Chapter 14.8: Lagrange Multipliers

D3: Optimization. I can locate and classify critical points of functions of two variables. I can find absolute maxima and minima on closed bounded sets. I can use the method of Lagrange multipliers to maximize and minimize functions of two or three variables subject to constraints. I can interpret the results of my calculations to solve problems.

Mechanics

1. Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

   Answer: The extreme values are 1 and 2.

2. Find the extreme values of the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x^4 + y^4 + z^4 = 1$.

   Answer: The extreme values are 1 and $\sqrt{3}$.

Applications

3. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

   Answer: $V(2, 2, 1) = 4$ cubic units

4. A niche restaurant in midtown Atlanta serves only garlic bread (denoted by g) and bunches of kale (denoted by k). The cost of producing these goods is given by the function $C(g, k) = 5g^2 + 2gk + 3k^2 + 10$. Assuming that the total amount of items to be produced is 40, compute the minimal production cost.

   Answer: $C_{min} = \frac{11230}{3} \approx \$3743.33$.

5. The height of a mountain is given by $h(x, y) = 300 - (3x^2 + 4xy + 3y^2)$. Compute the height of the lowest point on the mountain within $\sqrt{200}$ units of the origin. Where might this point occur? Are there multiple points where this occurs? *[Hint: First, justify why none of x, y, λ can be zero. Then solve for λ.]*

   Answer: The two minimums occur at $(x, y) = (\pm 10, \pm 10)$, where the height is -700 units. Warning: The two other solutions to the Lagrange system are $(x, y) = (\pm 10, \mp 10)$ and they are maximums at height 100.

Extensions

6. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin. *[Hint: It may be helpful algebraically to work with the square of the distance to the origin.]*

   Answer: The closest point is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and the farthest point is $(-1, -1, 2)$.

7. Find the maximum volume of a rectangular box that is inscribed in a sphere of radius r.

   Answer: $\dfrac{8r^3}{3\sqrt{3}}$