FOAM – Week 10 & 11

10.1 Algebra, Linear Equations, Linear Inequalities & Trigonometry

Algebra and Trigonometry

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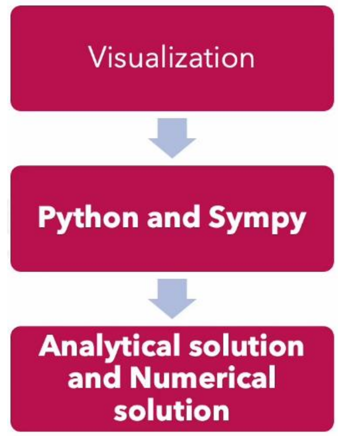


Arithmetic and Algebra

- Two foundational branches of mathematics
- Arithmetic – deals with computations of numbers using basic operations like addition, subtraction, multiplication, and division.

  $14 + 8(49 - 6^2 - 10)^3 \div 12 \times 3$
- Algebra uses numbers and variables (formally defined later) to solve problems.

  $X + y = 2$

  $3x - y = 5$
- First – discuss arithmetic of numbers

Arithmetic expression

- Combination of numbers and operations
- Operations
  - Addition, subtraction, multiplication, division, exponentiation
- Example

  Arithmetic expression: $14 + 8(49 - 6^2 - 10)^3 \div 12 \times 3$

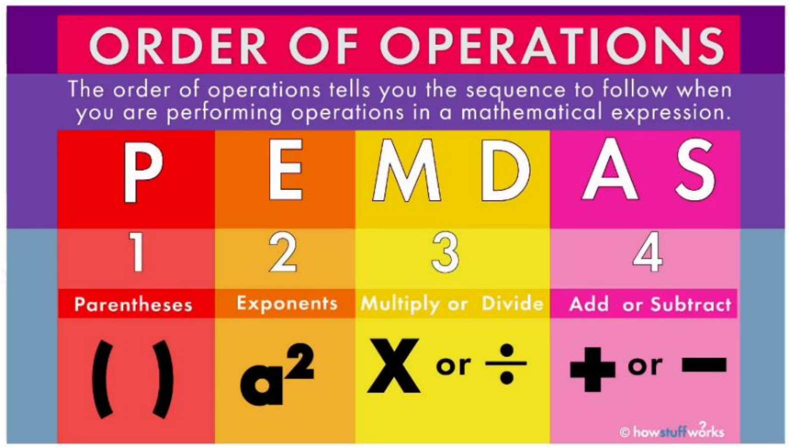
  Result: 68
- How do we compute arithmetic expressions accurately and consistently?

Order of operations

- Collection of rules applied to arithmetic and algebra for accurate and consistent results
- Rules applied in mathematics and computer programming languages
- We should follow a sequence of binary operations in the right order
- Order of operations dictates evaluation of mathematical expressions and solution of equations
- Pneumonic: PEMDAS

PEMDAS

- **P** – Parenthesis
- **E** - Exponential
- **M** – Multiplication
- **D** – Division
- **A** – Addition
- **S** – Subtraction



Example Illustration and introduction to Sympy

- **Example 1 -**

Compute

$$14 + 8 \times (49 - 36 - 10)^3 \div 12 \times 3$$

Solution

$$
\begin{aligned}
14 + 8 \times (49 - 36 - 10)^3 \div 12 \times 3 = \\
&= 14 + 8 \times \underbrace{(49 - 36 - 10)^3}_{1} \div 12 \times 3 \\
&= 14 + 8 \times (\underbrace{49 - 36}_{2} - 10)^3 \div 12 \times 3 \\
&= 14 + 8 \times \underbrace{(13 - 10)}_{3}{}^3 \div 12 \times 3 \\
&= 14 + 8 \times \underbrace{(3)^3}_{4} \div 12 \times 3 \\
&= 14 + \underbrace{8 \times 27}_{5} \div 12 \times 3 \\
&= 14 + \underbrace{216 \div 12}_{6} \times 3 \\
&= 14 + \underbrace{18 \times 3}_{7} \\
&= \underbrace{14 + 54}_{8} \\
&= 68
\end{aligned}
$$

Python interpreter

```
#Compute the expression given in Example 1
14+8*(49-36-10)**3/12*3
```

```
68.0
```

Sympy

```
from sympy import *
expression_string = "14+8*(49-36-10)**3/12*3"
simplified_expression = simplify(expression_string)
simplified_expression
```

68

- **Example 2 -**

Compute

$$\frac{14+8(41-6^2-10)^3}{12\times 3}$$

Solution

$$\begin{aligned}\frac{14+8(41-6^2-10)^3}{12\times 3} &= \frac{14+8(41-36-10)^3}{12\times 3}\\ &= \frac{14+8(-5)^3}{12\times 3}\\ &= \frac{14+8\times -125}{12\times 3}\\ &= \frac{14-1000}{12\times 3}\\ &= \frac{-986}{36}\\ &= \frac{-493}{18}\end{aligned}$$

Using python

```
[ ] (14+8*(41-6**2-10)**3)/(12*3)
```

```
-27.38888888888889
```

Sympy

```
from sympy import *
str_expr="(14+8*(41-6**2-10)**3)/(12*3)"
expr=simplify(str_expr)
expr
```

$-\frac{493}{18}$

- **Example 3 -**

Compute

$$\frac{\frac{3}{4}+\frac{5}{8}}{\frac{11}{12}-\frac{7}{24}}$$

Solution

$$\frac{\frac{3}{4}+\frac{5}{8}}{\frac{11}{12}-\frac{7}{24}}=\frac{\frac{6}{8}+\frac{5}{8}}{\frac{22}{24}-\frac{7}{24}}=\frac{\frac{11}{8}}{\frac{15}{24}}=\frac{11}{8}\times\frac{24}{15}=\frac{33}{15}=\frac{11}{5}$$

Sympy

```
[ ]
    from sympy import *
    str_expr="((3/4)+(5/8))/((11/12)-(7/24))"
    expr=simplify(str_expr)
    expr
```

$$\frac{11}{5}$$

10.2 Algebra, Linear Equations, Linear Inequalities & Trigonometry

Important properties

- Arithmetic and Algebra
- Order of Operations
- Cumulative, Associative and Distributive properties
- Properties applied to addition and multiplication
- Not applicable for subtraction and division
- Important to know for simplification of expressions, solution of equations

Cumulative Property

- A binary operation is commutative if changing the order of operands does not change the result
- Let us understand through examples

$$8 + 3 = 3 + 8$$

$$5 \times 4 = 4 \times 5$$

- Note that

$$8 - 3 \neq 3 - 8$$

$$5 \div 4 \neq 4 \div 5$$

- Formally

$$X + y = y + x$$

$$X \times y = y \times x$$

Associative Property -

- The associative property means that rearranging the parentheses in an expression will not change the result
- Let us understand through examples

$$13 + 42 + 22 = (13 + 42) + 22 = 55 + 22 = 77$$
$$13 + 42 + 22 = 13 + (42 + 22) = 55 + 22 = 77$$

$$8 \times 3 \times 15 = (8 \times 3) \times 15 = 24 \times 15 = 360$$
$$8 \times 3 \times 15 = 8 \times (3 \times 15) = 8 \times 45 = 360$$

- Formally

$$(x + y) + z = x + (y + z)$$

$$(x \times y) \times z = x \times (y \times z)$$

Distributive property –

- More specifically known as distributive property over addition
- According to the distributive property, multiplying the sum of two operands by a number will give the same results as multiplying each operand individually by the number and then adding the products together.
- Let us understand through examples

$$5 \times (4 + 6) = 5 \times 10 = 50$$
$$5\times (4 + 6) = 5\times 4 + 5 \times 6 = 20 + 30 = 50$$

- Formally

$$x \times (y + z) = x \times y + x \times z$$

10.3 Algebra, Linear Equations, Linear Inequalities & Trigonometry

Constants and Variables

- Constants and variables are used to represent quantities
- Use alphabets to represent both
- Constants – quantities that have fixed value
- Variables – quantities that have fixed value
- Variables – quantities whose values can represent a range of values. Can take any value in the specified range
- Commonly used alphabets for variables: x, y, z
- Commonly used alphabets for constants: a, b, c
- Example

$$ax + by = c$$

- Greek letters also used to represent constants and variables

Expressions and Equations

- Variables and constants can be combined to form expressions
- Examples

  $3x + 4y$

  $ax^2 + bx + c$

  xyz

  12

- Understand monomials and polynomials
- Multiple expressions that are connected with equality form equations
- Examples

  3 x + 4 y = 12

  xyz = 12

  $4x + 8y = 3x + 10y$

Expressions

- Variables and constants can be combined to form expressions
- Examples

  3x + 4y

  ax2 + bx + c

  xyz

  12

- Understand monomials and polynomials

  $a \sin(x)$

  $\sin(x) + \exp(y)$

Simplification of expressions

Example 1

Simplify the expression

$$12x + 9y + 10x - 4y$$

Solution

$$\begin{aligned} 12x + 9y + 10x - 4y &= 12x + 10x + 9y - 4y (\text{Commutative property}) \\ &= x(12 + 10) + y(9 - 4)(\text{Distributive property}) \\ &= 22x + 5y \end{aligned}$$

- End Solution

Example 2

Simplify the expression

$$12x + 9xy + 10xy - 4y$$

Solution

$$\begin{aligned} 12x + 9xy + 10xy - 4y &= 12x + (9 + 10)xy - 4y (\text{Distributive property}) \\ &= 12x + 19xy - 4y \end{aligned}$$

Exponentiation, Bases, and Exponents (or Powers)

- Exponentiation is a mathematical operation, written as b^n, involving two numbers, the base b and the exponent or power n.
- Pronounce as b raised to the power of n.
- When n is a positive integer, exponentiation corresponds to repeated multiplication of the base: that is, b^n is the product of multiplying the base n times

$$b^n = \underbrace{b \times b \times \cdots \times b}_{n \text{ times}}$$

- Example

$$4^5 = 4 \times 4 \times 4 \times 4 \times 4 = 1024$$

$$e^3 = e \times e \times e = 2.7182 \times 2.7182 \times 2.7182 = 20.0855$$

Rules for Exponentiation

$$\boxed{(a^n) \times (a^m) = a^{n+m}}$$

$$\begin{aligned} 5^3 \times 5^2 &= 125 \times 25 \\ &= 3125 \end{aligned}$$

Using the above rule, we can redo the calculation as

$$\begin{aligned} 5^3 \times 5^2 &= 5^{3+2} \\ &= 5^5 \\ &= 3125 \end{aligned}$$

Another rule is given by,

$$\boxed{\frac{a^n}{a^m} = a^{n-m}}$$

An accompanying example would be,

$$\frac{3^5}{3^1} = 3^{5-1} = 3^4 = 81$$

The third rule is,

$$\boxed{(ab)^n = a^n b^n}$$

$$(12 \times 12)^2 = 12^2 \times 12^2 = 144 \times 144 = 20736$$

The fourth rule would be,

$$\boxed{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$$

$$\left(\frac{4}{2}\right)^2 = \frac{4^2}{2^2} = \frac{16}{4} = 4$$

And finally, the fifth rule is given by,

$$\boxed{(a^m)^n = a^{mn}}$$

$$(2^3)^2 = 2^6 = 64$$

Examples –

Example 1

Simplify

$$\left(\frac{2a^4}{3b^5}\right)^3$$

Solution

$$\left(\frac{2a^4}{3b^5}\right)^3 = \frac{2a^4}{3b^5} \times \frac{2a^4}{3b^5} \times \frac{2a^4}{3b^5}$$
$$= \frac{2 \times 2 \times 2 \times a^4 \times a^4 \times a^4}{3 \times 3 \times 3 \times b^5 \times b^5 \times b^5}$$

Here Rule 1 is applied $(a^n) \times (a^m) = a^{n+m}$

$$= \frac{8a^{4+4+4}}{27b^{5+5+5}}$$
$$= \frac{8a^{12}}{27b^{15}}$$

- End Solution

Example 2

Simplify $(4x)^3$

Solution

Rule 3 is applied in this example $(ab)^n = a^n b^n$

$$(4x)^3 = 4^3 . x^3 = 64x^3$$

- End Solution

Example 3

Simplify $(6y^4)^3$

Solution

Rule 3 is applied here also.

$$(6y^4)^3 = 6^3 . y^{4\times 3} = 216y^{12}$$

- End Solution

10.4 Algebra, Linear Equations, Linear Inequalities & Trigonometry

Evaluation of Algebraic Expressions

- Determine the value of expression for a given value of each variable in the expression.
- We replace each variable with given values and then simplify the resulting expression using the order of operations.
- Example:

Given t = 5, p = -2

$$\left(\frac{2t}{3p^2}\right)^5$$

- Objective is to thoroughly use python and sympy to solve

Evaluation of Algebraic Expressions – Example

- **Example 1:**

Given $t = 4, p = -2$, evaluate

$$\left(\frac{2t}{3p^2}\right)^5$$

Solution

$$\left(\frac{2t}{3p^2}\right)^5 = \left(\frac{2\times 4}{3\times 4}\right)^5 = \left(\frac{8}{12}\right)^5 = \left(\frac{2}{3}\right)^5 = \left(\frac{32}{243}\right)$$

python module fractions –

```
from fractions import Fraction

t = 4
p = -2
numerator = (2*t)**5
denominator = (3*(p**2))**5
numerical_result = numerator/denominator
fractional_result = Fraction(numerator,denominator)
print(f"Result in decimals = {numerical_result}")
print(f"Result in fractions = {fractional_result}")
```

```
Result in decimals = 0.13168724279835392
Result in fractions = 32/243
```

python library sympy -

```
from sympy import *

t = Symbol('t')
p = Symbol('p')

expression_string="((2*t)/(3*(p**2)))**5"
init_printing()

simplified_expression = simplify(expression_string)
print(f"Simplified expression = ")
print(simplified_expression)

evaluated_expression = simplified_expression.subs({t:4, p: -2})
print(f"Evaluated expression at t=4 and p=-2 is ")
print(evaluated_expression)
```

```
Simplified expression =
32*t**5/(243*p**10)
Evaluated expression at t=4 and p=-2 is
32/243
```

- **Example 2:**

Given $x = 5, y = 2$, evaluate

$$\frac{12x + 9y}{5x - 4y}$$

Solution

$$\begin{aligned}\frac{12x + 9y}{5x - 4y} &= \frac{(12 \times 5) + (9 \times 2)}{(5 \times 5) - (4 \times 2)} \\ &= \frac{60 + 18}{25 - 8} \\ &= \frac{78}{17}\end{aligned}$$

Let's try out using sympy and fraction module.

python module fractions –

```
[ ] from fractions import Fraction
    x = 5
    y = 2
    numerator = (12*x)+(9*y)
    denominator = (5*x)-(4*y)
    numerical_result = numerator/denominator
    fractional_result = Fraction(numerator,denominator)
    print(f"Result in decimals = {numerical_result}")
    print(f"Result in fractions = {fractional_result}")
```

```
Result in decimals = 4.588235294117647
Result in fractions = 78/17
```

python library sympy -

```
from sympy import *
x,y = symbols('x y')
expression_string = "(12*x+9*y)/(5*x-4*y)"

simplified_expression = simplify(expression_string)
pprint(f"Simplified expression = ")
pprint(simplified_expression)

evaluated_expression = simplified_expression.subs({x:5, y: 2})
print(f"Evaluated expression at x=5 and y=2 is ")
pprint(evaluated_expression)
```

```
Simplified expression =
3⋅(4⋅x + 3⋅y)
─────────────
  5⋅x - 4⋅y
Evaluated expression at x=5 and y=2 is
78
──
17
```

11.1 Algebra, Linear Equations, Linear Inequalities & Trigonometry

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Recap –

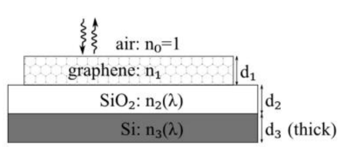
Given $t = 5, p = -2$, evaluate $\left(\frac{2t}{3p^2}\right)^5$

```python
from sympy import *

t = Symbol('t')
p = Symbol('p')

expression_string="((2*t)/(3*(p**2)))**5"
init_printing()

simplified_expression = simplify(expression_string)
print(f"Simplified expression = ")
print(simplified_expression)

evaluated_expression = simplified_expression.subs({t:4, p: -2})
print(f"Evaluated expression at t=4 and p=-2 is ")
print(evaluated_expression)
```

({t:4,p:-2})

```
Simplified expression =
32*t**5/(243*p**10)
Evaluated expression at t=4 and p=-2 is
32/243
```

Evaluation of expressions – Using Sympy in cutting edge research



$$A = r_{01} + r_{12}e^{2j\phi_1} + r_{23}e^{2j(\phi_1+\phi_2)} + r_{34}e^{2j(\phi_1+\phi_2+\phi_3)} + r_{12}r_{23}r_{34}e^{2j(\phi_1+\phi_3)}$$
$$+ r_{01}r_{12}r_{23}e^{2j\phi_2} + r_{01}r_{12}r_{34}e^{2j(\phi_2+\phi_3)} + r_{01}r_{23}r_{34}e^{2j\phi_3}$$
$$B = 1 + r_{01}r_{12}e^{2j\phi_1} + r_{01}r_{23}e^{2j(\phi_1+\phi_2)} + r_{01}r_{34}e^{2j(\phi_1+\phi_2+\phi_3)}$$
$$+ r_{01}r_{12}r_{23}r_{34}e^{2j(\phi_1+\phi_3)} + r_{12}r_{23}e^{2j\phi_2} + r_{12}r_{34}e^{2j(\phi_2+\phi_3)} + r_{23}r_{34}e^{2j\phi_3}$$
$$R = |A/B|^2$$

Later in the chapter, I used the same transfer-matrix method to derive reflectance formulas for more complicated multi-layer structures—such as graphene with a layer of an optically transparent impurity on top of it. Here I used a symbolic mathematics library (SymPy!) to calculate the correct result; getting it right and verifying it with pencil and paper would have been too tedious and too time-consuming for me.

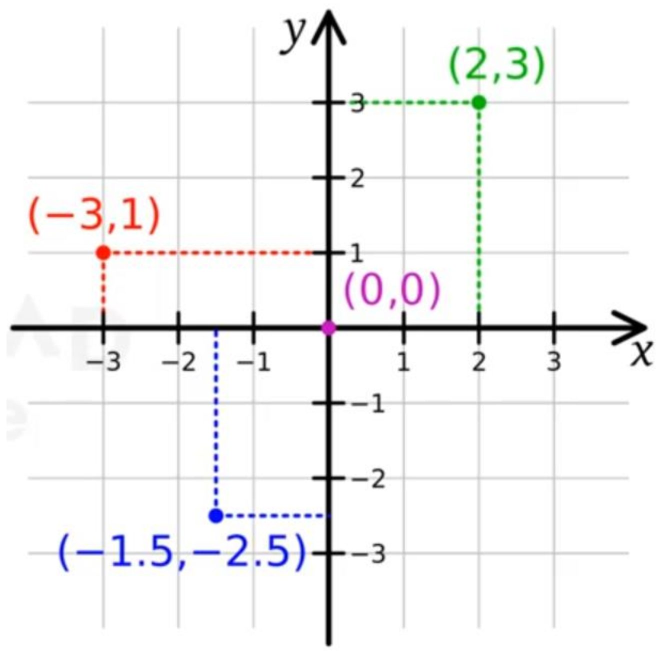
Courtesy: Dr. Dan Lenski. https://www.quora.com/What-is-the-longest-equation-known

11.2 Algebra, Linear Equations, Linear Inequalities & Trigonometry

Cartesian coordinates

- A coordinate system is used to uniquely used to specify points or positions in space.
- Focus on 2D plane for explanation for now

- Carsian coordinate system – most widely used
- Points specified with pair of numbers
- Numbers are signed distances from two fixed perpendicular oriented reference lines.
- Reference lines are called coordinate axis.
- The point where they meet is called origin
- One axis (mostly horizontal) called x axis, other referred as y axis

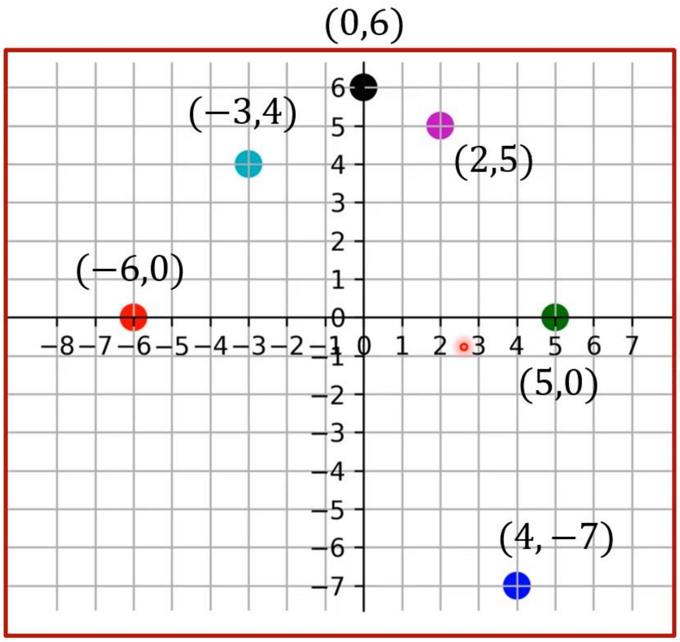


https://en.wikipedia.org/wiki/Cartesian_coordinate_system

Cartesian coordinates

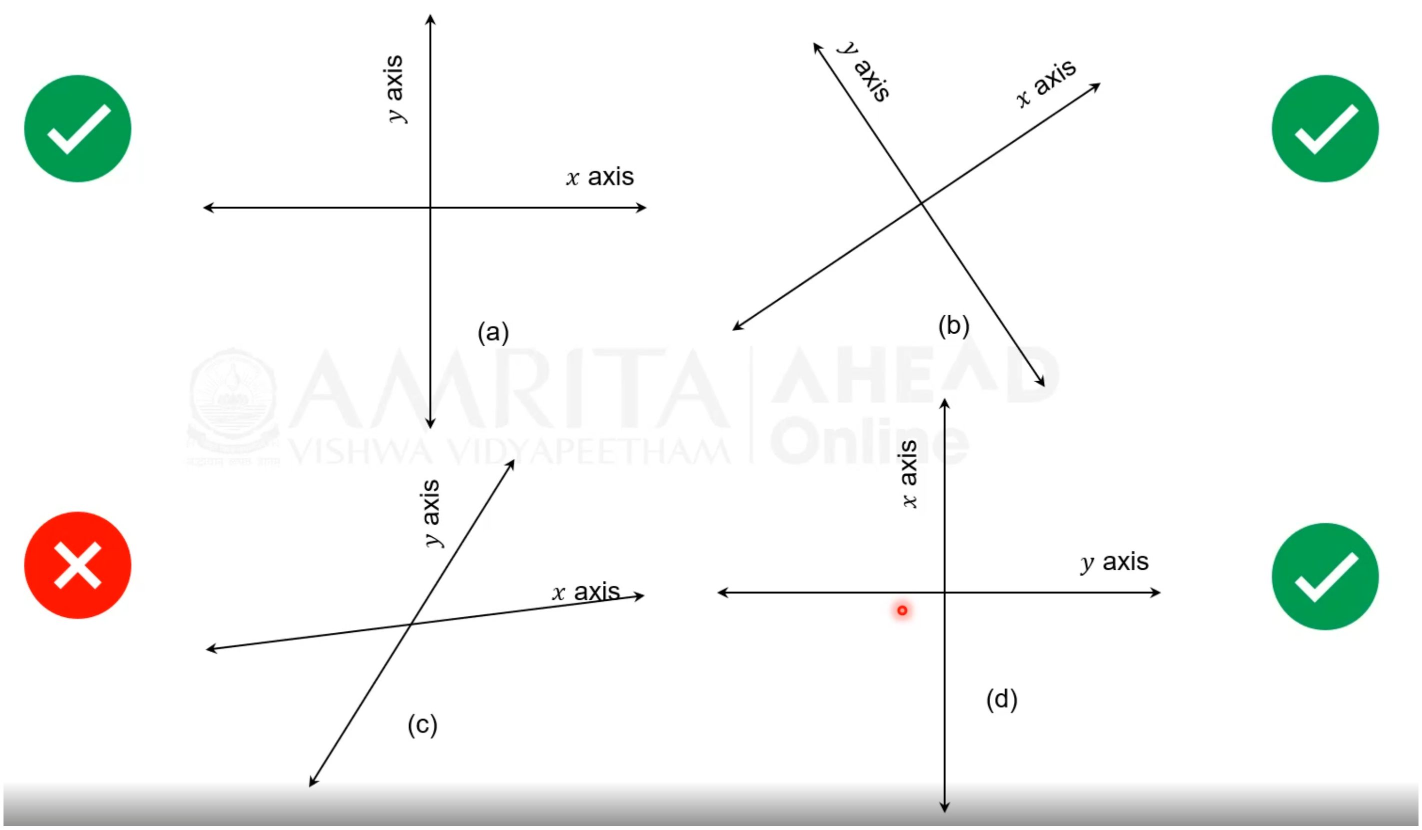
- First fix coordinate axes x and y as per our choice.
- Intersection point is the origin (0,0)
- Green point – 2 units along x axis and 3 unit far along y axis -> point represented by the pair (2,3).
- Red point (-3,1)
- Coordinate axis divide the coordinate plane into four region called quadrants.
- Sequentially numbered in counterclockwise direction I, II, III, IV.
- First quadrant: all numbers of the pairs positive
- Second quadrant: x negative, y positive
- Third quadrant: all negative
- Fourth quadrant: x positive, y negative

Example to practice

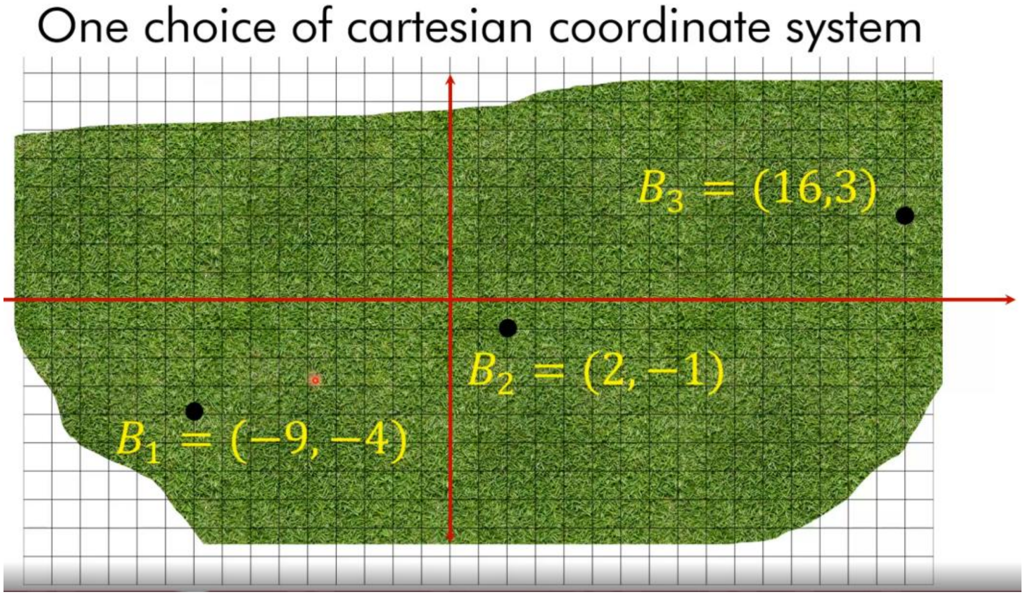


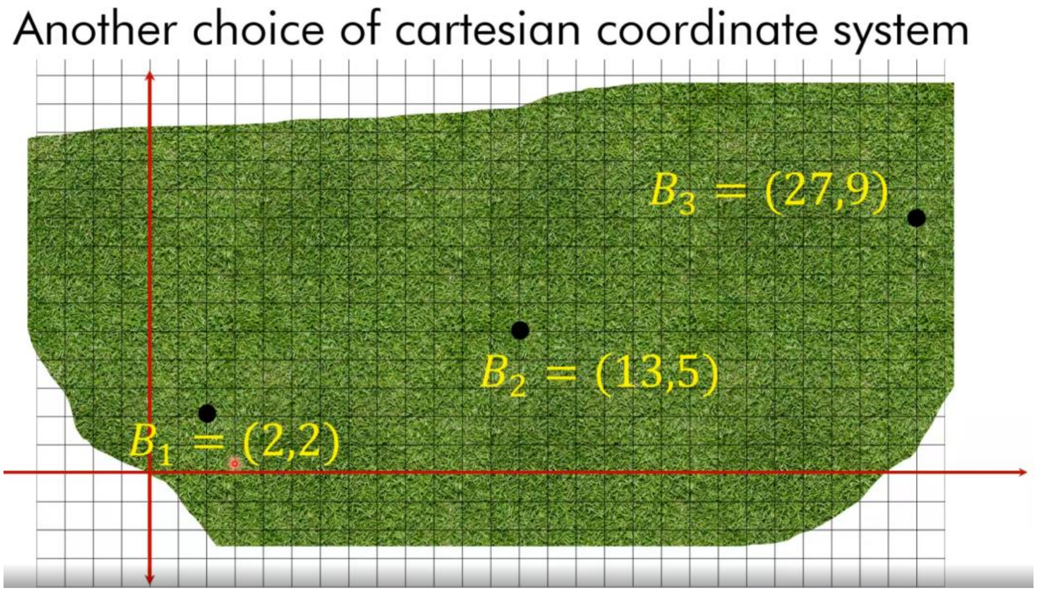

```python
import matplotlib.pyplot as plt
import numpy as np
x = [2, 5, -6, 0, 4, -3, 0]
y = [5, 0, 0, 6, -7, 4, 6]
color = ['m','g','r','y','b','c','k']
fig, ax = plt.subplots()
plt.scatter(x, y, s=100 , marker='o', c=color)
ax.spines['top'].set_color('none')
ax.spines['bottom'].set_position('zero')
ax.spines['left'].set_position('zero')
ax.spines['right'].set_color('none')
ax.set_xlim([-8,8])
ax.set_ylim([-8,8])
ax.set_xticks(range(-8, 8, 1))
ax.set_yticks(range(-8, 8, 1))
ax.axis('equal')
plt.grid()
plt.show()
```

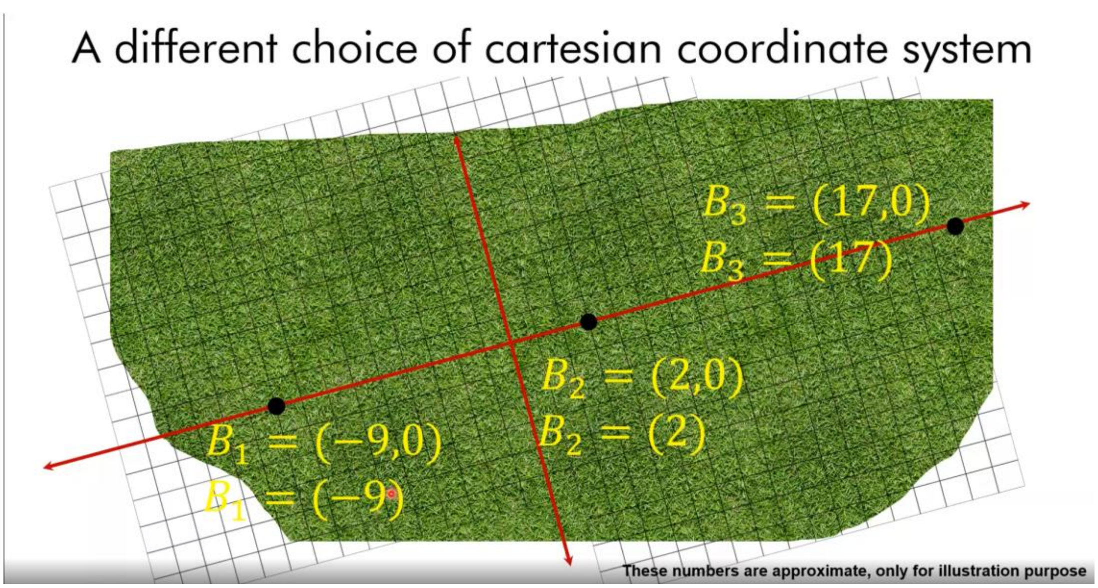
Example 2 – Which of the following cartesian systems?



Example 3 – A Conceptual Example





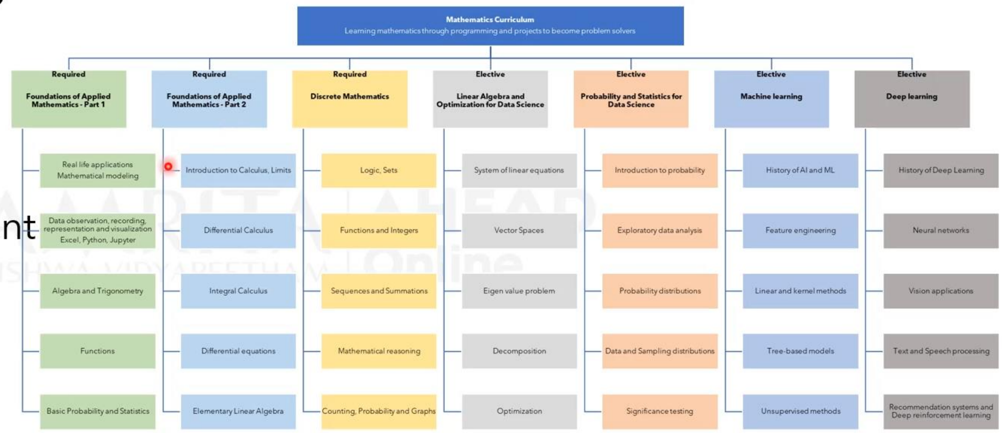


Coordinate transformations –

Coordinate transformations

- Fundamental to many data science and ML concepts

1. Dimensionality reduction
2. Principal Component Analysis
3. Singular Value Decomposition
4. Eigenvalues and Eigen matrix

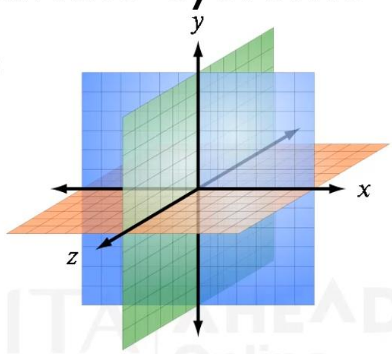


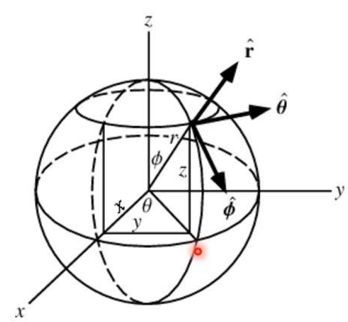
Coordinate systems -

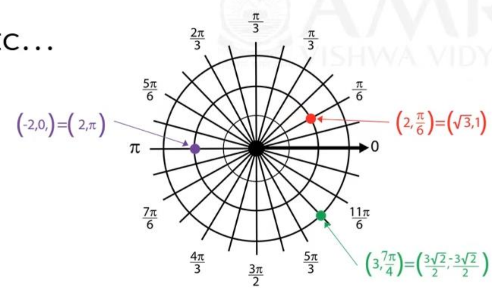
Coordinate systems

- Cartesian coordinate system (3D)
- Polar coordinate systems
- Spherical coordinate systems
- Curvilinear coordinate systems
- Generalize coordinate system

- Etc…







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