Chapter 14.7: Optimization

D3: Optimization. I can locate and classify critical points of functions of two variables. I can find absolute maxima and minima on closed bounded sets. I can use the method of Lagrange multipliers to maximize and minimize functions of two or three variables subject to constraints. I can interpret the results of my calculations to solve problems.

Mechanics

1. Find and classify all critical points for the function $x^3 + 3xy + y^3$.

   Answer: Saddle point at $(0, 0)$ and local minimum at $(0, -2)$.

2. Find all the local maxima, local minima, and saddle points of $f(x, y) = e^y(x^2 - y^2)$.

   Answer: Saddle point at $(0, 0)$ and local maximum at $(-1, -1)$.

3. Find the absolute maxima and minima of the function $f(x, y) = x^2 - xy + y^2 + 1$ on the closed triangular plate bounded by lines $x = 0$, $y = 4$, $y = x$ in the first quadrant.

   Answer: The absolute maximum is 17, achieved at $(0, 4)$ and $(4, 4)$, and the absolute minimum is 1, achieved at $(0, 0)$.

4. Give an example of a differentiable function $f(x, y)$ with no critical points.

   Answer: Many possible examples, e.g. any linear function $f(x, y) = ax + by + c$.

Applications

5. A terrible recipe for lemonade calls for you to just mix lemon juice, denoted by ℓ, and water, denoted by w (units in tonnes). Suppose that given a pair (ℓ, w), you are able to make $f(\ell, w) = \ell^2 - \ell w + w$ liters of lemonade. Given that have only 2 tonnes of lemon juice and 3 tonnes of water, what is the maximum amount of (a very acidic) lemonade can you make? How much of each ingredient is used?

   Answer: $f_{max} = 4$ is the unique maximum attained at $(\ell, w) = (2, 0)$. This means that the lemonade is pure lemon juice :)

6. In an alternate universe, Atlanta is famous for her extravagant beaches and pristine waters. In this universe, it is known that Atlanta's waters are extremely wavy, and the height of the water (relative to ground level) may be modeled by the function $h(x, y) = \sin(x)\cos(y)$, for $x \in (0, 2\pi)$ and $y \in (0, 2\pi)$. Visualize this using a 3D graphing software, and compute the height of the highest tides as well as the depth of the lowest troughs. At which coordinates (x, y) do these tides and troughs occur?

   Answer: Highest tides are at 1 occurring at $(3\pi/2, \pi)$, lowest troughs at -1 occurring at $(\pi/2, \pi)$.

7. Let us interpret local maxima as peaks of mountains, local minima as valleys and saddle points as passes between mountain peaks. Consider the statement: "It is impossible to have two mountain peaks without some sort of valley or pass connecting them. Therefore, if a function has two local maxima, there must also be a saddle point or a local minimum." Do you agree with this? Verify your answer by using software to graph the function $f(x, y) = 4x^2e^y - 2x^4 - e^4y$.

   Answer: The statement is false! The function given has precisely two local maximums and no other critical points.

Extensions

8. Can you conclude anything about $f(a, b)$, if f and its first and second partial derivatives are continuous around the critical point (a, b) and $f_{xx}(a, b)$ and $f_{yy}(a, b)$ have opposite signs? Justify your answer.

   Answer: Yes, this must be a saddle point because $f_{xx}(a, b)f_{yy}(a, b) < 0$ so $\det(Hf) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b) < 0$.

9. In each case, the origin is a critical point of f and $f_{xx}f_{yy} - (f_{xy})^2 = 0$ at the origin, so the Second Derivative Test fails at the origin. Use some other method to determine whether the function f has a maximum, a minimum, or neither at the origin.

   (a) $f(x, y) = x^2y^2$
   (b) $f(x, y) = 1 - x^2y^2$
   (c) $f(x, y) = xy^2$
   (d) $f(x, y) = x^3y^2$
   (e) $f(x, y) = x^3y^3$
   (f) $f(x, y) = x^4y^4$

   Answer:

   (a) Minimum is 0 at $(0, 0)$ since $f(x, y) > 0$ for all other (x, y).
   (b) Maximum is 1 at $(0, 0)$ since $f(x, y) < 1$ for all other (x, y).
   (c) Neither since $f(x, y) < 0$ for $x < 0$ and $f(x, y) > 0$ for $x > 0$.
   (d) Neither since $f(x, y) < 0$ for $x < 0$ and $f(x, y) > 0$ for $x > 0$.
   (e) Neither since $f(x, y) < 0$ for $x < 0$ and $y > 0$, but $f(x, y) > 0$ for $x > 0$ and $y > 0$.
   (f) Minimum is 0 at $(0, 0)$ since $f(x, y) > 0$ for all other (x, y).

10. Among all rectangular boxes of volume 27 cm^3, what are the dimensions of the box with the smallest surface area? What is the smallest possible surface area? (you may assume this occurs at a local min of the surface area function)

    Answer: The dimensions are $3 \times 3 \times 3$ and the surface area is 54.