Mathematical analysis I

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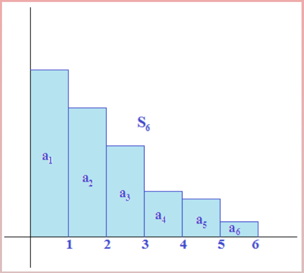
Subsection 3

Convergence of Series with Positive Terms

Positive Series

- A **positive series** $\sum a_n$ is one with $a_n > 0$, for all n;
- The terms can be thought of as areas of rectangles with width 1 and height a_n;
  The partial sum
  $$S_N = a_1 + \cdots + a_N$$
  is equal to the area of the first N rectangles;



- Clearly, the partial sums form an *increasing sequence* $S_N < S_{N+1}$;

Dichotomy and Integral Test

Dichotomy for Positive Series

If $S = \sum_{n=1}^{\infty} a_n$ is a positive series, then either

1. The partial sums S_N are bounded above, in which case S converges, or
2. The partial sums S_N are not bounded above, in which case S diverges.

The Integral Test

Let $a_n = f(n)$, where the function $f(x)$ is positive, decreasing and continuous for $x \geq 1$;

1. If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges;
2. If $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges;

Applying the Integral Test on the Harmonic Series

- **The Harmonic Series Diverges**: Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges;

  Consider the function $f(x) = \frac{1}{x}$; For $x \geq 1$, it is positive, decreasing and continuous, and, moreover, $f(n) = \frac{1}{n} = a_n$; So we check

  $$\int_1^{\infty} \frac{dx}{x} = \lim_{R \to \infty} \int_1^R \frac{dx}{x} = \lim_{R \to \infty} \ln R = \infty;$$

  Therefore, by the Integral Test, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges;

Another Application of the Integral Test

- Does $\displaystyle\sum_{n=1}^{\infty}\frac{n}{(n^2+1)^2}=\frac{1}{2^2}+\frac{2}{5^2}+\frac{3}{10^2}+\cdots$ converge?

  Consider the function $f(x)=\dfrac{x}{(x^2+1)^2}$; It is positive and continuous for $x\geq 1$; Is it also decreasing for $x\geq 1$? Let us compute its first derivative

  $$f'(x)=\frac{(x)'(x^2+1)^2-x[(x^2+1)^2]'}{[(x^2+1)^2]^2}=$$
  $$\frac{(x^2+1)^2-x\cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4}=\frac{(x^2+1)-4x^2}{(x^2+1)^3}=\frac{1-3x^2}{(x^2+1)^3}<0;$$

  Thus, the Integral Test is applicable and we get

  $$\int_1^{\infty}\frac{x}{(x^2+1)^2}dx=\lim_{R\to\infty}\int_1^R\frac{x}{(x^2+1)^2}dx\overset{u=x^2+1}{=}\lim_{R\to\infty}\int_2^{R^2+1}\frac{1}{2u^2}du=$$
  $$\lim_{R\to\infty}\left.\frac{-1}{2u}\right|_2^{R^2+1}=\lim_{R\to\infty}\left(\frac{1}{4}-\frac{1}{2R^2+2}\right)=\frac{1}{4};\text{ So, }\sum_{n=1}^{\infty}\frac{n}{(n^2+1)^2}\text{ converges;}$$

The p-Series

Convergence of the p-Series

The infinite series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges, if $p > 1$, and diverges, otherwise.

- If $p \le 0$, $\displaystyle\lim_{n\to\infty}\frac{1}{n^p} \neq 0$; By Divergence Test, p-series diverges;
- If $p > 0$, $f(x) = \dfrac{1}{x^p}$ is positive, decreasing and continuous on $[1, \infty)$; Thus, the Integral Test applies and

$$\int_1^{\infty}\frac{1}{x^p}dx = \begin{cases} \dfrac{1}{p-1}, & \text{if } p > 1 \\ \infty, & \text{if } p \le 1 \end{cases}$$

- Example: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}$ diverges, and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges;

Comparison Test

Comparison Test

Assume that for some $M > 0$, $0 \leq a_n \leq b_n$, for all $n \geq M$;

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges;
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges;

$\sqrt{n}\,3^n > 3^n, \; n > 1$

- Example: Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n}$ converge?

  Clearly, for all $n \geq 1$, we have $0 \leq \dfrac{1}{\sqrt{n}3^n} \leq \dfrac{1}{3^n}$; Moreover, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges since it is a geometric series with ration $\dfrac{1}{3} < 1$; Therefore, by Comparison $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n}$ also converges;

Example

- Does $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(n^2+3)^{1/3}}$ converge?

  Consider the function $f(x) = x^3 - x^2 - 3$; We show that for $x \geq 2$, $f(x) > 0$; Note $f(2) = 2^3 - 2^2 - 3 = 1 > 0$; Moreover, for $x \geq 2$ $f'(x) = 3x^2 - 2x = x(3x-2) > 0$, so f is increasing; Thus $f > 0$, all $x \geq 2$;

  We have shown, for $n \geq 2$, $f(n) = n^3 - n^2 - 3 > 0 \Rightarrow \underline{n^3 > n^2 + 3} \Rightarrow$ $n > (n^2+3)^{1/3} \Rightarrow \underline{\frac{1}{n} < \frac{1}{(n^2+3)^{1/3}}}$; But $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n}$ is the harmonic series that diverges; therefore, by Comparison $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(n^2+3)^{1/3}}$ also diverges;

Limit Comparison Test

Limit Comparison Test

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences and assume that $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists;

finite

1. If $L > 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges;
2. If $L = \infty$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ also converges;
3. If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges;

Example I

- Show that $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^4 - n - 1}$ converges;

Pick $a_n = \dfrac{n^2}{n^4 - n - 1}$ and $b_n = \dfrac{1}{n^2}$; Then

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^4 - n - 1} \cdot \frac{n^2}{1} =$$

$$\lim_{n\to\infty} \frac{1}{1 - \frac{1}{n^3} - \frac{1}{n^4}} = 1;$$

$L = 1 \neq 0$

finite

Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^4 - n - 1}$ also converges by the Limit Comparison Test;

Example II

- Show that $\displaystyle\sum_{n=3}^{\infty}\frac{1}{\sqrt{n^2+4}}$ diverges;

  Pick $a_n = \dfrac{1}{\sqrt{n^2+4}}$ and $b_n = \dfrac{1}{n}$; Then

$$L = \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+4}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{4}{n^2}}} = 1; \neq 0$$

divergent

Since $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n}$ diverges, $\displaystyle\sum_{n=3}^{\infty}\frac{1}{\sqrt{n^2+4}}$ also diverges by the Limit Comparison Test;

Subsection 4

Absolute and Conditional Convergence

Absolute Convergence

Absolute Convergence

The series $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges.

- Example: Verify that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ converges absolutely;
  We check
  $$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$
  which converges as a p-series with $p > 1$;

Absolute Convergence Implies Convergence

Theorem (Absolute Convergence Implies Convergence)

If $\sum |a_n|$ converges, then $\sum a_n$ also converges.

- Example: Verify that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converges;

  It was shown in the previous slide that $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right|$ converges;

  Therefore, by the Theorem, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ also converges;

Another Example

- Does $S = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \cdots$ converge absolutely?

  We have

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}},$$

  which is a p-series, with $p = \frac{1}{2} \leq 1$, and so diverges; Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is not absolutely convergent;

Conditional Convergence

- We saw than absolute convergence implies convergence:

$$\text{If } \sum |a_n| \text{ converges, then } \sum a_n \text{ also converges;}$$

- The converse is not true in general! I.e., the convergence of a series does not necessarily imply its absolute convergence;

Conditional Convergence

An infinite series $\sum a_n$ **converges conditionally** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Alternating Series

- An **alternating series** is an infinite series of the form

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots ,$$

where $a_n > 0$ and decrease to 0;

Leibniz Test for Alternating Series

Suppose $\{a_n\}$ is a positive sequence that is decreasing and converges to 0:

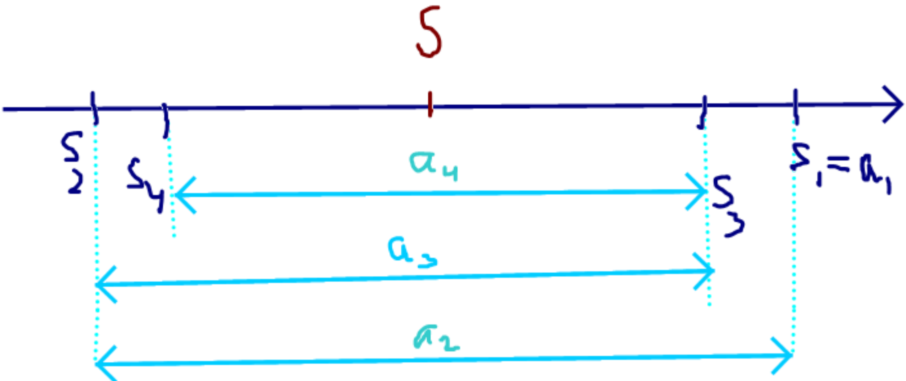
$$a_1 > a_2 > a_3 > \cdots > 0, \qquad \lim_{n\to\infty} a_n = 0;$$

Then the alternating series $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$ converges; Moreover, we have

$$0 < S < a_1 \qquad \text{and} \qquad S_{2N} < S < S_{2N+1}, \; N \geq 1;$$

$$S_{2N} < S < S_{2N+1} \quad , N \geq 1$$

$$S_1 = a_1 \ , \ S_2 = a_1 - a_2 \ , \ S_3 = a_1 - a_2 + a_3$$



Example

- Show that $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \cdots$ converges conditionally and that $0 \leq S \leq 1$;
  - We already saw that $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series;
  - On the other hand, S converges by the Leibniz Test, since $a_n = \frac{1}{\sqrt{n}}$ is a positive decreasing sequence converging to 0;
  - Therefore, S is conditionally convergent;
  - By the last part of the Leibniz Test, $0 < S < a_1 = 1$;

Error of Approximation of Alternating Series

Theorem

Let $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$, where a_n is a positive decreasing sequence that converges to 0; Then

$$|S - S_N| < a_{N+1};$$

I.e., the error committed when we approximate S by S_N is less than the size of the first omitted term a_{N+1};

i.e. id est "in other words"

Alternating Harmonic Series

- Show that $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges conditionally;
  Since $a_n = \frac{1}{n}$ is positive, decreasing and has limit 0, we get by the Leibniz Test that S converges;
  Moreover $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges (harmonic series);
  Thus, S is conditionally convergent;
- Show that $|S - S_6| < \frac{1}{7}$;
  By the approximation error theorem, we get that
  $|S - S_6| < a_{6+1} = a_7 = \frac{1}{7}$;
- Find an N, such that S_N approximates S with an error less than 10^{-3};
  We know that $|S - S_N| < a_{N+1}$; To make the error $|S - S_N| < 10^{-3}$ it suffices to arrange N so that
  $$a_{N+1} \leq 10^{-3} \Rightarrow \frac{1}{N+1} \leq 10^{-3} \Rightarrow N+1 \geq 1000 \Rightarrow N \geq 999;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$ abs. conv., S

How many terms should we take to obtain the sum with appr. error up to 0.001?

N-? $|S - S_N| < 10^{-3}$

$$|S - S_N| < a_{N+1}$$

N -? $a_{N+1} \leq 10^{-3}$ or $\frac{1}{(N+1)^2} \leq 10^{-3} \Rightarrow$

$$\Rightarrow (N+1)^2 \geq 1000, \quad N \geq \sqrt{1000} - 1$$

$$N \geq 30{,}6\ldots \quad N = 31$$

Subsection 5

The Ratio and Root Tests

The Ratio Test

$$\sum a_n$$

Theorem (Ratio Test)

Assume that $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists;

1. If $\rho < 1$, then $\sum a_n$ converges absolutely;
2. If $\rho > 1$, then $\sum a_n$ diverges;
3. If $\rho = 1$, then test is inconclusive.

d'Alembert Test

Applying the Ratio Test I

- Prove that $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges; $a_n = \frac{2^n}{n!}$, $a_{n+1} = \frac{2^{n+1}}{(n+1)!}$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| = \lim_{n\to\infty} \frac{2}{n+1} = 0;$$

  Since $\rho < 1$, the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges by the Ratio Test;

- Does the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converge?

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \right| = \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2} = \frac{1}{2};$$

  Since $\rho < 1$, the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges by the Ratio Test;

Applying the Ratio Test II

- Does the series $\sum_{n=0}^{\infty}(-1)^n \frac{n!}{1000^n}$ converge?

  $$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)!}{1000^{n+1}} \cdot \frac{1000^n}{(-1)^n n!}\right| =$$

  $$\lim_{n\to\infty}\frac{n+1}{1000} = +\infty;$$

  Since $\rho > 1$, the series $\sum_{n=0}^{\infty}(-1)^n \frac{n!}{1000^n}$ diverges by the Ratio Test;

If Ratio Test is Inconclusive Anything Can Happen

- Consider $\sum_{n=1}^{\infty} n^2$;
$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{n^2} = \lim_{n\to\infty}\frac{n^2+2n+1}{n^2} = 1;$$
So Ratio Test is inconclusive; However, $\lim_{n\to\infty} a_n \neq 0$, so the series $\sum_{n=1}^{\infty} n^2$ diverges by Divergence Test;

- Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$;
$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = \lim_{n\to\infty}\frac{n^2}{n^2+2n+1} = 1;$$
So Ratio Test is again inconclusive; However, $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p-series with $p = 2 > 1$ and, hence, it converges!

The Root Test

Theorem (Root Test)

Assume that $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists;

Cauchy Test

1. If $L < 1$, then $\sum a_n$ converges absolutely;
2. If $L > 1$, then $\sum a_n$ diverges;
3. If $L = 1$, the test is inconclusive.

- Example: Does $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$ converge?

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+3}\right)^n} = \lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2};$$

Since $L < 1$, the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$ converges by the Root Test;

Example. $\displaystyle\sum_{n=1}^{\infty} a^n\left(1+\frac{1}{n}\right)^{n^2}, \quad a \geq 0$

$$L=\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{a^n\left(1+\frac{1}{n}\right)^{n^2}}=$$

$$=a\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=a\cdot e$$

for a : $ae<1$ convergent

for a : $ae>1$ divergent

for $a=\frac{1}{e}$ the root test is inconclusive

Rearrangements, Stewart, p.737

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{(-1)^{n-1}}{n}+\dots=S\ ,\quad S>0$$

$$S' \quad 1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\dots+\frac{1}{2k-1}-\frac{1}{4k-2}-\frac{1}{4k}+\dots$$

$$S'_{3k}=\left(1-\frac{1}{2}-\frac{1}{4}\right)+\dots+\left(\frac{1}{2k-1}-\frac{1}{4k-2}-\frac{1}{4k}\right)$$

$$=\left(\frac{1}{2}-\frac{1}{4}\right)+\dots+\left(\frac{1}{4k-2}-\frac{1}{4k}\right)=$$

$$=\frac{1}{2}\left(1-\frac{1}{2}+\dots+\frac{1}{2k-1}-\frac{1}{2k}\right)=\frac{1}{2}\cdot S_{2k}$$

$$S'_{3k} = \frac{1}{2} S_{2k}$$

$$\downarrow \qquad \qquad \downarrow$$

$$S' = \frac{1}{2} S$$

$$S'_{3k+1} = S'_{3k} + \frac{1}{2k+1}$$

$$S'_{3k+2} = S'_{3k+1} - \frac{1}{4k+2}$$

$$S' = \frac{1}{2} S$$

11.7 Strategy for testing Series

Stewart, Calculus, p.739