Applications of 2nd order Linear ODE

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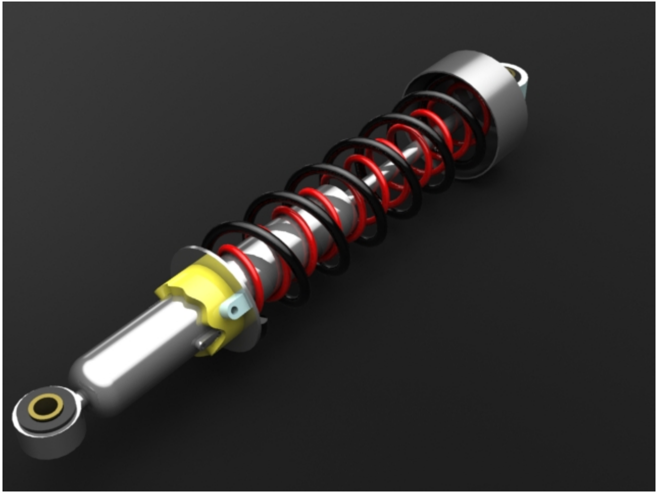
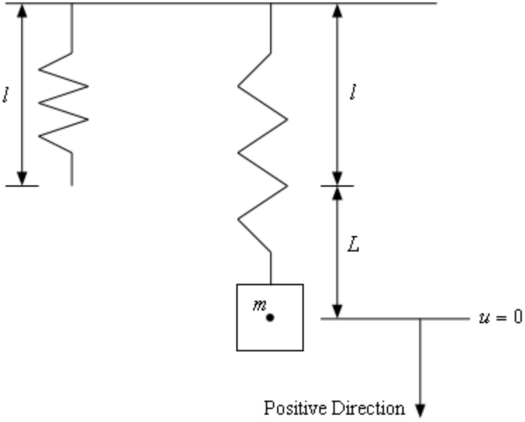
Mechanical Vibrations





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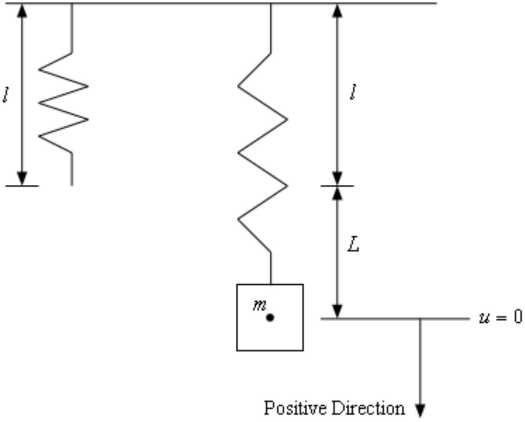

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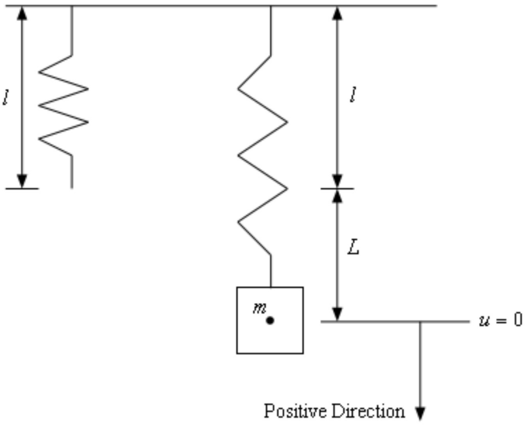

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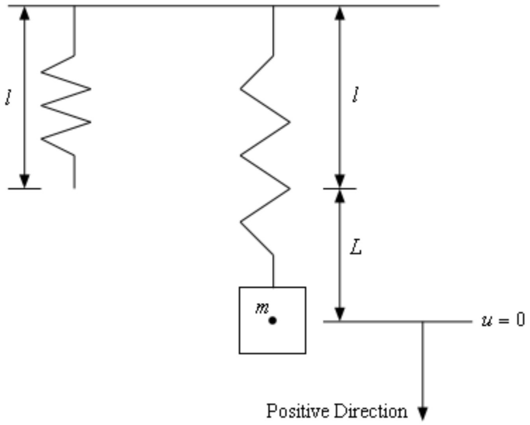

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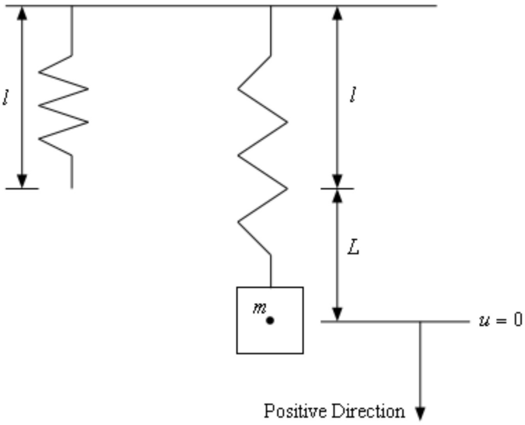

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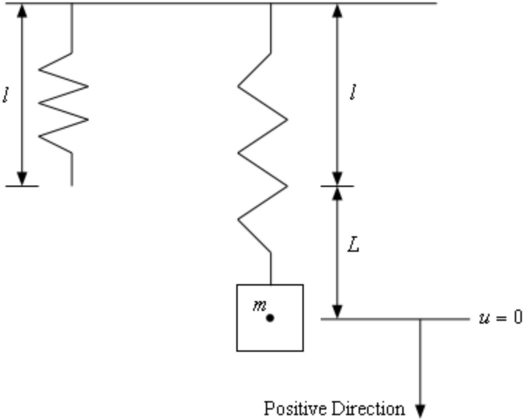

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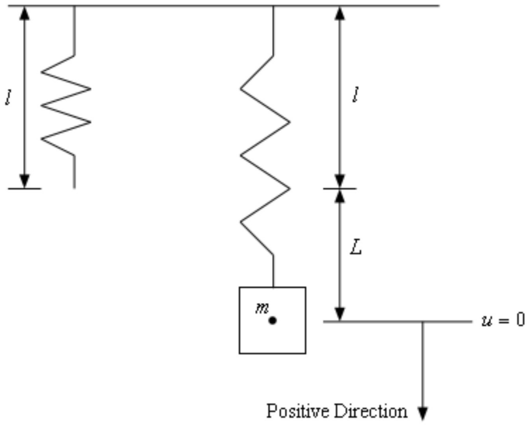

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The $u = 0$ position will correspond to this equlibrum position.

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There are four forces that we will assume act upon the object:

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- External Forces (may or may not be present), $F(t)$

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  - If $u > 0$, spring force increases in magnitude proportionaly, and vice versa if $u < 0$ it will decrease in magnitude (still being negative!).

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- **External Forces**, $F(t)$: If there are any other forces that we decide we want to act on our object we lump them in here and call it good. We typically call $F(t)$ the **forcing function**.

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\begin{aligned}
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So, in the end we get

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Note that we can use relation $mg = kL$ to determine the spring constant k.

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This is usually reduced to,

$$r_{1,2} = \pm\omega_0\, i,$$

where we introduce notation

$$\omega_0 = \sqrt{\frac{k}{m}},$$

which is called the **natural frequency**. Recall as well that $m > 0$ and $k > 0$ and so we can guarantee that this quantity will be complex.

The solution in this case is then

$$
\begin{aligned}
u(t) &= C_1 e^{0\cdot t}\cos(\omega_0 t) + C_2 e^{0\cdot t}\sin(\omega_0 t)\\
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We can write it in the following form,

$$u(t) = R\cos(\omega_0 t - \delta),$$

where

$$R = \sqrt{C_1^2 + C_2^2}$$

is the **amplitude** of the displacement and

$$\delta = \arctan\left(\frac{C_2}{C_1}\right)$$

is called the **phase shift** or phase angle of the displacement.

EXAMPLE

A 0.5 kg object stretches a spring 27.2 cm by itself. There is no damping and no external forces acting on the system. The spring is initially displaced 50 cm upwards from its equilibrium position and given an initial velocity of 1 m/sec downward. Find the displacement at any time t, $u(t)$.

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Clearly $m = 0.5$ and $L = 0.272$. Let's compute k using relation $mg = kL$:

$$k = \frac{mg}{L} = \frac{0.5 \cdot 9.8}{0.272} \approx 18.$$

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Compute the natural frequency:

$$\omega_0 = \sqrt{\frac{18}{0.5}} = \sqrt{36} = 6.$$

The general solution, along with its derivative, is then,

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\begin{aligned}
u(t) &= C_1 \cos(6t) + C_2 \sin(6t),\\
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-0.5 = u(0) = C_1 \quad &\implies \quad C_1 = -0.5\\
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Compute the amplitude and phase shift

$$R = \sqrt{C_1^2 + C_2^2} = \sqrt{\left(-\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{6}\right)} = \tfrac{\sqrt{10}}{6}$$
$$\delta = \arctan\left(\tfrac{C_2}{C_1}\right) = \arctan\left(\tfrac{1/6}{-1/2}\right) = \arctan\left(-\tfrac{1}{3}\right) \approx -0.32175$$

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Be careful: phase angle found is in the IV quadrant, but there is an angle in the II quadrant that works equally well:

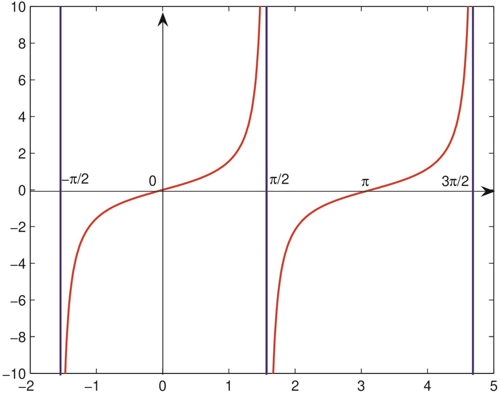
Which one to choose?

$$\delta_1 = -0.32175,$$
$$\delta_2 = \delta_1 + \pi = 2.81984$$

Take a look at the relations:

$$C_1 = R\cos\delta = \frac{\sqrt{10}}{6}\cos\delta$$
$$C_2 = R\sin\delta = \frac{\sqrt{10}}{6}\sin\delta$$
$$C_1 = -\frac{1}{2} < 0, \quad C_2 = \frac{1}{6} > 0$$

Choose δ_2.



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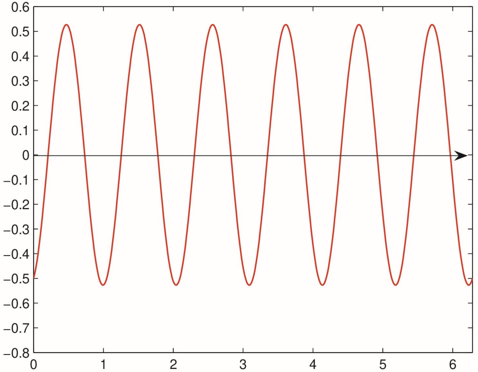

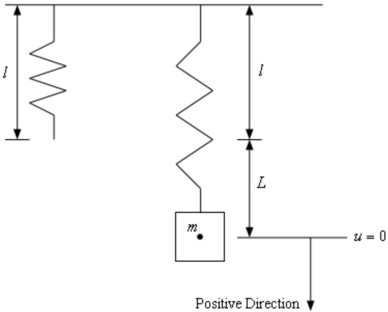

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Case 1. $D = \gamma^2 - 4km = 0$, which means we have two identical real roots

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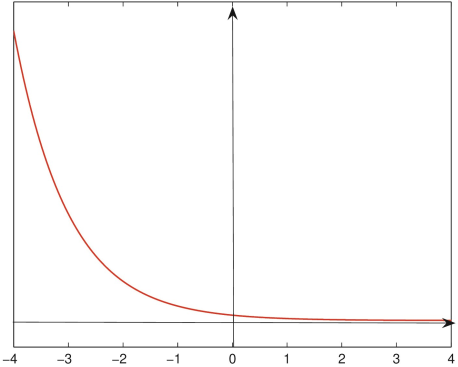
Notice that as $t \to \infty$ the displacement $u(t)$ will approach zero and so the damping in this case will do what it's supposed to do.



$$u(t) = C_1 e^{-\frac{\gamma}{2m}t} + C_2 t e^{-\frac{\gamma}{2m}t}.$$

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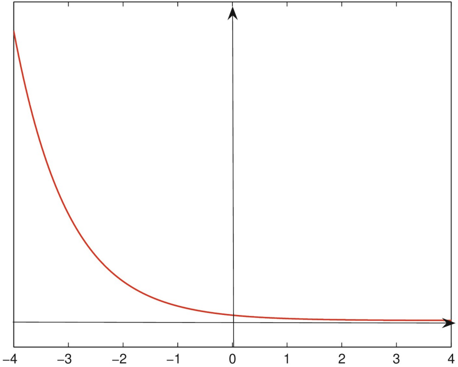

This case is called **critical damping**, and it will happen when the damping coefficient is

$$\begin{aligned} D &= \gamma^2 - 4mk = 0 \\ \gamma^2 &= 4mk \\ \gamma &= 2\sqrt{mk} = \gamma_{CR} \end{aligned}$$

The value of damping coefficient that gives the citical damping is called the **critical damping coefficient**, denoted by γ_{CR}

Case 2. $D = \gamma^2 - 4mk > 0$, in this case let's rewrite the roots:

$$
\begin{aligned}
r_{1,2} &= \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} \\
&= \frac{-\gamma \pm \gamma\sqrt{1 - \frac{4mk}{\gamma^2}}}{2m} \\
&= \frac{-\gamma}{2m}\left(1 \pm \sqrt{1 - \frac{4mk}{\gamma^2}}\right)
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From our assumption

$$
\begin{gathered}
\gamma^2 > 4mk \\
1 > \frac{4mk}{\gamma^2} \\
\sqrt{1 - \frac{4mk}{\gamma^2}} < 1
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Why this is important?

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From our assumption

$$\begin{gathered} \gamma^2 > 4mk \\ 1 > \frac{4mk}{\gamma^2} \\ \sqrt{1 - \frac{4mk}{\gamma^2}} < 1 \end{gathered}$$

Why this is important? Because it implies that $\boxed{r_{1,2} < 0}$

$$r_1 < 0 \quad \text{and} \quad r_2 < 0,$$

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In other words **the oscillations will fade out!**

So, once again the damper does what it is supposed to do.

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So, once again the damper does what it is supposed to do.

This will happen if

$$\begin{aligned} D = &\gamma^2 - 4mk > 0 \\ \gamma^2 &> 4mk \\ \gamma &> \gamma_{CR} \end{aligned}$$

and it is called **over damping**

Case 3. $D = \gamma^2 - 4mk < 0$, in this case we get complex roots:

$$r_{1,2} = \frac{-\gamma}{2m}\left(1 \pm \sqrt{1 - \frac{4mk}{\gamma^2}}\right) = \lambda \pm \mu i$$

with the real part guaranteed to be negative.

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Solution in this case is

$$\begin{aligned} u(t) &= C_1 e^{\lambda t}\cos(\mu t) + C_2 e^{\lambda t}\sin(\mu t) \\ &= e^{\lambda t}\left(C_1 \cos(\mu t) + C_2 \sin(\mu t)\right) \\ &= e^{\lambda t} R\cos(\mu t - \delta) \end{aligned}$$

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Since $\lambda < 0$, the displacement $u(t)$ will aproach zero as $t \to \infty$ and the damper will also work as it should do. This is called **under damping** and it happens if

$$\gamma < \gamma_{CR}$$

Take in the previuos problem $\gamma = 6$.

$$\frac{1}{2}u'' + 6u' + 18u = 0,$$
$$u(0) = -\frac{1}{2},$$
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Solution is

$$u(t) = -\frac{1}{2}e^{-6t} - 2te^{-6t}$$

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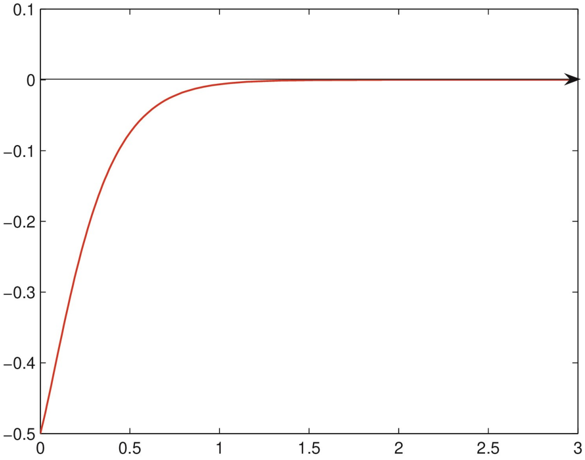

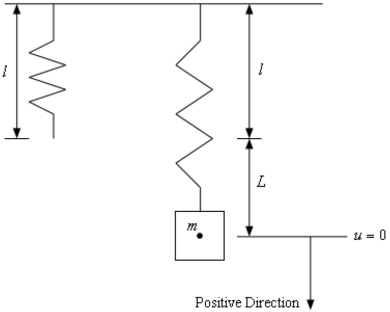

Take in the model problem $\gamma = 8.5$:

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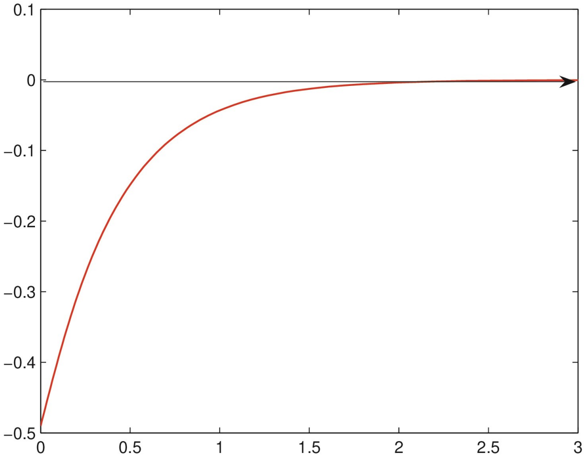

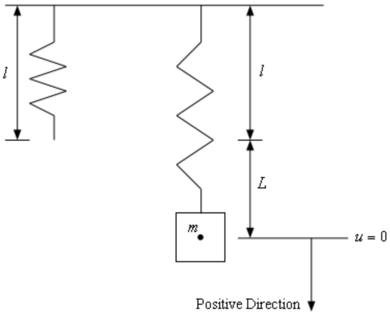

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Mechanical Vibrations. Free, Damped Vibrations

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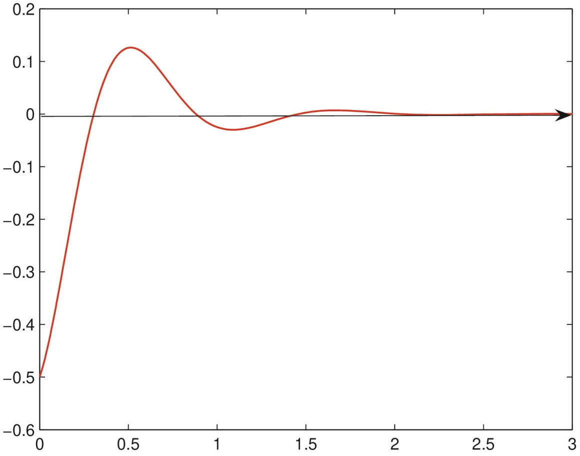

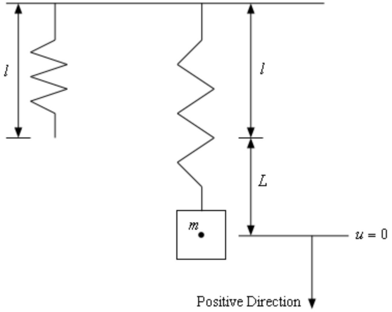

Consider the undamped case: $\gamma = 0$. The differential equation in this case is

$$mu'' + ku = F(t)$$

Mechanical Vibrations. Undamped forced vibrations

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This is a **nonhomogeneous 2nd order linear ODE**.

Solution is

$$u(t) = u_c(t) + U_p(t)$$

with u_c being the complimentary solution i.e solution of homogenuous ODE:

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and U_p is a particular solution of nonhomogenuous ODE.

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To get the particular solution we can use either undetermined coefficients or variation of parameters depending on which we find easier for a given forcing function.

There is a particular type of forcing function that we should take a look at since it leads to some interesting results.

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Mechanical Vibrations. Undamped forced vibrations

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The complimentary solution and the guess for particular solution is

$$\begin{aligned} u_c(t) &= C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \\ U_p(t) &= A \cos(\omega t) + B \sin(\omega t) \end{aligned}$$

where ω_0 is the natural frequency.

Mechanical Vibrations. Undamped forced vibrations

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Consider two cases: $\omega_0 \neq \omega$ and $\omega_0 = \omega$.

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Mechanical Vibrations. Undamped forced vibrations

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Using undetermined coefficients method we get the particular solution in the form

$$U_p(t) = \frac{F_0}{k - m\omega^2} \cos(\omega t)$$

$$U_p(t) = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \cos \qquad mk = \omega_0^2 \, (\omega t)$$

Mechanical Vibrations. Undamped forced vibrations

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And the final general solution is

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \cos(\omega t)$$

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Well in the first case, $\omega_0 \neq \omega$ our displacement function consists of two cosines and is nice and well behaved for all time.

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Mechanical Vibrations. Undamped forced vibrations

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In contrast, the second case, $\omega_0 = \omega$ we will have some serious issues at t increases. The addition of the t in the particular solution will mean that we are going to see an oscillation that grows in amplitude as $t \to \infty$.

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This case is called **resonance** and we would generally like to avoid this at all costs!

Example.

A $3kg$ object is attached to spring and will stretch the spring $392mm$ by itself. There is no damping in the system and a forcing function of the form $F(t) = 10\cos(\omega t)$ is attached to the object and the system will experience resonance. If the object is initially displaced $20cm$ downward from its equilibrium position and given a velocity of $10cm/s$ upward find the displacement at any time t.

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Compute $k = 75$ and $\omega_0 = 5$. Our IVP becomes

$$3u'' + 75u = 10\cos(5t)$$
$$u(0) = 0.2$$
$$u'(0) = -0.1$$

Solution is

$$u(t) = \frac{1}{5}\cos(5t) - \tfrac{1}{50}\sin(5t) + \frac{1}{3}t\sin(5t)$$
$$u(t) = 0.200998\cos(5t + 0.099669) + \frac{1}{3}t\sin(5t)$$

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Mechanical Vibrations. Undamped forced vibrations

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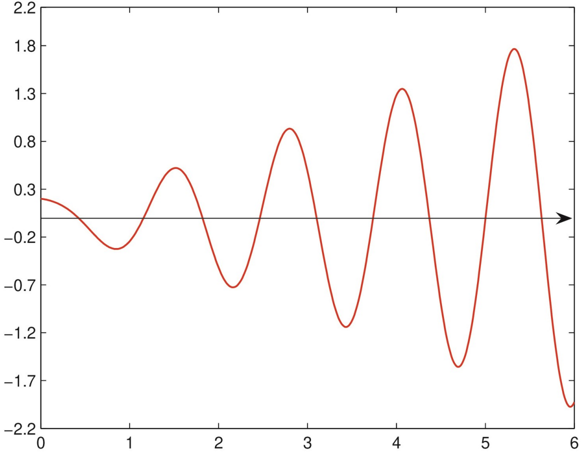

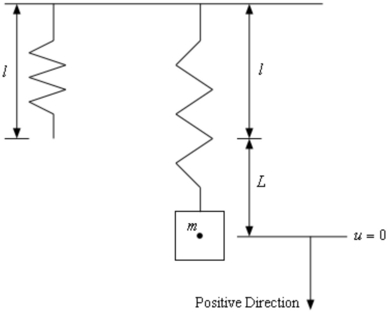

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$$u(t) = u_c(t) + U_p(t)$$

with u_c being the complimentary solution i.e solution of homogenuous ODE

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Mechanical Vibrations. Damped forced vibrations

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From free force example, we know that the complimentary solution will approach 0 as t increases.

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From free force example, we know that the complimentary solution will approach 0 as t increases.

Consider last example, and add in a damper that will exert a force of 45 Newtons when the velocity will be $50cm/s$.

$$F_d = \gamma u'$$

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and our IVP becomes

$$\begin{aligned} 3u'' + 90u' + 75u &= 10\cos(5t) \\ u(0) &= 0.2 \\ u'(0) &= -0.1 \end{aligned}$$

$$F_d = \gamma u' \quad \Rightarrow \quad 45 = \gamma \cdot 0.5 \quad \Rightarrow \quad \gamma = 90$$

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$$u_c(t) = C_1 e^{-0.8579t} + C_1 e^{-29.1421t}$$

Mechanical Vibrations. Damped forced vibrations

$$F_d = \gamma u' \quad \Rightarrow \quad 45 = \gamma \cdot 0.5 \quad \Rightarrow \quad \gamma = 90$$

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$$U_p(t) = A\cos(5t) + B\sin(5t)$$

which will gives us

$$U_p(t) = \frac{1}{45}\sin(5t)$$

$$F_d = \gamma u' \quad \Rightarrow \quad 45 = \gamma \cdot 0.5 \quad \Rightarrow \quad \gamma = 90$$

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$$u_c(t) = C_1 e^{-0.8579t} + C_1 e^{-29.1421t}$$

and the particular solution will be of the form

$$U_p(t) = A\cos(5t) + B\sin(5t)$$

which will gives us

$$U_p(t) = \frac{1}{45}\sin(5t)$$

General solution then is

$$u(t) = C_1 e^{-0.8579t} + C_1 e^{-29.1421t} + \frac{1}{45}\sin(5t)$$

Mechanical Vibrations. Damped forced vibrations

$$u(t) = C_1 e^{-0.8579t} + C_1 e^{-29.1421t} + \frac{1}{45}\sin(5t)$$