Linear Algebra and Analytic Geometry

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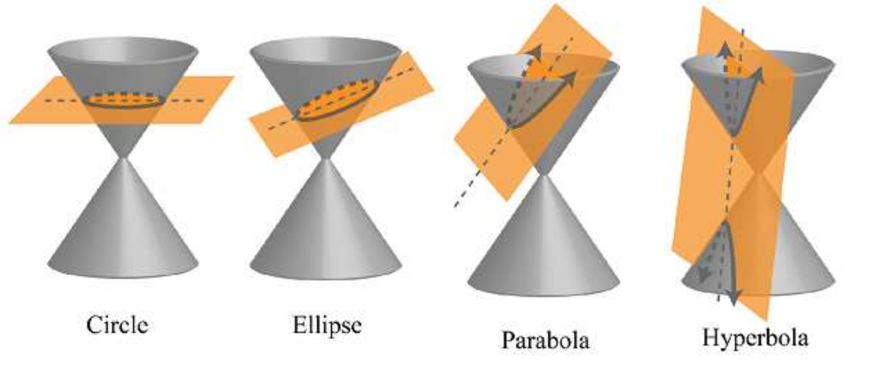
2020

- Parabolas
- Ellipses
- Hyperbolas
- Introduction to Polar Coordinates
- Polar Equations of the Conics
- Parametric Equations

Subsection 1

Parabolas

Conic Sections

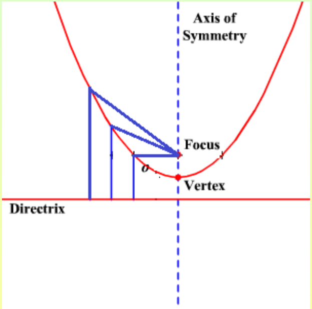


Definition of Parabola

Definition of Parabola

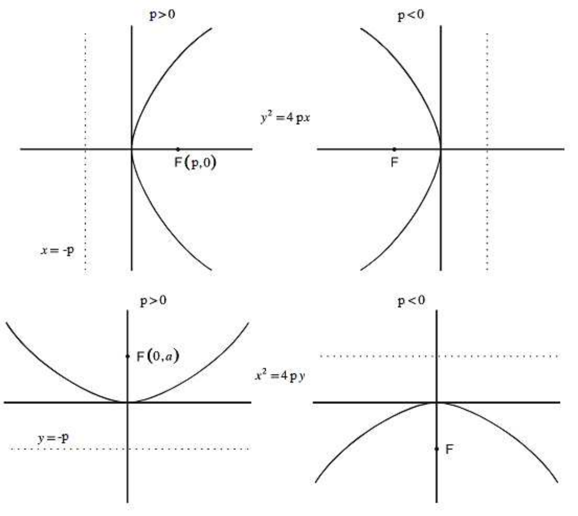
A **parabola** is the set of points in a plane that are equidistant from a fixed line, called the **directrix**, and a fixed point, called the **focus**, not on the directrix.

The line passing through the focus and perpendicular to the directrix is called the **axis of symmetry** of the parabola;



Standard Forms of the Equation of the Parabola

When the parabola has vertex at the origin, the standard forms of the equation are:

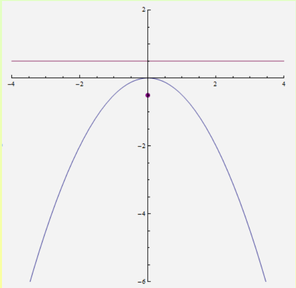


Example I

- Find the focus and directrix of the parabola given by the equation $y = -\frac{1}{2}x^2$;

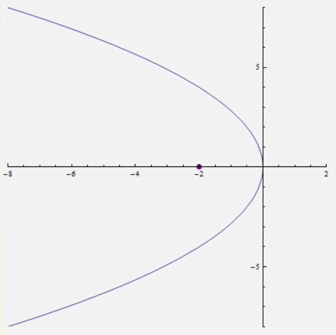
$$y = -\frac{1}{2}x^2 \quad \Rightarrow \quad x^2 = -2y \quad \Rightarrow \quad x^2 = 4(-\frac{1}{2})y;$$

This shows that $p = -\frac{1}{2}$, i.e., the focus is $(0, -\frac{1}{2})$ and the directrix $y = \frac{1}{2}$;



Example II

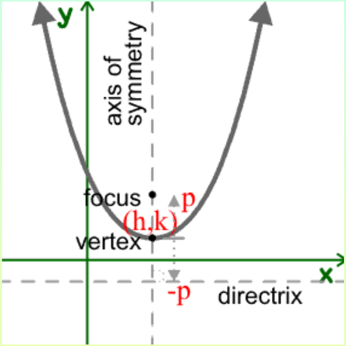
- Find the equation in standard form of the parabola with vertex at the origin and focus at $(-2, 0)$;

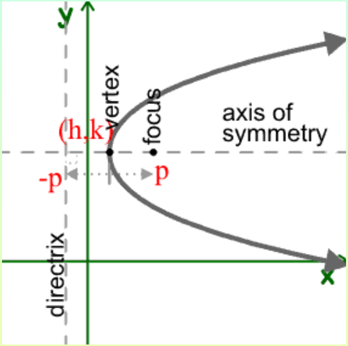


We have $p = -2$; Therefore, the equation is

$$y^2 = 4(-2)x \quad \Rightarrow \quad y^2 = -8x;$$

Standard Forms of the Equation of the Parabola





Equation is

$$(x-h)^2 = 4p(y-k);$$

Equation is

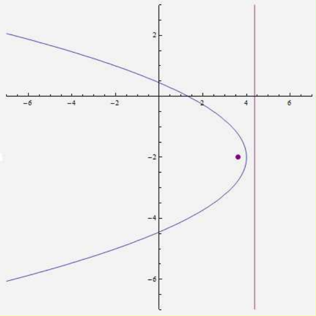
$$(y-k)^2 = 4p(x-h);$$

Example I

- Find the equation of the directrix and the coordinates of the vertex and of the focus of the parabola given by the equation $3x + 2y^2 + 8y - 4 = 0$;

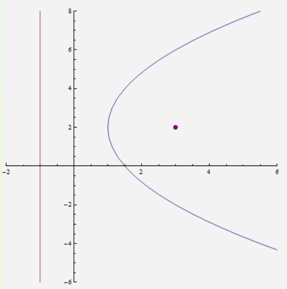
$$\begin{aligned} 3x + 2y^2 + 8y - 4 = 0 \quad &\Rightarrow \quad 2y^2 + 8y = -3x + 4 \\ \Rightarrow \quad 2(y^2 + 4y) = -3x + 4 \quad &\Rightarrow \quad 2(y^2 + 4y + 4) = -3x + 12 \\ \Rightarrow \quad 2(y+2)^2 = -3(x-4) \quad &\Rightarrow \quad (y+2)^2 = -\tfrac{3}{2}(x-4) \\ \Rightarrow \quad (y+2)^2 = 4(-\tfrac{3}{8})(x-4); & \end{aligned}$$

So $V = (4, -2)$, parabola opens left and $p = -\frac{3}{8}$; Therefore, directrix is $x = 4 + \frac{3}{8} \Rightarrow x = \frac{35}{8}$ and focus is at $(4 - \frac{3}{8}, -2) = (\frac{29}{8}, -2)$;



Example II

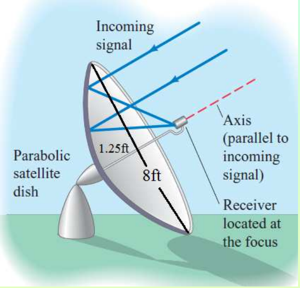
- Find an equation in the standard form of the parabola with directrix $x = -1$ and focus $(3, 2)$;



Directrix is vertical; Focus on the right of directrix, so equation has the form $(y-k)^2 = 4p(x-h)$; Therefore, since the distance from focus to directrix is 4, we get $p = 2$ and $(h, k) = (1, 2)$; These give equation $(y-2)^2 = 8(x-1)^2$;

Application: Focus of a Satellite Dish

A dish has a paraboloid shape; The signals it receives are reflected to a receiver at its focus; If the dish is 8 feet across at its opening and 1.25 feet deep at its center, find the location of the focus;



The dish may be modeled by the equation $y^2 = 4px$;
Since at $x = \frac{5}{4}$ feet, we have $y = 4$ feet, we obtain

$$4^2 = 4p\frac{5}{4} \quad \Rightarrow \quad p = \frac{16}{5} \text{ feet},$$

i.e., its focus is located $\frac{16}{5}$ feet above its vertex;

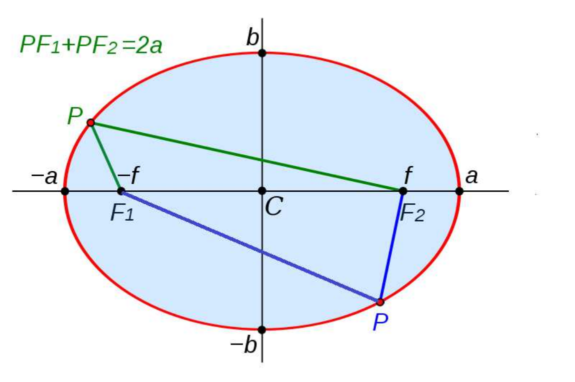
Subsection 2

Ellipses

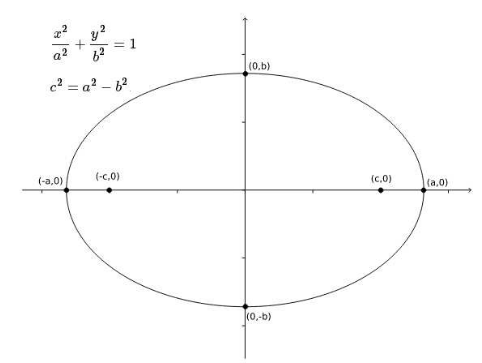
Definition of Ellipses

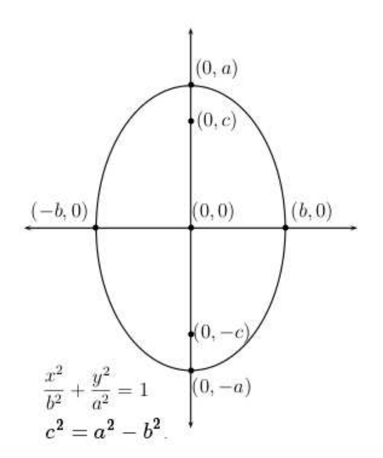
Definition of an Ellipse

An **ellipse** is the set of all points in the plane the sum of whose distances from two fixed points, called the **foci**, is a positive constant.



Standard Form of the Equation of an Ellipse



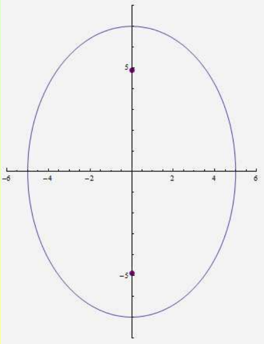


Example I

- Find the vertices and foci of the ellipse given by the equation $\dfrac{x^2}{25} + \dfrac{y^2}{49} = 1$; Sketch its graph;

  The y^2 term has a larger denominator, so the major axis is on the y-axis;

$$\begin{aligned} a^2 &= 49 \quad \Rightarrow \quad a = 7 \\ b^2 &= 25 \quad \Rightarrow \quad b = 5 \\ c^2 &= a^2 - b^2 = 24 \\ &\Rightarrow \quad c = 2\sqrt{6}; \end{aligned}$$



  Thus, the vertices are at $(0,7)$, $(0,-7)$, the foci are at $(0,2\sqrt{6})$, $(0,-2\sqrt{6})$;

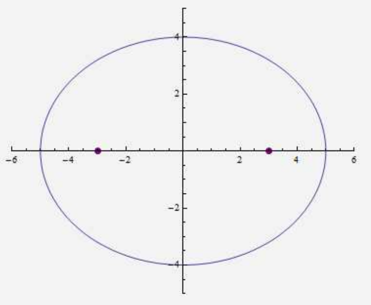
Example II

- Consider the ellipse with foci $(3, 0)$ and $(-3, 0)$ and major axis of length 10 as shown in the figure; Find an equation for this ellipse;

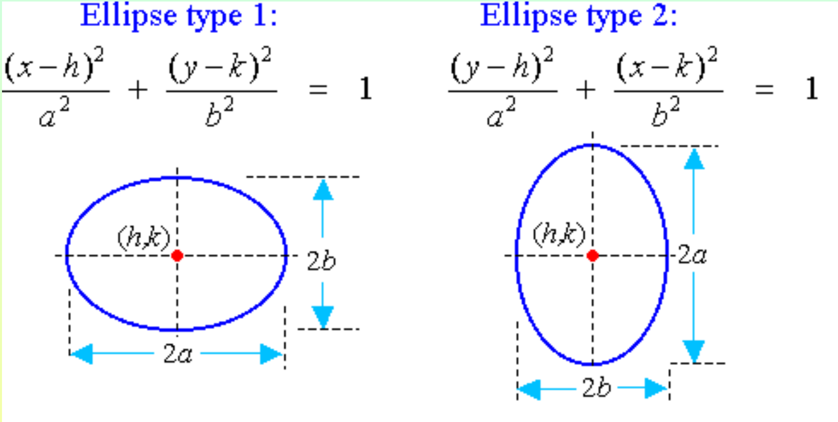
$$c = 3;$$
$$a = 5;$$
$$b^2 = a^2 - c^2 \quad \Rightarrow \quad b^2 = 16$$
$$\Rightarrow \quad b = 4;$$

Thus, an equation for this ellipse is

$$\frac{x^2}{25} + \frac{y^2}{16} = 1;$$



Standard Forms of the Equation of an Ellipse

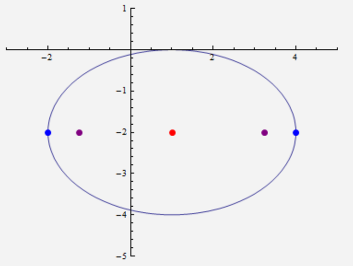


Example I

- Find the center, vertices and foci of the ellipse $4x^2 + 9y^2 - 8x + 36y + 4 = 0$; Then sketch the graph;

$$4x^2 + 9y^2 - 8x + 36y + 4 = 0 \quad \Rightarrow \quad 4x^2 + 9y^2 - 8x + 36y = -4$$
$$\Rightarrow \quad 4(x^2 - 2x) + 9(y^2 + 4y) = -4$$
$$\Rightarrow \quad 4(x^2 - 2x + 1) + 9(y^2 + 4y + 4) = -4 + 4 + 36$$
$$\Rightarrow \quad 4(x-1)^2 + 9(y+2)^2 = 36 \quad \Rightarrow \quad \frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1;$$

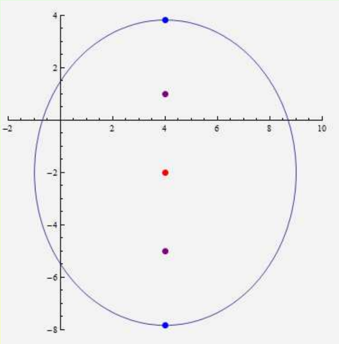
Thus, center is $(1, -2)$, $a = 3$ and, therefore, vertices are at $(4, -2)$ and $(-2, -2)$ and $c^2 = a^2 - b^2 = 5 \Rightarrow c = \sqrt{5}$, and, thus, foci are at $(1 + \sqrt{5}, -2)$ and $(1 - \sqrt{5}, -2)$;



Example II

- Find the standard form of the equation of the ellipse with center at $(4, -2)$, foci $F_2(4, 1)$ and $F_1(4, -5)$ and minor axis of length 10;

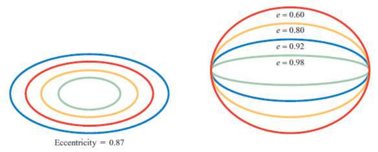
$$
\begin{aligned}
&(h, k) = (4, -2);\\
&c = 3;\\
&b = 5;\\
&a^2 = b^2 + c^2 = 34;\\
&\frac{(x-4)^2}{25} + \frac{(y+2)^2}{34} = 1;
\end{aligned}
$$



Eccentricity

Eccentricity of an Ellipse

The **eccentricity** e of an ellipse is the ratio of c to a, where c is the distance from the center to a focus and a is one-half the length of the major axis, i.e., $e = \frac{c}{a}$.



- Example: What is the eccentricity of the ellipse with equation $8x^2 + 9y^2 = 18$?

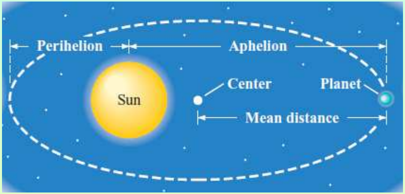
$$8x^2 + 9y^2 = 18 \quad \Rightarrow \quad \frac{4x^2}{9} + \frac{y^2}{2} = 1 \quad \Rightarrow \quad \frac{x^2}{(3/2)^2} + \frac{y^2}{(\sqrt{2})^2} = 1;$$

$$a = \frac{3}{2}; \qquad c = \sqrt{a^2 - b^2} = \sqrt{\frac{9}{4} - 2} = \frac{1}{2};$$

$$e = \frac{c}{a} = \frac{1/2}{3/2} = \frac{1}{3};$$

Application: The Earth's Orbit

Earth has a mean distance of 93 million miles and a perihelion distance of 91.5 million miles. Find an equation for Earth's orbit;



The mean distance gives $a = 93$; The distance from the Sun to the center of the Earth's orbit is

$$c = 93 - 91.5 = 1.5 \text{ million miles;}$$

Therefore, $b^2 = a^2 - c^2 = 8646.75$; Thus, an equation of the orbit is

$$\frac{x^2}{93^2} + \frac{y^2}{8646.75} = 1;$$

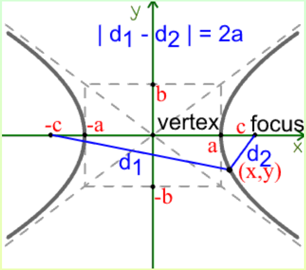
Subsection 3

Hyperbolas

Definition of a Hyperbola

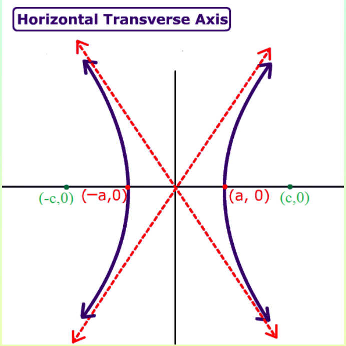
Definition of a Hyperbola

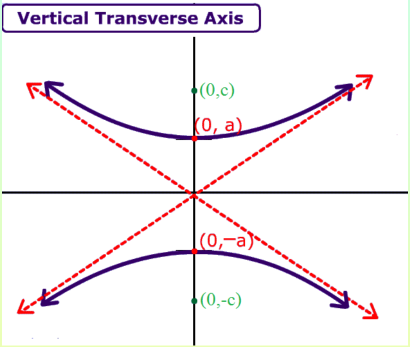
A **hyperbola** is the set of all points in the plane the difference between whose distances from two fixed points, called **foci**, is a positive constant.



- The axis joining the vertices is the **transverse axis**;
- The midpoint of the transverse axes is the **center**;
- The **conjugate axis** is the segment passing through the center and perpendicular to the transverse axis;

Standard Forms of the Equation of a Hyperbola





$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$
$$c^2 = a^2 + b^2;$$

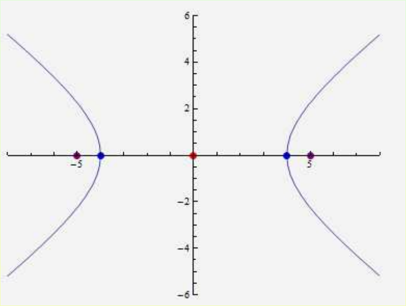
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1,$$
$$c^2 = a^2 + b^2;$$

Example

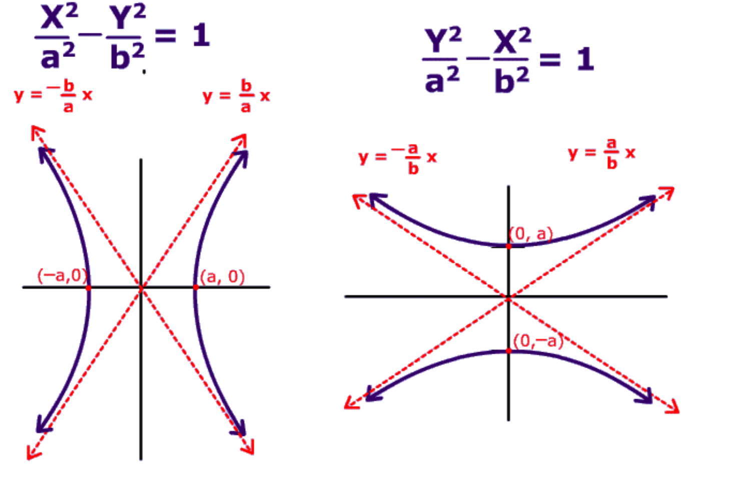
- Find the vertices and the foci of the hyperbola given by the equation $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$;

$$a = 4;$$
$$b = 3;$$
$$c = \sqrt{a^2 + b^2} = 5;$$

Vertices at $(-4, 0)$ and $(4, 0)$;
Foci at $(-5, 0)$ and $(5, 0)$;



Asymptotes



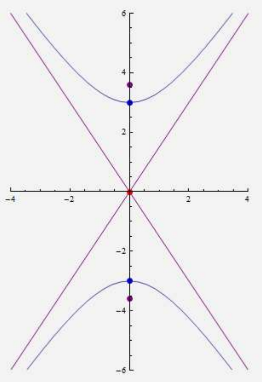
$\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$
$y=-\frac{b}{a}x$
$y=\frac{b}{a}x$
(−a,0)
(a, 0)
$\frac{Y^2}{a^2}-\frac{X^2}{b^2}=1$
$y=-\frac{a}{b}x$
$y=\frac{a}{b}x$
(0, a)
(0,−a)

Example

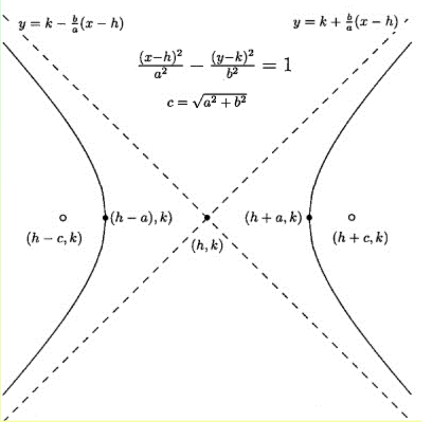
- Find the vertices, the foci and the asymptotes of the hyperbola given by $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$; Then sketch its graph;

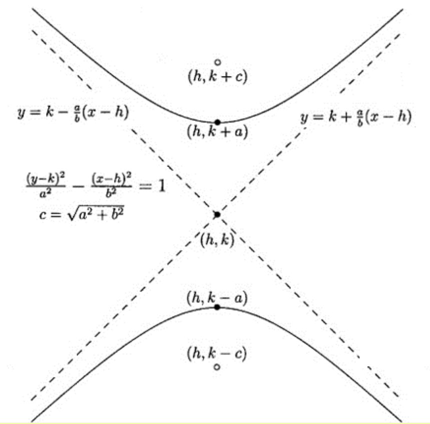
$$a = 3;$$
$$b = 2;$$
$$c = \sqrt{a^2 + b^2} = \sqrt{13};$$

Vertices at $(0, -3)$ and $(0, 3)$;
Foci at $(0, -\sqrt{13})$ and $(0, \sqrt{13})$;
Asymptotes $y = -\frac{3}{2}x$ and $y = \frac{3}{2}x$;



Standard Forms of the Equation of a Hyperbola



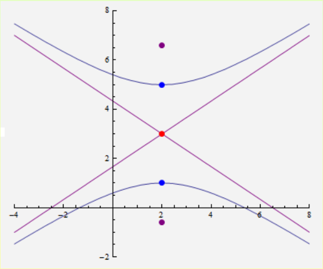


Example

- Find the center, vertices, foci and asymptotes of the hyperbola given by the equation $4x^2 - 9y^2 - 16x + 54y - 29 = 0$; Then sketch its graph;

$$4x^2 - 9y^2 - 16x + 54y - 29 = 0 \quad \Rightarrow \quad 4x^2 - 9y^2 - 16x + 54y = 29$$
$$\Rightarrow \quad 4(x^2 - 4x) - 9(y^2 - 6y) = 29$$
$$\Rightarrow \quad 4(x^2 - 4x + 4) - 9(y^2 - 6y + 9) = 29 + 16 - 81$$
$$\Rightarrow \quad 4(x-2)^2 - 9(y-3)^2 = -36 \quad \Rightarrow \quad \frac{(y-3)^2}{4} - \frac{(x-2)^2}{9} = 1;$$

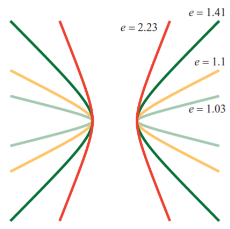
So $(h, k) = (2, 3)$, $a = 2$, $b = 3$, and $c = \sqrt{13}$;
These give that center is at $(2, 3)$, vertices are at $(2, 5)$ and $(2, 1)$, foci are at $(2, 3 + \sqrt{13})$ and $(2, 3 - \sqrt{13})$ and asymptotes are $y - 3 = -\frac{2}{3}(x - 2)$ and $y - 3 = \frac{2}{3}(x - 2)$;



Eccentricity

Eccentricity of a Hyperbola

The **eccentricity** e of a hyperbola is the ratio of c to a, where c is the distance from the center to a focus and a is one-half the length of the transverse axis, i.e., $e = \frac{c}{a}$.



- Example: What is an equation for a hyperbola centered at the origin with eccentricity $e = \frac{3}{2}$ and focus at $(6, 0)$?

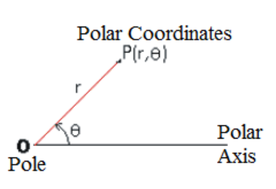
$$c = 6; \qquad \frac{c}{a} = \frac{3}{2} \Rightarrow a = 4; \qquad b^2 = c^2 - a^2 = 20;$$

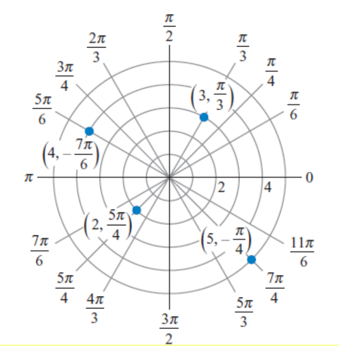
$$\frac{x^2}{16} - \frac{y^2}{20} = 1;$$

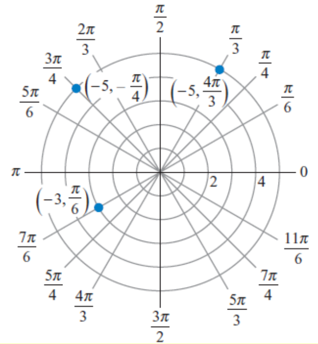
Subsection 4

Introduction to Polar Coordinates

Polar Coordinates







Polar Equations

- A **polar equation** is an equation in r and θ;
- A **solution** to a polar equation is an ordered pair (r, θ) that satisfies the equation;
- The **graph** of a polar equation is the set of all points whose ordered pairs are solutions of the equation;
- What is the graph of the polar equation $\theta = \frac{\pi}{6}$?
- What is the graph of $r = 2$?

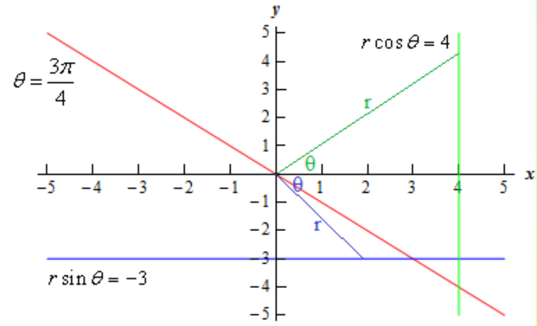
Polar Equation of a Line

The graph of a polar equation $\theta = \alpha$ is a line through the pole at an angle α from the polar axis;

Graph of $r = a$

The graph of a polar equation $r = a$ is a circle with center at the pole and radius a;

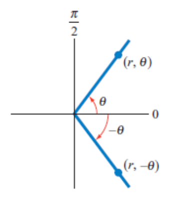
Graphs of $r \sin\theta = a$ and $r \cos\theta = a$

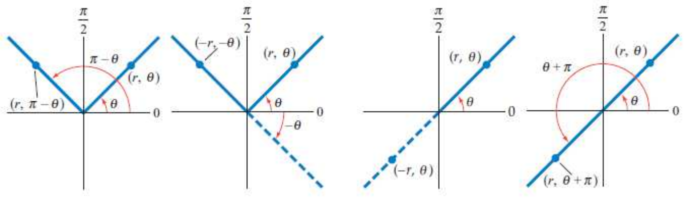


Graphs of $r \sin\theta = a$ and $r \cos\theta = a$

- The graph of $r \sin\theta = a$ is a horizontal line passing through the point $(a, \frac{\pi}{2})$;
- The graph of $r \cos\theta = a$ is a vertical line passing through the point $(a, 0)$;

Symmetries and Tests for Symmetry





Substitution	**Symmetry w.r.t**
$-\theta$ for θ	the line $\theta = 0$
$\pi - \theta$ for θ, $-r$ for r	the line $\theta = 0$
$\pi - \theta$ for θ	the line $\theta = \frac{\pi}{2}$
$-\theta$ for θ, $-r$ for r	the line $\theta = \frac{\pi}{2}$
$-r$ for r	the pole
$\pi + \theta$ for θ	the pole

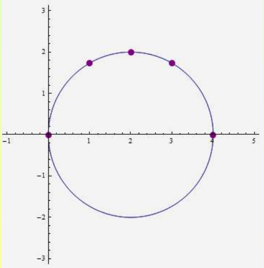
Example of Testing for Symmetry

Substitution	**Symmetry w.r.t**
$-\theta$ for θ	the line $\theta = 0$
$\pi - \theta$ for θ, $-r$ for r	the line $\theta = 0$

- Example: Show that the graph of $r = 4\cos\theta$ is symmetric with respect to $\theta = 0$; Graph the equation;

$$r = 4\cos(-\theta) \quad \Leftrightarrow \quad r = 4\cos\theta;$$

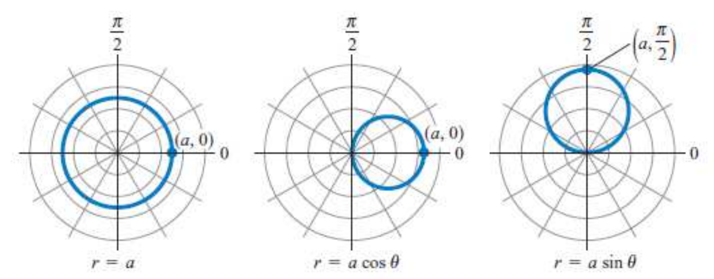
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	4	$2\sqrt{3}$	$2\sqrt{2}$	2	0
θ	0	$-\frac{\pi}{6}$	$-\frac{\pi}{4}$	$-\frac{\pi}{3}$	$-\frac{\pi}{2}$
r	4	$2\sqrt{3}$	$2\sqrt{2}$	2	0



Polar Equations of Circle

Polar Equations of a Circle

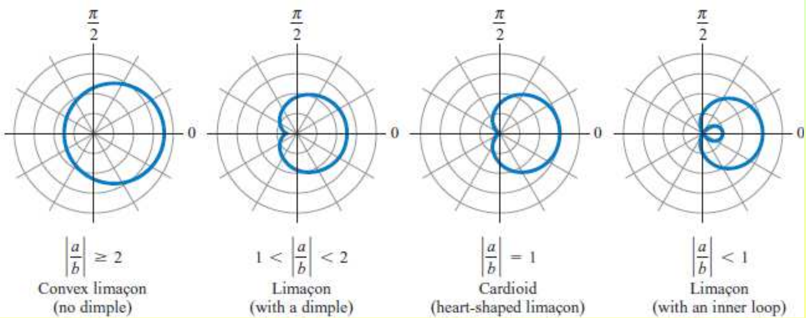
- The graph of $r = a$ is a circle with center at the pole and radius a;
- The graph of $r = a\cos\theta$ is a circle that is symmetric with respect to the line $\theta = 0$;
- The graph of $r = a\sin\theta$ is a circle that is symmetric with respect to the line $\theta = \frac{\pi}{2}$;



Polar Equations of Limaçons

Polar Equations of a Limaçon

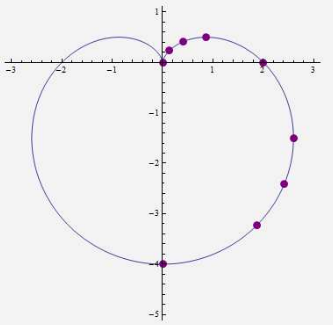
- The graph of the equation $r = a + b\cos\theta$ is a **limaçon** that is symmetric with respect to the line $\theta = 0$;
- The graph of the equation $r = a + b\sin\theta$ is a limaçon that is symmetric with respect to the line $\theta = \frac{\pi}{2}$;
- If $|a| = |b|$, then the graph is called a **cardioid**;



Example of a Limaçon

- Sketch the graph of $r = 2 - 2\sin\theta$;

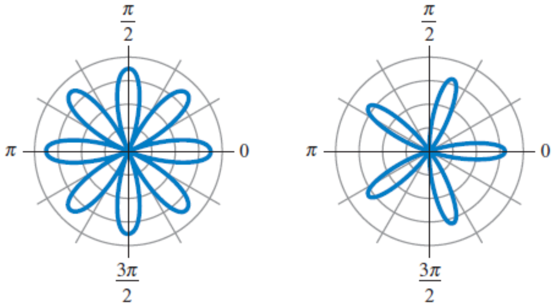
θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0
r	4	$2+\sqrt{3}$	$2+\sqrt{2}$	3	2
θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	
r	1	$2-\sqrt{2}$	$2-\sqrt{3}$	0	



Polar Equations of Rose Curves

Polar Equations of a Roses

The graphs of the equations $r = a\cos n\theta$ and $r = a\sin n\theta$ are **rose curves**; When n is even, the number of petals is $2n$; When n is odd the number of petals is n;



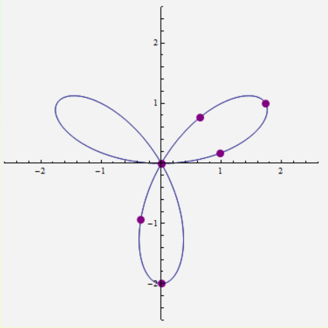
$r = a\cos 4\theta$
$n = 4$ is even, $2n = 8$ petals

$r = a\cos 5\theta$
$n = 5$ is odd, 5 petals

Example of a Rose Curve

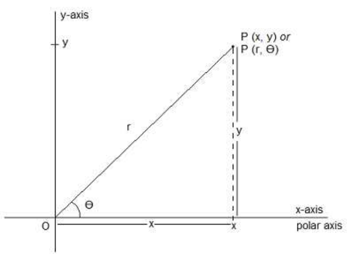
- Sketch the graph of $r = 2\sin 3\theta$;

θ	0	$\frac{\pi}{18}$	$\frac{\pi}{6}$	$\frac{5\pi}{18}$	$\frac{\pi}{3}$	$\frac{7\pi}{18}$	$\frac{\pi}{2}$
r	0	1	2	1	0	-1	-2



Transformations Between Rectangular and Polar

Transformations Between Rectangular and Polar Coordinates



- Given the point (r, θ) in polar coordinates, the transformation equations to change its representation into rectangular coordinates are

$$x = r\cos\theta \quad y = r\sin\theta;$$

- Given the point (x, y) in rectangular coordinates, the transformation equations to change its representation into polar coordinates are

$$r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x},\ x \neq 0,$$

where θ is chosen so that the point lies in the appropriate quadrant;

Transforming Coordinates

- Find the rectangular coordinates of the points whose polar coordinates are:
  - $(6, \frac{3\pi}{4})$;
    $$x = r\cos\theta = 6\cos\tfrac{3\pi}{4} = 6(-\tfrac{\sqrt{2}}{2}) = -3\sqrt{2};$$
    $$y = r\sin\theta = 6\sin\tfrac{3\pi}{4} = 6\tfrac{\sqrt{2}}{2} = 3\sqrt{2};$$
    $$(6, \tfrac{3\pi}{4}) \equiv (-3\sqrt{2}, 3\sqrt{2});$$
  - $(-4, 30^\circ)$;
    $$x = r\cos\theta = -4\cos 30^\circ = -4\cdot\tfrac{\sqrt{3}}{2} = -2\sqrt{3};$$
    $$y = r\sin\theta = -4\sin 30^\circ = -4\cdot\tfrac{1}{2} = -2;$$
    $$(-4, 30^\circ) \equiv (-2\sqrt{3}, -2);$$
- Find the polar coordinates of the point with rectangular coordinates $(-2, -2\sqrt{3})$;
  $$r = \sqrt{x^2+y^2} = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = 4;$$
  $$\tan\theta = \tfrac{y}{x} = \tfrac{-2\sqrt{3}}{-2} = \sqrt{3} \quad \Rightarrow \quad \theta = \tfrac{4\pi}{3};$$
  $$(-2, -2\sqrt{3}) \equiv (4, \tfrac{4\pi}{3});$$

Transforming Equations I

- Find a rectangular form of the equation $r^2 \cos 2\theta = 3$;

$$r^2 \cos 2\theta = 3 \quad \Rightarrow \quad r^2(2\cos^2\theta - 1) = 3 \quad \Rightarrow \quad 2r^2\cos^2\theta - r^2 = 3$$
$$\Rightarrow \quad 2(r\cos\theta)^2 - r^2 = 3 \quad \Rightarrow \quad 2x^2 - (x^2 + y^2) = 3$$
$$\Rightarrow \quad x^2 - y^2 = 3;$$

- Find a rectangular form of the equation $r = 8\cos\theta$;

$$r = 8\cos\theta \quad \Rightarrow \quad r^2 = 8r\cos\theta \quad \Rightarrow \quad x^2 + y^2 = 8x$$
$$\Rightarrow \quad x^2 - 8x + y^2 = 0 \quad \Rightarrow \quad x^2 - 8x + 16 + y^2 = 16$$
$$\Rightarrow \quad (x-4)^2 + y^2 = 4^2;$$

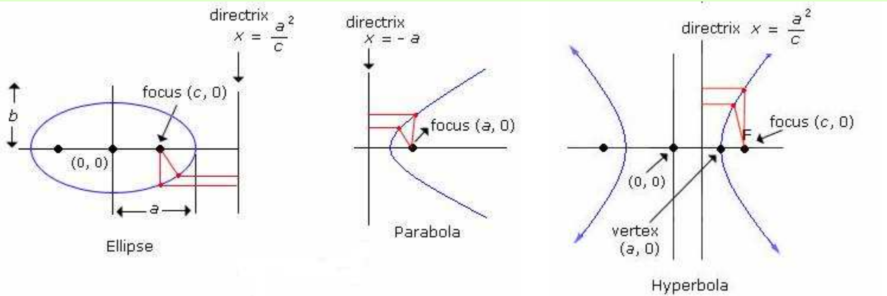
Subsection 5

Polar Equations of the Conics

Focus-Directrix Definitions of the Conics

Focus-Directrix Definitions of the Conics

Let F be a fixed point and D a fixed line on the plane; Consider the set of all points P, such that $\dfrac{d(P,F)}{d(P,D)} = e$, where e is a constant; The graph is a parabola for $e = 1$, an ellipse for $0 < e < 1$, and a hyperbola for $e > 1$.



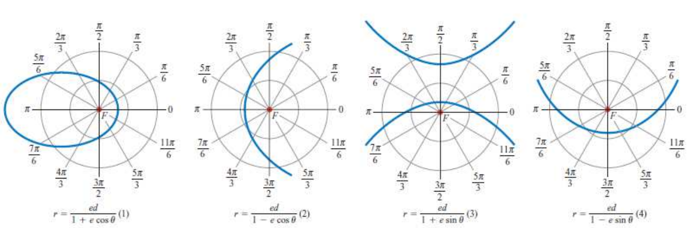
Standard Forms of Polar Equations of the Conics

Standard Forms of Polar Equations of the Conics

Suppose that the pole is the focus of a conic section of eccentricity e, with directrix d units from the focus; Then the equation of the conic is given by one of the following:

	Directrix right or above	Directrix left or below
Vertical Directrix	$r = \frac{ed}{1+e\cos\theta}$	$r = \frac{ed}{1-e\cos\theta}$
Horizontal Directrix	$r = \frac{ed}{1+e\sin\theta}$	$r = \frac{ed}{1-e\sin\theta}$

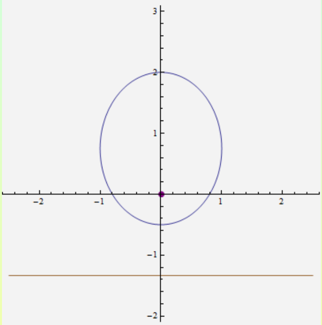
When the equation involves $\cos\theta$, the line $\theta = 0$ is an axis of symmetry; When it involves $\sin\theta$, the line $\theta = \frac{\pi}{2}$ is an axis of symmetry.



Example I

- What type is the conic that is given by the equation $r = \dfrac{4}{5 - 3\sin\theta}$?

$$r = \frac{4}{5 - 3\sin\theta}$$
$$\Rightarrow \quad r = \frac{4}{5(1 - \frac{3}{5}\sin\theta)}$$
$$\Rightarrow \quad r = \frac{\frac{4}{5}}{1 - \frac{3}{5}\sin\theta};$$



Thus, $e = \frac{3}{5} < 1$, showing that this is the equation of an ellipse;

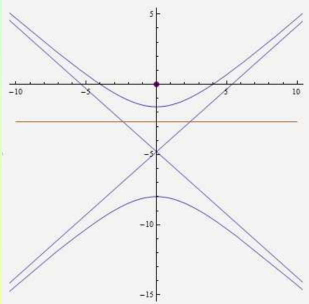
Note that the fact that the denominator has a "-" and a sine immediately reveals that the directrix is horizontal and lies below the focus located at the pole.

Example II

- Describe and sketch the graph of $r = \dfrac{8}{2 - 3\sin\theta}$;

$$r = \frac{8}{2-3\sin\theta} \;\Rightarrow\; r = \frac{8}{2(1-\frac{3}{2}\sin\theta)}$$
$$\Rightarrow \quad r = \frac{4}{1-\frac{3}{2}\sin\theta};$$

Thus, $e = \frac{3}{2} > 1$, showing that this is the equation of a hyperbola;

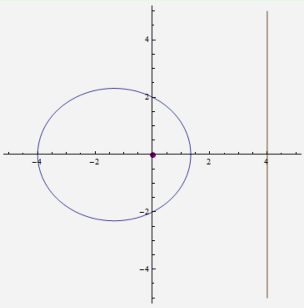


The fact that the denominator has a "-" and a sine immediately reveals that the directrix is horizontal and lies below the focus. Moreover, $ed = \frac{3}{2}d = 4 \Rightarrow d = \frac{8}{3}$;

Example III

- Describe and sketch the graph of $r = \dfrac{4}{2+\cos\theta}$;

$$r = \frac{4}{2+\cos\theta} \;\Rightarrow\; r = \frac{4}{2(1+\frac{1}{2}\cos\theta)}$$
$$\Rightarrow \quad r = \frac{2}{1+\frac{1}{2}\cos\theta};$$

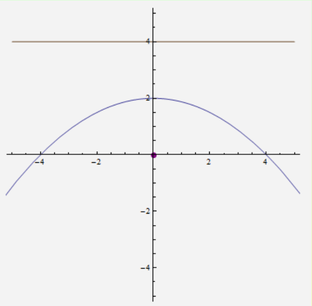


Thus, $e = \frac{1}{2} < 1$, showing that this is the equation of an ellipse;

The fact that the denominator has a "+" and a cosine immediately reveals that the directrix is vertical and lies to the right of the focus. Moreover, $ed = \frac{1}{2}d = 2 \Rightarrow d = 4$;

Example IV

- Find the polar equation of a parabola with vertex at $(2, \frac{\pi}{2})$ and focus at the pole;



The directrix is horizontal and lies above the pole; Therefore, the equation must involve the sine function and have a "+" sign, i.e., it is of the form $r = \frac{ed}{1+e\sin\theta}$; Since the conic is a parabola, $e = 1$; Since the distance from the focus to the directrix is 4, we have $d = 4$;

Therefore, the equation must be $r = \dfrac{4}{1+\sin\theta}$;

Subsection 6

Parametric Equations

Curves and Parametric Equations

Curve and Parametric Equations

Given an interval I, a **curve** is a set of ordered pairs (x, y), where

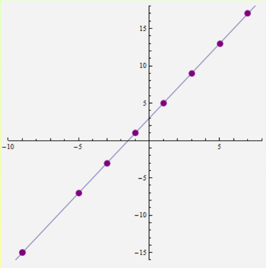
$$x = f(t), \qquad y = g(t), \qquad \text{for } t \in I;$$

The variable t is called the **parameter** and the equations $x = f(t)$ and $y = g(t)$ the **parametric equations** of the curve.

- Example: Consider the equations $\begin{cases} x = 2t - 1 \\ y = 4t + 1 \end{cases}$ for $t \in (-\infty, \infty)$;

  Plot a few points to reveal the curve:

t	$x = 2t - 1$	$y = 4t + 1$	(x, y)
-2	-5	-7	$(-5, -7)$
-1	-3	-3	$(-3, -3)$
0	-1	1	$(-1, 1)$
1	1	5	$(1, 5)$
2	3	9	$(3, 9)$

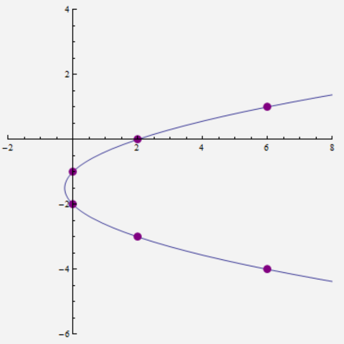


Example

Consider the equations $\begin{cases} x = t^2 + t \\ y = t - 1 \end{cases}$ for $t \in (-\infty, \infty)$;

Plot a few points to reveal the curve:

t	$x = t^2 + t$	$y = t - 1$	(x, y)
-3	6	-4	$(6, -4)$
-2	2	-3	$(2, -3)$
-1	0	-2	$(0, -2)$
0	0	-1	$(0, -1)$
1	2	0	$(2, 0)$
2	6	1	$(6, 1)$

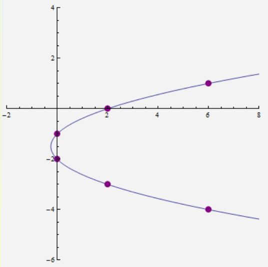


Eliminating the Parameter

- Consider again the equations $\begin{cases} x &= t^2 + t \\ y &= t - 1 \end{cases}$ for $t \in (-\infty, \infty)$;
  Solve the second for t: $t = y + 1$;
  Plug in this value in for t in the first equation:

  $$x = (y+1)^2 + (y+1) \Rightarrow x = y^2 + 2y + 1 + y + 1 \Rightarrow x = y^2 + 3y + 2;$$

  This clearly represents a parabola in Cartesian coordinates as we saw by plotting the parametric curve:



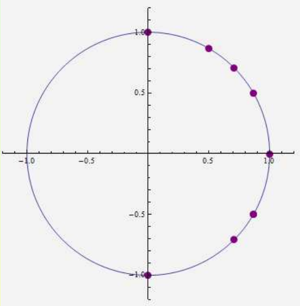
Example I

Eliminate the parameter and sketch the curve of the parametric equations
$\begin{cases} x = \sin t \\ y = \cos t \end{cases}$ for $0 \leq t \leq 2\pi$;

Square the first equation $x^2 = \sin^2 t$;
Square the second equation $y^2 = \cos^2 t$;
Add the two equations

$$x^2 + y^2 = \sin^2 t + \cos^2 t = 1;$$

Thus, we have a circle of radius 1 centered at the origin:

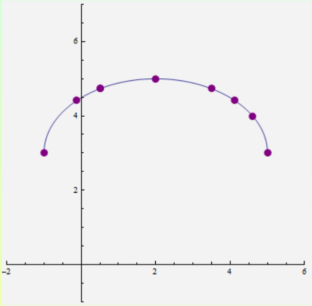


Example II

Eliminate the parameter and sketch the curve of the parametric equations

$$\begin{cases} x = 2 + 3\cos t \\ y = 3 + 2\sin t \end{cases} \quad \text{for } 0 \leq t \leq \pi;$$

Solve the first equation for $\cos t$ and square: $\cos^2(t) = (\frac{x-2}{3})^2$; Solve the second equation for $\sin t$ and square: $\sin^2 t = (\frac{y-3}{2})^2$; Add the two equations



$$\cos^2 t + \sin^2 t = (\frac{x-2}{3})^2 + (\frac{y-3}{2})^2$$

$$\Rightarrow \quad \frac{(x-2)^2}{9} + \frac{(y-3)^2}{4} = 1;$$

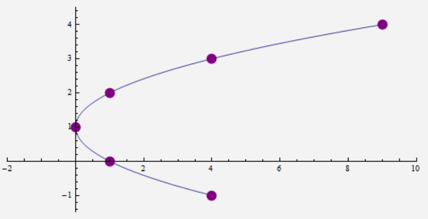
Thus, we have an ellipse with center $(2, 3)$ and length of major axis 6: Because $0 \leq t \leq \pi$, we actually get only the upper half of the ellipse!

Time as a Parameter

Consider the equations $\begin{cases} x &= t^2 \\ y &= t+1 \end{cases}$ for $-2 \leq t \leq 3$;

Plot a few points to reveal the curve:

t	$x = t^2$	$y = t+1$	(x, y)
-2	4	-1	$(4, -1)$
-1	1	0	$(1, 0)$
0	0	1	$(0, 1)$
1	1	2	$(1, 2)$
2	4	3	$(4, 3)$
3	9	4	$(9, 4)$

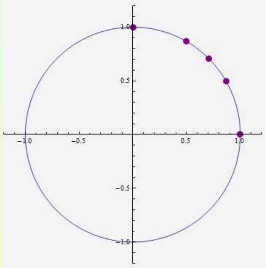


Example

Suppose the equations $\begin{cases} x = \sin t \\ y = \cos t \end{cases}$, for $0 \le t \le 2\pi$, describe the motion of a point in a plane; Describe the motion of the point.

Plot a few points to reveal the curve:

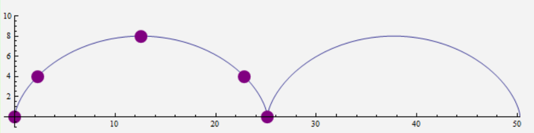
t	$x = \sin t$	$y = \cos t$	(x, y)
0	0	1	$(0, 1)$
$\frac{\pi}{2}$	1	0	$(1, 0)$
π	0	-1	$(0, -1)$
$\frac{3\pi}{2}$	-1	0	$(-1, 0)$
2π	0	1	$(0, 1)$

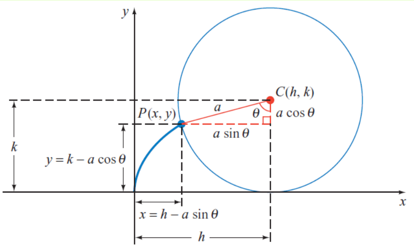


The point starts at $(0, 1)$ and rotates clockwise around the unit circle centered at the origin until it reaches back to its original position.

The Cycloid

- A **cycloid** is the curve traced by a point on the circumference of a circle of radius a that is rolling on a straight line without slipping;





$x = h - a\sin\theta; \quad y = k - a\cos\theta;$

Note $k = a$ and $h = a\theta$; Therefore, the parametric equations describing the cycloid are

$$\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{cases}, \quad \theta \geq 0;$$