Theorem 1.3.9
If k is an integer and $x^2 = k$ has a rational solution, then that solution is an integer.
Archimedian Property
An ordered field has the archimedian property if $\forall x \in \mathbb{R}$, there is a n s.t $x \leq n$.
Another consequence is there is a rational number between each pair of real numbers.
Sup and Inf
Theorem 1.5.1
Every non-empty subset of $\mathbb{R}$ that is bounded below has a G.L.B.
Sup Sup A is the smallest $M \in \mathbb{R}$ s.t $a \leq M$ for every $a \in A$
Inf Inf A is the largest $m \in \mathbb{R}$ s.t $m \leq a$ for every $a \in A$
Sequences
$|x - a| \leq \varepsilon \iff a - \varepsilon < x < a + \varepsilon$
Triangle Inequality
$|a + b| \leq |a| + |b|$, $||a| - |b|| \leq |a - b|$
Definition of a Limit
A sequence a_n converges if, for each $\varepsilon > 0$, there is a N s.t $|a_n - a| < \varepsilon$ whenever $n > N$
COR. 2.2.4 If a sequence converges its bounded
TH. 2.2.5 If $\lim a_n = a$ then $\lim |a_n| = |a|$
TH. 2.2.7 A sequence a_n converges to a iff, for each ε, there are only finitely many n for which $|a_n - a| \geq \varepsilon$
TH. 2.3.1 Let a_n and b_n be sequences of real numbers and suppose $\lim b_n = 0$. If $a \in \mathbb{R}$ and if there is a N_1 such that $|a_n - a| \leq b_n$ for all $n > N$, then $lim a_n = a$.
TH. 2.3.2 Let a_n be a sequence of real numbers s.t. $lim\, a_n = 0$, and let b_n be a bounded sequence Then the $lim\, a_n b_n = 0$.
TH. 2.3.3, Squeeze
If a_n, b_n, and c_n are sequences for which there is a number K s.t. $b_n < a_n < c_n$ for all $n > K$, and if $b_n \to a$ and $c_n \to a$, then $a_n \to a$.
The Main Limit Theorem
Suppose $a_n \to a$, $b_n \to b$, c is a real number, and $K \in \mathbb{N}$
(a)$ca_n \to ca$, (b) $a_n + b_n \to a + b$, (c) $a_n b_n \to ab$
(d)$a_n/b_n \to a/b$ if $b \neq 0$ and $b_n \neq 0$ for all n
(e) $a_n^k \to a^k$, (f) $a_n^{1/k} \to a^{1/k}$ if $a_n \geq 0$
TH. 2.3.8 If a_n and b_n are convergent sequences to a and b, and if there is a K s.t. $a_n < b_n$ whenever $n > K$, then $a \leq b$
TH. 2.4.1, Monotone Convergence
Each bounded monotone sequence converges
TH. 2.4.6
Each monotone sequence has a limit
TH. 2.4.7 Let a_n and b_n be sequences of $\mathbb{R}$
(a) if $a_n > 0$ for all n, then $lim\, a_n = \infty$ iff $lim\, 1/a_n = 0$
(b) if b_n is bounded below, then $lim\, a_n = \infty$ $\Rightarrow lim\, (a_n + b_n) = \infty$
(c)$lim\, a_n = \infty$ iff $lim\, (-a_n) = -\infty$
(d) if $a_n \leq b_n \, \forall n$, then $lim\, a_n = \infty \Rightarrow lim\, b_n = \infty$
(e) if there is a positive constant K s.t. $k \leq b_n \, \forall n$ then $lim\, a_n b_n = \infty$

Cauchy
Bolzano-Weierstrass
Every bounded sequence of real numbers has a convergent subsequence.
Cauchy Sequence A sequence is said to be Cauchy if, for every $\varepsilon > 0$, there is an N s.t. $|a_n - a_m| < \varepsilon$ whenever $n, m > N$
TH 2.5.8. A sequence of real numbers is Cauchy if and only if it converges.
Continuity
Def. 3.1.1 Let f be a function with $D \subset \mathbb{R}$ and let a be an element of D. F is continuous at a, if, for each $\varepsilon > 0$, there is a $\delta > 0$ s.t. $|f(x) - f(a)| < \varepsilon$ whenever $x \in D$ and $|x - a| < \delta$
Sequential Method Let F be a function on D and suppose $a \in D$. Then f is continuous at a iff whenever x_n is a sequence in D which converges to a, then $\{f(x_n)\}$ converges to $f(a)$
TH. 3.1.7 If r is a positive rational number, $f(x) = x^r$ is continuous on its natural domain
Combinations of Continuous Functions
Let f and g be fns with on D_f and D_g. Assume f and g are both continuous at a point $a \in= D_f \cap D_g$ and let c be a constant, then
(b)$f + g$ is continuous at a
(c) fg is continuous at a
(d)f/g is continuous at a
Theorem 3.2.1 If f is a continuous fn on a closed bounded interval I, then f is bounded on I, and it assumes both a minimum and maximum value on I
Intermediate Value Theorem
Let f be defined and cont on an interval containing the points a and b and assume $a < b$. If y is any number between $f(a)$ and $f(b)$, then there exists c with $a \leq c \leq b$ such that $f(c) = y$. *ie* takes on every value between bounds
TH 3.2.4 If f is a continuous function defined on a closed bounded interval $I = [a, b]$, then $f(I)$ is also a closed, bounded interval or a single point.
TH 3.2.5 If f is strictly monotone on I and its range $f(I)$ is an interval, then f is continuous on I.
TH 3.2.5 A cont., strictly monotone fn f on a closed interval I has a continuous inverse fn defined on $J = f(I)$. That is, there is a cont. fn. g with domain J s.t. $g(f(x)) = x$ for all $x \in I$ and $f(g(y)) = y$ for all $y \in J$
-Uniform Cont If f is a function with domain D then f is said to be uniformly cont on D if for each $\varepsilon > 0$, there is a $\delta > 0$ s.t. $|f(x) - f(a)| < \epsilon$ whenever $x, a, \in D$ and $|x - a| < \delta$.
-TH 3.3.4 If f is a cont. fn. on a closed bounded interval I, then f is uniformly cont. on I.
-TH 3.3.5 If f is uniformly cont. on its domain D, and if $\{x_n\}$ is any Cauchy sequence in D, then $\{f(x_n)\}$ is also Cauchy.
-TH 3.3.6 If f is a cont. fn. on a bounded interval I, which can be open, then f has a cont. extension to $\bar{I}$ iff f is uniformly cont. on I.

-Uniform Convergence Let f_n be a sequence of functions on a set $D \in \mathbb{R}$. Then:
(a) $\{f_n\}$ converges pointwise to f on D if for each $x \in D$ and each $\epsilon > 0$, there is a N s.t. $|f(x) - f_n(x)| < \epsilon$ whenever $n > N$
(b) $\{f_n\}$ converges uniformly on D to f if for each $\epsilon > 0$, there is a N s.t. $|f(x) - f_n(x)| < \epsilon$ whenever $x \in D$ and $n > N$
-TH 3.4.5 Let $\{f_n\}$ be a sequence of functions all of which are defined and cont. on a set D. If $\{f_n\}$ converges uniformly to f on D, then f is cont. on D.
-TH 3.4.6 Let $\{f_n\}$ be a sequence of fns defined on a set D. If there is a sequence of numbers b_n s.t. $b_n \to 0$ and $|f_n(x)| \leq b_n$ for all $x \in D$, then $\{f_n\}$ converges uniformly to 0 on D.
-TH 3.4.7 Let $\{f_n\}$ be a sequence of fns defined on a set D. If $\{f_n\}$ converges uniformly to 0 on D, then $\{f_n(x_n)\}$ converges to 0 for every sequence $\{x_n\}$ of points of D.
-TH 3.4.9 A sequence of fns $\{f_n\}$ on a set D is said to be uniformly Cauchy on D if for each $\epsilon > 0$, there is N s.t. $|f_n(x) - f_m(x)| < \epsilon$ whenever $x \in D$ and $n, m > N$
-TH 3.4.10 A sequence of functions $\{f_n\}$ on D is uniformly convergent on D iff it is uniformly Cauchy on D.
The Derivative
-Definition 4.1.1 Let I be an open interval, a a point of I, and f a fn defined on I except possibly at a. Then the limit of $f(x)$ as x approaches a is L if, for each $\epsilon > 0$, there is a $\delta > 0$ s.t $|f(x) - L| < \epsilon$ when $x \in I$ and $0 < |x - a| < \delta$
-Definition 4.1.6 Let f be a function defined on an open interval (a, b) where a could be $-\infty$ and b could be $+\infty$. The lim from the right is $\lim_{x \to a^+} f(x) = L$ if for every $\epsilon > 0$ there is a $m \in (a, b)$ s.t. $|f(x) - L| < \epsilon$ whenever $a < x < m$. For left limit, it is whenever $m < x < b$
-TH 4.1.10 Let (a, b) be a (possibly infinite) interval and let u be a^+ or b^- or a point in (a, b). If f is a fn defined on (a, b), then $\lim_{x \to u} f(x) = L$ iff $f(a_n) \to L$ whenever $\{a_n\}$ is a sequence of points in (a, b), distinct from u, with $a_n \to u$
-Main Limit Theorem Let (a, b) be a (possibly infinite) interval and let u be a^+ or b^- or a point in (a, b) and let c be a constant. Let f and g be functions defined on (a, b). If $\lim_{x \to u} f(x) = K$, and $\lim_{x \to u} g(x) = L$ then,
(a) $\lim_{x \to u} c = c$
(b) $\lim_{x \to u} cf(x) = cK$
(c) $\lim_{x \to u} f(x) + g(x) = K + L$
(d) $\lim_{x \to u} f(x)g(x) = KL$
(e) $\lim_{x \to u} f(x)/g(x) = K/L$ provided $L \neq 0$
-TH 4.1.12 Let (a, b) be a (possibly infinite) interval and let u be a^+ or b^-. If g is defined on (a, b) and $\lim_{x \to u} g(x) = L$, f is defined on an interval containing L and the image of g, and f is cont. at L, then $\lim_{x \to u} f(g(x)) = f(L)$.

-TH 4.1.14 If f is a fn defined on (a, b) then $\lim_{x \to a^+} f(x) = \infty$ if for each M, there is a $m \in (a, b)$ s.t $f(x) > M$ whenever $a < x < m$.
-TH 4.1.15 Let (a, b) be a (possibly infinite) interval and let u be a^+ or b^- or a point in (a, b). If f is positive on (a, b), then $\lim_{x \to u} f(x) = \infty$ iff $\lim_{x \to u} \frac{1}{f(x)} = 0$
-Def 4.2.1 Let f be a fn defined on an open interval containing $a \in \mathbb{R}$. If $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exist and is finite, we call it $f'(a)$.
f' is a new fn with d consisting of points in the domain of f at which f is differentiable.
or $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$
or $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
-4.2.5 If f is differentiable at a, it is cont. at a.
-Chain Rule Supose g is defined in an open interval I containing a and f is defined in an open interval containing $g(I)$. If g is differentiable at a and f is is differentiable at $g(a)$, then $f \circ g$ is differentiable at a and $(f \circ g)'(a) = f'(g(a))g'(a)$.
-TH 4.2.9 If f is cont. and strictly monotone on an open interval I containing a, f is diff. at a, and $f'(a) \neq 0$, then the inverse fn g of f is diff at $b = f(a)$ and $g'(b) = \frac{1}{f'(a)} = \frac{1}{f'(g(b))}$ ex 4.2.10
-CP A critical point is a point that satisfies: 1) c is an endpoint, 2) c is a stationary point, or c is a singular point, $f'(c)$ does not exist.
-TH 4.3.1 If f is cont. fn. on a closed bounded interval $[a, b]$ and $c \in [a, b]$ is a max/min, then c is a critical point.
-MVT If f is cont. on the closed interval $[a, b]$ and is diff. on the open interval (a, b), then there is at least one point $c \in (a, b)$ s.t. $f'(c) = \frac{f(b) - f(a)}{b - a}$
-TH 4.3.3 If f is diff. fn. on an open interval (a, b) and f' is 0 on (a, b), then f is constant.
-CO 4.3.4 If f and g are diff. on (a, b) and $f'(x) = g'(x)$ for all $x \in (a, b)$ then there is a constant c s.t. $f(x) = g(x) + c$.
-TH 4.3.5 If f is a fn. which is cont. on a closed interval $[a, b]$ and diff. on the open interval (a, b), then f is increasing on $[a, b]$ if $f'(x) > 0$ $\forall x \in (a, b)$ and decreasing if $f'(x) < 0$
-TH 4.3.6 Let f be cont. fn. on $[a, b]$ which is diff. on (a, b). Then f is non-decreasing on $[a, b]$ iff $f'(x) \geq 0$ and non-increasing if $f'(x) \leq 0$
-TH 4.3.9 If f is a diff. fn. on a (possibly infinite) open interval (a, b) and f' is bounded on (a, b), then f is uniformly cont. on (a, b).
-Cauchy MVT Let f and g be fns. which are cont. on a closed, bounded interval $[a, b]$ and diff on (a, b). Then there exist $c \in (a, b)$ s.t. $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$

-L'Hopital's Rule Let f and g be diff. fns. on a (possibly infinite) interval (a,b) and let u be a^+ or b^-. Suppose $g(x)$ and $g'(x)$ are non-zero on all (a,b) & (1)$\lim_{x\to u} f(x) = 0 = \lim_{x\to u} g(x)$ or (2) $\lim_{x\to u} f(x) = \infty = \lim_{x\to u} g(x)$, then $lim_{x\to u}\frac{f(x)}{g(x)} = lim_{x\to u}\frac{f'(x)}{g'(x)}$

-Note: Conditions (1) and (2) refer to the indeterminate form.

The Integral

-Upper and Lower Sums

-$U(f,P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$, LUB for sums of f and P

-$L(f,P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$, GLB for sums of f and P

-Refinement Let P and Q be partitions of a closed bounded interval $[a,b]$. Q is a refinement of P if $P \subset Q$. Partitions are refinements of themselves.

-TH 5.1.4 Let f be a bounded fn on a closed bounded interval $[a,b]$ and Q is a refinement of P, then $L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,Q)$

-TH 5.1.5 If P and Q are any two partitions of a closed bounded interval $[a,b]$ and f is a bounded fn on $[a,b]$ then $L(f,P) \le U(f,Q)$

-The Integral

$\int_a^{-b} f dx = inf\{U(f,Q) : Q \text{ is a partition of } [a,b]\}$

$\int_{-a}^{b} f dx = sup\{L(f,Q) : Q \text{ is a partition of } [a,b]\}$

-TH 5.1.7 The Riemann integral of f on $[a,b]$ exist iff, for each $\epsilon > 0$ there is a partition P of $[a,b]$ s.t. $U(f,P) - L(f,P) < \epsilon$

-TH 5.1.8 The Integral exists iff there is a sequence $\{P_n\}$ of partitions of $[a,b]$ s.t. $lim(U(f,P_n)) - L(f,P_n)) = 0$. In this case, $\int_a^b f(x)dx = limS_n(f)$ where, for each n, $S_n(f)$ may be chosen to be $U(f,P_n)$, $L(f,P_n)$, or any Riemann sum for f and the partition P_n

-TH 5.2.1 If f is a monotone fn on a closed bounded interval $[a,b]$ then f is integrable on $[a,b]$

-TH 5.2.2 If f is a cont. fn. on a closed bounded interval $[a,b]$, then f is integrable on $[a,b]$.

-TH 5.2.4 If f and g are integrable fn on $[a,b]$ and $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)dx \le \int_a^b g(x)dx$

-CO 5.2.5 Let f be an integrable fn on the closed bounded interval $I = [a,b]$ and $M = sup_I f$ and $m = inf_I f$. Then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

-TH 5.2.6 If f is integrable on $[a,b]$ then $|f|$ is also integrable on $[a,b]$ and $|\int_a^b f(x)dx| \le \int_a^b |f(x)|dx$

-Mean of Integrals If f is integrable fn. on a bounded interval $[a,b]$, then the mean of f on $[a,b]$ is the number $\frac{1}{b-a}\int_a^b f(x)dx$

-TH 5.2.7 If f is a cont. fn. on a closed bounded interval $[a,b]$, then there is a point $c \in [a,b]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$

-CO 5.2.9 With f and $a \le b \le c$, f is integrable on $[a,c]$ iff it is integrable on $[a,b]$ and on $[b,c]$.

-CO 5.2.9 With f and $a \le b \le c$, f is integrable on $[a,c]$ iff it is integrable on $[a,b]$ and on $[b,c]$.

-CO 5.2.10 If f is a bounded fn on a closed bounded interval $[a,b]$ and f is cont. except at finitely many points of $[a,b]$, then f is integrable.

-First FTC Let $[a,b]$ be a closed bounded interval and let f be a fn. which is cont. on $[a,b]$ and diff. on (a,b) with f' intergrable on $[a,b]$. Then $\int_a^b f'(x)dx = f(b) - f(a)$

-Second FTC Let f be a fn. which is integrable on a closed bounded interval $[b,c]$. For $a, x \in [b,c]$ define a fn. $F(x)$ by $F(x) = \int_a^x f(t)dt$. Then, F is cont. on $[b,c]$. At each point x of $[b,c]$ where f is cont. the fn F is diff. and $F'(x) = f(x)$.

-U-Sub Let g be a diff. fn. on an open interval I with g' intergrable on I and let $J = g(I)$. Let f be cont. on J. Then for any pair $a, b \in I$, $\int_a^b f(g(t))g'(t)dt = \int_{g(a)}^{g(b)} f(u)du$.

-Integration by Parts Suppose f and g are cont. fns. on a closed bounded interval $[a,b]$ and suppose that f and g are diff. on (a,b) with derivatives that are integrable on $[a,b]$. Then fg' and $f'g$ are integrable on $[a,b]$ and $\int_a^b f(x)g'(x)dx =$ $f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)dx$

Logs, Expos

-TH 5.4.2 For all $a, b \in (0, +\infty)$, $ln(ab) = ln(a) + ln(b)$

-TH 5.4.3 If $a > 0$ and r is rational, then $ln(a^r) = rln(a)$

-TH 5.4.4 The ln is strictly increasing on $(0, +\infty)$. Also, $lim_{x\to\infty} ln(x) = +\infty$ and $lim_{x\to 0} ln(x) = -\infty$

Exponential Functions

$exp'(x) = exp(x)$

$exp(a+b) = exp(a)exp(b)$

$exp(ra) = (exp(a))^r$

$a^x = exp(xln(a))$, $a^{xy} = (a^x)^y$

-TH 5.4.9 For each $a > 0$, we have $log_a x = \frac{ln(x)}{ln(a)}$

Infinite Series

Def 6.1.1 The series is said to converge to the number s if $lim_{n\to\infty} s_n = s$. Here we write $\sum_{k=1}^{\infty} a_k = s$. S is called the sum of the series. If $\{s_n\}$ diverges, then we say the series diverges.

Term Test If a series $a_1 + a_2 + a_3 + \cdots + a_k + \cdots$ converges, then $lim_{n\to\infty} a_n = 0$. **Note:** This is not iif. So $lim_{n\to\infty} a_n = 0 \not\Longrightarrow$ convergence. However, $lim_{n\to\infty} a_n \ne 0 \implies$ divergence

Geometric Series Thm If $a \ne 0$, the geometric series, $\sum_{k=0}^{\infty} ar^k$, converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges otherwise.

Non Negative Infinite Series An infinite series of non neg terms converges iif $\{S_n\}$ is bounded above.

EXTRA THEOREMS

-Uniform ConvergenceA sequence of fns $f_n : I \to \mathbb{R}$ converges uniformly to $f : I \to \mathbb{R}$ iff the sequence $a_n = sup_I |f_n - f|$ converges to 0

-Lower and Upper Integral Inequality If f and g are integrable fn on $[a,b]$ and $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_{--a}^{b} f(x)dx \le \int_{--a}^{b} g(x)dx$ and $\int_a^{--b} f(x)dx \le \int_a^{--b} g(x)dx$

-Constant K Integrability Let $f, g : [a,b] \to \mathbb{R}$. Suppose f is integrable and $\exists$ a constant K <0 s.t. $|g(x) - g(y)| < K|f(x) - f(y)|$ $\forall x, y, \in [a,b]$, then g is integrable.