Chapter 14.1: Multivariate Functions

G4: Surfaces. I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.

Mechanics

1. Algebraically describe the domains of each of the following functions. Then sketch them on (separate) xy-planes.

   (a) $f(x, y) = \sqrt{x - y - 1}$.

   (b) $f(x, y) = \sqrt{(x - 4)(y^2 - 1)}$.

   (c) $f(x, y) = \cos^{-1}(y - 4x^2)$.

   (d) $f(x, y) = \dfrac{1}{4 - x^2 - y^2}$.

   (e) $f(x, y) = \dfrac{1}{\ln(4 - x^2 - y^2)}$

   Answer:

   (a) $\{(x, y) \mid x - y \geq 1\}$

   (b) $\{(x, y) \mid x \geq 4, |y| \geq 1\} \cup \{(x, y) \mid x < 4, |y| < 1\}$

   (c) $\{(x, y) \mid 4x^2 - 1 \leq y \leq 4x^2 + 1\}$

   (d) All of $\mathbb{R}^2$ except the circle $x^2 + y^2 = 4$

   (e) All of the disk $x^2 + y^2 < 4$ except the circle $x^2 + y^2 = 3$.

2. For each of the surfaces (a)-(g), determine if the proposed descriptions of the level curves are correct. If not, give a correct descriptor. *[Note: consider a point as a circle/ellipse of radius 0]*

   (a) $z = 2x^2 - 3y^2$; Level curves are concentric ellipses.

   (b) $z = x^2 + y^2$; Level curves are concentric circles

   (c) $z = \dfrac{1}{x + y}$; Level curves are lines, whenever $x \neq -y$.

   (d) $z = 2x + 3y$; Level curves are parallel planes.

   (e) $z = \sqrt{25 - x^2 - y^2}$; Level curves are concentric circles, but only if $z > 5$ or $z < -5$

   (f) $z = \sqrt{x^2 + y^2}$; Level curves are concentric circles, but only if $z \geq 0$.

   (g) $z = xy$; Level curves are hyperbolas.

   Answer:

   (a) Not correct; level curves are hyperbolas

(b) Correct

(c) Correct

(d) Not correct; level curves are parallel lines

(e) Not correct; level curves are concentric circles, but only if $0 \le z \le 5$.

(f) Correct

(g) Correct

Applications

3. Multivariable functions are often used in economic models to describe how one should price an asset, or how to determine the utility of a product. For example, consider a *utility function* $u(x, y, z)$, where x, y, z represent three independent properties of an object (eg., price, quantity, quality), and u tells you how much you value that item. In this context, what economic significance do the level surfaces $u(x, y, z) = C$ have (assume C is a constant)? Give a example of how this phenomenon might manifest in your day-to-day life.

   Answer: The level surfaces tell you all combinations of price, quantity, and quality that you value the amount C.

Extensions

4. Find an equation for the level surface of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ passing through $(1, 1, 1)$. Sketch a plot of this level surface in $\mathbb{R}^3$.

   Answer: The sphere $3 = x^2 + y^2 + z^2$

5. Let $f(x, y) = (x - y)^2$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, and $x = y$, and describe the level curves. Use this information to produce a sketch of the graph of the surface. Confirm your sketch using a 3d graphing utility.

   Answer: When $x = 0$, the cross-section is the parabola $z = y^2$.

   When $y = 0$, the cross-section is the parabola $z = x^2$.

   When $x = y$, the cross-section is the line $z = 0$.

   The level curves are pairs of parallel lines $y = x \pm \sqrt{k}$.