Mathematical analysis I

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Subsection 6

The Chain Rule

Stewart, 14.5

The Chain Rule

- If $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t, then

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}, \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}.$$

St., example 3, p.926

Example: If $f(x, y) = e^x \sin y$, $x = st^2$, $y = s^2 t$, what are $\frac{\partial f}{\partial s}$, $\frac{\partial f}{\partial t}$?

We have

$$\frac{\partial f}{\partial x} = e^x \sin y, \quad \frac{\partial f}{\partial y} = e^x \cos y.$$

We also have

$$\frac{\partial x}{\partial s} = t^2, \quad \frac{\partial x}{\partial t} = 2st, \quad \frac{\partial y}{\partial s} = 2st, \quad \frac{\partial y}{\partial t} = s^2.$$

Therefore,

$$\frac{\partial f}{\partial s} = e^x \sin y \cdot t^2 + e^x \cos y \cdot 2st, \; \frac{\partial f}{\partial t} = e^x \sin y \cdot 2st + e^x \cos y \cdot s^2.$$

The Chain Rule: General Version

Stewart, p.927

- If f is a differentiable function of the n variables $x_1, x_2, \dots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \dots, t_m$, then f is a differentiable function of $t_1, \dots, t_m$ and, for all $i = 1, \dots, m$,

$$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}.$$

This may be expressed using the dot product:

$$\frac{\partial f}{\partial t_i} = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle \cdot \left\langle \frac{\partial x_1}{\partial t_i}, \frac{\partial x_2}{\partial t_i}, \dots, \frac{\partial x_n}{\partial t_i} \right\rangle.$$

see Example 4, 5, p.927

Using the Chain Rule

- Let $f(x, y, z) = xy + z$. Calculate $\frac{\partial f}{\partial s}$, where $x = s^2$, $y = st$, $z = t^2$. Compute the primary derivatives.

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial f}{\partial z} = 1.$$

Next, we get

$$\frac{\partial x}{\partial s} = 2s, \quad \frac{\partial y}{\partial s} = t, \quad \frac{\partial z}{\partial s} = 0.$$

Now apply the Chain Rule:

$$\begin{aligned} \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} \\ &= y \cdot 2s + x \cdot t + 1 \cdot 0 \\ &= (st) \cdot 2s + s^2 \cdot t = 3s^2 t. \end{aligned}$$

Evaluating the Derivative

Example 5, p.927

- If $f = x^4y + y^2z^3$, $x = rse^t$, $y = rs^2e^{-t}$ and $z = r^2s\sin t$, find $\frac{\partial f}{\partial s}$ when $r = 2, s = 1$ and $t = 0$.
  Note, first, that for $(r, s, t) = (2, 1, 0)$, we have $(x, y, z) = (2, 2, 0)$. Moreover,
  $$\frac{\partial f}{\partial x} = 4x^3y, \quad \frac{\partial f}{\partial y} = x^4 + 2yz^3, \quad \frac{\partial f}{\partial z} = 3y^2z^2.$$
  Thus, for $(r, s, t) = (2, 1, 0)$, we get $\frac{\partial f}{\partial x} = 64, \quad \frac{\partial f}{\partial y} = 16, \quad \frac{\partial f}{\partial z} = 0.$ Furthermore,
  $$\frac{\partial x}{\partial s} = re^t, \quad \frac{\partial y}{\partial s} = 2rse^{-t}, \quad \frac{\partial z}{\partial s} = r^2\sin t.$$
  Thus, for $(r, s, t) = (2, 1, 0)$, we get $\frac{\partial x}{\partial s} = 2, \quad \frac{\partial y}{\partial s} = 4, \quad \frac{\partial z}{\partial s} = 0.$
  Therefore, $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} = 64 \cdot 2 + 16 \cdot 4 + 0 \cdot 0 = 192.$

Polar Coordinates

see Stewart, ch. 10, 10.3, 10.4

- Let $f(x, y)$ be a function of two variables, and let (r, θ) be polar coordinates.
  - (a) Express $\frac{\partial f}{\partial \theta}$ in terms of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
  - (b) Evaluate $\frac{\partial f}{\partial \theta}$ at $(x, y) = (1, 1)$ for $f(x, y) = x^2 y$.

(a) Since $x = r\cos\theta$ and $y = r\sin\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$.
By the Chain Rule,

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}.$$

Since $x = r\cos\theta$ and $y = r\sin\theta$, we can write $\frac{\partial f}{\partial \theta}$ in terms of x and y alone: $\frac{\partial f}{\partial \theta} = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}$.

(b) Apply the preceding equation to $f(x, y) = x^2 y$:

$$\begin{aligned} \frac{\partial f}{\partial \theta} &= -y\frac{\partial}{\partial x}(x^2 y) + x\frac{\partial}{\partial y}(x^2 y) = -2xy^2 + x^3; \\ \left.\frac{\partial f}{\partial \theta}\right|_{(x,y)=(1,1)} &= -2\cdot 1\cdot 1^2 + 1^3 = -1. \end{aligned}$$

An Abstract Example on the Chain Rule

Example 6, p.928

- If $g(s,t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the PDE $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$.
  Notice that $g(s,t) = f(x,y)$, where $x = s^2 - t^2$ and $y = t^2 - s^2$. Thus, by the chain rule, we get

$$\begin{aligned} \frac{\partial g}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ &= 2s\frac{\partial f}{\partial x} - 2s\frac{\partial f}{\partial y}; \end{aligned}$$

$$\begin{aligned} \frac{\partial g}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\ &= -2t\frac{\partial f}{\partial x} + 2t\frac{\partial f}{\partial y}. \end{aligned}$$

Therefore,

$$\begin{aligned} t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} &= t(2s\frac{\partial f}{\partial x} - 2s\frac{\partial f}{\partial y}) + s(-2t\frac{\partial f}{\partial x} + 2t\frac{\partial f}{\partial y}) \\ &= 2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y} - 2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y} \\ &= 0. \end{aligned}$$

Implicit Differentiation: $y = y(x)$

Stewart, p.928 - 930

- Suppose that the equation $F(x, y) = 0$ defines y implicitly as a function of x.

  By the chain rule $\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$, whence

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}.$$

$F_y \neq 0$

Example 8, p.929

Example: Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$.

We have $F(x, y) = x^3 + y^3 - 6xy = 0$, whence

$$\frac{\partial F}{\partial x} = 3x^2 - 6y, \quad \frac{\partial F}{\partial y} = 3y^2 - 6x.$$

Therefore, $\frac{dy}{dx} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$.

Implicit Differentiation $z = z(x, y)$

Stewart,p.929-930

- Suppose that the equation $F(x, y, z) = 0$ defines z implicitly as a function of x and y.
  By the chain rule $\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$.
  But, we also have $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$, whence $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$, giving

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}. \quad \text{Similarly} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

$\frac{\partial F}{\partial z} \neq 0$

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.
We have $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1 = 0$, whence

$$\frac{\partial F}{\partial x} = 3x^2 + 6yz, \quad \frac{\partial F}{\partial y} = 3y^2 + 6xz, \quad \frac{\partial F}{\partial z} = 3z^2 + 6xy.$$

Therefore, $\frac{\partial z}{\partial x} = -\frac{3x^2+6yz}{3z^2+6xy} = -\frac{x^2+2yz}{z^2+2xy}$;
$\frac{\partial z}{\partial y} = -\frac{3y^2+6xz}{3z^2+6xy} = -\frac{y^2+2xz}{z^2+2xy}$.

Subsection 7

Optimization in Several Variables

Stewart, 14.7, p.946-956

Maxima and Minima

- A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$, when (x, y) is near (a, b). The z-value $f(a, b)$ is called the **local maximum value**.
- A function of two variables has a **local minimum** at (a, b) if $f(x, y) \geq f(a, b)$, when (x, y) is near (a, b). The z-value $f(a, b)$ is called the **local minimum value**.

Theorem Necessary Conditions

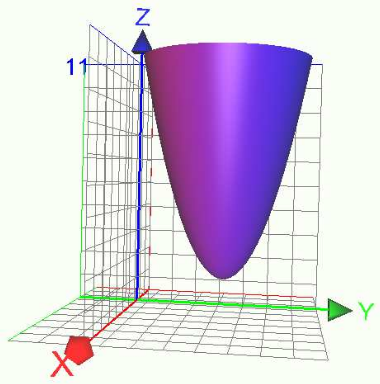
If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

- A point (a, b) is called a **critical point** of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.
- As was the case with functions of a single variable the critical points are candidates for local extrema. At a critical point the function may have a local maximum, a local minimum or neither.

Finding Critical Points

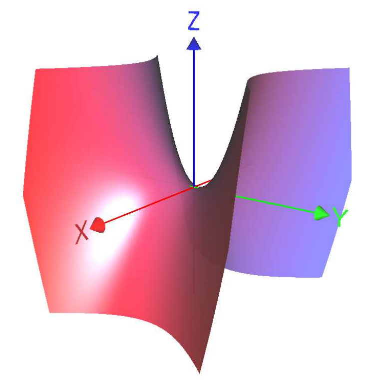
- Suppose $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Then, we have $f_x(x, y) = 2x - 2$ and $f_y = 2y - 6$. Therefore, f has a critical point $(x, y) = (1, 3)$. By rewriting $f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$, we see that $f(x, y) \geq 4 = f(1, 3)$. Therefore, f has an absolute minimum at $(1, 3)$ equal to 4.

Example 1, p.946



Another Example of Finding Critical Points

- Suppose $f(x,y) = y^2 - x^2$. Then, we have $f_x(x,y) = -2x$ and $f_y = 2y$. Therefore, f has a critical point $(x,y) = (0,0)$. Note, however, that for points on x-axis $f(x,0) = -x^2 \leq f(0,0)$ and for points on the y-axis $f(0,y) = y^2 \geq f(0,0)$. Thus, $f(0,0)$ can be neither a local max nor a local min.



The kind of point that occurs at $(0,0)$ is this case is called a **saddle point** because of its shape.

Example 2, p.947

Second Derivative Test

Stewart, p.947

Sufficient Conditions

- Suppose that $f_x(a,b) = 0$ and $f_y(a,b) = 0$ and that f has continuous second partial derivatives on a disk with center (a,b).

  Define
  $$D = D(a,b) = \overbrace{f_{xx}(a,b)}^{A}\,\overbrace{f_{yy}(a,b)}^{B} - [\overbrace{f_{xy}(a,b)}^{C}]^2 = \begin{vmatrix} A & C \\ C & B \end{vmatrix}$$

  Then, the following possibilities may occur:
  - If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum;
  - If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum;
  - If $D < 0$, then $f(a,b)$ is neither a local max nor a local min;
    In this case f has a **saddle point** at (a,b) and the graph of f crosses the tangent plane at (a,b);
  - If $D = 0$, the test is inconclusive;
    In this case, f could have a local min, a local max, a saddle point or none of the above.

Example I

- Find the local extrema and the saddle points of $f(x,y) = (x^2 + y^2)e^{-x}$.

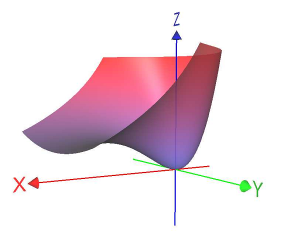
  We have $f_x(x,y) = 2xe^{-x} - (x^2 + y^2)e^{-x} = (2x - x^2 - y^2)e^{-x}$.
  Moreover, $f_{xx}(x,y) = (2 - 4x + x^2 + y^2)e^{-x}$ and $f_{xy}(x,y) = -2ye^{-x}$.

  Also $f_y(x,y) = 2ye^{-x}$ and $f_{yy}(x,y) = 2e^{-x}$.

  We now obtain $2ye^{-x} = 0$ implies $y = 0$ and, thus, $2x - x^2 = x(2-x) = 0$. This implies $x = 0$ or $x = 2$.

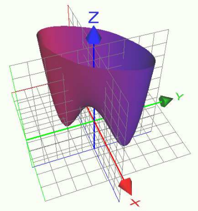
  Therefore, we get critical points $(0,0), (2,0)$.
  We compute

  $$D(0,0) = 2 \cdot 2 - 0^2 = 4 > 0$$
  $$f_{xx}(0,0) = 2 > 0$$
  $$D(2,0) = \frac{-2}{e^2}\frac{2}{e^2} - 0^2 = -\frac{4}{e^4} < 0$$



Example II

- Find the local extrema and the saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$.
  We have $f_x(x,y) = 4x^3 - 4y = 4(x^3 - y)$. Moreover, $f_{xx}(x,y) = 12x^2$ and $f_{xy}(x,y) = -4$.
  Also $f_y(x,y) = 4y^3 - 4x = 4(y^3 - x)$. Also, $f_{yy}(x,y) = 12y^2$.
  The system $\left\{ \begin{array}{rcl} x^3 - y & = & 0 \\ y^3 - x & = & 0 \end{array} \right\}$ gives $x^9 - x = 0$, and, thus, $x(x^8 - 1) = 0$. This implies $x = 0$ or $x^8 = 1$, whence $x = 0, x = \pm 1$.
  Therefore, we get critical points $(0,0), (-1,-1)$ and $(1,1)$.
  We compute
  $D(0,0) = 0 \cdot 0 - (-4)^2 = -16 < 0$
  $D(-1,-1) = 12 \cdot 12 - (-4)^2 = 128 > 0$
  $f_{xx}(-1,-1) = 12 > 0$
  $D(1,1) = 12 \cdot 12 - (-4)^2 = 128 > 0$
  $f_{xx}(1,1) = 12 > 0$



Example III

Example 5, p.950

- Find the shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$. The distance of $(1, 0, -2)$ from a point (x, y, z) is given by $d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$.
  If the point (x, y, z) is on the plane $x + 2y + z = 4$, then $z = 4 - x - 2y$, whence the distance formula becomes a function of two variables only

  $$d(x, y) = \sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2}.$$

  We want to minimize this function. We look instead at minimizing the square function $f(x, y) = d^2(x, y) = (x-1)^2 + y^2 + (6 - x - 2y)^2$. We compute partial derivatives and set them equal to zero to find critical points:

  $$f_x(x, y) = 2(x-1) - 2(6 - x - 2y) = 2(2x + 2y - 7) = 0$$
  $$f_y(x, y) = 2y - 4(6 - x - 2y) = 2(2x + 5y - 12) = 0;$$

Example III (Cont'd)

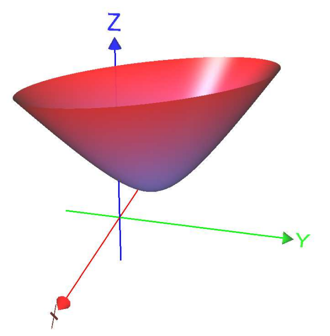
- We have

$$\left\{\begin{array}{rcl} 2x+2y & = & 7 \\ 2x+5y & = & 12 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} y = \frac{5}{3} \\ x = -\frac{5}{3}+\frac{7}{2} = \frac{11}{6}. \end{array}\right.$$

We can verify using the second derivative test that at $(\frac{11}{6}, \frac{5}{3})$ we have a minimum, but this is clear from the interpretation of $d(x,y)$.

Moreover, we can compute

$$z = 4 - x - 2y = 4 - \frac{11}{6} - \frac{10}{3} = -\frac{7}{6}.$$



Thus the point is $(\frac{11}{6}, \frac{5}{3}, -\frac{7}{6})$.

$d = \sqrt{(5/6)^2 + (5/3)^2 + (5/6)^2} = \frac{5}{6}\sqrt{6}$

Example IV

Example 6, p.950

- What is the max possible volume of a rectangular box without a lid that can be made of 12 square meters of cardboard?

  The volume equation is $V = \ell wh$ and the equation for the amount of cardboard gives $\ell w + 2\ell h + 2wh = 12$.

  The latter equation solved for h gives $h = \frac{12-\ell w}{2(\ell+w)}$.

  Therefore, the equation for the volume becomes $V = \frac{12\ell w - \ell^2 w^2}{2(\ell+w)}$.

  We compute V_ℓ using the quotient rule:

$$\begin{aligned}
V_\ell &= \frac{(12w-2\ell w^2)2(\ell+w)-2(12\ell w-\ell^2 w^2)}{4(\ell+w)^2} \\
&= \frac{(12w-2\ell w^2)(\ell+w)-(12\ell w-\ell^2 w^2)}{2(\ell+w)^2} \\
&= \frac{12w\ell+12w^2-2\ell^2 w^2-2\ell w^3-12\ell w+\ell^2 w^2}{2(\ell+w)^2} \\
&= \frac{12w^2-\ell^2 w^2-2\ell w^3}{2(\ell+w)^2} = \frac{w^2(12-\ell^2-2\ell w)}{2(\ell+w)^2}.
\end{aligned}$$

Example IV (Cont'd)

- By symmetry, we get

$$V_\ell = \frac{w^2(12 - \ell^2 - 2\ell w)}{2(\ell + w)^2}, \quad V_w = \frac{\ell^2(12 - w^2 - 2\ell w)}{2(\ell + w)^2}.$$

The system $\left\{ \begin{array}{rcl} 12 - 2\ell w - \ell^2 & = & 0 \\ 12 - 2\ell w - w^2 & = & 0 \end{array} \right\}$ gives $\ell^2 - w^2 = 0$ or $(\ell + w)(\ell - w) = 0$, yielding (since $\ell, w > 0$) $\ell = w$.
So $12 - 3\ell^2 = 0 \Rightarrow \ell^2 = 4 \Rightarrow \ell = 2$. Thus, since $h = \frac{12 - \ell w}{2(\ell + w)}$, we obtain that

$$\ell = 2, \quad w = 2 \quad \text{and} \quad h = 1.$$

The maximum volume is, therefore, 4 cubic meters.

Extreme Value Theorem

Stewart, p.951-952

Extreme Value Theorem: Functions of Two Variables

If f is continuous on a closed and bounded set $\mathcal{D}$ in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in $\mathcal{D}$.

- To find those absolute extrema in a closed and bounded set $\mathcal{D}$, we use

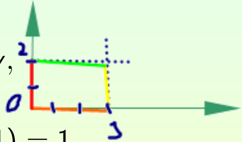
The Closed and Bounded Region Method

1. Find the values of f at the critical points of f in $\mathcal{D}$;
2. Find the extreme values of f on the boundary of $\mathcal{D}$;
3. The largest of the values from the previous steps is the absolute maximum value and the smallest of these values is the absolute minimum value.

Finding Absolute Extrema in Closed Bounded Set

Example 7, p.952

- Find the absolute extrema of $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $\mathcal{D} = \{(x, y) : 0 \le x \le 3, 0 \le y \le 2\}$.

  Compute the partial derivatives: $f_x(x, y) = 2x - 2y$, $f_y(x, y) = -2x + 2$.

  Therefore, the only critical point is $(1, 1)$ and $f(1, 1) = 1$.

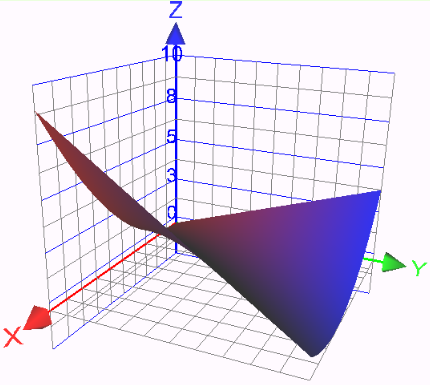
  On the boundary, we have

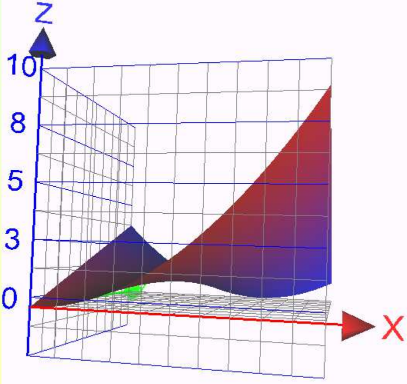
  - If $0 \le x \le 3, y = 0$, then $f(x, 0) = x^2$ has min $f(0, 0) = 0$ and max $f(3, 0) = 9$.
  - If $x = 3, 0 \le y \le 2$, then $f(3, y) = 9 - 4y$ has min $f(3, 2) = 1$ and max $f(3, 0) = 9$.
  - If $0 \le x \le 3, y = 2$, then $f(x, 2) = (x - 2)^2$ has min $f(2, 2) = 0$ and max $f(0, 2) = 4$.
  - If $x = 0, 0 \le y \le 2$, then $f(0, y) = 2y$ has min $f(0, 0) = 0$ and max $f(0, 2) = 4$.

Illustration of $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $\mathcal{D}$

- Thus, on the boundary, the min value is $f(0, 0) = f(2, 2) = 0$ and the max value is $f(3, 0) = 9$.

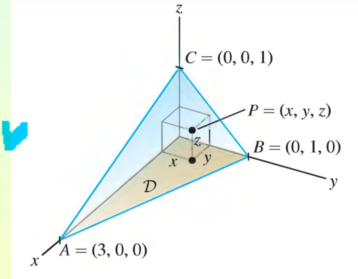
  Since $f(1, 1) = 1$ these are also the absolute extrema on $\mathcal{D}$.





Application

- What is the max possible volume of a box inscribed in the tetrahedron bounded by the coordinate planes and the plane $\frac{1}{3}x + y + z = 1$?



The volume equation is $V = xyz$. Since the (x, y, z) is a point on $\frac{1}{3}x+y+z = 1$, we must have $z = 1-\frac{1}{3}x-y$. Therefore, $V = xy(1-\frac{1}{3}x-y) = xy-\frac{1}{3}x^2y-xy^2$.
We get:
$\frac{\partial V}{\partial x} = y - \frac{2}{3}xy - y^2 = y(1 - \frac{2}{3}x - y)$,
$\frac{\partial V}{\partial y} = x - \frac{1}{3}x^2 - 2xy = x(1 - \frac{1}{3}x - 2y)$.
Therefore,

$$\left\{ \begin{array}{rcl} \frac{2}{3}x + y & = & 1 \\ \frac{1}{3}x + 2y & = & 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} \frac{4}{3}x + 2y & = & 2 \\ \frac{1}{3}x + 2y & = & 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} x & = & 1 \\ y & = & \frac{1}{3} \end{array} \right\}.$$

Since the maximum cannot occur on the boundary, we get that the maximum volume is $1 \cdot \frac{1}{3} - \frac{1}{3} \cdot 1^2 \cdot \frac{1}{3} - 1 \cdot (\frac{1}{3})^2 = \frac{1}{9}$ cubic meters.

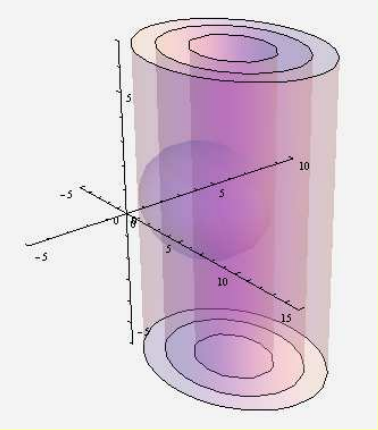
Subsection 8

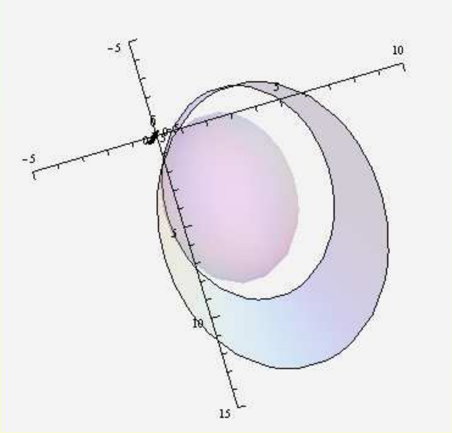
Lagrange Multipliers

Stewart, 14.8

Illustration of General Idea of Lagrange Multipliers

Problem: Maximize or minimize an objective function $f(x, y, z) = (x-5)^2 + 3(y-3)^2$ subject to a constraint $g(x, y, z) = (x-4)^2 + 3(y-2)^2 + 4(z-1)^2 = 20 = k$.





Lagrange Multipliers

- **Problem:** Maximize or minimize an objective function $f(x, y, z)$ subject to a constraint $g(x, y, z) = k$.
  Example: Maximize the volume $V(\ell, w, h) = \ell wh$ subject to $S(\ell, w, h) = \ell w + 2\ell h + 2wh = 12$.

The Method of Lagrange Multipliers

(a) Find all values of (x, y, z) and λ (a parameter called a **Lagrange multiplier**), such that

$$\left\{ \begin{array}{rcl} \nabla f(x, y, z) & = & \lambda \nabla g(x, y, z) \\ g(x, y, z) & = & k \end{array} \right\} \tag{1}$$

(b) Evaluate f at all (x, y, z) found in (a): The largest value is the max of f and the smallest value is the min of f.

- Recall that $\nabla f = \langle f_x, f_y, f_z \rangle$ and $\nabla g = \langle g_x, g_y, g_z \rangle$.
  So, the System (1) may be rewritten in the form:

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad f_z = \lambda g_z, \quad g = k.$$

$L(x,y,z,\lambda) = f(x,y,z) - \lambda\,(g(x,y,z)-k)$ Lagrange function

Example I: Lagrange Multiplier Method

Example 2, p.960

- Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

  Set $g(x,y) = x^2 + y^2$ and we want $g(x,y) = 1$.

  We get the system

$$\left\{ \begin{array}{lcl} f_x(x,y) & = & \lambda g_x(x,y) \\ f_y(x,y) & = & \lambda g_y(x,y) \\ g(x,y) & = & 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} 2x & = & \lambda 2x \\ 4y & = & \lambda 2y \\ x^2 + y^2 & = & 1 \end{array} \right\} \Rightarrow$$

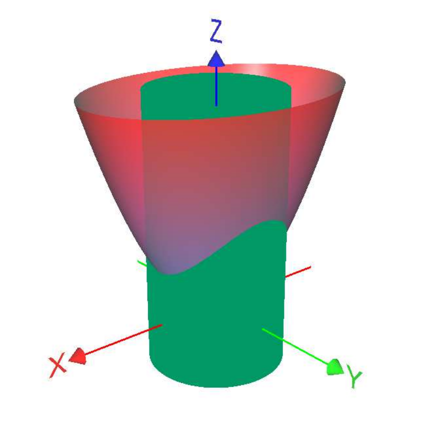
$$\left\{ \begin{array}{lcl} x = 0 & \text{or} & \lambda = 1 \\ y = 0 & \text{or} & \lambda = 2 \end{array} \right.$$

  Therefore, we get for (x,y) the values $(0,\pm 1)$ and $(\pm 1, 0)$.

  Since $f(0,\pm 1) = 2$ and $f(\pm 1, 0) = 1$, f has max 2 and min 1, subject to $x^2 + y^2 = 1$.

Example I Illustrated

- The extreme values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.
  Max: $f(0, \pm 1) = 2$ and Min: $f(\pm 1, 0) = 1$.



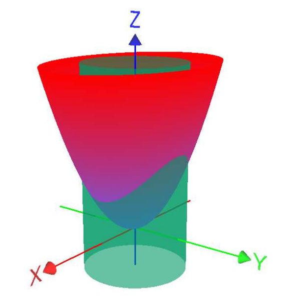
Example I Modified

- Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$.

  Recall the method for finding extreme values on a closed and bounded region!

  First, we find critical points of f: We have $f_x = 2x$ and $f_y = 4y$; Thus, the only critical point is $(x, y) = (0, 0)$ and $f(0, 0) = 0$.

  Then we compute min and max on the boundary: We did this using Lagrange multipliers and found min $f(\pm 1, 0) = 1$ and max $f(0, \pm 1) = 2$.
  Therefore, on the disk $x^2 + y^2 \leq 1$, f has absolute min $f(0, 0) = 0$ and absolute max $f(0, \pm 1) = 2$.



Example II: Lagrange Multiplier Method

- Find the extreme values of $f(x,y) = 2x + 5y$ on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

  Set $g(x,y) = \frac{x^2}{16} + \frac{y^2}{9}$ and we want $g(x,y) = 1$.

  We get the system

$$\left\{ \begin{array}{rcl} f_x(x,y) & = & \lambda g_x(x,y) \\ f_y(x,y) & = & \lambda g_y(x,y) \\ g(x,y) & = & 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} 2 & = & \lambda \frac{x}{8} \\ 5 & = & \lambda \frac{2y}{9} \\ \frac{x^2}{16} + \frac{y^2}{9} & = & 1 \end{array} \right\} \Rightarrow$$

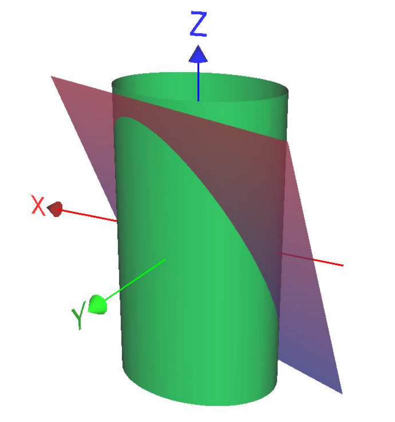
$$\left\{ \begin{array}{l} x = \frac{16}{\lambda} \\ y = \frac{45}{2\lambda} \\ \frac{16^2}{16\lambda^2} + \frac{45^2}{36\lambda^2} = 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x = \frac{16}{\lambda} \\ y = \frac{45}{2\lambda} \\ \frac{64}{4\lambda^2} + \frac{225}{4\lambda^2} = 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x = \pm\frac{32}{17} \\ y = \pm\frac{45}{17} \\ \lambda = \pm\frac{17}{2} \end{array} \right.$$

  Therefore, we get for (x,y) the values $(\frac{32}{17}, \frac{45}{17})$ and $(-\frac{32}{17}, -\frac{45}{17})$.

  We compute $f(\frac{32}{17}, \frac{45}{17}) = 17$ and $f(-\frac{32}{17}, -\frac{45}{17}) = -17$.

Example II Illustrated

- The extreme values of $f(x, y) = 2x + 5y$ on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.
  Max: $f(\frac{32}{17}, \frac{45}{17}) = 17$ and Min: $f(-\frac{32}{17}, -\frac{45}{17}) = -17$.



Example III: Lagrange Multiplier Method

- Find the points on the sphere $x^2+y^2+z^2=4$ with smallest and largest square distance from the point $(3,1,-1)$.
  Set $f(x,y,z)=(x-3)^2+(y-1)^2+(z+1)^2$ be the square distance from (x,y,z) to $(3,1,-1)$ and $g(x,y,z)=x^2+y^2+z^2$ so that $g(x,y,z)=4$.
  We get the system

$$\left\{\begin{array}{rcl} f_x(x,y,z) & = & \lambda g_x(x,y,z) \\ f_y(x,y,z) & = & \lambda g_y(x,y,z) \\ f_z(x,y,z) & = & \lambda g_z(x,y,z) \\ g(x,y,z) & = & 4 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} 2(x-3) & = & \lambda 2x \\ 2(y-1) & = & \lambda 2y \\ 2(z+1) & = & \lambda 2z \\ x^2+y^2+z^2 & = & 4 \end{array}\right\}$$

$$\Rightarrow \left\{\begin{array}{rcl} \frac{1}{\lambda-1} & = & -\frac{1}{3}x \\ \frac{1}{\lambda-1} & = & -y \\ \frac{1}{\lambda-1} & = & z \\ x^2+y^2+z^2 & = & 4 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x & = & -3z \\ y & = & -z \\ x^2+y^2+z^2 & = & 4 \end{array}\right\}$$

Example III: Lagrange Multiplier Method (Cont'd)

- The system gives

$$\left\{ \begin{array}{rcl} x & = & -3z \\ y & = & -z \\ 9z^2+z^2+z^2 & = & 4 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x = \mp \frac{6}{\sqrt{11}} \\ y = \mp \frac{2}{\sqrt{11}} \\ z = \pm \frac{2}{\sqrt{11}} \end{array} \right\}$$

Therefore, we get

$$\begin{array}{l} (x,y,z) = (\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}) \text{ or} \\ (x,y,z) = (-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}) \end{array}.$$

f has smallest value at one of those points and the largest at the other.

$f(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}) = \frac{165-44\sqrt{11}}{11} = 15 - 11\sqrt{11}$,

$f(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}) = \frac{165+44\sqrt{11}}{11} = 15 + 11\sqrt{11}$.

Lagrange Multipliers with Two Constraints

- **Problem:** Maximize or minimize an objective function $f(x,y,z)$ subject to the constraints $g(x,y,z)=k$ and $h(x,y,z)=c$.

The Method of Lagrange Multipliers Revisited

(a) Find all values of (x,y,z) and λ,μ (two parameters called **Lagrange multipliers**), such that

$$\left\{\begin{array}{rcl} \nabla f(x,y,z) & = & \lambda\nabla g(x,y,z)+\mu\nabla h(x,y,z) \\ g(x,y,z) & = & k \\ h(x,y,z) & = & c \end{array}\right\} \qquad (2)$$

(b) Evaluate f at all (x,y,z) resulting from (a): The largest value is the max of f and the smallest value is the min of f.

- Since $\nabla f=\langle f_x,f_y,f_z\rangle$, $\nabla g=\langle g_x,g_y,g_z\rangle$ and $\nabla h=\langle h_x,h_y,h_z\rangle$ the System (2) may be rewritten in the form:

$$f_x=\lambda g_x+\mu h_x,\quad f_y=\lambda g_y+\mu h_y,\quad f_z=\lambda g_z+\mu h_z,\quad g=k,\quad h=c.$$

$L(x,y,z,\lambda,\mu)=f(x,y,z)-\lambda(g(x,y,z)-k)-\mu(h(x,y,z)-c)$

Example IV: Lagrange Multiplier Method

Example 5, p. 962

- Find the extreme values of $f(x, y, z) = x + 2y + 3z$ on the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.
  Set $g(x, y, z) = x - y + z$ and $h(x, y, z) = x^2 + y^2$ so that $g(x, y, z) = 1$ and $h(x, y, z) = 1$.
  We get the system

$$\left\{ \begin{array}{rcl} f_x(x,y,z) & = & \lambda g_x(x,y,z) + \mu h_x(x,y,z) \\ f_y(x,y,z) & = & \lambda g_y(x,y,z) + \mu h_y(x,y,z) \\ f_z(x,y,z) & = & \lambda g_z(x,y,z) + \mu h_z(x,y,z) \\ g(x,y,z) & = & 1 \\ h(x,y,z) & = & 1 \end{array} \right\} \Rightarrow$$

$$\left\{ \begin{array}{rcl} 1 & = & \lambda + \mu 2x \\ 2 & = & -\lambda + \mu 2y \\ 3 & = & \lambda \\ x - y + z & = & 1 \\ x^2 + y^2 & = & 1 \end{array} \right\} \Rightarrow$$

Example IV: Lagrange Multiplier Method (Cont'd)

$$\left\{\begin{array}{l} 1=\lambda+\mu 2x \\ 2=-\lambda+\mu 2y \\ 3=\lambda \\ x-y+z=1 \\ x^2+y^2=1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \lambda=3 \\ x=\ -\frac{1}{\mu} \\ y=\frac{5}{2\mu} \\ x-y+z=1 \\ \frac{1}{\mu^2}+\frac{25}{4\mu^2}=1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \lambda=3 \\ \mu=\ \pm\frac{\sqrt{29}}{2} \\ x=\ \mp\frac{2}{\sqrt{29}} \\ y=\ \pm\frac{5}{\sqrt{29}} \\ z=1\pm\frac{7}{\sqrt{29}} \end{array}\right.$$

Therefore, we get for (x, y, z) the values $(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1+\frac{7}{\sqrt{29}})$ and $(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1-\frac{7}{\sqrt{29}})$.

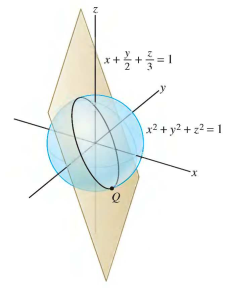
The max of f occurs at the first point and is $3+\sqrt{29}$.

Example V: Lagrange Multiplier Method

- The intersection of the plane $x + \frac{1}{2}y + \frac{1}{3}z = 0$ with the unit sphere $x^2 + y^2 + z^2 = 1$ is a great circle. Find the point on this great circle with the largest x coordinate.

  Set $f(x,y,z) = x$, $g(x,y,z) = x + \frac{1}{2}y + \frac{1}{3}z$ and $h(x,y,z) = x^2 + y^2 + z^2$ so that $g(x,y,z) = 0$ and $h(x,y,z) = 1$. We get the system

$$\left\{ \begin{array}{lcl} f_x(x,y,z) & = & \lambda g_x(x,y,z) + \mu h_x(x,y,z) \\ f_y(x,y,z) & = & \lambda g_y(x,y,z) + \mu h_y(x,y,z) \\ f_z(x,y,z) & = & \lambda g_z(x,y,z) + \mu h_z(x,y,z) \\ g(x,y,z) & = & 0 \\ h(x,y,z) & = & 1 \end{array} \right\}.$$



Example V: Lagrange Multiplier Method (Cont'd)

- Since $f(x,y,z)=x$, $g(x,y,z)=x+\frac{1}{2}y+\frac{1}{3}z$ and $h(x,y,z)=x^2+y^2+z^2$, we get

$$\left\{\begin{array}{rcl} f_x(x,y,z) & = & \lambda g_x(x,y,z)+\mu h_x(x,y,z) \\ f_y(x,y,z) & = & \lambda g_y(x,y,z)+\mu h_y(x,y,z) \\ f_z(x,y,z) & = & \lambda g_z(x,y,z)+\mu h_z(x,y,z) \\ g(x,y,z) & = & 0 \\ h(x,y,z) & = & 1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 1=\lambda+2\mu x \\ 0=\frac{1}{2}\lambda+2\mu y \\ 0=\frac{1}{3}\lambda+2\mu z \\ x+\frac{1}{2}y+\frac{1}{3}z=0 \\ x^2+y^2+z^2=1 \end{array}\right\}.$$

$\mu \neq 0$ $y=\frac{3}{2}z$

Note that μ cannot be zero. The second and third equations yield $\lambda = -4\mu y$ and $\lambda = -6\mu z$. Thus, $-4\mu y=-6\mu z$, i.e., since $\mu \neq 0$, $y=\frac{3}{2}z$. Applying $x+\frac{1}{2}y+\frac{1}{3}z=0$, we get $x=-\frac{13}{12}z$. Finally, we substitute into $x^2+y^2+z^2=1$ to get $(-\frac{13}{12}z)^2+(\frac{3}{2}z)^2+z^2=1$, whence $\frac{637}{144}z^2=1$, yielding $z=\pm\frac{12}{7\sqrt{13}}$. $\Rightarrow$ 1) $z=\frac{12}{7\sqrt{13}}$ 2) $z=-\frac{12}{7\sqrt{13}}$

Therefore, we obtain the critical points $(-\frac{\sqrt{13}}{7},\frac{18}{7\sqrt{13}},\frac{12}{7\sqrt{13}})$ $(\frac{\sqrt{13}}{7},-\frac{18}{7\sqrt{13}},-\frac{12}{7\sqrt{13}})$. We conclude that the max x occurs at the second point and is equal to $\frac{\sqrt{13}}{7}$.