## Chapter 13.4: Curvature and Normals

G3: Geometry of Curves. I can compute the arc length of a curve in two or three dimensions and apply arc length to solve problems. I can compute normal vectors and curvature for curves in two and three dimensions. I can interpret these objects geometrically and in applications.

## **Mechanics**

- 1. Find  $\mathbf{T}, \mathbf{N}$  and  $\kappa$  for the curve  $\mathbf{r}(t) = \cos(2t)\mathbf{i} \sin(2t)\mathbf{j} + 6t\mathbf{k}$ , for  $t \geq 0$ . What do you notice about  $\kappa$ ? Explain (perhaps with a picture) why this happens.
- 2. Compute the unit tangent vector, unit normal vector, and curvature of the curve  $\mathbf{r}(t) = \langle \sqrt{2}t, 1+t, e^t \rangle$  for all  $t \in \mathbb{R}$ .
- 3. Compute **N** for the curve  $\mathbf{r}(t) = \langle t, (1/3)t^3 \rangle, t \in \mathbb{R}$  for  $t \neq 0$ . Does **N** exist at t = 0? Graph the curve, along with its normal vectors at the times t = -1, -0.5, 0.5, 1 and explain what is happening to **N** as t passes through (0, 0)

## **Applications**

- 4. You are an engineer overseeing the construction of a certain bridge on campus. The blueprint shows that the bridge has a side view profile which looks like the parabola  $y = x^2$ . Unfortunately, the material that the bridge is supposed to be built with is extremely rigid, and can only support curves with  $\kappa \leq 1.5$  units. Can this bridge be safely built with this material? [Hint: Where is the curvature the greatest?]
- 5. Imagine that you are an ant travelling along the space curve

$$\mathbf{r}_1(t) = \left(\frac{3}{2}t^2 + 2t, 4t - 1, -3t^2 + 10t\right)$$

while your ant-friend is travelling along a different space curve

$$\mathbf{r}_2(t) = \left(2t^2 - 3t + 10, -\frac{1}{2}t^2 + 9t, -2t^2\right)$$

Assuming you are both looking "forwards" and are on the same scale of time, is there a time t when you are both looking in the same direction? If so, at what time?

## **Extensions**

- 6. For a smooth curve  $\mathbf{r}(t)$ , define its binormal vector  $\mathbf{B}(t)$  at a time t to be  $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ , where the  $\times$  is the vector cross product. Compute  $\mathbf{B}$  for  $\mathbf{r}(t) = (t, 3\cos t, 3\sin t)$ .
- 7. Give an example of a parametric curve in  $\mathbb{R}^2$  which has  $\mathbf{N}(t) = \left(\frac{-3}{\sqrt{e^{2t}+9}}, \frac{e^t}{\sqrt{e^{2t}+9}}\right)$ . You may want to use the fact that  $||(e^t, 3)|| = \sqrt{e^{2t}+9}$ . [Hint: First deduce a possible  $\mathbf{T}$ , then use the given fact, and integrate.]