Chapter 14.3: Partial Derivatives

Mechanics

1. Find all first and second partial derivatives for $f(x, y) = e^x + x\ln(y)$.

2. Find f_x, f_y, f_z, and f_{xzz} for the function $f(x, y, z) = x\sin(yz)$.

3. Find the total derivative Df at the given point for each function below. Remember that Df is the matrix of (partial) derivatives of the function and if f is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ then Df is a $m \times n$ matrix.

   (a) $f(x) = 2x^3 + 7$ at $x = 2$.
   (b) $\mathbf{f}(t) = \langle 2\cos(t), 2\sin(t), t\rangle$ at $t = \pi/2$.
   (c) $f(x, y) = \sqrt{y - x}$ at $(x, y) = (1, 2)$.
   (d) $f(x, y, z) = e^{2y-x} + z^2 + 4$ at $(x, y, z) = (1, 2, 3)$.
   (e) $\mathbf{f}(s, t) = \langle 2s + 3t, t - s\rangle$ at $(s, t) = (1, 1)$.
   Note: The graph of this function is a surface (in this case all of $\mathbb{R}^2$) parameterized by two variables just like the graph of the function in (b) is a curve parameterized by one variable - we'll see these more later! Another way of thinking about this is that this is a *change of variables* for $\mathbb{R}^2$ between the system of coordinates (s, t) and (x, y).

Applications

4. The speed of sound C traveling through ocean water is a function of temperature, salinity, and depth. It may be modeled by the function

$$C(T, S, D) = 1450 + 4.5T - 0.05T^2 + 0.0003T^3 + (1.5 - 0.01T)(S - 35) + 0.015D,$$

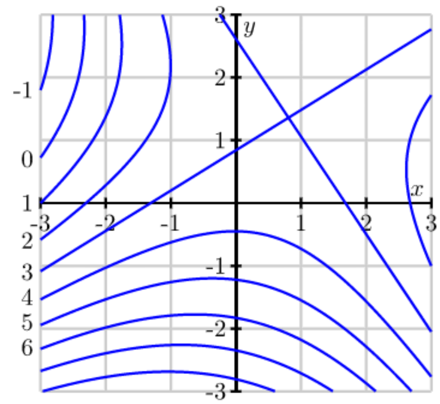
   where C is the speed of sound in meters/second, T is the temprature in degrees Celsius, S is the salinity in grams/liter of water, and D is the depth below the ocean surface in meters.

   (a) State the units in which each of the partial derivatives C_T, C_S, and C_D are expressed and explain the physical meaning of each.
   (b) Find the partial derivatives C_T, C_S, and C_D.
   (c) Evaluate each of the three partial derivatives at the point where $T = 10, S = 35$, and $D = 100$. What does the sign of each partial derivative tell us about the behavior of the function C at the point $(10, 35, 100)$?

5. Recall from last week's worksheet that a utility function is a multivariable function $u(x, y, z)$, where x, y, z represent three independent properties of an object (eg., price, quantity, quality), and u tells you how much you value that item. The *marginal utility functions* are the partial derivatives u_x, u_y and u_z. What is the economic interpretation of the marginal utilities?

Extensions

6. Below is a contour plot for a function $f(x, y)$, with values for some of the contours (level curves) indicated on the *left* of the figure.



   (a) Find the sign of the partial derivatives $f_x(-2, -1)$ and $f_y(-2, -1)$.

   (b) At the point $(0, -1/2)$, which is larger? f_x or f_y?

   (c) Find all (x, y) where $f_x(x, y) = 0$.

   (d) Locate, if possible, one point (x, y) where $f_x(x, y) < 0$.

7. The fifth-order partial derivative $\partial^5 f/\partial x^2 \partial y^3$ is zero for each of the following functions. To show this as quickly as possible, which variable would you differentiate with respect to first: x or y?

   Try to answer without writing anything down. Why did you make the choice you did?

   (a) $f(x, y) = y^2 x^4 e^x + 2$

   (b) $f(x, y) = y^2 + y(\sin(x) - x^4)$

   (c) $f(x, y) = x^2 + 5xy + \sin(x) + 7e^x$

   (d) $f(x, y) = xe^{y/2}$

8. Let A be any 2×2 matrix, and let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$. Compute the total derivative $D\mathbf{f}$. What do you notice? What familiar family of functions from Calc 1 does this remind you of? Can you generalize this result?