Comparative Difficulty of X-Sudoku and Sudoku Stripe

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The Puzzles

In addition to ordinary sudoku rules…

- X-sudoku
  - Requires that both main diagonals contain the digits 1-9
- Sudoku Stripe
  - Requires that the digits 1-9 occur in ascending or descending order at least once
  - Can occur in any row, column, or region

Examples

4	7	8	1	9	5	3	2	*6*
5	*2*	6	4	8	3	7	*1*	9
9	3	*1*	6	7	2	*4*	5	8
2	5	9	*8*	6	*7*	1	3	4
3	6	4	2	*5*	1	9	8	7
1	8	7	*3*	4	*9*	5	6	2
7	1	*2*	9	3	8	*6*	4	5
6	*9*	3	5	2	4	8	*7*	1
8	4	5	7	1	6	2	9	*3*

Solved X-Sudoku

6	2	7	8	9	1	3	4	5
5	4	3	6	7	2	1	9	8
9	1	8	3	4	5	7	2	6
4	5	6	*1*	*2*	*3*	8	7	9
7	8	9	*4*	*5*	*6*	2	3	1
2	3	1	*7*	*8*	*9*	6	5	4
3	6	5	9	1	7	4	8	2
8	7	2	5	6	4	9	1	3
1	9	4	2	3	8	5	6	7

Solved Sudoku Stripe

Figure 1: Example Solutions

Motivation

- The x-condition
  - Expressible more compactly in plain English
  - Is easier to check in a proposed solution
  - Is stricter (only one way to satisfy)
- The stripe condition
  - Expressible less compactly in plain English
  - Is harder to check in a proposed solution
  - Is less strict (many ways to satisfy)

Hypothesis

Stripe sudoku is harder for a SAT solver than x-sudoku is.

Explication

- Harder how?
  - Solver may search longer to find a model or determine none exists
  - Metrics: max decision level, total decisions, conflict clauses added
- What basis for comparison?
  - Same puzzles, same solver, similar encodings
  - Require the solver to make at least one decision
  - Compare average and maximum results across the entire dataset

The Dataset

- 49,151 nonequivalent 9 x 9 sudoku puzzles with exactly 17 hints each
- Maintained by Gordon Royle at the University of Western Australia
- http://staffhome.ecm.uwa.edu.au/~00013890/sudokumin.php

The Solvers

- zChaff
  - Maintained by Princeton University
  - https://www.princeton.edu/~chaff/zchaff.html
- CaDiCaL
  - Maintained by Armin Biere
  - https://github.com/arminbiere/cadical

Minimal Encoding

$$Sudoku = Cell \bigwedge Row \bigwedge Column \bigwedge Region \bigwedge Hints \tag{1}$$

$$Cell = \bigwedge_{r=1}^{n} \bigwedge_{c=1}^{n} \bigvee_{v=1}^{n} (r, c, v) \tag{2}$$

$$Row = \bigwedge_{r=1}^{n} \bigwedge_{v=1}^{n} \bigvee_{c_i=1}^{n-1} \bigvee_{c_j=c_i+1}^{n} \neg(r, c_i, v) \vee \neg(r, c_j, v) \tag{3}$$

$$Column = \bigwedge_{c=1}^{n} \bigwedge_{v=1}^{n} \bigvee_{r_i=1}^{n-1} \bigvee_{r_j=r_i+1}^{n} \neg(r_i, c, v) \vee \neg(r_j, c, v) \tag{4}$$

$$Region = \bigwedge_{r_b=1}^{\sqrt{n}} \bigwedge_{c_b=1}^{\sqrt{n}} \bigwedge_{n=1}^{n} \bigwedge_{r_i=1}^{\sqrt{n}} \bigwedge_{c_i=1}^{\sqrt{n}} \bigwedge_{r_j=r_i+1}^{\sqrt{n}} \bigwedge_{c_j=1}^{\sqrt{n}} \neg(r_b * \sqrt{n} + r_i, c_b * \sqrt{n} + c_i, v) \vee \neg(r_b * \sqrt{n} + r_j, c_b * \sqrt{n} + c_j, v) \tag{5}$$

$$Hints = \bigwedge^{\#Hints} (r, c, v) \tag{6}$$

X Encoding

$$Diagonal\_Left\_Right = \bigwedge_{i_i=1}^{n} \bigwedge_{v=1}^{n} \bigwedge_{i_j=i_i+1}^{n} \neg(i_i, i_i, v) \vee \neg(i_j, i_j, v)$$

$$Diagonal\_Right\_Left = \bigwedge_{i_i=1}^{n} \bigwedge_{v=1}^{n} \bigwedge_{i_j=i_i+1}^{n} \neg(i_i, n - i_i, v) \vee \neg(i_j, n - i_j, v)$$

Stripe Encoding

$$Stripe = RowsStripe \vee ColumnsStripe \vee RegionsStripe$$

$$RowsStripe = \bigvee_{r=1}^{n} (\bigwedge_{c=1}^{n} (r, c, c)) \vee (\bigwedge_{c=1}^{n} (r, c, n-c))$$

$$RowsStripe = \bigvee_{r=1}^{n} row_r$$

$$Ascending = \bigwedge_{c=1}^{n} (r, c, c) \leftrightarrow (r, 1, 1) \wedge y_1$$

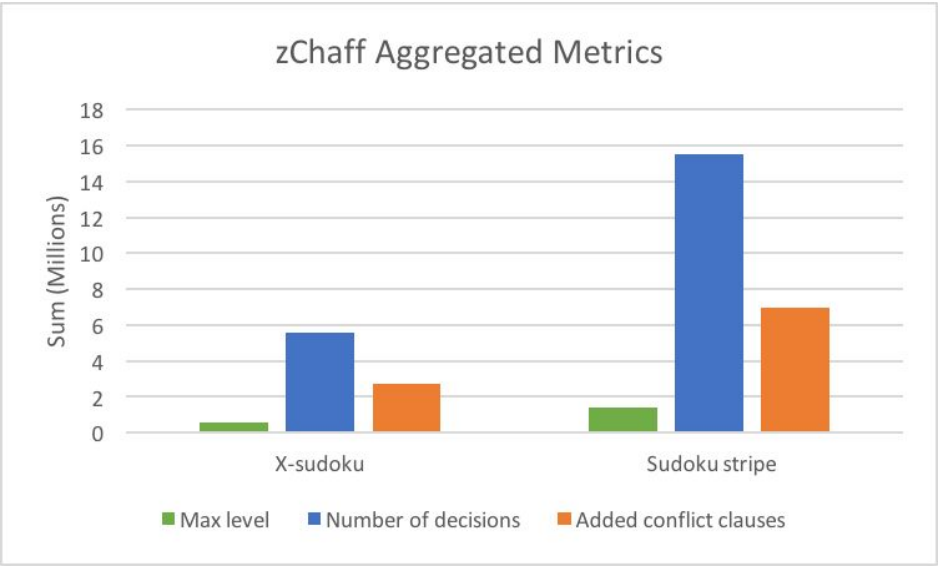
$$y_1 \rightarrow \bigwedge_{c=2}^{n-1} (r, c, c) \leftrightarrow \neg y_1 \vee (\bigwedge_{c=2}^{n-1} (r, c, c)) \leftrightarrow \bigwedge_{c=2}^{n-1} (\neg y_1 \vee (r, c, c))$$

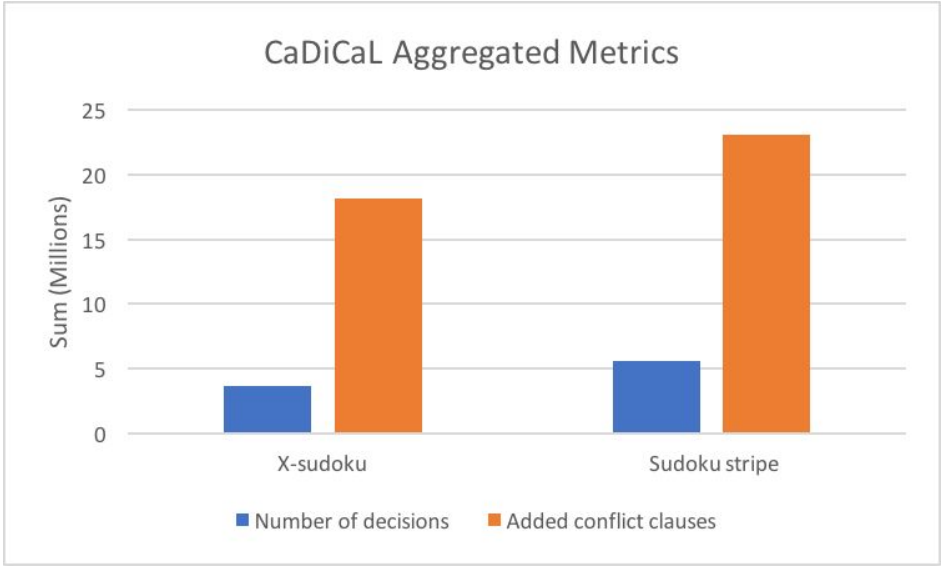
$$Descending = \bigwedge_{c=1}^{n} (r, c, n-c) \leftrightarrow (r, 1, n-1) \wedge y_2$$

$$y_2 \rightarrow \bigwedge_{c=2}^{n-1} (r, c, n-c) \leftrightarrow \neg y_2 \vee (\bigwedge_{c=2}^{n-1} (r, c, n-c)) \leftrightarrow \bigwedge_{c=2}^{n-1} (\neg y_2 \vee (r, c, n-c))$$

$$row_r \rightarrow ((r, 1, 1) \wedge y_1) \vee ((r, 1, n) \wedge y_2) \leftrightarrow (\neg row_r \vee (r, 1, 1) \vee y_2) \wedge (\neg row_r \vee (r, 1, n) \wedge y_1)$$

Results: Aggregated





Results: Statistics

zChaff Statistics

	Maximum depth		Number of decisions		Added conflict clauses	
	x	stripe	x	stripe	x	stripe
Mean	17.78	28.51	169.37	319.64	83.09	142.76
Standard deviation	9.10	9.06	223.79	249.05	118.11	135.93
Max	67	77	7876	3535	2900	1791

Results: Statistics

CaDiCaL Statistics

	Number of decisions		Added conflict clauses	
	x	stripe	x	stripe
Mean	112.72	171.84	553.62	704.84
Standard deviation	161.77	173.01	1033.87	932.15
Max	2715	2849	20669	19236

Conclusion

- Relative to a fixed dataset and SAT-solver:
  - Sudoku stripe seems more difficult than x-sudoku in the average case.
    - But how strong is this experiment's support?
  - Sudoku stripe is not necessarily more difficult in the worst case.
    - The hardest x-instance may be as hard or harder than the hardest stripe-instance.

Future Work

- Different Solvers
- Different settings
- Different puzzles

Github

Clone our repository and try our experiment for yourself!

https://github.com/huntermcknight/x-stripe

YouTube

Watch our presentation at:

https://youtu.be/msFuuGF-dZs