AM II

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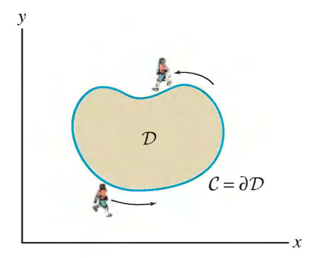
Subsection 1

Green's Theorem

Simple Closed Curves and Boundary Orientation

- Consider a domain $\mathcal{D}$ whose boundary $\mathcal{C}$ is a simple closed curve, i.e., a closed curve that does not intersect itself.
  We follow standard usage and denote the boundary curve $\mathcal{C}$ by $\partial\mathcal{D}$.



- The counterclockwise orientation of $\partial\mathcal{D}$ is called the **boundary orientation**.

  When you traverse the boundary in this direction, the domain lies to your left as in the figure.

Line Integrals: A Reminder

- Recall the two notations for the line integral of $\boldsymbol{F} = \langle F_1, F_2 \rangle$:

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} \quad \text{and} \quad \int_{\mathcal{C}} F_1 dx + F_2 dy.$$

- If $\mathcal{C}$ is parametrized by $\boldsymbol{c}(t) = (x(t), y(t))$, for $a \leq t \leq b$, then

$$dx = x'(t)dt, \quad dy = y'(t)dt.$$

  So we get that

$$\int_{\mathcal{C}} F_1 dx + F_2 dy = \int_a^b (F_1(x(t), y(t))x'(t) + F_2(x(t), y(t))y'(t))dt.$$

- We will assume that:
  - The components of all vector fields have continuous partial derivatives;
  - $\mathcal{C}$ is smooth ($\mathcal{C}$ has a parametrization with derivatives of all orders) or piece wise smooth (a finite union of smooth curves joined together at corners).

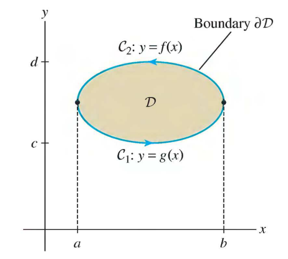
Green's Theorem

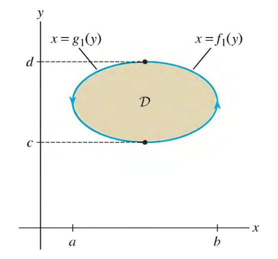
Theorem (Green's Theorem)

Let $\mathcal{D}$ be a domain whose boundary $\partial\mathcal{D}$ is a simple closed curve, oriented counterclockwise. Then

$$\oint_{\partial\mathcal{D}} F_1 dx + F_2 dy = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$$

- We only prove the case in which $\partial\mathcal{D}$ can be described as:
  - the union of two graphs $y = g(x)$ and $y = f(x)$ with $g(x) \leq f(x)$ and
  - the union of two graphs $x = g_1(y)$ and $x = f_1(y)$, with $g_1(y) \leq f_1(y)$.



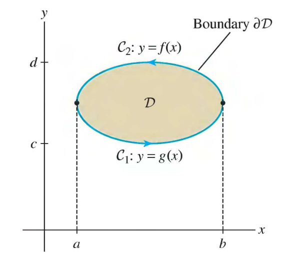


Green's Theorem (Cont'd)

- Green's Theorem is obtained by adding the two equations

$$\oint_{\partial \mathcal{D}} F_1 dx = -\iint_{\mathcal{D}} \frac{\partial F_1}{\partial y} dA, \quad \oint_{\partial \mathcal{D}} F_2 dy = \iint_{\mathcal{D}} \frac{\partial F_2}{\partial x} dA.$$

To prove the first, we write $\oint_{\partial \mathcal{D}} F_1 dx = \int_{\mathcal{C}_1} F_1 dx + \int_{\mathcal{C}_2} F_1 dx$, where $\mathcal{C}_1$ is the graph of $y = g(x)$ and $\mathcal{C}_2$ is the graph of $y = f(x)$. To compute these line integrals, we parameterize the graphs from left to right using t as parameter: $\boldsymbol{c}_1(t) = (t, g(t))$, $a \leq t \leq b$, and $\boldsymbol{c}_2(t) = (t, f(t))$, $a \leq t \leq b$.



Since $\mathcal{C}_2$ is oriented from right to left, the line integral over $\partial \mathcal{D}$ is the difference $\oint_{\partial \mathcal{D}} F_1 dx = \int_{\boldsymbol{c}_1} F_1 dx - \int_{\boldsymbol{c}_2} F_1 dx$.

Green's Theorem (Conclusion)

- In both parametrizations, $x = t$, so $dx = dt$. So we get

$$\oint_{\partial\mathcal{D}} F_1 dx = \int_a^b F_1(t, g(t))dt - \int_a^b F_1(t, f(t))dt.$$

Now, the key step is to apply the Fundamental Theorem of Calculus to $\frac{\partial F_1}{\partial y}(t, y)$ as a function of y with t held constant:

$$F_1(t, f(t)) - F_1(t, g(t)) = \int_{g(t)}^{f(t)} \frac{\partial F_1}{\partial y}(t, y)dy.$$

Substituting the integral on the right in the preceding equation, we get

$$\oint_{\partial\mathcal{D}} F_1 dx = -\int_a^b \int_{g(t)}^{f(t)} \frac{\partial F_1}{\partial y}(t, y)dy\,dt = -\iint_{\mathcal{D}} \frac{\partial F_1}{\partial y} dA.$$

The other equation is proved in a similar fashion, by expressing $\partial\mathcal{D}$ as the union of the graphs of $x = f_1(y)$ and $x = g_1(y)$.

Green's Theorem: Conservative Vector Fields

- Recall that if $\boldsymbol{F} = \nabla V$, then the cross-partial condition is satisfied:

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0.$$

- In this case, Green's Theorem merely confirms what we already know: The line integral of a conservative vector field around any closed curve is zero.

Example: Verifying Green's Theorem

- Verify Green's Theorem for the line integral along the unit circle $\mathcal{C}$, oriented counterclockwise: $\oint_{\mathcal{C}} xy^2 dx + x dy$.

  We evaluate the line integral directly. We use the standard parametrization of the unit circle: $x = \cos\theta,\ y = \sin\theta$. Then $dx = -\sin\theta d\theta,\ dy = \cos\theta d\theta$. The integrand in the line integral is

$$\begin{array}{rcl} xy^2 dx + x dy & = & \cos\theta \sin^2\theta(-\sin\theta d\theta) + \cos\theta(\cos\theta d\theta) \\ & = & (-\cos\theta\sin^3\theta + \cos^2\theta) d\theta. \end{array}$$

  So we get

$$\begin{array}{rcl} \oint_{\mathcal{C}} xy^2 dx + x dy & = & \int_0^{2\pi} (-\cos\theta\sin^3\theta + \cos^2\theta) d\theta \\ & = & \int_0^{2\pi} (-\cos\theta\sin^3\theta + \frac{1}{2}(1 + \cos 2\theta)) d\theta \\ & = & -\frac{\sin^4\theta}{4} \mid_0^{2\pi} + \frac{1}{2}(\theta + \frac{1}{2}\sin 2\theta) \mid_0^{2\pi} \\ & = & 0 + \frac{1}{2}(2\pi + 0) = \pi. \end{array}$$

Example: Verifying Green's Theorem (Cont'd)

- We now evaluate the line integral using Green's Theorem.
  We have $F_1 = xy^2,\ F_2 = x$. So $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}x - \frac{\partial}{\partial y}xy^2 = 1 - 2xy$.
  According to Green's Theorem,

$$\oint_{\mathcal{C}} xy^2 dx + xdy = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dA = \iint_{\mathcal{D}} (1 - 2xy)dA,$$

where $\mathcal{D}$ is the disk $x^2 + y^2 \leq 1$ enclosed by $\mathcal{C}$. The integral of $2xy$ over $\mathcal{D}$ is zero:

$$\begin{aligned} \textstyle\iint_{\mathcal{D}} (-2xy)dA &= -2\textstyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xydydx \\ &= -\textstyle\int_{-1}^{1} xy^2 \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = 0. \end{aligned}$$

Therefore,

$$\iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dA = \iint_{\mathcal{D}} 1dA = \text{Area}(\mathcal{D}) = \pi.$$

Example

- Verify Green's Theorem for the line integral along the unit circle $\mathcal{C}$, oriented counterclockwise: $\oint_{\mathcal{C}} xydx + ydy$.

  We evaluate the line integral directly. We use the standard parametrization of the unit circle: $x = \cos\theta$, $y = \sin\theta$. Then $dx = -\sin\theta d\theta$, $dy = \cos\theta d\theta$. The integrand in the line integral is

$$\begin{array}{rcl} xydx + ydy & = & \cos\theta\sin\theta(-\sin\theta d\theta) + \sin\theta(\cos\theta d\theta) \\ & = & (-\cos\theta\sin^2\theta + \sin\theta\cos\theta)d\theta \\ & = & (-\sin^2\theta + \sin\theta)\cos\theta d\theta. \end{array}$$

  So we get

$$\begin{array}{rcl} \oint_{\mathcal{C}} xydx + ydy & = & \int_0^{2\pi}(-\sin^2\theta + \sin\theta)\cos\theta d\theta \\ & = & (-\frac{1}{3}\sin^3\theta + \frac{1}{2}\sin^2\theta)\,|_0^{2\pi} \\ & = & 0. \end{array}$$

Example (Cont'd)

- We now evaluate the line integral using Green's Theorem. We have $F_1 = xy,\ F_2 = y$. So $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x} y - \frac{\partial}{\partial y} xy = -x$. According to Green's Theorem,

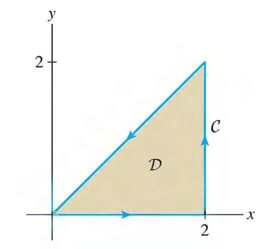
$$\oint_{\mathcal{C}} xydx + yxdy = \iint_{\mathcal{D}} (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})dA = \iint_{\mathcal{D}} -xdA,$$

where $\mathcal{D}$ is the disk $x^2 + y^2 \leq 1$ enclosed by $\mathcal{C}$. We calculate:

$$\begin{aligned}
\iint_{\mathcal{D}} -xdA &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} -xdydx \\
&= \int_{-1}^{1} -xy \,\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \\
&= \int_{-1}^{1} (-x\sqrt{1-x^2} - x\sqrt{1-x^2})dx \\
&= \int_{-1}^{1} -2x\sqrt{1-x^2}dx \\
&= \frac{2}{3}(1-x^2)^{3/2} \,\Big|_{-1}^{1} = 0.
\end{aligned}$$

Example: Line Integral Using Green's Theorem

- Compute the circulation of $\boldsymbol{F} = \langle \sin x, x^2 y^3 \rangle$ around the triangular path $\mathcal{C}$ in the figure.
  To compute the line integral directly, we would have to parametrize all three sides.
  Instead, we apply Green's Theorem to the domain $\mathcal{D}$ enclosed by the triangle.



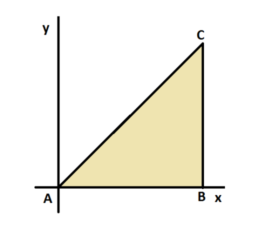
This domain is described by $0 \le x \le 2$, $0 \le y \le x$. Applying Green's Theorem, we obtain

$$\begin{aligned}
\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial x} x^2 y^3 - \frac{\partial}{\partial y} \sin x = 2xy^3; \\
\oint_{\mathcal{C}} \sin x dx + x^2 y^3 dy &= \iint_{\mathcal{D}} 2xy^3 dA = \int_0^2 \int_0^x 2xy^3 dy\, dx \\
&= \int_0^2 \left(\frac{1}{2} xy^4 \Big|_0^x\right) dx = \frac{1}{2} \int_0^2 x^5 dx \\
&= \frac{1}{12} x^6 \Big|_0^2 = \frac{16}{3}.
\end{aligned}$$

Example

- Apply Green's Theorem to evaluate $\oint_{\mathcal{C}} e^{2x+y}dx + e^{-y}dy$, where $\mathcal{C}$ is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$ oriented counterclockwise.

  To compute the line integral directly, we would have to parametrize all three sides.



  We apply Green's Theorem to the domain $\mathcal{D}$ enclosed by the triangle. This domain is described by $0 \le x \le 1$, $0 \le y \le x$. Applying Green's Theorem, we obtain

$$
\begin{aligned}
\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial x}e^{-y} - \frac{\partial}{\partial y}e^{2x+y} = -e^{2x+y}; \\
\oint_{\mathcal{C}} e^{2x+y}dx + e^{-y}dy &= -\iint_{\mathcal{D}} e^{2x+y}dA = -\int_0^1\int_0^x e^{2x+y}dy\,dx \\
&= -\int_0^1 e^{2x+y}\Big|_0^x dx = \int_0^1 (e^{2x} - e^{3x})dx \\
&= \left(\frac{1}{2}e^{2x} - \frac{1}{3}e^{3x}\right)\Big|_0^1 = \frac{1}{2}e^2 - \frac{1}{3}e^3 - \frac{1}{6}.
\end{aligned}
$$

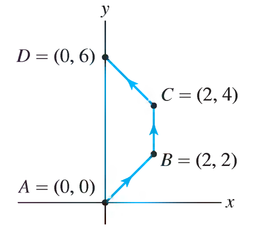
Example

- Apply Green's Theorem to evaluate $\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s}$, where $\boldsymbol{F} = \langle x+y, x^2-y \rangle$ and $\mathcal{C}$ is the boundary of the region enclosed by $y = x^2$, $y = \sqrt{x}$, $0 \le x \le 1$.
  We apply Green's Theorem to the domain $\mathcal{D}$ enclosed by the given curves. This domain is described by $0 \le x \le 1$, $x^2 \le y \le \sqrt{x}$.
  Applying Green's Theorem, we obtain

$$\begin{aligned}
\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial x}(x^2-y) - \frac{\partial}{\partial y}(x+y) = 2x-1; \\
\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \iint_{\mathcal{D}} (2x-1)dA = \int_0^1 \int_{x^2}^{\sqrt{x}} (2x-1)dy\,dx \\
&= \int_0^1 (2xy-y) \Big|_{x^2}^{\sqrt{x}} dx \\
&= \int_0^1 (2x^{3/2} - x^{1/2} - 2x^3 + x^2)dx \\
&= \left(\frac{4}{5}x^{5/2} - \frac{2}{3}x^{3/2} - \frac{1}{2}x^4 + \frac{1}{3}x^3\right) \Big|_0^1 \\
&= \frac{4}{5} - \frac{2}{3} - \frac{1}{2} + \frac{1}{3} \\
&= -\frac{1}{30}.
\end{aligned}$$

Example

- Evaluate $I = \int_{\mathcal{C}} (\sin x + y)dx + (3x + y)dy$ for the nonclosed path $ABCD$.
  We apply Green's Theorem to the quadrangle $\mathcal{D}$ enclosed by the path $ABCDA$.



Note that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}(3x + y) - \frac{\partial}{\partial y}(\sin x + y) = 3 - 1 = 2.$
So we have

$$\oint_{\mathcal{C}+\overline{DA}} (\sin x + y)dx + (3x + y)dy = \iint_{\mathcal{D}} (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})dA$$

$$\int_{\mathcal{C}} (\sin x + y)dx + (3x + y)dy + \int_{\overline{DA}} (\sin x + y)dx + (3x + y)dy = \iint_{\mathcal{D}} (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})dA$$

$$I = \iint_{\mathcal{D}} (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})dA + \int_{\overline{AD}} (\sin x + y)dx + (3x + y)dy$$

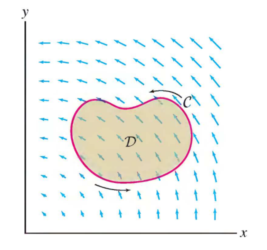
$$I = \iint_{\mathcal{D}} 2dA + \int_0^6 [(\sin 0 + y)\frac{dx}{dy} + (3 \cdot 0 + y)]dy$$

$$I = 2\text{Area}(\mathcal{D}) + \int_0^6 ydy = 2 \cdot 8 + \frac{1}{2}y^2 \mid_0^6 = 16 + 18 = 34.$$

Area Enclosed by a Curve

- Green's Theorem applied to $\boldsymbol{F} = \langle -y, x \rangle$ leads to a formula for the area of the domain $\mathcal{D}$ enclosed by a simple closed curve $\mathcal{C}$.
  We have

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(-y) = 2.$$



By Green's Theorem, $\oint_{\mathcal{C}} -ydx + xdy = \iint_{\mathcal{D}} 2dxdy = 2\text{Area}(\mathcal{D})$.
We obtain

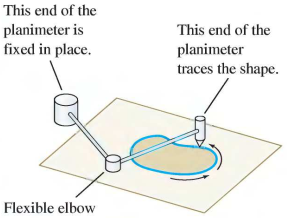
$$\text{Area enclosed by } \mathcal{C} = \frac{1}{2}\oint_{\mathcal{C}} xdy - ydx.$$

Planimeter

- The formula

$$\text{Area enclosed by } \mathcal{C} = \frac{1}{2}\oint_{\mathcal{C}} x\,dy - y\,dx$$

  tells us how to compute an enclosed area by making measurements only along the boundary.
- It is the mathematical basis of the **planimeter**, a device that computes the area of an irregular shape when you trace the boundary with a pointer at the end of a movable arm.



Example: Computing Area via Green's Theorem

- Compute the area of the ellipse $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$ using a line integral.

  We parametrize the boundary of the ellipse by

$$x = a\cos\theta, \quad y = b\sin\theta, \quad 0 \le \theta \le 2\pi.$$

  Now we get

$$\begin{aligned}
xdy - ydx &= (a\cos\theta)(b\cos\theta d\theta) - (b\sin\theta)(-a\sin\theta d\theta) \\
&= ab(\cos^2\theta + \sin^2\theta)d\theta = abd\theta; \\
\text{Enclosed area} &= \frac{1}{2}\oint_{\mathcal{C}} xdy - ydx \\
&= \frac{1}{2}\int_0^{2\pi} abd\theta = \pi ab.
\end{aligned}$$

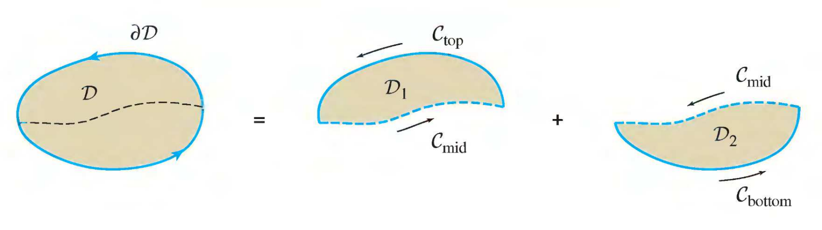
  This is the standard formula for the area of an ellipse.

Additivity of Circulation

Claim: Circulation around a closed curve has an additivity property:

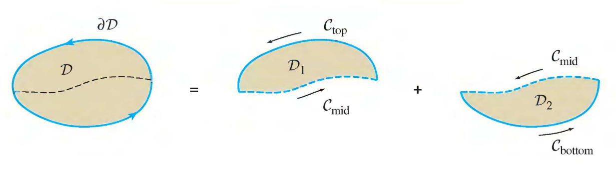
If we decompose a domain $\mathcal{D}$ into two (or more) non-overlapping domains $\mathcal{D}_1$ and $\mathcal{D}_2$ that intersect only on part of their boundaries, then

$$\oint_{\partial\mathcal{D}} \boldsymbol{F}\cdot d\boldsymbol{s} = \oint_{\partial\mathcal{D}_1} \boldsymbol{F}\cdot d\boldsymbol{s} + \oint_{\partial\mathcal{D}_2} \boldsymbol{F}\cdot d\boldsymbol{s}.$$



Additivity of Circulation (Cont'd)

- We verify the equation $\oint_{\partial\mathcal{D}} \boldsymbol{F}\cdot d\boldsymbol{s} = \oint_{\partial\mathcal{D}_1} \boldsymbol{F}\cdot d\boldsymbol{s} + \oint_{\partial\mathcal{D}_2} \boldsymbol{F}\cdot d\boldsymbol{s}$.



Note first that $\oint_{\partial\mathcal{D}} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{\mathcal{C}_{\text{top}}} \boldsymbol{F}\cdot d\boldsymbol{s} + \int_{\mathcal{C}_{\text{bot}}} \boldsymbol{F}\cdot d\boldsymbol{s}$, with $\mathcal{C}_{\text{top}}$ and $\mathcal{C}_{\text{bot}}$ as shown. Then observe that the dashed segment $\mathcal{C}_{\text{mid}}$ occurs in both $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$ but with opposite orientations. If $\mathcal{C}_{\text{mid}}$ is oriented right to left, then

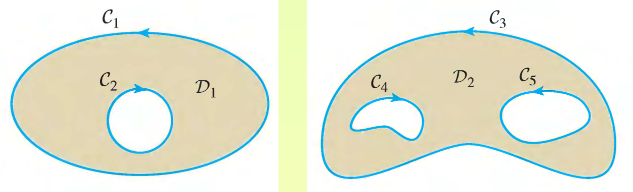
$$\begin{aligned}
\oint_{\partial\mathcal{D}_1} \boldsymbol{F}\cdot d\boldsymbol{s} &= \int_{\mathcal{C}_{\text{top}}} \boldsymbol{F}\cdot d\boldsymbol{s} - \int_{\mathcal{C}_{\text{mid}}} \boldsymbol{F}\cdot d\boldsymbol{s};\\
\int_{\partial\mathcal{D}_2} \boldsymbol{F}\cdot d\boldsymbol{s} &= \int_{\mathcal{C}_{\text{bot}}} \boldsymbol{F}\cdot d\boldsymbol{s} + \int_{\mathcal{C}_{\text{mid}}} \boldsymbol{F}\cdot d\boldsymbol{s}.
\end{aligned}$$

We add these two equations to get

$$\oint_{\partial\mathcal{D}_1} \boldsymbol{F}\cdot d\boldsymbol{s} + \oint_{\partial\mathcal{D}_2} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{\mathcal{C}_{\text{top}}} \boldsymbol{F}\cdot d\boldsymbol{s} + \int_{\mathcal{C}_{\text{bot}}} \boldsymbol{F}\cdot d\boldsymbol{s} = \oint_{\partial\mathcal{D}} \boldsymbol{F}\cdot d\boldsymbol{s}.$$

More General Domains and Boundary Orientations

- We consider domains whose boundary consists of more than one simple closed curve.



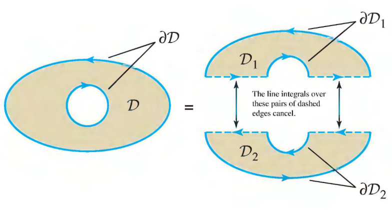
- As before, $\partial\mathcal{D}$ denotes the boundary of $\mathcal{D}$ with its boundary orientation. Recall this means that the region lies to the left as the curve is traversed in the direction specified by the orientation.
- For the domains in the figures, $\partial\mathcal{D}_1 = \mathcal{C}_1 + \mathcal{C}_2$, $\partial\mathcal{D}_2 = \mathcal{C}_3 + \mathcal{C}_4 - \mathcal{C}_5$. Here the curve $\mathcal{C}_5$ occurs with a minus sign because it is oriented counterclockwise, but the boundary orientation requires a clockwise orientation.

Green's Theorem for General Domains

- Green's Theorem remains valid for more general domains of this type:

$$\oint_{\partial\mathcal{D}} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$$

This equality is proved by decomposing $\mathcal{D}$ into smaller domains each of which is bounded by a simple closed curve. Consider, for instance, the region $\mathcal{D}$ in the figure. We decompose $\mathcal{D}$ into domains $\mathcal{D}_1$ and $\mathcal{D}_2$.



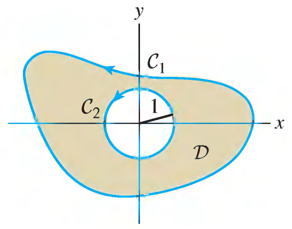
Then $\partial\mathcal{D} = \partial\mathcal{D}_1 + \partial\mathcal{D}_2$ because the edges common to $\partial\mathcal{D}_1$, $\partial\mathcal{D}_2$ occur with opposite orientation and therefore cancel. The previous version of Green's Theorem applies to both $\mathcal{D}_1$ and $\mathcal{D}_2$:

$$\begin{aligned}
\oint_{\partial\mathcal{D}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_{\partial\mathcal{D}_1} \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{\partial\mathcal{D}_2} \boldsymbol{F} \cdot d\boldsymbol{s} \\
&= \iint_{\mathcal{D}_1} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA + \iint_{\mathcal{D}_2} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA \\
&= \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.
\end{aligned}$$

Example

- Calculate $\oint_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{s}$, where $\boldsymbol{F} = \langle x - y, x + y^3 \rangle$ and $\mathcal{C}_1$ is the outer boundary curve oriented counterclockwise. Assume that the domain $\mathcal{D}$ in the figure has area 8.

  We cannot compute the line integral over $\mathcal{C}_1$ directly because the curve $\mathcal{C}_1$ is not specified.



  However, $\partial\mathcal{D} = \mathcal{C}_1 - \mathcal{C}_2$. So Green's Theorem yields

$$\oint_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{s} - \oint_{\mathcal{C}_2} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$$

  We have

$$\begin{aligned} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial x}(x + y^3) - \frac{\partial}{\partial y}(x - y) = 1 - (-1) = 2; \\ \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA &= \iint_{\mathcal{D}} 2 dA = 2\mathrm{Area}(\mathcal{D}) = 2 \cdot 8 = 16. \end{aligned}$$

  Thus we get $\oint_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{s} - \oint_{\mathcal{C}_2} \boldsymbol{F} \cdot d\boldsymbol{s} = 16$.

Example (Cont'd)

- We showed that $\oint_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{s} = \oint_{\mathcal{C}_2} \boldsymbol{F} \cdot d\boldsymbol{s} + 16$.

  To compute the second integral, parametrize the unit circle $\mathcal{C}_2$ by $\boldsymbol{c}(t) = \langle \cos\theta, \sin\theta \rangle$. Then

$$\begin{array}{rcl} \boldsymbol{F} \cdot \boldsymbol{c}'(t) & = & \langle x - y, x + y^3 \rangle \cdot \langle -\sin\theta, \cos\theta \rangle \\ & = & \langle \cos\theta - \sin\theta, \cos\theta + \sin^3\theta \rangle \cdot \langle -\sin\theta, \cos\theta \rangle \\ & = & -\sin\theta\cos\theta + \sin^2\theta + \cos^2\theta + \sin^3\theta\cos\theta \\ & = & 1 - \sin\theta\cos\theta + \sin^3\theta\cos\theta. \end{array}$$

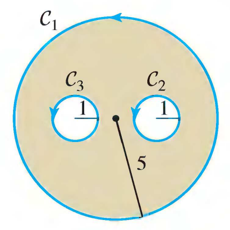
  The integrals of $\sin\theta\cos\theta$ and $\sin^3\theta\cos\theta$ over $[0, 2\pi]$ are both zero. So, we get

$$\oint_{\mathcal{C}_2} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_0^{2\pi} (1 - \sin\theta\cos\theta + \sin^3\theta\cos\theta) d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

  Finally, we obtain $\oint_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{s} = 16 + 2\pi$.

Example

- Calculate $\oint_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{s}$, if $\oint_{\mathcal{C}_2} \boldsymbol{F} \cdot d\boldsymbol{s} = 3\pi$, $\oint_{\mathcal{C}_3} \boldsymbol{F} \cdot d\boldsymbol{s} = 4\pi$ and $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 9$.
  We apply the genral version of Green's Theorem.



$$\int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

$$\int_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{s} - \int_{\mathcal{C}_2} \boldsymbol{F} \cdot d\boldsymbol{s} - \int_{\mathcal{C}_3} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

$$\int_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\mathcal{C}_2} \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{\mathcal{C}_3} \boldsymbol{F} \cdot d\boldsymbol{s} + \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

$$\int_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{s} = 3\pi + 4\pi + \iint_{\mathcal{D}} 9 dA$$

$$\int_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{s} = 3\pi + 4\pi + 9\text{Area}(\mathcal{D})$$

$$\int_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{s} = 3\pi + 4\pi + 9(25\pi - \pi - \pi)$$

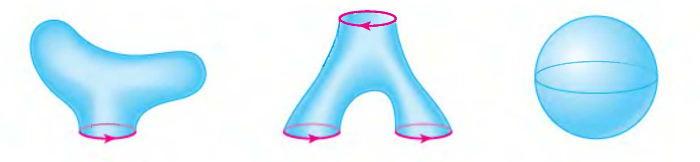
$$\int_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{s} = 214\pi.$$

Subsection 2

Stokes' Theorem

Closed Surfaces

- The figure shows three surfaces with different types of boundaries.



(A) Boundary consists of a single closed curve (B) Boundary consists of three closed curves (C) Closed surface (the boundary is empty)

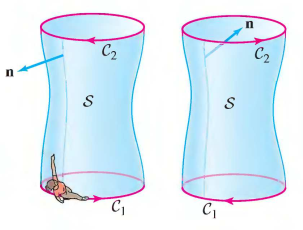
- The boundary of a surface is denoted $\partial\mathcal{S}$.
- Observe that:
  - The boundary in (A) is a single, simple closed curve;
  - The boundary in (B) consists of three closed curves;
  - The surface in (C) is called a **closed surface** because its boundary is empty. In this case, we write $\partial\mathcal{S} = \emptyset$.

Boundary Orientation

- An **orientation** is a continuously varying choice of unit normal vector at each point of a surface $\mathcal{S}$.
- When $\mathcal{S}$ is oriented, we can specify an orientation of $\partial\mathcal{S}$, called the **boundary orientation**:

  Imagine that you are a unit normal vector walking along the boundary curve. The boundary orientation is the direction for which the surface is on your left as you walk.

  Example: The boundary of the surface consists of two curves, $\mathcal{C}_1$ and $\mathcal{C}_2$. On the left, the normal vector points to the outside. The woman (normal vector) is walking along $\mathcal{C}_1$ and has the surface to her left, so she is walking in the positive direction.



  The boundary orientations on the right are reversed because the opposite normal has been selected to orient the surface.

The Curl

- The **curl** of the vector field $\boldsymbol{F} = \langle F_1, F_2, F_3 \rangle$ is the vector field defined by the *symbolic* determinant

rot $(\vec{F})$

$$\begin{aligned} \operatorname{curl}(\boldsymbol{F}) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}. \end{aligned}$$

- In more compact form, the curl is the symbolic cross product

$$\operatorname{curl}(\boldsymbol{F}) = \nabla \times \boldsymbol{F},$$

op. Hamilton

where ∇ is the del "operator" (or "nabla"): $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$.

- In terms of components, $\operatorname{curl}(\boldsymbol{F})$ is the vector field

$$\operatorname{curl}(\boldsymbol{F}) = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle.$$

Linearity of the Curl

- The curl obeys the linearity rules:

$$\operatorname{curl}(\boldsymbol{F} + \boldsymbol{G}) = \operatorname{curl}(\boldsymbol{F}) + \operatorname{curl}(\boldsymbol{G})$$

  and

$$\operatorname{curl}(c\boldsymbol{F}) = c \operatorname{curl}(\boldsymbol{F}), \quad c \text{ any constant.}$$

- These can be checked directly or, alternatively, follow by the multilinearity of the determinant.

Example: Calculating the Curl

- Calculate the curl of $\boldsymbol{F} = \langle xy, e^x, y+z \rangle$.

  We compute the curl as a symbolic determinant:

$$
\begin{aligned}
\operatorname{curl}(\boldsymbol{F}) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & e^x & y+z \end{vmatrix} \\
&= \left(\tfrac{\partial}{\partial y}(y+z) - \tfrac{\partial}{\partial z}e^x\right)\boldsymbol{i} - \left(\tfrac{\partial}{\partial x}(y+z) - \tfrac{\partial}{\partial z}xy\right)\boldsymbol{j} \\
&\qquad + \left(\tfrac{\partial}{\partial x}e^x - \tfrac{\partial}{\partial y}xy\right)\boldsymbol{k} \\
&= \boldsymbol{i} + (e^x - x)\boldsymbol{k}.
\end{aligned}
$$

Example: Conservative Vector Fields Have Zero Curl

- Verify the following:

$$\text{If } \boldsymbol{F} = \nabla V, \text{ then } \operatorname{curl}(\boldsymbol{F}) = \mathbf{0}. \text{ That is, } \operatorname{curl}(\nabla V) = \mathbf{0}.$$

The curl of a vector field is zero if

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = 0, \quad \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = 0, \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0.$$

But these equations are equivalent to the cross-partials condition that is satisfied by every conservative vector field $\boldsymbol{F} = \nabla V$.

Stokes' Theorem

- Assume that $\mathcal{S}$ is an oriented surface with parametrization $G : \mathcal{D} \to \mathcal{S}$, where $\mathcal{D}$ is a domain in the plane bounded by smooth, simple closed curves, and G is one-to-one and regular, except possibly on the boundary of $\mathcal{D}$.
- More generally, $\mathcal{S}$ may be a finite union of surfaces of this type.

Theorem (Stokes' Theorem)

For surfaces $\mathcal{S}$ as described above,

$$\oint_{\partial \mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_{\mathcal{S}} \mathrm{curl}(\boldsymbol{F}) \cdot d\boldsymbol{S}.$$

The integral on the left is defined relative to the boundary orientation of $\partial \mathcal{S}$. If $\mathcal{S}$ is closed ($\partial \mathcal{S} = \emptyset$), then the surface integral on the right is zero.

Proof of Stokes' Theorem

- Each side of the equation is equal to a sum over the components of $\boldsymbol{F}$:

$$\begin{aligned}\oint_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{s} &= \oint_{\mathcal{C}} F_1 dx + F_2 dy + F_3 dz\\ \iint_{\mathcal{S}} \operatorname{curl}(\boldsymbol{F})\cdot d\boldsymbol{S} &= \iint_{\mathcal{S}} \operatorname{curl}(F_1\boldsymbol{i})\cdot d\boldsymbol{S} + \iint_{\mathcal{S}} \operatorname{curl}(F_2\boldsymbol{j})\cdot d\boldsymbol{S}\\ &\quad + \iint_{\mathcal{S}} \operatorname{curl}(F_3\boldsymbol{k})\cdot d\boldsymbol{S}.\end{aligned}$$

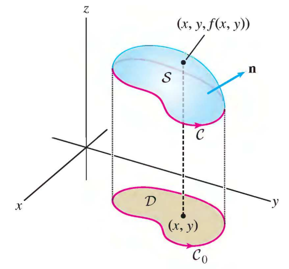
The proof consists of showing that the F_1-, F_2- and F_3-terms are separately equal.

We will prove only the case of $\mathcal{S}$ being the graph of a function $z = f(x,y)$ lying over a domain $\mathcal{D}$ in the xy-plane. Furthermore, we will carry the details only for the F_1-terms. The calculation for F_2-components is similar, and so is that of the F_3-terms.

Claim: $\oint_{\mathcal{C}} F_1 dx = \iint_{\mathcal{S}} \operatorname{curl}(F_1(x,y,z)\boldsymbol{i})\cdot d\boldsymbol{S}$.

Stokes' Theorem: Proof of the Claim

Claim: $\oint_{\mathcal{C}} F_1 dx = \iint_{\mathcal{S}} \text{curl}(F_1(x,y,z)\boldsymbol{i}) \cdot d\boldsymbol{S}$. Orient $\mathcal{S}$ with upward-pointing normal. Let $\mathcal{C} = \partial\mathcal{S}$ be the boundary curve. Let $\mathcal{C}_0$ be the boundary of $\mathcal{D}$ in the xy-plane. Let $\boldsymbol{c}_0 = (x(t), y(t))$, $a \leq t \leq b$ be a counterclockwise parametrization of $\mathcal{C}_0$. The boundary curve $\mathcal{C}$ projects onto $\mathcal{C}_0$. So $\mathcal{C}$ has parametrization $\boldsymbol{c}(t) = (x(t), y(t), f(x(t), y(t)))$.



Thus $\oint_{\mathcal{C}} F_1(x,y,z)dx = \int_a^b F_1(x(t), y(t), f(x(t), y(t)))\frac{dx}{dt}dt$. The integral on the right is precisely the integral we obtain by integrating $F_1(x, y, f(x,y))dx$ over the curve $\mathcal{C}_0$ in the plane $\mathbb{R}^2$. In other words, $\oint_{\mathcal{C}} F_1(x,y,z)dx = \int_{\mathcal{C}_0} F_1(x,y,f(x,y))dx$. By Green's Theorem applied to the integral on the right,

$$\oint_{\mathcal{C}} F_1(x,y,z)dx = -\iint_{\mathcal{D}} \frac{\partial}{\partial y} F_1(x,y,f(x,y))dA.$$

Stokes' Theorem: Proof of the Claim (Cont'd)

- We have $\oint_{\mathcal{C}} F_1(x,y,z)dx = -\iint_{\mathcal{D}} \frac{\partial}{\partial y} F_1(x,y,f(x,y))dA$. By the Chain Rule,
  $\frac{\partial}{\partial y} F_1(x,y,f(x,y)) = F_{1y}(x,y,f(x,y)) + F_{1z}(x,y,f(x,y))f_y(x,y)$.
  So finally we obtain

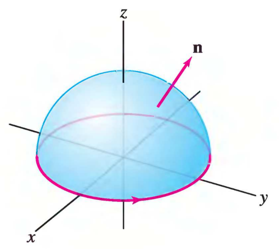
$$\oint_{\mathcal{C}} F_1 dx = -\iint_{\mathcal{D}} (F_{1y}(x,y,f(x,y)) + F_{1z}(x,y,f(x,y))f_y(x,y))dA.$$

To finish the proof, we compute the surface integral of $\operatorname{curl}(F_1\boldsymbol{i})$ using the parametrization $G(x,y) = (x,y,f(x,y))$ of $\mathcal{S}$:

$$\begin{array}{rcl} \boldsymbol{n} & = & \langle -f_x(x,y), -f_y(x,y), 1\rangle \quad \text{(upward normal)} \\ \operatorname{curl}(F_1\boldsymbol{i})\cdot\boldsymbol{n} & = & \langle 0, F_{1z}, -F_{1y}\rangle \cdot \langle -f_x(x,y), -f_y(x,y), 1\rangle \\ & = & -F_{1z}(x,y,f(x,y))f_y(x,y) - F_{1y}(x,y,f(x,y)) \\ \iint_{\mathcal{S}} \operatorname{curl}(F_1\boldsymbol{i})\cdot d\boldsymbol{S} & = & -\iint_{\mathcal{D}} (F_{1z}(x,y,z)f_y(x,y) \\ & & \qquad + F_{1y}(x,y,f(x,y)))dA. \end{array}$$

Example: Verifying Stokes' Theorem

- Verify Stokes' Theorem for $\boldsymbol{F} = \langle -y, 2x, x+z\rangle$ and the upper hemisphere with outward-pointing normal vectors $\mathcal{S} = \{(x,y,z) : x^2+y^2+z^2 = 1, z \geq 0\}$.
  We will show that both the line integral and the surface integral in Stokes' Theorem are equal to 3π.



  Compute the line integral around the boundary curve. The boundary of $\mathcal{S}$ is the unit circle oriented in the counterclockwise direction with parametrization $\boldsymbol{c}(t) = (\cos t, \sin t, 0)$. Thus,

$$
\begin{aligned}
\boldsymbol{c}'(t) &= \langle -\sin t, \cos t, 0\rangle; \\
\boldsymbol{F}(\boldsymbol{c}(t)) &= \langle -\sin t, 2\cos t, \cos t\rangle; \\
\boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) &= \langle -\sin t, 2\cos t, \cos t\rangle \cdot \langle -\sin t, \cos t, 0\rangle \\
&= \sin^2 t + 2\cos^2 t = 1 + \cos^2 t; \\
\oint_{\partial\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_0^{2\pi} (1+\cos^2 t) dt = 2\pi + \pi = 3\pi.
\end{aligned}
$$

Example: Verifying Stokes' Theorem (Cont'd)

- Compute the curl.

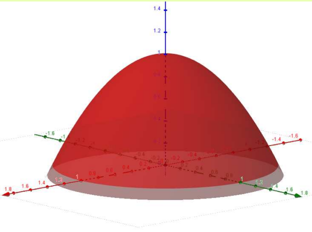
$$\begin{aligned}
\operatorname{curl}(\boldsymbol{F}) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j}, & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & 2x & x+z \end{vmatrix} \\
&= (\tfrac{\partial}{\partial y}(x+z) - \tfrac{\partial}{\partial z}2x)\boldsymbol{i} - (\tfrac{\partial}{\partial x}(x+z) - \tfrac{\partial}{\partial z}(-y))\boldsymbol{j} \\
&\qquad + (\tfrac{\partial}{\partial x}2x - \tfrac{\partial}{\partial y}(-y))\boldsymbol{k} \\
&= \langle 0, -1, 3 \rangle .
\end{aligned}$$

Use spherical coordinates: $G(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$. Then $\boldsymbol{n} = \sin\phi \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$. Therefore, $\operatorname{curl}(\boldsymbol{F}) \cdot \boldsymbol{n} = \sin\phi \langle 0, -1, 3 \rangle \cdot \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle = -\sin\theta \sin^2\phi + 3\cos\theta \sin\phi$. So

$$\begin{aligned}
\iint_{\mathcal{S}} \operatorname{curl}(\boldsymbol{F}) \cdot d\boldsymbol{S} &= \int_0^{\pi/2} \int_0^{2\pi} (-\sin\theta \sin^2\phi + 3\cos\phi \sin\phi) d\theta d\phi \\
&= 0 + 2\pi \int_0^{\pi/2} 3\cos\phi \sin\phi d\phi = \\
&= 2\pi (\tfrac{3}{2} \sin^2\phi) \Big|_0^{\pi/2} = 3\pi .
\end{aligned}$$

Example

- Verify Stokes' Theorem for $\boldsymbol{F} = \langle 2xy, x, y+z\rangle$ and the surface $\mathcal{S} = \{(x,y,z) : z = 1-x^2-y^2, x^2+y^2 \leq 1\}$, with upward pointing normals.

  The boundary of $\mathcal{S}$ is the unit circle oriented in the counterclockwise direction: $\boldsymbol{c}(t) = (\cos t, \sin t, 0)$.
  Thus,

$$\begin{array}{rcl} \boldsymbol{c}'(t) & = & \langle -\sin t, \cos t, 0\rangle; \\ \boldsymbol{F}(\boldsymbol{c}(t)) & = & \langle 2\sin t\cos t, \cos t, \sin t\rangle; \\ \boldsymbol{F}(\boldsymbol{c}(t))\cdot\boldsymbol{c}'(t) & = & \langle 2\sin t\cos t, \cos t, \sin t\rangle\cdot\langle -\sin t, \cos t, 0\rangle \\ & = & -2\sin^2 t\cos t + \cos^2 t; \\ \oint_{\partial\mathcal{S}} \boldsymbol{F}\cdot d\boldsymbol{s} & = & \int_0^{2\pi}(-2\sin^2 t\cos t + \cos^2 t)dt \\ & = & [-\frac{2}{3}\sin^3 t + \frac{1}{2}(t+\frac{1}{2}\sin 2t)]_0^{2\pi} \\ & = & \frac{1}{2}\cdot 2\pi = \pi. \end{array}$$

Example (Cont'd)

- Compute the curl.

$$\begin{aligned}\mathrm{curl}(\boldsymbol{F}) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j}, & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x & y+z \end{vmatrix} \\ &= (\tfrac{\partial}{\partial y}(y+z) - \tfrac{\partial}{\partial z}x)\boldsymbol{i} - (\tfrac{\partial}{\partial x}(y+z) - \tfrac{\partial}{\partial z}2xy)\boldsymbol{j} \\ &\qquad + (\tfrac{\partial}{\partial x}x - \tfrac{\partial}{\partial y}2xy)\boldsymbol{k} \\ &= \langle 1, 0, 1-2x \rangle.\end{aligned}$$

Use cylindrical coordinates: $G(r,\theta) = (r\cos\theta, r\sin\theta, 1-r^2)$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$. Then

$$\boldsymbol{n} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle.$$ Therefore,

$\mathrm{curl}(\boldsymbol{F}) \cdot \boldsymbol{n} = \langle 1, 0, 1-2r\cos\theta \rangle \cdot \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle = r$. So

$$\iint_{\mathcal{S}} \mathrm{curl}(\boldsymbol{F}) \cdot d\boldsymbol{S} = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 r\,dr = 2\pi \cdot \frac{r^2}{2}\Big|_0^1 = \pi.$$

Example

- Use Stokes' Theorem to compute the flux of curl($\boldsymbol{F}$), where $\boldsymbol{F} = \langle e^{z^2} - y, e^{z^3} + x, \cos(xz)\rangle$ through the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, with outward pointing normal.

  We have

$$\begin{aligned} \boldsymbol{c}(t) &= (\cos t, \sin t, 0); \\ \boldsymbol{c}'(t) &= \langle -\sin t, \cos t, 0\rangle; \\ \mathbf{F}(\boldsymbol{c}(t)) &= \langle 1 - \sin t, 1 + \cos t, 1\rangle; \\ \mathbf{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) &= \langle 1 - \sin t, 1 + \cos t, 1\rangle \cdot \langle -\sin t, \cos t, 0\rangle \\ &= -\sin t + \sin^2 t + \cos t + \cos^2 t \\ &= 1 + \cos t - \sin t. \end{aligned}$$

So
$$\begin{aligned} \iint_{\mathcal{S}} \mathrm{curl}(\boldsymbol{F}) \cdot d\boldsymbol{S} &= \oint_{\partial\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{s} \\ &= \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) dt \\ &= \int_0^{2\pi} (1 + \cos t - \sin t) dt \\ &= (t + \sin t + \cos t) \,\Big|_0^{2\pi} = 2\pi. \end{aligned}$$

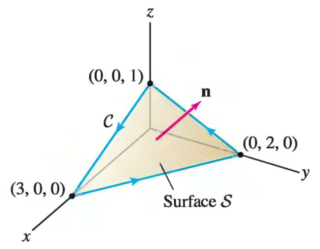
A Special Case

- curl($\boldsymbol{F}$) contains the partial derivatives $\frac{\partial F_1}{\partial y}$ and $\frac{\partial F_1}{\partial z}$ but not the partial $\frac{\partial F_1}{\partial x}$.
- So if $F_1 = F_1(x)$ is a function of x alone, then $\frac{\partial F_1}{\partial y} = \frac{\partial F_1}{\partial z} = 0$, and F_1 does not contribute to the curl.
- The same holds for the other components.
- In summary, if each of F_1, F_2, and F_3 depends only on its corresponding variable x, y, or z, then

$$\mathrm{curl}(\langle F_1(x), F_2(y), F_3(z)\rangle) = 0.$$

Example

- Use Stokes' Theorem to show that
$$\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} = 0,$$
where $\boldsymbol{F} = \langle \sin(x^2), e^{y^2} + x^2, z^4 + 2x^2 \rangle$ and $\mathcal{C}$ is the boundary of the triangle shown with the indicated orientation.

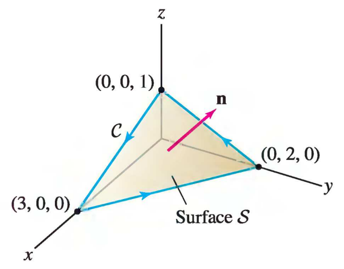


We apply Stokes' Theorem $\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_{\mathcal{S}} \text{curl}(\boldsymbol{F}) \cdot d\boldsymbol{S}$. We show that the integral on the right is zero.

By the preceding slide, the first component $\sin(x^2)$ does not contribute to the curl since it depends only on x. Similarly, e^{y^2} and z^4 drop out of the curl. So we have

$$\begin{aligned}
&\text{curl}(\langle \sin x^2, e^{y^2} + x^2, z^4 + 2x^2 \rangle) \\
&= \text{curl}(\langle \sin x^2, e^{y^2}, z^4 \rangle) + \text{curl}(\langle 0, x^2, 2x^2 \rangle) \\
&= \langle 0, -\tfrac{\partial}{\partial x} 2x^2, \tfrac{\partial}{\partial x} x^2 \rangle = \langle 0, -4x, 2x \rangle.
\end{aligned}$$

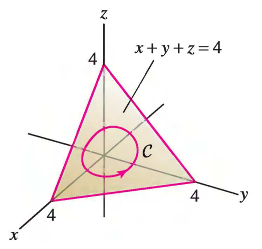
Example (Cont'd)



- Now we see that $\mathcal{C}$ is the boundary of the triangular surface $\mathcal{S}$ contained in the plane $\frac{x}{3}+\frac{y}{2}+z=1$. Therefore, $\boldsymbol{u}=\langle\frac{1}{3},\frac{1}{2},1\rangle$ is a normal vector to this plane. But $\boldsymbol{u}$ and curl($\boldsymbol{F}$) are orthogonal: $\text{curl}(\boldsymbol{F})\cdot\boldsymbol{u}=\langle 0,-4x,2x\rangle\cdot\langle\frac{1}{3},\frac{1}{2},1\rangle= -2x+2x=0$. In other words, the normal component of curl($\boldsymbol{F}$) along $\mathcal{S}$ is zero. Since the surface integral of a vector field is equal to the surface integral of the normal component, we conclude that $\iint_{\mathcal{S}}\text{curl}(\boldsymbol{F})\cdot d\boldsymbol{S}=0$.

Example

- Let $\boldsymbol{F} = \langle -z^2, 2zx, 4y - x^2 \rangle$ and $\mathcal{C}$ be the simple closed curve in the plane $x + y + z = 4$ that encloses a region of area 16.
  Calculate $\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s}$, where $\mathcal{C}$ is oriented in the counterclockwise direction viewed from above.



$$
\begin{aligned}
\operatorname{curl}(\boldsymbol{F}) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z^2 & 2zx & 4y - x^2 \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(4y - x^2) - \frac{\partial}{\partial z} 2zx\right)\boldsymbol{i} \\
&\quad - \left(\frac{\partial}{\partial x}(4y - x^2) - \frac{\partial}{\partial z}(-z^2)\right)\boldsymbol{j} + \left(\frac{\partial}{\partial x} 2zx - \frac{\partial}{\partial y}(-z^2)\right)\boldsymbol{k} \\
&= \langle 4 - 2x, 2x - 2z, 2z \rangle; \\
\boldsymbol{n} &= \langle 1, 1, 1 \rangle; \\
\operatorname{curl}(\boldsymbol{F}) \cdot \boldsymbol{e_n} &= \langle 4 - 2x, 2x - 2z, 2z \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle = \frac{4}{\sqrt{3}}.
\end{aligned}
$$

Example

- We obtained

$$\mathrm{curl}(\boldsymbol{F}) \cdot \boldsymbol{e_n} = \frac{4}{\sqrt{3}}.$$

So we get

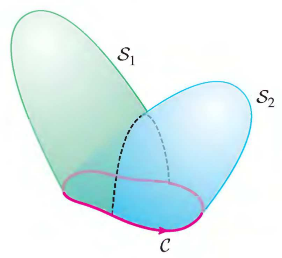
$$\begin{aligned}\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \iint_{\mathcal{S}} \mathrm{curl}(\boldsymbol{F}) \cdot d\boldsymbol{S} = \iint_{\mathcal{S}} (\mathrm{curl}(\boldsymbol{F}) \cdot \boldsymbol{e_n}) dS \\ &= \iint_{\mathcal{S}} \frac{4}{\sqrt{3}} dS = \frac{4}{\sqrt{3}} \mathrm{Area}(\mathcal{S}) = \frac{64}{\sqrt{3}}.\end{aligned}$$

Surface Independence for Curl Vector Fields

Theorem (Surface Independence for Curl Vector Fields)

If $\boldsymbol{F} = \mathrm{curl}(\boldsymbol{A})$, then the flux of $\boldsymbol{F}$ through a surface $\mathcal{S}$ depends only on the oriented boundary $\partial\mathcal{S}$ and not on the surface itself:

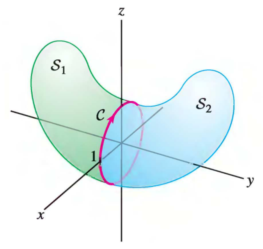
$$\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S} = \oint_{\partial\mathcal{S}} \boldsymbol{A} \cdot d\boldsymbol{s}.$$



In particular, if $\mathcal{S}$ is closed ($\partial\mathcal{S} = \emptyset$), then $\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S} = 0$.

Example

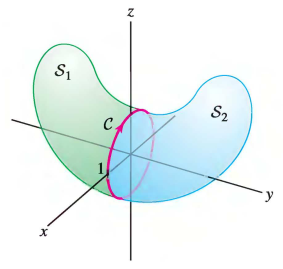
- Let $\boldsymbol{F} = \text{curl}(\boldsymbol{A})$, where
$$\boldsymbol{A} = \langle y + z, \sin(xy), e^{xyz} \rangle.$$
Find the flux of $\boldsymbol{F}$ through the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ in the figure, whose common boundary $\mathcal{C}$ is the unit circle in the xz-plane.



With $\mathcal{C}$ oriented in the direction of the arrow, $\mathcal{S}_1$ lies to the left. By the preceding theorem, $\iint_{\mathcal{S}_1} \boldsymbol{F} \cdot d\boldsymbol{S} = \oint_{\mathcal{C}} \boldsymbol{A} \cdot d\boldsymbol{s}$. We compute the line integral on the right. Note that $\boldsymbol{c}(t) = \langle \cos t, 0, \sin t \rangle$ traces $\mathcal{C}$ in the direction of the arrow. We have

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{c}(t)) &= \langle 0 + \sin t, \sin(0), e^0 \rangle = \langle \sin t, 0, 1 \rangle; \\
\boldsymbol{A}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) &= \langle \sin t, 0, 1 \rangle \cdot \langle -\sin t, 0, \cos t \rangle = -\sin^2 t + \cos t; \\
\oint_{\mathcal{C}} \boldsymbol{A} \cdot d\boldsymbol{s} &= \int_0^{2\pi} (-\sin^2 t + \cos t) dt \\
&= -\tfrac{1}{2}(t - \tfrac{1}{2} \sin 2t) \Big|_0^{2\pi} = -\pi.
\end{aligned}$$

Example (Cont'd)



- We have $\iint_{\mathcal{S}_1} \boldsymbol{F} \cdot d\boldsymbol{S} = \oint_{\mathcal{C}} \boldsymbol{A} \cdot d\boldsymbol{s}$ and $\oint_{\mathcal{C}} \boldsymbol{A} \cdot d\boldsymbol{s} = -\pi$.
  We conclude that $\iint_{\mathcal{S}_1} \boldsymbol{F} \cdot d\boldsymbol{S} = -\pi$. On the other hand, $\mathcal{S}_2$ lies on the right as you traverse $\mathcal{C}$. Therefore $\mathcal{S}_2$ has oriented boundary $-\mathcal{C}$. So
  $$\iint_{\mathcal{S}_2} \boldsymbol{F} \cdot d\boldsymbol{S} = \oint_{-\mathcal{C}} \boldsymbol{A} \cdot d\boldsymbol{s} = -\oint_{\mathcal{C}} \boldsymbol{A} \cdot d\boldsymbol{s} = \pi.$$

Subsection 3

Divergence Theorem

The Divergence of a Vector Field

- The **divergence** of a vector field $\boldsymbol{F} = \langle F_1, F_2, F_3 \rangle$ is defined by

$$\mathrm{div}(\boldsymbol{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

- We often write the divergence as a symbolic dot product:

$$\nabla \cdot \boldsymbol{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle F_1, F_2, F_3 \rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

- Unlike the gradient and curl, the divergence is a scalar function.
- Like the gradient and curl, the divergence obeys the linearity rules:

$$\begin{aligned} \mathrm{div}(\boldsymbol{F} + \boldsymbol{G}) &= \mathrm{div}(\boldsymbol{F}) + \mathrm{div}(\boldsymbol{G}); \\ \mathrm{div}(c\boldsymbol{F}) &= c\,\mathrm{div}(\boldsymbol{F}), \quad c \text{ any constant.} \end{aligned}$$

Example

(a) Find the divergence of $\boldsymbol{F} = \sin(x+z)\boldsymbol{i} - ye^{xz}\boldsymbol{k}$.

(b) Evaluate the divergence of $\boldsymbol{F} = \langle e^{xy}, xy, z^4 \rangle$ at $P = (1, 0, 2)$.

(a) We have

$$\operatorname{div}(\boldsymbol{F}) = \frac{\partial}{\partial x}\sin(x+z) + \frac{\partial}{\partial y}0 + \frac{\partial}{\partial z}(-ye^{xz}) = \cos(x+z) - xye^{xz}.$$

(b) We have

$$\operatorname{div}(\boldsymbol{F}) = \frac{\partial}{\partial x}e^{xy} + \frac{\partial}{\partial y}xy + \frac{\partial}{\partial z}z^4 = ye^{xy} + x + 4z^3.$$

Therefore

$$\operatorname{div}(\boldsymbol{F})(P) = \operatorname{div}(\boldsymbol{F})(1, 0, 2) = 0 \cdot e^0 + 1 + 4 \cdot 2^3 = 33.$$

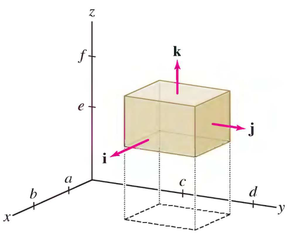
The Divergence Theorem

Theorem (Divergence Theorem)

Let $\mathcal{S}$ be a closed surface that encloses a region $\mathcal{W}$ in $\mathbb{R}^3$. Assume that $\mathcal{S}$ is piecewise smooth and is oriented by normal vectors pointing to the outside of $\mathcal{W}$. Let $\boldsymbol{F}$ be a vector field whose domain contains $\mathcal{W}$. Then

$$\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_{\mathcal{W}} \operatorname{div}(\boldsymbol{F}) dV.$$

- We prove the Divergence Theorem in the special case that $\mathcal{W}$ is a box $[a, b] \times [c, d] \times [e, f]$.
  We write each side as a sum over components.



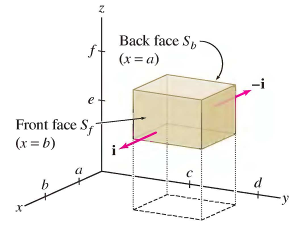
The Divergence Theorem (Cont'd)

- We have, by linearity,

$$\begin{aligned}&\iint_{\partial\mathcal{W}}(F_1\boldsymbol{i}+F_2\boldsymbol{j}+F_3\boldsymbol{k})\cdot d\boldsymbol{S}\\ &= \iint_{\partial\mathcal{W}}F_1\boldsymbol{i}\cdot d\boldsymbol{S}+\iint_{\partial\mathcal{W}}F_2\boldsymbol{j}\cdot d\boldsymbol{S}+\iint_{\partial\mathcal{W}}F_3\boldsymbol{k}\cdot d\boldsymbol{S};\\ &\iiint_{\mathcal{W}}\mathrm{div}(F_1\boldsymbol{i}+F_2\boldsymbol{j}+F_3\boldsymbol{k})dV\\ &= \iiint_{\mathcal{W}}\mathrm{div}(F_1\boldsymbol{i})dV+\iiint_{\mathcal{W}}\mathrm{div}(F_2\boldsymbol{j})dV+\iiint_{\mathcal{W}}\mathrm{div}(F_3\boldsymbol{k})dV.\end{aligned}$$

  We show that corresponding terms are equal. We do the $\boldsymbol{i}$-component. Assume $\boldsymbol{F}=F_1\boldsymbol{i}$. The surface integral over boundary $\mathcal{S}$ of the box is the sum of the integrals over the six faces.

  However, $\boldsymbol{F}=F_1\boldsymbol{i}$ is orthogonal to the normal vectors to the top and bottom as well as the two side faces because $\boldsymbol{F}\cdot\boldsymbol{j}=\boldsymbol{F}\cdot\boldsymbol{k}=0$. Therefore, the surface integrals over these faces are zero. Nonzero contributions come only from the front and back faces, which we denote $\mathcal{S}_f$ and $\mathcal{S}_b$.



The Divergence Theorem (Conclusion)

- So we get $\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{\mathcal{S}_f} \boldsymbol{F} \cdot d\boldsymbol{S} + \iint_{\mathcal{S}_b} \boldsymbol{F} \cdot d\boldsymbol{S}$.
  To evaluate these integrals, we parametrize $\mathcal{S}_f$ and $\mathcal{S}_b$ by $G_f(y,z) = (b,y,z)$, $c \le y \le d$, $e \le z \le f$, $G_b(y,z) = (a,y,z)$, $c \le y \le d$, $e \le z \le f$. The normal vectors for these parametrizations are $\frac{\partial G_f}{\partial y} \times \frac{\partial G_f}{\partial z} = \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$, $\frac{\partial G_b}{\partial y} \times \frac{\partial G_b}{\partial z} = \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$. The outward-pointing normal for $\mathcal{S}_b$ is $-\boldsymbol{i}$. So we have

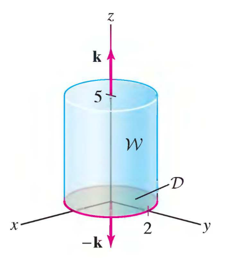
$$\begin{aligned} &\iint_{\mathcal{S}_f} \boldsymbol{F} \cdot d\boldsymbol{S} + \iint_{\mathcal{S}_b} \boldsymbol{F} \cdot d\boldsymbol{S} \\ &= \int_e^f \int_c^d F_1(b,y,z)\,dy\,dz - \int_e^f \int_c^d F_1(a,y,z)\,dy\,dz \\ &= \int_e^f \int_c^d (F_1(b,y,z) - F_1(a,y,z))\,dy\,dz. \end{aligned}$$

  By the FTC, $F_1(b,y,z) - F_1(a,y,z) = \int_a^b \frac{\partial F_1}{\partial x}(x,y,z)\,dx$. Since $\operatorname{div}(\boldsymbol{F}) = \operatorname{div}(F_1 \boldsymbol{i}) = \frac{\partial F_1}{\partial x}$, we obtain:

$$\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S} = \int_e^f \int_c^d \int_a^b \frac{\partial F_1}{\partial x}(x,y,z)\,dx\,dy\,dz = \iiint_{\mathcal{W}} \operatorname{div}(\boldsymbol{F})\,dv.$$

Example: Verifying the Divergence Theorem

- Verify the Divergence Theorem for $\boldsymbol{F} = \langle y, yz, z^2\rangle$ and the cylinder in the figure.



We must verify that the flux $\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\mathcal{S}$ is the boundary of the cylinder, is equal to the integral of div($\boldsymbol{F}$) over the cylinder. We compute the flux through $\mathcal{S}$ first: It is the sum of three surface integrals over the side, top and bottom.

To integrate over the side, we use the standard parametrization of the cylinder: $G(\theta, z) = (2\cos\theta, 2\sin\theta, z)$, $0 \leq \theta < 2\pi$, $0 \leq z \leq 5$. Then $\boldsymbol{n} = \boldsymbol{T}_\theta \times \boldsymbol{T}_z = \langle -2\sin\theta, 2\cos\theta, 0\rangle \times \langle 0, 0, 1\rangle = \langle 2\cos\theta, 2\sin\theta, 0\rangle$.

Moreover, $\boldsymbol{F}(G(\theta, z)) = \langle y, yz, z^2\rangle = \langle 2\sin\theta, 2z\sin\theta, z^2\rangle$. So

$$\begin{aligned} \boldsymbol{F} \cdot d\boldsymbol{S} &= \langle 2\sin\theta, 2z\sin\theta, z^2\rangle \cdot \langle 2\cos\theta, 2\sin\theta, 0\rangle d\theta dz \\ &= (4\cos\theta\sin\theta + 4z\sin^2\theta) d\theta dz; \\ \iint_{\text{side}} \boldsymbol{F} \cdot d\boldsymbol{S} &= \int_0^5 \int_0^{2\pi} (4\cos\theta\sin\theta + 4z\sin^2\theta) d\theta dz \\ &= \int_0^5 [2\sin^2\theta + 2z(\theta - \tfrac{1}{2}\sin 2\theta)]_0^{2\pi} dz = 4\pi \tfrac{z^2}{2} \Big|_0^5 = 50\pi. \end{aligned}$$

Example: Verifying the Divergence Theorem (Cont'd)

- We next integrate over the top and bottom of the cylinder. The top of the cylinder is at height $z = 5$. So we can parametrize the top by $G(x, y) = (x, y, 5)$ for (x, y) in the disk $\mathcal{D}$ of radius 2: $\mathcal{D} = \{(x, y) : x^2 + y^2 \leq 4\}$. Then $\boldsymbol{n} = \boldsymbol{T}_x \times \boldsymbol{T}_y = \langle 1, 0, 0 \rangle \times \langle 0, 1, 0 \rangle = \langle 0, 0, 1 \rangle$. Note that $\boldsymbol{F}(G(x, y)) = F(x, y, 5) = \langle y, 5y, 5^2 \rangle$. So $\boldsymbol{F}(G(x, y)) \cdot \boldsymbol{n} = \langle y, 5y, 5^2 \rangle \cdot \langle 0, 0, 1 \rangle = 25$. Finally,

$$\iint_{\text{top}} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{\mathcal{D}} 25 dA = 25 \text{Area}(\mathcal{D}) = 25(4\pi) = 100\pi.$$

  Along the bottom disk of the cylinder, we have $z = 0$ and $\boldsymbol{F}(x, y, 0) = \langle y, 0, 0 \rangle$. Thus, $\boldsymbol{F}$ is orthogonal to the vector $-\boldsymbol{k}$ normal to the bottom disk. So the integral along the bottom is zero.

Example: Verifying the Divergence Theorem (Conclusion)

- The total flux is $\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S} = 50\pi + 100\pi + 0 = 150\pi$.

  We finally compute the integral of divergence:

$$\begin{array}{rcl} \operatorname{div}(\boldsymbol{F}) & = & \operatorname{div}(\langle y, yz, z^2 \rangle) \\ & = & \frac{\partial}{\partial x} y + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z} z^2 \\ & = & 0 + z + 2z = 3z. \end{array}$$

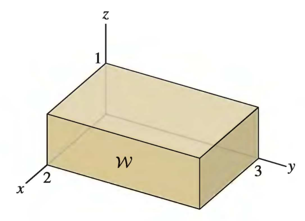
  The cylinder $\mathcal{W}$ consists of all points (x, y, z) for $0 \leq z \leq 5$ and (x, y) in the disk $\mathcal{D}$. So the integral of the divergence is:

$$\begin{array}{rcl} \iiint_{\mathcal{W}} \operatorname{div}(\boldsymbol{F}) dV & = & \iint_{\mathcal{D}} \int_0^5 3z dz dA \\ & = & \iint_{\mathcal{D}} \frac{75}{2} dA \\ & = & (\frac{75}{2})(\text{Area}(\mathcal{D})) \\ & = & (\frac{75}{2})(4\pi) \\ & = & 150\pi. \end{array}$$

Example: Using the Divergence Theorem

- Use the Divergence Theorem to evaluate $\iint_{\mathcal{S}} \langle x^2, z^4, e^z \rangle \cdot d\boldsymbol{S}$, where $\mathcal{S}$ is the boundary of the box $\mathcal{W}$ in the figure.

  First, compute the divergence:



$$\operatorname{div}(\langle x^2, z^4, e^z \rangle) = \frac{\partial}{\partial x} x^2 + \frac{\partial}{\partial y} z^4 + \frac{\partial}{\partial z} e^z = 2x + e^z.$$

Then apply the Divergence Theorem and use Fubini's Theorem:

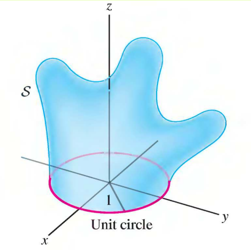
$$\begin{aligned}
\iint_{\mathcal{S}} \langle x^2, z^4, e^z \rangle \cdot d\boldsymbol{S} &= \iiint_{\mathcal{W}} (2x + e^z) dV \\
&= \int_0^2 \int_0^3 \int_0^1 (2x + e^z) dz\, dy\, dx \\
&= 3 \int_0^2 2x dx + 6 \int_0^1 e^z dz \\
&= 3x^2 \mid_0^2 + 6e^z \mid_0^1 \\
&= 12 + 6(e - 1) = 6e + 6.
\end{aligned}$$

Example

- Use the Divergence Theorem to evaluate the flux $\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F} = \langle 0, 0, \frac{z^3}{3} \rangle$ and $\mathcal{S}$ is the sphere $x^2 + y^2 + z^2 = 1$.
  We have $\text{div}(\boldsymbol{F}) = z^2$.
  Now apply the Divergence Theorem using spherical coordinates:

$$\begin{aligned}
\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S} &= \iiint_{\mathcal{B}} \text{div}(\boldsymbol{F}) dV \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^1 (\rho \cos\phi)^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi} d\theta \int_0^{\pi} \cos^2\phi \sin\phi \, d\phi \int_0^1 \rho^4 d\rho \\
&= \theta \,\Big|_0^{2\pi} \left(-\frac{1}{3}\cos^3\phi\right) \Big|_0^{\pi} \, \frac{1}{5}\rho^5 \,\Big|_0^1 \\
&= 2\pi \cdot \frac{2}{3} \cdot \frac{1}{5} = \frac{4\pi}{15}.
\end{aligned}$$

Example



- Let $\mathcal{S}_1$ be the closed surface consisting of $\mathcal{S}$ together with the unit disk. If

$$\iint_{\mathcal{S}_1} \langle x, 2y, 3z \rangle \cdot d\boldsymbol{S} = 72,$$

find the volume enclosed by $\mathcal{S}_1$.
The key is to notice that

$$\operatorname{div}(\boldsymbol{F}) = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}2y + \frac{\partial}{\partial z}3z = 1 + 2 + 3 = 6.$$

Thus, using the Divergence Theorem, we get:

$$\begin{aligned} \text{Volume} &= \iiint_{\mathcal{W}} dV = \tfrac{1}{6} \iiint_{\mathcal{W}} 6dV \\ &= \tfrac{1}{6} \iiint_{\mathcal{W}} \operatorname{div}(\langle x, 2y, 3z \rangle) dV \\ &= \tfrac{1}{6} \iint_{\mathcal{S}_1} \langle x, 2y, 3z \rangle \cdot d\boldsymbol{S} = \tfrac{1}{6} \cdot 72 = 12. \end{aligned}$$

Example

(a) Show that if $\mathcal{W}$ ia a region in $\mathbb{R}^3$ with a smooth boundary $\mathcal{S}$, then

$$\text{Volume}(\mathcal{W}) = \frac{1}{3} \iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S},$$

where $\boldsymbol{F} = \langle x, y, z \rangle$.

(b) Use Part (a) to calculate the volume of the unit ball as a surface integral over the unit sphere.

(a) Note that $\text{div}(\boldsymbol{F}) = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 1 + 1 + 1 = 3$.
Thus, we have

$$\begin{aligned}
\text{Volume}(\mathcal{W}) &= \iiint_{\mathcal{W}} 1 dV = \tfrac{1}{3} \iiint_{\mathcal{W}} 3 dV \\
&= \tfrac{1}{3} \iiint_{\mathcal{W}} \text{div}(\langle x, y, z \rangle) dV \\
&= \tfrac{1}{3} \iint_{\mathcal{S}} \langle x, y, z \rangle \cdot d\boldsymbol{S} \\
&= \tfrac{1}{3} \iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S}.
\end{aligned}$$

Example (Part (b))

(b) We use spherical coordinates $x = r\sin\phi\cos\theta,\ y = r\sin\phi\sin\theta,$ $z = r\sin\phi$.

$$
\begin{aligned}
G(\theta,\phi) &= (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi);\\
\boldsymbol{F}(G(\theta,\phi)) &= \langle\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle;\\
\boldsymbol{n} &= \sin\phi\langle\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle;\\
\boldsymbol{F}(G(\theta,\phi))\cdot\boldsymbol{n} &= \sin\phi\langle x,y,z\rangle\cdot\langle x,y,z\rangle\\
&= R^2\sin\phi = \sin\phi.
\end{aligned}
$$

Now we get

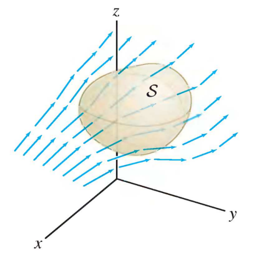
$$
\begin{aligned}
\text{Volume}(\mathcal{B}) &= \tfrac{1}{3}\iint_{\mathcal{S}}\langle x,y,z\rangle\cdot d\boldsymbol{S}\\
&= \tfrac{1}{3}\iint_{\mathcal{D}}\boldsymbol{F}(G(\theta,\phi))\cdot\boldsymbol{n}dA\\
&= \tfrac{1}{3}\int_0^{2\pi}\int_0^{\pi}\sin\phi d\phi d\theta\\
&= \tfrac{1}{3}\int_0^{2\pi}d\theta\int_0^{\pi}\sin\phi d\phi\\
&= \tfrac{1}{3}\cdot 2\pi\cdot 2 = \tfrac{4\pi}{3}.
\end{aligned}
$$

Example: A Vector Field with Zero Divergence

- Compute the flux of

$$\boldsymbol{F} = \langle z^2 + xy^2, \cos(x+z), e^{-y} - zy^2 \rangle$$

through the boundary of the surface $\mathcal{S}$ in the figure.



Although $\boldsymbol{F}$ is rather complicated, its divergence is zero:

$$\begin{aligned} \mathrm{div}(\boldsymbol{F}) &= \frac{\partial}{\partial x}(z^2 + xy^2) + \frac{\partial}{\partial y}\cos(x+z) + \frac{\partial}{\partial z}(e^{-y} - zy^2) \\ &= y^2 - y^2 = 0. \end{aligned}$$

The Divergence Theorem shows that the flux is zero. Letting $\mathcal{W}$ be the region enclosed by $\mathcal{S}$, we have

$$\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_{\mathcal{W}} \mathrm{div}(\boldsymbol{F}) dV = \iiint_{\mathcal{W}} 0 dV = 0.$$

The Inverse Square Field

- The Divergence Theorem is a powerful tool for computing the flux of electrostatic fields.
- This is due to the special properties of the inverse-square vector field

$$\boldsymbol{F}_{\text{isq}} = \frac{\boldsymbol{e}_r}{r^2},$$

where

$$\boldsymbol{e}_r = \frac{\langle x, y, z \rangle}{r} = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}.$$

Divergence of the Inverse-Square Field

- The inverse-square vector field $\boldsymbol{F}_{\text{isq}} = \frac{\boldsymbol{e}_r}{r^2}$ has zero divergence $\text{div}(\frac{\boldsymbol{e}_r}{r^2}) = 0$.
  Write the field as
  $$\boldsymbol{F}_{\text{isq}} = \langle F_1, F_2, F_3 \rangle = \frac{1}{r^2}\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \rangle = \langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \rangle.$$
  We have
  $$\begin{array}{rcl} \frac{\partial F_1}{\partial x} & = & \frac{\partial}{\partial x}\frac{x}{r^3} = \frac{1}{r^3} + x\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-3/2} \\ & = & \frac{1}{r^3} - \frac{3x^2}{\sqrt{(x^2+y^2+z^2)^{5/2}}} = \frac{1}{r^3} - \frac{3x^2}{r^5} = \frac{r^2-3x^2}{r^5}. \end{array}$$
  Similarly, $\frac{\partial F_2}{\partial y} = \frac{r^2-3y^2}{r^5}$ and $\frac{\partial F_3}{\partial z} = \frac{r^2-3z^2}{r^5}$.
  Thus, we compute
  $$\begin{array}{rcl} \text{div}(\boldsymbol{F}_{\text{isq}}) & = & \frac{r^2-3x^2}{r^5} + \frac{r^2-3y^2}{r^5} + \frac{r^2-3z^2}{r^5} \\ & = & \frac{3r^2-3(x^2+y^2+z^2)}{r^5} = \frac{3r^2-3r^2}{r^5} = 0. \end{array}$$

Flux of the Inverse Square Field

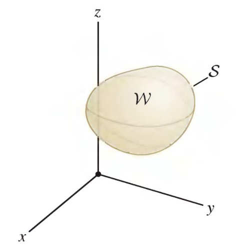
- The next theorem shows that the flux of $\boldsymbol{F}_{\text{isq}}$ through a closed surface $\mathcal{S}$ depends only on whether $\mathcal{S}$ contains the origin.

Theorem (Flux of the Inverse-Square Field)

The flux of $\boldsymbol{F}_{\text{isq}} = \frac{\boldsymbol{e}_r}{r^2}$ through closed surfaces is given by

$$\iint_{\mathcal{S}} \left(\frac{\boldsymbol{e}_r}{r^2}\right) \cdot d\boldsymbol{S} = \begin{cases} 4\pi, & \text{if } \mathcal{S} \text{ encloses the origin} \\ 0, & \text{if } \mathcal{S} \text{ does not enclose the origin} \end{cases}$$

- First, suppose $\mathcal{S}$ does not contain the origin. Then the region $\mathcal{W}$ enclosed by $\mathcal{S}$ is contained in the domain of $\boldsymbol{F}_{\text{isq}}$. So we can apply the Divergence Theorem. We know that $\text{div}(\boldsymbol{F}_{\text{isq}}) = 0$. Therefore $\iint_{\mathcal{S}} \left(\frac{\boldsymbol{e}_r}{r^2}\right) \cdot d\boldsymbol{S} = \iiint_{\mathcal{W}} \text{div}(\boldsymbol{F}_{\text{isq}}) dV = \iiint_{\mathcal{W}} 0 dV = 0.$

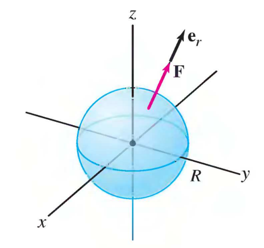


Flux of the Inverse Square Field (Cont'd)

- Next, let $\mathcal{S}_R$ be the sphere of radius R centered at the origin.

  We cannot use the Divergence Theorem because $\mathcal{S}_R$ contains a point (the origin) where $\boldsymbol{F}_{\text{isq}}$ is not defined.

  However, we can compute the flux of $\boldsymbol{F}_{\text{isq}}$ through $\mathcal{S}_R$ using spherical coordinates.

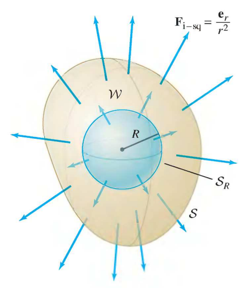


  Recall that the outward-pointing normal vector in spherical coordinates is $\boldsymbol{n} = \boldsymbol{T}_\phi \times \boldsymbol{T}_\theta = (R^2 \sin\phi)\boldsymbol{e}_r$. The inverse-square field on $\mathcal{S}_R$ is simply $\boldsymbol{F}_{\text{isq}} = \frac{\boldsymbol{e}_r}{R^2}$. Thus,

$$
\begin{aligned}
\boldsymbol{F}_{\text{isq}} \cdot \boldsymbol{n} &= \frac{\boldsymbol{e}_r}{R^2} \cdot (R^2 \sin\phi \boldsymbol{e}_r) = \sin\phi(\boldsymbol{e}_r \cdot \boldsymbol{e}_r) = \sin\phi; \\
\iint_{\mathcal{S}_R} \boldsymbol{F}_{\text{isq}} \cdot d\boldsymbol{S} &= \int_0^{2\pi} \int_0^{\pi} \boldsymbol{F}_{\text{isq}} \cdot \boldsymbol{n} \, d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\phi d\theta \\
&= 2\pi \int_0^{\pi} \sin\phi \, d\phi = 4\pi.
\end{aligned}
$$

Flux of the Inverse Square Field (Conclusion)

- To extend this result to any surface $\mathcal{S}$ containing the origin, choose a sphere $\mathcal{S}_R$ whose radius $R > 0$ is so small that $\mathcal{S}_R$ is contained inside $\mathcal{S}$. Let $\mathcal{W}$ be the region between $\mathcal{S}_R$ and $\mathcal{S}$. The oriented boundary of $\mathcal{W}$ is the difference $\partial\mathcal{W} = \mathcal{S} - \mathcal{S}_R$. This means that $\mathcal{S}$ is oriented by outward-pointing normals and $\mathcal{S}_R$ by inward-pointing normals.



By the Divergence Theorem (in a more general form than previously stated),

$$\begin{aligned}
\iint_{\partial\mathcal{W}} \boldsymbol{F}_{\text{isq}} \cdot d\boldsymbol{S} &= \iint_{\mathcal{S}} \boldsymbol{F}_{\text{isq}} \cdot d\boldsymbol{S} - \iint_{\mathcal{S}_R} \boldsymbol{F}_{\text{isq}} \cdot d\boldsymbol{S} \\
&= \iiint_{\mathcal{W}} \text{div}(\boldsymbol{F}_{\text{isq}}) dV \\
&= \iiint_{\mathcal{W}} 0 dV = 0.
\end{aligned}$$

So the fluxes through $\mathcal{S}$ and $\mathcal{S}_R$ are equal. Hence they both equal 4π.

Subsection 4

The Fundamental Theorems of Calculus

The Fundamental Theorem of Calculus

- We have studied several "Fundamental Theorems".
  Each of these is a relation of the type

$$\begin{matrix}\text{Integral of a derivative} \\ \text{on an oriented domain}\end{matrix} = \begin{matrix}\text{Integral over the oriented} \\ \text{boundary of the domain}\end{matrix}$$

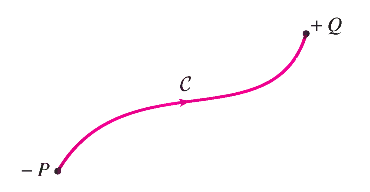
- In single-variable calculus, the **Fundamental Theorem of Calculus** (**FTC**) relates the integral of $f'(x)$ over an interval $[a, b]$ to the "integral" of $f(x)$ over the boundary of $[a, b]$ consisting of two points a and b:

$$\underbrace{\int_a^b f'(x)dx}_{\text{Integral of derivative over } [a,b]} = \underbrace{f(b) - f(a)}_{\text{"Integral" over the boundary of } [a,b]}$$

  The boundary of $[a, b]$ is oriented by assigning a plus sign to b and a minus sign to a.

The Fundamental Theorem of Line Integrals

- The **Fundamental Theorem for Line Integrals** generalizes the Fundamental Theorem of Calculus:
  Instead of an interval $[a, b]$ (a path from a to b along the x-axis), we take any path from points P to Q in $\mathbb{R}^3$.

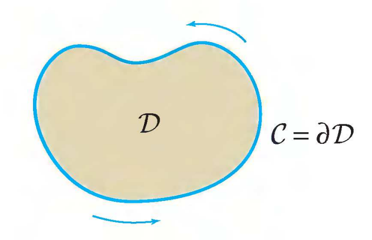


Instead of $f'(x)$ we use the gradient:

$$\underbrace{\int_{\mathcal{C}} \nabla V \cdot d\boldsymbol{s}}_{\text{Integral of derivative over a curve}} \quad = \quad \underbrace{V(Q) - V(P)}_{\text{“Integral” over the boundary } \partial\mathcal{C} = Q - P}$$

Green's Theorem

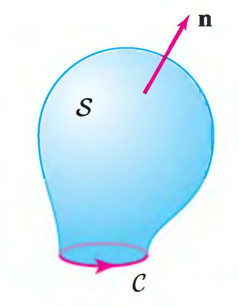
- **Green's Theorem** is a two-dimensional version of the Fundamental Theorem of Calculus that relates the integral of a derivative over a domain $\mathcal{D}$ in the plane to an integral over its boundary curve $\mathcal{C} = \partial\mathcal{D}$:



$$\underbrace{\iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA}_{\text{Integral of derivative over domain}} \qquad = \qquad \underbrace{\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s}}_{\text{Integral over boundary curve}}$$

Stokes' Theorem

- **Stokes' Theorem** extends Green's Theorem:
  Instead of a domain in the plane (a flat surface), we allow any surface in $\mathbb{R}^3$.
  The appropriate derivative is the curl.

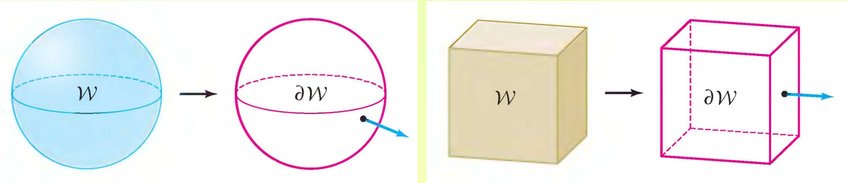


$$\underbrace{\iint_{\mathcal{S}} \text{curl}(\boldsymbol{F}) \cdot d\boldsymbol{S}}_{\text{Integral of derivative over surface}} = \underbrace{\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s}}_{\text{Integral over boundary curve}}$$

The Divergence Theorem

- **The Divergence Theorem** follows the same pattern.

  For $\mathcal{S}$ a closed surface that encloses a 3D region $\mathcal{W}$, i.e., $\mathcal{S}$ the boundary of $\mathcal{W}$,



$$\underbrace{\iiint_{\mathcal{W}} \operatorname{div}(\boldsymbol{F})\, dV}_{\text{Integral of derivative over 3D region}} = \underbrace{\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S}}_{\text{Integral over boundary surface}}$$