional is truth-functionally true.

Theorem 3.12 A sentence **P** of SL is ***truth-functionally false*** if and only if $\{\mathbf{P}\}$ is truth-functionally inconsistent.

Theorem 3.13 A sentence **P** of SL is ***truth-functionally true*** if and only if $\{\sim \mathbf{P}\}$ is truth-functionally inconsistent.

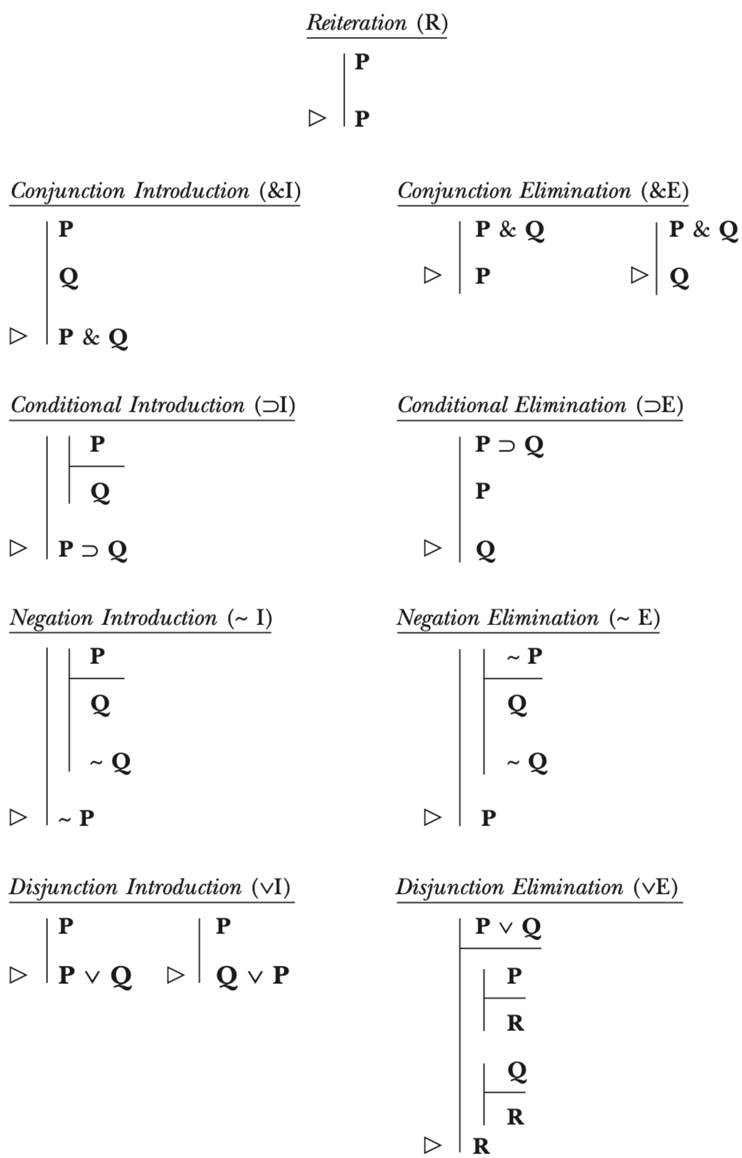
Theorem 3.14 A sentence **P** of SL is ***truth-functionally indeterminate*** if and only if both $\{\sim \mathbf{P}\}$ and $\{\mathbf{P}\}$ are truth-functionally consistent.

Theorem 3.15 Sentences **P** and **Q** of SL are ***truth-functionally equivalent*** if and only if $\{\sim(\mathbf{P} \equiv \mathbf{Q})\}$ is truth-functionally inconsistent.

Theorem 3.16 A set Γ of sentences of SL truth-functionally entails a sentence **P** of SL if and only if $\Gamma \cup \{\sim \mathbf{P}\}$ is truth-functionally inconsistent.

4 The Derivation System SD

Definition 4.1 The following are the derivation rules for SD.



Biconditional Introduction (≡I)

$$\begin{array}{l|l} & \begin{array}{|l} \mathbf{P} \\ \hline \mathbf{Q} \end{array} \\ & \begin{array}{|l} \mathbf{Q} \\ \hline \mathbf{P} \end{array} \\ & \mathbf{P} \equiv \mathbf{Q} \end{array}$$

Biconditional Elimination (≡E)

$$\begin{array}{l|l} & \mathbf{P} \equiv \mathbf{Q} \\ & \mathbf{P} \\ \triangleright & \mathbf{Q} \end{array} \qquad \begin{array}{l|l} & \mathbf{P} \equiv \mathbf{Q} \\ & \mathbf{Q} \\ \triangleright & \mathbf{P} \end{array}$$

Definition 4.2 A ***derivation in SD*** is a series of sentences of SL, each of which is either an assumption or is obtained from previous sentences by one of the rules of SD.

Definition 4.3 A sentence **P** of SL is ***derivable in SD*** from a set Γ of sentences of SL if and only if there is a derivation in SD in which all the primary assumptions are members of Γ and **P** occurs in the scope of only those assumptions.

Definition 4.4 An argument of SL is ***valid in SD*** if and only if the conclusion of the argument is derivable in SD from the set consisting of the premises. An argument of SL is ***invalid in SD*** if and only if it is not valid in SD.

Definition 4.5 A sentence **P** of SL is a ***theorem in SD*** if and only if **P** is derivable in SD from the empty set.

Definition 4.6 Sentences **P** and **Q** are ***equivalent in SD*** if and only if **Q** is derivable from $\{\mathbf{P}\}$ and **P** is derivable from $\{\mathbf{Q}\}$.

Definition 4.7 A set Γ of sentences of SL is ***inconsistent in SD*** if and only if there is a sentence **P** such that both **P** and $\sim$ **P** are derivable in SD from Γ. A set Γ is ***consistent in SD*** if and only if it is not inconsistent in SD.

Notation 4.8 '**P** is derivable from Γ' is notated ' $\Gamma \vdash \mathbf{P}$'. '**P** is not derivable from Γ' is notated '$\Gamma \nvdash \mathbf{P}$.'

Definition 4.9 The following consists of all the derivation rules of $SD+$ that are additive to SD, as well as the rules of replacement of $SD+$.

Modus Tollens (MT)

$$\begin{array}{l|l} & \mathbf{P} \supset \mathbf{Q} \\ & \sim \mathbf{Q} \\ \triangleright & \sim \mathbf{P} \end{array}$$

Hypothetical Syllogism (HS)

$$\begin{array}{l|l} & \mathbf{P} \supset \mathbf{Q} \\ & \mathbf{Q} \supset \mathbf{R} \\ \triangleright & \mathbf{P} \supset \mathbf{R} \end{array}$$

Disjunctive Syllogism (DS)

$$\begin{array}{l|l} & \mathbf{P} \vee \mathbf{Q} \\ & \sim \mathbf{P} \\ \triangleright & \mathbf{Q} \end{array} \quad \text{or} \quad \begin{array}{l|l} & \mathbf{P} \vee \mathbf{Q} \\ & \sim \mathbf{Q} \\ \triangleright & \mathbf{P} \end{array}$$

Commutation (Com)

$\mathbf{P} \,\&\, \mathbf{Q} \lhd\rhd \mathbf{Q} \,\&\, \mathbf{P}$
$\mathbf{P} \vee \mathbf{Q} \lhd\rhd \mathbf{Q} \vee \mathbf{P}$

Association (Assoc)

$\mathbf{P} \,\&\, (\mathbf{Q} \,\&\, \mathbf{R}) \lhd\rhd (\mathbf{P} \,\&\, \mathbf{Q}) \,\&\, \mathbf{R}$
$\mathbf{P} \vee (\mathbf{Q} \vee \mathbf{R}) \lhd\rhd (\mathbf{P} \vee \mathbf{Q}) \vee \mathbf{R}$

Implication (Impl)

$\mathbf{P} \supset \mathbf{Q} \lhd\rhd \sim \mathbf{P} \vee \mathbf{Q}$

Double Negation (DN)

$\mathbf{P} \lhd\rhd \sim\sim \mathbf{P}$

De Morgan (DeM)

$\sim (\mathbf{P} \,\&\, \mathbf{Q}) \lhd\rhd \sim \mathbf{P} \vee \sim \mathbf{Q}$
$\sim (\mathbf{P} \vee \mathbf{Q}) \lhd\rhd \sim \mathbf{P} \,\&\, \sim \mathbf{Q}$

Idempotence (Idem)

$\mathbf{P} \lhd\rhd \mathbf{P} \,\&\, \mathbf{P}$
$\mathbf{P} \lhd\rhd \mathbf{P} \vee \mathbf{P}$

Transposition (Trans)

$\mathbf{P} \supset \mathbf{Q} \lhd\rhd \sim \mathbf{Q} \supset \sim \mathbf{P}$

Exportation (Exp)

$\mathbf{P} \supset (\mathbf{Q} \supset \mathbf{R}) \lhd\rhd (\mathbf{P} \,\&\, \mathbf{Q}) \supset \mathbf{R}$

Distribution (Dist)

$\mathbf{P} \,\&\, (\mathbf{Q} \vee \mathbf{R}) \lhd\rhd (\mathbf{P} \,\&\, \mathbf{Q}) \vee (\mathbf{P} \,\&\, \mathbf{R})$
$\mathbf{P} \vee (\mathbf{Q} \,\&\, \mathbf{R}) \lhd\rhd (\mathbf{P} \vee \mathbf{Q}) \,\&\, (\mathbf{P} \vee \mathbf{R})$

Equivalence (Equiv)

$\mathbf{P} \equiv \mathbf{Q} \lhd\rhd (\mathbf{P} \supset \mathbf{Q}) \,\&\, (\mathbf{Q} \supset \mathbf{P})$
$\mathbf{P} \equiv \mathbf{Q} \lhd\rhd (\mathbf{P} \,\&\, \mathbf{Q}) \vee (\sim \mathbf{P} \,\&\, \sim \mathbf{Q})$

5 Sentential Logic: Metatheory

Theorem 5.1 Every sentence of SL has an equal number of left and right parentheses.

Definition 5.2 A ***truth-function*** is an n-place function (with $n \in \mathbb{Z}^+$) that maps each combination of n truth-values to a single truth-value.

Metatheorem 5.3 Every truth-function can be expressed by a sentence of SL that contains no sentential connectives other than '$\sim$', ' $\vee$', and '&.'

Metatheorem 5.4 (*Soundness Metatheorem for SD*) For any set Γ of sentences of SL and any sentence $\mathbf{P}$ of SL, if $\Gamma \vdash \mathbf{P}$ in SD then $\Gamma \vDash \mathbf{P}$.

Metatheorem 5.5 If $\Gamma \vDash \mathbf{P}$, then for every superset Γ of Γ, $\Gamma' \vDash \mathbf{P}$.

Metatheorem 5.6 If $\Gamma \cup \{\mathbf{Q}\} \vDash \mathbf{R}$, then $\Gamma \vDash \mathbf{Q} \supset \mathbf{R}$.

Metatheorem 5.7 If $\Gamma \vDash \mathbf{Q}$ and $\Gamma \vDash \sim \mathbf{Q}$ for some sentence $\mathbf{Q}$, then Γ is truth-functionally inconsistent.

Metatheorem 5.8 If $\Gamma \cup \{\mathbf{Q}\}$ is truth-functionally inconsistent, then $\Gamma \vDash \sim \mathbf{Q}$.

Metatheorem 5.9 For every sentence $\mathbf{P}$ of SL, if $\Gamma \vDash \mathbf{P}$ then $\Gamma \vdash \mathbf{P}$ in SD.

Metatheorem 5.10 For any set Γ of sentences of SL, if Γ is consistent in SD then Γ is truth-functionally consistent.

Metatheorem 5.11 If $\Gamma \vDash \mathbf{P}$ then $\Gamma \cup \{\sim \mathbf{P}\}$ is truth-functionally inconsistent.

Definition 5.12 A set Γ of sentences in SL is ***maximally consistent in SD*** if and only if Γ is consistent in SD and, for every sentence $\mathbf{P}$ of SL that is not a member of Γ, $\Gamma \cup \{\sim\mathbf{P}\}$ is inconsistent in SD.

Metatheorem 5.13 (The *Maximal Consistency Lemma*) If Γ is a set of sentences of SL that is consistent in SD, then Γ is a subset of at least one set of sentences that is maximally consistent in SD.

Metatheorem 5.14 If a set Γ of sentences of SL is inconsistent in SD, then some finite subset of Γ is also inconsistent in SD.

Metatheorem 5.15 If Γ is inconsistent in SD, then every superset of Γ is inconsistent in SD.

Metatheorem 5.16 (*Consistency Lemma*) Every set of sentences of SL that is maximally consistent in SD is truth-functionally consistent.

Metatheorem 5.17 If $\Gamma \vdash \mathbf{P}$ and Γ^* is a maximally consistent superset of Γ, then $\mathbf{P}$ is a member of Γ^*.

Metatheorem 5.18 If Γ^* is maximally consistent in SD and $\mathbf{P}$ and $\mathbf{Q}$ are sentences of SL, then:

- $\sim\mathbf{P} \in \Gamma^*$ if and only if $\mathbf{P} \notin \Gamma^*$.
- $\mathbf{P}$ & $\mathbf{Q} \in \Gamma^*$ if and only if both $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$.
- $\mathbf{P} \vee \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \in \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$.
- $\mathbf{P} \supset \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \notin \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$ or $\mathbf{Q} \in \Gamma^*$.
- $\mathbf{P} \equiv \mathbf{Q} \in \Gamma^*$ if and only if either $\mathbf{P} \in \Gamma^*$ and $\mathbf{Q} \in \Gamma^*$, or $\mathbf{P} \notin \Gamma^*$ and $\mathbf{Q} \notin \Gamma^*$.

Metatheorem 5.19 A set Γ of sentences of SL is truth-functionally consistent if and only if every finite subset of Γ is truth-functionally consistent.

6 Predicate Logic: Syntax and Symbolization

Definition 6.1 The vocabulary of PL consists of:

- ***Sentence Letters:*** The capital Roman letters 'A' through 'Z', with or without positive-integer subscripts.
- ***Predicates*** The capital Roman letters 'A' through 'Z', with or without positive-integer subscripts, followed by one or more primes.
- ***Individual Terms***:

- ***Individual constants***: The lowercase Roman letters 'a' through 'v', with or without positive-integer subscripts.
- ***Individual variables***: The lowercase Roman letters 'w' through 'z', with or without positive-integer subscripts.

- ***Truth-functional connectives:*** $\sim$, &, $\vee$, $\supset$, $\equiv$.
- ***Quantifier symbols***: $\forall, \exists$
- ***Punctuation marks***: ( )

Definition 6.2 An ***expression*** of PL is a sequence of not necessarily distinct elements of the vocabulary of PL.

Definition 6.3 A ***quantifier of PL*** is an expression of PL of the form $(\forall \mathbf{x})$ or $(\exists \mathbf{x})$. An expression of the first from is a ***universal quantifier***, and one of the second form is an ***existential quantifier***.

Definition 6.4 ***Atomic formulas of PL*** consist of every expression of PL that is either a sentence letter of PL or an **n**-place predicate of PL followed by **n** individual terms of PL.

Definition 6.5 The following is a recursive definition of ***formula of PL***:

- Every atomic formula of PL is a formula of PL.
- If $\mathbf{P}$ is a formula of PL, so is $\sim \mathbf{P}$.
- If $\mathbf{P}$ and $\mathbf{Q}$ are formulas of PL, so are $(\mathbf{P} \mathbin{\&} \mathbf{Q})$, $(\mathbf{P} \vee \mathbf{Q})$, $(\mathbf{P} \supset \mathbf{Q})$, and $(\mathbf{P} \equiv \mathbf{Q})$.
- If $\mathbf{P}$ is a formula of PL that contains at least one occurrence of $\mathbf{x}$ and no $\mathbf{x}$-quantifier, then $(\forall \mathbf{x})\mathbf{P}$ and $(\exists \mathbf{x})\mathbf{P}$ and are both formulas of PL.
- Nothing is a formula of PL unless it can be formed by repeated applications of the previous clauses.

Definition 6.6 The terms ***subformula*** and ***main logical operator*** are defined as follows:

- Every formula is a subformula of itself.
- IF $\mathbf{P}$ is an atomic formula of PL, then $\mathbf{P}$ contains no logical operator, and hence no main logical operator, and $\mathbf{P}$ has no immediate subformula.
- If $\mathbf{P}$ is a formula of PL of the form $\sim\mathbf{Q}$, then the tilde that precedes $\mathbf{Q}$ is the main logical operator of $\mathbf{P}$, and $\mathbf{Q}$ is the immediate subformula of $\mathbf{P}$.
- If $\mathbf{P}$ is a formula of PL of the form $(\mathbf{Q} \mathbin{\&} \mathbf{R})$, $(\mathbf{Q} \vee \mathbf{R})$, $(\mathbf{Q} \supset \mathbf{R})$, or $(\mathbf{Q} \equiv \mathbf{R})$, then the binary connective between $\mathbf{Q}$ and $\mathbf{R}$ is the main logical operator of $\mathbf{P}$, and $\mathbf{Q}$ and $\mathbf{R}$ are the immediate subformulas of $\mathbf{P}$.

- If $\mathbf{P}$ is a formula of PL of the form $(\forall\mathbf{x})\mathbf{Q}$ or of the form $(\exists\mathbf{x})\mathbf{Q}$, then the quantifier that occurs before $\mathbf{Q}$ is the main logical operator of $\mathbf{P}$, and $\mathbf{Q}$ is the immediate subformula of $\mathbf{P}$.

Definition 6.7 The ***subformulas*** of a formula $\mathbf{P}$ of PL are

- $\mathbf{P}$ itself
- the immediate subformulas of $\mathbf{P}$
- the subformulas of $\mathbf{P}$'s immediate subformulas.

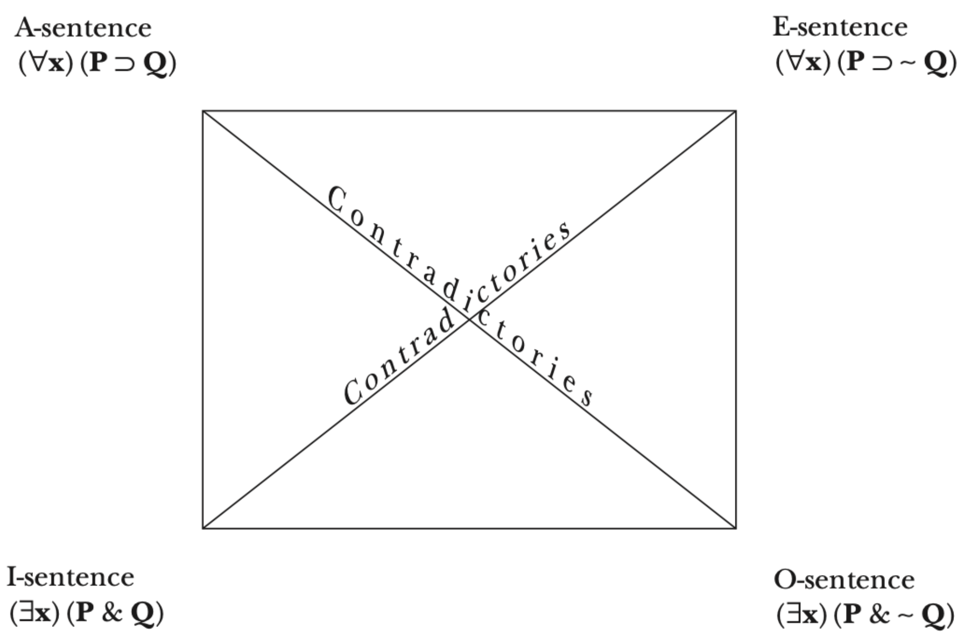
Definition 6.8 The ***scope of a quantifier*** in a formula $\mathbf{P}$ of PL is the quantifier itself and the subformula $\mathbf{Q}$ that immediately follows the quantifier.

Definition 6.9 A ***bound variable*** is a variable $\mathbf{x}$ that occurs in a formula $\mathbf{P}$ of PL that is within the scope of an $\mathbf{x}$-quantifier.

Definition 6.10 A ***free variable*** is a variable $\mathbf{x}$ that occurs in a formula of $\mathbf{p}$ of PL that is not bound.

Definition 6.11 A formula $\mathbf{P}$ of PL is a ***sentence of PL*** if and only if no occurrence of a variable in $\mathbf{P}$ is free.

Definition 6.12 If $\mathbf{P}$ is a sentence of PL of the form $(\forall\mathbf{x})\mathbf{Q}$ or $(\exists\mathbf{x})\mathbf{Q}$, and $\mathbf{a}$ is an individual constant, then $\mathbf{Q}(\mathbf{a}/\mathbf{x})$ is a substitution instance of $\mathbf{P}$. The constant $\mathbf{a}$ is the ***instantiating constant***.



The following table displays equivalent sentence forms. Here **P** is a formula containing at least one free occurrence of **x** and **Q** is a sentence of *PL* in which **x** *does not* occur.

$(\exists x)\mathbf{P} \supset \mathbf{Q}$	$(\forall x)(\mathbf{P} \supset \mathbf{Q})$
$(\forall x)\mathbf{P} \supset \mathbf{Q}$	$(\exists x)(\mathbf{P} \supset \mathbf{Q})$
$\mathbf{Q} \supset (\exists x)\mathbf{P}$	$(\exists x)(\mathbf{Q} \supset \mathbf{P})$
$\mathbf{Q} \supset (\forall x)\mathbf{P}$	$(\forall x)(\mathbf{Q} \supset \mathbf{P})$
$(\exists x)\mathbf{P} \vee \mathbf{Q}$	$(\exists x)(\mathbf{P} \vee \mathbf{Q})$
$(\forall x)\mathbf{P} \vee \mathbf{Q}$	$(\forall x)(\mathbf{P} \vee \mathbf{Q})$
$\mathbf{Q} \vee (\exists x)\mathbf{P}$	$(\exists x)(\mathbf{Q} \vee \mathbf{P})$
$\mathbf{Q} \vee (\forall x)\mathbf{P}$	$(\forall x)(\mathbf{Q} \vee \mathbf{P})$
$(\exists x)\mathbf{P} \,\&\, \mathbf{Q}$	$(\exists x)(\mathbf{P} \,\&\, \mathbf{Q})$
$(\forall x)\mathbf{P} \,\&\, \mathbf{Q}$	$(\forall x)(\mathbf{P} \,\&\, \mathbf{Q})$
$\mathbf{Q} \,\&\, (\exists x)\mathbf{P}$	$(\exists x)(\mathbf{Q} \,\&\, \mathbf{P})$
$\mathbf{Q} \,\&\, (\forall x)\mathbf{P}$	$(\forall x)(\mathbf{Q} \,\&\, \mathbf{P})$

Definition 6.13 The language PLE is an expansion of PL that includes (in addition to all the vocabulary of PL) a two-place predicate that is defined as the identity predicate, and functors.

Definition 6.14 The ***identity predicate*** is symbolized as '=' and denotes that two objects are identical to each other. The identity predicate is ***transitive***, ***symmetric***, and ***reflexive***.

Definition 6.15 A ***function*** is an operation that takes one or more element of a set as arguments and returns a single value. We require that the functions we symbolize with functors have the following characteristics:

- An **n**-place function must yield one and only one value for each **n**-tuple of arguments.
- The value of a function for an **n**-tuple of members of a UD must be a member of that UD.

Definition 6.16 ***Complex terms*** are terms of the form $f(\mathbf{t}_1,\mathbf{t}_2, \ldots, \mathbf{t}_n)$.

INDIVIDUAL TERMS OF *PLE*

	Open	*Closed*
Simple	Individual variables	Individual constants
Complex	Individual term formed from a functor and *at least one* individual variable—for example, f(x), f(a,x), $g(f$(a),y), $g(h$(x,y),a)	Individual term formed from a functor and containing *no* individual variable—for example, f(a), g(a,b), $f(g$(a,f(a,c)))

Definition 6.17 In addition to the vocabulary of PL, the vocabulary of PLE also includes

- =": The two-place identity predicate
- functors of PLE: Lowercase italicized Roman letters $a, b, c \ldots$, with or without a numeric subscript, followed by **n** primes.
- Individual terms of PLE:
  - Individual constants are individual terms of PLE
  - Individual variables are individual terms of PLE
  - Expressions of the form $f(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n)$, where f is an **n**-place functor and $\mathbf{t}_1$, $\mathbf{t}_2, \ldots, \mathbf{t}_n$ are individual terms of PLE.
- If **P** is a sentence of PLE of the form $(\forall \mathbf{x})\mathbf{Q}$ or $(\exists \mathbf{x})\mathbf{Q}$ and **t** is a closed individual term, then $\mathbf{Q}(\mathbf{t}/\mathbf{x})$ is a ***substitution instance*** of **P**. The individual term **t** is the ***instantiating individual term.***

7 Predicate Logic: Semantics

Definition 7.1 The basic semantic concept of PL is ***interpretation***.

Definition 7.2 The ***extension*** of a predicate is the set of **n**-tuples of the UD that satisfy the predicate.

Definition 7.3 An ***interpretation*** for PL specifies a nonempty set as a UD and assigns a truth-value to each sentence letter of PL, a member of the UD to each individual constant of PL, and a set of **n**-tuples of members of the UD to each **n**0place predicate of PL.

Definition 7.4 A ***variable assignment for an interpretation I*** assigns to each individual variable of PL a member of the UD. We use the notation $\mathbf{d_I}$ to denote a variable assignment for interpretation **I**, and $\mathbf{d_I}(\mathbf{x})$ to denote the value $\mathbf{d_I}$ assigns to **x**.

Definition 7.5 The ***variant of assignment*** $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ means the variable assignment for interpretation **I** that assigns the same values as $\mathbf{d_I}$ to all variables other than **x**, and assigns the value **u** to **x**.

Definition 7.6 The notation $\text{den}_{\mathbf{I},\mathbf{d_I}}$ is read 'the denotation with respect to interpretation **I** and variable interpretation $\mathbf{d_I}$ and is defined as follows:

- If **t** is a variable, then $\text{den}_{\mathbf{I},\mathbf{d_I}} = \mathbf{d_I}(\mathbf{t})$.
- If **t** is an individual constant then $\text{den}_{\mathbf{I},\mathbf{d_I}} = \mathbf{I}(\mathbf{t})$.

Definition 7.7 The following is the recursive definition of the concept of satisfaction of formulas. Let **I** be an interpretation, $\mathbf{d_I}$ a variable assignment for **I**, and **P** a formula of PL.Then

1. If **P** is a sentence letter, $\mathbf{d_I}$ satisfies **P** on interpretation **I** if and only if $\mathbf{I}(\mathbf{P}) = \mathbf{T}$.

2. If $\mathbf{P}$ is an atomic formula of the form $\mathbf{At}_1 \ldots \mathbf{t}_n$ (where $\mathbf{A}$ is an $\mathbf{n}$-place predicate), $\mathbf{d_I}$ satisfies $\mathbf{P}$ on interpretation $\mathbf{I}$ if and only if $<\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_1), \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_1), \ldots, \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_n)> \in \mathbf{I}(\mathbf{A})$.

3. If $\mathbf{P}$ is a formula of the form $\sim\mathbf{Q}$, $\mathbf{d_I}$ satisfies $\mathbf{P}$ on interpretation $\mathbf{I}$ if and only if $\mathbf{d_I}$ does not satisfy $\mathbf{Q}$ on interpretation $\mathbf{I}$.

4. If $\mathbf{P}$ is a formula of the form $\mathbf{Q}$ & $\mathbf{R}$, $\mathbf{d_I}$ satisfies $\mathbf{P}$ on interpretation $\mathbf{I}$ if and only if $\mathbf{d_I}$ satisfies $\mathbf{Q}$ on interpretation $\mathbf{I}$ and $\mathbf{d_I}$ satisfies $\mathbf{R}$ on interpretation $\mathbf{I}$.

5. If $\mathbf{P}$ is of the form $\mathbf{Q} \vee \mathbf{R}$, $\mathbf{d_I}$ satisfies $\mathbf{P}$ on interpretation $\mathbf{I}$ if and only if either $\mathbf{d_I}$ satisfies $\mathbf{Q}$ on interpretation $\mathbf{I}$ or $\mathbf{d_I}$ satisfies $\mathbf{R}$ on interpretation $\mathbf{I}$.

6. If $\mathbf{P}$ is a formula of the form $\mathbf{Q} \supset \mathbf{R}$, $\mathbf{d_I}$ satisfies $\mathbf{P}$ on interpretation $\mathbf{I}$ if and only if either $\mathbf{d_I}$ does not satisfy $\mathbf{Q}$ on interpretation $\mathbf{I}$ or $\mathbf{d_I}$ satisfies $\mathbf{R}$ on interpretation $\mathbf{I}$.

7. If $\mathbf{P}$ is a formula of the form $\mathbf{Q} \equiv \mathbf{R}$, $\mathbf{d_I}$ satisfies $\mathbf{P}$ on interpretation $\mathbf{I}$ if and only if either $\mathbf{d_I}$ satisfies $\mathbf{Q}$ on interpretation $\mathbf{I}$ and $\mathbf{d_I}$ satisfies $\mathbf{R}$ on interpretation $\mathbf{I}$, or $\mathbf{d_I}$ does not satisfy $\mathbf{Q}$ on interpretation $\mathbf{I}$ and $\mathbf{d_I}$ does not satisfy $\mathbf{R}$ on interpretation $\mathbf{d_I}$.

8. If $\mathbf{P}$ is a formula of the form $(\forall\mathbf{x})\mathbf{Q}$, $\mathbf{d_I}$ satisfies $\mathbf{P}$ on interpretation $\mathbf{I}$ if and only if each member of $\mathbf{u}$ of the UD is such that $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies $\mathbf{Q}$ on interpretation $\mathbf{I}$.

9. If $\mathbf{P}$ is a formula of the form $(\exists\mathbf{x})\mathbf{Q}$, $\mathbf{d_I}$ satisfies $\mathbf{P}$ on interpretation $\mathbf{I}$ if and only if there is at least one member $\mathbf{u}$ of the UD such that $\mathbf{d_I}[\mathbf{u}/\mathbf{x}]$ satisfies $\mathbf{Q}$ on interpretation $\mathbf{I}$.

10. If $\mathbf{P}$ is an atomic formula of the form $\mathbf{t}_1 = \mathbf{t}_2$, then $\mathbf{d_I}$ satisfies $\mathbf{P}$ on interpretation $\mathbf{I}$ if and only if $\text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_1) = \text{den}_{\mathbf{I},\mathbf{d_I}}(\mathbf{t}_2)$.

Definition 7.8 A sentence $\mathbf{P}$ is ***true on an interpretation I*** if and only if every variable assignment $\mathbf{d_I}$ satisfies $\mathbf{P}$ on $\mathbf{I}$. A sentence $\mathbf{P}$ of PL is **false on an interpretation I** if and only if no variable assignment $\mathbf{d_I}$ satisfies $\mathbf{P}$ on $\mathbf{I}$.

Definition 7.9 A sentence $\mathbf{P}$ of PL is ***quantificationally true*** if and only if $\mathbf{P}$ is true on every interpretation.

Definition 7.10 A sentence $\mathbf{P}$ of PL is ***quantificationally false*** if and only if $\mathbf{P}$ is false on every interpretation.

Definition 7.11 A sentence $\mathbf{P}$ of PL is ***quantificationally indeterminate*** if and only if $\mathbf{P}$ is neither quantificationally true nor quantificationally false.

Definition 7.12 Sentences $\mathbf{P}$ and $\mathbf{Q}$ of PL are ***quantificationally equivalent*** if and only if there is no interpretation on which $\mathbf{P}$ and $\mathbf{Q}$ have different truth-values.

Definition 7.13 A set of sentences of PL is ***quantificationally consistent*** if and only if there is at least one interpretation on which all the members of the set are true. A set of sentences of PL is ***quantificationally inconsistent*** if and only if the set is not quantificationally consistent.

Definition 7.14 A set Γ of sentences of PL ***quantificationally entails*** a sentence $\mathbf{P}$ of PL if and only if there is no interpretation on which every member of Γ is true and $\mathbf{P}$ is false.

Definition 7.15 An argument of PL is ***quantificationally valid*** if and only if there is no interpretation on which every premise is true and the conclusion is false. An argument of PL is ***quantificationally invalid*** if and only if the argument is not quantificationally valid.

8 Predicate Logic: Derivations

The following are the elimination and introduction rules unique to PD. Because every sentence of SL is a sentence of PL, all of the derivation rules for SD are also in PD.

Universal Elimination ($\forall$E)

$$\begin{array}{l|l} & (\forall \mathbf{x})\mathbf{P} \\ \rhd & \mathbf{P}(\mathbf{a}/\mathbf{x}) \end{array}$$

Existential Introduction ($\exists$I)

$$\begin{array}{l|l} & \mathbf{P}(\mathbf{a}/\mathbf{x}) \\ \rhd & (\exists \mathbf{x})\mathbf{P} \end{array}$$

Universal Introduction ($\forall$I)

$$\begin{array}{l|l} & \mathbf{P}(\mathbf{a}/\mathbf{x}) \\ \rhd & (\forall \mathbf{x})\mathbf{P} \end{array}$$

provided that

(i) **a** does not occur in an open assumption.
(ii) **a** does not occur in $(\forall \mathbf{x})\mathbf{P}$.

Existential Elimination ($\exists$E)

$$\begin{array}{l|l} & (\exists \mathbf{x})\mathbf{P} \\ & \begin{array}{|l} \mathbf{P}(\mathbf{a}/\mathbf{x}) \\ \hline \mathbf{Q} \end{array} \\ \rhd & \mathbf{Q} \end{array}$$

provided that

(i) **a** does not occur in an open assumption.
(ii) **a** does not occur in $(\exists \mathbf{x})\mathbf{P}$.
(iii) **a** does not occur in $\mathbf{Q}$.

Definition 7.16 The definitions of derivability, validity, theorem, equivalence, and inconsistency in PD are completely analogous to those of SD.

The following are some useful strategies in constructing derivations.

- If the current goal sentence can be obtained by Reiteration, use that rule, otherwise,

- If the current goal sentence can be obtained by using a non-subderivation rule, or a series of such rules, do so; otherwise
- Try to obtain the goal sentence by using an appropriate subderivation rule.
- When using a negation rule, try to use an already accessible negation (if there is one) as the $\sim Q$ that the negation rules require be derived.
- When using Universal Elimination use goal sentences as guides when choosing the instantiating constant.
- When the goal to be derived is an existentially quantified sentence make a substitution instance of that sentence a subgoal, with the intent of applying Existential Introduction to that subgoal to obtain the goal.
- When the current goal is a universally quantified sentence make a substitution instance of that quantified sentence a subgoal, with the intent of applying Universal Introduction to that subgoal Make sure the two restrictions on the instantiating constant for the use of Universal Introduction are met. Be sure to choose an instantiating constant that does not occur in the universally quantified sentence that is the goal and that does not occur in any assumption that will be open when Universal Introduction is applied to derive that goal.
- When one of the accessible assumptions is an existentially quantified sentence, consider using Existential Elimination to obtain the current goal. Set up an Existential Elimination subderivation, and continue working within that subderivation until a sentence that does not contain the constant used to form the substitution instance that is the assumption of that subderivation is derived.
- When contradictory sentences are available within an Existential Elimination subderivation but cannot be moved out of that subderivation without violating the restrictions on Existential Elimination, derive another sentence-one that is contradictory to a sentence accessible outside the Existential Elimination subderivation and that does not contain the instantiating constant for this use of Existential Elimination. That sentence will be derivable by the appropriate negation strategy (using the contradictory sentences that are available within the Existential Elimination subderivation).
- There will often be more than one plausible strategy, and often more than one will lead to success. Rather than trying to figure out which of these is the most promising it is often wise to just pick one and pursue it.

$PD+$ contains all of the rules of $SD+$ and PD along with QN.

Quantifier Negation (QN)

$\sim(\forall \mathbf{x})\mathbf{P} \lhd\rhd (\exists \mathbf{x}) \sim \mathbf{P}$

$\sim(\exists \mathbf{x})\mathbf{P} \lhd\rhd (\forall \mathbf{x}) \sim \mathbf{P}$

PDE extends PD to include the following rules for PLE :

Identity Introduction (=I)

$\triangleright \mid (\forall \mathbf{x})\mathbf{x} = \mathbf{x}$

Identity Elimination (=E)

$$\begin{array}{l|l} & \mathbf{t}_1 = \mathbf{t}_2 \\ & \mathbf{P} \\ \triangleright & \mathbf{P}(\mathbf{t}_1//\mathbf{t}_2) \end{array} \qquad \text{or} \qquad \begin{array}{l|l} & \mathbf{t}_1 = \mathbf{t}_2 \\ & \mathbf{P} \\ \triangleright & \mathbf{P}(\mathbf{t}_2//\mathbf{t}_1) \end{array}$$

where $\mathbf{t}_1$ and $\mathbf{t}_2$ are closed terms.

The notation $\mathbf{P}(\mathbf{t}_1//\mathbf{t}_2)$ is read '$\mathbf{P}$ with one ore more occurrences of $\mathbf{t}_2$ replaced by $\mathbf{t}_1$'.

In PDE, we have to modify the rules for PD to account for the identity predicate and functions introduced in PLE :

Here are the quantifier rules, modified as appropriate for the system *PDE*.

Universal Elimination ($\forall$E)

$$\begin{array}{l|l} & (\forall \mathbf{x})\mathbf{P} \\ \triangleright & \mathbf{P}(\mathbf{t}/\mathbf{x}) \end{array}$$

Existential Introduction ($\exists$I)

$$\begin{array}{l|l} & \mathbf{P}(\mathbf{t}/\mathbf{x}) \\ \triangleright & (\exists \mathbf{x})\mathbf{P} \end{array}$$

where $\mathbf{t}$ is a closed term

Universal Introduction ($\forall$I)

$$\begin{array}{l|l} & \mathbf{P}(\mathbf{a}/\mathbf{x}) \\ \triangleright & (\forall \mathbf{x})\mathbf{P} \end{array}$$

provided that:
(i) $\mathbf{a}$ does not occur in an open assumption.
(ii) $\mathbf{a}$ does not occur in $(\forall \mathbf{x})\mathbf{P}$.

Existential Elimination ($\exists$E)

$$\begin{array}{l|l} & (\exists \mathbf{x})\mathbf{P} \\ & \begin{array}{|l} \mathbf{P}(\mathbf{a}/\mathbf{x}) \\ \hline \mathbf{Q} \end{array} \\ \triangleright & \mathbf{Q} \end{array}$$

provided that:
(i) $\mathbf{a}$ does not occur in an open assumption.
(ii) $\mathbf{a}$ does not occur in $(\exists \mathbf{x})\mathbf{P}$.
(iii) $\mathbf{a}$ does not occur in $\mathbf{Q}$.

where $\mathbf{a}$ is an individual constant.