

## Chapter 13.4: Curvature and Normals

**G3: Geometry of Curves.** I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.

### Mechanics

1. Find  $\mathbf{T}$ ,  $\mathbf{N}$  and  $\kappa$  for the curve  $\mathbf{r}(t) = \cos(2t)\mathbf{i} - \sin(2t)\mathbf{j} + 6t\mathbf{k}$ , for  $t \geq 0$ . What do you notice about  $\kappa$ ? Explain (perhaps with a picture) why this happens.

**Answer:**  $\mathbf{T}(t) = \frac{1}{\sqrt{10}}(-\sin(2t)\mathbf{i} - \cos(2t)\mathbf{j} + 3\mathbf{k}),$

$\mathbf{N}(t) = -\cos(2t)\mathbf{i} + \sin(2t)\mathbf{j}$

$\kappa(t) = \frac{1}{10}$ . The curvature is constant.

2. Compute the unit tangent vector, unit normal vector, and curvature of the curve  $\mathbf{r}(t) = \langle \sqrt{2}t, 1+t, e^t \rangle$  for all  $t \in \mathbb{R}$ .

**Answer:**  $\mathbf{T}(t) = \frac{1}{\sqrt{3+e^{2t}}} \langle \sqrt{2}, 1, e^t \rangle,$

$\mathbf{N}(t) = \frac{1}{\sqrt{9+3e^{2t}}} \langle -\sqrt{2}e^t, -e^t, 3 \rangle,$

$\kappa(t) = \frac{\sqrt{3}e^t}{(3+e^{2t})^{3/2}}.$

3. Compute  $\mathbf{N}$  for the curve  $\mathbf{r}(t) = \langle t, (1/3)t^3 \rangle, t \in \mathbb{R}$  for  $t \neq 0$ .

Does  $\mathbf{N}$  exist at  $t = 0$ ? Graph the curve, along with its normal vectors at the times  $t = -1, -0.5, 0.5, 1$  and explain what is happening to  $\mathbf{N}$  as  $t$  passes through  $(0, 0)$

**Answer:**  $\mathbf{N} = \left\langle \frac{-t^2}{\sqrt{1+t^4}}, \frac{1}{\sqrt{1+t^4}} \right\rangle$  if  $t > 0$  and  $\left\langle \frac{t^2}{\sqrt{1+t^4}}, \frac{-1}{\sqrt{1+t^4}} \right\rangle$  if  $t < 0$ .

The normal vector does not exist when  $t = 0$ ; as  $t$  passes from negative to positive values the normal vector changes which side of the curve it is on.

### Applications

4. You are an engineer overseeing the construction of a certain bridge on campus. The blueprint shows that the bridge has a side view profile which looks like the parabola  $y = x^2$ . Unfortunately, the material that the bridge is supposed to be built with is extremely rigid, and can only support curves with  $\kappa \leq 1.5$  units. Can this bridge be safely built with this material? [*Hint: Where is the curvature the greatest?*]

**Answer:** It cannot; the greatest curvature is  $\kappa = 2$  units.

5. Imagine that you are an ant travelling along the space curve

$$\mathbf{r}_1(t) = \left( \frac{3}{2}t^2 + 2t, 4t - 1, -3t^2 + 10t \right)$$

while your ant-friend is travelling along a different space curve

$$\mathbf{r}_2(t) = \left( 2t^2 - 3t + 10, -\frac{1}{2}t^2 + 9t, -2t^2 \right)$$

Assuming you are both looking “forwards” and are on the same scale of time, is there a time  $t$  when you are both looking in the same direction? If so, at what time?

**Answer:** Yes, at  $t = 5$ .

## Extensions

6. For a smooth curve  $\mathbf{r}(t)$ , define its *binormal vector*  $\mathbf{B}(t)$  at a time  $t$  to be  $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ , where the  $\times$  is the vector cross product. Compute  $\mathbf{B}$  for  $\mathbf{r}(t) = (t, 3 \cos t, 3 \sin t)$ .

**Answer:**  $\mathbf{B}(t) = \frac{1}{\sqrt{10}} \langle 3, \sin(t), -\cos(t) \rangle$

7. Give an example of a parametric curve in  $\mathbb{R}^2$  which has  $\mathbf{N}(t) = \left( \frac{-3}{\sqrt{e^{2t}+9}}, \frac{e^t}{\sqrt{e^{2t}+9}} \right)$ . You may want to use the fact that  $\|(e^t, 3)\| = \sqrt{e^{2t}+9}$ . [*Hint: First deduce a possible  $\mathbf{T}$ , then use the given fact, and integrate.*]

**Answer:**  $\mathbf{r}(t) = \langle -e^t, -3t \rangle$