Mathematical analysis I

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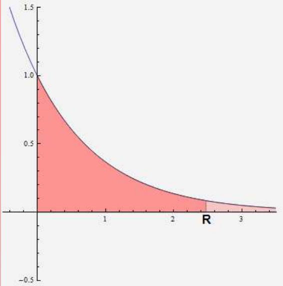
Improper Integrals

Overview of Improper Integrals

Suppose, we wanted to determine the amount of area under the graph of $f(x) = e^{-x}$ over the unbounded interval $[0, \infty)$; This is given by the **improper integral**

$$\int_0^\infty e^{-x} dx;$$



It is called improper because it represents the area of an unbounded region;

To compute such an integral, we first introduce an artificial bound $R > 0$ and we compute instead the proper integral

$$\int_0^R e^{-x} dx = \left. -e^{-x} \right|_0^R = (-e^{-R} - (-1)) = 1 - e^{-R};$$

Finally, we "push" R towards $+\infty$:

$$\int_0^\infty e^{-x} dx = \lim_{R \to \infty} \int_0^R e^{-x} dx = \lim_{R \to \infty} (1 - e^{-R}) = 1 - 0 = 1;$$

Formal Definitions

Definitions of Improper Integrals

- If, for some fixed a, the function $f(x)$ is integrable on $[a, b]$ for all $b > a$, then define the **improper integral of** $f(x)$ **over** $[a, \infty)$ by
$$\int_a^{\infty} f(x)dx = \lim_{R\to\infty}\int_a^R f(x)dx;$$
The integral **converges** if the limits exists and is finite, and it **diverges**, otherwise;
- If, for some fixed a, the function $f(x)$ is integrable on $[b, a]$ for all $b < a$, then define the **improper integral of** $f(x)$ **over** $(-\infty, a]$ by
$$\int_{-\infty}^{a} f(x)dx = \lim_{R\to-\infty}\int_R^a f(x)dx;$$
The integral **converges** if the limits exists and is finite, and it **diverges**, otherwise;

Formal Definitions (Cont'd)

Definition of Third Type of Improper Integral

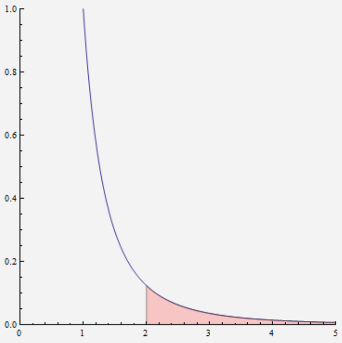
- If, for all $a < 0, b > 0$, the function $f(x)$ is integrable on $[a, 0], [0, b]$, then we define the **improper integral of** $f(x)$ **over** $(-\infty, \infty)$ by

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx;$$

The integral **converges** if both integrals on the right converge and it **diverges**, otherwise;

Example of Improper Integral I

Show that $\int_2^\infty \frac{1}{x^3}dx$ converges and compute its value;



We first calculate

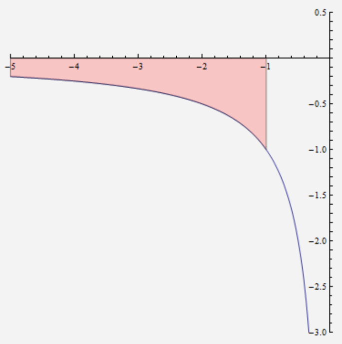
$$\int_2^R \frac{1}{x^3}dx = \left.\frac{-1}{2x^2}\right|_2^R = \frac{1}{8} - \frac{1}{2R^2};$$

Therefore, we obtain:

$$\int_2^\infty \frac{1}{x^3}dx = \lim_{R\to\infty}\int_2^R \frac{1}{x^3}dx = \lim_{R\to\infty}\left(\frac{1}{8} - \frac{1}{2R^2}\right) = \frac{1}{8} - 0 = \frac{1}{8};$$

Example of Improper Integral II

Determine whether $\int_{-\infty}^{-1} \frac{1}{x} dx$ converges;
If so, compute its value;



We first calculate

$$\int_{R}^{-1} \frac{1}{x} dx = \ln|x|\Big|_{R}^{-1} = -\ln|R|;$$

Therefore, we obtain:

$$\int_{-\infty}^{-1} \frac{1}{x} dx = \lim_{R \to -\infty} \int_{R}^{-1} \frac{1}{x} dx = \lim_{R \to -\infty} [-\ln|R|] = \lim_{R \to \infty} [-\ln R] = -\infty;$$

The p-Integral

- Show that, for $a > 0$, $\displaystyle\int_a^\infty \frac{1}{x^p}dx = \begin{cases} \dfrac{a^{1-p}}{p-1}, & \text{if } p > 1 \\ \text{diverges}, & \text{if } p \le 1 \end{cases}$;

$$\int_a^\infty \frac{1}{x^p}dx = \lim_{R\to\infty}\int_a^R x^{-p}dx = \lim_{R\to\infty}\left.\frac{x^{1-p}}{1-p}\right|_a^R = \lim_{R\to\infty}\left(\frac{R^{1-p}}{1-p} - \frac{a^{1-p}}{1-p}\right);$$

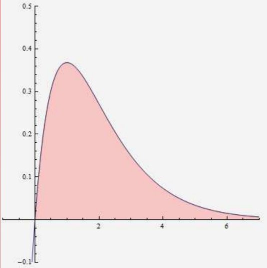
  - If $p > 1$, then $1 - p < 0$, so $\lim_{R\to\infty} R^{1-p} = 0$ and, therefore, $\displaystyle\int_a^\infty \frac{1}{x^p}dx = \frac{a^{1-p}}{p-1}$;
  - If $p < 1$, then $1 - p > 0$, so $\lim_{R\to\infty} R^{1-p} = \infty$; Therefore, the integral diverges;
  - If $p = 1$, then $\displaystyle\int_a^\infty \frac{1}{x}dx = \lim_{R\to\infty}\int_a^R \frac{1}{x}dx = \lim_{R\to\infty}(\ln R - \ln a) = \infty$;

Using L'Hôpital's Rule

- Recall how L'Hôpital's Rule is applied:

$$\lim_{x\to\infty}\frac{x+1}{e^x}\left(=\frac{\infty}{\infty}\right)\overset{\text{L'Hôpital}}{=}\lim_{x\to\infty}\frac{(x+1)'}{(e^x)'}=\lim_{x\to\infty}\frac{1}{e^x}=0;$$

- Calculate $\int_0^\infty xe^{-x}dx$;



$$\int xe^{-x}dx=\int x(-e^{-x})'dx\overset{\text{By Parts}}{=}$$
$$-xe^{-x}-\int -e^{-x}dx=$$
$$-xe^{-x}-e^{-x}=-\frac{x+1}{e^x};$$

$$\int_0^\infty xe^{-x}dx=\lim_{R\to\infty}\int_0^R xe^{-x}dx=\lim_{R\to\infty}\left(1-\frac{R+1}{e^R}\right)=1-0=1;$$

Application: Escape Velocity

- The earth exerts a gravitational force of magnitude $F(r) = G\frac{M_e m}{r^2}$ on an object of mass m at distance r from its center;
  - Find the work required to move the object infinitely far from the earth;

$$W = \int_{r_e}^{\infty} F(r)dr = \int_{r_e}^{\infty} G\frac{M_e m}{r^2}dr = \lim_{R\to\infty}\int_{r_e}^{R} G\frac{M_e m}{r^2}dr =$$

$$GM_e m \lim_{R\to\infty}\int_{r_e}^{R}\frac{1}{r^2}dr = GM_e m \lim_{R\to\infty}\left(-\frac{1}{r}\right)\Big|_{r_e}^{R} =$$

$$GM_e m \lim_{R\to\infty}\left[\frac{1}{r_e} - \frac{1}{R}\right] = G\frac{M_e m}{r_e}\ \text{J};$$

- Calculate the escape velocity v_{esc} on the earth's surface;
  The escape velocity must provide kinetic energy at least as big as the work required to move the object infinitely far from the earth;

$$\frac{1}{2}mv_{\mathrm{esc}}^2 \geq G\frac{M_e m}{r_e} \quad \Rightarrow \quad v_{\mathrm{esc}}^2 \geq \frac{2GM_e}{r_e} \quad \Rightarrow \quad v_{\mathrm{esc}} \geq \sqrt{\frac{2GM_e}{r_e}};$$

Application: Perpetual Annuity

- An investment pays a dividend continuously at a rate of \$6,000 per year; Compute the present value of the income stream if the interest rate is 4% and the dividends continue forever.

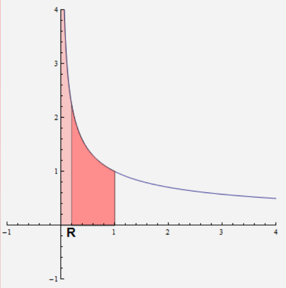
$$\text{PV} = \int_0^\infty Pe^{-rt}dt = \lim_{T\to\infty}\int_0^T 6000e^{-0.04t}dt =$$

$$\lim_{T\to\infty} -\frac{6000}{0.04}e^{-0.04t}\Big|_0^T = -150000\lim_{T\to\infty}(e^{-0.04T}-1) =$$

$$-15000\cdot(-1) = \$150,000;$$

Improper Integrals for Infinite Discontinuities at Endpoints

We determine the amount of area under the graph of $f(x) = \dfrac{1}{\sqrt{x}}$ over the interval $[0, 1]$;



This is given by the **improper integral**

$$\int_0^1 \frac{1}{\sqrt{x}} dx;$$

It is improper because it represents the area of an unbounded region;

To compute such an integral, we first introduce an artificial bound $0 < R < 1$ and we compute instead the proper integral

$$\int_R^1 x^{-1/2} dx = 2\sqrt{x}\Big|_R^1 = 2\sqrt{1} - 2\sqrt{R} = 2 - 2\sqrt{R};$$

Finally, we "push" R towards 0 from the right:

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{R \to 0^+} \int_R^1 \frac{1}{\sqrt{x}} dx = \lim_{R \to 0^+} (2 - 2\sqrt{R}) = 2 - 0 = 2;$$

Definitions of Integrals with Infinite Discontinuities

Integrants with Infinite Discontinuities

- If $f(x)$ is continuous on $[a, b)$ but discontinuous at $x = b$, we define

$$\int_a^b f(x)dx = \lim_{R \to b^-} \int_a^R f(x)dx;$$

- If $f(x)$ is continuous on $(a, b]$ but discontinuous at $x = a$, we define

$$\int_a^b f(x)dx = \lim_{R \to a^+} \int_R^b f(x)dx;$$

- In both cases the improper integral **converges** if the limit exists and it **diverges** otherwise;

Examples of Improper Integrals

- Calculate $\int_0^9 \frac{1}{\sqrt{x}} dx$;

$$\int_0^9 \frac{1}{\sqrt{x}} dx = \lim_{R \to 0^+} \int_R^9 x^{-1/2} dx =$$

$$\lim_{R \to 0^+} [2\sqrt{x}\Big|_R^9] = \lim_{R \to 0^+} (6 - 2\sqrt{R}) = 6 - 0 = 6;$$

- Calculate $\int_0^{1/2} \frac{1}{x} dx$;

$$\int_0^{1/2} \frac{1}{x} dx = \lim_{R \to 0^+} \int_R^{1/2} \frac{1}{x} dx =$$

$$\lim_{R \to 0^+} [\ln x\Big|_R^{1/2}] = \lim_{R \to 0^+} (\ln \frac{1}{2} - \ln R) = \infty;$$

p-Integral Revisited

- Show that, for $a > 0$, $\displaystyle\int_0^a \frac{1}{x^p}dx = \begin{cases} \dfrac{a^{1-p}}{1-p}, & \text{if } p < 1 \\ \text{diverges}, & \text{if } p \geq 1 \end{cases}$;

$$\int_0^a \frac{1}{x^p}dx = \lim_{R\to 0^+}\int_R^a x^{-p}dx = \lim_{R\to 0^+}\left.\frac{x^{1-p}}{1-p}\right|_R^a = \lim_{R\to 0^+}\left(\frac{a^{1-p}}{1-p} - \frac{R^{1-p}}{1-p}\right);$$

  - If $p < 1$, then $1 - p > 0$, so $\lim_{R\to 0^+} R^{1-p} = 0$; Therefore, $\displaystyle\int_0^a \frac{1}{x^p}dx = \frac{a^{1-p}}{1-p}$;
  - If $p > 1$, then $1 - p < 0$, so $\lim_{R\to 0^+} R^{1-p} = \infty$ and, therefore, the integral diverges;
  - If $p = 1$, then $\displaystyle\int_0^a \frac{1}{x}dx = \lim_{R\to 0^+}\int_R^a \frac{1}{x}dx = \lim_{R\to 0^+}(\ln a - \ln R) = \infty$;

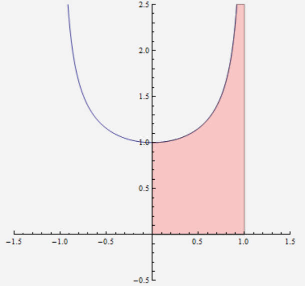
An Additional Example

- Evaluate $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$;

  First, recall the formula $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$;

  We compute

$$\int_0^R \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x\Big|_0^R =$$
$$\sin^{-1} R - \sin^{-1} 0 = \sin^{-1} R;$$



  Therefore, we get

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{R\to 1^-}\int_0^R \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{R\to 1^-}\sin^{-1} R = \frac{\pi}{2};$$

The Comparison Test for Improper Integrals

Comparison Test for Improper Integrals

Assume that $f(x) \geq g(x) \geq 0$ for $x \geq a$;

- If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ also converges;
- If $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ also diverges;

The Comparison Test may also be applied for improper integrals with infinite discontinuities at the endpoints;

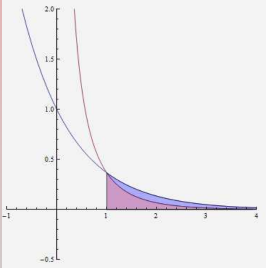
Applying the Comparison Test I

- Show that $\int_1^\infty \frac{e^{-x}}{x} dx$ converges;

  Note that for $x \geq 1$, we get $0 \leq \frac{1}{x} \leq 1 \quad \Rightarrow \quad 0 \leq \frac{e^{-x}}{x} \leq e^{-x}$;

  Therefore, by the comparison test, to show that $\int_1^\infty \frac{e^{-x}}{x} dx$ converges, it suffices to show that $\int_1^\infty e^{-x} dx$ converges; Here is the computation:

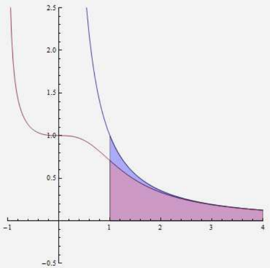
$$\int_1^\infty e^{-x} dx = \lim_{R\to\infty} \int_1^R e^{-x} dx = \lim_{R\to\infty} [(-e^{-x})|_1^R] = \lim_{R\to\infty} (e^{-1} - e^{-R}) = \frac{1}{e};$$



Applying the Comparison Test II

- Show that $\int_1^{\infty} \frac{1}{\sqrt{x^3+1}} dx$ converges;
  Note that for $x \geq 1$, we get
  $x^3 \leq x^3 + 1 \Rightarrow \sqrt{x^3} \leq \sqrt{x^3+1} \Rightarrow 0 \leq \frac{1}{\sqrt{x^3+1}} \leq \frac{1}{\sqrt{x^3}}$; By the comparison test, to show that $\int_1^{\infty} \frac{1}{\sqrt{x^3+1}} dx$ converges, it suffices to show that $\int_1^{\infty} \frac{1}{\sqrt{x^3}} dx$ converges;

  This is, however, true, since this is a p-integral, with $p = \frac{3}{2} > 1$;



Applying the Comparison Test III

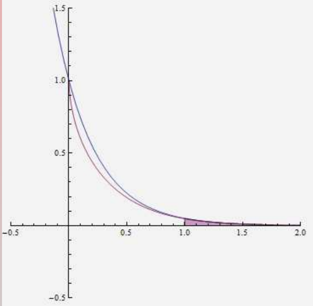
- Does $\int_1^\infty \frac{1}{\sqrt{x}+e^{3x}}dx$ converge?

  Note that for $x \geq 1$, we get $e^{3x} \leq \sqrt{x}+e^{3x} \Rightarrow 0 \leq \frac{1}{\sqrt{x}+e^{3x}} \leq \frac{1}{e^{3x}}$;

  By the comparison test, to show $\int_1^\infty \frac{1}{\sqrt{x}+e^{3x}}dx$ converges, it suffices to show $\int_1^\infty \frac{1}{e^{3x}}dx$ converges;

$$\int_1^\infty \frac{1}{e^{3x}}dx = \lim_{R\to\infty}[-\tfrac{1}{3}e^{-3x}\big|_1^R] =$$

$$\lim_{R\to\infty}\left[\frac{1}{3e^3} - \frac{1}{3e^{3R}}\right] = \frac{1}{3e^3};$$



Applying the Comparison Test IV

- Does $\displaystyle\int_0^{1/2} \frac{1}{x^8+x^2}dx$ converge?
  Note that for $0 < x \le \frac{1}{2}$, we get
  $x^8 \le x^2 \Rightarrow x^8 + x^2 \le x^2 + x^2 = 2x^2 \Rightarrow 0 \le \dfrac{1}{2x^2} \le \dfrac{1}{x^8+x^2}$; By the comparison test, to show $\displaystyle\int_0^{1/2} \frac{1}{x^8+x^2}dx$ diverges, it suffices to show $\displaystyle\frac{1}{2}\int_0^{1/2} \frac{1}{x^2}dx$ diverges;

  This is, however, true, since this is a p-integral, with $p = 2 > 1$;