Area of a Parallelogram

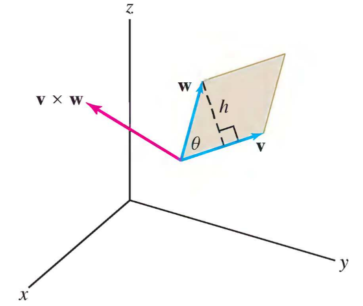
- Consider the parallelogram $\mathcal{P}$ spanned by nonzero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ with a common basepoint.

$\mathcal{P}$ has:

  - base $b = \|\boldsymbol{v}\|$;
  - height $h = \|\boldsymbol{w}\| \sin\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

Therefore, $\mathcal{P}$ has area

$$A = bh = \|\boldsymbol{v}\|\|\boldsymbol{w}\| \sin\theta = \|\boldsymbol{v} \times \boldsymbol{w}\|.$$



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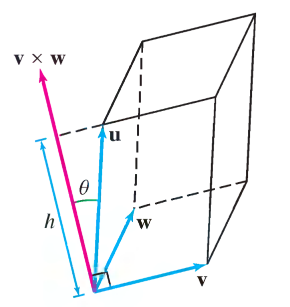
Volume of a Parallelepiped

- Consider the parallelepiped $\boldsymbol{P}$ spanned by three nonzero vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

The base of $\boldsymbol{P}$ is the parallelogram spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$. So the area of the base is $\|\boldsymbol{v} \times \boldsymbol{w}\|$.
The height is $h = \|\boldsymbol{u}\| \cdot |\cos\theta|$, where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v} \times \boldsymbol{w}$.



Therefore,

$$\text{Volume of } \boldsymbol{P} = (\text{area of base})(\text{height}) = \|\boldsymbol{v} \times \boldsymbol{w}\| \cdot \|\boldsymbol{u}\| \cdot |\cos\theta|.$$

Thus,

$$\text{Volume of } \boldsymbol{P} = |\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})|.$$

The quantity $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$ is called the ~~**vector**~~ **triple product**.
scalar

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mixt product of u,v,w

$\vec{u} \cdot (\vec{v} \times \vec{w})$

scalar triple product

mixt triple product

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The ~~Vector~~ Triple Product

- The scalar triple product $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$ can be expressed as a determinant.

  Suppose $\boldsymbol{u} = \langle a_1, a_2, a_3 \rangle$, $\boldsymbol{v} = \langle b_1, b_2, b_3 \rangle$ and $\boldsymbol{w} = \langle c_1, c_2, c_3 \rangle$. Then we have:

$$
\begin{aligned}
\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) &= \boldsymbol{u} \cdot \left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \boldsymbol{k} \right) \\
&= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= \det \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix}.
\end{aligned}
$$

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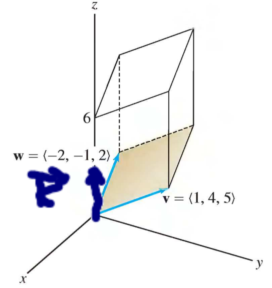
Example

- Let $\boldsymbol{v} = \langle 1, 4, 5 \rangle$ and $\boldsymbol{w} = \langle -2, -1, 2 \rangle$. Calculate:
  (a) The area A of the parallelogram spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$.
  (b) The volume V of the parallelepiped in the figure.



Both the area and the volume require computing the cross product

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} 4 & 5 \\ -1 & 2 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} 1 & 5 \\ -2 & 2 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} 1 & 4 \\ -2 & -1 \end{vmatrix} \boldsymbol{k} = \langle 13, -12, 7 \rangle.$$

(a) The area of the parallelogram spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$ is $A = \|\boldsymbol{v} \times \boldsymbol{w}\| = \sqrt{13^2 + (-12)^2 + 7^2} = \sqrt{362}$.

(b) The vertical leg of the parallelepiped is the vector $6\boldsymbol{k}$. So $V = |(6\boldsymbol{k}) \cdot (\boldsymbol{v} \times \boldsymbol{w})| = |\langle 0, 0, 6 \rangle \cdot \langle 13, -12, 7 \rangle| = 6(7) = 42$.

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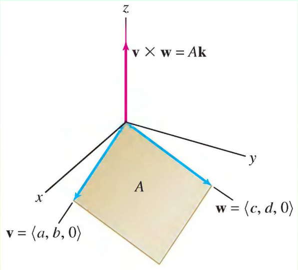
Parallelograms on the Plane

$\mathbb{R}^2$

$\mathbb{R}^3$

- We can compute the area A of the parallelogram spanned by vectors $\boldsymbol{v} = \langle a, b \rangle$ and $\boldsymbol{w} = \langle c, d \rangle$ by regarding $\boldsymbol{v}$ and $\boldsymbol{w}$ as vectors in space with zero component in the z-direction.



We write $\boldsymbol{v} = \langle a, b, 0 \rangle$ and $\boldsymbol{w} = \langle c, d, 0 \rangle$. The cross product $\boldsymbol{v} \times \boldsymbol{w}$ is a vector pointing in the z-direction:

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a & b & 0 \\ c & d & 0 \end{vmatrix} = \begin{vmatrix} b & 0 \\ d & 0 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a & b \\ c & d \end{vmatrix} \boldsymbol{k} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \boldsymbol{k}.$$

Thus, the parallelogram spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$ has area

$$A = \|\boldsymbol{v} \times \boldsymbol{w}\| = \left| \det \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix} \right|.$$, v, w, Oxy($\mathbb{R}^2$)

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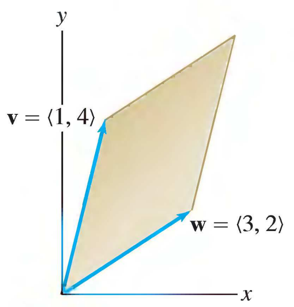
Example

- Compute the area A of the parallelogram in the figure



We have

determinant

$$A = \left|\det\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix}\right| = \left|\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix}\right| = |1 \cdot 2 - 3 \cdot 4| = |-10| = 10.$$

absolute value

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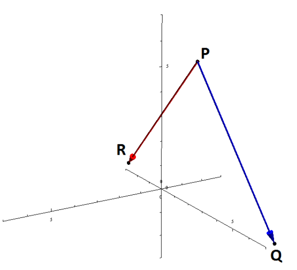
Area of Triangle

- Find the area of a triangle with vertices

$$P = (1, 4, 6), Q = (-2, 5, -1), R = (1, -1, 1).$$

This triangle has sides $\overrightarrow{PQ} = \langle -3, 1, -7 \rangle$ and $\overrightarrow{PR} = \langle 0, -5, -5 \rangle$.

$\vec{PQ} \times \vec{PR} =$



$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

Its area is

$$A = \frac{1}{2} \|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \frac{1}{2} \|\langle -40, -15, 15 \rangle\| = \frac{1}{2} 5\sqrt{82}.$$

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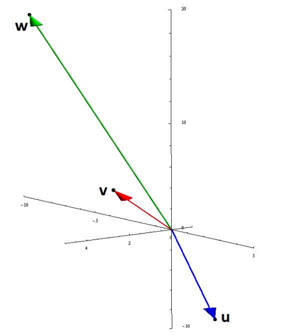
An Example of Co-Planar Vectors

- Show that the vectors $\boldsymbol{u} = \langle 1, 4, -7 \rangle$, $\boldsymbol{v} = \langle 2, -1, 4 \rangle$ and $\boldsymbol{w} = \langle 0, -9, 18 \rangle$ are co-planar.

  We show that the scalar triple product

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = 0.$$



$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} + (-7) \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix} = 0.$$

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Stewart, p.812, Theorem 11, prop.5 and 6

$\vec{a}\ \vec{b}\ \vec{c}$

$\vec{a}\cdot(\vec{b}\times\vec{c})$

$(\vec{a}\times\vec{b})\cdot\vec{c}$

scalar triple product

mixt product

$\vec{a}\times(\vec{b}\times\vec{c})$

vector triple product

double vector product

Stewart, p.813

Stewart,p.814, ex. 18

$\vec{a}\times(\vec{b}\times\vec{c})\neq(\vec{a}\times\vec{b})\times\vec{c}$

NOT associative

Stewart, p.816, ex.50,51