Chapter 14.6: Linearization and Tangent Planes

D2: Tangent Planes and Linear Approximations. I can find equations for tangent planes to surfaces and linear approximations of functions at a given point and apply these to solve problems.

Mechanics

1. Find the linearization of $f(x,y) = e^{2y-x}$ at $(1,2)$. Without doing any more calculations, find an equation of the tangent plane of the surface $f(x,y)$ at $(1,2)$.

   Answer: $L(x,y) = e^3 - e^3(x-1) + 2e^3(y-2)$

   tangent plane: $z = e^3 - e^3(x-1) + 2e^3(y-2)$

2. Find the linearization of $f(x,y,z) = \arctan(xyz)$ at $(1,1,0)$.

   Answer: $L(x,y,z) = z$

3. Use the linearization to approximate $f(2.95, 7.1)$ for the function $f(x,y) = \sqrt{x^2+y}$, knowing that $f(3,7) = 4$.

   Answer: $f(2.95, 7.1) \approx 4 - 1/40$

4. Find an equation of the tangent plane to the unit sphere $x^2+y^2+z^2 = 1$ at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right)$.

   Answer: $\sqrt{2}\left(x - \frac{1}{\sqrt{2}}\right) + \left(y - \frac{1}{2}\right) + \left(z - \frac{1}{2}\right) = 0$

Applications

5. Suppose you are shining a flashlight on a smooth surface. The *angle of incidence*, θ_i is the angle at which a light beam hits the surface, measured with respect to the surface normal (i.e., the normal vector to the tangent plane at point of contact). The *angle of reflection* θ_r is the angle of the reflected light beam measured with respect to the surface normal. The *law of reflection* states that in a vacuum, we must have $\theta_i = \theta_r$. Draw a labeled picture to convince yourself that this is reasonable.

   Now, consider the paraboloid $z = x^2 + y^2$, and a light ray traveling along the path $\mathbf{r}(t) = (-2,-3,2)t + (3,4,0)$. Compute the angle of reflection at the point $(1,1,2)$ [*Hint: How can one find the angle between two vectors?*].

   Answer: $\theta_r = \arccos(\frac{4}{\sqrt{17}}) \approx 0.245$ rad

Extensions

6. Use software to graph the function $z = x^{1/3}y^{1/3}$. Examine the graph at $(0, 0)$ - does it look like the function has a tangent plane there? Use this to deduce a necessary condition for a function $f(x, y)$ to have a tangent plane.

   Answer: No, there are two different tangent planes. The function cannot have a cusp at the point with the tangent plane.