Linear Algebra and Analytic Geometry

Conf.univ.,dr. Elena Cojuhari

elena.cojuhari@mate.utm.md

Technical University of Moldova

2020

Subsection 1

Systems of Linear Equations

Linear Equations

- A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where b and the coefficients $a_1, \ldots, a_n$ are real or complex numbers.
Example: The equations

$$4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2(\sqrt{6} - x_1) + x_3$$

are both linear because they can be rearranged algebraically:

$$3x_1 - 5x_2 = -2 \quad \text{and} \quad 2x_1 + x_2 - x_3 = 2\sqrt{6}.$$

The equations

$$4x_1 - 5x_2 = x_1x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 6$$

are not linear because of the presence of x_1x_2 in the first equation and $\sqrt{x_1}$ in the second.

Linear Systems and Solutions

- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables - say, $x_1, \dots, x_n$.

  Example:

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & 1.5x_3 & = & 8 \\ x_1 & & & - & 4x_3 & = & -7 \end{array}$$

- A **solution** of the system is a list $(s_1, s_2, \dots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \dots, s_n$ are substituted for $x_1, \dots, x_n$, respectively.

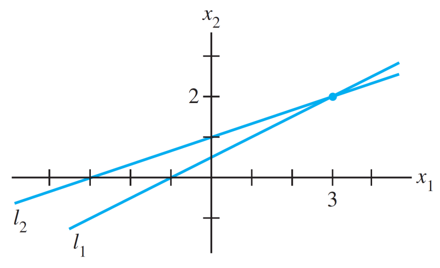
  Example: $(5, 6.5, 3)$ is a solution of the system above because, when these values are substituted for x_1, x_2, x_3, respectively, the equations simplify to $8 = 8$ and $-7 = -7$.

Solution Sets and Equivalent Systems

- The set of all possible solutions is called the **solution set** of the linear system.
- Two linear systems are called **equivalent** if they have the same solution set:
  - Each solution of the first system is a solution of the second system; and
  - Each solution of the second system is a solution of the first.

Example

- Consider the system $\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$.
- Finding the solution set is easy because it amounts to finding the intersection of two lines.
- A pair of numbers (x_1, x_2) satisfies both equations in the system if and only if the point (x_1, x_2) lies on both lines.



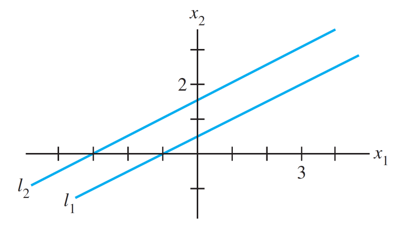
- In the system above, the solution is the single point $(3, 2)$.

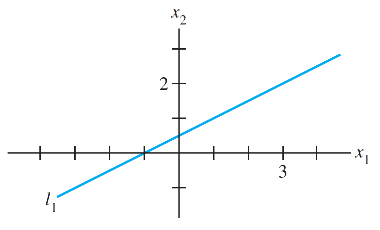
Example

- Two lines need not intersect in a single point: they could be parallel, or they could coincide and hence "intersect" at every point on the line.

  Example: The figures below show the graphs that correspond to the following systems:

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 3 \end{cases} \qquad \begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 1 \end{cases}$$





Types of Linear Systems

- A system of linear equations has
    1. no solution; or
    2. exactly one solution; or
    3. infinitely many solutions.
- A system of linear equations is said to be:
    - **consistent** if it has either one solution or infinitely many solutions;
    - **inconsistent** if it has no solution.

Matrix Representation of a Linear System

- The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**.
- Consider the system $\left\{\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}\right.$
- The matrix $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$ is called the **coefficient matrix** (or **matrix of coefficients**) of the system.
- The matrix $\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$ is called the **augmented matrix** of the system.
- An augmented matrix of a system consists of the coefficient matrix with an added column containing the right side constants.

Size of a Matrix

- The size of a matrix tells how many rows and columns it has.
- If m and n are positive integers, an $m \times n$ **matrix** is a rectangular array of numbers with m rows and n columns.

  Example: The augmented matrix

  $$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

  is a 3×4 (read "3 by 4") matrix.

Elementary Row Operations

Elementary Row Operations

1. (**Replacement**) Replace one row by the sum of itself and a multiple of another row.
2. (**Interchange**) Interchange two rows.
3. (**Scaling**) Multiply all entries in a row by a nonzero constant.

Example:

$$\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 5 \\ 0 & -7 \end{bmatrix};$$
$$\begin{bmatrix} 0 & 5 \\ 1 & 8 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 8 \\ 0 & 5 \end{bmatrix};$$
$$\begin{bmatrix} 3 & 12 \\ 2 & 3 \end{bmatrix} \xrightarrow{R_1 \leftarrow \frac{1}{3} R_1} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}.$$

- Two matrices A, B are called **row equivalent** (denoted $A \sim B$) if there is a sequence of elementary row operations that transforms one matrix into the other.

Reversibility

- Row operations are reversible.
  - If two rows are interchanged, they can be returned to their original positions by another interchange.
  - If a row is scaled by a nonzero constant c, then multiplying the new row by $\frac{1}{c}$ produces the original row.
  - Consider a replacement operation involving two rows - say, rows 1 and 2 - and suppose that c times row 1 is added to row 2.
    To "reverse" this operation, add $-c$ times row 1 to (new) row 2 and obtain the original row 2.

Example:

$$\begin{bmatrix} 0 & 5 \\ 1 & 8 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 8 \\ 0 & 5 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 5 \\ 1 & 8 \end{bmatrix};$$

$$\begin{bmatrix} 3 & 12 \\ 2 & 3 \end{bmatrix} \xrightarrow{R_1 \leftarrow \frac{1}{3} R_1} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \xrightarrow{R_1 \leftarrow 3R_1} \begin{bmatrix} 3 & 12 \\ 2 & 3 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 5 \\ 0 & -7 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix};$$

Row Equivalence and Solutions

- We are interested in row operations on the augmented matrix of a system of linear equations.
- Suppose a system is changed to a new one via row operations.
- By considering each type of row operation, we can see that any solution of the original system remains a solution of the new system.
- Conversely, since the original system can be produced via row operations on the new system, each solution of the new system is also a solution of the original system.
- This discussion justifies the following statement.

> If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Example

- Solve the linear system $\left\{\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}\right.$
- Form the augmented matrix and use elementary row operations to produce a simpler equivalent system:

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + 4R_1} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{1}{2}R_2}$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 + 3R_2]{R_1 \leftarrow R_1 + 2R_2} \left[\begin{array}{rrr|r} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[R_2 \leftarrow R_2 + 4R_3]{R_1 \leftarrow R_1 + 7R_3}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]. \quad \text{So } (x_1, x_2, x_3) = (29, 16, 3).$$

Determining Consistency

- Determine whether the linear system $\left\{\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 15 \\ 2x_1 & + & x_2 & - & 2x_3 & = & -13 \\ -x_1 & + & 2x_2 & + & x_3 & = & -6 \end{array}\right.$ is consistent.
- We have

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 15 \\ 2 & 1 & -2 & -13 \\ -1 & 2 & 1 & -6 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 15 \\ 0 & 3 & -4 & -43 \\ 0 & 1 & 2 & 9 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 15 \\ 0 & 1 & 2 & 9 \\ 0 & 3 & -4 & -43 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 3R_2} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 15 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & -10 & -70 \end{array}\right]$$

Thus we get $\left\{\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 15 \\ & & x_2 & + & 2x_3 & = & 9 \\ & & & - & 10x_3 & = & -70 \end{array}\right.$ which is consistent.

Determining Consistency

- Determine whether the linear system
$$\left\{\begin{array}{rcrcrcl} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & 1 \end{array}\right.$$ is consistent.
- We have

$$\left[\begin{array}{rrr|r} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{2}R_1}$$
$$\left[\begin{array}{rrr|r} 1 & -\frac{3}{2} & 1 & \frac{1}{2} \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 5R_1} \left[\begin{array}{rrr|r} 1 & -\frac{3}{2} & 1 & \frac{1}{2} \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + \frac{1}{2}R_2}$$
$$\left[\begin{array}{rrr|r} 1 & -\frac{3}{2} & 1 & \frac{1}{2} \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{array}\right], \quad \text{which is inconsistent!}$$

Subsection 2

Row Reduction and Echelon Forms

Echelon and Reduced Echelon Forms

- A **nonzero row** or **column** in a matrix means a row or column that contains at least one nonzero entry.
- A **leading entry** of a row refers to the leftmost nonzero entry.

Definition

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

3. The leading entry in each nonzero row is 1.
4. Each leading 1 is the only nonzero entry in its column.

Echelon and Reduced Echelon Matrices

- An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form).
- Property 2 says that the leading entries form an echelon ("steplike") pattern that moves down and to the right through the matrix.

  Example: The matrices

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

  are in echelon form.

  The second matrix is in reduced echelon form, but the first is not.

Example

- The following matrices are in echelon form, where the leading entries ($\square$) may have any nonzero value and the starred entries ($*$) may have any value (including zero).

$$\begin{bmatrix} \square & * & * & * \\ 0 & \square & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \square & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \square & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \square & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \square & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \square & * \end{bmatrix}$$

- The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below and above each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Reduction and Uniqueness of the Reduced Echelon Form

- Any nonzero matrix may be **row reduced** (that is, transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations.
- However, the reduced echelon form one obtains from a matrix is unique:

Theorem (Uniqueness of the Reduced Echelon Form)

Each matrix is row equivalent to one and only one reduced echelon matrix.

- If a matrix A is row equivalent to an echelon matrix U, we call U an **echelon form** (or **row echelon form**) **of** A;
- If U is in reduced echelon form, we call U **the reduced echelon form of** A.

Pivot Positions and Pivot Columns

- When row operations on a matrix produce an echelon form, further row operations to obtain the reduced echelon form do not change the positions of the leading entries.
- Since the reduced echelon form is unique, the leading entries are always in the same positions in any echelon form obtained from a given matrix.
- These leading entries correspond to leading 1's in the reduced echelon form.

Definition

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.
A **pivot column** is a column of A that contains a pivot position.

Example (Step 1)

- Row reduce the matrix $A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$ to echelon form, and locate the pivot columns of A.
- The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or pivot, must be placed in this position.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Example (Step 2)

- Create zeros below the pivot, 1, by adding multiples of the first row to the rows below:

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 + 2R_1]{R_2 \leftarrow R_2 + R_1} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- The pivot position in the second row must be as far left as possible - namely, in the second column.
- Choose the 2 in this position as the next pivot.

Example (Step 3)

- Create zeros below the pivot, 2, by adding multiples of the second row to the rows below:

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \xrightarrow[R_4 \leftarrow R_4 + \frac{3}{2}R_2]{R_3 \leftarrow R_3 - \frac{5}{2}R_2} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

- There is no way to create a leading entry in column 3!
- However, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Row Reduction Algorithm

1. Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.
2. Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
3. Use row replacement operations to create zeros in all positions below the pivot.
4. Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it.

   Apply steps 1-3 to the submatrix that remains.

   Repeat the process until there are no more nonzero rows to modify.
5. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot.

   If a pivot is not 1, make it 1 by a scaling operation.

Linear Systems: Basic and Free Variables

- Suppose that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form $\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$.
- There are three variables because the augmented matrix has four columns.
- The associated system of equations is $\left\{\begin{array}{ccccccc} x_1 & & & - & 5x_3 & = & 1 \\ & & x_2 & + & x_3 & = & 4 \\ & & & & 0 & = & 0 \end{array}\right.$.
- The variables x_1 and x_2 corresponding to pivot columns in the matrix are called **basic variables**.
- The other variable, x_3, is called a **free variable**.

Solutions of Linear Systems

- Whenever a system is consistent the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables.
- This operation is possible because the reduced echelon form places each basic variable in one and only one equation.
- In the example above

$$\left\{\begin{array}{ccccccc} x_1 & & & - & 5x_3 & = & 1 \\ & & x_2 & + & x_3 & = & 4 \\ & & & & 0 & = & 0 \end{array}\right. \quad \Rightarrow \quad \left\{\begin{array}{l} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{array}\right.$$

- The statement "x_3 is free" means that you are free to choose any value for x_3.

Example

- Find the general solution of the linear system whose augmented matrix has been reduced to $\left[\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$.
- The matrix is in echelon form, but we want the reduced echelon form before solving for the basic variables.

$$\left[\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \xrightarrow[R_2 \leftarrow R_2 + R_3]{R_1 \leftarrow R_1 + 2R_3} \left[\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$$

Example (Cont'd)

- There are five variables because the augmented matrix has six columns.
- The associated system now is

$$\begin{cases} x_1 + 6x_2 + 3x_4 = 0 \\ x_3 - 4x_4 = 5 \\ x_5 = 7 \end{cases}$$

- The pivot columns of the matrix are 1, 3, and 5, so the basic variables are x_1, x_3 and x_5.
- The remaining variables, x_2 and x_4, must be free.
- Solving for the basic variables, we obtain $\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$

Parametric Descriptions of Solution Sets

- The descriptions of the solution set given above is a **parametric description** in which the free variables act as parameters.
- **Solving a system** amounts to finding a parametric description of the solution set or determining that the solution set is empty.
  - Whenever a system is consistent and has free variables, the solution set has many parametric descriptions.
    For consistency, we make the (arbitrary) convention of always using the free variables as the parameters for describing a solution set.
  - Whenever a system is inconsistent, the solution set is empty, even when the system has free variables.
    In this case, the solution set has no parametric representation.

Existence and Uniqueness Questions: Example

- Determine the existence and uniqueness of the solutions to the system

$$\begin{cases} \quad\quad\;\; 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15 \end{cases}$$

- We reduce the augmented matrix to an echelon form

$$\left[\begin{array}{rrrrr|r} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rrrrr|r} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{rrrrr|r} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & -2 & 4 & -4 & -2 & 6 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 + R_3}$$

$$\left[\begin{array}{rrrrr|r} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 1 & -2 & 2 & 2 & 1 \\ 0 & -2 & 4 & -4 & -2 & 6 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + 2R_2} \left[\begin{array}{rrrrr|r} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 1 & -2 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 & 8 \end{array}\right]$$

Example (Cont'd)

- The echelon form is $\left[\begin{array}{ccccc|c} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 1 & -2 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 & 8 \end{array}\right]$.

  The basic variables are x_1, x_2 and x_5 and the free x_3 and x_4.

  There is no equation such as $0 = 1$ that would indicate an inconsistent system, so we could use back-substitution to find a solution.

  The solution is not unique because there are free variables.

  Since each different choice of x_3 and x_4 determines a different solution, the system has infinitely many solutions.

Existence and Uniqueness Questions

- When a system is in echelon form and contains no equation of the form $0 = b$, with b nonzero, every nonzero equation contains a basic variable with a nonzero coefficient.
  - If the basic variables are completely determined (with no free variables), then there is a unique solution;
  - If one of the basic variables may be expressed in terms of one or more free variables, there are infinitely many solutions.

Theorem (Existence and Uniqueness Theorem)

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column - that is, if and only if an echelon form of the augmented matrix has no row of the form $[0 \ \dots \ 0 \ b]$, $b \neq 0$. If a linear system is consistent, then the solution set contains either:

(i) a unique solution, when there are no free variables, or

(ii) infinitely many solutions, when there is at least one free variable.

Using Row Reduction to Solve a Linear System

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form.

   Decide whether the system is consistent.

   If there is no solution, stop;

   otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Subsection 3

Vector Equations

Vectors in $\mathbb{R}^2$ and Equality

- A matrix with only one column is called a **column vector**, or simply a **vector**.

  Example: The following are vectors with two entries

  $$\boldsymbol{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

  where w_1 and w_2 are any real numbers.
- The set of all vectors with two entries is denoted by $\mathbb{R}^2$ (read "r-two").
- The $\mathbb{R}$ stands for the real numbers that appear as entries, and the exponent 2 for the number of entries.
- Two vectors in $\mathbb{R}^2$ are **equal** if and only if their corresponding entries are equal.

  Example: $\begin{bmatrix} 4 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$ are not equal, because vectors in $\mathbb{R}^2$ are ordered pairs of real numbers.

Sum and Scalar Multiplication

- Given two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^2$, their **sum** is the vector $\boldsymbol{u}+\boldsymbol{v}$ obtained by adding corresponding entries of $\boldsymbol{u}$ and $\boldsymbol{v}$.

  Example:

  $$\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1+2 \\ -2+5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

- Given a vector $\boldsymbol{u}$ and a real number c, the **scalar multiple** of $\boldsymbol{u}$ by c is the vector $c\boldsymbol{u}$ obtained by multiplying each entry in $\boldsymbol{u}$ by c.

  Example: If $\boldsymbol{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $c = 5$, then $c\boldsymbol{u} = 5\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$.

- The number c in $c\boldsymbol{u}$ is called a **scalar**.
- Scalars are written in lightface type to distinguish them from vectors which are written using boldface.

Example

- Given $\boldsymbol{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, find $4\boldsymbol{u}$, $(-3)\boldsymbol{v}$, and $4\boldsymbol{u} + (-3)\boldsymbol{v}$.
- We have

$$4\boldsymbol{u} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}, \qquad (-3)\boldsymbol{v} = \begin{bmatrix} -6 \\ 15 \end{bmatrix}.$$

- Moreover

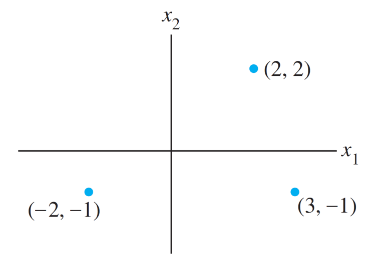
$$4\boldsymbol{u} + (-3)\boldsymbol{v} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}.$$

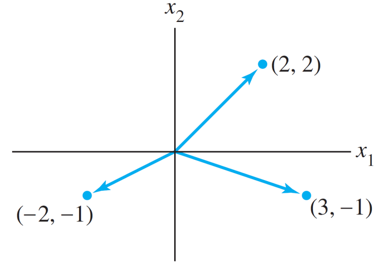
Notation

- Sometimes, for convenience (and also to save space), we may write a column vector such as $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ in the form $(3, -1)$.
- In this case, the parentheses and the comma distinguish the vector $(3, -1)$ from the 1×2 row matrix $[3 \ \ -1]$, written with brackets and no comma.
- Thus $\begin{bmatrix} 3 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 3 & -1 \end{bmatrix}$ because the matrices have different shapes, even though they have the same entries.

Geometry of Vectors

- Consider a rectangular coordinate system in the plane.
- We can identify a point (a, b) with the column vector $\begin{bmatrix} a \\ b \end{bmatrix}$.
- So we may regard $\mathbb{R}^2$ as the set of all points in the plane.



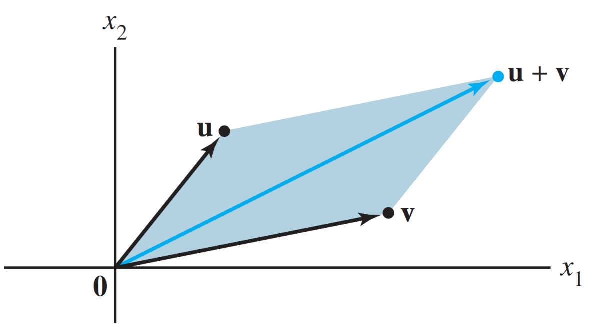


- The geometric visualization of a vector such as $(3, -1)$ is often aided by including an arrow (directed line segment) from the origin $(0, 0)$ to the point $(3, -1)$.

The Parallelogram Rule for Addition

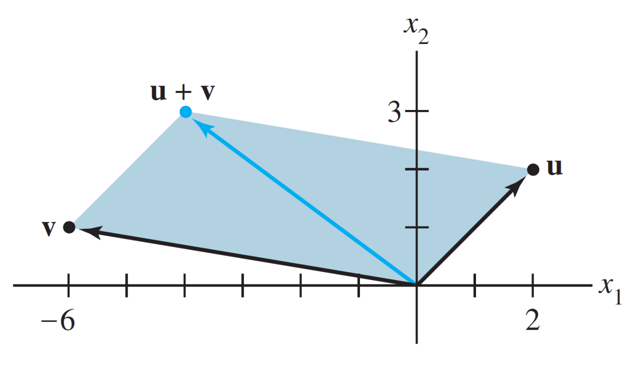
Parallelogram Rule for Addition

If $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^2$ are represented as points in the plane, then $\boldsymbol{u} + \boldsymbol{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\boldsymbol{u}, \mathbf{0}$ and $\boldsymbol{v}$.



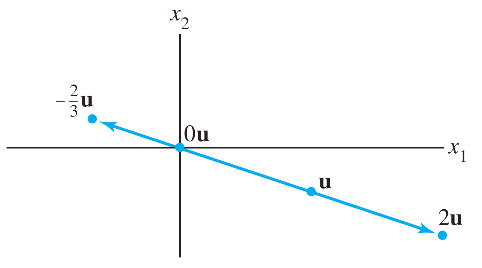
Example

- The vectors $\boldsymbol{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$ and $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ are displayed below:



Example

- Let $\boldsymbol{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Display the vectors $\boldsymbol{u}$, $2\boldsymbol{u}$, and $-\frac{2}{3}\boldsymbol{u}$ on a graph.
- The vectors $\boldsymbol{u}$, $2\boldsymbol{u} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ and $-\frac{2}{3}\boldsymbol{u} = \begin{bmatrix} -2 \\ \frac{2}{3} \end{bmatrix}$ are shown below.

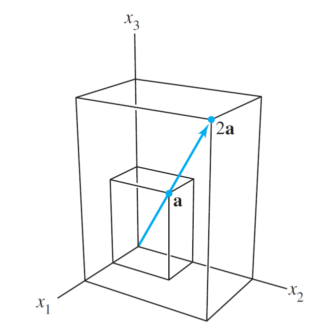


- The arrow for $2\boldsymbol{u}$ is twice as long as the arrow for u, and the arrows point in the same direction.
- The arrow for $-\frac{2}{3}\boldsymbol{u}$ is two-thirds the length of the arrow for $\boldsymbol{u}$, and the arrows point in opposite directions.

Vectors in $\mathbb{R}^3$

- Vectors in $\mathbb{R}^3$ are 3×1 column matrices with three entries.
- They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin sometimes included for visual clarity.

Example: The vectors $\boldsymbol{a} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ and $2\boldsymbol{a}$ are displayed on the right:



Vectors in $\mathbb{R}^n$

- If n is a positive integer, $\mathbb{R}^n$ (read "r-n") denotes the collection of all lists (or **ordered** n**-tuples**) of n real numbers, usually written as $n \times 1$ column matrices, such as $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$.
- The vector whose entries are all zero is called the **zero vector** and is denoted by $\mathbf{0}$.
- **Equality** of vectors in $\mathbb{R}^n$ and the operations of **scalar multiplication** and **vector addition** in $\mathbb{R}^n$ are defined entry by entry just as in $\mathbb{R}^2$.

Algebraic Properties of $\mathbb{R}^n$

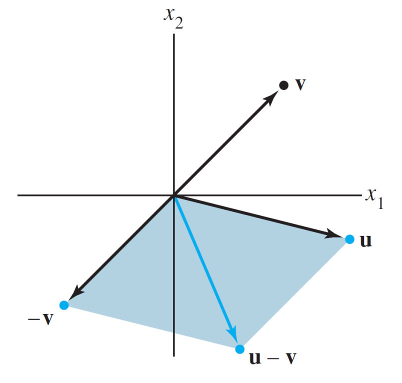
- For all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in $\mathbb{R}^n$ and all scalars c and d:
  (i) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$;
  (ii) $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$;
  (iii) $\boldsymbol{u} + \mathbf{0} = \mathbf{0} + \boldsymbol{u} = \boldsymbol{u}$;
  (iv) $\boldsymbol{u} + (-\boldsymbol{u}) = -\boldsymbol{u} + \boldsymbol{u} = \mathbf{0}$, where $-\boldsymbol{u}$ denotes $(-1)\boldsymbol{u}$;
  (v) $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$;
  (vi) $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$;
  (vii) $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$;
  (viii) $1\boldsymbol{u} = \boldsymbol{u}$.
- We show (v) to give a flavor of a proof:

$$c(\boldsymbol{u} + \boldsymbol{v}) = c\left(\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right) = c\begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix} = \begin{bmatrix} c(u_1 + v_1) \\ \vdots \\ c(u_n + v_n) \end{bmatrix} =$$

$$\begin{bmatrix} cu_1 + cv_1 \\ \vdots \\ cu_n + cv_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix} + \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix} = c\boldsymbol{u} + c\boldsymbol{v}.$$

Subtraction of Vectors

- For simplicity of notation, a vector such as $\boldsymbol{u} + (-1)\boldsymbol{v}$ is often written as $\boldsymbol{u} - \boldsymbol{v}$.
- The figure below shows $\boldsymbol{u} - \boldsymbol{v}$ as the sum of $\boldsymbol{u}$ and $-\boldsymbol{v}$.



Linear Combinations

- Given vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\boldsymbol{y}$ defined by

$$\boldsymbol{y} = c_1\boldsymbol{v}_1 + \cdots + c_p\boldsymbol{v}_p$$

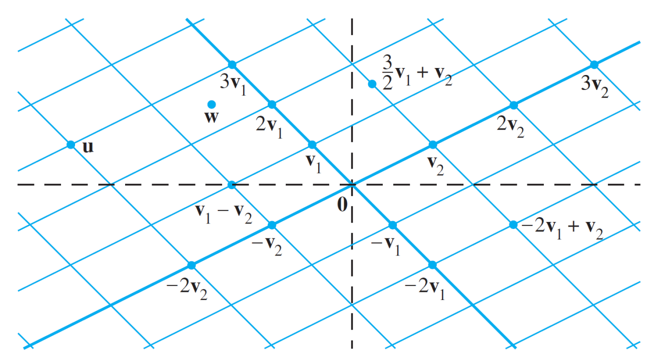
  is called a **linear combination** of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ with **weights** $c_1, \ldots, c_p$.
- Property (ii) permits us to omit parentheses when forming such a linear combination.
- The weights in a linear combination can be any real numbers, including zero.

  Example: Some linear combinations of vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are

$$\sqrt{3}\boldsymbol{v}_1 + \boldsymbol{v}_2, \quad \frac{1}{2}\boldsymbol{v}_1 = \frac{1}{2}\boldsymbol{v}_1 + 0\boldsymbol{v}_2, \quad \boldsymbol{0} = 0\boldsymbol{v}_1 + 0\boldsymbol{v}_2.$$

Example

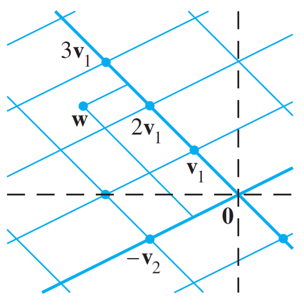
- Selected linear combinations of $\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are shown:



- Parallel grid lines are drawn through integer multiples of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.
- Estimate the linear combinations of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ that generate the vectors $\boldsymbol{u}$ and $\boldsymbol{w}$.

Example (Cont'd)

- The parallelogram rule shows that $\boldsymbol{u}$ is the sum of $3\boldsymbol{v}_1$ and $-2\boldsymbol{v}_2$, i.e., $\boldsymbol{u} = 3\boldsymbol{v}_1 - 2\boldsymbol{v}_2$.
  This expression for $\boldsymbol{u}$ can be interpreted as instructions for traveling from the origin to $\boldsymbol{u}$ along two straight paths:
  - First, travel 3 units in the $\boldsymbol{v}_1$ direction to $3\boldsymbol{v}_1$;
  - Then travel -2 units in the $\boldsymbol{v}_2$ direction (parallel to the line through $\boldsymbol{v}_2$ and $\mathbf{0}$).
- Next, although the vector $\boldsymbol{w}$ is not on a grid line, $\boldsymbol{w}$ appears to be about halfway between two pairs of grid lines, at the vertex of a parallelogram determined by $\frac{5}{2}\boldsymbol{v}_1$ and $-\frac{1}{2}\boldsymbol{v}_2$.
  Thus a reasonable estimate for $\boldsymbol{w}$ is $\boldsymbol{w} = \frac{5}{2}\boldsymbol{v}_1 - \frac{1}{2}\boldsymbol{v}_2$.



Example

- Let $\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\boldsymbol{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$.
- Determine whether $\boldsymbol{b}$ can be generated (or written) as a linear combination of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$.
- That is, determine whether weights x_1 and x_2 exist such that

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 = \boldsymbol{b}.$$

- If this vector equation has a solution, find it.

Example (Cont'd)

- Use the definitions of scalar multiplication and vector addition to rewrite the vector equation

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

- We obtain

$$\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

- Therefore, we must have $\left\{ \begin{array}{rcr} x_1 + 2x_2 & = & 7 \\ -2x_1 + 5x_2 & = & 4 \\ -5x_1 + 6x_2 & = & -3 \end{array} \right.$.

Example (Cont'd)

- To solve this system, we row reduce the augmented matrix of the system:

$$\left[\begin{array}{rr|r} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 + 5R_1]{R_2 \leftarrow R_2 + 2R_1} \left[\begin{array}{rr|r} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{1}{9}R_2}$$
$$\left[\begin{array}{rr|r} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 16R_2]{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right].$$

- Thus the solution is $x_1 = 3$ and $x_2 = 2$.
- Hence $\boldsymbol{b}$ is a linear combination of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, with weights $x_1 = 3$ and $x_2 = 2$, i.e., $3\boldsymbol{a}_1 + 2\boldsymbol{a}_2 = \boldsymbol{b}$.

Vector Equations and Linear Systems

- The original vectors $\boldsymbol{a}_1, \boldsymbol{a}_2$ and $\boldsymbol{b}$ are the columns of the augmented matrix that we row reduced.
- For brevity, we write this matrix in a way that identifies its columns - namely,
$$\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{b} \end{bmatrix}.$$
- It is clear how to write this augmented matrix immediately from the vector equation
$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 = \boldsymbol{b}$$
without going through any intermediate steps:

  Take the vectors in the order in which they appear in this equation and put them into the columns of a matrix.

Vector Equations and Linear Systems

- The discussion above is easily modified to establish the following fundamental fact:

> A vector equation
>
> $$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = b$$
>
> has the same solution set as the linear system whose augmented matrix is
>
> $$\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n & \boldsymbol{b} \end{bmatrix}.$$
>
> In particular, $\boldsymbol{b}$ can be generated by a linear combination of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix above.

Spanned or Generated Sets

Definition

If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ is denoted by Span$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by** $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$.
That is, Span$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ is the collection of all vectors that can be written in the form

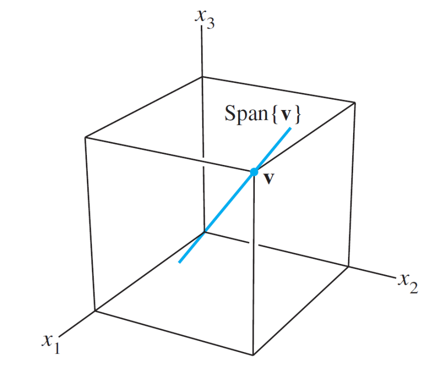
$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p$$

with $c_1, \ldots, c_p$ scalars.

- Asking whether a vector $\boldsymbol{b}$ is in Span$\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ amounts to asking whether the vector equation $x_1\boldsymbol{v}_1 + \cdots + x_p\boldsymbol{v}_p = \boldsymbol{b}$ has a solution.
- Equivalently, asking whether the linear system with augmented matrix $[\boldsymbol{v}_1 \ \boldsymbol{v}_2 \ \cdots \ \boldsymbol{v}_p \ \boldsymbol{b}]$ has a solution.

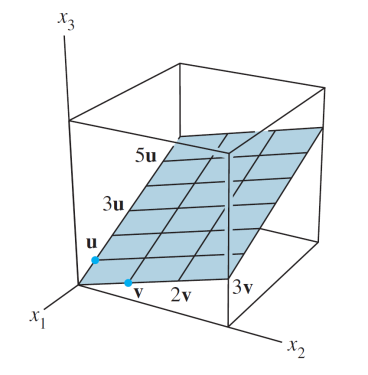
Geometry of Span$\{\boldsymbol{v}\}$

- Let $\boldsymbol{v}$ be a nonzero vector in $\mathbb{R}^3$.
- Then Span$\{\boldsymbol{v}\}$ is the set of all scalar multiples of $\boldsymbol{v}$.
- This is the set of points on the line in $\mathbb{R}^3$ through $\boldsymbol{v}$ and $\mathbf{0}$.



Geometry of Span$\{\boldsymbol{u}, \boldsymbol{v}\}$

- If $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero vectors in $\mathbb{R}^3$, with $\boldsymbol{v}$ not a multiple of $\boldsymbol{u}$, then Span$\{\boldsymbol{u}, \boldsymbol{v}\}$ is the plane in $\mathbb{R}^3$ that contains $\boldsymbol{u}, \boldsymbol{v}$ and $\mathbf{0}$.
- In particular, Span$\{\boldsymbol{u}, \boldsymbol{v}\}$ contains the line in $\mathbb{R}^3$ through $\boldsymbol{u}$ and $\mathbf{0}$ and the line through $\boldsymbol{v}$ and $\mathbf{0}$.



Example

- Let $\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\boldsymbol{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$.
  Then $\text{Span}\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ is a plane through the origin in $\mathbb{R}^3$.
  Is $\boldsymbol{b}$ in that plane?
- We must find whether the equation $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 = \boldsymbol{b}$ has a solution.
  To answer this, row reduce the augmented matrix $[\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{b}]$.
  We have

$$\left[\begin{array}{rr|r} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 + 2R_1} \left[\begin{array}{rr|r} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 6R_2} \left[\begin{array}{rr|r} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{array}\right]$$

The third equation is $0 = -2$, which shows that the system has no solution.
Since the vector equation $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 = \boldsymbol{b}$ has no solution, $\boldsymbol{b}$ is not in $\text{Span}\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$.

Subsection 4

The Matrix Equation $A\boldsymbol{x} = \boldsymbol{b}$

Product of a Matrix and a Vector

Definition

If A is an $m \times n$ matrix, with columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, and if $\boldsymbol{x}$ is in $\mathbb{R}^n$, then the **product of A and $\boldsymbol{x}$**, denoted by $A\boldsymbol{x}$, is **the linear combination of the columns of A using the corresponding entries in $\boldsymbol{x}$ as weights**; that is,

$$A\boldsymbol{x} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n.$$

- Note that $A\boldsymbol{x}$ is defined only if the number of columns of A equals the number of entries in $\boldsymbol{x}$.

Example

(a)

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix};$$

(b)

$$\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 32 \\ -20 \end{bmatrix} + \begin{bmatrix} -21 \\ 0 \\ 14 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}.$$

Example

- For $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ in $\mathbb{R}^m$, write the linear combination $3\boldsymbol{v}_1 - 5\boldsymbol{v}_2 + 7\boldsymbol{v}_3$ as a matrix times a vector.
- Place $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ into the columns of a matrix A and place the weights $3, -5, 7$ into a vector $\boldsymbol{x}$.

  That is,

$$3\boldsymbol{v}_1 - 5\boldsymbol{v}_2 + 7\boldsymbol{v}_3 = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = A\boldsymbol{x},$$

where $A = [\boldsymbol{v}_1 \ \boldsymbol{v}_2 \ \boldsymbol{v}_3]$.

The Matrix Equation of a Linear System

- We saw how to write a system of linear equations as a vector equation involving a linear combination of vectors.
- For example, the system $\begin{cases} x_1 + 2x_2 - x_3 &= 4 \\ \quad -5x_2 + 3x_3 &= 1 \end{cases}$ is equivalent to

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

- The linear combination on the left side is a matrix times a vector, so it becomes

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

- This equation has the form $A\mathbf{x} = \mathbf{b}$ and is called a **matrix equation**, to distinguish it from a vector equation.
- Notice the matrix in A is just the matrix of coefficients of the system.

Linear Systems, Vector Equations and Matrix Equations

Theorem

If A is an $m \times n$ matrix, with columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, and if $\boldsymbol{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\boldsymbol{x} = \boldsymbol{b}$$

has the same solution set as the vector equation

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n & \boldsymbol{b} \end{bmatrix}.$$

- Thus, a system of linear equations may now be viewed in three different but equivalent ways:
  - as a matrix equation;
  - as a vector equation;
  - as a system of linear equations.

Consistency of a Linear System

- The definition of $A\boldsymbol{x}$ leads directly to the following useful fact.

> The equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if $\boldsymbol{b}$ is a linear combination of the columns of A.

- We previously considered the existence question, "Is $\boldsymbol{b}$ in $\text{Span}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$?"
- Equivalently, "Is $A\boldsymbol{x} = \boldsymbol{b}$ consistent?"
- A harder existence problem is to determine whether the equation $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for all possible $\boldsymbol{b}$.

Example

- Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.
  Is the equation $A\boldsymbol{x} = \boldsymbol{b}$ consistent for all possible b_1, b_2, b_3?
- Row reduce the augmented matrix for $A\boldsymbol{x} = \boldsymbol{b}$:

$$\left[\begin{array}{rrr|c} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 + 3R_1]{R_2 \leftarrow R_2 + 4R_1} \left[\begin{array}{rrr|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_2} \left[\begin{array}{rrrc} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{array}\right]$$

The third entry in column 4 equals $b_1 - \frac{1}{2}b_2 + b_3$. The equation $A\boldsymbol{x} = \boldsymbol{b}$ is not consistent for every $\boldsymbol{b}$ because some choices of $\boldsymbol{b}$ can make $b_1 - \frac{1}{2}b_2 + b_3$ nonzero.

Characterization Theorem

- A set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p\}$ in $\mathbb{R}^m$ **spans** (or **generates**) $\mathbb{R}^m$ if every vector in $\mathbb{R}^m$ is a linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$, that is, if $\text{Span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\} = \mathbb{R}^m$.

Theorem

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent.

(a) For each $\boldsymbol{b}$ in $\mathbb{R}^m$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.

(b) Each $\boldsymbol{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

Computing Ax Using the Definition

- Compute $A\boldsymbol{x}$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.
- We have from the definition:

$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}.$$

The Row-Vector Rule for Computing Ax

If the product $A\boldsymbol{x}$ is defined, then the i-th entry in $A\boldsymbol{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\boldsymbol{x}$.

Example: We have

$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix};$$

$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix}.$$

Computing $A\boldsymbol{x}$ Using the Row-Vector Rule

(a)

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 3 + (-1) \cdot 7 \\ 0 \cdot 4 + (-5) \cdot 3 + 3 \cdot 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \cdot 4 + (-3) \cdot 7 \\ 8 \cdot 4 + 0 \cdot 7 \\ (-5) \cdot 4 + 2 \cdot 7 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}.$$

The Identity Matrix

- The matrix with 1's on the diagonal and 0's elsewhere is called an **identity matrix** and is denoted by I.
- Note that we have

$$\begin{aligned} I\boldsymbol{x} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 \\ 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 \\ 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 \end{bmatrix} \\ &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{x}. \end{aligned}$$

- There is an analogous $n \times n$ identity matrix, sometimes written as I_n.
- As above, $I_n\boldsymbol{x} = \boldsymbol{x}$, for every $\boldsymbol{x}$ in $\mathbb{R}^n$.

Properties of the Matrix-Vector Product $A\mathbf{x}$

Theorem

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

- We deal with the case $n = 3$. Consider $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$, and $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^3$. For $i = 1, 2, 3$, let u_i and v_i be the ith entries in $\mathbf{u}$ and $\mathbf{v}$.

(a) We compute $A(\mathbf{u} + \mathbf{v})$ as a linear combination of the columns of A using the entries in $\mathbf{u} + \mathbf{v}$ as weights.

$$\begin{aligned} A(\mathbf{u}+\mathbf{v}) &= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} \\ &= (u_1 + v_1)\mathbf{a}_1 + (u_2 + v_2)\mathbf{a}_2 + (u_3 + v_3)\mathbf{a}_3 \\ &= (u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) + (v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3) \\ &= A\mathbf{u} + A\mathbf{v}. \end{aligned}$$

Properties of the Matrix-Vector Product $A\boldsymbol{x}$ (Part (b)

(b) Compute $A(c\boldsymbol{u})$ as a linear combination of the columns of A using the entries in $c\boldsymbol{u}$ as weights.

$$\begin{aligned}
A(c\boldsymbol{u}) &= \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{bmatrix} \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} \\
&= (cu_1)\boldsymbol{a}_1 + (cu_2)\boldsymbol{a}_2 + (cu_3)\boldsymbol{a}_3 \\
&= c(u_1\boldsymbol{a}_1) + c(u_2\boldsymbol{a}_2) + c(u_3\boldsymbol{a}_3) \\
&= c(u_1\boldsymbol{a}_1 + u_2\boldsymbol{a}_2 + u_3\boldsymbol{a}_3) \\
&= c(A\boldsymbol{u}).
\end{aligned}$$

Subsection 5

Solution Sets of Linear Systems

Homogeneous Linear Systems

- A system of linear equations is said to be **homogeneous** if it can be written in the form $A\boldsymbol{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.
- Such a system $A\boldsymbol{x} = \mathbf{0}$ always has at least one solution, namely, $\boldsymbol{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$).
- This zero solution is usually called the **trivial solution**.
- The important question is whether $A\boldsymbol{x} = \mathbf{0}$ has a **nontrivial solution**.
- The Existence and Uniqueness Theorem of a previous section leads immediately to the following fact:

> The homogeneous equation $A\boldsymbol{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Example

- Determine if the following homogeneous system has a nontrivial solution and describe the solution set.

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 0 \\ -3x_1 - 2x_2 + 4x_3 = 0 \\ 6x_1 + x_2 - 8x_3 = 0 \end{cases}$$

- We row reduce the augmented matrix to echelon form:

$$\left[\begin{array}{rrr|r} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 + R_1} \left[\begin{array}{rrr|r} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right]$$
$$\xrightarrow{R_3 \leftarrow R_3 + 3R_2} \left[\begin{array}{rrr|r} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since x_3 is a free variable, $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions (one for each choice of x_3).

Example (Cont'd)

- To describe the solution set, continue the row reduction to reduced echelon form:

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{5}{3}R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \left\{\begin{array}{rcl} x_1 - \frac{4}{3}x_3 & = & 0 \\ x_2 & = & 0 \\ 0 & = & 0 \end{array}\right.$$

Solve for the basic variables x_1 and x_2 and obtain $x_1 = \frac{4}{3}x_3$, $x_2 = 0$, with x_3 free.

As a vector, the general solution has the form

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3 \boldsymbol{v}, \text{ where } \boldsymbol{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}.$$

Thus, every solution in this case is a scalar multiple of $\boldsymbol{v}$, i.e., the solution set is a line through $\mathbf{0}$ in $\mathbb{R}^3$.

Example

- Describe all solutions of the homogeneous "system" $10x_1 - 3x_2 - 2x_3 = 0$.
- We solve for the basic variable x_1 in terms of the free variables. The general solution is $x_1 = 0.3x_2 + 0.2x_3$, with x_2 and x_3 free. As a vector, the general solution is

$$\begin{aligned}\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0.3x_2 + 0.3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.2x_3 \\ 0 \\ x_3 \end{bmatrix} \\ &= x_2 \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix} = x_2\boldsymbol{u} + x_3\boldsymbol{v}.\end{aligned}$$

Thus, every solution is a linear combination of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Since neither $\boldsymbol{u}$ nor $\boldsymbol{v}$ is a scalar multiple of the other, the solution set is a plane through the origin.

Parametric Vector Form of Solution Set

- The equation $10x_1 - 3x_2 - 2x_3 = 0$ is an implicit description of the solution plane.
- Solving this equation amounts to finding an explicit description of the plane as the set spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$.
- The equation $\boldsymbol{x} = x_2\boldsymbol{u} + x_3\boldsymbol{v}$ is called a **parametric vector equation** of the plane.
- Sometimes such an equation is written as

$$\boldsymbol{x} = s\boldsymbol{u} + t\boldsymbol{v}, \quad s, t \text{ in } \mathbb{R},$$

  to emphasize that the parameters vary over all real numbers.
- In a previous example, the equation $\boldsymbol{x} = x_3\boldsymbol{v}$ (with x_3 free), or $\boldsymbol{x} = t\boldsymbol{v}$ (with t in $\mathbb{R}$), is a parametric vector equation of a line.
- Whenever a solution set is described explicitly with vectors, we say that the solution is in **parametric vector form**.

Solving a Nonhomogeneous System

- Describe all solutions of $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}.$$

- We produce the reduced echelon form of the augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \overset{\text{done}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left\{ \begin{array}{rcl} x_1 - \frac{4}{3}x_3 & = & -1 \\ x_2 & = & 2 \\ 0 & = & 0 \end{array} \right.$$

Thus $x_1 = -1 + \frac{4}{3}x_3,\ x_2 = 2$ and x_3 is free.
As a vector, the general solution as the form

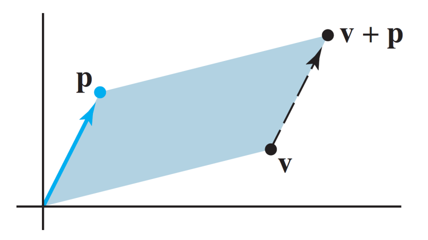
$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}.$$

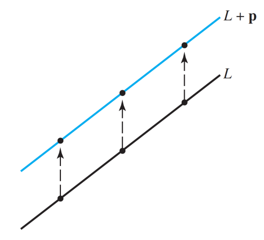
Solving a Nonhomogeneous System (Cont'd)

- The equation $\boldsymbol{x} = \boldsymbol{p} + x_3\boldsymbol{v}$, or, writing t as a general parameter, $\boldsymbol{x} = \boldsymbol{p} + t\boldsymbol{v}$ (t in $\mathbb{R}$) describes the solution set in parametric vector form.
- Recall from the previous example that the solution set of $A\boldsymbol{x} = \boldsymbol{0}$ has the parametric vector equation $\boldsymbol{x} = t\boldsymbol{v}$ (t in $\mathbb{R}$) [with the same $\boldsymbol{v}$].
- Thus the solutions of $A\boldsymbol{x} = \boldsymbol{b}$ are obtained by adding the vector $\boldsymbol{p}$ to the solutions of $A\boldsymbol{x} = \boldsymbol{0}$.
- The vector $\boldsymbol{p}$ itself is just one particular solution of $A\boldsymbol{x} = \boldsymbol{b}$ [corresponding to $t = 0$].

Geometry: Translations

- To describe the solution set of $A\boldsymbol{x} = \boldsymbol{b}$ geometrically, we can think of vector addition as a **translation**.
- Given $\boldsymbol{v}$ and $\boldsymbol{p}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, the effect of adding $\boldsymbol{p}$ to $\boldsymbol{v}$ is to move $\boldsymbol{v}$ in a direction parallel to the line through $\boldsymbol{p}$ and $\mathbf{0}$.
- We say that $\boldsymbol{v}$ is **translated by** $\boldsymbol{p}$ to $\boldsymbol{v} + \boldsymbol{p}$.

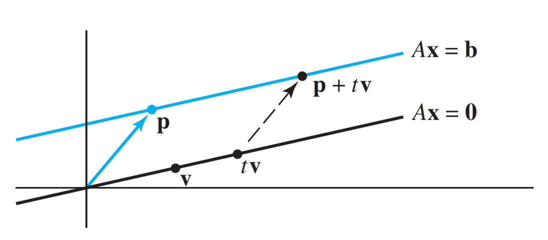




- If each point on a line L in $\mathbb{R}^2$ or $\mathbb{R}^3$ is translated by a vector $\boldsymbol{p}$, the result is a line parallel to L.

Geometry and Solution Sets

- Suppose L is the line through $\mathbf{0}$ and $\boldsymbol{v}$, described by $\boldsymbol{x} = t\boldsymbol{v}$.
- Adding $\boldsymbol{p}$ to each point on L produces the translated line described by $\boldsymbol{x} = \boldsymbol{p} + t\boldsymbol{v}$.
- Note that $\boldsymbol{p}$ is on the line described by the latter equation.
- We call the line $\boldsymbol{x} = \boldsymbol{p} + t\boldsymbol{v}$ the **equation of the line through $\boldsymbol{p}$ parallel to $\boldsymbol{v}$**.
- Thus the solution set of $A\boldsymbol{x} = \boldsymbol{b}$ is a line through $\boldsymbol{p}$ parallel to the solution set of $A\boldsymbol{x} = \mathbf{0}$.

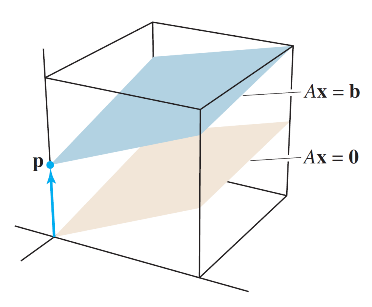


Homogeneous and Nonhomogeneous Systems

Theorem

Suppose the equation $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for some given $\boldsymbol{b}$, and let $\boldsymbol{p}$ be a solution. Then the solution set of $A\boldsymbol{x} = \boldsymbol{b}$ is the set of all vectors of the form $\boldsymbol{w} = \boldsymbol{p} + \boldsymbol{v}_h$, where $\boldsymbol{v}_h$ is any solution of the homogeneous equation $A\boldsymbol{x} = \mathbf{0}$.

- In case there are two free variables, we have the following picture:



Writing a Solution Set in Parametric Vector Form

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\boldsymbol{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\boldsymbol{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Subsection 6

Linear Independence

Linear Independence

Definition

An indexed set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_p\boldsymbol{v}_p = \mathbf{0}$$

has only the trivial solution.
The set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p = \mathbf{0}.$$

- The last equation is called a **linear dependence relation** among $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ when the weights are not all zero.
- An indexed set is linearly dependent if and only if it is not linearly independent.

Example

- Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.
  - (a) Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
  - (b) If possible, find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$.

(a) We must determine if there is a nontrivial solution of $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$. We reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Clearly, x_1 and x_2 are basic variables and x_3 is free. Each nonzero value of x_3 determines a nontrivial solution of the equation. Hence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.

Example (Part (b))

(b) To find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$, completely row reduce the augmented matrix and write the new system:

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow -\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - 4R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left\{\begin{array}{rcl} x_1 - 2x_3 & = & 0 \\ x_2 + x_3 & = & 0 \\ 0 & = & 0 \end{array}\right.$$

Thus $x_1 = 2x_3$, $x_2 = -x_3$, and x_3 is free.

Choose any nonzero value for x_3, say $x_3 = 5$.

Then $x_1 = 10$ and $x_2 = -5$.

Substitute these values into the original equation to obtain

$$10\mathbf{v}_1 - 5\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}.$$

Linear Independence of Matrix Columns

- Suppose that we begin with a matrix $A = \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_n \end{bmatrix}$ instead of a set of vectors.
- The matrix equation $A\boldsymbol{x} = \mathbf{0}$ can be written as

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \mathbf{0}.$$

- Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $A\boldsymbol{x} = \mathbf{0}$.
- Thus we have the following important fact:

> The columns of a matrix A are linearly independent if and only if the equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.

Example

- Determine if the columns of the matrix $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are linearly independent.
- To study $A\mathbf{x} = \mathbf{0}$, row reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 5 & 8 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - 5R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{array}\right].$$

At this point, it is clear that there are three basic variables and no free variables.

So the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, and the columns of A are linearly independent.

The Case of a Single Vector

- A set containing only one vector - say, $\boldsymbol{v}$ - is linearly independent if and only if $\boldsymbol{v}$ is not the zero vector.
- This is because the vector equation $x_1\boldsymbol{v} = \mathbf{0}$ has only the trivial solution when $\boldsymbol{v} \neq \mathbf{0}$.
- The zero vector is linearly dependent because $x_1\mathbf{0} = \mathbf{0}$ has many nontrivial solutions.

The Case of Two Vectors

- Determine if the following sets of vectors are linearly independent.

$$(a) \quad \boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}; \qquad (\mathrm{b}) \quad \boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

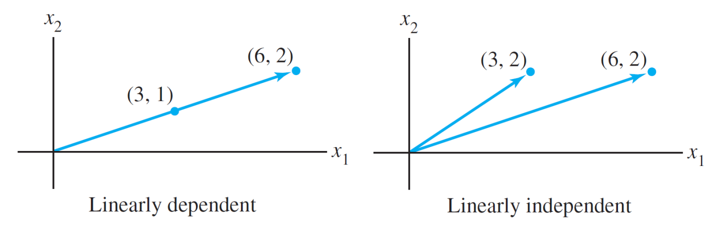
(a) Notice that $\boldsymbol{v}_2$ is a multiple of $\boldsymbol{v}_1$: $\boldsymbol{v}_2 = 2\boldsymbol{v}_1$. Hence $-2\boldsymbol{v}_1 + \boldsymbol{v}_2 = \mathbf{0}$. This shows that $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is linearly dependent.

(b) The vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are certainly not multiples of one another. Suppose c and d satisfy $c\boldsymbol{v}_1 + d\boldsymbol{v}_2 = \mathbf{0}$. If $c \neq 0$, then we can solve for $\boldsymbol{v}_1$ in terms of $\boldsymbol{v}_2$: $\boldsymbol{v}_1 = -\frac{d}{c}\boldsymbol{v}_2$. This result is impossible because $\boldsymbol{v}_1$ is not a multiple of $\boldsymbol{v}_2$. So c must be zero. Similarly, d must also be zero. Thus $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a linearly independent set.

The Case of Two Vectors (Geometry)

- The arguments in the preceding example show that you can always decide by inspection when a set of two vectors is linearly dependent.
- Simply check whether at least one of the vectors is a scalar times the other.

A set of two vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.



A Characterization of Linear Dependence

Theorem (Characterization of Linearly Dependent Sets)

An indexed set $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.
In fact, if S is linearly dependent and $\boldsymbol{v}_1 \neq \mathbf{0}$, then some $\boldsymbol{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{j-1}$.

- Suppose that $\boldsymbol{v}_j$ in S equals a linear combination of the other vectors:

$$\boldsymbol{v}_j = x_1\boldsymbol{v}_1 + \cdots + x_{j-1}\boldsymbol{v}_{j-1} + x_{j+1}\boldsymbol{v}_{j+1} + \cdots + x_p\boldsymbol{v}_p.$$

Then subtracting $\boldsymbol{v}_j$ from both sides of the equation, we get

$$x_1\boldsymbol{v}_1 + \cdots + x_{j-1}\boldsymbol{v}_{j-1} - 1 \cdot \boldsymbol{v}_j + x_{j+1}\boldsymbol{v}_{j+1} + \cdots + x_p\boldsymbol{v}_p = \mathbf{0}.$$

This is a linear dependence relation with a nonzero weight (-1) for $\boldsymbol{v}_j$. Thus S is linearly dependent.

A Characterization of Linear Dependence (Converse)

- Conversely, suppose S is linearly dependent. If $\boldsymbol{v}_1$ is zero, then it is a (trivial) linear combination of the other vectors in S. Otherwise, $\boldsymbol{v}_1 \neq \mathbf{0}$, and there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p = \mathbf{0}.$$

Let j be the largest subscript for which $c_j \neq 0$. If $j = 1$, then $c_1\boldsymbol{v}_1 = \mathbf{0}$, which is impossible because $\boldsymbol{v}_1 \neq \mathbf{0}$. So $j > 1$, and

$$\begin{aligned}
& c_1\boldsymbol{v}_1 + \cdots + c_j\boldsymbol{v}_j + 0\boldsymbol{v}_{j+1} + \cdots + 0\boldsymbol{v}_p = \mathbf{0} \\
\Rightarrow \quad & c_j\boldsymbol{v}_j = -c_1\boldsymbol{v}_1 - \cdots - c_{j-1}\boldsymbol{v}_{j-1} \\
\Rightarrow \quad & \boldsymbol{v}_j = (-\tfrac{c_1}{c_j})\boldsymbol{v}_1 + \cdots + (-\tfrac{c_{j-1}}{c_j})\boldsymbol{v}_{j-1}.
\end{aligned}$$

Example

- Let $\boldsymbol{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. Describe the set spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$, and explain why a vector $\boldsymbol{w}$ is in Span$\{\boldsymbol{u}, \boldsymbol{v}\}$ if and only if $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is linearly dependent.
- The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent because neither vector is a multiple of the other. So they span a plane in $\mathbb{R}^3$. In fact, Span$\{\boldsymbol{u}, \boldsymbol{v}\}$ is the x_1x_2-plane (with $x_3 = 0$).

  If $\boldsymbol{w}$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$, then $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is linearly dependent, by the preceding theorem.

  Conversely, suppose that $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is linearly dependent. By the same theorem, some vector in $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a linear combination of the preceding vectors (since $\boldsymbol{u} \neq \mathbf{0}$). That vector must be $\boldsymbol{w}$, since $\boldsymbol{v}$ is not a multiple of $\boldsymbol{u}$. So $\boldsymbol{w}$ is in Span$\{\boldsymbol{u}, \boldsymbol{v}\}$.

A Sufficient Condition for Dependence

Theorem

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

- Let $A = \begin{bmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_p \end{bmatrix}$. Then A is $n \times p$, and the equation $A\boldsymbol{x} = \boldsymbol{0}$ corresponds to a system of n equations in p unknowns. If $p > n$, there are more variables than equations, so there must be a free variable.

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

Hence $A\boldsymbol{x} = \boldsymbol{0}$ has a nontrivial solution. So the columns of A are linearly dependent.

Example

- The vectors
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$
are linearly dependent by the preceding theorem, because there are three vectors in the set and there are only two entries in each vector.
- Indeed we have
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ -1 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \mathbf{0}.$$
- Notice, however, that none of the vectors is a multiple of one of the other vectors.

Another Sufficient Condition for Dependence

Theorem

If a set $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

- By renumbering the vectors, we may suppose $\boldsymbol{v}_1 = 0$.

  Then the equation

$$1\boldsymbol{v}_1 + 0\boldsymbol{v}_2 + \cdots + 0\boldsymbol{v}_p = \mathbf{0}$$

  shows that S is linearly dependent.

Example

- Determine by inspection if the given set is linearly dependent.

$$(a)\ \begin{bmatrix}1\\7\\6\end{bmatrix}, \begin{bmatrix}2\\0\\9\end{bmatrix}, \begin{bmatrix}3\\1\\5\end{bmatrix}, \begin{bmatrix}4\\1\\8\end{bmatrix}\ (b)\ \begin{bmatrix}2\\3\\5\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\8\end{bmatrix}\ (c)\ \begin{bmatrix}-2\\4\\6\\10\end{bmatrix}, \begin{bmatrix}3\\-6\\-9\\15\end{bmatrix}.$$

(a) The set contains four vectors, each of which has only three entries. So the set is linearly dependent.

(b) The same theorem does not apply here because the number of vectors does not exceed the number of entries in each vector. Since the zero vector is in the set, the set is linearly dependent by the preceding theorem.

(c) Compare the corresponding entries of the two vectors. The second vector seems to be $-\frac{3}{2}$ times the first vector. This relation holds for the first three pairs of entries, but fails for the fourth pair. Thus neither of the vectors is a multiple of the other. Hence they are linearly independent.

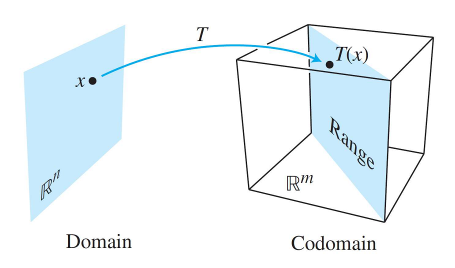
Subsection 7

Introduction to Linear Transformations

Transformations

- A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\boldsymbol{x}$ in $\mathbb{R}^n$ a vector $T(\boldsymbol{x})$ in $\mathbb{R}^m$.
- The set $\mathbb{R}^n$ is called the **domain** of T, and $\mathbb{R}^m$ is called the **codomain** of T.
- The notation $T : \mathbb{R}^n \to \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.

- For $\boldsymbol{x}$ in $\mathbb{R}^n$, the vector $T(\boldsymbol{x})$ in $\mathbb{R}^m$ is called the **image** of $\boldsymbol{x}$ (under the action of T).
- The set of all images $T(\boldsymbol{x})$ is called the **range** of T.



Matrix Transformations

- We focus on mappings associated with matrix multiplication.
- For each $\boldsymbol{x}$ in $\mathbb{R}^n$, $T(\boldsymbol{x})$ is computed as $A\boldsymbol{x}$, where A is an $m \times n$ matrix.
- For simplicity, we sometimes denote such a matrix transformation by $\boldsymbol{x} \mapsto A\boldsymbol{x}$.
- Observe that the domain of T is $\mathbb{R}^n$ when A has n columns.
- The codomain of T is $\mathbb{R}^m$ when each column of A has m entries.
- The range of T is the set of all linear combinations of the columns of A, because each image $T(\boldsymbol{x})$ is of the form $A\boldsymbol{x}$.

Example

- Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\boldsymbol{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\boldsymbol{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$.

  Define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\boldsymbol{x}) = A\boldsymbol{x}$, so that

$$T(\boldsymbol{x}) = A\boldsymbol{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}.$$

(a) Find $T(\boldsymbol{u})$, the image of $\boldsymbol{u}$ under the transformation T.
(b) Find an $\boldsymbol{x}$ in $\mathbb{R}^2$ whose image under T is $\boldsymbol{b}$.
(c) Is there more than one $\boldsymbol{x}$ whose image under T is $\boldsymbol{b}$?
(d) Determine if $\boldsymbol{c}$ is in the range of the transformation T.

Example (Cont'd)

(a) $T(\boldsymbol{u}) = A\boldsymbol{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$.

(b) We need to solve $T(\boldsymbol{x}) = \boldsymbol{b}$ for $\boldsymbol{x}$, i.e., we must solve the equation $\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$. We row reduce the augmented matrix:

$$\left[\begin{array}{rr|r} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - 3R_1} \left[\begin{array}{rr|r} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{1}{14} R_2}$$

$$\left[\begin{array}{rr|r} 1 & -3 & 3 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 4 & -2 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 4R_2]{R_1 \leftarrow R_1 + 3R_2} \left[\begin{array}{rr|r} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{array}\right].$$

Hence $x_1 = \frac{3}{2}$, $x_2 = -\frac{1}{2}$.

Example (Cont'd)

(c) Any $\boldsymbol{x}$ whose image under T is $\boldsymbol{b}$ must satisfy the equation in (b). From the row reduced echelon form, it is clear that the equation has a unique solution. So there is exactly one $\boldsymbol{x}$ whose image is $\boldsymbol{b}$.

(d) The vector $\boldsymbol{c}$ is in the range of T if $\boldsymbol{c}$ is the image of some $\boldsymbol{x}$ in $\mathbb{R}^2$, that is, if $\boldsymbol{c} = T(\boldsymbol{x})$ for some $\boldsymbol{x}$. This is just another way of asking if the system $A\boldsymbol{x} = \boldsymbol{c}$ is consistent. To find the answer, row reduce the augmented matrix:

$$\left[\begin{array}{rr|r} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - 3R_1} \left[\begin{array}{rr|r} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{rr|r} 1 & -3 & 3 \\ 0 & 4 & 8 \\ 0 & 14 & -7 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{4}R_2} \left[\begin{array}{rr|r} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 14R_2} \left[\begin{array}{rr|r} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{array}\right].$$

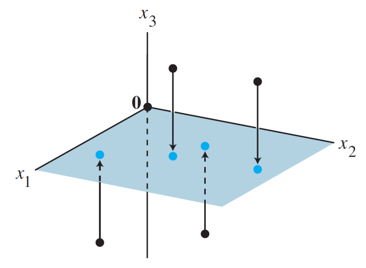
The third equation, $0 = -35$, shows that the system is inconsistent. So $\boldsymbol{c}$ is not in the range of T.

Projections

- Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
- Then, we have for $\boldsymbol{x} \mapsto A\boldsymbol{x}$:

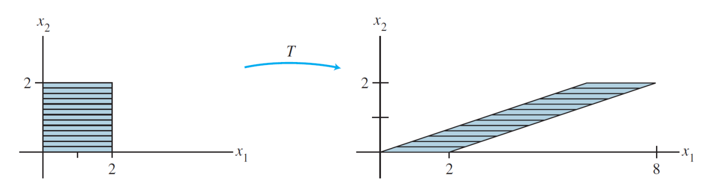
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

- So the transformation $\boldsymbol{x} \mapsto A\boldsymbol{x}$ **projects** points in $\mathbb{R}^3$ onto the x_1x_2-plane.



Shear Transformations

- Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.
- The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is called a **shear transformation**.
- It can be shown that if T acts on each point in the 2×2 square, then the set of images forms the shaded parallelogram.



- For instance,

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}.$$

Linear Transformations

- We have seen that if A is $m \times n$, then the transformation $\boldsymbol{x} \mapsto A\boldsymbol{x}$ has the properties

$$A(\boldsymbol{u}+\boldsymbol{v}) = A\boldsymbol{u} + A\boldsymbol{v} \quad \text{and} \quad A(c\boldsymbol{u}) = cA\boldsymbol{u},$$

  for all $\boldsymbol{u}, \boldsymbol{v}$ in $\mathbb{R}^n$ and all scalars c.
- These properties, written in function notation, identify the most important class of transformations in linear algebra.

Definition

A transformation (or mapping) T is **linear** if:

(i) $T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$, for all $\boldsymbol{u}, \boldsymbol{v}$ in the domain of T;

(ii) $T(c\boldsymbol{u}) = cT(\boldsymbol{u})$, for all scalars c and all $\boldsymbol{u}$ in the domain of T.

- Every matrix transformation is a linear transformation.

Preservation of Sums and Scalar Products

- Linear transformations preserve the operations of vector addition and scalar multiplication.
- Property (i) says that the result $T(\boldsymbol{u}+\boldsymbol{v})$ of first adding $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ and then applying T is the same as first applying T to $\boldsymbol{u}$ and to $\boldsymbol{v}$ and then adding $T(\boldsymbol{u})$ and $T(\boldsymbol{v})$ in $\mathbb{R}^m$.

$$\begin{tikzcd} \mathbb{R}^n \times \mathbb{R}^n \arrow[r, "+"] \arrow[d, "T \times T"'] & \mathbb{R}^n \arrow[d, "T"] \\ \mathbb{R}^m \times \mathbb{R}^m \arrow[r, "+"'] & \mathbb{R}^m \end{tikzcd} \qquad \begin{tikzcd} (\boldsymbol{u}, \boldsymbol{v}) \arrow[r, "+"] \arrow[d, "T \times T"'] & \boldsymbol{u}+\boldsymbol{v} \arrow[d, "T"] \\ (T(\boldsymbol{u}), T(\boldsymbol{v})) \arrow[r, "+"'] & \begin{array}{c} T(\boldsymbol{u}+\boldsymbol{v}) = \\ T(\boldsymbol{u})+T(\boldsymbol{v}) \end{array} \end{tikzcd}$$

Properties of Linear Transformations

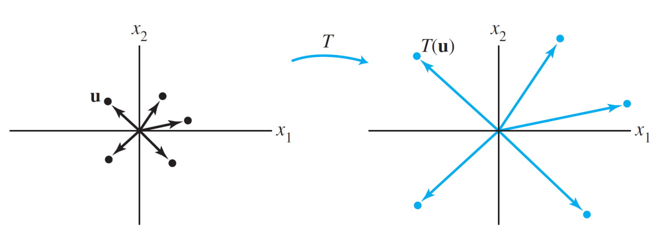
- These two properties lead easily to the following useful facts for a linear transformation T:
  - $T(\mathbf{0}) = \mathbf{0}$;
  - $T(c\boldsymbol{u} + d\boldsymbol{v}) = cT(\boldsymbol{u}) + dT(\boldsymbol{v})$, for all vectors $\boldsymbol{u}, \boldsymbol{v}$ in the domain of T and all scalars c, d.
- These are proven as follows:
  - $T(c\boldsymbol{u} + d\boldsymbol{v}) = T(c\boldsymbol{u}) + T(d\boldsymbol{v}) = cT(\boldsymbol{u}) + dT(\boldsymbol{v})$.
  - $T(\mathbf{0}) = T(0\boldsymbol{u}) = 0T(\boldsymbol{u}) = \mathbf{0}$.
- Observe that if a transformation satisfies the second property for all $\boldsymbol{u}, \boldsymbol{v}$ and c, d, it must be linear.
- Repeated application of the second property produces a useful generalization:

$$T(c_1\boldsymbol{v}_1 + \cdots + c_p\boldsymbol{v}_p) = c_1T(\boldsymbol{v}_1) + \cdots + c_pT(\boldsymbol{v}_p).$$

Dilations

- Given a scalar r, define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\boldsymbol{x}) = r\boldsymbol{x}$.
- T is called a **contraction** when $0 \leq r \leq 1$ and a **dilation** when $r > 1$.
- Let $r = 3$, and show that T is a linear transformation.
- Let $\boldsymbol{u}, \boldsymbol{v}$ be in $\mathbb{R}^2$ and let c, d be scalars. Then

$$\begin{aligned} T(c\boldsymbol{u} + d\boldsymbol{v}) &= 3(c\boldsymbol{u} + d\boldsymbol{v}) = 3c\boldsymbol{u} + 3d\boldsymbol{v} \\ &= c(3\boldsymbol{u}) + d(3\boldsymbol{v}) = cT(\boldsymbol{u}) + dT(\boldsymbol{v}). \end{aligned}$$



Rotations

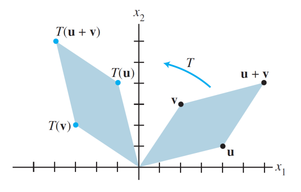
- Define a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(\boldsymbol{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

Find the images of $\boldsymbol{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

- We compute

$$T(\boldsymbol{u}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix},$$
$$T(\boldsymbol{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix},$$
$$T(\boldsymbol{u} + \boldsymbol{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}.$$



- T rotates $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{u} + \boldsymbol{v}$ counterclockwise about the origin through 90°.

Subsection 8

The Matrix of a Linear Transformation

Example

- The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
  Suppose T is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ such that
  $$T(\boldsymbol{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \text{ and } T(\boldsymbol{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}.$$
  With no additional information, find a formula for the image of an arbitrary $\boldsymbol{x}$ in $\mathbb{R}^2$.
- Write $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2$.
  Since T is a linear transformation,
  $$\begin{aligned} T(\boldsymbol{x}) &= x_1 T(\boldsymbol{e}_1) + x_2 T(\boldsymbol{e}_2) \\ &= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 + 0 \end{bmatrix}. \end{aligned}$$

Matrix of a Linear Transformation

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\boldsymbol{x}) = A\boldsymbol{x} \text{ for all } \boldsymbol{x} \text{ in } \mathbb{R}^n.$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\boldsymbol{e}_j)$, where $\boldsymbol{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$:
$A = \begin{bmatrix} T(\boldsymbol{e}_1) & \cdots & T(\boldsymbol{e}_n) \end{bmatrix}$.

- Write $\boldsymbol{x} = I_n\boldsymbol{x} = \begin{bmatrix} \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \end{bmatrix} \boldsymbol{x} = x_1\boldsymbol{e}_1 + \cdots + x_n\boldsymbol{e}_n$.
  Use the linearity of T to compute

$$\begin{aligned} T(\boldsymbol{x}) &= T(x_1\boldsymbol{e}_1 + \cdots + x_n\boldsymbol{e}_n) = x_1 T(\boldsymbol{e}_1) + \cdots + x_n T(\boldsymbol{e}_n) \\ &= \begin{bmatrix} T(\boldsymbol{e}_1) & \cdots & T(\boldsymbol{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A\boldsymbol{x}. \end{aligned}$$

Matrix of a Linear Transformation (Cont'd)

- For the uniqueness of A, assume that

$$T(\boldsymbol{x}) = A\boldsymbol{x} = B\boldsymbol{x}, \text{ for all } \boldsymbol{x} \text{ in } \mathbb{R}^n.$$

Let

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_n \end{bmatrix}, \quad B = \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_n \end{bmatrix}.$$

Plugging in $\boldsymbol{x} = \boldsymbol{e}_1$, we get

$$T(\boldsymbol{e}_1) = A\boldsymbol{e}_1 = B\boldsymbol{e}_1 \Rightarrow \boldsymbol{a}_1 = \boldsymbol{b}_1.$$

Plugging in $\boldsymbol{x} = \boldsymbol{e}_2$, we get $\boldsymbol{a}_2 = \boldsymbol{b}_2$.

Continuing, we get that $\boldsymbol{a}_i = \boldsymbol{b}_i$, for all $i = 1, \ldots, n$.

This shows that $A = \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_n \end{bmatrix} = B$.

- The matrix A is called the **standard matrix for the linear transformation** T.

Example

- For $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ in $\mathbb{R}^m$, write the linear combination $3\boldsymbol{v}_1 - 5\boldsymbol{v}_2 + 7\boldsymbol{v}_3$ as a matrix times a vector.
- Place $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ into the columns of a matrix A and place the weights $3, -5, 7$ into a vector $\boldsymbol{x}$.

  That is,

$$3\boldsymbol{v}_1 - 5\boldsymbol{v}_2 + 7\boldsymbol{v}_3 = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = A\boldsymbol{x},$$

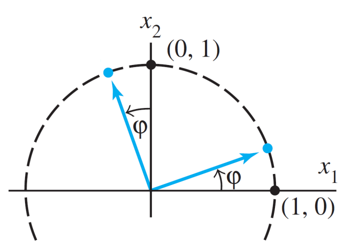
  where $A = [\boldsymbol{v}_1 \ \ \boldsymbol{v}_2 \ \ \boldsymbol{v}_3]$.

Example

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle φ, with counterclockwise rotation for a positive angle.
  We could show geometrically that such a transformation is linear.
  Find the standard matrix A of this transformation.
- We have that:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ rotates into } \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix};$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ rotates into } \begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}.$$



Thus, we get

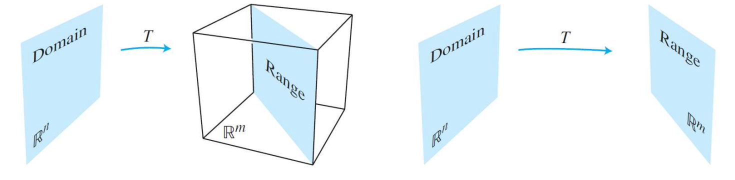
$$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}.$$

Onto Mappings

Definition

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\boldsymbol{b}$ in $\mathbb{R}^m$ is the image of at least one $\boldsymbol{x}$ in $\mathbb{R}^n$.

- Equivalently, T is onto $\mathbb{R}^m$ when the range of T is all of the codomain $\mathbb{R}^m$.
- That is, T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if, for each $\boldsymbol{b}$ in the codomain $\mathbb{R}^m$, there exists at least one solution of $T(\boldsymbol{x}) = \boldsymbol{b}$.



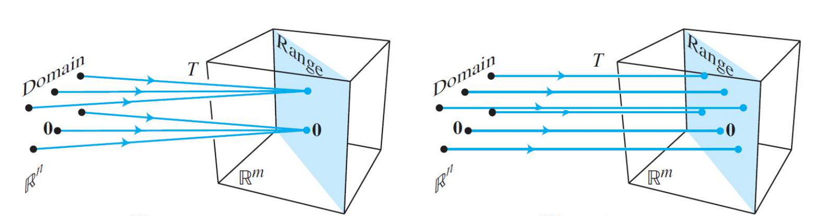
T is *not* onto $\mathbb{R}^m$ $\qquad$ T is onto $\mathbb{R}^m$

One-to-One Mappings

Definition

A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one-to-one** $\mathbb{R}^m$ if each $\boldsymbol{b}$ in $\mathbb{R}^m$ is the image of at most one $\boldsymbol{x}$ in $\mathbb{R}^n$.

- Equivalently, T is one-to-one if, for each $\boldsymbol{b}$ in the codomain $\mathbb{R}^m$, there exists at most one solution of $T(\boldsymbol{x}) = \boldsymbol{b}$.



T is *not* one-to-one

T is one-to-one

Example

- Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

  - Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$?
  - Is T a one-to-one mapping?

- Since A happens to be in echelon form, we can see at once that A has a pivot position in each row.

  By a previous theorem, for each $\boldsymbol{b}$ in $\mathbb{R}^3$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ is consistent. In other words, the linear transformation T maps $\mathbb{R}^4$ (its domain) onto $\mathbb{R}^3$.

  However, since the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a free variable (because there are four variables and only three basic variables), each $\boldsymbol{b}$ is the image of more than one $\boldsymbol{x}$. That is, T is not one-to-one.

Characterization of One-to-One Transformations

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\boldsymbol{x}) = \mathbf{0}$ has only the trivial solution.

- Since T is linear, $T(\mathbf{0}) = \mathbf{0}$.

  If T is one-to-one, then the equation $T(\boldsymbol{x}) = \mathbf{0}$ has at most one solution. Hence it has only the trivial solution.

  If T is not one-to-one, then there is a $\boldsymbol{b}$ that is the image of at least two different vectors in $\mathbb{R}^n$ - say, $\boldsymbol{u}$ and $\boldsymbol{v}$. That is, $T(\boldsymbol{u}) = \boldsymbol{b}$ and $T(\boldsymbol{v}) = \boldsymbol{b}$. But then, since T is linear,

$$T(\boldsymbol{u} - \boldsymbol{v}) = T(\boldsymbol{u}) - T(\boldsymbol{v}) = \boldsymbol{b} - \boldsymbol{b} = \mathbf{0}.$$

The vector $\boldsymbol{u} - \boldsymbol{v}$ is not zero, since $\boldsymbol{u} \neq \boldsymbol{v}$. Hence the equation $T(\boldsymbol{x}) = \mathbf{0}$ has more than one solution.

Onto and One-to-one in Terms of Matrix

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

(a) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

(b) T is one-to-one if and only if the columns of A are linearly independent.

(a) By a previous theorem, the columns of A span $\mathbb{R}^m$ if and only if for each $\boldsymbol{b}$ in $\mathbb{R}^m$ the equation $A\boldsymbol{x} = \boldsymbol{b}$ is consistent. In other words, if and only if for every $\boldsymbol{b}$, the equation $T(\boldsymbol{x}) = \boldsymbol{b}$ has at least one solution. This is true if and only if T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

(b) The equations $T(\boldsymbol{x}) = \mathbf{0}$ and $A\boldsymbol{x} = \mathbf{0}$ are the same except for notation. So, by the preceding theorem, T is one-to-one if and only if $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution. This happens if and only if the columns of A are linearly independent, as remarked previously.

Example

- In this example column vectors are written in rows, such as $\boldsymbol{x} = (x_1 \ x_2)$, and $T(\boldsymbol{x})$ is written as $T(x_1, x_2)$ instead of the more formal $T((x_1, x_2))$.
- Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$.
  Show that T is a one-to-one linear transformation.
  Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?
- We have

$$T(\boldsymbol{x}) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

  So T is indeed a linear transformation, with its standard matrix being the one shown above.
  The columns of A are linearly independent because they are not multiples. By a previous theorem T is one-to-one.

Example (Cont'd)

- We obtained $T(\boldsymbol{x}) = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

  To decide if T is onto $\mathbb{R}^3$, examine the span of the columns of A.
  Since A is 3×2, the columns of A span $\mathbb{R}^3$ if and only if A has 3 pivot positions, by a previous theorem.
  This is impossible, since A has only 2 columns.
  So the columns of A do not span $\mathbb{R}^3$.
  Hence, the associated linear transformation is not onto $\mathbb{R}^3$.