Mathematics for Computer Science

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Lecture 1

Sets

Mathematics deals with objects of different kinds:

- numbers (natural, integers, rational, real, etc);
- points (in plane or space), vectors;
- lines, planes, triangles, circles, angles etc;
- equations;
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Definition

If an object x is an element of a set S, we write $x \in S$.
If x is not an element of S, then we write $x \notin S$.

Elements of a set

Elements of a set can be even other sets:

Example

$$C = \big\{\{x, y\}, \{x, z\}, \{y, z\}\big\}, \quad \text{e.g. } \{x, z\} \in C.$$

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Example

1. If Q is the set of all quadrangles, and A is a parallelogram, then $A \in Q$. If C is a circle, then $C \notin Q$.
2. If G is the set of all even numbers, then $16 \in G$, and $3 \notin G$.
3. If L is the set of all solutions of the equation $x^2 = 1$, then $1 \in L$, while $2 \notin L$.
4. If $C([0, 1])$ is the set of all continuous functions on $[0, 1]$, then $f(x) = \sin x \in C([0, 1])$, while $g(x) = \ln x \notin C([0, 1])$.

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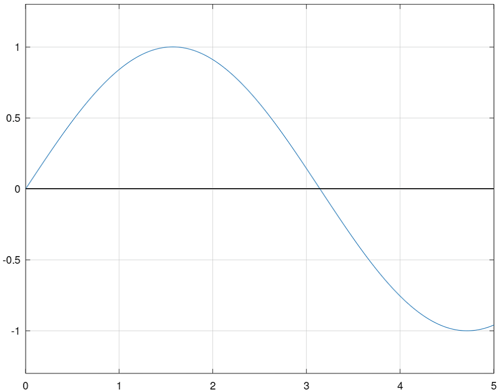

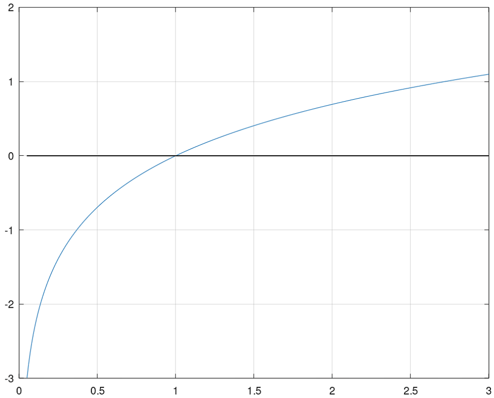

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3. If there are more elements, then periods are used (implied list):

$$\{0, 1, 2, 3, \ldots\}, \quad \{2, 4, 6, \ldots, 20\}, \quad \{1, 4, 9, \ldots, 100\}.$$

The meaning should be clear from the context. In this descriptive or explicit method, an element may be listed more than once.

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Moreover, **the order in which the elements appear is irrelevant!**
Thus, the following all describe the same set:

$$\{1, 2, 3\}, \quad \{2, 3, 1\}, \quad \{1, 1, 3, 2, 3\}$$

Specifying a set

4. By giving (specifying) a rule which determines if a given object is in the set or not.

1. $\{x \mid x \text{ is a natural number}\}$
2. $\{x \mid x \text{ is a natural number and } x > 0\}$
3. $\{y \mid y \text{ solves } (y+1)\cdot(y-3) = 0\}$
4. $\{p \mid p \text{ is an even prime number}\}$.
5. $\{f \mid f : [0,1] \to \mathbb{R},\ f \text{ is continuous}\}$
6. $\{p \mid p \text{ is a polynomial of degree } 5\}$
7. $\{g \mid g(x) = \frac{p(x)}{q(x)}, p \text{ and } q \text{ are polynomials}\}$
8. $\{\alpha \mid \alpha \in \mathbb{R} \text{ and } p(\alpha) = 0, \text{ where } p \text{ is a given polynomial}\}$

Set-builder notation

The general situation can be described as follows:

A set is determined by a defining property P of its elements, written

$$\{x \mid P(x)\},$$

where $P(x)$ means that x has the property described by P and x serves as a variable for objects. Any other letter, or symbol except P, would have done equally well.

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Set-builder notation can specify the form of the elements of a set:

$$\{3x - 1 \mid x \in \mathbb{Z}\} = \{\dots, -7, -4, -1, 2, 5, 8, 11, \dots\}$$

Subset

Definition

The set A is a **subset** of B, written $A \subseteq B$, if every element of A is also an element of B.

If a set A is not a subset of B we write $A \nsubseteq B$.

Clearly, $S \subseteq S$ for any set S.

Example

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C},$$

$$\{2\} \subseteq \{1, 2, 9, 36\} \subseteq \{n^2 \mid n \in \{0, 1, \dots, 10\} \subseteq \mathbb{N}\}.$$

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Similarly,

$$\varnothing \neq \{\varnothing\} \neq \{\{\varnothing\}\} \neq \{\{\varnothing\}, \varnothing\}.$$

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4. $2^{2^{\{a\}}} = \Big\{\varnothing, \{\varnothing\}, \{\{a\}\}, \{\varnothing, \{a\}\}\Big\}$.

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Two sets A and B are equal, if $A \subseteq B$ and $B \subseteq A$.
If A and B are equal, we write $A = B$.

To prove that two sets A and B are equal, we must show that A is a subset of B, and that B is a subset of A.

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Want to show that $A = B$.
We will show that $A \subseteq B$ and $B \subseteq A$.

Equality of sets

$$A = \{x \in \mathbb{R} \mid x^2 = 1\}$$
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Proof.

First, need to prove that "$A \subseteq B$". Let $x \in A$.
Then, by the definition of set A, x solves the equation $x^2 = 1$,

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Next prove that "$B \subseteq A$": Let $x \in B$.
Then, by definition of set B, $x = 1$ or $x = -1$.
In either case, x is a real number and solves the equation $x^2 = 1$,
hence, it fulfills the defining properties of A.
This implies that $x \in A$. $\square$

Proper subset

Definition

A set X is a **proper subset** of Y, written $X \subsetneq Y$,
if X is a subset of Y and $X \neq Y$.

Example

Let $E = \{n \in \mathbb{Z} \mid n \text{ is even}\}$. Then,

- $E \subsetneq \mathbb{Z}$. Indeed, $E \subseteq \mathbb{Z}$ and $E \neq \mathbb{Z}$ since, for example, $1 \in \mathbb{Z}$, but $1 \notin E$.
- $\mathbb{N} \not\subseteq E$ since $1 \in \mathbb{N}$, but $1 \notin E$.
- $E \not\subseteq \mathbb{N}$ since $-2 \in E$, but $-2 \notin \mathbb{N}$.

Set operations

Definition

The **union** of two sets A and B is the set

$$A \cup B = \{x \mid x \in A \quad \textbf{or} \quad x \in B\}.$$

Definition

The **intersection** of two sets A and B is the set

$$A \cap B = \{x \mid x \in A \quad \textbf{and} \quad x \in B\}.$$

Definition

The **difference** of two sets A and B is the set

$$A \backslash B = \{x \mid x \in A \quad \textbf{and} \quad x \notin B\}.$$

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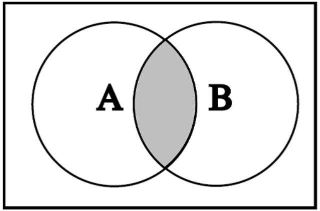
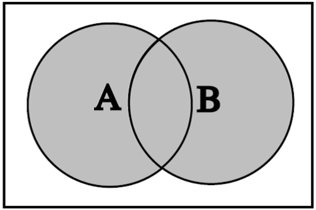
The reason that these constructions are important is that it is typically the case that complicated events described in English words can be broken down into simpler events using these constructions.



$A \cap B$



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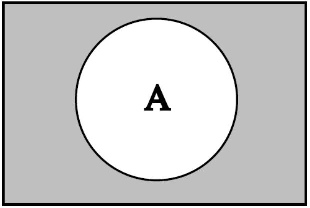

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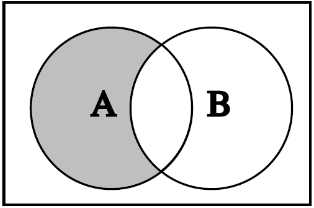

$A \setminus B$

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The operations we can perform with sets:

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Here are some properties that can be easily proved

$$A \cap A = A, \quad A \cup A = A$$
$$A \cap \emptyset = \emptyset, \quad A \cup \emptyset = A, \quad A \cap A^c = \emptyset,$$
$$A \cap B = B \cap A, \quad A \cup B = B \cup A$$
$$(A \cap B) \cap C = A \cap (B \cap C), \quad (A \cup B) \cup C = A \cup (B \cup C),$$
$$A \cup (A \cap B) = A, \quad A \cap (A \cup B) = A,$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$
$$(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c.$$

The last line identities are called **De Morgan's (duality) laws** for sets.

Cartesian product

Definition

Let X and Y be sets. **Cartesian product** or **product** of X and Y, denoted $X \times Y$, is the set of all **ordered pairs** (x, y), where $x \in X$ and $y \in Y$. In other words,

$$X \times Y = \{(x, y) \mid x \in X \textbf{ and } y \in Y\}.$$

Example

Let $X = \{a, b, c\}$ and $Y = \{2, 5\}$. Then,

$$X \times Y = \{(a, 2), (a, 5), (b, 2), (b, 5), (c, 2), (c, 5)\}.$$

Clearly, $(2, b) \notin X \times Y$ and neither $(c, 3) \notin X \times Y$.
On the other hand, $(2, b) \in Y \times X$.

Note that $X \times Y \neq Y \times X$. Also, $X \times X = X^2$.

Naive set theory

All definitions and properties discussed so far constitute the NAIVE SET THEORY.

Naive set theory is a non-formalized theory: uses natural language to talk about sets.

Words **and**, **or**, **if ... then**, **not**, **for every** are NOT subject to rigorous definition.

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The first development of set theory was a naive set theory.

It was created at the end of the 19th century by **Georg Cantor** in order to allow mathematicians to work with infinite sets consistently.

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So, our assumption that $S \in S$ is not true. Thus, the opposite is true.

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Assume $S \notin S$. Therefore, S does not satisfy the property characterizing S, i.e. $x \notin x$ is not satisfied. Thus, the opposite property holds, i.e. $x \in x$.
Thus, $S \in S$. **Contradiction again!**

Barber's paradox

Suppose there is a town with just one male barber; and that every man in the town keeps himself clean-shaven: some by shaving themselves, some by attending the barber.

Barber's paradox

Suppose there is a town with just one male barber; and that every man in the town keeps himself clean-shaven: some by shaving themselves, some by attending the barber.

It seems reasonable to imagine that the barber obeys the following rule:
He shaves all and only those men who do not shave themselves.

Does the barber shave himself?

Asking this, we discover that the situation presented is in fact impossible:

If the barber does not shave himself, he must abide by the rule and shave himself.

If he does shave himself, according to the rule, he will not shave himself.

Berry's paradox

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PARADOX!

Conclusion

Self referencing is not a good idea! Yet.

Russell's paradox solution

In order to solve the Russell's paradox, change the definition of a set with a given property. Instead of:

$$S = \{x \mid P(x)\}$$

define

$$S = \{x \mid x \in \mathcal{U} \text{ and } P(x)\},$$

where $\mathcal{U}$ is some initial given set, called **universal set** (or just Universe).

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Basically, universal set is a set which is so big that it contains all of the mathematical objects that we want to talk about.

Repeat Russel's paradox arguments, but this time with

$$S = \{x \mid x \in \mathcal{U} \text{ and } x \notin x\}$$

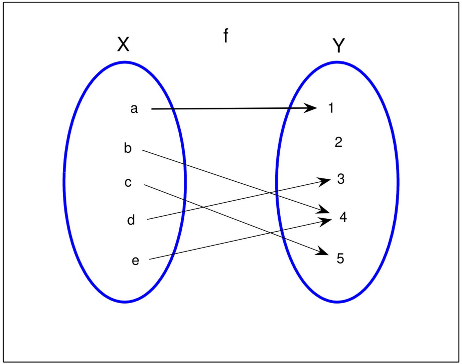
and conclude that there is no paradox!

Functions

Definition

A function $f : X \to Y$ is a relation (law) between two sets X and Y, that relates **every** element of X to **exactly one** element of Y.

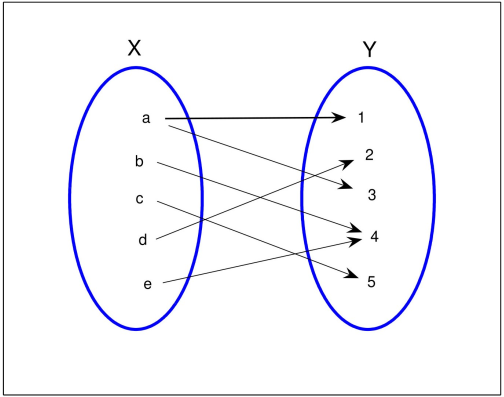
The set X is called the **domain** and the set Y is called the **codomain**.

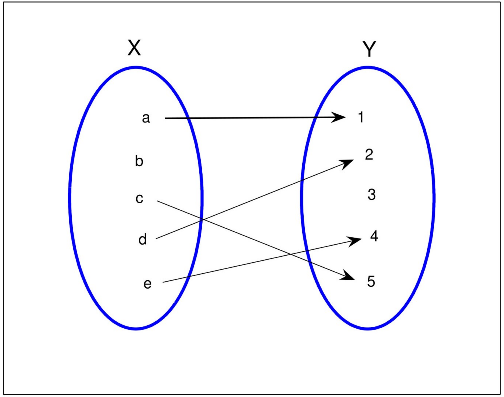


$$f(a) = 1,$$
$$f(b) = 4,$$
$$f(c) = 5,$$
$$f(d) = 3,$$
$$f(e) = 4.$$

Functions

The relations below are NOT functions.





$f(a) = 1$ and $f(a) = 3$

$f(b)$ not defined

Functions

There are several ways to define a function.

Most often it is defined by formulas:

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$$f_3(x, n) = \text{the length } n \text{ sequence } (\underbrace{x, \ldots, x}_{n \text{ copies of } x})$$

Functions

A function can be defined by a table that shows its values or a graph:

α	β	$f_4(\alpha, \beta)$
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F	T	T
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x	$f_5(x)$
1.0	0.7892
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1.3	1.1866
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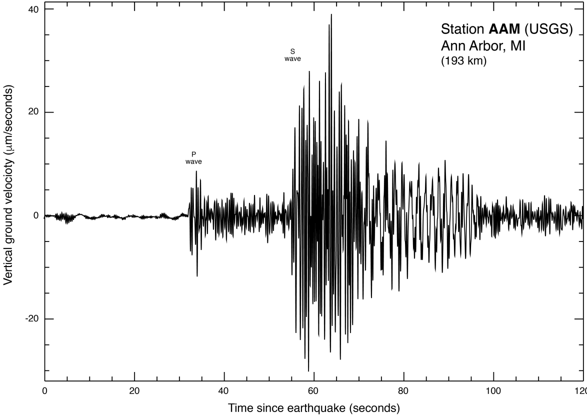

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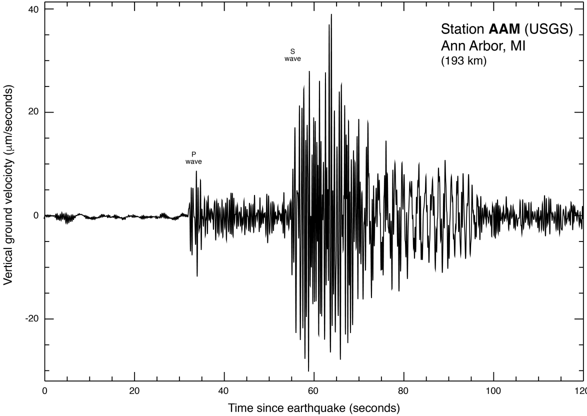

Note that f_4 can also be described by a formula

$$f_4(\alpha, \beta) = \alpha \vee \beta = \alpha \,\mathrm{OR}\, \beta$$

A function can be specified by a procedure/specification for computing its value.

Given a binary string y, define $f_6(y)$ to be the length of a left to right search of the bits in the binary string y until a digit 1 appears:

$$\begin{aligned} f_6(0010) &= 3, \\ f_6(100) &= 1, \\ f_6(00000111010) &= 6, \\ f_6(00000) &\text{ is undefined.} \end{aligned}$$

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Another type of interesting functions for Computer Science are the so-called **recursive functions**.

Bijections or bijective functions

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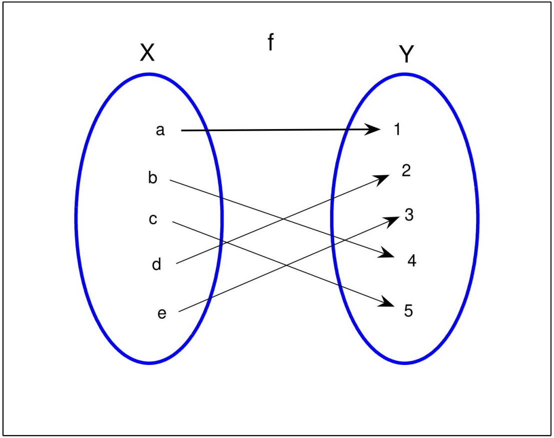
A bijection or bijective function is a function $f : X \to Y$ that maps **exactly** one element of the domain to **each** element of the codomain.



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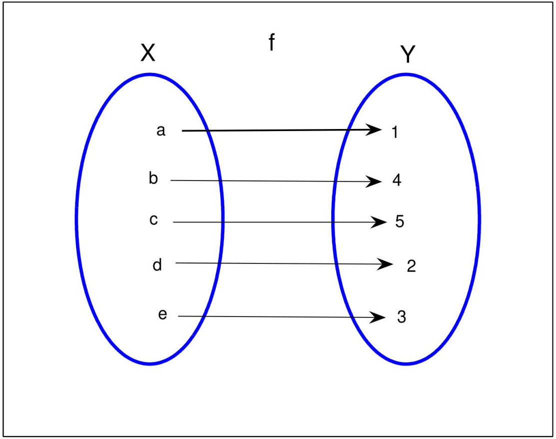
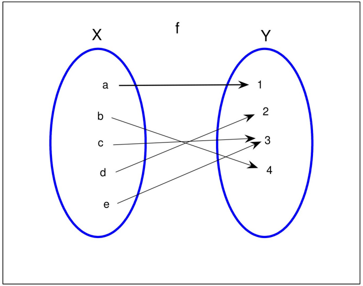


Not bijections



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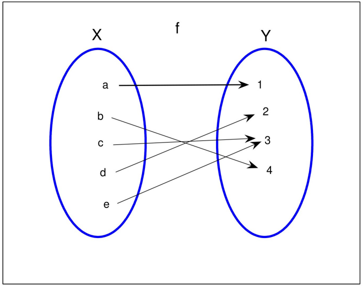

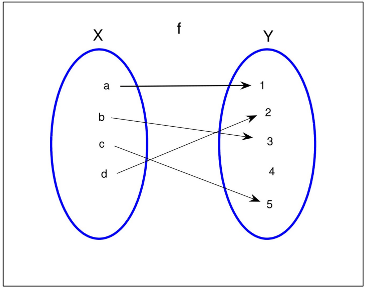

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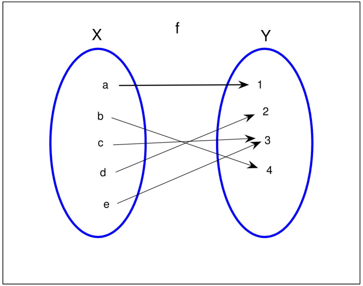
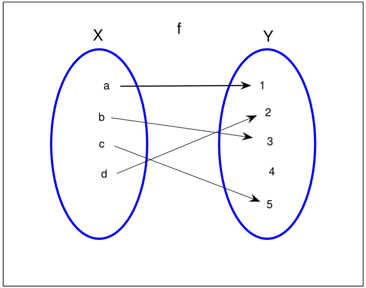
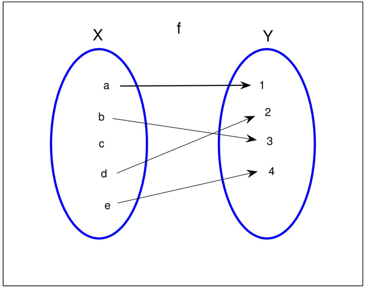


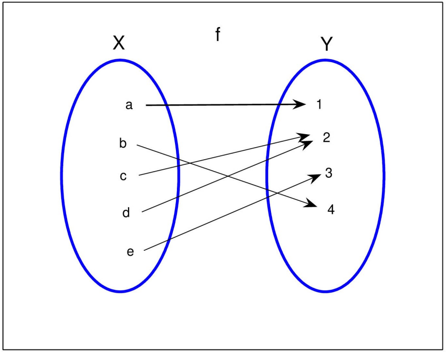




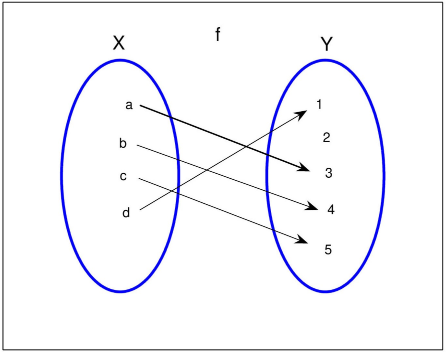
Injections and surjections

A function $f : X \to Y$ is called

- **surjection** if every element of Y is mapped to **at least one** time.
- **injection** if every element of Y is mapped to **at most one** time.
- **bijection** if every element of Y is mapped to **exactly one** time.



Surjection, not injection



Injection, not surjection

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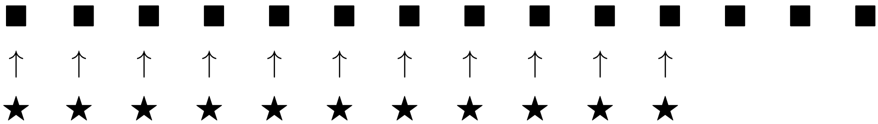

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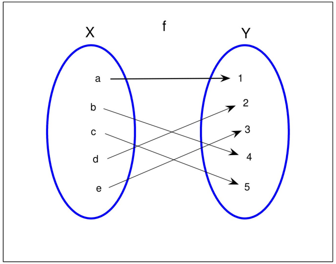
\bigstar & \bigstar & \bigstar & \bigstar & \bigstar & \bigstar & \bigstar & \bigstar & \bigstar & \bigstar & \bigstar & & &
\end{array}$$

Define a function $f : S \to D$ that pairs every star with a (distinct) dot.
Clearly, each dot must be paired with at most one star.



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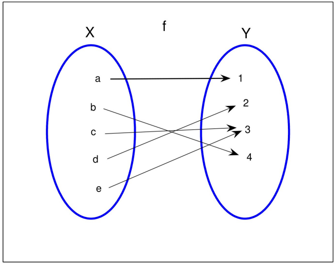
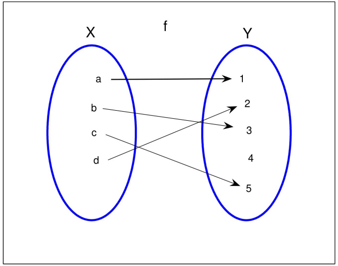
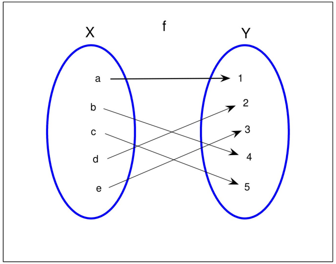


Surjection



Injection

Set Cardinality and functions



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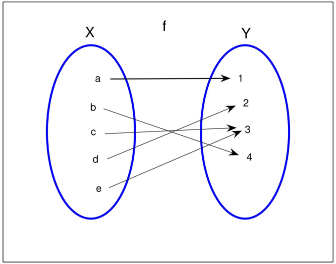

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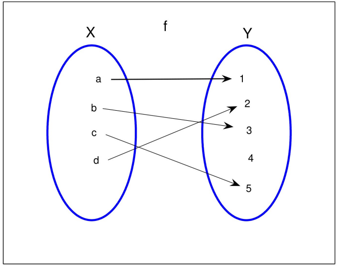

Injection

Theorem (Mapping Rule)

If X and Y are finite sets then:

1. $f : X \to Y$ *is surjection, if and only if* $|X| \geqslant |Y|$.
2. $f : X \to Y$ *is injection, if and only if* $|X| \leqslant |Y|$.
3. $f : X \to Y$ *is bijection, if and only if* $|X| = |Y|$.

But how about Mapping Rule applied to infinite sets?

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In general, what is infinity? How many elements are in infinite sets?

Are all infinite sets of the same size?

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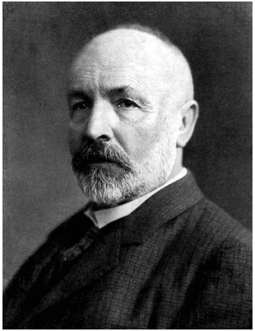

Georg Cantor, 1845–1918

George Cantor extended the Mapping Rule Theorem to infinite sets.

Two infinite sets are having the "same size" if and only if there is a bijection between them.

Mathematical community in 19th century doubted the relevance of Cantor's ideas.

Infinite sets

Theorem (Sroder-Berstein)

For any sets X and Y, if

$$X \text{ surj } Y \quad \text{and} \quad Y \text{ surj } X,$$

then

$$X \text{ bij } Y, \quad (\text{ in other words } |X| = |Y|).$$

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Theorem

For all sets X, Y,

$$X \textit{ surj } Y \quad \textit{OR} \quad Y \textit{ surj } X.$$

Infinite sets

1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\cdots$
$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	
$\bigstar$	$\bigstar$	$\bigstar$	$\bigstar$	$\bigstar$	$\bigstar$	$\bigstar$	$\bigstar$	$\bigstar$	$\bigstar$	$\bigstar$	$\cdots$
A	B	C	D	E	F	G	H	I	J	K	$\cdots$

Infinite sets

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\begin{array}{cccccccccccc}
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Finite Sets versus Infinite Sets

Important: Infinity is different!

A basic property of finite sets that does not carry over to infinite sets is that adding something new makes the set bigger.

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$$|X \cup \{b\}| = |X| + 1.$$

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Lemma

Let X be a set and $b \notin X$. Then set X is infinite if and only if

$$X \;\mathit{bij}\; X \cup \{b\}.$$

Countable sets

Definition

A set X is called **infinitely countable** if and only if $\mathbb{N} \overset{\text{bij}}{\sim} X$.
In other words, if and only if there exists a bijection $f : \mathbb{N} \to X$.

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- Any finite set is countable;
- $\mathbb{N}$ is countable.

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Proposition

If $A \subseteq B$ and set B is countable, then A is also countable.

Countable sets

Proposition

If given two sets $A \subseteq B$, then $|A| \leqslant |B|$.

Questions

Where do we have more elements?

1. $\{0, 1, 2, 3, 4, 5, \ldots\}$ or $\{1, 2, 3, 4, 5, 6, \ldots\}$?

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2. $\{0, 1, 2, 3, 4, 5, \ldots\}$ or $\{0, 2, 4, 6, 8, 10, \ldots\}$?

Countable sets

Proposition

If given two sets $A \subseteq B$, then $|A| \leqslant |B|$.

Questions

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Can Computer Science solve any problem?

Countable sets

Set $\mathbb{N}$:	0	1	2	3	4	5	6	...
Set $\mathbb{Z}^+$:	1	2	3	4	5	6	7	...

Countable sets

$$\begin{array}{lccccccccc} \text{Set } \mathbb{N}: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \dots \\ \text{Set } \mathbb{Z}^+: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots \end{array}$$

There is a bijection $f : \mathbb{N} \to \mathbb{Z}^+$.

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$$\forall n \in \mathbb{N}, \quad f(n) = n + 1.$$

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Beware!

Make sure that this is indeed a bijection!

Countable sets

Set $\mathbb{N}$:		0	1	2	3	4	5	6	7	8	...
Set $\mathbb{Z}$:	...	-4	-3	-2	-1	0	1	2	3	4	...

Countable sets

Set $\mathbb{N}$:		0	1	2	3	4	5	6	7	8	$\dots$
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First, rearrange elements of $\mathbb{Z}$.

Countable sets

Set $\mathbb{N}$:	0	1	2	3	4	5	6	7	8	$\ldots$
Set $\mathbb{Z}$:	0	-1	1	-2	2	-3	3	-4	4	$\ldots$

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Countable sets

Set $\mathbb{N}$:	0	1	2	3	4	5	6	7	8	...
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	...
Set $\mathbb{Z}$:	0	-1	1	-2	2	-3	3	-4	4	...

There is a bijection $f : \mathbb{N} \to \mathbb{Z}$.

$$f(n) = \begin{cases} n/2, & n \text{ is even,} \\ -(n+1)/2, & n \text{ is odd.} \end{cases}$$

Thus, set $\mathbb{Z}$ is countable.

Countable sets

$$\begin{array}{lccccccccccc} \text{Set } \mathbb{N}: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \dots \\ \text{Set } \mathbb{Z}: & 0 & -1 & 1 & -2 & 2 & -3 & 3 & -4 & 4 & \dots \end{array}$$

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Definition

The cardinal number of $\mathbb{N}$ is denoted by $\aleph_0$ (pronounced "alef zero"). $\aleph_0$ is the first so-called **transfinite number**.

$$\aleph_0 = |\mathbb{N}|.$$

Union of countable sets

Countability of a set means that you can list its elements as a sequence:

$$\begin{array}{rcccccccc} \text{Set } \mathbb{N}: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \dots \\ \text{Set } A: & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \dots \end{array}$$

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Proposition

Union of two countable sets is a countable set.

$$
\begin{array}{lcccccccc}
\text{Set } A: & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \ldots \\
\text{Set } B: & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & \ldots
\end{array}
$$

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$$\aleph_0 + \aleph_0 = \aleph_0.$$

Union of countable sets

Proposition

Union of three countable sets is a countable set.

$$
\begin{array}{llllllll}
\text{Set } A: & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \dots \\
\text{Set } B: & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & \dots \\
\text{Set } C: & c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & \dots
\end{array}
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\text{Set } C: & c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & \dots
\end{array}
$$

Proposition

If $A_i,\ i = 1, 2, 3, \dots, n$ are countable sets, then

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

is also a countable set.

Cartesian product

Definition

Given two sets X and Y, we call **cartesian product** or **product set** of X and Y (denoted by $X \times Y$), the set of all **ordered pairs**:

$$X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}.$$

Example

1. $X = \{a, b\}$
   $X \times \mathbb{N} = \{(a, 0), (b, 0), (a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3), \ldots\}$

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2. $\mathbb{N} \times \mathbb{N} = \{(n, m) \mid n, m \in \mathbb{N}\}$;
   $\mathbb{N} \times \mathbb{N} = \{(0, 0), (0, 1), (0, 2), \dots, (1, 0), (1, 1), (1, 2), \dots, \}$
   $(3, 19) \in \mathbb{N} \times \mathbb{N}$,
   $(4521, 178) \in \mathbb{N} \times \mathbb{N}$,
   $(10^6, 2^{2018}) \in \mathbb{N} \times \mathbb{N}$.

Countable sets

We can add to a countable set a finite number of elements and it will remain countable.

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How about $\mathbb{N} \times \mathbb{N}$? Is it a countable set?
Arrange elements of $\mathbb{N} \times \mathbb{N}$ in a matrix form:

$$\begin{array}{ccccccc}
(0,0) & (0,1) & (0,2) & (0,3) & (0,4) & (0,5) & \cdots \\
(1,0) & (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,0) & (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
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How about $\mathbb{N} \times \mathbb{N}$? Is it a countable set?
Arrange elements of $\mathbb{N} \times \mathbb{N}$ in a matrix form:

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Clearly, $\mathbb{N} \times \mathbb{N}$ is a countable set.

Similarly, if A and B are two countable sets, then $A \times B$ is also a countable set.

Countable Sets

So far, we have shown that

$$\aleph_0 = |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Z}^+| = |\mathbb{N} \times \mathbb{N}|.$$

How about $\mathbb{Q}$?

How about $\mathbb{Q}$? Let's start with $\mathbb{Q}^+$.

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Arrange positive rational numbers $\mathbb{Q}^+$ in matrix form and count them:

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\begin{matrix}
1/1 & 2/1 & 3/1 & 4/1 & 5/1 & 6/1 & 7/1 & \cdots \\
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$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Therefore, $|\mathbb{Q}^+| = \aleph_0$ and

$$|\mathbb{Q}| = |\mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+| = \aleph_0.$$

Countable Union of Countable Sets

Proposition

A countable union of countable sets is a countable set.
If $A_i, \;\; i = 1, 2, 3, \ldots$ are countable sets, then

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots \cup A_n \cup \ldots$$

is also a countable set.

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is also a countable set.

Set A_1:	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	$\dots$
Set A_2:	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	a_{27}	$\dots$
Set A_3:	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}	a_{37}	$\dots$
Set A_4:	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	a_{46}	a_{47}	$\dots$
Set A_5:	a_{51}	a_{52}	a_{53}	a_{54}	a_{55}	a_{56}	a_{57}	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

Lecture Outline

- Naive Set Theory:
  - Sets, Equality of sets, Subsets (proper subsets), Power set;
  - Set Operations, Properties, Cartesian product;
  - Paradoxes of naive set theory.
- Functions (bijections, injections, surjections);
- Cardinal number of a set;
- Mapping Rule;
- Infinite sets;
- Countable Sets;
- Union of countable sets;
- Cartesian product of countable sets;
- Countability of rational numbers.