Chapter 13.4: Curvature and Normals

G3: Geometry of Curves. I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.

Mechanics

1. Find $\mathbf{T}, \mathbf{N}$ and κ for the curve $\mathbf{r}(t) = \cos(2t)\mathbf{i} - \sin(2t)\mathbf{j} + 6t\mathbf{k}$, for $t \geq 0$. What do you notice about κ? Explain (perhaps with a picture) why this happens.

   Answer: $\mathbf{T}(t) = \dfrac{1}{\sqrt{10}}(-\sin(2t)\mathbf{i} - \cos(2t)\mathbf{j} + 3\mathbf{k})$,

   $\mathbf{N}(t) = -\cos(2t)\mathbf{i} + \sin(2t)\mathbf{j}$

   $\kappa(t) = \dfrac{1}{10}$. The curvature is constant.

2. Compute the unit tangent vector, unit normal vector, and curvature of the curve $\mathbf{r}(t) = \langle \sqrt{2}t, 1 + t, e^t \rangle$ for all $t \in \mathbb{R}$.

   Answer: $\mathbf{T}(t) = \dfrac{1}{\sqrt{3 + e^{2t}}}\langle \sqrt{2}, 1, e^t \rangle$,

   $\mathbf{N}(t) = \dfrac{1}{\sqrt{9 + 3e^{2t}}}\langle -\sqrt{2}e^t, -e^t, 3 \rangle$,

   $\kappa(t) = \dfrac{\sqrt{3}e^t}{(3 + e^{2t})^{3/2}}$.

3. Compute $\mathbf{N}$ for the curve $\mathbf{r}(t) = \langle t, (1/3)t^3 \rangle, t \in \mathbb{R}$ for $t \neq 0$.

   Does $\mathbf{N}$ exist at $t = 0$? Graph the curve, along with its normal vectors at the times $t = -1, -0.5, 0.5, 1$ and explain what is happening to $\mathbf{N}$ as t passes through $(0, 0)$

   Answer: $\mathbf{N} = \langle \dfrac{-t^2}{\sqrt{1 + t^4}}, \dfrac{1}{\sqrt{1 + t^4}} \rangle$ if $t > 0$ and $\langle \dfrac{t^2}{\sqrt{1 + t^4}}, \dfrac{-1}{\sqrt{1 + t^4}} \rangle$ if $t < 0$.

   The normal vector does not exist when $t = 0$; as t passes from negative to positive values the normal vector changes which side of the curve it is on.

Applications

4. You are an engineer overseeing the construction of a certain bridge on campus. The blueprint shows that the bridge has a side view profile which looks like the parabola $y = x^2$. Unfortunately, the material that the bridge is supposed to be built with is extremely rigid, and can only support curves with $\kappa \leq 1.5$ units. Can this bridge be safely built with this material? [*Hint: Where is the curvature the greatest?*]

   Answer: It cannot; the greatest curvature is $\kappa = 2$ units.

5. Imagine that you are an ant travelling along the space curve

$$\mathbf{r}_1(t) = \left(\frac{3}{2}t^2 + 2t, 4t - 1, -3t^2 + 10t\right)$$

while your ant-friend is travelling along a different space curve

$$\mathbf{r}_2(t) = \left(2t^2 - 3t + 10, -\frac{1}{2}t^2 + 9t, -2t^2\right)$$

Assuming you are both looking "forwards" and are on the same scale of time, is there a time t when you are both looking in the same direction? If so, at what time?

Answer: Yes, at $t = 5$.

Extensions

6. For a smooth curve $\mathbf{r}(t)$, define its *binormal vector* $\mathbf{B}(t)$ at a time t to be $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, where the $\times$ is the vector cross product. Compute $\mathbf{B}$ for $\mathbf{r}(t) = (t, 3\cos t, 3\sin t)$.

Answer: $\mathbf{B}(t) = \dfrac{1}{\sqrt{10}}\langle 3, \sin(t), -\cos(t)\rangle$

7. Give an example of a parametric curve in $\mathbb{R}^2$ which has $\mathbf{N}(t) = \left(\frac{-3}{\sqrt{e^{2t}+9}}, \frac{e^t}{\sqrt{e^{2t}+9}}\right)$. You may want to use the fact that $\|(e^t, 3)\| = \sqrt{e^{2t}+9}$. [*Hint: First deduce a possible* $\mathbf{T}$*, then use the given fact, and integrate.*]

Answer: $\mathbf{r}(t) = \langle -e^t, -3t\rangle$