Homework 5

Due December 3rd, 2023

Problem 5.1

Consider a continuous random variable X with the following density function:

$$f(x) = \begin{cases} k(x+2), & if \quad x \in [3, 10] \\ 0, & otherwise \end{cases}$$

a) Find parameter k.

b) Find cumulative distribution function F_X.

c) Find the probability that $X^2 + 45 \leqslant 14X$.

d) Find $P(X \leqslant 7 \mid X > 3)$.

e) Find $P(X > 8)$

e) Find expected value $E(X)$, variance $V(X)$ and standard deviation $D(X)$.

Problem 5.2

Choose independently two numbers X and Y at random from the interval $[0, 1]$ with uniform density. Note that the point (X, Y) is then chosen at random in the unit square. Find the probability that

a) $X + Y < 1/3$.

b) $XY < 1/3$.

c) $\max\{X, Y\} > 1/3$.

d) $\min\{X, Y\} < 1/3$.

e) $X < 1/3$ and $1 - Y > 1/3$.

f) $X^2 + Y^2 > 1/3$.

Problem 5.3

Choose independently two numbers X and Y at random from the interval $[0, 1]$ with uniform density. Note that the point (X, Y) is then chosen at random in the unit square. Find the cumulative distribution and density functions for the following continuous random variables

a) $Z = X - Y$.

b) $W = XY$.

c) $V = X^3$.

Problem 5.4

Suppose you are watching a radioactive source that emits particles at a rate described by the exponential density

$$f(t) = \lambda e^{-\lambda t},$$

where $\lambda = 1$, so that the probability $P(X \leqslant T)$ that a particle will appear in the next T seconds is

$$P(X \leqslant T) = \int_0^T \lambda e^{-\lambda t} dt.$$

Find the probability that a particle (not necessarily the first) will appear

a) within the next second.

b) within the next 5 seconds.

c) between 2 and 5 seconds from now.

d) after 5 seconds from now.

Problem 5.5

Assume that a new light bulb will burn out after t hours, where t is chosen from $[0, \infty)$ with an exponential density

$$f(t) = \lambda e^{-\lambda t}.$$

In this context, λ is often called the failure rate of the bulb.

a) Assume that $\lambda = 0.01$, and find the probability that the bulb will not burn out before T hours. This probability is often called the reliability of the bulb.

b) For what T is the reliability of the bulb $= 1/2$?

c) Find the probability that the bulb burns out in the forty-nineth hour, given that it lasts 48 hours.

Problem 5.6

Take a stick of unit length and break it into three pieces, choosing the break points at random. (The break points are assumed to be chosen simultaneously.) What is the probability that the three pieces can be used to form a triangle?

Hint: The sum of the lengths of any two pieces must exceed the length of the third, so each piece must have length $< 1/2$.

Problem 5.7

A baker blends 600 raisins into a dough mix and, from this, makes 500 cookies.

a) Find the probability that a randomly picked cookie will have no raisins.

b) Find the probability that a randomly picked cookie will have exactly two raisins.

c) Find the probability that a randomly chosen cookie will have at least five raisins in it.

Problem 5.8

Let X be a random variable normally distributed with parameters $\mu = 50$, $\sigma = 10$. Estimate

a) $P(X > 70)$.

b) $P(X \leqslant 45)$.

c) $P(X > 30)$.

d) $P(60 \leqslant X < 90)$.