Chapter 15.7: Cylindrical and Spherical Coordinates

I3: Change of Variables. I can use polar, cylindrical, and spherical coordinates to transform double and triple integrals and can sketch regions based on given polar, cylindrical, and spherical iterated integrals. I can use general change of variables to transform double and triple integrals for easier calculation. I can choose the most appropriate coordinate system to evaluate a specific integral.

During studio, focus on *setting up* the integrals. However, don't forget to carry out the actual integration in your own time. Both the correct triple integral and correct final answer are given in the answers for this worksheet.

Mechanics

1. Use spherical coordinates to verify that the volume of a sphere with radius ρ is $\frac{4}{3}\pi\rho^3$.
2. Use cylindrical coordinates to compute $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}(x^2+y^2)\,dz\,dx\,dy$
3. Find the volume of the solid that lies within the sphere $x^2+y^2+z^2=4$, above the xy-plane, and below the cone $z=\sqrt{x^2+y^2}$.
4. Find the volume of the region bounded above by the paraboloid $z=9-x^2-y^2$, below by the xy-plane, and lying *outside* the cylinder $x^2+y^2=1$.
5. Suppose $a\geq 0$. Find the volume of the region cut from the solid sphere $\rho\leq a$ by the half-planes $\theta=0$ and $\theta=\pi/6$ in the first octant.
6. When might you prefer to use cylindrical coordinates over spherical ones? In other words, is there a particular type of symmetry whose presence suggests that cylindrical coordinates may be more useful?

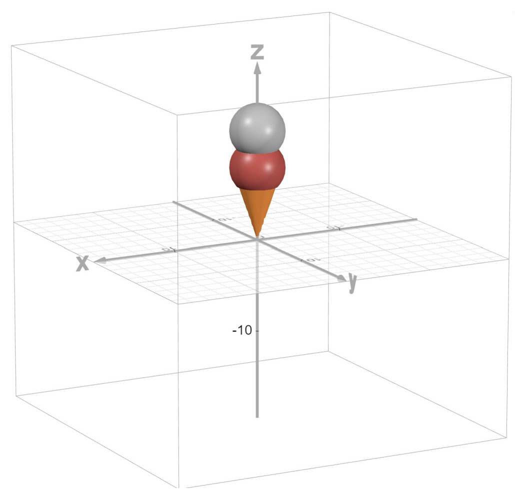
Applications

7. Let D be the right circular cylinder whose base is the circle $r=2\sin\theta$ in the xy-plane and whose top is the plane $z=4-y$. Recall that $r=2\sin\theta$ describes a circle centered at $(0,1)$ with radius 1 in the xy-plane. Using cylindrical coordinates,
   (a) find the volume of the region D.
   (b) find the $\bar{x}$ component of the centroid of the region. *[Hint: Use symmetry.]*

8. A double-scoop ice cream cone can be modeled by the region bound by the three component surfaces

$$\begin{array}{rll} \text{Cone: } & 9x^2 + 9y^2 - z^2 = 0 & \text{for } 0 \le z \le 6 \\ \text{Scoop One: } & x^2 + y^2 + (z-8)^2 = 9 & \text{for } 6 \le z \le 10 \\ \text{Scoop Two: } & x^2 + y^2 + (z-12)^2 = 9 & \text{for } 10 \le z \end{array}$$

Compute the volume of the entire dessert (including what is inside the cone, presumably more ice cream). *[Hint: Do not compute the volume of the scoops as they are presented. Translate each scoop down so they are centered at the origin.]*



Extensions

9. Let B be the unit ball given by $x^2 + y^2 + z^2 \le 1$. Compute the average distance of a point in B to the origin. Before you do any calculations, do you expect the average to be less than or greater than 0.5? Why?

10. Find the volume of the solid that is between the spheres $\rho = \sqrt{2}$ and $\rho = 2$, but outside of the circular cylinder $x^2 + y^2 = 1$. It will be helpful to draw a cross-section in a plane $\theta = c$ for this problem and to use symmetry.