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Linear Algebra

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Building blocks, operations and terminologies

- Entities
  - Scalars
  - Vectors
  - Matrices
  - Tensors
- Algebra and operations of these entities
  - Addition, subtraction, multiplication, inverse
  - Inner products (dot products) and outer products
  - ...
- Algorithms
  - Gauss elimination, LU decomposition
  - QR decomposition
  - Eigen value/vector computation
  - Eigen value decomposition, SVD
  - ...

Scalars and Vectors

- Different interpretations of entities
  - Physicist
  - Computer science
  - Mathematician

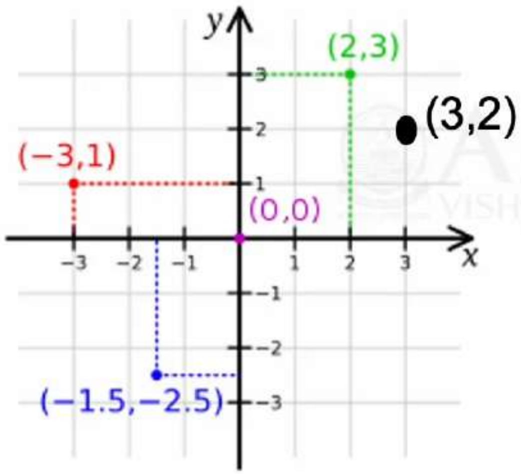
Focus from this perspective

- Scalars are single numbers – more on this later
- Vectors – central concept in linear algebra
  - Vectors are ordered list of numbers
  - Vectors are members of objects known as vector spaces

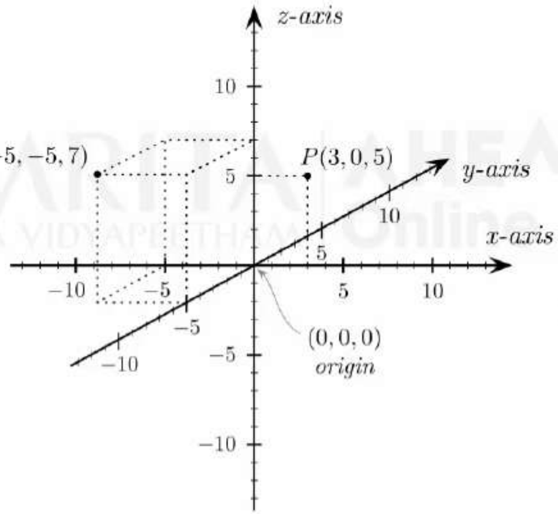
*https://youtu.be/fNk_zzaMoSs

Vectors

- Vectors are ordered list of numbers
- Examples
  - Coordinates
  - Feature list, say for a house



(2,3)



(-5,-5,7)

Features of house =
(Cost, Stories, Square foot)

(30 lakhs, 1 story, 1500 sq ft) or (30,1,1500)

(80 lakhs, 2 story, 2300 sq ft) or (80,2,2300)

(2.1 cr, 2 story, 3500 sq ft) or (210, 2, 3500)

Representation of Vectors

- Vectors are written in various forms
- Vertical or horizontal format
  - Horizontal – row vector
  - Vertical – column vector

$$(2,3,1) \quad \Bigg| \quad \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \quad \Bigg| \quad \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad \Bigg| \quad \begin{bmatrix} 5 & 3 & 2 \end{bmatrix}$$

- Format may have to be preserved sometimes, sometimes not

Vector Notation

- Vector - denoted by bold variables or with overhead arrow/tilde
- Vector elements - entries or components and are usually un-bold
- Elements can be represented by indexing the vector variable

$$\boldsymbol{x} = (p, q)$$
$$x_1 = p, x_2 = q$$

$$\boldsymbol{x} = (x, y, z)$$
$$x_1 = x, x_2 = y, x_3 = z$$

$$\vec{x} = (p, q)$$

$$\boldsymbol{b} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$
$$b_1 = 2, b_2 = 3, b_3 = 1$$

$$\vec{b} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$\boldsymbol{x} = x_i$$
$$i = 1,2, \cdots 5$$

$$\boldsymbol{y} = [y_1 \ y_2 \ y_3]$$

$$\boldsymbol{y} = y_j$$
$$j = 1,2,3$$

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Properties of Vector

- Introducing commonly used vector properties
- Length, Size, Shape
- Sum, Max, Min
- Norms
  - L0-norm
  - L1-norm
  - L2-norm (magnitude)
  - L∞-norm

Properties of Vector

- Consider the example

$$\boldsymbol{x} = \begin{bmatrix} -5 \\ 3 \\ 4 \end{bmatrix}; \boldsymbol{y} = [4 \quad 2 \quad 1]$$

- Size, Shape
  - Size of $\boldsymbol{x}$ → 3; Size of $\boldsymbol{y}$ → 3
  - Shape of $\boldsymbol{x}$ → 3 X 1 (3 rows and 1 column)
  - Shape of $\boldsymbol{y}$ → 1 X 3 (1 row and 3 columns)

- Sum
  - Sum of vector $\boldsymbol{x}$ $$\sum_{i=1}^{3} x_i = x_1 + x_2 + x_3 = -5 + 3 + 4 = 2$$
  - Sum of vector $\boldsymbol{y}$ $$\sum_{i=1}^{3} y_i = y_1 + y_2 + y_3 = 4 + 2 + 1 = 7$$

Properties of Vector

- Consider the example

$$\boldsymbol{x} = \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}; \boldsymbol{y} = \begin{bmatrix} 4 & 2 & 1 \end{bmatrix}$$

- Maximum, Minimum
  - Maximum of vector

$$\max(\boldsymbol{x}) = \max(-5, 3, 2) = 3$$
$$\max(\boldsymbol{y}) = \max(4, 2, 1) = 4$$

  - Minimum of vector

$$\min(\boldsymbol{x}) = \min(-5,3,2) = -5$$
$$\min(\boldsymbol{y}) = \min(4,2,1) = 1$$

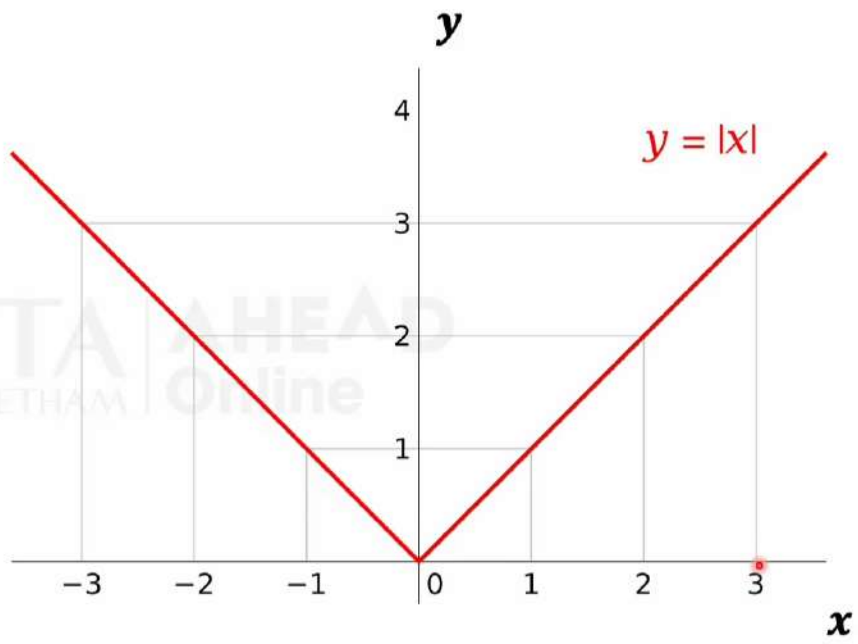
- Python commands available in NumPy - size, shape, sum, max, min

Absolute value

- Absolute value of a number x is denoted by $|x|$
$$abs(x) = |x|$$

- Examples
$$|3| = 3$$
$$|-3| = 3$$

- Absolute function
$$y = f(x) = |x|$$



Comparing Vectors

Features of house =
(Cost, Stories, Square foot)

(30 lakhs, 1 story, 1500 sq ft) or (30,1,1500)

(80 lakhs, 2 story, 2300 sq ft) or (80,2,2300)

(2.1 cr, 2 story, 3500 sq ft) or (210, 2, 3500)

- Assume 3 vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ \vdots \\ x_{999} \\ x_{1000} \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \vdots \\ \vdots \\ y_{999} \\ y_{1000} \end{bmatrix} \quad z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ \vdots \\ \vdots \\ z_{999} \\ z_{1000} \end{bmatrix}$$

Properties of Vector

- Consider the example

$$\boldsymbol{x} = \begin{bmatrix} -5 \\ 3 \\ 4 \end{bmatrix}; \boldsymbol{y} = \begin{bmatrix} 4 & 2 & 1 \end{bmatrix}$$

- Norms
  - L1-norm

$$\|\boldsymbol{x}\|_1 = |-5| + |3| + |4| = 12$$
$$\|\boldsymbol{y}\|_1 = |4| + |2| + |1| = 7$$

  - L2-norm (magnitude)

$$\|\boldsymbol{x}\|_2 = \sqrt[2]{(-5)^2 + 3^2 + 4^2} = \sqrt{50} = 7.07$$
$$\|\boldsymbol{y}\|_2 = \sqrt[2]{4^2 + 2^2 + 1^2} = \sqrt{21} = 4.58$$

  - L∞-norm

$$\|\boldsymbol{x}\|_\infty = \max(|-5|, |3|, |4|) = \max(5,3,4) = 5$$
$$\|\boldsymbol{y}\|_\infty = \max(|4|, |2|, |1|) = 4$$

COLAB DEMO

```
import numpy as np
x=np.array([[-5],[3],[4]])
y=np.array([[4,2,1]])
```

```
x
```

```
y
```

```
print(np.size(x)) #Return number of elements
```

```
print(np.shape(x)) #Returns the shape
```

COLAB DEMO

```
[6] print(np.max(x))  #Returns maximum of the array

    4

[7] print(np.min(x)) #Returns minimum of array

    -5

    print(np.sum(x)) #Returns sum of array elements

    2

[ ] print(np.linalg.norm(x)) #L2-Norm, Euclidean norm, Most popular
```

COLAB DEMO

```
[9] print(np.linalg.norm(x)) #L2-Norm, Euclidean norm, Most popular

    7.0710678118654755

[10] print(np.linalg.norm(x,1)) #L1-norm

    12.0

print(np.linalg.norm(x,np.inf)) # L infinity norm
```

Run cell (Ctrl+Enter)
cell has not been executed in this session

```
[ ] print(np.size(y))
    print(np.shape(y))
    print(np.max(y))
    print(np.min(y))
```

COLAB DEMO

```python
print(np.size(y))
print(np.shape(y))
print(np.max(y))
print(np.min(y))
print(np.sum(y))
print(np.linalg.norm(y))
print(np.linalg.norm(y,1))
print(np.linalg.norm(y,ord=np.inf))
```

```
3
(1, 3)
4
1
7
4.58257569495584
4.0
7.0
```

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Vector Algebra

- Addition
- Subtraction
- Scaling
- Multiplication
  - Inner product
  - Outer product
  - Element by element multiplication
- No division

Vector Addition and Subtraction

- Consider two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$

$$\boldsymbol{x} = \begin{bmatrix} -5 \\ 3 \\ 4 \end{bmatrix} \qquad \boldsymbol{x} + \boldsymbol{y} = \begin{bmatrix} -5+1 \\ 3+3 \\ 4+1 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \\ 5 \end{bmatrix}$$

$$\boldsymbol{y} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad \boldsymbol{x} - \boldsymbol{y} = \begin{bmatrix} -5-1 \\ 3-3 \\ 4-1 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 3 \end{bmatrix}$$

Commutative Property of Vectors

- Consider two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$

$$\boldsymbol{u} = \boldsymbol{x} + \boldsymbol{y}$$
$$\boldsymbol{v} = \boldsymbol{y} + \boldsymbol{x}$$
$$\boldsymbol{u} = \boldsymbol{v}$$

$$\boldsymbol{x} = \begin{bmatrix} -5 \\ 3 \\ 4 \end{bmatrix} \qquad \boldsymbol{x} + \boldsymbol{y} = \begin{bmatrix} -5 + 1 \\ 3 + 3 \\ 4 + 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \\ 5 \end{bmatrix}$$

$$\boldsymbol{y} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad \boldsymbol{y} + \boldsymbol{x} = \begin{bmatrix} 1 + -5 \\ 3 + 3 \\ 1 + 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \\ 5 \end{bmatrix}$$

Changing the order of addition does not change the result

Associative Property

- Consider three vectors $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ $\quad \boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \boldsymbol{y} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \boldsymbol{z} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$$\boldsymbol{u} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$$
$$\boldsymbol{v} = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$$
$$\boldsymbol{u} = \boldsymbol{v}$$

$$\boldsymbol{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1+2 \\ 3+1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix}$$

$$\boldsymbol{v} = \left(\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+1 \\ 3+3 \\ 4+1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix}$$

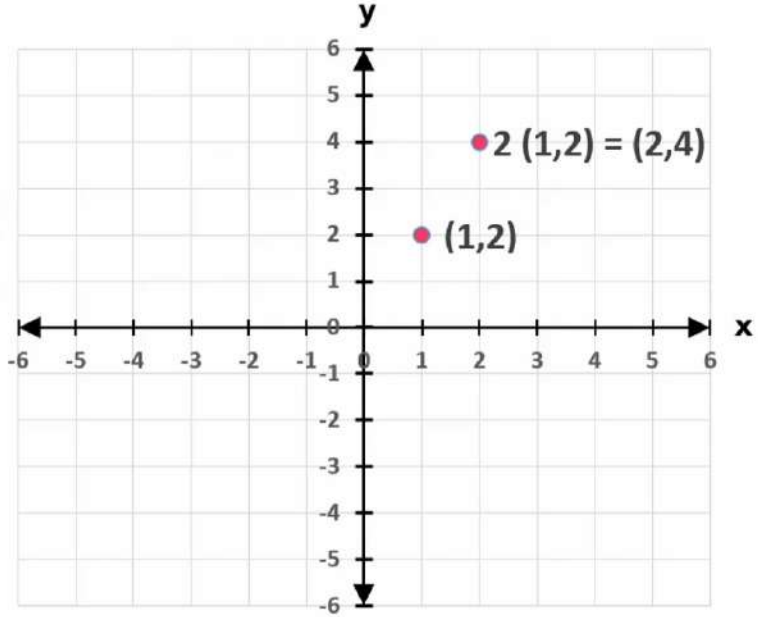
No matter how a set of three or more vector are grouped together, the sum remains the same

Scalar Multiplication

- Consider vector $\boldsymbol{x}$ and scalar k , $k\boldsymbol{x}$ is the result of scalar multiplication

$$\boldsymbol{x} = \begin{bmatrix} -5 \\ 3 \\ 4 \end{bmatrix} \qquad k = 3$$

$$k\boldsymbol{x} = 3\boldsymbol{x} = 3\begin{bmatrix} -5 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3\times-5 \\ 3\times3 \\ 3\times4 \end{bmatrix} = \begin{bmatrix} -15 \\ 9 \\ 12 \end{bmatrix}$$



Linear Combination

$$\boldsymbol{u} = \begin{bmatrix} 30 \\ 1 \\ 1500 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 80 \\ 2 \\ 2300 \end{bmatrix}$$

$$2\boldsymbol{u} = \begin{bmatrix} 2\times30 \\ 2\times1 \\ 2\times1500 \end{bmatrix} = \begin{bmatrix} 60 \\ 2 \\ 3000 \end{bmatrix} \qquad 3\boldsymbol{v} = \begin{bmatrix} 3\times80 \\ 3\times2 \\ 3\times2300 \end{bmatrix} = \begin{bmatrix} 240 \\ 6 \\ 6900 \end{bmatrix}$$

Vector Addition

$$\boldsymbol{2u} + \boldsymbol{3v} = \begin{bmatrix} 60 \\ 2 \\ 3000 \end{bmatrix} + \begin{bmatrix} 240 \\ 6 \\ 6900 \end{bmatrix} = \begin{bmatrix} 60+240 \\ 2+6 \\ 3000+6900 \end{bmatrix} = \begin{bmatrix} 300 \\ 8 \\ 9900 \end{bmatrix}$$

Scalar Multiplication

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Inner Product (or Dot product)

- Dot product of two vectors $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$ is the number $\boldsymbol{v} \cdot \boldsymbol{w}$

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2$$

$$\boldsymbol{v} = (-2, 3)$$

$$\boldsymbol{w} = (1, 2)$$

$$\boldsymbol{v} \cdot \boldsymbol{w} = -2\times1 + 3\times2 = 4$$

$$\boldsymbol{w} \cdot \boldsymbol{v} = 1\times-2 + 2\times3 = 4$$

$$\boldsymbol{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\boldsymbol{v} \cdot \boldsymbol{w} = 4\times-1 + 2\times2 = 0$$

$$\boldsymbol{w} \cdot \boldsymbol{v} = -1\times4 + 2\times2 = 0$$

Inner Product (or Dot product)

$\boldsymbol{v} = (1, 2, 6)$

$\boldsymbol{w} = (-2, 1, 1)$

$\boldsymbol{v} \cdot \boldsymbol{w} = 1\times-2 + 2\times1 + 6\times1$

$\boldsymbol{v} \cdot \boldsymbol{w} = -2 + 2 + 6 = 6$

Vectors having 3 elements

$\boldsymbol{v} = (1, -1\ , 2, 3)$

$\boldsymbol{w} = (-2, 4, 1, 0)$

$\boldsymbol{v} \cdot \boldsymbol{w} = 1\times-2 + -1\times4 + 2\times1 + 3\times0$

$\boldsymbol{v} \cdot \boldsymbol{w} = -2 - 4 + 2 + 0 = -4$

Vectors having 4 elements

Example

 ₹ 200

 ₹ 350

 ₹ 7500

Let prices for each unit be $\boldsymbol{p} = (p_1, p_2, p_3)$

$\boldsymbol{p} = (200, 250, 7500)$

Let quantity $\boldsymbol{q} = (q_1, q_2, q_3)$

Negative when we buy

Positive when we sell

$\boldsymbol{q} = (20, 10, -1)$

$\text{Income} = q_1 p_1 + q_2 p_2 + q_3 p_3 = \boldsymbol{q} \cdot \boldsymbol{p}$

$200 \times 20 + 250 \times 10 + 7500 \times -1 = 0$

Length and Unit Vector

- Length $\|\boldsymbol{v}\|$ of a vector $\boldsymbol{v}$ is the square root of $\boldsymbol{v} \cdot \boldsymbol{v}$

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = (v_1^2 + v_2^2 + \cdots + v_n^2)^{\frac{1}{2}}$$

- $\boldsymbol{n}$ is the dimension or size
- Unit vector is a vector whose length equals 1

$$\boldsymbol{v} = \left(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}, \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\right) \qquad \|\boldsymbol{v}\| = \sqrt{\left(\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right)} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

- Is $\boldsymbol{u} = (1,1)$ a unit vector ? No. Length is $\sqrt{2}$

COLAB Demo

```
import numpy as np
v=np.array([-2, 3])
w=np.array([1 ,2])
print(np.dot(v,w))
```

```
import numpy as np
v=np.array([4, 2])
w=np.array([-1 ,2])
print(np.dot(v,w))
```

COLAB Demo

```
[2] print(np.dot(v,w))
```

```
0
```

```
import numpy as np
v=np.array([1,2,6])
w=np.array([-2,1,1])
print(np.dot(v,w))
```

```
[ ] import numpy as np
v=np.array([1,-1,2,3])
w=np.array([-2,4,1,0])
```

COLAB Demo

```
import numpy as np
v=np.array([1,-1,2,3])
w=np.array([-2,4,1,0])
print(np.dot(v,w))
```

```
[ ] import numpy as np
    v=np.array([1/np.sqrt(2),1/np.sqrt(2)])
    print(np.dot(v,v))
```

```
[ ] import numpy as np
```

COLAB Demo

```
[5] import numpy as np
    v=np.array([1/np.sqrt(2),1/np.sqrt(2)])
    print(np.dot(v,v))
```

```
0.9999999999999998
```

```
import numpy as np
v=np.array([1,1])
print(np.dot(v,v)) #Length or Euclidean norm
```

```
2
```