BINARY INTEGERS

A *binary integer* x is a finite sequence of the digits 0 and 1, which we write symbolically as

$$x = (a_m a_{m-1} \cdots a_2 a_1 a_0)_2$$

where I insert the parentheses with subscript $()_2$ in order to make clear that the number is binary. The above has the decimal equivalent

$$x = a_m 2^m + a_{m-1} 2^{m-1} + \cdots + a_1 2^1 + a_0$$

For example, the binary integer $x = (110101)_2$ has the decimal value

$$x = 2^5 + 2^4 + 2^2 + 2^0 = 53$$

The binary integer $x = (111 \cdots 1)_2$ with m ones has the decimal value

$$x = 2^{m-1} + \cdots + 2^1 + 1 = 2^m - 1$$

DECIMAL TO BINARY INTEGER CONVERSION

Given a decimal integer x we write

$$\begin{aligned} x &= (a_m a_{m-1} \cdots a_2 a_1 a_0)_2 \\ &= a_m 2^m + a_{m-1} 2^{m-1} + \cdots + a_1 2^1 + a_0 \end{aligned}$$

Divide x by 2, calling the quotient x_1. The remainder is a_0, and

$$x_1 = a_m 2^{m-1} + a_{m-1} 2^{m-2} + \cdots + a_1 2^0$$

Continue the process. Divide x_1 by 2, calling the quotient x_2. The remainder is a_1, and

$$x_2 = a_m 2^{m-2} + a_{m-1} 2^{m-3} + \cdots + a_2 2^0$$

After a finite number of such steps, we will obtain all of the coefficients a_i, and the final quotient will be zero.

Try this with a few decimal integers.

EXAMPLE

The following shortened form of the above method is convenient for hand computation. Convert $(11)_{10}$ to binary.

$$\begin{array}{llllllll}
\lfloor 2\sqrt{11} \rfloor & = 5 & = x_1 & & & & & a_0 = 1 \\
& \lfloor 2\sqrt{5} \rfloor & = 2 & = x_2 & & & & a_1 = 1 \\
& & \lfloor 2\sqrt{2} \rfloor & = 1 & = x_3 & & & a_2 = 0 \\
& & & \lfloor 2\sqrt{1} \rfloor & = 0 & = x_4 & & a_3 = 1
\end{array}$$

In this, the notation $\lfloor b \rfloor$ denotes the largest integer $\leq b$, and the notation $2\sqrt{n}$ denotes the quotient resulting from dividing 2 into n. From the above calculation, $(11)_{10} = (1011)_2$.

BINARY FRACTIONS

A *binary fraction* x is a sequence (possibly infinite) of the digits 0 and 1:

$$\begin{aligned} x &= (.a_1 a_2 a_3 \cdots a_m \cdots)_2 \\ &= a_1 2^{-1} + a_2 2^{-2} + a_3 2^{-3} + \cdots \end{aligned}$$

For example, $x = (.1101)_2$ has the decimal value

$$\begin{aligned} x &= 2^{-1} + 2^{-2} + 2^{-4} \\ &= .5 + .25 + .0625 = 0.8125 \end{aligned}$$

Recall the formula for the geometric series

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}, \quad r \neq 1$$

Letting $n \to \infty$ with $|r| < 1$, we obtain the formula

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1 - r}, \quad |r| < 1$$

Using this,

$$\begin{aligned}(.0101010101010 \cdots)_2 &= 2^{-2} + 2^{-4} + 2^{-6} + \cdots \\ &= 2^{-2}\left(1 + 2^{-2} + 2^{-4} + \cdots\right)\end{aligned}$$

which sums to the fraction $1/3$.

Also,

$$\begin{aligned}&(.11001100110011 \cdots)_2 \\ &= 2^{-1} + 2^{-2} + 2^{-5} + 2^{-6} + \cdots\end{aligned}$$

and this sums to the decimal fraction $0.8 = \frac{8}{10}$.

DECIMAL TO BINARY FRACTION CONVERSION

In

$$
\begin{aligned}
x_1 &= (.a_1 a_2 a_3 \cdots a_m \cdots)_2 \\
&= a_1 2^{-1} + a_2 2^{-2} + a_3 2^{-3} + \cdots
\end{aligned}
$$

we multiply by 2. The integer part will be a_1; and after it is removed we have the binary fraction

$$
\begin{aligned}
x_2 &= (.a_2 a_3 \cdots a_m \cdots)_2 \\
&= a_2 2^{-1} + a_3 2^{-2} + a_4 2^{-3} + \cdots
\end{aligned}
$$

Again multiply by 2, obtaining a_2 as the integer part of $2x_2$. After removing a_2, let x_3 denote the remaining number. Continue this process as far as needed.

For example, with $x = \frac{1}{5}$, we have

$$
\begin{gathered}
x_1 = .2; \quad 2x_1 = .4; \quad x_2 = .4 \text{ and } a_1 = 0 \\
2x_2 = .8; \quad x_3 = .8 \text{ and } a_2 = 0 \\
2x_3 = 1.6; \quad x_4 = .6 \text{ and } a_2 = 1
\end{gathered}
$$

Continue this to get the pattern

$$
(.2)_{10} = (.00110011001100 \cdots)_2
$$