Practice set 4

Problem 4.1

Another approximation to $I(f) = \int_a^b f(x)\,dx$ is obtained if we replace $f(x)$ by the constant $f\left(\frac{a+b}{2}\right)$ on the entire interval $[a, b]$. Show that this leads to the numerical integration formula, called **midpoint rule**:

$$M_1(f) = (b-a)\, f\left(\frac{a+b}{2}\right)$$

In analogy to the derivation of the composite trapesoidal and Simpson's rules, generalize to the numerical integration formula

$$M_n(f) = h\,[f(x_1) + f(x_2) + \ldots + f(x_{n-1}) + f(x_n)]$$

where $h = \frac{b-a}{2}$ and $x_j = a + \left(j - \frac{1}{2}\right) h$. This rule is called **composite midpoint rule**.

Problem 4.2

As another approximation to $I(f) = \int_a^b f(x)\,dx$ replace $f(x)$ by the degree 4 interpolation polynomial at five evenly spaced points $x_j = a + j\,h, j = 0, 1, 2, 3, 4$. Show that this leads to the approximation formula

$$B_4(f) = \frac{2h}{45}\left[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\right]$$

This formula is called the **Boole's rule**.

Problem 4.3

Consider the numerical integration of

$$I = \int_0^1 \frac{e^x + e^{-x}}{2}\,dx$$

using the trapezoidal rule $T_n(f)$ and Simpson's rule $S_n(f)$. Give an exact error formulas $E_n^T(f)$ and $E_n^S(f)$. Using the formulas for E_n^T and E_n^S, determine how large n should be chosen in order to have $|I - T_n| \le 10^{-8}$ and $|I - S_n| \le 10^{-8}$.

Problem 4.4

Repeat previuos problem for

$$I = \int_{-1}^2 e^{3x-1}\,dx.$$

Problem 4.5

Consider using the trapezoidal rule T_n to approximate

$$I = \int_0^\pi \frac{dx}{2 + \cos(x)}$$

Then $I - T_n \approx Ch^2$. Answer True or False. Give appropriate explanations for your answer.

Problem 4.6

Using error formula for trapezoidal rule estimate the number n of subdivisions to evaluate the following integral

$$I = \int_1^3 \ln x\,dx$$

to the accuracy of 10^{-5}.

Problem 4.7

Following is a table of numerical integrals I_n approximating an integral I, where and also are given the successive differencies $I_n - I_{\frac{1}{2}n}$. Show that the table values seem to be consistent with an error of the form

$$I - I_n \approx \frac{c}{n^p}$$

for some constant c and some p. Estimate p and c. Then estimate $I - I_{512}$. How large n should be chosen if the error is to be less than 10^{-10}?

n	I_n	$I_n - I_{\frac{1}{2}n}$	*Ratio*
2	0.402368927062		
4	0.400431916045	$-1.937E-3$	
8	0.400077249447	$-3.547E-4$	
16	0.400013713469	$-6.354E-5$	
32	0.400002427846	$-1.129E-5$	
64	0.400000429413	$-1.998E-6$	
128	0.400000075924	$-3.535E-7$	
256	0.400000013423	$-6.250E-8$	
512	0.400000002373	$-1.105E-8$	

Problem 4.8

Following is a table of numerical integrals I_n approximating an integral $I = \int_0^1 f(x)\,dx$ with Simpson's rule. Show that the table values seem to be consistent with an error of the form

$$I - I_n \approx \frac{c}{n^p}.$$

Find value of p and comment on anything unusual in the above numerical results.

n	*Error*	*Ratio*
2	$2.860E-2$	
4	$1.012E-2$	
8	$3.587E-3$	
16	$1.268E-3$	
32	$4.485E-4$	

Problem 4.9

Let

$$I_h(f) = \frac{3}{4}h\,f(0) + 3f(2h).$$

Find the degree of precission of the approximation

$$I_h(f) \approx \int_0^{2h} f(x)dx$$

Problem 4.10

For the formula

$$\int_0^2 f(x)dx \approx w_1 f(x_1) + w_2 f(2)$$

determine the weights w_1, w_2 and x_2 such that the quadrature formula is exact for all polynomials of as large degree as possible.

Problem 4.11

Using the table below estimate the value of $f'(0.5)$ using forward, backward and centered difference formulas. What is the best estimate and argue your answer. Also, estimate $f''(0.5)$.

x	$f(x)$
0.3	7.3891
0.4	7.4633
0.5	7.5383
0.6	7.6141
0.7	7.6906