# SHEAVES

**definitions:** A **presheaf** $\mathcal{P} = \{\mathcal{P}(U); \rho_V^U\}$ of groups (sets, R-modules,...) over a topological space $X$ consists of:

1. a group (set, R-module,...) $\mathcal{P}(U)$ for each open set $U \subset X$, and
2. a homomorphism (map,...) $\rho_V^U : \mathcal{P}(U) \to \mathcal{P}(V)$ for each pair $V \subset U$ of open subsets of $X$ such that
   a. $\rho_U^U = \text{id}$, and
   b. $\rho_W^U = \rho_W^V \circ \rho_V^U$ for $W \subset V \subset U$.

A presheaf $\mathcal{P}$ is **complete** if whenever an open set $U$ can be written as $\bigcup_\alpha U_\alpha$ with $U_\alpha$ open, then

1. if $s, t \in \mathcal{P}(U)$ and $\rho_{U_\alpha}^U(s) = \rho_{U_\alpha}^U(t)$ for all $\alpha$ then $s = t$, and
2. if $s_\alpha \in \mathcal{P}(U_\alpha)$ and $\rho_{U_\alpha \cap U_\beta}^{U_\alpha}(s_\alpha) = \rho_{U_\alpha \cap U_\beta}^{U_\beta}(s_\beta)$ for all $\alpha, \beta$ then there exists $s \in \mathcal{P}(U)$ such that $s_\alpha = \rho_{U_\alpha}^U(s)$.

Remark on the definition of a presheaf: A presheaf can be defined in categorical terms. To this end given any topological space $X$ we can define the category OPEN($X$) of open subsets of $X$. The objects in this catagory are all open sets in $X$ and the morphisms are the inclusion maps. More specifically given two objects (open sets) $U$ and $V$ if $U \subset V$ then the set of morphisms from $U$ to $V$ just contains the inclusion map $i : U \to V$ otherwise it is empty. Now we can describe a presheaf as a contravariant functor from OPEN($X$) to GROUPS (SETS, R-MODULES,...).
Given a presheaf $\mathcal{P}$ on a space $X$ an element of $\mathcal{P}(U)$ is usually called a **section** (of $\mathcal{P}$) over $U$ and the maps $\rho_V^U$ are called **restriction maps**. The reason for this terminology will become clear later.

Remark on the definition of a complete presheaf: Condition 1. for a presheaf to be complete just says that a section is determined locally. Condition 2. just says that given a bunch of sections that agree "on their common domains" we can piece them together to get a section over the union of their domains.

**examples:** these are all examples of complete presheaves of groups.

1. Let $X$ be any topological space and $G$ any group. The **constant presheaf** $G$ assigns to any open set $U$ the group $G(U) = G$. The restriction maps are all just the identity map on $G$, $i : G \to G$. Some important examples of constant presheaves are when $G$ is the integers $\mathbf{Z}$, the rationals $\mathbf{Q}$, the reals $\mathbf{R}$ and the complex numbers $\mathbf{C}$.
2. Let $M$ be any smooth manifold.
   a. The **presheaf of smooth functions** assigns to each open $U$
   $$C^\infty(U) = \{f : U \to \mathbf{R} | f \text{ is smooth}\}$$
   and for each inclusion $V \subset U$ the restriction map is just natural restriction of functions (if $f \in C^\infty(U)$ then $\rho_V^U(f) = f|_V$).
   b. The **presheaf of non-zero smooth functions** assigns to each open set $U$
   $$C^*(U) = \{f : U \to \mathbf{R} | \text{f is smooth and never zero}\}$$
   This is a group under multiplication. The restriction maps are as in 2.a.
   c. The **presheaf of smooth p-forms** assigns to each open set $U$
   $$\mathcal{A}^p(U) = \{\text{smooth p-forms over U}\}$$
   Again restriction maps are as in 2.a.
3. Let $M$ be a complex manifold. In all the examples below the restriction maps are just natural restriction of functions.
   a. The **presheaf of holomorphic functions** assigns to each open set $U$
   $$\mathcal{O}(U) = \{\text{holomorphic functions on } U\}$$
   b. The **presheaf of non-zero holomorphic functions** assigns to each open set $U$
   $$\mathcal{O}^*(U) = \{\text{never zero holomorphic functions on } U\}$$

this is a group under multiplication.

c. The **presheaf of holomorphic p-forms** assigns to each open set $U$

$$\Omega^p(U) = \{\text{holomorphic p-forms on } U\}$$

d. The **presheaf of smooth forms of type** $(p, q)$ assigns to each open set $U$

$$\mathcal{A}^{p,q}(U) = \{\text{smooth forms of type } (p, q) \text{ on } U\}$$

e. The **presheaf of meromorphic functions** assigns to each open set $U$

$$\mathcal{M}(U) = \{\text{meromorphic functions on } U\}$$

A meromorphic function on $U$ is a thing that is locally a quotient of holomorphic functions (note since $0/0$ is undefined a meromorphic functions is not exactly a function).

f. The **presheaf of non-zero meromorphic functions** assigns to each open set $U$

$$\mathcal{M}^*(U) = \{\text{not identically zero (or infinite) meromorphic functions on } U\}$$

this is a group under multiplication.

4. Let $E \to M$ be a holomorphic vector bundle over $M$. In all the examples below the restriction maps are just natural restriction of functions.

a. The **presheaf of holomorphic sections of** $E$ assigns to each open set $U$

$$\mathcal{O}(E)(U) = \{\text{holomorphic sections of } E \text{ over } U\}$$

b. The **presheaf of smooth** $E$**-valued** $(p, q)$**-forms** assigns to each open set $U$

$$\mathcal{A}^{p,q}(E)(U) = \{\text{smooth } E\text{-valued } (p, q)\text{-forms over } U\}$$

**example:** of a noncomplete presheaf. We take our space to be $\mathbf{C}$. Consider the presheaf that assigns to each open set $U$

$$\mathcal{B}(U) = \{\text{bounded holomorphic functions on } U\}$$

To see that this is not a complete presheaf consider the functions $f_i \in \mathcal{B}(U_i)$ where $U_i =$ the ball of radius $i$ and $f_i(z) = z$. Since $\mathbf{C} = \bigcup_i U_i$ if $\mathcal{B}$ were complete then there would exist a bounded holomorphic function $f$ on $\mathbf{C}$ such that $f|_{U_i}(z) = z$ but Liouville's theorem says that any such $f$ must be constant.

**definition:** a **map of presheaves** $\alpha : \mathcal{F} \to \tilde{\mathcal{F}}$ over $X$ is a collection of homomorphisms

$$\{\alpha_U : \mathcal{F}(U) \to \tilde{\mathcal{F}}(U)\}$$

such that $\alpha_V \circ \rho_V^U = \tilde{\rho}_V^U \circ \alpha_U$ for all $V \subset U$.

**example:** On a complex manifold we have the following map of presheaves

$$\exp : \mathcal{O} \to \mathcal{O}^*$$

where if $f$ is a holomorphic map on $U$ then $\exp(f) = e^{2\pi i f}$ is a never zero holomorphic function on $U$.

**definitions:** a **sheaf** $\mathcal{S}$ of groups (sets, R-modules,...) over a topological space $X$ is a topological space $\mathcal{S}$ together with a map $\pi : \mathcal{S} \to X$ (called the **projection map**) satisfying

1. $\pi$ is a local homeomorphism and onto $X$,
2. $\mathcal{S}_x \equiv \pi^{-1}(x)$ is a group (set, R-module,...) for each $x \in X$ called the **stalk** over $x$, and
3. the operations in the stalks is continuous
   (i.e. $\mathcal{S} \circ \mathcal{S} \to \mathcal{S} : (s, t) \mapsto s - t$ is continuous where $\mathcal{S} \circ \mathcal{S} = \{(s, t) \in \mathcal{S} \times \mathcal{S} | \pi(s) = \pi(t)\}$)

a map $f : U \to \mathcal{S}$ such that $\pi \circ f =$id is called a **section of** $\mathcal{S}$ **over** $U$.

**note:**

1. $\Gamma(\mathcal{S}, U) = \{\text{sections of } \mathcal{S} \text{ over } U\}$ has the same algebraic structure as $\mathcal{S}$.
2. A section is an open map since $\pi$ is a local homeomorphism.
3. If $f(x) = g(x)$ where $f$ and $g$ are sections of $\mathcal{S}$ over $U$ then $f = g$ on some neighborhood of $x$.

**examples:**

1. Let $X$ be any topological space and $G$ a group with the discrete topology. The $\mathcal{G} = X \times G$ is called the **constant sheaf**.
2. For example 2. we shall outline a construction that produces a sheaf out of a presheaf. Given a presheaf $\mathcal{P}$ over $X$ we will construct the **sheaf of germs of** $\mathcal{P}$ and denote it $\alpha(\mathcal{P}) = \mathcal{S}$. A stalk over $x \in X$ is

$$\mathcal{S}_x = \lim_{\rightarrow} \mathcal{P}(U)$$

where the direct limit is taken over all open sets $U$ containing $x$. More concretely let

$$\mathcal{S}_x^* = \coprod \mathcal{P}(U)$$

where the coproduct is over all $U$ containing $x$. Define an equivalence relation on $\mathcal{S}_x^*$ by saying $f \sim g$, where $f \in \mathcal{P}(U)$ and $g \in \mathcal{P}(V)$, if there exists an open set $W \subset U \cap V$ such that $x \in W$ and $\rho_W^U(f) = \rho_W^V(g)$. Denote the equivalence class of $f$ by $[f]_x$. Now let

$$\mathcal{S}_x = \mathcal{S}_x^* / \sim$$

and

$$\mathcal{S} = \coprod_x \mathcal{S}_x$$

Topologize $\mathcal{S}$ by the basis $\{O_f\}$ where for each open set $U$ and each $f \in \mathcal{P}(U)$, $O_f = \{[f]_x | x \in U\}$.
So for each example of a presheaf we get a sheaf of germs.

We have just described how to get a sheaf from a presheaf. Going the other way, from a sheaf to a presheaf, is even easier. Given a sheaf $\mathcal{S}$ consider the presheaf $\beta(\mathcal{S})$ of sections of $\mathcal{S}$ defined by $\beta(\mathcal{S})(U) = \Gamma(\mathcal{S}, U)$ and the restriction maps are just the natural restriction of sections to a smaller set. These two operations $\alpha$ and $\beta$ are more or less inverses of each other but before we can state the appropriate theorem we need one more definition.

**definition:** a **sheaf homomorphism** $\psi : \mathcal{S} \to \mathcal{S}'$ is a continuous map such that $\pi = \pi' \circ \psi$ and is a homomorphism on each stalk.

**Theorem:** For any sheaf $\mathcal{S}$ we have

$$\alpha(\beta(\mathcal{S})) \cong \mathcal{S}$$

If $\mathcal{P}$ is a complete presheaf we have

$$\beta(\alpha(\mathcal{P})) \cong \mathcal{P}$$

If $\mathcal{P}$ is not complete then $\beta(\alpha(\mathcal{P}))$ is called the **completion of** $\mathcal{P}$.

Though we shall not prove the above theorem we will give the isomorphisms involved. To show $\alpha(\beta(\mathcal{S})) \cong \mathcal{S}$ we exhibit a map $\Phi$ from $\alpha(\beta(\mathcal{S}))$ to $\mathcal{S}$. Given $[f]_x$ a germ of a section of $\mathcal{S}$ in $\alpha(\beta(\mathcal{S}))$ then $\Phi([f]_x) = f(x)$. To show $\beta(\alpha(\mathcal{P})) \cong \mathcal{P}$ we exhibit a map $\Psi$ from $\mathcal{P}$ to $\beta(\alpha(\mathcal{P}))$. So given an element $f$ in the group $\mathcal{P}(U)$ then $\Psi(f) = \sigma_f$ an element of $\Gamma(\alpha(\mathcal{P}), U) = \beta(\alpha(\mathcal{P}))$ defined by $\sigma_f : U \to \alpha(\mathcal{P}) : p \mapsto [f]_p$.

**definition:** a sequence of sheaves and sheaf homomorphisms

$$\ldots \to \mathcal{S}_{i-1} \xrightarrow{f_i} \mathcal{S}_i \xrightarrow{f_{i+1}} \mathcal{S}_{i+1} \to \ldots$$

is called **exact** if $\mathrm{im} f_i = \ker f_{i+1}$ for all $i$. Another way to say this is for each $x$ the sequence

$$\ldots \to (\mathcal{S}_{i-1})_x \to (\mathcal{S}_i)_x \to (\mathcal{S}_{i+1})_x \to \ldots$$

is an exact sequence of groups.

**examples:**

1. On a complex manifold we have
$$0 \to \mathbf{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$
where $i$ is just inclusion and $\exp[f]_m = [e^{2\pi i f}]_m$. This is an exact sequence. The only difficult part to check is that exp is onto. Given an element $[f]_m \in \mathcal{O}^*$ choose a function, say $f$, to represent the germ on a small enough open set to define a logarithm. The element $[\frac{1}{2\pi i}\log f]_m \in \mathcal{O}$ will map to $[f]_m$. Note it was very important that we could pick a small open set to define the logarithm. If we considered the same sequence on the presheaf level exp would not be onto.
2. Similarly on a smooth manifold we have
$$0 \to \mathbf{Z} \xrightarrow{i} \mathcal{A} \xrightarrow{\exp} \mathcal{A}^* \to 0$$
3. On any topological space given two sheaves such that
$$0 \to \mathcal{S} \to \mathcal{S}'$$
is exact we can construct a **quotient sheaf** $\mathcal{S}'' = \mathcal{S}'/\mathcal{S}$ by $\mathcal{S}'' = \alpha(\beta(\mathcal{S}')/\beta(\mathcal{S}))$. By using this definition we avoid having to specify a topology on the quotient space (it is automatically taken care of by the map $\alpha$). It is now easy to check
$$0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{S}'' \to 0$$
is exact. A specific example of the above construction is when we have a complex manifold we get
$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0.$$
4. On a smooth manifold Poincaré's lemma tells us
$$0 \to \mathbf{R} \to C^\infty \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \xrightarrow{d} \dots$$
is exact.
5. Similarly on a complex manifold we have
$$0 \to \Omega^p \to \mathcal{A}^{p,0} \xrightarrow{\bar\partial} \mathcal{A}^{p,1} \xrightarrow{\bar\partial} \dots$$
and
$$0 \to \mathbf{C} \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots$$
are exact.

**Theorem:**

1. Given an exact sequence of presheaves
$$0 \to \mathcal{P} \to \mathcal{Q} \to \mathcal{R} \to 0$$
(i.e. whenever you plug in an open set into the above sequence it becomes an exact sequence of groups) the following sequence of sheaves is exact
$$0 \to \alpha(\mathcal{P}) \to \alpha(\mathcal{Q}) \to \alpha(\mathcal{R}) \to 0.$$
2. Given an exact sequence of sheaves
$$0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{S}'' \to 0$$
the following sequence is exact
$$0 \to \beta(\mathcal{S}) \to \beta(\mathcal{S}') \to \beta(\mathcal{S}'')$$

Note that exactness is lost at the last place.

Exactness of a sequence of sheaves is a local property while exactness of a sequence of presheaves is a global property. Sheaf cohomology is a measure of the failure of a local property to extend to a global one (i.e. is a measure of the failure of the last sequence in the last theorem to be exact). There are some sheaves for which we can pass form a sequence of sheaves to a sequence of presheaves without loosing exactness. We consider those now.

**definitions:** a sheaf $\mathcal{S}$ over $X$ is **soft** if for any closed subset $S \subset X$ the restriction map $\Gamma(X, \mathcal{S}) \to \Gamma(S, \mathcal{S})$ is onto. Said a different way, any section over $S$ can be extended to a global section.

A sheaf $\mathcal{S}$ is **fine** if for any locally finite open cover $\{U_\alpha\}$ of $X$ there exists a family of sheaf morphisms $\{\eta_\alpha : \mathcal{S} \to \mathcal{S}\}$ such that

1. $\sum \eta_\alpha = 1$ and
2. $\eta_\alpha$ has the closure of its support contained in $U_\alpha$ for all $\alpha$.

the family of sheaf morphisms $\{\eta_\alpha\}$ described above is called a **sheaf partition of unity**.

**examples:**

1. using a normal smooth partition of unity on a manifold it is easy to see that $C^\infty$, $\mathcal{A}^p$ and $\mathcal{A}^{p,q}$ are all fine and hence soft sheaves.
2. constant sheaves are not soft (hence not fine). To see this consider the closed subset $\{x, y\}$ in some path component of some space. A section of $\mathbf{Z}$, say, over this set is just two integers. If these integers are not the same then this section clearly cannot be extended over the whole space since sections are locally constant.
3. $\mathcal{O}$ is not soft (hence not fine). To see this consider the section $\log z$ of $\mathcal{O}$ defined on some suitable subset of the complex plane. This section cannot be extended to the whole plane.

**theorem:**

1. fine sheaves are soft.
2. if

$$0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{S}'' \to 0$$

is an exact sequence of sheaves and $\mathcal{S}$ is soft, then

$$0 \to \beta(\mathcal{S}) \to \beta(\mathcal{S}') \to \beta(\mathcal{S}'') \to 0$$

is also exact.

## COHOMOLOGY OF SHEAVES

**definitions:** Given a sheaf $\mathcal{S}$ and a locally finite open cover $\mathcal{U} = \{U_\alpha\}$ of $M$ we can define:

1. a **p-chain**, $\sigma = (U_{\alpha_0}, \ldots, U_{\alpha_p})$, (associated to the open cover $\mathcal{U}$) is a (p+1)-tuple of distinct open sets (techniquely we should well order the index set of $\mathcal{U}$ and require $\alpha_0 < \alpha_1 < \ldots \alpha_p$) with nonempty intersection.
2. the **support** of a p-chain $\sigma = (U_{\alpha_0}, \ldots, U_{\alpha_p})$ is $|\sigma| = U_{\alpha_0} \cap \ldots \cap U_{\alpha_p}$.
3. a **p-cochain**, $f$, with coefficients in the sheaf $\mathcal{S}$ is an assignment to each p-chain $\sigma$ a section of $\mathcal{S}$ over $|\sigma|$. Let $f_{\alpha_0 \ldots \alpha_p}$ denote the section of $\mathcal{S}$ that $f$ assigns to $\sigma = (U_{\alpha_0}, \ldots, U_{\alpha_p})$.
4. denote by $C^p(\mathcal{U}, \mathcal{S})$ the set of all p-cochains. Note that

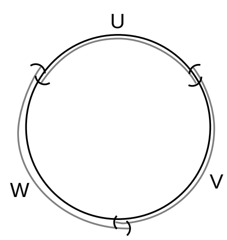
$$C^p(\mathcal{U}, \mathcal{S}) = \prod_{\alpha_0 \neq \ldots \alpha_p} \Gamma(\mathcal{S}, U_{\alpha_0} \cap \ldots \cap U_{\alpha_p})$$

To better understand the definitions of the $C^p(\mathcal{U}, \mathcal{S})$ it is helpful to consider the **nerve** of a locally finite open cover $\mathcal{U}$. We will not go into much detail on nerves since we will not use them in any essential way. Given $\mathcal{U}$ as above the nerve of $\mathcal{U}$, $\mathcal{N}(\mathcal{U})$, is a simplicial complex with

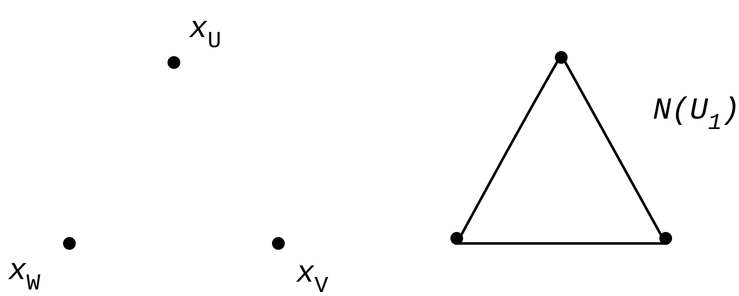
1. One vertex for each $U_\alpha \in \mathcal{U}$,
2. One edge for each pair of open sets $U_{\alpha_0}, U_{\alpha_1} \in \mathcal{U}$ with $U_{\alpha_0} \cap U_{\alpha_1} \neq \emptyset$,
3. In general we get one p-simplex for each (p+1)-tupel, $U_{\alpha_0}, \ldots, U_{\alpha_p}$, of open sets from $\mathcal{U}$ with nonempty intersection.

We will not describe how to assemble the simplices but this will hopefully be apparent from the examples below.

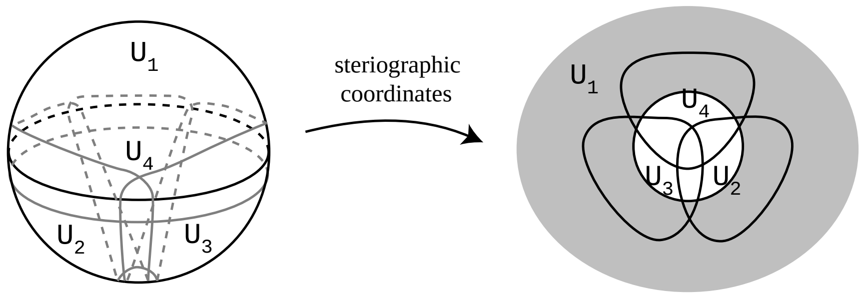
1. Consider $S^1$ with the open cover $\mathcal{U}_1 = \{U, V, W\}$



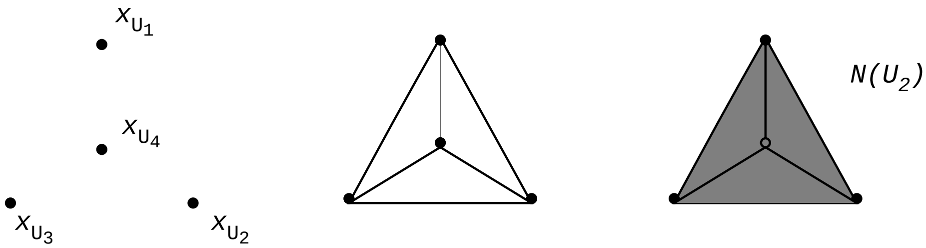

We build up the nerve $\mathcal{N}(\mathcal{U}_1)$ one skeleta at a time below



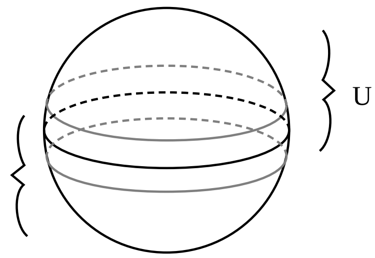

2. Consider $S^2$ with the open cover $\mathcal{U}_2 = \{U_0, U_1, U_2, U_3\}$



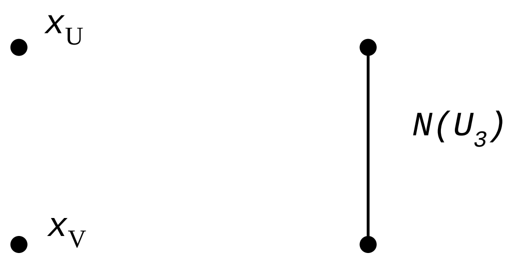
The nerve $\mathcal{N}(\mathcal{U}_2)$ is constructed below




3. Consider $S^2$ with the open cover $\mathcal{U}_3 = \{U, V\}$




The nerve $\mathcal{N}(\mathcal{U}_3)$ is




Now consider the constant sheaf $\mathbf{Z}$ on each of the spaces in the above examples. $C^p(\mathcal{U}_i, \mathbf{Z})$ is just the set of maps from the $p$-skeleton of $\mathcal{N}(\mathcal{U}_i)$ to the integers. Thus these groups $C^p(\mathcal{U}_i, \mathbf{Z})$ are just the simplicial $p$-cochain groups associated to $\mathcal{N}(\mathcal{U}_i)$. So we can think of the above definitions as a generalization of normal $p$-cochains.

**definitions:** to turn $C^0(\mathcal{U}, \mathcal{S}), C^1(\mathcal{U}, \mathcal{S}), \ldots$ into a cochain complex we define the **coboundary operator** $\delta : C^p \to C^{p+1}$ by

$$(\delta f)_{\alpha_0 \ldots \alpha_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \rho^{U_{\alpha_0} \ldots \hat{U}_{\alpha_j} \ldots U_{\alpha_{p+1}}}_{U_{\alpha_0} \ldots U_{\alpha_{p+1}}} f_{\alpha_0 \ldots \hat{\alpha_j} \ldots \alpha_{p+1}}$$

a p-cochain $f$ is called a **p-cocycle** if $\delta f = 0$. It is standard to check that $\delta^2 = 0$. Let

$$Z^p(\mathcal{U}, \mathcal{S}) = \ker \delta \subset C^p(\mathcal{U}, \mathcal{S})$$

**examples:**

1. Let $f \in C^0(\mathcal{U}, \mathcal{S})$ so $f_U$ is a section of $\mathcal{S}$ over $U$ and

$$(\delta f)_{UV} = f_V|_{U \cap V} - f_U|_{U \cap V}$$

so if $f$ is a 0-cocycle then $f$ assigns sections of $\mathcal{S}$ to each open set and all the sections agree on the "over laps" hence $f$ defines a global section of $\mathcal{S}$. Thus $Z^0(\mathcal{U}, \mathcal{S}) = \Gamma(\mathcal{S}, M)$.

2. Let $g \in C^1(\mathcal{U}, \mathcal{S})$ then

$$(\delta g)_{UVW} = g_{VW} - g_{UW} + g_{UV}$$

so if $g$ is a 1-cocycle and written multiplicatively we get

$$g_{VW}g_{UV} = g_{UW}$$

Later we will see this has something to do with line bundles.

We are finally ready to define the **p-cohomology of $M$ with coefficients in $\mathcal{S}$ relative to the open cover $\mathcal{U}$** as

$$H^p(\mathcal{U}, \mathcal{S}) = \frac{Z^p(\mathcal{U}, \mathcal{S})}{\delta(C^{p-1}(\mathcal{U}, \mathcal{S}))}$$

**exercise:** Compute $H^p(\mathcal{U}_i, \mathbf{Z})$ for examples 1.-3. above.

We now wish to get something that is independent of the open cover. So given two locally finite open covers $\mathcal{U} = \{U_\alpha\}_{\alpha \in I_U}$ and $\mathcal{V} = \{V_\alpha\}_{\alpha \in I_V}$ we say $\mathcal{V}$ is a refinment of $\mathcal{U}$ if for all $V_\alpha$ there is a $U_{\alpha'}$ such that $V_\alpha \subset U_{\alpha'}$. We will think of a refinement as a function $\phi : I_V \to I_U$ such that $V_\alpha \subset U_{\phi(\alpha)}$ for all $\alpha \in I_V$. Now $\phi$ gives us a map

$$\rho_\phi : C^p(\mathcal{U}, \mathcal{S}) \to C^p(\mathcal{V}, \mathcal{S})$$

by

$$(\rho_\phi f)_{\alpha_0 \ldots \alpha_p} = \rho|_{\cap V_{\alpha_i}}^{\cap U_{\alpha_i}} f_{\phi(\alpha_0)\ldots\phi(\alpha_p)}.$$

Clearly we have $\delta \circ \rho_\phi = \rho_\phi \circ \delta$ so we get a map

$$\rho : H^p(\mathcal{U}, \mathcal{S}) \to H^p(\mathcal{V}, \mathcal{S}).$$

(given two refinement maps $\phi$ and $\phi'$ for the same refinement, $\rho_\phi$ and $\rho_{\phi'}$ will be chain homotopic thus inducing the same map $\rho$ on cohomology)
The $p^{th}$ **Čech cohomology group of $M$ with coefficients in $\mathcal{S}$** is

$$\check{H}^p(M; \mathcal{S}) = \lim_{\rightarrow} H^p(\mathcal{U}, \mathcal{S})$$

where the direct limit is over all locally finite open covers $\mathcal{U}$. In general it is very difficult to calculate with direct limits there is however one case where it is easy

**Theorem:** (actually trivial, but important, observation)

$$\check{H}^0(M; \mathcal{S}) = \Gamma(\mathcal{S}, M)$$

This is true since $H^0(\mathcal{U}, \mathcal{S}) = \Gamma(\mathcal{S}, M)$ for any cover $\mathcal{U}$. In most cases we will not try and compute the direct limit. We can get around computing it by using the following important theorem.

**Leray's Theorem:** If $\mathcal{U}$ is a locally finite open cover of $M$ such that $\check{H}^q(U_{\alpha_0} \cap \ldots U_{\alpha_p}; \mathcal{S}) = 0$ for all $q > 0$ and all $\alpha_0, \ldots \alpha_p$, then $\check{H}^*(M; \mathcal{S}) = \mathcal{H}^*(\mathcal{U}, \mathcal{S})$.

Morally Leray's theorem says that one should pick an open cover with "small" sets to compute the Čech cohomology. This theorem at first glance might appear "recursive," i.e. we didn't want to compute something hard so we reduced it to computing an infinite number of hard things (that all the cohomology of the intersections is trivial), but as we will see later we can actually do quite a few computations with it. For now let's look at several techniques useful for computations (and other things).

**Theorem:** If $\mathcal{S}$ is soft then $\check{H}^p(M; \mathcal{S}) = 0$ for all $p > 0$.

We shall only consider the proof of this when $\mathcal{S}$ is fine (remember that fine sheaves are soft). If $\mathcal{S}$ is fine and $\mathcal{U}$ is a Leray cover for $\mathcal{S}$ then let $\eta_\alpha$ be a sheaf partition of unity and define a map $h_p : C^p(\mathcal{U}, \mathcal{S}) \to C^{p-1}(\mathcal{U}, \mathcal{S})$ by

$$(h_p(f))_{U_0 \ldots U_{p-1}} = \sum_\alpha \eta_\alpha f_{U_\alpha U_0 \ldots U_{p-1}}.$$

It can be shown that if $\delta f = 0$ then $\delta h_p(f) = f$. It is a good exercise to prove this for $p = 1$. The theorem clearly follows.

**note:** The above theorem says that all sheaf cohomology groups with coefficients in $C^\infty, \mathcal{A}^p$ and $\mathcal{A}^{p,q}$ are trivial.

Now given an exact sequence of sheaves over $M$

$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$$

we clearly have induced maps

$$\alpha : C^p(\mathcal{U}, \mathcal{A}) \to C^p(\mathcal{U}, \mathcal{B})$$

$$\beta : C^p(\mathcal{U}, \mathcal{B}) \to C^p(\mathcal{U}, \mathcal{C})$$

for any open cover $\mathcal{U}$. It is then easy to check that these maps commute with $\delta$ and hence give maps $\alpha^*$ and $\beta^*$ on the Čeck cohomology groups. Now consider $f \in [f] \in H^p(\mathcal{U}, \mathcal{C})$ so $\delta f = 0$. Since $\beta$ is onto we can find a refinement $\mathcal{U}'$ of $\mathcal{U}$ and an element $g \in C^p(\mathcal{U}', \mathcal{B})$ such that $\beta g = f$. Now $\beta\delta g = \delta f = 0$ so we can find an element $h \in C^{p+1}(\mathcal{U}', \mathcal{A})$ such that $\alpha h = \delta g$. $h$ is closed since $\alpha\delta h = \delta^2 g = 0$ and $\alpha$ is one-to-one. So we can define a map $\delta : C^p(M; \mathcal{C}) \to C^{p+1}(M; \mathcal{A})$ by $\delta[f] = [h]$. With a little diagram chasing one can show that $\delta$ is well-defined. So finally, given the short exact sequence above we get a long exact sequence

$$\ldots \to \check{H}^p(M; \mathcal{A}) \xrightarrow{\alpha^*} \check{H}^p(M; \mathcal{B}) \xrightarrow{\beta^*} \check{H}^p(M; \mathcal{C}) \xrightarrow{\delta} \check{H}^{p+1}(M; \mathcal{A}) \to \ldots$$

**Theorem:**

1. If $M$ is simplicial, then
$$H^*_{\text{simplicial}}(M; \mathbf{Z}) = \check{H}^*(M; \mathbf{Z}).$$
2. If $M$ is a smooth manifold, then
$$H^*_{\text{DeRham}}(M) = \check{H}^*(M; \mathbf{R}).$$
3. If $M$ is a complex manifold, then
$$H^{p,q}_{\bar{\partial}}(M) = \check{H}^q(M; \Omega^p).$$

To prove part 1. we start with a simplicial complex $K$ such that $|K|$ (the underlying topological space) is homeomorphic to $M$. The star of a vertex $x$, $\mathrm{st}(x)$, is just the union of all simplices that have $x$ as a vertex. Take as an open cover of $M$ all open sets of the form $U = \mathrm{int}(\mathrm{st}(x))$ for some vertex $x$. This cover $\mathcal{U}$ is clearly locally finite and can be shown to be "good." It is easy to see that the nerve of this cover $\mathcal{N}(\mathcal{U})$ is isomorphic to $K$ as a simplicial complex (drawing a few pictures should make this clear). Thus $C^p(\mathcal{U}, \mathbf{Z})$ is the same as the simplical p-cochain group for $K$ by the comments we made earlier on nerves. It is also easy to see that the coboundary maps are the same. Thus part 1. follows. It would be a good exercise to explicitly write down the map between the cochain groups for the two theories.

The proof of part 2. is a good illustration of how to use long exact sequences in cohomology computations (the proof of part 3. is almost identical). We start with the following exact sequence

$$0 \to \mathbf{R} \to \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \to \ldots$$

Set $\mathcal{Z}^p = \ker d \subset \mathcal{A}^p$ then the exactness of the the above sequence is equivalent the exactness of the following

$$0 \to \mathbf{R} \to \mathcal{A}^0 \xrightarrow{d} \mathcal{Z}^1 \to 0$$

$$0 \to \mathcal{Z}^1 \to \mathcal{A}^1 \xrightarrow{d} \mathcal{Z}^2 \to 0$$

$$\ldots$$

$$0 \to \mathcal{Z}^p \to \mathcal{A}^p \xrightarrow{d} \mathcal{Z}^{p+1} \to 0$$

Since $\mathcal{A}^n$ is soft the long exact sequences associated to the above short exact sequences turn into a bunch of isomorphisms

$$\check{H}^n(M;\mathcal{Z}^m) \cong \check{H}^{n+1}(M;\mathcal{Z}^{m-1})$$

and the sequence

$$0 \to \Gamma(\mathcal{Z}^{p-1}) \to \Gamma(\mathcal{A}^{p-1}) \xrightarrow{d} \Gamma(\mathcal{Z}^p) \to \check{H}^1(M;\mathcal{Z}^{p-1}) \to 0.$$

So we have $\check{H}^p(M;\mathbf{R}) \cong \check{H}^{p-1}(M;\mathcal{Z}^1) \cong \check{H}^{p-2}(M;\mathcal{Z}^2) \cong \ldots \check{H}^1(M;\mathcal{Z}^{p-1}) \cong \frac{\Gamma(\mathcal{Z}^p)}{d\Gamma(\mathcal{A}^{p-1})} = H^p_{\mathrm{DR}}(M)$.

**computation:**

1. $\check{H}^q(M;\mathcal{O}) = 0$ for $q > \dim M$.
   since $\mathcal{O} = \Omega^0$ part 3. of the above theorem gives $\check{H}^q(M;\mathcal{O}) \cong H^{p,q}_{\overline{\partial}}(M) = 0$ by the $\overline{\partial}$-Poincaré lemma.
2. $\check{H}^q(\mathbf{C};\mathcal{O}) = 0 \qquad q > 0$
   once again by the $\overline{\partial}$-Poincaré lemma.
3. $\check{H}^q(\mathbf{C};\mathbf{Z}) = 0 \qquad q > 0$
   since $\mathbf{C}$ is contractable.
4. $\check{H}^q(\mathbf{C};\mathcal{O}^*) = 0 \qquad q > 0$
   this easily follows from examples 2. and 3. using the long exact sequence associated to
   $$0 \to \mathbf{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0.$$
5. $\check{H}^q(\mathbf{C}^n \times (\mathbf{C}^*)^m;\mathcal{O}) = 0 \qquad q > 0$
   this is just another version of the $\overline{\partial}$-Poincaré lemma.
6. $\check{H}^q(\mathbf{C}P^n;\mathcal{O}) = \begin{cases} \mathbf{C} & q = 0; \\ 0 & \text{otherwise.} \end{cases}$

   We will just show this for $n = 1$. $\mathbf{C}P^1 = \{[u,v] | (u,v) \in \mathbf{C}^2 - \{(0,0)\}\}$. Now let $U$ be the $v = 1$ chart and $V$ be the $u = 1$ chart. $U = V = \mathbf{C}$ and $U \cap V = \mathbf{C}^*$ thus $\{U,V\}$ is a Leray cover for $\mathcal{O}$ by examples 2. and 5. If $u$ is the variable in $U$ and $v$ is the variable in $V$ then $u = 1/v$. Now
   $$\begin{aligned} C^0(\{U,V\},\mathcal{O}) &= \{(f,g) | f \in \mathcal{O}(U), g \in \mathcal{O}(V)\} \\ C^1(\{U,V\},\mathcal{O}) &= \{h \in \mathcal{O}(U \cap V)\}. \end{aligned}$$
   If $f = \sum_{n=0}^{\infty} a_n u^n$ and $g = \sum_{n=0}^{\infty} b_n v^n = \sum_{n=0}^{\infty} b_n u^{-n}$ then $\delta(f,g) = g - f = 0$ implies that $a_n = b_n = 0$ for $n > 0$ and $a_0 = b_0$. Thus we have $\check{H}^0(M;\mathcal{O}) = \mathbf{C}$. If $h \in C^1$ then $h \in \mathcal{O}(U \cap V)$ so $h = \sum_{-\infty}^{\infty} a_n u^n$. Thus if we set $f = \sum_{n=0}^{\infty} a_n u^n$ and $g = \sum_{n=0}^{\infty} a_{-n} v^n$ then clearly $\delta(f,g) = h$. Thus $H^1(M;\mathcal{O}) = 0$.
7. $\check{H}^q(\mathbf{C}P^n;\Omega^p) = \begin{cases} \mathbf{C} & p = q \leq n; \\ 0 & \text{otherwise.} \end{cases}$

   Once again we will only show this for $n = 1$. Using the same cover as above if $(\omega,\eta) \in C^0$ then $\omega \in \Omega^1(U)$ and $\eta \in \Omega^1(V)$. So $\omega = (\sum_{n=0}^{\infty} a_n u^n)du$ and $\eta = (\sum_{n=0}^{\infty} b_n v^n)dv = -(\sum_{n=0}^{\infty} b_n u^{-n-2})du$. The last inequality is true since $dv = d(1/u) = -(1/u^2)du$. Thus if $\delta(\omega,\eta) = 0$ then $a_n = b_n = 0$ for all $n$. So $\check{H}^0(\mathbf{C}P^1;\Omega^1) = 0$. Now if $\nu \in C^1$ then $\nu \in \Omega^1(U \cap V)$ so $\nu = (\sum_{-\infty}^{\infty} a_n u^n)du$. Since $C^2 = 0$ clearly $\delta\nu = 0$ so we just need to see which $\nu$'s are in $\delta(C^0)$. Let $\omega = -(\sum_{n=0}^{\infty} a_n u^n)du$ and $\eta = (\sum_{n=0}^{\infty} a_{-n-2} v^n)$. Then $\delta(\omega,\eta) = \nu$ implies that $a_{-1} = 0$. Hence $\check{H}^1(\mathbf{C}P^1;\Omega^1) = \mathbf{C}$.

## Line Bundles and Divisors

### Line Bundles

**definition:** A **holomorphic complex line bundle**, $\pi : L \to M$, over a complex manifold $M$ is a complex manifold $L$ and a projection map $\pi$ such that

1. $\pi$ is a holomorphic surjection,
2. $\pi^{-1}(m) = \mathbf{C}$ for all $m \in M$,
3. there is an open cover $\{U_\alpha\}$ of $M$ and maps $\{\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbf{C}\}$ such that $\pi_1 \circ \phi_\alpha = \pi$, where $\pi_1 : U_\alpha \times \mathbf{C} \to U_\alpha$ is projection, and $\phi_\alpha$ is a linear isomorphism on each fiber.

We can define a smooth complex line bundle similarly by replacing holomorphic with smooth in the above definition.

The open sets and functions $\{U_\alpha, \phi_\alpha\}$ in condition 3. of the definition are called a **local trivialization** of $L$. For each pair of open sets $U_\alpha$, $U_\beta$ with non-empty intersection we have a map

$$\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbf{C} \to (U_\alpha \cap U_\beta) \times \mathbf{C}$$

which is the identity on the first factor $(U_\alpha \cap U_\beta)$ and a linear transformation on the second factor $\mathbf{C}$. Thus we can define

$$g_{\alpha\beta} : (U_\alpha \cap U_\beta) \to \mathrm{GL}(1, \mathbf{C})$$

by

$$g_{\alpha\beta}(x) = \phi_\alpha \circ \phi_\beta^{-1}|_{\{x\}\times\mathbf{C}}.$$

These maps $g_{\alpha\beta}$ are called **transition functions** for $L$. Since $\mathrm{GL}(1,\mathbf{C}) = \mathbf{C}^*$ it is clear that $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. Moreover it can be easily checked (just compose the appropriate $\phi_\alpha$'s and think a little) that the $g_{\alpha\beta}$'s satisfy

$$\left.\begin{array}{ll} g_{\alpha\beta}g_{\beta\alpha} = 1 & \text{on } U_\alpha \cap U_\beta \\ g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma} & \text{on } U_\alpha \cap U_\beta \cap U_\gamma \end{array}\right\} \qquad (*)$$

Now take $\{U_\alpha, \phi_\alpha\}$ and $\{U_\alpha, \phi'_\alpha\}$ two trivializations of the same line bundle $L$ (note by taking refiniments of open covers if necessary we are loosing no generality in assuming our two trivializations are with respect to the same open cover). Let $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ be their corresponding transition functions. As above $\phi'_\alpha\phi_\alpha^{-1} : U_\alpha \times \mathbf{C} \to U_\alpha \times \mathbf{C}$ is the identity on the first factor and a linear map on the second, let that linear map be denoted by $\lambda_\alpha$. Clearly $\lambda_\alpha \in \mathcal{O}^*(U_\alpha)$ and

$$g'_{\alpha\beta} = \lambda_\alpha g_{\alpha\beta} \lambda_\beta^{-1}. \qquad (**)$$

**definition:** Let $L$ and $L'$ be two line bundles over $M$. A **bundle homomorphism** $\psi : L \to L'$ is a holomorphic fiber preserving map that is linear on each fiber. If $\psi$ is an isomorphism on each fiber then it is called a **bundle isomorphism**.

**Theorem:**

1. Two line bundles $L$ and $L'$ are isomorphic if and only if there is a local trivialization of each (relative to the same open cover) such that their transition functions are related by $(**)$.
2. Given an open cover $\{U_\alpha\}$ of $M$ and holomorphic functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbf{C}$ that satisfy $(*)$ then there exist a unique (up to isomorphism) line bundle $L$ over $M$ with $\{g_{\alpha\beta}\}$ as transition functions (relative to $\{U_\alpha\}$).

Part 1. is trivial (Given two isomorphic line bundles over $M$ find an open cover that trivializes both and then use the bundle isomorphism to get the $\lambda_\alpha$'s needed in $(**)$. Conversely, given two line bundles with a local trivialization for each related by $(**)$ then one can construct the bundle isomorphism over the local trivializations).

For part two we just note that $L = \coprod (U_\alpha \times \mathbf{C})/\sim$ is the desired bundle, where $(x, y) \in U_\beta \times \mathbf{C}$ is equivalent to $(x, g_{\alpha\beta}(x)(y)) \in U_\alpha \times \mathbf{C}$.

**Theorem:** $H^1(M;\mathcal{O}^*) = \{\text{holomorphic line bundles on } M\}$

Given a line bundle $L \to M$ there is a local trivialization $\{U_\alpha\}$ that satisfy Leray's thereom for the sheaf $\mathcal{O}^*$. Let $\{g_{\alpha\beta}\}$ be the transition functions for $L$ thus $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. If we define $g(U_\alpha, U_\beta) = g_{\alpha\beta}$ then $g$ is a 1-cochain. $\delta g(U_\alpha, U_\beta, U_\gamma) = g_{\beta\gamma} g_{\alpha\gamma}^{-1} g_{\alpha\beta} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$ by $(*)$ thus $g$ is a 1-cocycle and hence represents an element in $H^1(\{U_\alpha\}, \mathcal{O}^*) = H^1(M;\mathcal{O}^*)$. It is easy to see that this is well-defined because two different sets of transition functions for $L$ will be related by $(**)$ and hence the cocycles that they give will differ by a coboundary hence representing the same element in $H^1(M;\mathcal{O}^*)$. It is left as an easy exercise to check that this is a one-to-one and onto correspondence.

Notice that the above theorem only identifys $H^1(M;\mathcal{O}^*)$ with the set of holomorphic line bundles. We would like to say these are isomorphic as groups. Thus we define the **Picard group** of $M$ to be

$$\mathrm{Pic}(M) = \{\text{holomorphic line bundles on } M\}$$

with the following operations: Let $L, L' \in \mathrm{Pic}(M)$ and $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ be their transition functions relative to some open cover $\{U_\alpha\}$.

1. define $L \otimes L'$ as the line bundle associated to the transition functions $\{g_{\alpha\beta} \cdot g'_{\alpha\beta}\}$.
2. define $L^*$ as the line bundle associated to the transition functions $\{g_{\alpha\beta}^{-1}\}$.

Clearly $\mathrm{Pic}(M)$ is a group under $\otimes$ with inverses given by $^*$. It is also easy to check that the identifications defined in the theorem above actually show

$$\mathrm{Pic}(M) \cong H^1(M;\mathcal{O}^*)$$

as groups.

The exact sequence

$$0 \to \mathbf{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

gives us the following piece of an exact sequence

$$\ldots \to H^1(M;\mathcal{O}) \to H^1(M;\mathcal{O}^*) \xrightarrow{\delta} H^2(M;\mathbf{Z}) \to H^2(M;\mathcal{O}) \to \ldots$$

So we have

$$\delta : \mathrm{Pic}(M) \to H^2(M;\mathbf{Z})$$

Thus given any holomorphic line bundle $L$ we can associate to it an element of the second cohomology of $M$. Define the **first Chern class** of a line bundle $L$ by $c_1(L) = \delta(L)$. We also set $\mathrm{Pic}_0(M) = \{L \in \mathrm{Pic}(M) | c_1(L) = 0\}$.

The above discussion could have been done with smooth complex line bundles instead of holomophic ones just by replacing $\mathcal{O}^*$ by $\mathcal{A}^*$ thus we have

**Theorem:** $H^1(M;\mathcal{A}^*) = \{\text{smooth complex line bundles on } M\}$

And the exact sequence

$$0 \to \mathbf{Z} \to \mathcal{A} \to \mathcal{A}^* \to 0$$

gives us

$$\ldots \to H^1(M;\mathcal{A}) \to H^1(M;\mathcal{A}^*) \to H^2(M;\mathbf{Z}) \to H^2(M;\mathcal{A}) \to \ldots$$

But we saw earlier that $\mathcal{A}$ is a soft sheaf thus $H^n(M;\mathcal{A}) = 0$ for all $n > 0$. Thus we get

$$\delta : \{\text{smooth line bundles on } M\} \xrightarrow{\cong} H^2(M;\mathbf{Z})$$

So if we define the **first Chern class** of a smooth line bundle $L$ to be $c_1(L) = \delta(L)$ it is clear that this actually determines the bundle $L$.

**NOTE:** Since $\mathcal{O}$ is not a soft sheaf a holomorphic line bundle **is not** determined, up to holomorphic equivalence, by its first Chern class. It does, however, turn out that it is determined up to smooth equivalence by the first Chern class.

Earlier we computed $H^p(\mathbf{C}P^n;\mathcal{O}) = 0$ if $p > 0$. Thus $\mathrm{Pic}(\mathbf{C}P^n) \cong H^2(\mathbf{C}P^n;\mathbf{Z}) \cong \mathbf{Z}$ with the first isomorphism given by $c_1(\cdot)$. We will see later that the **universal bundle** $J$ over $\mathbf{C}P^n$ generates $\mathrm{Pic}(\mathbf{C}P^n)$. The universal bundle is the line bundle on $\mathbf{C}P^n$ whose fiber over a given point is just the line in $\mathbf{C}^{n+1}$ that the point represents. So

$$J \subset \mathbf{C}P^n \times \mathbf{C}^{n+1}$$

with $J_{\mathbf{z}} = \{\lambda(z_0, z_1, \ldots z_n) | \lambda \in \mathbf{C}\}$ where $\mathbf{z} = [z_0, z_1, \ldots z_n]$. To see that $J$ generates the Picard group we need to compute its Chern class which we will delay for a little while.

We have just seen that on some comlex manifolds the first Chern class does determine a holomorphic line bundle but this is far from true in general. This fact leads us to define one of the fundamental invariants of complex manifolds used in the classification of (complex) surfaces. The **irregularity** of $M$ is

$$q(M) = \dim H^1(M;\Omega^0)$$

but since a 0-form is essentially a function we know $\mathcal{O} = \Omega^0$. Thus we could also define the irregularity of $M$ to be $q(M) = \dim H^1(M;\mathcal{O})$. Notice that $q(M)$ is just the dimension of the kernal of $c_1(\cdot)$ and so measures the failure of the first Chern class to determine a holomorphic line bundle over $M$, in fact

$$q(M) = 0 \Leftrightarrow \mathrm{Pic}_0(M) \text{ is trivial.}$$

To get a little feel for $q(m)$ consider the following
**example:** Let $\Sigma$ be a Riemannian surface. Hodge theory tells us

$$H^1(\Sigma;\mathbf{C}) \cong H^1(\Sigma;\mathcal{O}) \oplus H^0(\Sigma;\Omega^1)$$

and that $H^1(\Sigma;\mathcal{O}) \cong H^0(\Sigma;\Omega^1)$. Thus $b_1 = \dim H^1(\Sigma;\mathbf{C}) = 2\dim H^1(\Sigma;\mathcal{O}) = 2q(\Sigma)$. Notice that this implies that $q(\Sigma) = \mathrm{genus}(\Sigma)$. Hence holomorphic line bundles on a Riemannian surface $\Sigma$ are determined by their first Chern class if and only if the genus of $\Sigma$ is 0 if and only if $\Sigma = \mathbf{C}P^1$.

## Divisors

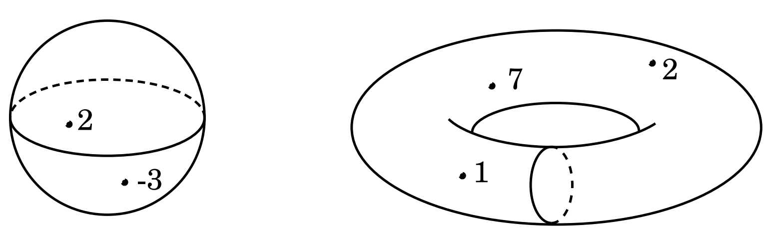
**definition:** A **divisor**, $D$, on $M$ is a locally finite formal linear combination of irreducible analytic hypersurfaces of $M$

$$D = \sum a_i V_i$$

where $a_i$ is an integer and $V_i$ is an irreducible hypersurface. $D$ is called **effective**, denoted $D \geq 0$, if $a_i \geq 0$ for all $i$.

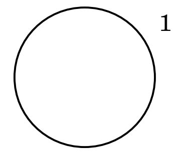
**examples:**

1. On a Riemannian surface a divisor is just a finite set of points labeled by integers.




   Later we shall see that we should think of a divisor as labeling (with multiplicities) the zeros and poles of a meromorphic (twisted) function.

2. Consider $\mathbf{C}P^1 \subset \mathbf{C}P^2$ the zero set of $Z_0 + Z_1 + Z_2$ where $[Z_0 : Z_1 : Z_2]$ are homogeneous coordinates for $\mathbf{C}P^2$. In the Kirby picture of $\mathbf{C}P^2$




we can see the $\mathbf{C}P^1$ as a spanning disk to the knot in the 0-handle union the core of the 2-handle. $D = n\mathbf{C}P^1$ is a divisor on $\mathbf{C}P^2$.

**note:** the set of divisors on $M$, $\mathrm{Div}(M)$, has a natural group structure (under addition).

Let $V$ be an irreducible analytic hypersurface and $f$ a locally defining function for $V$ at $p$. If $g$ is a holomorphic function defined near $p$ then define the **order** of $g$ along $V$ at $p$, $\mathrm{ord}_{V,p}(g)$, to be the largest integer $a$ such that $g = f^a h$ where $h$ is holomorphic. Actually it is easy to see that $\mathrm{ord}_{V,p}(g)$ is independent of $p$ so set $\mathrm{ord}_V(g) = \mathrm{ord}_{V,p}(g)$ for any $p$ on $V$. Clearly we have

$$\mathrm{ord}_V(gh) = \mathrm{ord}_V(g) + \mathrm{ord}_V(h).$$

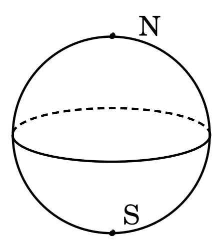
Now if $g$ is a meromorphic function on $M$ locally $g = h/h'$ with $h$ and $h'$ holomorphic and relatively prime. So we can define $\mathrm{ord}_V(g) = \mathrm{ord}_V(h) - \mathrm{ord}_V(h')$. If $a = \mathrm{ord}_V(g)$ then we say $g$ has a zero of order $a$ along $V$ if $a > 0$ or $g$ has a pole of order $-a$ if $a < 0$. The **divisor**, $(g)$, of a meromorphic function $g$ is

$$(g) = \sum_{V \text{ irred.}} \mathrm{ord}_V(g)V.$$

Clearly $(g)$ is effective if and only if $g$ is holomorphic.

**examples:**

1. On $\mathbf{C}P^1 = \{[Z_0 : Z_1]\}$ consider the function $f([Z_0 : Z_1]) = \frac{Z_0^3}{Z_0 Z_1^2}$.




   The divisor associated to $f$ is $(f) = 2N - 2S$.
2. $S^2 \times S^2 = \mathbf{C}P^1 \times \mathbf{C}P^1 \subset \mathbf{C}P^3$. We would like to describe a function on $\mathbf{C}P^1 \times \mathbf{C}P^1$. We will do this by describing one on $\mathbf{C}P^3$ and restricting it. So we need to see the above inclusion explicitly. To this end consider the coordinates $([Z_0 : Z_1], [W_0 : W_1])$ on $\mathbf{C}P^1 \times \mathbf{C}P^1$ and the coordinates $[X_0 : X_1 : X_2 : X_3]$ on $\mathbf{C}P^3$. The inclusion is $i([Z_0 : Z_1], [W_0 : W_1]) = [Z_0W_0 : Z_0W_1 : Z_1W_0 : Z_1W_1]$ Notice that the image of the inclusion is the zero set of $X_0X_3 - X_1X_2$ thus $\mathbf{C}P^1 \times \mathbf{C}P^1$ is the degree 2 hypersurface in $\mathbf{C}P^3$. Now $\mathbf{C}P^1 \times \{\mathrm{pt}\}$ is given by $([Z_0 : Z_1], [1 : 0])$ and hence is $[Z_0 : 0 : Z_1 : 0]$ in $\mathbf{C}P^3$. Similarly $\{\mathrm{pt}\} \times \mathbf{C}P^1$ is given by $([1 : 0], [W_0 : W_1])$ and hence is $[W_0 : W_1 : 0 : 0]$ in $\mathbf{C}P^3$. Now consider the function $f([X_0 : X_1 : X_2 : X_3]) = \frac{X_0 - X_2}{X_0 - X_1}$ on $\mathbf{C}P^3$. From the above it should be clear that the divisor associated to $f|_{\mathbf{C}P^1 \times \mathbf{C}P^1}$ is $(f|_{\mathbf{C}P^1 \times \mathbf{C}P^1}) = \mathbf{C}P^1 \times \{\mathrm{pt}\} - \{\mathrm{pt}\} \times \mathbf{C}P^1$.

**Theorem:** $\mathrm{Div}(M) \cong H^0(M; \mathcal{M}^*/\mathcal{O}^*)$ as groups.

Given a divisor $D = \sum a_i V_i$ we can find a locally finite open cover $\{U_\alpha\}$ of $M$ such that $U_\alpha \cap V_i$ has a locally defining function $g_{i\alpha} \in \mathcal{O}^*(U_\alpha)$. Set $f_\alpha = \prod g_{i\alpha}^{a_i}$. Clearly $f_\alpha$ is in $\mathcal{M}^*$ and $f_\alpha/f_\beta$ is in $\mathcal{O}^*(U_\alpha \cap U_\beta)$. $\{f_\alpha\}$ is in $C^0(\{U_\alpha\}, \mathcal{M}^*)$ but since the $g_{i\alpha}$'s are only well defined up to a nonzero holomorphic function the divisor $D$ only gives us a well-defined element in $C^0(\{U_\alpha\}, \mathcal{M}^*/\mathcal{O}^*)$. Now $\delta(\{f_\alpha\})(U_\alpha, U_\beta) = f_\beta/f_\alpha$ which is in $\mathcal{O}^*$ thus is 0 in $\mathcal{M}^*/\mathcal{O}^*$. Thus $\{f_\alpha\}$ defines an element of $H^0(M; \mathcal{M}^*/\mathcal{O}^*)$ (note for this we should have started with a cover that satisfied Leray's theorem).

Now given a global section $f$ of $\mathcal{M}^*/\mathcal{O}^*$ we can take a Leray open cover $\{U_\alpha\}$ and get meromorphic functions $f_\alpha \not\equiv 0$ on $U_\alpha$ with $f_\alpha/f_\beta$ in $\mathcal{O}^*(U_\alpha \cap U_\beta)$. Thus for any irreducible hypersurface $V$ we have $\text{ord}_V(f_\alpha) = \text{ord}_V(f_\beta)$ if $V \cap U_\alpha \cap U_\beta \neq \emptyset$. So we get a divisor $D = \sum \text{ord}_V(f_\alpha)V$ where for each $V$ we choose an $\alpha$ such that $U_\alpha \cap V \neq \emptyset$.

**Theorem:**

1. There is a natural homomorphism $[\cdot] : \text{Div}(M) \to \text{Pic}(M)$.
2. $\ker[\cdot] = \{(f) | f \in \mathcal{M}^*(M)\}$
3. $L \in \text{Pic}(M)$ is $[D]$ for some divisor $D$ if and only if $L$ has a global meromorphic section $s : M \to L$, in which case we can take $D = (s)$. Moreover $L$ is associated to an effective divisor if and only if $L$ has a global holomorphic section.

Given a divisor $D$ let $\{f_\alpha\}$ be locally defing funcitons for $D$ relative to the open cover $\{U_\alpha\}$. Set $g_{\alpha\beta} = f_\alpha/f_\beta$. Clearly $g_{\alpha\beta} \in \mathcal{O}^*$ and $g_{\alpha\beta}g_{\beta\alpha} = 1$ and $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$. Thus the $g_{\alpha\beta}$'s define a line bundle, denote it $[D]$. To see that this line bundle depends only on $D$ consider another set of locally defining functions $f'_\alpha$. Let $h_\alpha = f_\alpha/f'_\alpha$. Now since $h_\alpha \in \mathcal{O}^*(U_\alpha)$ and $g'_{\alpha\beta} = h_\alpha g_{\alpha\beta} h_\beta^{-1}$ an earlier theorem tells us that the bunbles associated to $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ are isomorphic, hence $[D]$ is well-defined. It is easy to see from the construction that $[\cdot]$ is a homomorphism.

If $D = (f)$ for some meromorphic function $f$ then locally defining fuctions for $D$ relative to some open cover $\{U_\alpha\}$ are $f_\alpha = f|_{U_\alpha}$. Thus the line bundle $[D]$ has as transition functions $g_{\alpha\beta} = f_\alpha/f_\beta = 1$ and hence is trivial. Now given a divisor $D$ such that $[D]$ is trivial we want to find a global fuction $f$ such that $(f) = D$. Since $[D]$ is trivial we know there exist some local trivialization $\{U_\alpha, f_\alpha\}$ such that $g_{\alpha\beta} = \lambda_\alpha^{-1}\lambda_\beta$ for some $\lambda_\alpha \in \mathcal{O}^*(U_\alpha)$. In other words $f_\alpha/f_\beta = \lambda_\beta/\lambda_\alpha$ or $f_\alpha\lambda_\alpha = f_\beta\lambda_\beta$ and thus we can piece together a global meromorphic function $f$. Clearly, $(f) = D$.

So far we have only defined $(f)$ for a function on $M$ but it is straight forward to generalise this to sections of a line bundle (using local trivializations of the line bundle and checking that this is well-defined). Now given a divisor $D$ let $\{f_\alpha \in \mathcal{M}(U_\alpha)\}$ be locally defining functions for $D$. Note that $g_{\alpha\beta}f_\beta = \frac{f_\alpha}{f_\beta}f_\beta = f_\alpha$ and thus the $f_\alpha$'s give a global meromorphic section $s_f$ of $[D]$ with $D = (s_f)$. Conversly, consider $L$ a line bundle with transition functions $\{g_{\alpha\beta}\}$ and a global section $s$. Let $s_\alpha = s|_{U_\alpha}$. So $s_\alpha/s_\beta = g_{\alpha\beta}$ and thus $L = [(s)]$.

**definition:** Two divisors $D, D'$ are said to be **linearly equivalent**, $D \sim D'$, if $D = D' + (f)$ for some meromorphic function $f$. Equivalently if $[D] = [D']$.

The above theorem says that $\text{Div}(M)/\sim \hookrightarrow \text{Pic}(M)$, it also identifies the image.

**Remarks:**

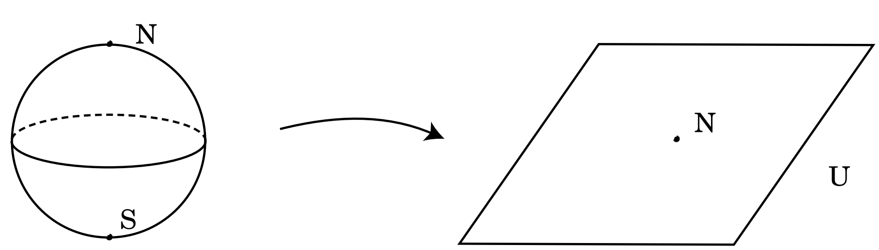
1. If $M$ is a complex submanifold of $\mathbf{C}P^n$ for some $n$ then we actually get $\text{Div}(M)/\sim \cong \text{Pic}(M)$.
2. The maps $(\cdot)$ and $[\cdot]$ can also be seen as the first two maps in the exact sequence

$$\ldots \to H^0(M;\mathcal{M}^*) \to H^0(M;\mathcal{M}^*/\mathcal{O}^*) \to H^1(M;\mathcal{O}^*) \to H^1(M;\mathcal{M}^*) \to \ldots$$

   Which is gotten from

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0.$$

**example:** Consider the divisor $D = N + S$ on $\mathbf{C}P^1$




Let $[X_0 : X_1]$ be homogeneous coordinates on $\mathbf{C}P^1$. In these coordinates $N = [0 : 1]$ and $S = [1 : 0]$. We have the charts $U = \{[X_0 : X_1] | X_1 \neq 0\}$ and $V = \{[X_0 : X_1] | X_0 \neq 0\}$, with non-homogeneous corrdinates $u = X_0/X_1$ and $v = X_1/X_0$ respectively. On $U$ consider the function $f_U(u) = u$ and on $V$

consider the function $f_V(v) = v$. Clearly $\{f_U, f_V\}$ is a locally defining set of functions for $D$. Thus the line bundle $[D]$ is trivial over $U$ and $V$ and has as transition function $g_{UV} = f_V/f_U = v/u = \frac{1}{u^2}$. Notice that we would have gotten the same line bundle if we had taken $D' = 2N$ because then the locally defining functions would be $f_U(u) = u^2$ and $f_V(v) = 1$ and hence the transition function for $[D']$ is still $g_{UV} = \frac{1}{u^2}$. More generally we can see any line bundle over $\mathbf{C}P^1$ as comming from $D_n = nN$ for some $n$.

Our definition of the first Chern class of a line bundle $L$, $c_1(L) = \delta(L)$ is not exactly well suited for computations. So now we would like to relate $c_1(L)$ to a couple of (hopefully) better understood things. But first we need a few definitions.

Given a line bundle $L$ we can find a connection (covariant derivative), $\nabla$, on it. $\nabla$ can be written in a local trivialization over $U_\alpha$ as $d + \theta_\alpha$, where $d$ is the normal DeRham derivative and $\theta_\alpha$ is a 1-form (with values in $\mathcal{U}_1$). The curvature $\Theta$ is defined (locally) as $d\theta_\alpha + \theta_\alpha \wedge \theta_\alpha = d\theta_\alpha$ since $\mathcal{U}_1$ is abelian. Thus $\Theta$ is a closed 2-form and hence represents an element of $H^2_{DR}(M)$.

Given a divisor $D = \sum a_i V_i$ we can think of it as defining an element in $H_{2n-2}(M^n; \mathbf{Z})$. Let $\eta_D$ be the Poincaré dual of $D$. So $\eta_D \in H^2(M; \mathbf{Z})$. We are now ready for

**Theorem:**

1. $c_1(L) = [\frac{\sqrt{-1}}{2\pi}\Theta]$
2. if $L = [D]$ then $c_1(L) = \eta_D$.

**examples:**

1. Remember: $\text{Pic}(\mathbf{C}P^n) \cong H^2(\mathbf{C}P^n; \mathbf{Z}) \cong \mathbf{Z}$. We would now like to prove an earlier claim that $J$ the universal line bundle over $\mathbf{C}P^n$ generates $\text{Pic}(\mathbf{C}P^n)$. To this end consider $H = \mathbf{C}P^{n-1} \subset \mathbf{C}P^n$ this is called the **hyperplane divisor**. We know $H$ generates $H_2(\mathbf{C}P^n; \mathbf{Z})$ and hence $\eta_H$ generates $H^2(\mathbf{C}P^n; \mathbf{Z})$ thus under the above isomorphism $[H]$, called the **hyperplane bundle**, should generate $\text{Pic}(\mathbf{C}P^n)$. Now we are have left to show

$$J = [H]^* = [-H]$$

Indeed if $[Z_0 : \ldots : Z_n]$ are homogeneous coordinates and $U_0 = \{[Z_0 : \ldots : Z_n] | Z_0 \neq 0\}$ then consider the section $e_0$ of $J$ over $U_0$ given by

$$e_0(\mathbf{Z}) = (1, Z_1/Z_0, \ldots, Z_n/Z_0).$$

To see this is actually a section of $J$ remember we can think of $J$ as a subbundle of $\mathbf{C}P^n \times \mathbf{C}^{n+1}$ where $J_{\mathbf{Z}}$ is the line in $\mathbf{C}^{n+1}$ that $\mathbf{Z}$ represents. $e_0$ is clearly holomorphic and nonzero on $U_0$. It also extends to a moromorphic section of $J$ over all of $\mathbf{C}P^n$ with a pole along $H$. Thus $J = [-H]$.

2. Let $M$ be a compact complex manifold and $V$ a smooth analytic hypersurface in $M$. We can define the **normal bundle** to $V$ as

$$N_V = \frac{T_M|_V}{T_V}$$

Where $T_M$ and $T_V$ are the holomorphic tangent bundles over $M$ and $V$ respectively. The **conormal bundle** is just $N_V^*$. We can think of it as containing cotangent vectors on $M$ that are zero along $T_V \subset T_M$. We now have the following formula (the baby adjunction formula)

$$N_V^* = [-V]|_V$$

To see that this formula is true let $\{f_\alpha\}$ be locally defining functions for $V$. Thus $[V]$ has as transition functions $g_{\alpha\beta} = f_\alpha/f_\beta$. Now $f_\alpha$ is identically zero on $V \cap U_\alpha$ thus $df_\alpha$ is a section of $N_V^*$; moreover, $df_\alpha$ is never zero since $V$ is smooth. On $V \cap U_\alpha \cap U_\beta$ we have $df_\beta = d(g_{\alpha\beta} f_\alpha) = (dg_{\alpha\beta}) f_\alpha + g_{\alpha\beta} df_\alpha = g_{\alpha\beta} df_\alpha$ since $f_\alpha = 0$ on $V$. Thus the $df_\alpha$'s define a global never zero section of $N_V^* \otimes [V]|_V$ showing that it is trivial.

3. Let $M$ be an $n$ dimensional compact complex manifold. The **canonical bundle** is

$$K_M = \wedge^n T_M^*$$

where $T^*_M$ is the holomorphic cotangent bundle. Note $\mathcal{O}(K_M) = \Omega^n(M)$

i. $K_{\mathbf{C}P^n} = [-(n+1)H]$

ii. the **geometric genus** of $M$ is $p_g(M) = \dim H^0(M; \mathcal{O}(K_M))$. On a Riemannian surface $\Sigma$, $p_g(\Sigma) = q(\Sigma) = \text{genus}(\Sigma)$. So geometric genus actually corresponds to genus for a surface. Thus our generalization of genus is the number of independent holomorphic sections of the canonical bundle.

iii. $c_1(T_M) = -c_1(K_M)$

iv. If $V$ is a smooth analytic hypersurface in $M$ we have the **adjunction formula**

$$K_V = (K_M \otimes [V])|_V$$

To see that i. is true in the chart $U_0 = \{[z_0 : \ldots : z_n] | z_0 \neq 0\}$ consider the $n$-form $\omega = \frac{dw_1}{w_1} \wedge \ldots \wedge \frac{dw_n}{w_n}$ where $w_i = \frac{z_i}{z_0}$. $\omega$ in non-zero and has a pole along each hyper plane $z_i = 0$. If we now change coordinates to another chart ($U_j$ say) then we can see that $\omega$ also has a pole along the hypersurface $Z_0 = 0$ too, thus proving i. In the chart $U_j$ consider the oordinates $w'_i = \frac{z_i}{z_j}$. We now have

$$w_i = \frac{w'_i}{w'_0}, \quad \text{for } i \neq j \qquad w_j = \frac{1}{w'_0}$$

hence

$$\frac{dw_i}{w_i} = \frac{dw'_i}{w'_i} - \frac{dw'_0}{w'_0} \quad \text{for } i \neq j \qquad \frac{dw_j}{w_j} = \frac{-dw'_0}{w'_0}$$

and thus $\omega = (-1)^j \frac{dw'_0}{w'_0} \wedge \ldots \wedge \widehat{\frac{dw'_j}{w'_j}} \wedge \ldots \wedge \frac{dw'_n}{w_n}$ on $U_j$ showing it does have a pole along $z_0 = 0$ as claimed.

To show iii. is true just note the sting of well known first Chern class identities (remember we are using the holomorphic tangent and cotangent bundles below)

$$c_1(TM) = -c_1(T^*M) = -c_1(\bigwedge\nolimits^n T^*M) = -c_1(K_M)$$

Finally to see iv. consider the exact sequence of vector bundles

$$0 \to N^*_V \to T^*M|_V \to T^*_V \to 0.$$

Thus $T^*M|_V = T^*V \oplus N^*_V$ and simple linear algebre shows $(\bigwedge^n T^*M)|_V = \bigwedge^{n-1} T^*V \otimes N^*_V$. Hence using the baby adjunction formula from above you get iv.