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Introduction to Complex Analysis

Subsection 1

Exponential Functions

Complex Exponential Function

- We repeat the definition of the complex exponential function:

Definition (Complex Exponential Function)

The function e^z defined by

$$e^z = e^x \cos y + ie^x \sin y$$

is called the **complex exponential function**.

- This function agrees with the real exponential function when z is real: in fact, if $z = x + 0i$,
$$e^{x+0i} = e^x(\cos 0 + i \sin 0) = e^x(1 + i \cdot 0) = e^x.$$
- The complex exponential function also shares important differential properties of the real exponential function:
  - e^x is differentiable everywhere;
  - $\frac{d}{dx} e^x = e^x$, for all x.

Analyticity of e^z

Theorem (Analyticity of e^z)

The exponential function e^z is entire and its derivative is $\frac{d}{dz}e^z = e^z$.

- We use the criterion based on the real and imaginary parts. $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$ are continuous real functions and have continuous first-order partial derivatives, for all (x,y). In addition, the Cauchy-Riemann equations in u and v are easily verified: $\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$. Therefore, the exponential function e^z is entire. The derivative of an analytic function f is given by $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$. So the derivative of e^z is: $\frac{d}{dz}e^z = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = e^z$.
- Since the real and imaginary parts of an analytic function are harmonic conjugates, we can show the only entire function f that agrees with the real exponential function e^x for real input and that satisfies $f'(z) = f(z)$ is the complex exponential function e^z.

Derivatives of Exponential Functions

- Find the derivative of each of the following functions:
  - (a) $iz^4(z^2 - e^z)$
  - (b) $e^{z^2-(1+i)z+3}$
- We use the various rules for complex derivatives:
  - (a)

$$\begin{aligned} \frac{d}{dz}(iz^4(z^2 - e^z)) &= \frac{d}{dz}(iz^4)(z^2 - e^z) + iz^4\frac{d}{dz}(z^2 - e^z) \\ &= 4iz^3(z^2 - e^z) + iz^4(2z - e^z) \\ &= 6iz^5 - iz^4e^z - 4iz^3e^z. \end{aligned}$$

  - (b)

$$\begin{aligned} \frac{d}{dz}(e^{z^2-(1+i)z+3}) &= e^{z^2-(1+i)z+3} \cdot \frac{d}{dz}(z^2 - (1+i)z + 3) \\ &= e^{z^2-(1+i)z+3} \cdot (2z - 1 - i). \end{aligned}$$

Modulus, Argument, and Conjugate

- If we express the complex number $w = e^z$ in polar form:
$$w = e^x \cos y + ie^x \sin y = r(\cos\theta + i\sin\theta),$$
we see that $r = e^x$ and $\theta = y + 2n\pi$, for $n = 0, \pm 1, \pm 2, \ldots$.
- Because r is the modulus and θ is an argument of w, we have:
$$|e^z| = e^x, \quad \arg(e^z) = y + 2n\pi, n = 0, \pm 1, \pm 2, \ldots.$$
- We know from calculus that $e^x > 0$, for all real x, whence $|e^z| > 0$. This implies that $e^z \neq 0$, for all complex z, i.e., $w = 0$ is not in the range of $w = e^z$.
- Note, however, that e^z may be a negative real number: E.g., if $z = \pi i$, then $e^{\pi i}$ is real and $e^{\pi i} < 0$.
- A formula for the conjugate of the complex exponential e^z is found using the even-odd properties of the real cosine and sine functions: $\overline{e^z} = e^x \cos y - ie^x \sin y = e^x \cos(-y) + ie^x \sin(-y) = e^{x-iy} = e^{\overline{z}}$. Therefore, for all complex z, $\overline{e^z} = e^{\overline{z}}$.

Algebraic Properties

Theorem (Algebraic Properties of e^z)

If z_1 and z_2 are complex numbers, then:

(i) $e^0 = 1$;

(ii) $e^{z_1} e^{z_2} = e^{z_1 + z_2}$;

(iii) $\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$.

(iv) $(e^{z_1})^n = e^{n z_1}$, $n = 0, \pm 1, \pm 2, \ldots$.

(i) Clearly, $e^{0+0i} = e^0(\cos 0 + i \sin 0) = 1$.

(ii) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Hence $e^{z_1} e^{z_2} = (e^{x_1} \cos y_1 + ie^{x_1} \sin y_1)(e^{x_2} \cos y_2 + ie^{x_2} \sin y_2) = e^{x_1+x_2}(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + ie^{x_1+x_2}(\sin y_1 \cos y_2 + \cos y_1 \sin y_2)$. Using the addition formulas for the real cosine and sine functions, we get $e^{z_1} e^{z_2} = e^{x_1+x_2} \cos(y_1 + y_2) + ie^{x_1+x_2} \sin(y_1 + y_2)$. The right-hand side is $e^{z_1+z_2}$.

- The proofs of (iii) and (iv) are similar.

Periodicity

- The most striking difference between the real and complex exponential functions is the periodicity of e^z.
- We say that a complex function f is **periodic** with **period** T if

$$f(z+T) = f(z), \quad \text{for all complex } z.$$

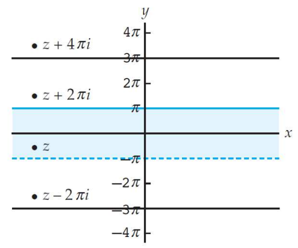
- The real exponential function is not periodic, but the complex exponential function is because it is defined using the real cosine and sine functions, which are periodic.
- We have

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z(\cos 2\pi + i \sin 2\pi) = e^z.$$

The complex exponential function e^z is periodic with a pure imaginary period $2\pi i$.

That is, for $f(z) = e^z$, we have $f(z+2\pi i) = f(z)$, for all z.

The Fundamental Region

- We saw that, for all values of z, $e^{z+2\pi i} = e^z$.
- Thus, we also have $e^{(z+2\pi i)+2\pi i} = e^{z+2\pi i} = e^z$.
- By repeating, we find that $e^{z+2n\pi i} = e^z$, for $n = 0, \pm 1, \pm 2, \ldots$.
- This means that $-2\pi i, 4\pi i, 6\pi i$, and so on, are also periods of e^z.
- If e^z maps the point z onto the point w, then it also maps the points $z \pm 2\pi i, z \pm 4\pi i, z \pm 6\pi i$, and so on, onto the point w.
- Thus, e^z is not one-to-one, and all values e^z are assumed in any infinite horizontal strip of width 2π in the z-plane. That is, all values are assumed in $-\infty < x < \infty, y_0 < y \leq y_0 + 2\pi$, y_0 a real constant.
- 

  The infinite horizontal strip defined by: $-\infty < x < \infty$, $-\pi < y \leq \pi$, is called the **fundamental region** of the complex exponential function.

The Exponential Mapping

- Since all values of the complex exponential function e^z are assumed in the fundamental region, the image of this region under the mapping $w = e^z$ is the same as the image of the entire complex plane.
- Note that this region consists of the collection of vertical line segments $z(t) = a + it$, $-\pi < t \leq \pi$, where a is any real number.
- The image of $z(t) = a + it$, $-\pi < t \leq \pi$, under $w = e^z$ is parametrized by $w(t) = e^{z(t)} = e^{a+it} = e^a e^{it}$, $-\pi < t \leq \pi$, and, thus, $w(t)$ defines a circle centered at the origin with radius e^a.
- Because a can be any real number, the radius e^a of this circle can be any nonzero positive real number.
- Thus, the image of the fundamental region under the exponential mapping consists of the collection of all circles centered at the origin with nonzero radius, i.e., the image of the fundamental region $-\infty < x < \infty$, $-\pi < y \leq \pi$, under $w = e^z$ is the set of all complex w with $w \neq 0$, or, equivalently, the set $|w| > 0$.

Using Horizontal Lines to Determine the Image

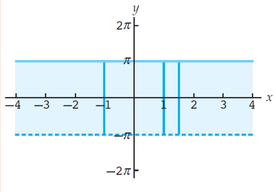
- The image can also be found by using horizontal lines in the fundamental region.
- Consider the horizontal line $y = b$. This line can be parametrized by $z(t) = t + ib$, $-\infty < t < \infty$. So its image under $w = e^z$ is given by $w(t) = e^{z(t)} = e^{t+ib} = e^t e^{ib}$, $-\infty < t < \infty$.
- Defining a new parameter $s = e^t$, and observing that $0 < s < \infty$, the image is given by $W(s) = e^{ib}s$, $0 < s < \infty$, which, is the set consisting of all points $w \neq 0$ in the ray emanating from the origin and containing the point $e^{ib} = \cos b + i \sin b$.
- Thus, the image of the horizontal line $y = b$ under the mapping $w = e^z$ is the set of all points $w \neq 0$ in the ray emanating from the origin and making an angle of b radians with the positive u-axis, i.e., the set of all w, satisfying $\arg(w) = b$.

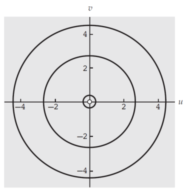
Exponential Mapping Properties

Exponential Mapping Properties

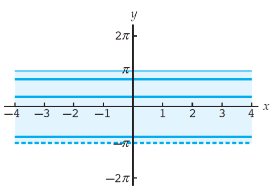
(i) $w = e^z$ maps the fundamental region $-\infty < x < \infty$, $-\pi < y \leq \pi$, onto the set $|w| > 0$.

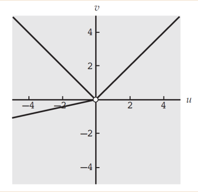
(ii) $w = e^z$ maps the vertical line segment $x = a$, $-\pi < y \leq \pi$, onto the circle $|w| = e^a$.





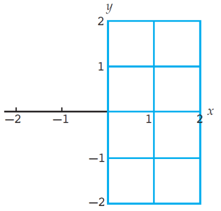
(iii) $w = e^z$ maps the horizontal line $y = b$, $-\infty < x < \infty$, onto the ray $\arg(w) = b$.

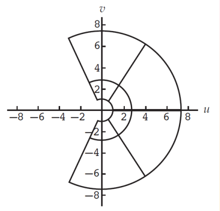




Exponential Mapping of a Grid

- Find the image of the grid shown below left under $w = e^z$.





The grid consists of the vertical line segments $x = 0, 1, 2$, $-2 \le y \le 2$, and the horizontal line segments $y = -2, -1, 0, 1, 2$, $0 \le x \le 2$.

Using the properties of the exponential mapping, we have that:

- the image of the vertical line segment $x = 0$, $-2 \le y \le 2$, is the circular arc $|w| = e^0 = 1$, $-2 \le \arg(w) \le 2$.
- The segments $x = 1$ and $x = 2$, $-2 \le y \le 2$, map onto the arcs $|w| = e$ and $|w| = e^2$, $-2 \le \arg(w) \le 2$, respectively.
- The horizontal segment $y = 0$, $0 \le x \le 2$, maps onto the portion of the ray emanating from the origin defined by $\arg(w) = 0$, $1 \le |w| \le e^2$.
- The segments $y = -2, -1, 1, 2$ map onto the segments defined by $\arg(\mathrm{w}) = -2$, $\arg(w) = -1$, $\arg(w) = 1$, $\arg(w) = 2$, $1 \le |w| \le e^2$.

The end result is shown on the right.

Subsection 2

Logarithmic Functions

Complex Logarithm

- In real analysis, the natural logarithm function $\ln x$ is often defined as an inverse function of the real exponential function e^x. We use $\log_e x$ to represent the real logarithmic function.
- The situation is different in complex analysis because the complex exponential function e^z is not a one-to-one function on its domain $\mathbb{C}$.
- Given a fixed nonzero complex number z, the equation $e^w = z$ has infinitely many solutions e.g., $\frac{1}{2}\pi i$, $\frac{5}{2}\pi i$, and $-\frac{3}{2}\pi i$ are all solutions to $e^w = i$.
- In general, if $w = u + iv$ is a solution of $e^w = z$, then $|e^w| = |z|$ and $\arg(e^w) = \arg(z)$. Thus, $e^u = |z|$ and $v = \arg(z)$, or, equivalently, $u = \log_e |z|$ and $v = \arg(z)$. Therefore, given a nonzero complex number z we have shown that: if $e^w = z$, then $w = \log_e |z| + i\arg(z)$.
- This set of values defines a multiple-valued function $w = G(z)$, called the **complex logarithm** of z and denoted by $\ln z$.

Definition of the Complex Logarithmic Function

Definition (Complex Logarithm)

The multiple-valued function $\ln z$ defined by:

$$\ln z = \log_e |z| + i\arg(z)$$

is called the **complex logarithm**.

- The notation $\ln z$ will always be used to denote the multiple valued complex logarithm.
- By switching to exponential notation $z = re^{i\theta}$, we obtain the following alternative description of the complex logarithm:

$$\ln z = \log_e r + i(\theta + 2n\pi), \;\; n = 0, \pm 1, \pm 2, \ldots.$$

- The complex logarithm can be used to find all solutions to the exponential equation $e^w = z$, when z is a nonzero complex number.

Solving Exponential Equations I

- Find all complex solutions to the equation $e^w = i$.

  For each equation $e^w = z$, the set of solutions is given by $w = \ln z$, where $w = \log_e |z| + i\arg(z)$. For $z = i$, we have $|z| = 1$ and $\arg(z) = \frac{\pi}{2} + 2n\pi$. Thus, we get $w = \ln i = \log_e 1 + i(\frac{\pi}{2} + 2n\pi)$, whence

$$w = \frac{(4n+1)\pi}{2} i, \ n = 0, \pm 1, \pm 2, \ldots.$$

  Therefore, each of the values: $w = \ldots, -\frac{7\pi}{2} i, -\frac{3\pi}{2} i, \frac{\pi}{2} i, \frac{5\pi}{2} i, \ldots$ satisfies the equation $e^w = i$.

Solving Exponential Equations II

- Find all complex solutions to the equation $e^w = 1 + i$ and to the equation $e^w = -2$.
- For $z = 1 + i$, we have $|z| = \sqrt{2}$ and $\arg(z) = \frac{\pi}{4} + 2n\pi$. Thus, we get

$$\begin{aligned} w &= \ln(1+i) = \log_e \sqrt{2} + i(\tfrac{\pi}{4} + 2n\pi) \\ &= \tfrac{1}{2}\log_e 2 + \tfrac{(8n+1)\pi}{4} i, \;\; n = 0, \pm 1, \pm 2, \ldots. \end{aligned}$$

- Since $z = -2$, we have $|z| = 2$ and $\arg(z) = \pi + 2n\pi$. Thus, $w = \ln(-2) = \log_e 2 + i(\pi + 2n\pi)$. That is,

$$w = \log_e 2 + (2n+1)\pi i, \;\; n = 0, \pm 1, \pm 2, \ldots.$$

Logarithmic Identities

- Complex logarithm satisfies the following identities, which are analogous to identities for the real logarithm:

Theorem (Algebraic Properties of $\ln z$)

If z_1 and z_2 are nonzero complex numbers and n is an integer, then

(i) $\ln(z_1 z_2) = \ln z_1 + \ln z_2$;

(ii) $\ln \frac{z_1}{z_2} = \ln z_1 - \ln z_2$;

(iii) $\ln z_1^n = n \ln z_1$.

- $\ln z_1 + \ln z_2 = \log_e |z_1| + i\arg(z_1) + \log_e |z_2| + i\arg(z_2) = \log_e |z_1| + \log_e |z_2| + i(\arg(z_1) + \arg(z_2))$.
  The real logarithm satisfies $\log_e a + \log_e b = \log_e (ab)$, for $a > 0$ and $b > 0$, so $\log_e |z_1 z_2| = \log_e |z_1| + \log_e |z_2|$. Also, $\arg(z_1) + \arg(z_2) = \arg(z_1 z_2)$. Therefore, $\ln z_1 + \ln z_2 = \log_e |z_1 z_2| + i\arg(z_1 z_2) = \ln(z_1 z_2)$.
- Parts (ii) and (iii) are similar.

Principal Value of Complex Logarithm

- The complex logarithm of a positive real has infinitely many values.
- Example: $\ln 5$ is the set of values $\log_e 5 + 2n\pi i$, where n is any integer, whereas $\log_e 5$ has a single value $\log_e 5 = 1.6094$. The unique value of $\ln 5$ corresponding to $n = 0$ is the same as $\log_e 5$.
- In general, this value of the complex logarithm is called the **principal value of the complex logarithm** since it is found by using the principal argument $\mathrm{Arg}(z)$ in place of the argument $\arg(z)$.
- We denote the principal value of the logarithm by the symbol $\mathrm{Ln}z$, which, thus, defines a function, whereas $\ln z$ is multi-valued.

Definition (Principal Value of the Complex Logarithm)

The complex function $\mathrm{Ln}z$ defined by:

$$\mathrm{Ln}z = \log_e |z| + i\mathrm{Arg}(z)$$

is called the **principal value of the complex logarithm**.

Computing the Principal Value of the Complex Logarithm

- Compute the principal value of the complex logarithm $\mathrm{Ln} z$ for

$$\text{(a) } z = i \quad \text{(b) } z = 1 + i \quad \text{(c) } z = -2$$

(a) For $z = i$, we have $|z| = 1$ and $\mathrm{Arg}(z) = \frac{\pi}{2}$. So we get

$$\mathrm{Ln} i = \log_e 1 + \frac{\pi}{2} i = \frac{\pi}{2} i.$$

(b) For $z = 1 + i$, we have $|z| = \sqrt{2}$ and $\mathrm{Arg}(z) = \frac{\pi}{4}$. Thus,

$$\mathrm{Ln}(1 + i) = \log_e \sqrt{2} + \frac{\pi}{4} i = \frac{1}{2} \log_e 2 + \frac{\pi}{4} i.$$

(c) For $z = -2$, we have $|z| = 2$ and $\mathrm{Arg}(z) = \pi$, whence

$$\mathrm{Ln}(-2) = \log_e 2 + \pi i.$$

- Warning! The algebraic identities for the complex logarithm are not necessarily satisfied by the principal value of the complex logarithm.

Lnz as an Inverse Function

- Because Lnz is one of the values of the complex logarithm $\ln z$, it follows that: $e^{\mathrm{Ln} z} = z$, for all $z \neq 0$.
- This suggests that the logarithmic function Lnz is an inverse function of e^z.
- Because the complex exponential function is not one-to-one on its domain, this statement is not accurate.
- The relationship between these functions is similar to the relationship between the squaring function z^2 and the principal square root function $z^{1/2} = \sqrt{|z|}e^{i\mathrm{Arg}(z)/2}$.
- The exponential function must first be restricted to a domain on which it is one-to-one in order to have a well-defined inverse function.
- In fact, e^z is a one-to-one function on the fundamental region $-\infty < x < \infty$, $-\pi < y \leq \pi$.

Lnz as an Inverse Function (Cont'd)

- We show that if the domain of e^z is restricted to the fundamental region, then Lnz is its inverse function.

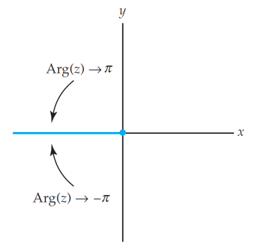
  Consider a point $z = x + iy$, $-\infty < x < \infty$, $-\pi < y \leq \pi$. We have $|e^z| = e^x$ and $\arg(e^z) = y + 2n\pi$, n an integer. Thus, y is an argument of e^z. Since z is in the fundamental region, we also have $-\pi < y \leq \pi$, whence y is the principal argument of e^z, i.e., $\mathrm{Arg}(e^z) = y$. In addition, for the real logarithm we have $\log_e e^x = x$, and so $\mathrm{Ln} e^z = \log_e |e^z| + i\mathrm{Arg}(e^z) = \log_e e^x + iy = x + iy$. Thus, we have shown that $\mathrm{Ln} e^z = z$, if $-\infty < x < \infty$ and $-\pi < y \leq \pi$.

Lnz as an Inverse Function of e^z

If the complex exponential $f(z) = e^z$ is defined on the fundamental region $-\infty < x < \infty$, $-\pi < y \leq \pi$, then f is one-to-one and the inverse function of f is the principal value of the complex logarithm $f^{-1}(z) = \mathrm{Ln} z$.

Discontinuities of $\mathrm{Ln}\, z$

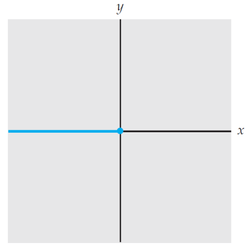
- The principal value of the complex logarithm $\mathrm{Ln}\, z$ is discontinuous at $z = 0$ since this function is not defined there.
- $\mathrm{Ln}\, z$ turns out to also be discontinuous at every point on the negative real axis.
- This may be intuitively clear since the value of $\mathrm{Ln}\, z$ for a point z near the negative x-axis in the second quadrant has imaginary part close to π, whereas the value of a nearby point in the third quadrant has imaginary part close to $-\pi$.



- The function $\mathrm{Ln}\, z$ is, however, continuous on the set consisting of the complex plane excluding the non-positive real axis.

Continuity

- Recall that a complex function $f(z) = u(x, y) + iv(x, y)$ is continuous at a point $z = x + iy$ if and only if both u and v are continuous real functions at (x, y).
- The real and imaginary parts of $\mathrm{Ln} z$ are $u(x, y) = \log_e |z| = \log_e \sqrt{x^2 + y^2}$ and $v(x, y) = \mathrm{Arg}(z)$, respectively.
- From calculus, we know that the function $u(x, y) = \log_e \sqrt{x^2 + y^2}$ is continuous at all points in the plane except $(0, 0)$ and the function $v(x, y) = \mathrm{Arg}(z)$ is continuous on $|z| > 0$, $-\pi < \arg(z) < \pi$.
- Therefore, it follows that $\mathrm{Ln} z$ is a continuous function on the domain $|z| > 0$, $-\pi < \arg(z) < \pi$, i.e., f_1 defined by: $f_1(z) = \log_e r + i\theta$ is continuous on the domain where $r = |z| > 0$ and $-\pi < \theta = \arg(z) < \pi$.



Analyticity

- Since the function f_1 agrees with the principal value of the complex logarithm $\mathrm{Ln} z$ where they are both defined, it follows that f_1 assigns to the input z one of the values of the multiple-valued function $F(z) = \ln z$.
- I.e., we have shown that the function f_1 is a branch of the multiple-valued function $F(z) = \ln z$.
- This branch is called the **principal branch of the complex logarithm**. The nonpositive real axis is a branch cut for f_1 and the point $z = 0$ is a branch point.
- The branch f_1 is an analytic function on its domain:

Theorem (Analyticity of the Principal Branch of $\ln z$)

The principal branch f_1 of the complex logarithm is an analytic function and its derivative is given by: $f_1'(z) = \dfrac{1}{z}$.

- We prove that f_1 is analytic by using polar coordinates.

Analyticity (Proof)

- Because f_1 is defined on the domain $r > 0$ and $-\pi < \theta < \pi$, if z is a point in this domain, then we can write $z = re^{i\theta}$, with $-\pi < \theta < \pi$. Since the real and imaginary parts of f_1 are $u(r,\theta) = \log_e r$ and $v(r,\theta) = \theta$, respectively, we find that: $\frac{\partial u}{\partial r} = \frac{1}{r}$, $\frac{\partial v}{\partial \theta} = 1$, $\frac{\partial v}{\partial r} = 0$, and $\frac{\partial u}{\partial \theta} = 0$. Thus, u and v satisfy the Cauchy-Riemann equations in polar coordinates $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$. Because u, v, and the first partial derivatives of u and v are continuous at all points in the domain, it follows that f_1 is analytic in this domain. In addition, the derivative of f_1 is given by: $f_1'(z) = e^{-i\theta}(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}) = \frac{1}{re^{i\theta}} = \frac{1}{z}$.
- Because $f_1(z) = \text{Ln}z$, for each point z in the domain, it follows that $\text{Ln}z$ is differentiable in this domain, and that its derivative is given by f_1'. That is, if $|z| > 0$ and $-\pi < \arg(z) < \pi$ then:

$$\frac{d}{dz}\text{Ln}z = \frac{1}{z}.$$

Derivatives of Logarithmic Functions I

- Find the derivatives of the function $z\mathrm{Ln}z$ in an appropriate domain.

  The function $z\mathrm{Ln}z$ is differentiable at all points where both of the functions z and $\mathrm{Ln}z$ are differentiable. Because z is entire and $\mathrm{Ln}z$ is differentiable on the domain $|z| > 0, -\pi < \arg(z) < \pi$, $z\mathrm{Ln}z$ is differentiable on the domain defined by $|z| > 0, -\pi < \arg(z) < \pi$:

$$\frac{d}{dz}[z\mathrm{Ln}z] = \frac{d}{dz}z \cdot \mathrm{Ln}z + z\frac{d}{dz}\mathrm{Ln}z = \mathrm{Ln}z + z\frac{1}{z} = \mathrm{Ln}z + 1.$$

Derivatives of Logarithmic Functions II

- Find the derivatives of the function $\mathrm{Ln}(z+1)$ in an appropriate domain.

  The function $\mathrm{Ln}(z+1)$ is a composition of the functions $\mathrm{Ln} z$ and $z+1$. Because the function $z+1$ is entire, it follows from the chain rule that $\mathrm{Ln}(z+1)$ is differentiable at all points $w = z+1$ such that $|w| > 0$ and $-\pi < \arg(w) < \pi$. To determine the corresponding values of z for which $\mathrm{Ln}(z+1)$ is not differentiable, we first solve for z in terms of w to obtain $z = w-1$. The equation $z = w-1$ defines a linear mapping of the w-plane onto the z-plane given by translation by -1. Under this mapping the non-positive real axis is mapped onto the ray emanating from $z=-1$ and containing the point $z=-2$. Thus, $\mathrm{Ln}(z+1)$ is differentiable at all points z that are not on this ray.

$$\frac{d}{dz}\mathrm{Ln}(z+1) = \frac{1}{z+1}\cdot 1 = \frac{1}{z+1}.$$

Logarithmic Mapping

- The complex logarithmic mapping $w = \text{Ln} z$ can be understood in terms of the exponential mapping $w = e^z$ since these functions are inverses of each other.
- Recall $w = e^z$ maps the fundamental region $-\infty < x < \infty$, $-\pi < y \leq \pi$, in the z-plane onto the set $|w| > 0$ in the w-plane. Hence, that inverse mapping $w = \text{Ln} z$ maps the set $|z| > 0$ in the z-plane onto the region $-\infty < u < \infty$, $-\pi < v \leq \pi$, in the w-plane.
- We summarize the relevant properties of the logarithmic mapping:

Logarithmic Mapping Properties

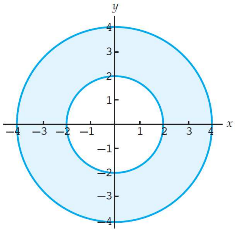
(i) $w = \text{Ln} z$ maps the set $|z| > 0$ onto the region $-\infty < u < \infty$, $-\pi < v \leq \pi$.

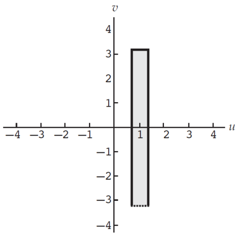
(ii) $w = \text{Ln} z$ maps the circle $|z| = r$ onto the vertical line segment $u = \log_e r$, $-\pi < v \leq \pi$.

(iii) $w = \text{Ln} z$ maps the ray $\arg(z) = \theta$ onto the horizontal line $v = \theta$, $-\infty < u < \infty$.

Example Involving the Logarithmic Mapping

- Find the image of the annulus $2 \leq |z| \leq 4$ under the logarithmic mapping $w = \text{Ln} z$.
  The boundary circles $|z| = 2$ and $|z| = 4$ of the annulus map onto the vertical line segments $u = \log_e 2$ and $u = \log_e 4$, $-\pi < v \leq \pi$. In a similar manner, each circle $|z| = r$, $2 \leq r \leq 4$, maps onto a vertical line segment $u = \log_e r$, $-\pi < v \leq \pi$. Since the real log is increasing, $u = \log_e r$ takes on all values in $\log_e 2 \leq u \leq \log_e 4$ when $2 \leq r \leq 4$. Therefore, the image of $2 \leq |z| \leq 4$ is the rectangular region $\log_e 2 \leq u \leq \log_e 4$, $-\pi < v \leq \pi$:





Other Branches of $\ln z$

- The principal branch of the complex logarithm f_1 is just one of many possible branches of the multiple valued function $F(z) = \ln z$.
- We can define other branches of F by simply changing the interval defining θ to a different interval of length 2π.
- For example, $f_2(z) = \log_e r + i\theta$, $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, defines a branch of F whose branch cut is the non-positive imaginary axis.
  For the branch f_2 we have $f_2(1) = 0$, $f_2(2i) = \log_e 2 + \frac{1}{2}\pi i$, and $f_2(-1-i) = \frac{1}{2}\log_e 2 + \frac{5}{4}\pi i$.
- It can also be shown that any branch

$$f_k(z) = \log_e r + i\theta,\ \theta_0 < \theta < \theta_0 + 2\pi,$$

of $F(z) = \ln z$ is analytic on its domain, and its derivative is given by:

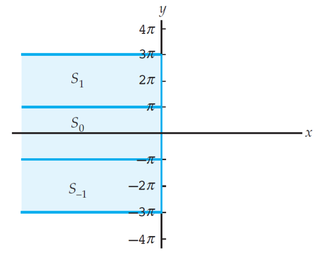
$$f_k'(z) = \frac{1}{z}.$$

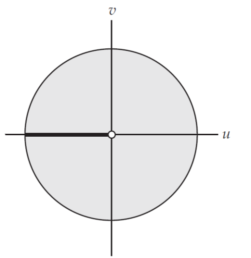
Comparisons with Real Analysis

- Although the complex exponential and logarithmic functions are similar to the real exponential and logarithmic functions in many ways, it is important to keep in mind their differences:
  - The real exponential function is one-to-one, but the complex exponential is not.
  - $\log_e x$ is a single-valued function, but $\ln z$ is multiple-valued.
  - Many properties of real logarithms apply to the complex logarithm, such as $\ln(z_1 z_2) = \ln z_1 + \ln z_2$, but these properties do not always hold for the principal value $\mathrm{Ln} z$.

Riemann Surfaces

- Consider the mapping $w = e^z$ on the half-plane $x \leq 0$.
- Each half-infinite strip S_n defined by $(2n-1)\pi < y \leq (2n+1)\pi$, $x \leq 0$, for $n = 0, \pm 1, \pm 2, \ldots$ is mapped onto the punctured unit disk $0 < |w| \leq 1$:

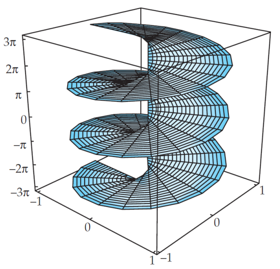




Thus, $w = e^z$ describes an infinite-to-one covering of the punctured unit disk. To visualize this covering, we imagine there being a different image disk B_n for each half-infinite strip S_n.

Riemann Surfaces (Cont'd)

- Cut each disk B_n open along the segment $-1 \leq u < 0$. We construct a Riemann surface for $w = e^z$ by attaching, for each n, the cut disk B_n to the cut disk B_{n+1} along the edge that represents the image of the half-infinite line $y = (2n+1)\pi$. In xyz-space, the images $\ldots, z_{-1}, z_0, z_1, \ldots$ of z in $\ldots, B_{-1}, B_0, B_1, \ldots$, respectively, lie directly above the point $w = e^z$ in the xy-plane.



- By projecting the points of the Riemann surface vertically down onto the xy-plane we see the infinite-to-one nature of the mapping $w = e^z$.
- The multiple-valued function $F(z) = \ln z$ may be visualized by considering all points in the Riemann surface lying directly above a point in the xy-plane.

Subsection 3

Complex Powers

Complex Powers

- Complex powers, such as $(1+i)^i$, are defined in terms of the complex exponential and logarithmic functions.
- Recall from that $z = e^{\ln z}$, for all nonzero complex numbers z.
- Thus, when n is an integer, z^n can be written as

$$z^n = (e^{\ln z})^n = e^{n \ln z}.$$

- This formula, which holds for integer exponents n, suggests the following definition for the complex power z^α, for any complex exponent α:

Definition (Complex Powers)

If α is a complex number and $z \neq 0$, then the complex power z^α is defined to be:

$$z^\alpha = e^{\alpha \ln z}.$$

Complex Power Function

- $z^\alpha = e^{\alpha \ln z}$ gives an infinite set of values because the complex logarithm $\ln z$ is multiple-valued.
- When n is an integer, the expression is single-valued (in agreement with fact that z^n is a function when n is an integer).

  To see this, note $z^n = e^{n \ln z} = e^{n[\log_e |z| + i\arg(z)]} = e^{n \log_e |z|} e^{n\arg(z) i}$. If $\theta = \mathrm{Arg}(z)$, then $\arg(z) = \theta + 2k\pi$, where k is an integer. So $e^{n\arg(z)i} = e^{n(\theta + 2k\pi)i} = e^{n\theta i} e^{2nk\pi i}$. But, by definition, $e^{2nk\pi i} = \cos(2nk\pi) + i \sin(2nk\pi)$. Because n and k are integers, we have $2nk\pi$ is an even multiple of π, and so $\cos(2nk\pi) = 1$ and $\sin(2nk\pi) = 0$. Consequently, $e^{2nk\pi i} = 1$ and we get $z^n = e^{n \log_e |z|} e^{n\mathrm{Arg}(z) i}$, which is single-valued.
- In general, $z^\alpha = e^{\alpha \ln z}$ defines a multiple-valued function.
- It is called a **complex power function**.

Computing Complex Powers

- Find the values of the given complex power:

$$\text{(a)} \quad i^{2i} \qquad\qquad \text{(b)} \quad (1+i)^i.$$

(a) We have seen that $\ln i = \frac{(4n+1)\pi}{2} i$. Thus, we obtain:

$$i^{2i} = e^{2i \ln i} = e^{2i[(4n+1)\pi i/2]} = e^{-(4n+1)\pi},$$

for $n = 0, \pm 1, \pm 2, \ldots$.

(b) We have also seen that $\ln(1+i) = \frac{1}{2}\log_e 2 + \frac{(8n+1)\pi}{4} i$, for $n = 0, \pm 1, \pm 2, \ldots$. Thus, we obtain:

$$(1+i)^i = e^{i \ln(1+i)} = e^{i[(\log_e 2)/2 + (8n+1)\pi i/4]},$$

or

$$(1+i)^i = e^{-(8n+1)\pi/4 + i(\log_e 2)/2},$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Properties of Complex Powers

- Complex powers satisfy the following properties that are analogous to properties of real powers:
  - $z^{\alpha_1} z^{\alpha_2} = z^{\alpha_1 + \alpha_2}$;
  - $\dfrac{z^{\alpha_1}}{z^{\alpha_2}} = z^{\alpha_1 - \alpha_2}$;
  - $(z^{\alpha})^n = z^{n\alpha}$, for $n = 0, \pm 1, \pm 2, \ldots$.
- Each of these properties can be derived from the definition of complex powers and the algebraic properties of the complex exponential function e^z:
  - For example, by the definition, $z^{\alpha_1} z^{\alpha_2} = e^{\alpha_1 \ln z} e^{\alpha_2 \ln z}$. By using properties of the exponential, $z^{\alpha_1} z^{\alpha_2} = e^{\alpha_1 \ln z + \alpha_2 \ln z} = e^{(\alpha_1 + \alpha_2) \ln z}$. By the definition, $e^{(\alpha_1 + \alpha_2) \ln z} = z^{\alpha_1 + \alpha_2}$. Thus, $z^{\alpha_1} z^{\alpha_2} = z^{\alpha_1 + \alpha_2}$.

Principal Value of a Complex Power

- The complex power z^α is, in general, multiple-valued because it is defined using the multiple-valued complex logarithm $\ln z$.
- We can assign a unique value to z^α by using the principal value of the complex logarithm $\mathrm{Ln} z$ in place of $\ln z$.
- This value of the complex power is called the **principal value** of z^α.
- Example: Since $\mathrm{Ln} i = \frac{\pi}{2} i$, the principal value of i^{2i} is $i^{2i} = e^{2i \mathrm{Ln} i} = e^{2i \frac{\pi}{2} i} = e^{-\pi}$.

Definition (Principal Value of a Complex Power)

If α is a complex number and $z \neq 0$, then the function defined by:

$$z^\alpha = e^{\alpha \mathrm{Ln} z}$$

is called the **principal value of the complex power** z^α.

- Notation: z^α will be used to denote both the multiple-valued power function $F(z) = z^\alpha$ and the **principal value power function**.

Computing the Principal Value of a Complex Power

- Find the principal value of each complex power:

$$\text{(a)} \quad (-3)^{i/\pi} \qquad\qquad \text{(b)} \quad (2i)^{1-i}.$$

(a) For $z = -3$, we have $|z| = 3$ and $\operatorname{Arg}(-3) = \pi$, and so $\operatorname{Ln}(-3) = \log_e 3 + i\pi$. Thus, we obtain:

$$(-3)^{i/\pi} = e^{(i/\pi)\operatorname{Ln}(-3)} = e^{(i/\pi)(\log_e 3 + i\pi)} = e^{-1 + i(\log_e 3)/\pi}.$$

Finally, since $e^{-1+i(\log_e 3)/\pi} = e^{-1}[\cos \frac{\log_e 3}{\pi} + i \sin \frac{\log_e 3}{\pi}]$, $(-3)^{i/\pi} = e^{-1}[\cos \frac{\log_e 3}{\pi} + i \sin \frac{\log_e 3}{\pi}]$.

(b) For $z = 2i$, we have $|z| = 2$ and $\operatorname{Arg}(z) = \frac{\pi}{2}$, and so $\operatorname{Ln} 2i = \log_e 2 + i\frac{\pi}{2}$. Thus, we obtain:

$$(2i)^{1-i} = e^{(1-i)\operatorname{Ln} 2i} = e^{(1-i)(\log_e 2 + i\pi/2)} = e^{\log_e 2 + \pi/2 - i(\log_e 2 - \pi/2)}.$$

Since $(2i)^{1-i} = e^{\log_e 2 + \pi/2}[\cos(\log_e 2 - \frac{\pi}{2}) - i \sin(\log_e 2 - \frac{\pi}{2})]$, we finally get $(2i)^{1-i} = e^{\log_e 2 + \pi/2}[\cos(\log_e 2 - \frac{\pi}{2}) - i \sin(\log_e 2 - \frac{\pi}{2})]$.

Analyticity

- In general, the principal value of a complex power z^α is not a continuous function on the complex plane because the function $\text{Ln}z$ is not continuous on the complex plane.
- The function $e^{\alpha z}$ is continuous on the entire complex plane and the function $\text{Ln}z$ is continuous on the domain $|z| > 0$, $-\pi < \arg(z) < \pi$, so z^α is continuous on the domain $|z| > 0$, $-\pi < \arg(z) < \pi$.
- Using polar coordinates $r = |z|$ and $\theta = \arg(z)$, we have found that $f_1(z) = e^{\alpha(\log_e r + i\theta)}$, $-\pi < \theta < \pi$ is a branch of $F(z) = z^\alpha = e^{\alpha \ln z}$.
- It is called the **principal branch of the complex power** z^α. Its branch cut is the non-positive real axis, and $z = 0$ is a branch point.
- The branch f_1 agrees with the principal value z^α on the domain $|z| > 0$, $-\pi < \arg(z) < \pi$. Consequently, the derivative of f_1 can be found using the chain rule:
$$f_1'(z) = \frac{d}{dz} e^{\alpha \text{Ln}z} = e^{\alpha \text{Ln}z} \frac{d}{dz}[\alpha \text{Ln}z] = e^{\alpha \text{Ln}z} \frac{\alpha}{z}.$$
Using the principal value $z^\alpha = e^{\alpha \text{Ln}z}$, we find $f_1'(z) = \frac{\alpha z^\alpha}{z} = \alpha z^{\alpha - 1}$.

Derivative of a Power Function

- Find the derivative of the principal value z^i at the point $z = 1 + i$.

  Because the point $z = 1 + i$ is in the domain $|z| > 0$, $-\pi < \arg(z) < \pi$, it follows that $\frac{d}{dz} z^i = iz^{i-1}$, and so, $\frac{d}{dz} z^i \Big|_{z=1+i} = \; iz^{i-1}\Big|_{z=1+i} = i(1+i)^{i-1}$. We can rewrite this value as:

$$i(1+i)^{i-1} = i(1+i)^i(1+i)^{-1} = i(1+i)^i \frac{1}{1+i} = \frac{1+i}{2}(1+i)^i.$$

  Moreover, the principal value of $(1+i)^i$ is:
  $(1+i)^i = e^{-\pi/4 + i(\log_e 2)/2}$, and so

$$\frac{d}{dz} z^i \Bigg|_{z=1+i} = \frac{1+i}{2} e^{-\pi/4 + i(\log_e 2)/2}.$$

Remarks

(i) There are some properties of real powers that are not satisfied by complex powers. One example of this is that for complex powers, $(z^{\alpha_1})^{\alpha_2} \neq z^{\alpha_1 \alpha_2}$ unless α_2 is an integer.

(ii) As with complex logarithms, some properties that hold for complex powers do not hold for principal values of complex powers.

For example, we can prove that $(z_1 z_2)^\alpha = z_1^\alpha z_2^\alpha$, for any nonzero complex numbers z_1 and z_2. However, this property does not hold for principal values of these complex powers:

If $z_1 = -1$, $z_2 = i$, and $\alpha = i$, then the principal value of $(-1 \cdot i)^i$ is $e^{i \operatorname{Ln}(-i)} = e^{\pi/2}$. On the other hand, the product of the principal values of $(-1)^i$ and i^i is $e^{i \operatorname{Ln}(-1)} e^{i \operatorname{Ln} i} = e^{-\pi} e^{-\pi/2} = e^{-3\pi/2}$.

Subsection 4

Complex Trigonometric Functions

Complex Sine and Cosine Functions

- If x is a real variable, then
$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x.$$
- By adding these equations and simplifying, we get:
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}.$$
- If we subtract the two equations, then we obtain
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$
- The formulas for the real cosine and sine functions can be used to define the complex sine and cosine functions.

Definition (Complex Sine and Cosine Functions)

The **complex sine and cosine functions** are defined by:
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

The Complex Tangent, Cotangent, Secant, and Cosecant

- The complex sine and cosine functions agree, by definition, with the real sine and cosine functions for real input.
- Analogous to real trigonometric functions, we next define the complex tangent, cotangent, secant, and cosecant functions using the complex sine and cosine:

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z},$$
$$\sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}.$$

- These functions also agree with their real counterparts for real input.

Values of Complex Trigonometric Functions

- Express the value of the given trigonometric function in the form $a+ib$.

$$\text{(a)}\quad \cos i \qquad \text{(b)}\quad \sin(2+i) \qquad \text{(c)}\quad \tan(\pi-2i).$$

(a) $\cos i = \dfrac{e^{i\cdot i}+e^{-i\cdot i}}{2} = \dfrac{e^{-1}+e}{2}$.

(b) $\sin(2+i) = \dfrac{e^{i(2+i)}-e^{-i(2+i)}}{2i} = \dfrac{e^{-1+2i}-e^{1-2i}}{2i} = \dfrac{e^{-1}(\cos 2+i\sin 2)-e(\cos(-2)+i\sin(-2))}{2i}$.

(c) $\tan(\pi-2i) = \dfrac{(e^{i(\pi-2i)}-e^{-i(\pi-2i)})/2i}{(e^{i(\pi-2i)}+e^{-i(\pi-2i)})/2} = \dfrac{e^{i(\pi-2i)}-e^{-i(\pi-2i)}}{(e^{i(\pi-2i)}+e^{-i(\pi-2i)})i} = \dfrac{e^2-e^{-2}}{(e^2+e^{-2})i} = -\dfrac{e^2-e^{-2}}{e^2+e^{-2}}i$.

Identities

- We now list some of the more useful of the trigonometric identities:
  - $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$;
  - $\cos^2 z + \sin^2 z = 1$;
  - $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$;
  - $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$.
- Because of the sum/difference formulas, we also have the double-angle formulas:

$$\sin 2z = 2\sin z \cos z \quad \text{and} \quad \cos 2z = \cos^2 z - \sin^2 z.$$

- We only verify $\cos^2 z + \sin^2 z = 1$:

$$\begin{aligned} \cos^2 z + \sin^2 z &= \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 \\ &= \frac{e^{2iz}+2+e^{-2iz}}{4} - \frac{e^{2iz}-2+e^{-2iz}}{4} = 1. \end{aligned}$$

- Some properties of the real trigonometric functions are not satisfied by their complex counterparts:
  E.g., $|\sin x| \leq 1$ and $|\cos x| \leq 1$, for all real x, but $|\cos i| > 1$ and $|\sin(2+i)| > 1$.

Periodicity

- We know that the complex exponential function is periodic with a pure imaginary period of $2\pi i$, i.e., $e^{z+2\pi i} = e^z$, for all complex z.
- Replacing z with iz, we get $e^{iz+2\pi i} = e^{i(z+2\pi)} = e^{iz}$.
- Thus, e^{iz} is a periodic function with real period 2π.
- Similarly, $e^{-i(z+2\pi)} = e^{-iz}$, i.e., e^{-iz} is periodic with period of 2π.
- It now follows that:
$$\sin(z+2\pi) = \frac{e^{i(z+2\pi)} - e^{-i(z+2\pi)}}{2i} = \frac{e^{iz} - e^{-iz}}{2i} = \sin z.$$
- A similar statement also holds for the complex cosine function.
- Thus, the complex sine and cosine are periodic functions with a real period of 2π.
- The periodicity of the secant and cosecant functions follows immediately from the definitions.
- Moreover, the complex tangent and cotangent are periodic with a real period of π.

Trigonometric Equations

- Since the complex sine and cosine functions are periodic, there are always infinitely many solutions to equations of the form $\sin z = w$ or $\cos z = w$.
- One approach to solving such equations is to use the definition in conjunction with the quadratic formula:

  Example: Find all solutions to the equation $\sin z = 5$.

  $\sin z = 5$ is equivalent to the equation $\frac{e^{iz} - e^{-iz}}{2i} = 5$. By multiplying this equation by e^{iz} and simplifying we obtain $e^{2iz} - 10ie^{iz} - 1 = 0$. This equation is quadratic in e^{iz}, i.e., $(e^{iz})^2 - 10i(e^{iz}) - 1 = 0$. By the quadratic formula that the solutions are given by $e^{iz} = \frac{10i + (-96)^{1/2}}{2} = 5i \pm 2\sqrt{6}i = (5 \pm 2\sqrt{6})i$. In order to find the values of z, we must solve the two resulting exponential equations using the complex logarithm.

Trigonometric Equations (Cont'd)

- We must solve $e^{iz} = (5 \pm 2\sqrt{6})i$ using the complex logarithm.
  - If $e^{iz} = (5 + 2\sqrt{6})i$, then $iz = \ln(5i + 2\sqrt{6}i)$ or $z = -i \ln[(5 + 2\sqrt{6})i]$. Because $(5 + 2\sqrt{6})i$ is a pure imaginary number and $5 + 2\sqrt{6} > 0$, we have $\arg[(5 + 2\sqrt{6})i] = \frac{1}{2}\pi + 2n\pi$. Thus, $z = -i \ln[(5 + 2\sqrt{6})i] = -i[\log_e(5 + 2\sqrt{6}) + i(\frac{\pi}{2} + 2n\pi)]$ or $z = \frac{(4n+1)\pi}{2} - i \log_e(5 + 2\sqrt{6})$, for $n = 0, \pm 1, \pm 2, \ldots$.
  - Similarly, if $e^{iz} = (5 - 2\sqrt{6})i$, then $z = -i \ln[(5 - 2\sqrt{6})i]$. Since $(5 - 2\sqrt{6})i$ is a pure imaginary number and $5 - 2\sqrt{6} > 0$, it has an argument of $\frac{\pi}{2}$, and so:
    $z = -i \ln[(5 - 2\sqrt{6})i] = -i[\log_e(5 - 2\sqrt{6}) + i(\frac{\pi}{2} + 2n\pi)]$ or
    $z = \frac{(4n+1)\pi}{2} - i \log_e(5 - 2\sqrt{6})$ for $n = 0, \pm 1, \pm 2, \ldots$.

$\sin z$ and $\cos z$ in terms of x and y

- To find a formula in terms of x and y for the modulus of the sine and cosine functions, we replace z by $x + iy$ in $\sin z$:

$$\begin{aligned}\sin z &= \frac{e^{-y+ix} - e^{y-ix}}{2i} \\ &= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i} \\ &= \sin x\frac{e^{y} + e^{-y}}{2} + i\cos x\frac{e^{y} - e^{-y}}{2}.\end{aligned}$$

- Since the real hyperbolic sine and cosine functions are defined by $\sinh y = \dfrac{e^{y} - e^{-y}}{2}$ and $\cosh y = \dfrac{e^{y} + e^{-y}}{2}$, we can rewrite as

$$\sin z = \sin x \cosh y + i \cos x \sinh y.$$

- A similar computation gives

$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

Modulus of Sine and Cosine

- By the expression for $\sin z$:
$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}.$$
- Recall $\cos^2 x + \sin^2 x = 1$ and $\cosh^2 y = 1 + \sinh^2 y$:
$$\begin{aligned} |\sin z| &= \sqrt{\sin^2 x(1+\sinh^2 y) + \cos^2 x \sinh^2 y} \\ &= \sqrt{\sin^2 x + (\cos^2 x + \sin^2 x)\sinh^2 y}, \end{aligned}$$
$$|\sin z| = \sqrt{\sin^2 x + \sinh^2 y}.$$
- Similarly, for the modulus of the complex cosine function:
$$|\cos z| = \sqrt{\cos^2 x + \sinh^2 y}.$$
- Since $\sinh x$ is unbounded, the complex sine and cosine functions are not bounded on the complex plane, i.e., there does not exist a real constant M so that $|\sin z| < M$, for all z in $\mathbb{C}$, nor does there exist a real constant M so that $|\cos z| < M$, for all z in $\mathbb{C}$.

Zeros

- The zeros of the real sine occur at integer multiples of π and the zeros of the real cosine occur at odd integer multiples of $\frac{\pi}{2}$.
- These zeros of the real sine and cosine functions are also zeros of the complex sine and cosine, respectively.
- To find all zeros, we must solve $\sin z = 0$ and $\cos z = 0$.
- $\sin z = 0$ is equivalent to $|\sin z| = 0$, i.e., $\sqrt{\sin^2 x + \sinh^2 y} = 0$, which is equivalent to: $\sin^2 x + \sinh^2 y = 0$.
- Since $\sin^2 x$ and $\sinh^2 y$ are nonnegative real numbers, we must have $\sin x = 0$ and $\sinh y = 0$.
  - $\sin x = 0$ occurs when $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$.
  - $\sinh y = 0$ occurs only when $y = 0$.

  So, the only solutions of $\sin z = 0$ in the complex plane are the real numbers $z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, i.e., the zeros of the complex sine function are the same as the zeros of the real sine.
- Similarly, the only zeros of the complex cosine function are the real numbers $z = \frac{(2n+1)\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$.

Analyticity

- The derivatives of the complex sine and cosine functions are found using the chain rule:

$$\begin{aligned} \frac{d}{dz}\sin z &= \frac{d}{dz}\frac{e^{iz}-e^{-iz}}{2i} = \frac{ie^{iz}+ie^{-iz}}{2i} \\ &= \frac{e^{iz}+e^{-iz}}{2} = \cos z. \end{aligned}$$

  Since this derivative is defined for all complex z, $\sin z$ is entire.
- Similarly, $\frac{d}{dz}\cos z = -\sin z$.
- The derivatives of $\sin z$ and $\cos z$ can then be used to compute the derivatives of all of the complex trigonometric functions:

$$\begin{array}{lll} \frac{d}{dz}\sin z = \cos z & \frac{d}{dz}\cos z = -\sin z & \frac{d}{dz}\tan z = \sec^2 z \\ \frac{d}{dz}\cot z = -\csc^2 z & \frac{d}{dz}\sec z = \sec z\tan z & \frac{d}{dz}\csc z = -\csc z\cot z \end{array}$$

- The sine and cosine functions are entire, but the tangent, cotangent, secant, and cosecant functions are only analytic at those points where the denominator is nonzero.

Trigonometric Mapping

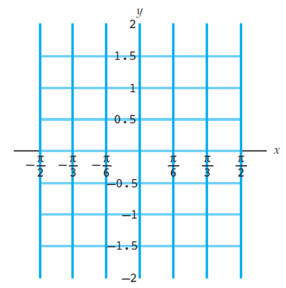
- Since $\sin z$ is periodic with a real period of 2π, it takes on all values in any infinite vertical strip $x_0 < x \leq x_0 + 2\pi$, $-\infty < y < \infty$.
- This allows us to study the mapping $w = \sin z$ on the entire complex plane by analyzing it on any one of these strips.
- Consider the strip $-\pi < x \leq \pi$, $-\infty < y < \infty$.
- Observe that $\sin z$ is not one-to-one on this region, e.g., $z_1 = 0$ and $z_2 = \pi$ are in this region and $\sin 0 = \sin \pi = 0$.
- From $\sin(-z + \pi) = \sin z$, it follows that the image of the strip $-\pi < x \leq -\frac{\pi}{2}$, $-\infty < y < \infty$, is the same as the image of the strip $\frac{\pi}{2} < x \leq \pi$, $-\infty < y < \infty$, under $w = \sin z$.
- Therefore, we need only consider the mapping $w = \sin z$ on the region $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $-\infty < y < \infty$, to gain an understanding of this mapping on the entire z-plane.
- One can show that the complex sine function is one-to-one on the domain $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $-\infty < y < \infty$.

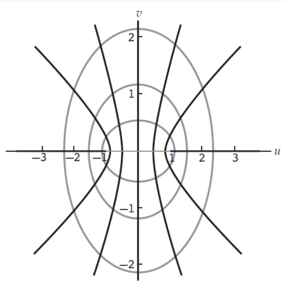
The Mapping $w = \sin z$

- Describe the image of the region $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $-\infty < y < \infty$, under the complex mapping $w = \sin z$.
  We determine the image of vertical lines $x = a$ with $-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$.
  - Assume that $a \neq -\frac{\pi}{2}, 0, \frac{\pi}{2}$. The image of the vertical line $x = a$ is given by: $u = \sin a \cosh y$, $v = \cos a \sinh y$, $-\infty < y < \infty$. Since $-\frac{\pi}{2} < a < \frac{\pi}{2}$ and $a \neq 0$, it follows that $\sin a \neq 0$ and $\cos a \neq 0$, whence $\cosh y = \frac{u}{\sin a}$ and $\sinh y = \frac{v}{\cos a}$. The identity $\cosh^2 y - \sinh^2 y = 1$ gives: $(\frac{u}{\sin a})^2 - (\frac{v}{\cos a})^2 = 1$. It represents a hyperbola with vertices at $(\pm \sin a, 0)$ and slant asymptotes $v = \pm(\frac{\cos a}{\sin a})u$.
    Because the point $(a, 0)$ is on the line $x = a$, the point $(\sin a, 0)$ must be on the image of the line. Therefore, the image of the vertical line $x = a$, with $-\frac{\pi}{2} < a < \frac{\pi}{2}$ and $a \neq 0$, under $w = \sin z$ is the branch of the hyperbola that contains the point $(\sin a, 0)$.
    Because $\sin(-z) = -\sin z$, for all z, the image of the line $x = -a$ is a branch of the hyperbola containing the point $(-\sin a, 0)$.
    Therefore, the pair $x = a$ and $x = -a$, with $-\frac{\pi}{2} < a < \frac{\pi}{2}$ and $a \neq 0$, are mapped onto the full hyperbola.

The Mapping $w = \sin z$ (Cont'd)

- Summary:



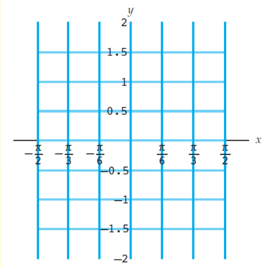


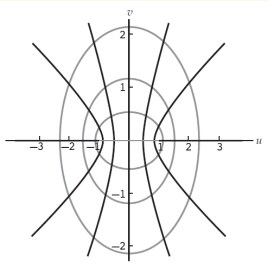
  - The image of the line $x = -\frac{\pi}{2}$ is the set of points $u \leq -1$ on the negative real axis.
  - The image of the line $x = \frac{\pi}{2}$ is the set of points $u \geq 1$ on the positive real axis.
  - The image of the line $x = 0$ is the imaginary axis $u = 0$.

In summary, the image of the infinite vertical strip $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $-\infty < y < \infty$, under $w = \sin z$, is the entire w-plane.

Following Horizontal Line Segments

- The image could also be found using horizontal line segments $y = b$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, instead of vertical lines.
- The images are: $u = \sin x \cosh b$, $v = \cos x \sinh b$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
  - When $b \neq 0$, we get $\left(\frac{u}{\cosh b}\right)^2 + \left(\frac{v}{\sinh b}\right)^2 = 1$, which is an ellipse with u-intercepts at $(\pm \cosh b, 0)$ and v-intercepts at $(0, \pm \sinh b)$.
  - If $b > 0$, then the image of the segment $y = b$ is the upper-half of the ellipse and the image of the segment $y = -b$ the bottom-half.

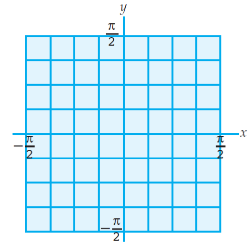


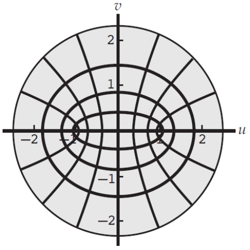


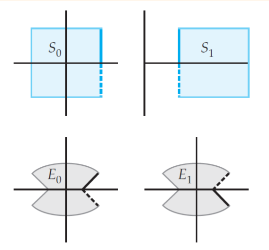
  - Observe that if $b = 0$, then the image of the line segment $y = 0$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, is the line segment $-1 \le u \le 1$, $v = 0$ on the real axis.

Riemann Surface I

- Since the complex sine function is periodic, the mapping $w = \sin z$ is not one-to-one on the complex plane. A Riemann surface helps visualize $w = \sin z$.
- Consider the mapping on the square S_0 defined by $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.





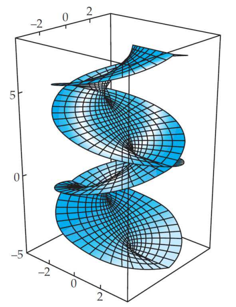


S_0 is mapped onto the elliptical region E.

- Similarly, the adjacent square S_1 defined by $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, also maps onto E.

Riemann Surface II

- A Riemann surface is constructed by starting with two copies of E, E_0 and E_1, representing the images of S_0 and S_1, respectively. We cut E_0 and E_1 open along the line segments in the real axis from 1 to $\cosh\left(\frac{\pi}{2}\right)$ and from -1 to $-\cosh\left(\frac{\pi}{2}\right)$.
- Part of the Riemann surface consists of the two elliptical regions E_0 and E_1 with the black segments glued together and the dashed segments glued together.
- To complete the Riemann surface, we take for every integer n an elliptical region E_n representing the image of the square S_n defined by $\frac{(2n-1)\pi}{2} \leq x \leq \frac{(2n+1)\pi}{2}$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Each region E_n is cut open, as E_0 and E_1 were, and E_n is glued to E_{n+1} along their boundaries in a manner analogous to that used for E_0 and E_1.



Subsection 5

Complex Hyperbolic Functions

Complex Hyperbolic Sine and Cosine

- The real hyperbolic sine and hyperbolic cosine functions are defined using the real exponential by $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$.

Definition (Complex Hyperbolic Sine and Cosine)

The **complex hyperbolic sine** and **hyperbolic cosine functions** are defined by:

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}.$$

- These agree with the real hyperbolic functions for real input.
- Unlike the real hyperbolic functions, the complex hyperbolic functions are periodic and have infinitely many zeros.
- The **complex hyperbolic tangent**, **cotangent**, **secant**, and **cosecant**:

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}$$
$$\operatorname{sech} z = \frac{1}{\cosh z}, \quad \operatorname{csch} z = \frac{1}{\sinh z}.$$

Derivatives of Complex Hyperbolic Functions

- The hyperbolic sine and cosine functions are entire because the functions e^z and e^{-z} are entire.
- Moreover, we have:

$$\frac{d}{dz}\sinh z = \frac{d}{dz}\frac{e^z - e^{-z}}{2} = \frac{e^z + e^{-z}}{2} = \cosh z.$$

- A similar computation for $\cosh z$ yields

$$\frac{d}{dz}\cosh z = \sinh z.$$

- Derivatives of Complex Hyperbolic Functions

$$\frac{d}{dz}\sinh z = \cosh z, \qquad \frac{d}{dz}\cosh z = \sinh z, \qquad \frac{d}{dz}\tanh z = \operatorname{sech}^2 z,$$
$$\frac{d}{dz}\coth z = -\operatorname{csch}^2 z, \qquad \frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \tanh z, \qquad \frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z \coth z.$$

Relation To Sine and Cosine

- The real trigonometric and the real hyperbolic functions share many similar properties, e.g., $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\sinh x = \cosh x$.
- There is a simple connection between the complex trigonometric and hyperbolic functions: Replace z with iz in the definition of $\sinh z$:

$$\sinh(iz) = \frac{e^{iz} - e^{-iz}}{2} = i\frac{e^{iz} - e^{-iz}}{2i} = i\sin z,$$

or $-i\sinh(iz) = \sin z$.
- Substituting iz for z in $\sin z$, we find $\sinh z = -i\sin(iz)$.
- After repeating this process for $\cos z$ and $\cosh z$, we obtain:

$$\begin{array}{lcl} \sin z = -i\sinh(iz) & \text{and} & \cos z = \cosh(iz), \\ \sinh z = -i\sin(iz) & \text{and} & \cosh z = \cos(iz). \end{array}$$

- Other relations can be similarly derived:

$$\tan(iz) = \frac{\sin(iz)}{\cos(iz)} = i\frac{\sinh z}{\cosh z} = i\tanh z.$$

Obtaining Hyperbolic Identities

- Some of the more commonly used hyperbolic identities:
  - $\sinh(-z) = -\sinh z$ and $\cosh(-z) = \cosh z$;
  - $\cosh^2 z - \sinh^2 z = 1$;
  - $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$;
  - $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$.
- Example: Verify that $\cosh(z_1 + z_1) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$, for all complex z_1 and z_2.

  We have $\cosh(z_1 + z_2) = \cos(iz_1 + iz_2)$. So by a trigonometric identity and additional applications of the preceding identities,

$$\begin{aligned}
\cosh(z_1 + z_2) &= \cos(iz_1 + iz_2) \\
&= \cos iz_1 \cos iz_2 - \sin iz_1 \sin iz_2 \\
&= \cos iz_1 \cos iz_2 + (-i \sin iz_1)(-i \sin iz_2) \\
&= \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2.
\end{aligned}$$

Real versus Complex Trig and Hyperbolic Trig Functions

(i) In real analysis, the exponential function was just one of a number of apparently equally important elementary functions. In complex analysis, however, the complex exponential function assumes a much greater role: All of the complex elementary functions can be defined solely in terms of the complex exponential and logarithmic functions. The exponential and logarithmic functions can be used to evaluate, differentiate, integrate, and map using elementary functions.

(ii) As functions of a real variable x, $\sinh x$ and $\cosh x$ are not periodic. In contrast, the complex functions $\sinh z$ and $\cosh z$ are periodic. Moreover, $\cosh x$ has no zeros and $\sinh x$ has a single zero at $x = 0$. The complex functions $\sinh z$ and $\cosh z$, on the other hand, both have infinitely many zeros.

Subsection 6

Inverse Trigonometric and Hyperbolic Functions

Inverse Sine

- The complex sine function is periodic with a real period of 2π.
- Moreover, the sine function maps the complex plane onto the complex plane: $\text{Range}(\sin z) = \mathbb{C}$.
- Thus, for any complex number z there exists infinitely many solutions w to the equation $\sin w = z$. Rewrite as $\frac{e^{iw} - e^{-iw}}{2i} = z$ or $e^{2iw} - 2ize^{iw} - 1 = 0$. Use the quadratic formula to solve $e^{iw} = iz + (1 - z^2)^{1/2}$.. This expression involves the two square roots of $1 - z^2$. We solve for w using the complex logarithm: $iw = \ln\left[iz + (1 - z^2)^{1/2}\right]$ or $w = -i \ln\left[iz + (1 - z^2)^{1/2}\right]$.

Definition (Inverse Sine)

The multiple-valued function $\sin^{-1} z$ defined by:

$$\sin^{-1} z = -i \ln\left[iz + (1 - z^2)^{1/2}\right]$$

is called the **inverse sine** or **arcsine**, sometimes written $\arcsin z$.

Values of Inverse Sine

- Find all values of $\sin^{-1}\sqrt{5}$.

  Set $z = \sqrt{5}$ in the formula defining $\sin^{-1} z$:
  $\sin^{-1}\sqrt{5} = -i \ln[i\sqrt{5} + (1 - (\sqrt{5})^2)^{1/2}] = -i \ln[i\sqrt{5} + (-4)^{1/2}]$.
  The two square roots $(-4)^{1/2}$ of -4 are found to be $\pm 2i$. So
  $\sin^{-1}\sqrt{5} = -i \ln[i\sqrt{5} \pm 2i] = -i \ln[(\sqrt{5} \pm 2)i]$. Because $(\sqrt{5} \pm 2)i$ is a pure imaginary number with positive imaginary part, we have
  $|(\sqrt{5} \pm 2)i| = \sqrt{5} \pm 2$ and $\arg([(\sqrt{5} \pm 2)i]) = \frac{\pi}{2}$. Thus, we have
  $\ln[(\sqrt{5} \pm 2)i] = \log_e(\sqrt{5} \pm 2) + i(\frac{\pi}{2} + 2n\pi)$ for $n = 0, \pm 1, \pm 2, \ldots$.
  Observe that $\log_e(\sqrt{5} - 2) = \log_e \frac{1}{\sqrt{5}+2} = \log_e 1 - \log_e(\sqrt{5} + 2) = 0 - \log_e(\sqrt{5} + 2)$, and so $\log_e(\sqrt{5} \pm 2) = \pm \log_e(\sqrt{5} + 2)$.
  Therefore, $-i \ln[(\sqrt{5} \pm 2)i] = -i[\log_e(\sqrt{5} \pm 2) + i(\frac{\pi}{2} + 2n\pi)] = -i[\pm \log_e(\sqrt{5} + 2) + i\frac{(4n+1)\pi}{2}]$, and so
  $\sin^{-1}\sqrt{5} = \frac{(4n+1)\pi}{2} \pm i \log_e(\sqrt{5} + 2)$, for $n = 0, \pm 1, \pm 2, \ldots$.

Inverse Cosine and Tangent

- Similarly, we may solve the equations $\cos w = z$ and $\tan w = z$.

Definition (Inverse Cosine and Inverse Tangent)

The multiple-valued function $\cos^{-1} z$ defined by:

$$\cos^{-1} z = -i \ln\left[z + i(1-z^2)^{1/2}\right]$$

is called the **inverse cosine**. The multiple-valued function $\tan^{-1} z$ defined by:

$$\tan^{-1} z = \frac{i}{2} \ln\left(\frac{i+z}{i-z}\right)$$

is called the **inverse tangent**.

- The inverse cosine and inverse tangent are multiple-valued functions since they are defined in terms of the complex logarithm $\ln z$.
- The expression $(1-z^2)^{1/2}$ represents the two square roots of the complex number $1-z^2$.
- Every value of $w = \cos^{-1} z$ satisfies the equation $\cos w = z$, and, similarly, every value of $w = \tan^{-1} z$ satisfies the equation $\tan w = z$.

Defining a Univalued Inverse Function

- The inverse sine and inverse cosine are multiple-valued functions that can be made single-valued by specifying a single value of the square root to use for the expression $(1-z^2)^{1/2}$ and a single value of the complex logarithm.
- The inverse tangent can be made single-valued by just specifying a single value of $\ln z$.
- Example: If $z=\sqrt{5}$, then the principal square root of $1-(\sqrt{5})^2=-4$ is $2i$, and $\mathrm{Ln}(i\sqrt{5}+2i)=\log_e(\sqrt{5}+2)+\frac{\pi i}{2}$. Using the definition, we get $f(\sqrt{5})=\frac{\pi}{2}-i\log_e(\sqrt{5}+2)$. Thus, the value of the function f at $z=\sqrt{5}$ is the value of $\sin^{-1}\sqrt{5}$ associated to $n=0$ and the square root $2i$ in the preceding example.

Branches and Analyticity

- Determining domains of branches of inverse trigonometric functions is complicated and not discussed.
- The derivatives of branches can be found using implicit differentiation: Suppose that f_1 is a branch of $F(z) = \sin^{-1} z$. If $w = f_1(z)$, then $z = \sin w$. By differentiating both sides with respect to z and applying the chain rule, $1 = \cos w \cdot \frac{dw}{dz}$, or $\frac{dw}{dz} = \frac{1}{\cos w}$. From the trigonometric identity $\cos^2 w + \sin^2 w = 1$, $\cos w = (1 - \sin^2 w)^{1/2}$, and, since $z = \sin w$, $\cos w = (1 - z^2)^{1/2}$. After substituting this expression for $\cos w$, we obtain $f_1'(z) = \frac{dw}{dz} = \frac{1}{(1-z^2)^{1/2}}$.

  If we let $\sin^{-1} z$ denote the branch f_1, then this formula may be restated as:
$$\frac{d}{dz} \sin^{-1} z = \frac{1}{(1 - z^2)^{1/2}}.$$
  We must use the same branch of the square root function that defined $\sin^{-1} z$ when finding values of its derivative.

Derivatives of Branches $\sin^{-1} z, \cos^{-1} z$ and $\tan^{-1} z$

- In a similar manner, derivatives of branches of the inverse cosine and the inverse tangent can be found.
- In the following formulas, the symbols $\sin^{-1} z$, $\cos^{-1} z$, and $\tan^{-1} z$ represent branches of the corresponding multiple-valued functions, so, the formulas for the derivatives hold only on the domains of these branches:

Derivatives of Branches $\sin^{-1} z, \cos^{-1} z$ and $\tan^{-1} z$

$$\frac{d}{dz}\sin^{-1} z = \frac{1}{(1-z^2)^{1/2}}, \frac{d}{dz}\cos^{-1} z = \frac{-1}{(1-z^2)^{1/2}}, \frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}$$

- Again, when finding the value of a derivative, we must use the same square root as is used to define the branch.
- The formulas are similar to those for the derivatives of the real inverse trigonometric functions. The difference is the specific choice of a branch of the square root function.

Derivative of a Branch of Inverse Sine

- Let $\sin^{-1} z$ represent a branch of the inverse sine obtained by using the principal branches of the square root and the logarithm. Find the derivative of this branch at $z = i$.

  Note that this branch is differentiable at $z = i$ because $1 - i^2 = 2$ is not on the branch cut of the principal branch of the square root function, and because $i(i) + (1 - i^2)^{1/2} = -1 + \sqrt{2}$ is not on the branch cut of the principal branch of the complex logarithm.

  We have:

$$\frac{d}{dz} \sin^{-1} z \Big|_{z=i} = \frac{1}{(1 - z^2)^{1/2}} \Big|_{z=i} = \frac{1}{(1 - i^2)^{1/2}} = \frac{1}{2^{1/2}}.$$

  Using the principal branch of the square root, we obtain $2^{1/2} = \sqrt{2}$. Therefore, the derivative is $\frac{1}{\sqrt{2}}$ or $\frac{1}{2}\sqrt{2}$.

Inverse Hyperbolic Functions

- The inverses of the hyperbolic functions are also defined in terms of the complex logarithm because the hyperbolic functions are defined in terms of the complex exponential.

Definition (Inverse Hyperbolic Sine, Cosine, and Tangent)

The multiple-valued functions $\sinh^{-1} z$, $\cosh^{-1} z$, and $\tanh^{-1} z$, defined by:

$$\sinh^{-1} z = \ln\left[z + (z^2+1)^{1/2}\right], \qquad \cosh^{-1} z = \ln\left[z + (z^2-1)^{1/2}\right],$$

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

are called the **inverse hyperbolic sine**, the **inverse hyperbolic cosine**, and the **inverse hyperbolic tangent**, respectively.

- The expressions given in the definition allow us to solve equations involving the complex hyperbolic functions.
- In particular, if $w = \sinh^{-1} z$, then $\sinh w = z$; if $w = \cosh^{-1} z$, then $\cosh w = z$; and if $w = \tanh^{-1} z$, then $\tanh w = z$.

Analyticity

- Branches of the inverse hyperbolic functions are defined by choosing branches of the square root and complex logarithm.
- The derivative of a branch can be found using implicit differentiation: Here $\sinh^{-1} z$, $\cosh^{-1} z$ and $\tanh^{-1} z$ represent branches of the corresponding multiple-valued functions.

Derivatives of Branches $\sinh^{-1} z, \cosh^{-1} z, \tanh^{-1} z$

$$\frac{d}{dz}\sinh^{-1} z = \frac{1}{(z^2+1)^{1/2}}, \qquad \frac{d}{dz}\cosh^{-1} z = \frac{1}{(z^2-1)^{1/2}},$$
$$\frac{d}{dz}\tanh^{-1} z = \frac{1}{1-z^2}.$$

- We must be consistent in our use of branches when evaluating derivatives.
- The formulas are the same as the ones for the derivatives of the real inverse hyperbolic functions except for the choice of branch.

Computing Inverse Hyperbolic Cosine

- Let $\cosh^{-1} z$ represent the branch of the inverse hyperbolic cosine obtained by using the branch $f_2(z) = \sqrt{r}e^{i\theta/2}$, $0 < \theta < 2\pi$, of the square root and the principal branch of the complex logarithm. Find the values: (a) $\cosh^{-1} \frac{\sqrt{2}}{2}$ (b) $\frac{d}{dz} \cosh^{-1} z\Big|_{z=\sqrt{2}/2}$.

(a) We use $\cosh^{-1} z = \ln\left[z + (z^2 - 1)^{1/2}\right]$ with $z = \frac{1}{2}\sqrt{2}$ and the stated branches of the square root and logarithm. When $z = \frac{1}{2}\sqrt{2}$, we have that $z^2 - 1 = -\frac{1}{2}$. Since $-\frac{1}{2}$ has exponential form $\frac{1}{2}e^{i\pi}$, the square root given by the branch f_2 is: $f_2(\frac{1}{2}e^{i\pi}) = \sqrt{\frac{1}{2}}e^{i\pi/2} = \frac{1}{\sqrt{2}}i = \frac{\sqrt{2}}{2}i$. The value of our branch of the inverse cosine is then given by: $\cosh^{-1} \frac{\sqrt{2}}{2} = \ln\left[z + (z^2 - 1)^{1/2}\right] = \ln\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right]$. Because $|\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i| = 1$ and $\mathrm{Arg}(\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i) = \frac{\pi}{4}$, the principal branch of the logarithm is $\log_e 1 + i\frac{\pi}{4} = \frac{\pi i}{4}$. Therefore, $\cosh^{-1} \frac{\sqrt{2}}{2} = \frac{\pi i}{4}$.

Computing The Derivative of Inverse Hyperbolic Cosine

(b) We have

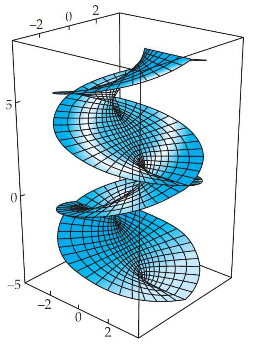
$$\begin{aligned} \frac{d}{dz}\cosh^{-1} z\bigg|_{z=\sqrt{2}/2} &= \frac{1}{(z^2-1)^{1/2}}\bigg|_{z=\sqrt{2}/2} \\ &= \frac{1}{[(\sqrt{2}/2)^2-1]^{1/2}} \\ &= \frac{1}{(-1/2)^{1/2}}. \end{aligned}$$

After using f_2 to find the square root in this expression we obtain:

$$\frac{d}{dz}\cosh^{-1} z\bigg|_{z=\sqrt{2}/2} = \frac{1}{\sqrt{2}i/2} = -\sqrt{2}i.$$

The Riemann Surface of the Sine Revisited

- The multiple-valued function $F(z) = \sin^{-1} z$ can be visualized using the Riemann surface constructed for $\sin z$ previously.



In order to see the image of a point z_0 under the multiple-valued mapping $w = \sin^{-1} z$, we imagine that z_0 is lying in the xy-plane. We then consider all points on the Riemann surface lying directly over z_0.

Each of these points on the surface corresponds to a unique point in one of the squares S_n described previously.

Thus, this infinite set of points in the Riemann surface represents the infinitely many images of z_0 under $w = \sin^{-1} z$.