Assignment 1

Control Theory group 48

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1 – Theoretical Model of a DC Motor

(a)

The physical laws that describe the behaviour of the system of a DC motor are as the following:

Newton's 2nd Law: $J\ddot{\theta} + b\dot{\theta} = Ki$

Kirchoff's Law: $JL\frac{di}{dt} + Ri = V - K\dot{\theta}$

With J the moment of inertia, $\dot{\theta}$ the angular velocity, and b is the constant coefficient of viscous friction which we assume for the motor as is proportional to angular velocity. Furthermore, L is the internal impedance of the motor and R is the internal resistance of the motor. K is the back electromotive force constant as well as the motor torque constant as they are in consitent units for the motor and i is the current through the motor. Now we apply the Laplace transform to the equations to get:

$$s(Js + b)\Theta(s) = KI(s)$$

$$(Ls + R)I(s) = V(s) - Ks\Theta(s)$$

Which we can manipulate to find the transfer function:

$\implies I(s) = \frac{V(s) - Ks\Theta(s)}{Ls+R}$

$\implies s(Js + b)\Theta(s) = k[\frac{V(s) - Ks\Theta(s)}{Ls+R}]$

$\implies (Ls + R)s(Js + b)\Theta(s) = KV(s) - K^2\Theta(s)$

$\implies \Theta(s)[K^2 + s(Js + b)(Ls + R)] = KV(s)$

$\implies H(s) = \frac{\dot{\Theta}(s)}{V(s)} = \frac{K}{(Js+b)(Ls+R)+K^2}$

By inspection we can see that there are no zeros. Furthermore, we assume that impedance, L, is negligible. Thus, the Transfer Function is modified to:

$H(s) = \frac{K}{(Js+b)(R)+K^2}$

And the poles can be found by the following process:

$\implies H(s) = \frac{K}{JRs+(bR+K^2)}$

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Setting the denominator to zero:

$s = \frac{-bR-K^2}{JR}$ is the one pole we have in this CT system.

(b)

The physical meaning of the model's input is the input voltage with units of Volts. The physical meaning of the model's output is rotational velocity with units of radians per second. The states we will define later are angle and angular velocity with units of radians and radians per second respectively.

(c)

We model the ideal system of the DC motor with the only friction coming from the coefficient of viscous friction. We assume K doesn't depend on i or the rotational speed, and that resistance, R, is independent of i. We also assume that the wheels will not slide on the surface they are operating on. Finally, as is mentioned in part a, we assume that the internal impedance of the motor is negligible.

(d)

We discretise the Transfer Function by using the Backwards Euler method ($s \implies \frac{z-1}{Tz}$) on the transfer function as follows:

$H(s) = \frac{K}{(Js+b)(R)+K^2}$

$\implies H(z) = \frac{K}{JR(\frac{1}{T}\frac{z-1}{z})+(bR+K^2)}$

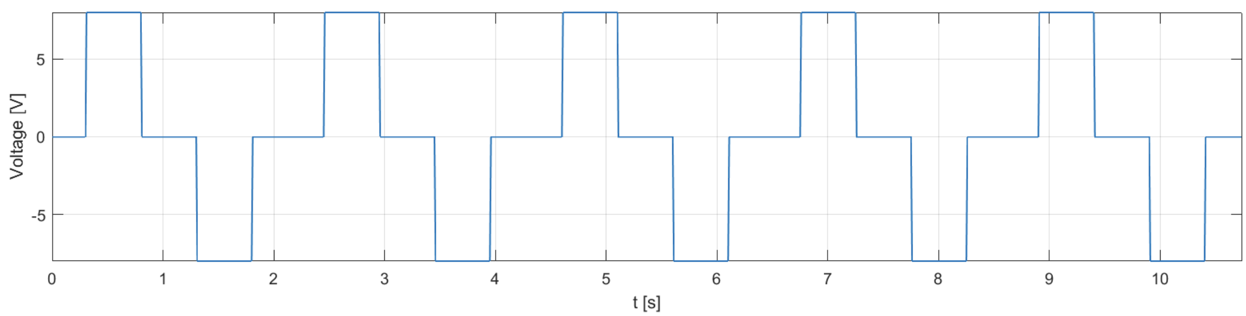
$\implies H(z) = \frac{KTz}{JR(z-1)+(bR+K^2)Tz}$

$\implies H(z) = \frac{KTz}{(JR+T(bR+K^2))z-JR}$

2 – DC-motor-plus-wheel Identification

(a)

We designed our excitation signal as a periodic square wave-based function with period of 2.15 sec. From 0 to 500 ms we have a positive 8V applied. From 500 ms to 1 sec we have no voltage applied. From 1 sec to 1.5 sec we have a negative 8V applied and from 1.5 sec to 2.15 sec we have no voltage applied. We designed this signal as such because the unit impulse gives the correct harmonics and the uneven period makes it so we don't neglect every other harmonic (if the function was symmetric it could be expressed entirely by sines or cosines). The plot of the excitation signal is given below – it is repeated for five periods.



(b)

We can manipulate our $H(z)$ to the form $\frac{b_1 z}{z+a_0}$ with the unknowns to be found recursively by identification using the least squares set up below:

$$\begin{bmatrix} \omega(2) \\ \dots \\ \omega(k) \\ \dots \\ \omega(N) \end{bmatrix} = \begin{bmatrix} -\omega(1) & v(2) \\ \dots & \dots \\ -\omega(k-1) & v(k) \\ \dots & \dots \\ -\omega(N-1) & v(N) \end{bmatrix} \begin{bmatrix} a_0 \\ b_1 \end{bmatrix}$$

The ω and v are the measured output angular frequency and input voltage respectively. Note that we omit b_0 as it doesn't exist in our theoretical approach even though it can exist in the real world. We derive the frequencies for our poles and zeros from the following equations:

$$z = e^{-s} = e^{-\frac{f}{f_s}}$$

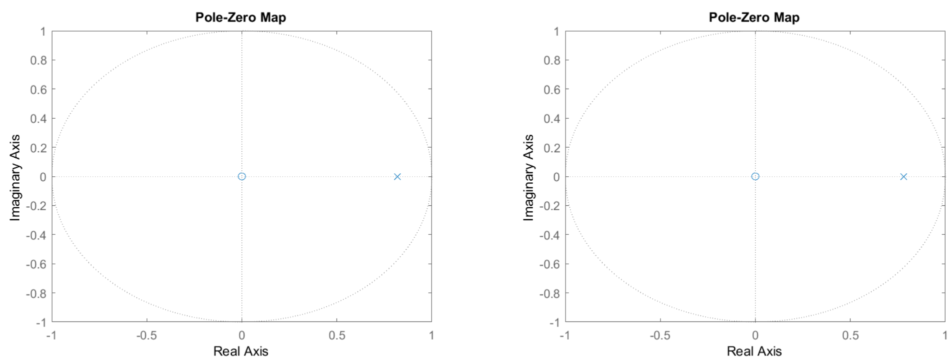
With f_s the sampling frequency and f our desired frequency. Thus,

$$ln(z) = -\frac{f}{f_s} \implies f = -f_s \cdot ln(z)$$

We find a zero of 0 ($-\infty$ rad/s) and a pole of 0.819 (19.97 rad/s).

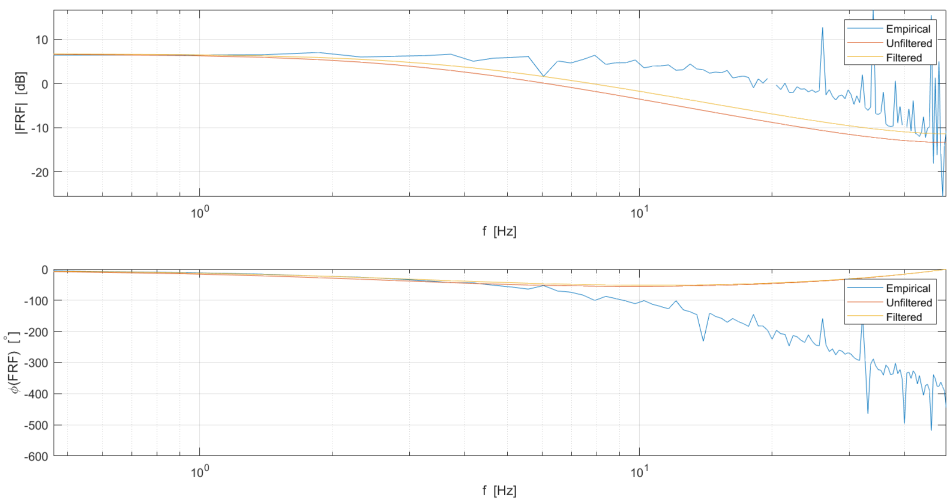
(c)

We prefer to filter the signal before fitting the model because the noise present can skew the fit of the model. After filtering out the noise we can fit a more accurate model than if we filtered it after fitting the model. We filter angular frequency and voltage based off the frequencies observed to correspond to the noise in the signal. We use a Butterworth filter for this purpose. We constructed various filters with cutoff frequencies ranging from 2 to 10 Hz with orders between 2nd and 20th. Eventually we settled on a 6th order filter with a cutoff frequency of 10 Hz. After this filtering, we found the same zero, but a pole shifted left to 0.78 (24.85 rad/s). The plots with the unfiltered and filtered poles and zeros are shown respectively below.

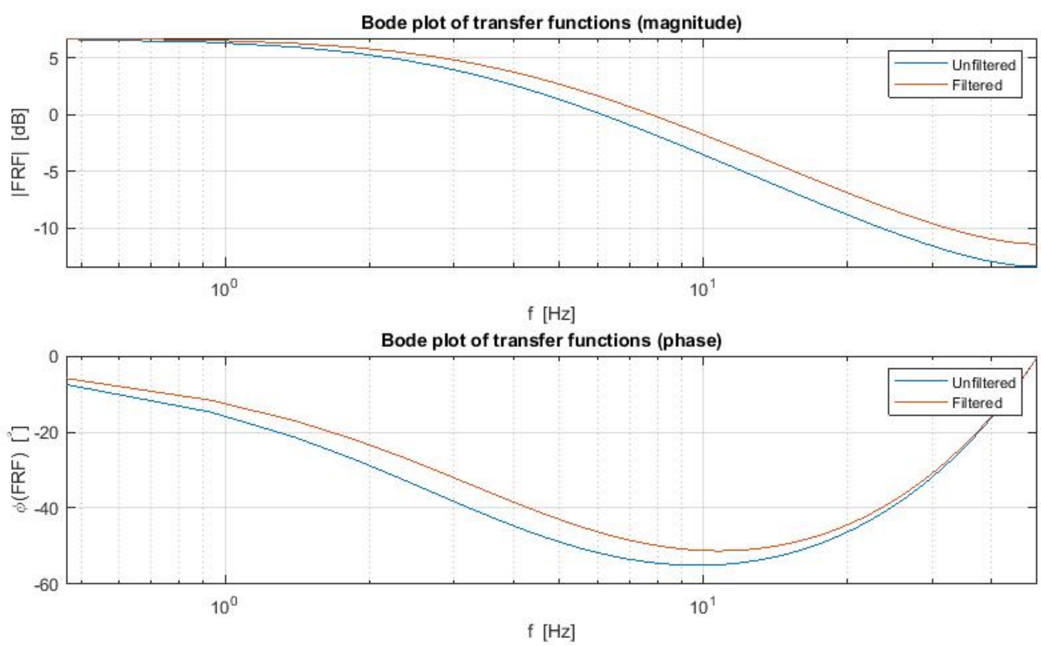


(d)

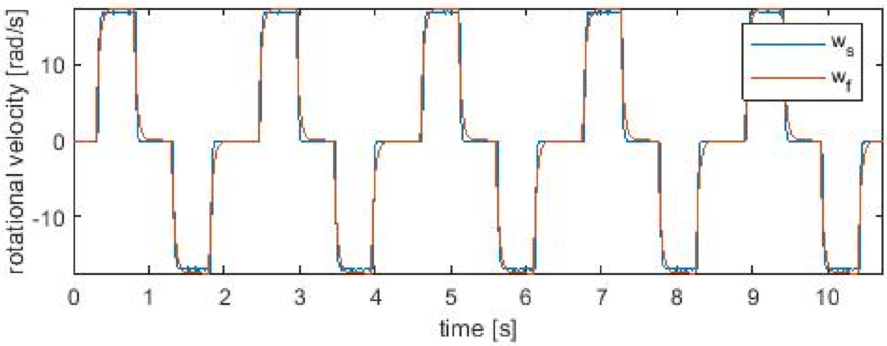
Our empirical transfer function is calculated and plotted by lines 81 through 101 in our MATLAB code titled '`identification_motor.m`' and the bode plot with the empirical transfer function, our model without filtering, and our model with filtering can be seen below:

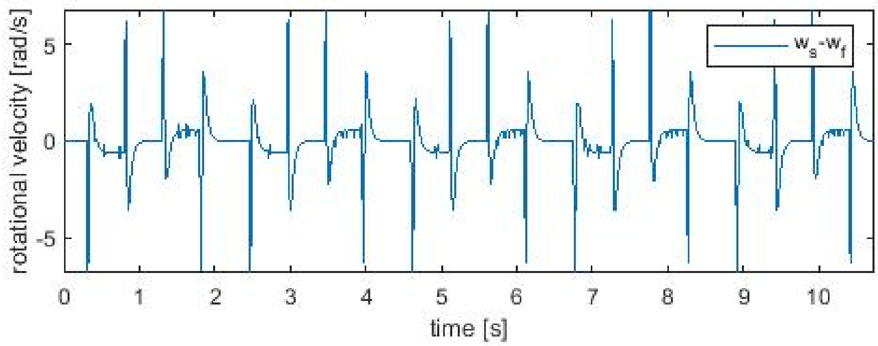


To better see the difference between the filtered and unfiltered models, these two are plotted against each other below.

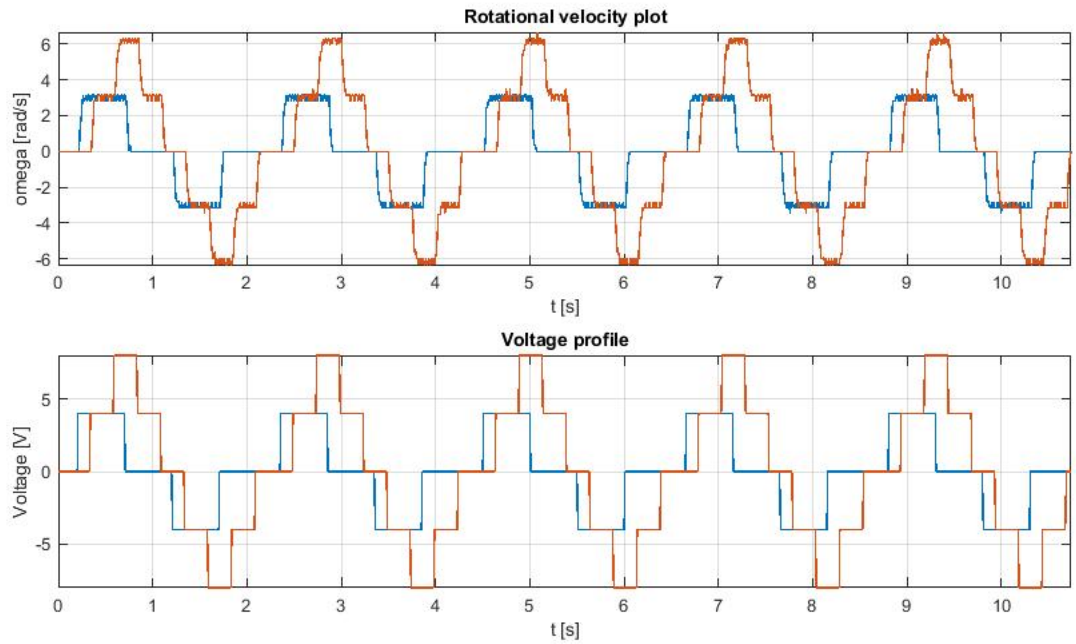


The measured response (w_s) of our system with our previously defined step input with the simulated response of our final identified model (w_f) are plotted below. Also plotted is the difference.

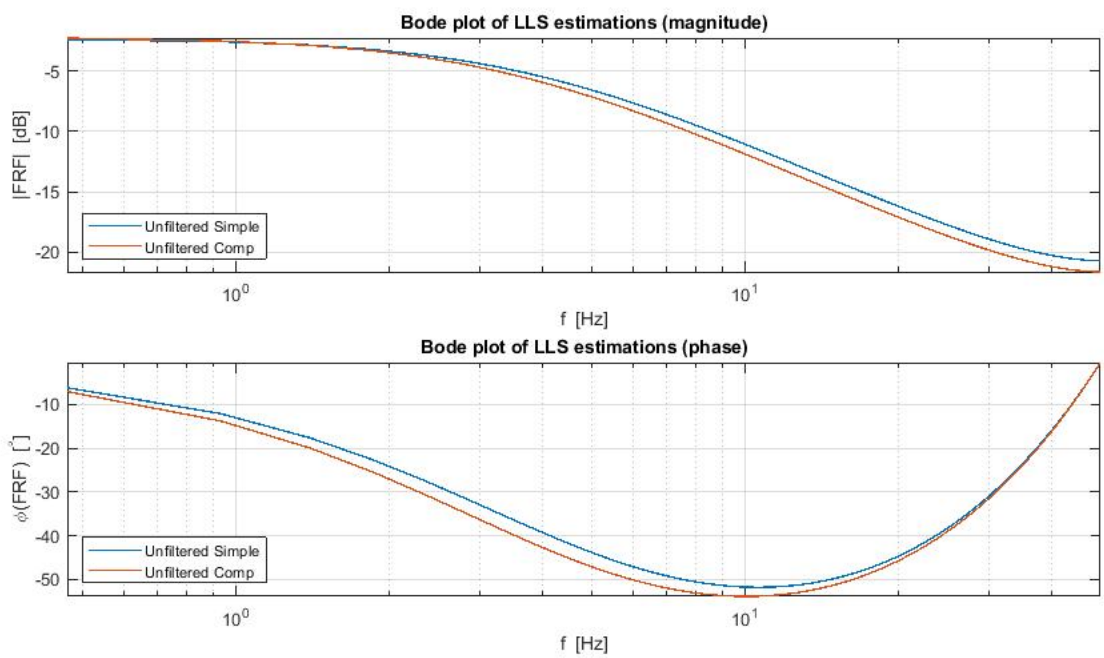


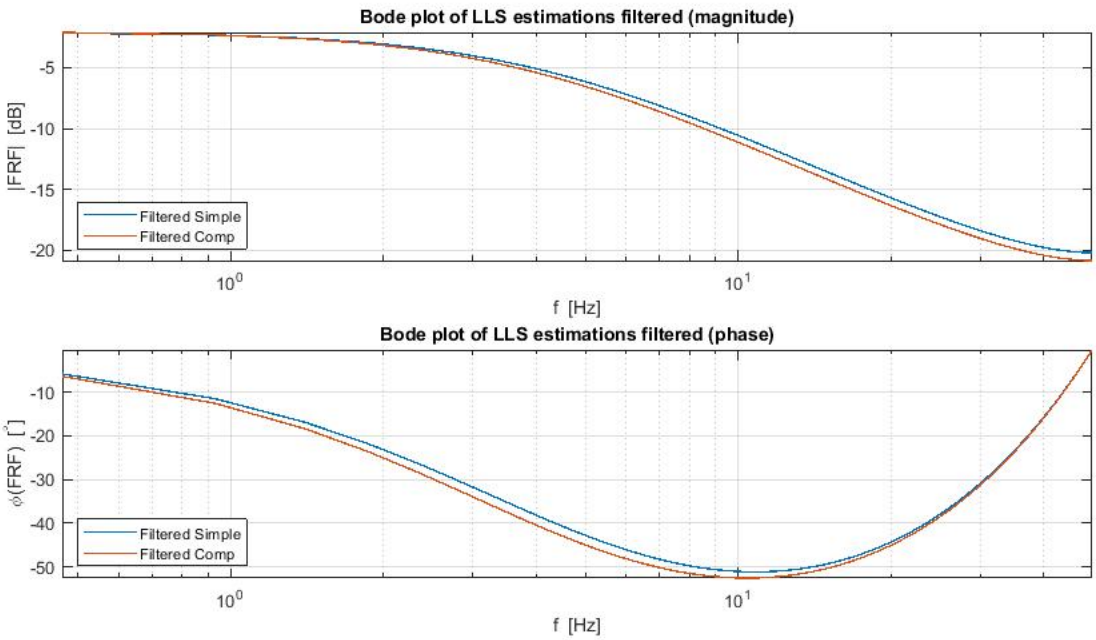


Finally, we test the linearity of the system. We do this by testing the response with two of our defined signals 250 ms out of phase with each other as well as the superposition of those two inputs. The superposition principle states that for inputs $v_1(t)$ and $v_2(t)$ and outputs $\omega_1(t)$ and $\omega_2(t)$ that if $v_1(t) \rightarrow \omega_1(t)$ and $v_2(t) \rightarrow \omega_2(t)$ then $v_1(t) + v_2(t) \rightarrow \omega_1(t) + \omega_2(t)$. See the two voltage and angular velocity profiles below.



We find the same identification as before with each of the identical but out of phase signals, but with the superimposed signal (designated below as 'comp' for "compound signal") we find the break in linearity.





As we can see, though the identified transfer functions are quite similar, there is a difference that can be seen between them. It can also be seen that with filtering, the two transfer functions approach each other though they still remain distinct. The numerical values of the transfer functions are tabulated below.

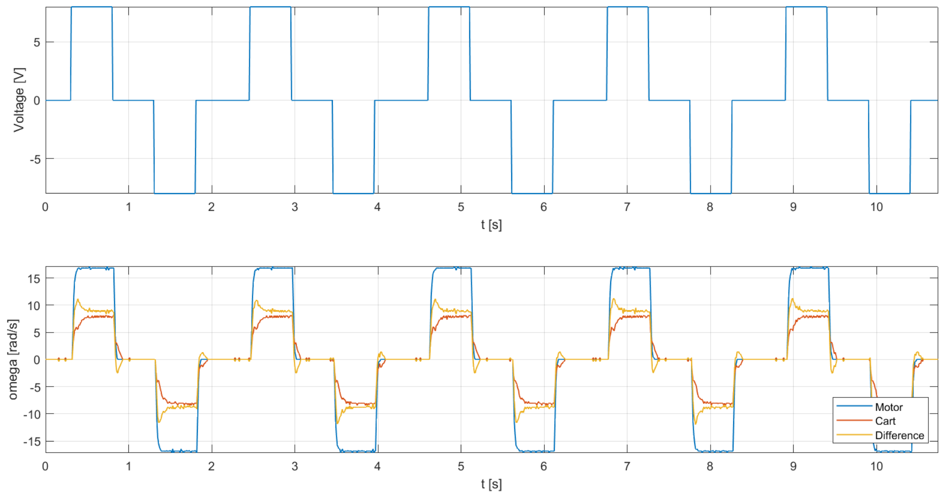
	simple	comp
unfiltered	$\frac{0.1643z}{z-0.7862}$	$\frac{0.1495z}{z-0.8081}$
filtered	$\frac{0.1741z}{z-0.7782}$	$\frac{0.1630z}{z-0.7934}$

The violations in linearity most likely have to do with the relationship between a constant voltage and a constant angular velocity for the system not being completely linear as we have assumed for our experiments.

3 – Cart Identification

(a)

We used the same form of the step input to identify the cart as we did to identify the motors except that we applied only a peak voltage of 4 V to each as it was more manageable. We use the transfer function to plot the response (ω). We can see the comparison of the responses plotted below.



The response is of a similar form but you can see that the rise time is significantly higher in the cart. This is likely due to the larger mass that must be accelerated to the same velocity (and thus angular velocity of the wheels).

(b)

We proceeded the process of identifying the cart by applying another filter. Again, we use a Butterworth filter to filter angular frequency and voltage based off the frequencies observed to correspond to the noise in the signal. We constructed various filters with cutoff frequencies ranging from 2 to 10 Hz with orders between 2nd and 20th. Eventually we settled on a 6th order filter with a cutoff frequency of 10 Hz. Again we find a zero at zero and this time, a filtered pole of 0.8434 (17.03 rad/s). We expected the pole to move to the center as a result of our perceived increase in b, but it turns out that it moved outward. We figure that this is due to the larger increase in J. As s moves to zero ($s = e^{-b/J}$), z moves to 1. Our final discretized transfer function for the cart is seen as $H(z) = \frac{0.3173z}{z-0.8434}$. The following table summarizes our results.

	motor	cart
unfiltered	$\frac{0.3513z}{z-0.8134}$	$\frac{0.2930z}{z-0.8535}$
filtered	$\frac{0.4806z}{z-0.7798}$	$\frac{0.3173z}{z-0.8434}$

Notice the decrease in the coefficient of the numerator, K. For the motor the coefficient was 0.4806 whereas for the cart it is 0.3173. This tracks as the response angular velocity is lower for the cart than the wheel because of the added and resistance.