Assignment 3

Control Theory group 48

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1 – State Feedback Controller

(a)

We start with a differential equation:

$$v = -\frac{dx}{dt}$$
$$v(s) = -X(s) \cdot s$$
$$TF(s) = \frac{X(s)}{V(s)} = -\frac{1}{s}$$

We then use forward Euler ($s \implies \frac{1}{T_s}(z-1)$) to discretize the Transfer Function.

$$TF(z) = -\frac{1}{\frac{1}{T_s}(z-1)} = -\frac{T_s}{z-1}$$

$$\frac{X(z)}{V(z)} = -\frac{T_s}{z-1}$$

Now, to find the state-space model, we then use the MATLAB command `tf2ss` to get our A, B, C, and D matrices:

$$A = 1,\ B = 1,\ C = -T_s,\ D = 0$$

The following state equations (with state vector, m and output x) are given by these matrices:

. $m[k+1] = m[k] + v[k]$

. $x[k] = -T_s \cdot m[k]$

However, upon inspection, these do not properly yield position as the output x. It can be seen that by the equation $H(z) = C(I(z) - A)^{-1}B + D$ which yields a transfer function from a state space representation that the following matrices yield the same transfer function:

$$A = 1,\ B = Ts,\ C = -1,\ D = 0$$

The following state equations that these new matrices yield do give an appropriate output, namely (with state vector, x and output y):

- $x[k+1] = x[k] + T_s \cdot v[k]$
- $y[k] = -x[k]$

(b)

For a Closed Loop State Feedback System:

- $x[k+1] = (A - BK)x[k] + B \cdot v[k]$
- $y[k] = (C - DK)x[k] + D \cdot v[k]$

So for our system:

- $x[k+1] = (1 - T_s \cdot K)m[k] - T_s \cdot v[k]$
- $y[k] = -x[k]$

To get back to the Transfer Function:

$$TF(z) = \frac{X(z)}{V(z)} = (C-DK)(zI-A+BK)^{-1}B+D = -1(z-1+T_s \cdot K)^{-1} \cdot T_s = -\frac{T_s}{z-(1-T_s \cdot K)}$$

For $K < 0$ and $K > \frac{2}{T_s}$ the system is unstable as the pole is driven out of the unit circle. Optimal K is within the unit circle, but as close as possible to 0.

2 – Validation of the principles of the Kalman filter on our system

(a)

Again, the general measurement equation is:

$$y[k] = Cx[k] + Dv[k]$$

Which, in our case is:

$$y[k] = -x[k]$$

(b)

Kalman gain L_{k+1} as a function of the state estimate covariance, $\hat{P}_{k|k}$, is found below starting with the equation for $\hat{P}_{k+1|k}$:

$$\hat{P}_{k+1|k} = A\hat{P}_{k|k}A^T + Q_k = \hat{P}_{k|k} + Q_k$$

We then introduce innovation covariance:

$$S_{k+1} = C\hat{P}_{k+1|k}C^T + R_{k+1} = \hat{P}_{k|k} + Q_k + R_{k+1}$$

Optimal Kalman gain is thus:

$$L_{k+1} = \hat{P}_{k+1|k}C^T S_{k+1}^{-1} = (\hat{P}_{k|k} + Q_k)\frac{1}{\hat{P}_{k|k} + Q_k + R_{k+1}} = \frac{\hat{P}_{k|k} + Q_k}{\hat{P}_{k|k} + Q_k + R_{k+1}}$$

It can be seen from the above expression that as Q goes to infinity, L_{k+1} goes to 1. As R goes to infinity, L_{k+1} goes to zero.

(c)

The next state estimate covariance ($\hat{P}_{k+1|k+1}$) as a function of the previous state estimate covariance ($\hat{P}_{k|k}$), is defined below.

$\hat{P}_{k+1|k+1} = (I - L_{k+1} \cdot C)\hat{P}_{k+1|k} = (1 - \frac{\hat{P}_{k|k}+Q_k}{\hat{P}_{k|k}+Q_k+R_{k+1}} \cdot 1) \cdot (\hat{P}_{k|k} + Q_k)$

$= \hat{P}_{k|k} + Q_k - \frac{(\hat{P}_{k|k}+Q_k)^2}{\hat{P}_{k|k}+Q_k+R_{k+1}} = \frac{(\hat{P}_{k|k}+Q_k)^2+(\hat{P}_{k|k}+Q_k)R_{k+1}-(\hat{P}_{k|k}+Q_k)^2}{\hat{P}_{k|k}+Q_k+R^{k+1}}$

$= \frac{(\hat{P}_{k|k}+Q_k)R_{k+1}}{\hat{P}_{k|k}+Q_k+R_{k+1}}$

Thus is can be seen from the above equation that as Q goes to infinity, $\hat{P}_{k+1|k+1}$ goes to R_{k+1} and as R goes to infinity, $\hat{P}_{k+1|k+1}$ goes to $\hat{P}_{k|k} + Q_k$.

(d)

To find $\hat{P}_\infty$ we assume $\hat{P}_{k+1|k+1} = \hat{P}_{k|k}$. Thus,

. $\quad \hat{P}_\infty = \frac{R_{k+1}(\hat{P}_\infty+Q_k)}{\hat{P}_\infty+Q_k+R_{k+1}}$

$\implies \hat{P}_\infty^2 + \hat{P}_\infty(Q_k + R_{k+1}) = \hat{P}_\infty R_{k+1} + R_{k+1}Q_k$

$\implies \hat{P}_\infty^2 + \hat{P}_\infty Q_k - R_{k+1}Q_k = 0$

$\Longrightarrow \hat{P}_\infty = \frac{-Q_k \pm \sqrt{Q_k^2 + 4R_{k+1}Q_k}}{2}$

Given that only positive values are allowed and rearranging a bit, we find:

$$\hat{P}_\infty = Q_k\left(-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{R_{k+1}}{Q_k}}\right)$$

Now, using this result to solve for L_∞,

. $\quad L_\infty = \hat{P}_\infty C^T S_{k+1}^{-1} = Q_k\left(-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{R_{k+1}}{Q_k}}\right) \cdot 1 \cdot \frac{1}{\hat{P}_{k|k} + Q_k + R_{k+1}}$

. $\quad = \frac{Q_k\left(-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{R_{k+1}}{Q_k}}\right)}{Q_k\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{R_{k+1}}{Q_k}}\right) + R_{k+1}}$

Now, we define the ratio ρ as $\frac{Q_k}{R_{k+1}}$ and we find

$$L_\infty = \frac{-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\rho}}}{\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\rho}}\right) + \frac{1}{\rho}}$$

Thus, L_∞ only depends on ρ.

(e)

For the LQE in general: $det(zI - (A - LC)) = 0$
Given A = 1, C = 1, and L derived from $L_\infty = \frac{\hat{P}_\infty + Q}{\hat{P}_\infty + Q + R}$

$$z - 1 + \frac{\hat{P}_\infty + Q}{\hat{P}_\infty + Q + R} = 0$$

Using our derived $\hat{P}_\infty = Q(-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{R}{Q}})$ we find:

$$z - 1 + \frac{Q(-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{R}{Q}}) + Q}{Q(-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{R}{Q}}) + Q + R} = 0$$

$$\Longrightarrow z = 1 - \frac{-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\rho}}}{-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\rho}} + \frac{1}{\rho}} = \frac{2}{\frac{1}{2}\rho + \sqrt{\frac{1}{4}\rho^2 + 1} + 1}$$

With ρ as before as $\frac{Q}{R}$. With $\frac{1}{2}\rho + \sqrt{\frac{1}{4}\rho^2 + 1} + 1 > 1$ always, then $z < 1$ always. This makes the system always stable. If ρ increases, z decreases (poles move closer to the origin) and the response is faster. Also in this case, the estimate is more noisey and we should trust the measurement. If ρ decreases, z increases and the pole moves closer to 1 (but never reaches it). The response time in this case is slower. Then, the estimate is less noisey and we should trust the model.

3 – State Estimator and Feedback Controller Implementation

(a)

Choice of appropriate K,Q,R, and $\hat{P}_{0|0}$: We tested four values of K (0.5, 1, 2, and 4) and we found the best response with K=2 (see the next section). We chose Q and R based on measurements. R comes from the measurement noise. Amplitude between the peaks is defined as 6σ and $R = \sigma^2$.

$$6\sigma = 0.104m \implies R = (\frac{0.104m}{6})^2 = 3 \times 10^{-4}m^2$$

Our first estimation of Q was derived from

$$Q = BQB^T = 0.01s(\frac{0.1m/s}{6})^2 0.01s = 3 \times 10^{-8}$$

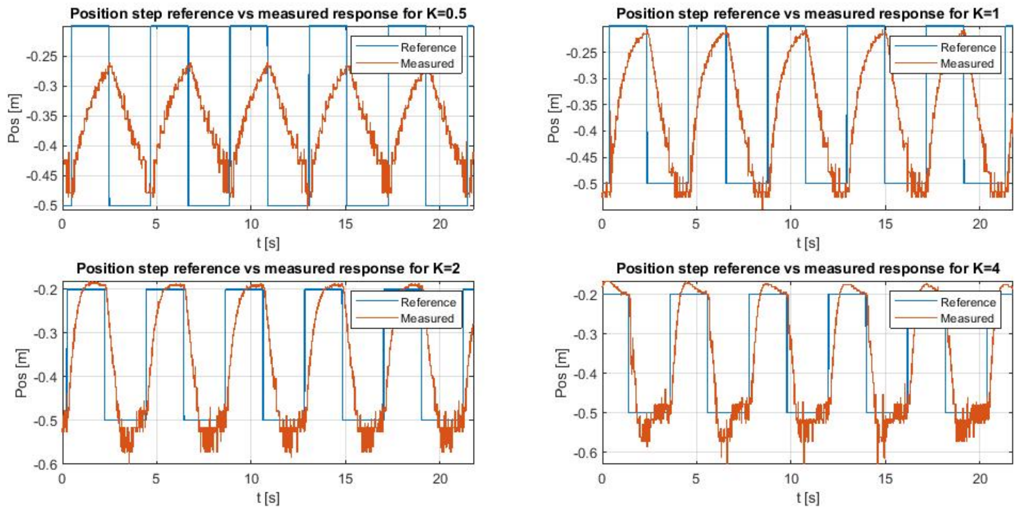
We raise it to 3×10^{-7} to have a nice minimum ρ of 0.001. $\hat{P}_{0|0}$ is derived from our $\hat{P}_{k+1|k+1}$ where $\hat{P}_{k|k} = 0$:

$$\hat{P}_{k+1|k+1} = \frac{QR}{Q+R}$$

For R much greater than Q we find $\hat{P}_{0|0} = Q$ and throughout the assignment we modify mostly Q (and thus $\hat{P}_{0|0}$) to experiment with different ρ values.

(b)

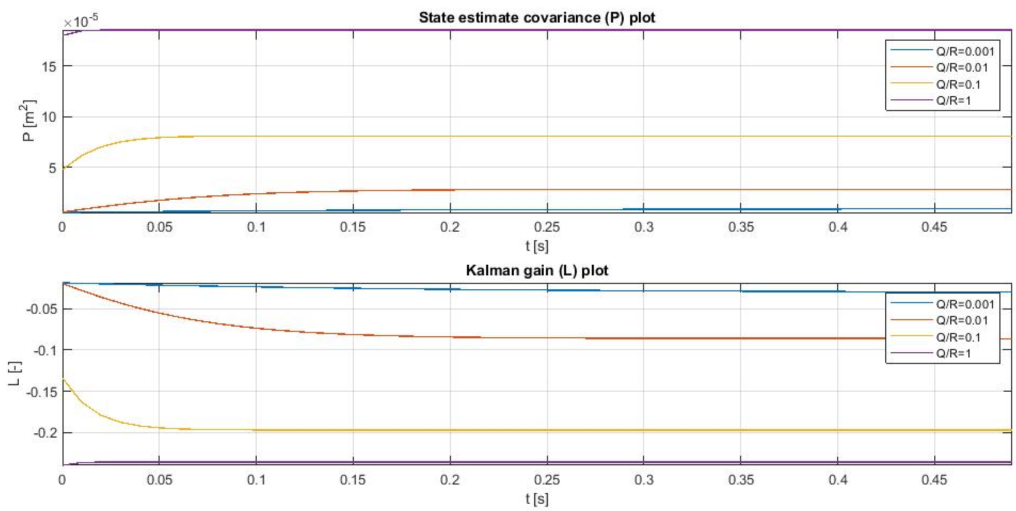
We measured position responses from K values ranging from 0.5 to 4. The results are graphed below.



As we can see, at a K of 1 (and below), the rise time is too long and is thus unsatisfactory. For a K of 4, there is too much overshoot. Thus we settle on a K value of 2. This fits with our predictions in 1 (b) as we see that the system is more stable with K closer to 0.

(c)

The evolution of $\hat{P}_{k|k}$ and L_k with time for different values of Q/R is plotted below.



We can see a quicker evolution with a higher value of ρ. With the higher two of the values of ρ, we find P and L evolve to P_∞ and L_∞, but the lower two values don't reach it in this time period.

(d)

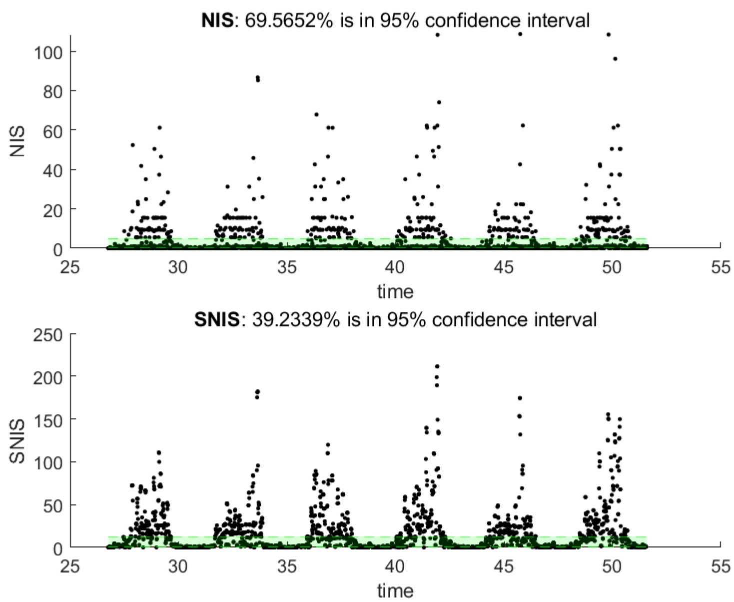
NIS and SNIS are defined by:

$$NIS_k = \nu^T S_k^{-1} \nu_k$$

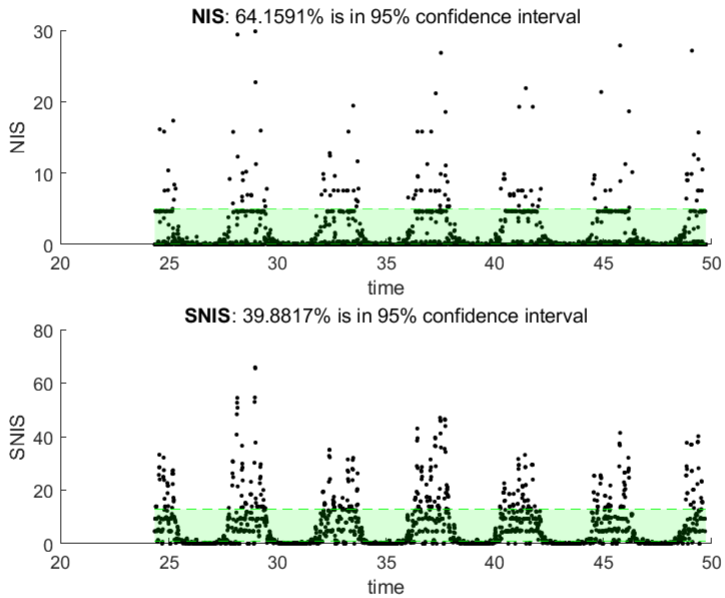
$$SNIS_k = \sum_{j=k-M+1}^{k} \nu^T S_j^{-1} \nu_j$$

SNIS sums the last M NIS-values. The NIS and SNIS for different values of $\frac{Q}{R}$ are tabulated below:

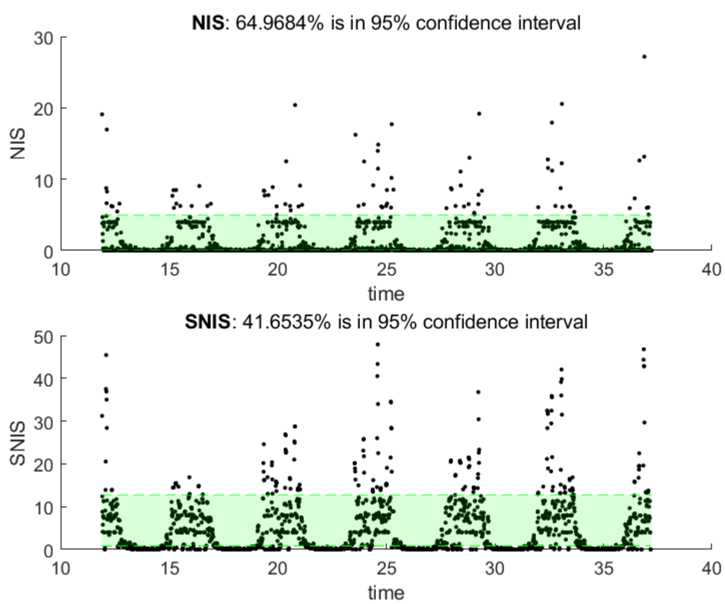
For $\frac{Q}{R} = 0.001$:



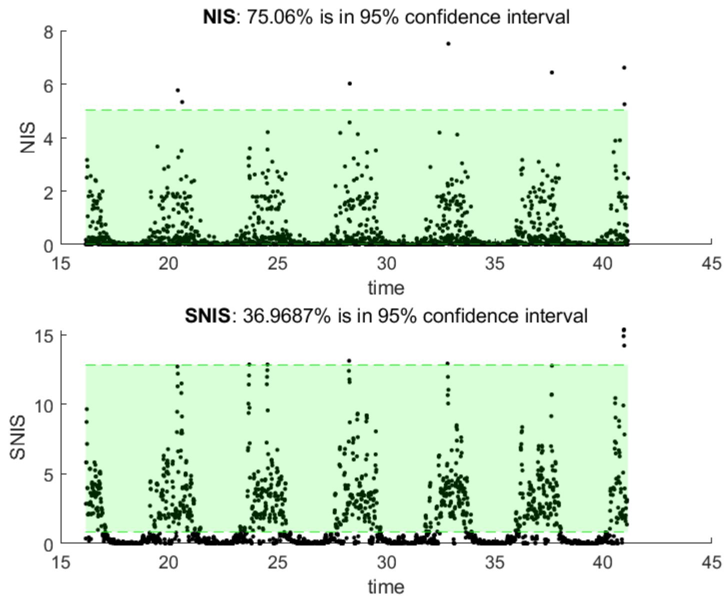
For $\frac{Q}{R} = 0.01$:



For $\frac{Q}{R} = 0.1$:



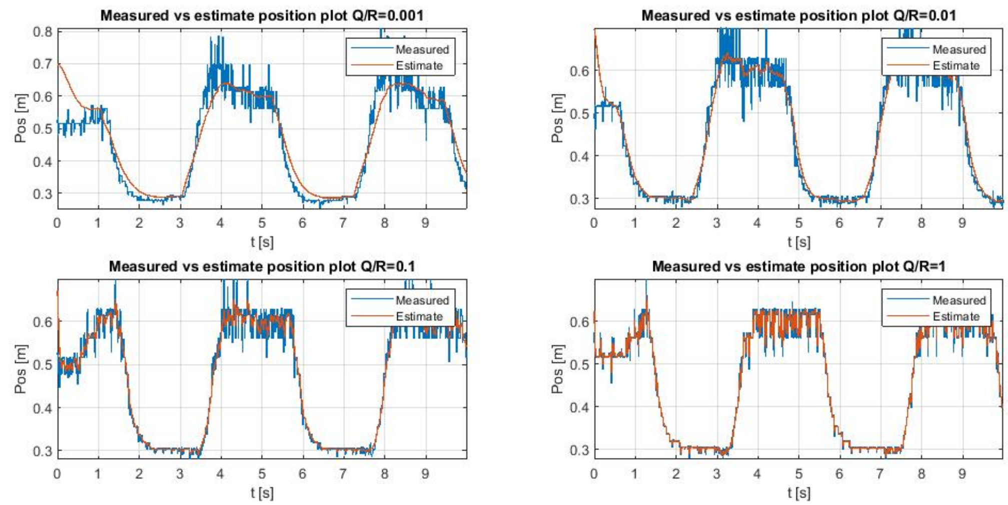
For $\frac{Q}{R} = 1$:

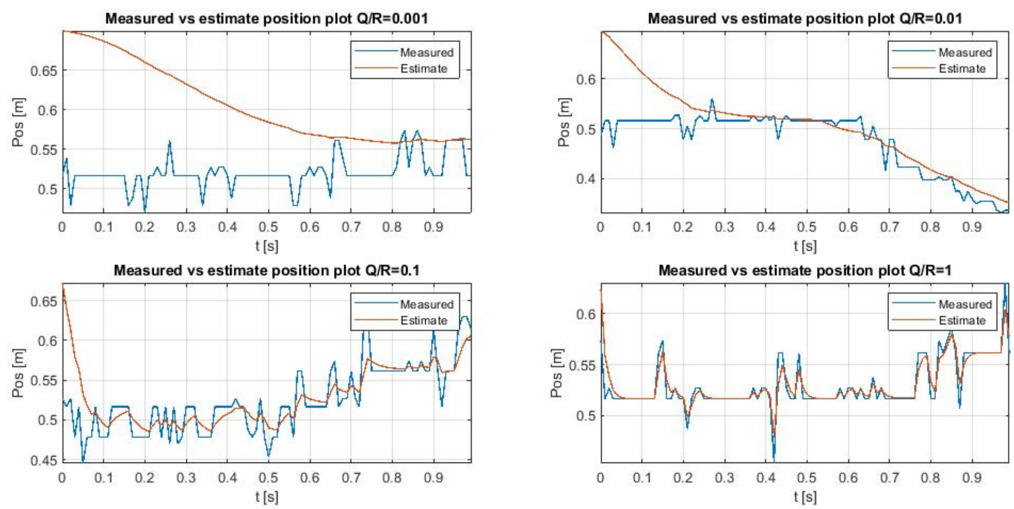


NIS and SNIS being innovation variables, have a great discrepancy in our test case. This is due to the transient profile of the test case. This increases the mean value, so some points close to zero are excluded (which shouldn't happen). This reduces drastically the NIS and SNIS. Thus, we can conclude that our test case is not convenient for NIS and SNIS estimation of our Kalman filter. As a possible profile, a step input or a step profile like ours, but with bigger period, is advised.

(e)

For our experiments with an incorrect initial state estimate, we start the cart at -0.5m, code the initial state as -0.7m, and set the input reference position signal as a step signal with period 4.2s with 2s at -0.3m and 2.2s at -0.6m. The results are graphed below:





It can be seen that the higher the ration of Q/R, the quicker the system corrects for the incorrect initial state estimate. This matches our conclusion in 2 (e). The higher ρ corresponds to a faster response time, but if more affected by the noise of the system.

(f)

Using pole placement we determined the desired L we should use such that the closed-loop pole of the estimator is 10x slower than the closed-loop pole of the state feedback controller we have chosen. Our current pole is:

. $H(z) = -\frac{T_s}{z-(1-KT_s)}$ with $T_s = 0.01s$ and K = 2/s

. $H(z) = -\frac{0.01}{z-(1-0.02)}$ Thus our pole is z = 0.98.

. $f_{controller} = -f_s \cdot ln(0.98) = 2.02$ Hz. To make an estimator 10 x slower:

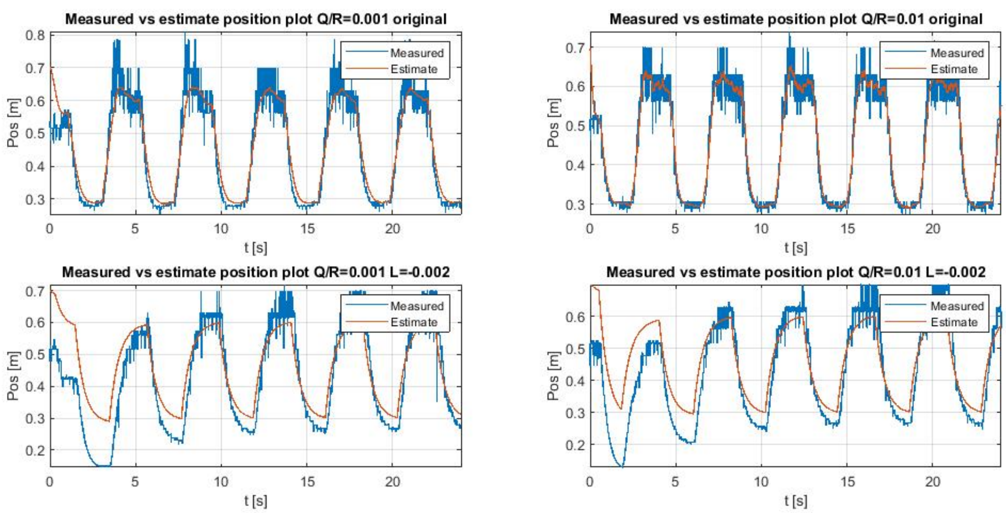
. $f_{est} = \frac{f_{cont}}{10} = 0.202$ Hz

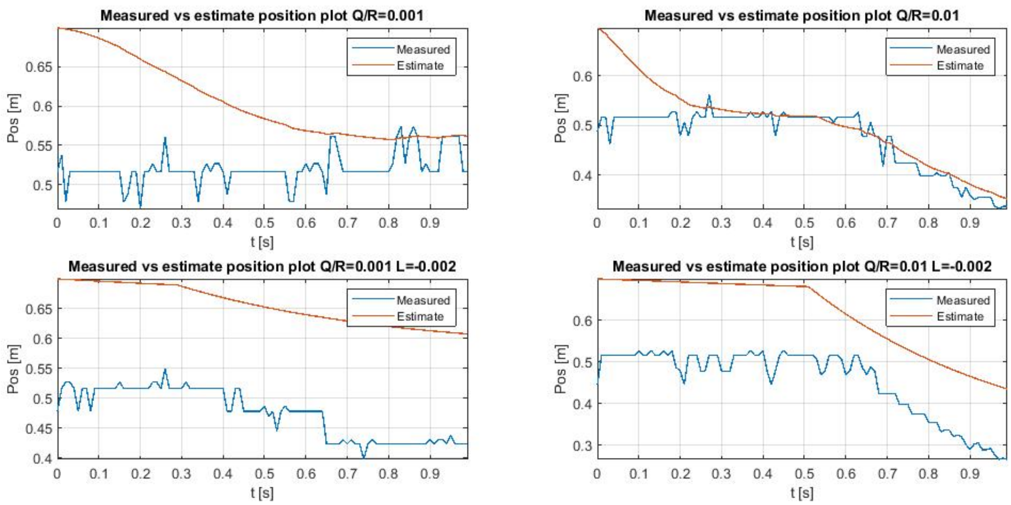
. $z_{est} = e^{-\frac{f_{est}}{f_s}} = e^{-\frac{0.202}{100}} = 0.998$

. $A - LC = 0.998$ with A = 1 and C = -1:

. $1 + L = 0.998 \implies L = -0.002$

Thus, the L we input into our controller is an L of -0.002. We use the same inputs as in 3 (e) for our tests and find the following results.





The 'original' graphs are taken from our previous experiment for comparison. As we can see, the measured and estimated positions never fully converge for the fixed L input. Also, the cart drifts in the right direction to align itself with the position reference, but never aligns itself with the correct peaks and troughs of the input signal. If we were to design a state estimator using pole placement, we would you place the closed-loop poles such that the estimator is two to six times faster. For example, 10 Hz: $z_{est} = e^{-\frac{10}{100}} = 0.905$. Thus, the controller dominates.