Chapter 14.3: Partial Derivatives

Mechanics

1. Find all first and second partial derivatives for $f(x, y) = e^x + x\ln(y)$.

   Answer:
$$f_x = e^x + \ln(y), \quad f_y = \frac{x}{y}$$
$$f_{xx} = e^x, \quad f_{xy} = \frac{1}{y} = f_{yx}, \quad f_{yy} = -\frac{x}{y^2}$$

2. Find f_x, f_y, f_z, and f_{xzz} for the function $f(x, y, z) = x\sin(yz)$.

   Answer:
$$f_x = \sin(yz), \quad f_y = xz\cos(yz), \quad f_z = xy\cos(yz), \quad f_{xzz} = -y^2\sin(yz)$$

3. Find the total derivative Df at the given point for each function below. Remember that Df is the matrix of (partial) derivatives of the function and if f is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ then Df is a $m \times n$ matrix.

   (a) $f(x) = 2x^3 + 7$ at $x = 2$.

   (b) $\mathbf{f}(t) = \langle 2\cos(t), 2\sin(t), t\rangle$ at $t = \pi/2$.

   (c) $f(x, y) = \sqrt{y - x}$ at $(x, y) = (1, 2)$.

   (d) $f(x, y, z) = e^{2y-x} + z^2 + 4$ at $(x, y, z) = (1, 2, 3)$.

   (e) $\mathbf{f}(s, t) = \langle 2s + 3t, t - s\rangle$ at $(s, t) = (1, 1)$.
   Note: The graph of this function is a surface (in this case all of $\mathbb{R}^2$) parameterized by two variables just like the graph of the function in (b) is a curve parameterized by one variable - we'll see these more later! Another way of thinking about this is that this is a *change of variables* for $\mathbb{R}^2$ between the system of coordinates (s, t) and (x, y).

   Answer:

   (a) $Df(2) = f'(2) = [24]$

   (b) $D\mathbf{f}(\pi/2) = \mathbf{f}'(\pi/2) = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

   (c) $Df(1, 2) = \begin{bmatrix} -1/2 & 1/2 \end{bmatrix}$

   (d) $Df(1, 2, 3) = \begin{bmatrix} -e^3 & 2e^3 & 6 \end{bmatrix}$

   (e) $D\mathbf{f}(1, 1) = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$

Applications

4. The speed of sound C traveling through ocean water is a function of temperature, salinity, and depth. It may be modeled by the function
$$C(T, S, D) = 1450 + 4.5T - 0.05T^2 + 0.0003T^3 + (1.5 - 0.01T)(S - 35) + 0.015D,$$

where C is the speed of sound in meters/second, T is the temprature in degrees Celsius, S is the salinity in grams/liter of water, and D is the depth below the ocean surface in meters.

(a) State the units in which each of the partial derivatives C_T, C_S, and C_D are expressed and explain the physical meaning of each.

(b) Find the partial derivatives C_T, C_S, and C_D.

(c) Evaluate each of the three partial derivatives at the point where $T = 10, S = 35$, and $D = 100$. What does the sign of each partial derivative tell us about the behavior of the function C at the point $(10, 35, 100)$?

Answer:

(a) C_T: (meters/second)/ degree Celsius - this gives the change in speed for each one degree C of temperature increase. C_S (meters/second)/(grams/liter) - this gives the change in speed for each one gram/liter increase in salinity C_D: (meters/second)/meter - this gives the change in speed for each one meter increase in depth below the surface

(b) $C_T = 4.5 - 0.1T + 0.0009T^2 - 0.01(S - 35)$ $C_S = 1.5 - 0.01T$ $C_D = 0.015$

(c) At $(T, S, D) = (10, 35, 100)$, we have $C_T = 3.59, C_S = 1.4, C_D = 0.015$. This tells us that if we increase the temperature, salinity, or depth from these conditions the speed of sound will increase as well.

5. Recall from last week's worksheet that a utility function is a multivariable function $u(x, y, z)$, where x, y, z represent three independent properties of an object (eg., price, quantity, quality), and u tells you how much you value that item. The *marginal utility functions* are the partial derivatives u_x, u_y and u_z. What is the economic interpretation of the marginal utilities?

   Answer: The marginal utility functions tell you how much your utility (value) changes when you change one of the properties of the object, while keeping the other two properties fixed.

Extensions

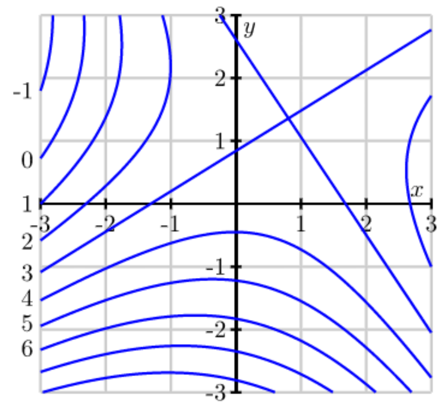
6. Below is a contour plot for a function $f(x, y)$, with values for some of the contours (level curves) indicated on the *left* of the figure.

(a) Find the sign of the partial derivatives $f_x(-2,-1)$ and $f_y(-2,-1)$.

(b) At the point $(0,-1/2)$, which is larger? f_x or f_y?

(c) Find all (x,y) where $f_x(x,y)=0$.

(d) Locate, if possible, one point (x,y) where $f_x(x,y)<0$.



Answer:

(a) $f_x(-2,-1)>0$ and $f_y(-2,-1)<0$
(b) $f_y(0,-1/2)>f_x(0,-1/2)$
(c) $f_x(x,y)=0$ along the tops of the arcs labeled $4,5,6,\ldots$
(d) One such point is $(1,-3/2)$

7. The fifth-order partial derivative $\partial^5 f/\partial x^2\partial y^3$ is zero for each of the following functions. To show this as quickly as possible, which variable would you differentiate with respect to first: x or y?

   Try to answer without writing anything down. Why did you make the choice you did?

   (a) $f(x,y)=y^2x^4e^x+2$
   (b) $f(x,y)=y^2+y(\sin(x)-x^4)$
   (c) $f(x,y)=x^2+5xy+\sin(x)+7e^x$
   (d) $f(x,y)=xe^{y/2}$

   Answer: Note this does not have a definitive right answer - some differences may arise and that's good! Discuss!

   (a) First y since $\partial^3 f/\partial y^3=0$ and the y-partial derivatives are easier
   (b) First y, since $\partial^3 f/\partial y^3=0$
   (c) First y, since $\partial^2 f/\partial y^2=0$
   (d) First x, since $\partial^2 f/\partial x^2=0$ and the x-partial derivatives are easier.

   A common theme is to work with the variable with lower powers/simpler expressions first when taking mixed partials.

8. Let A be any 2×2 matrix, and let $\mathbf{f}:\mathbb{R}^2\to\mathbb{R}^2$ be given by $\mathbf{f}(\mathbf{x})=A\mathbf{x}$. Compute the total derivative $D\mathbf{f}$. What do you notice? What familiar family of functions from Calc 1 does this remind you of? Can you generalize this result?