Chapter 15.4: Polar Coordinates

I3: Change of Variables. I can use polar, cylindrical, and spherical coordinates to transform double and triple integrals and can sketch regions based on given polar, cylindrical, and spherical iterated integrals. I can use general change of variables to transform double and triple integrals for easier calculation. I can choose the most appropriate coordinate system to evaluate a specific integral.

Mechanics

1. Use polar coordinates to compute the integral:

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-x^2-y^2}\ dx\ dy$$

2. Evaluate $\iint_D y^2 + 3x\ dA$ where D is the region in the 3rd quadrant between $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

3. Use a double integral to determine the volume of the solid that is inside the cylinder $x^2 + y^2 = 16$, below $z = 2x^2 + 2y^2$, and above the xy-plane.

4. Give an example of region and function you would *not* want to use polar coordinates to integrate. Justify your answer.

Applications

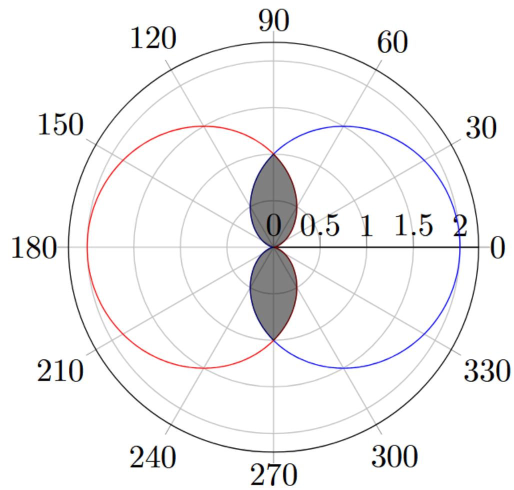
5. The previous worksheet featured a problem involving a peculiar Pringles can. This problem rectifies that inconsistency.

   A true can of Pringles chips may be modeled by the cylinder $x^2 + y^2 = 1$ bounded above and below like $0 \leq z \leq 5$. Assuming that the Pringles container is filled up with chips until the surface $z = x^2 - y^2 + 3$, are there more chips or air in the can? *[Hint: Use the identity* $\cos^2\theta - \sin^2\theta = \cos 2\theta$*].*

6. In the town of Churchill in northern Canada, the density of polar bears around the town dump is given by $p(x, y) = e^{x^2+y^2}$ bears per square unit. Use polar coordinates to compute the average number of polar bears in the region $1 \leq x^2 + y^2 \leq 2$.

Extensions

7. Find the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$. *[Hint: Use symmetry to restrict your calculation to only the first quadrant.]*



8. An integral of great importance in statistics is the Gaussian integral $I = \displaystyle\int_0^\infty e^{-x^2}\,dx$.

   The function $f(x) = e^{-x^2}$ has no elementary antiderivative, so this integral is hard to compute in the usual way. Fortunately, polar coordinates provide a solution.

   Notice that $I^2 = \left(\displaystyle\int_0^\infty e^{-x^2}\,dx\right)\left(\displaystyle\int_0^\infty e^{-y^2}\,dy\right) = \displaystyle\int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dxdy.$

   (a) The domain of the above double integral is the first quadrant $[0,\infty) \times [0,\infty)$. Describe this region using polar coordinates, and transform I^2 into an (improper) polar integral.

   (b) Evaluate your double integral to compute the value of I^2. Use this to find the value of the original Gaussian integral I.

   You can find some history of this integral here.