Numerical Analysis / Numerical Methods

Spring 2023 Lectures 3-4 Review

Review of interpolation and approximation

PURPOSES OF INTERPOLATION

1. Replace a set of data points

$$\{(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\}$$

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2. Approximate functions (at least continuous) with simpler ones, usually polynomials or ‘piecewise polynomials’. Here interpolation is used to form polynomials that accurately approximate continuous functions, and next question is how accurately we can do it?

Polynomial interpolation

Once data set with $n+1$ points $\{(x_i, y_i)\}_{i=0}^{n}$ is given, there is a UNIQUE polynomial $P_n(x)$ of degree at most n that passes them:

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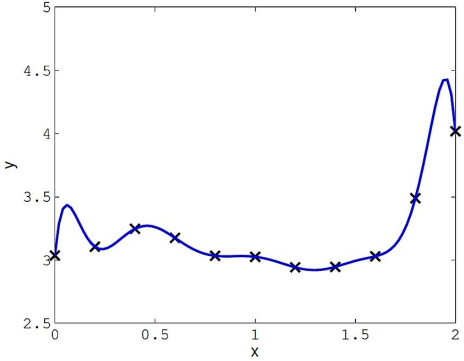

Polynomial of degree 10 passing 11 data points

Polynomial interpolation

This UNIQUE polynomial can be obtained by (see Lecture 3 Reading):

1. Lagrange formula:

$$P_n(x) = \sum_{i=0}^{n} y_i L_i(x), \quad \text{where } L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}.$$

2. Newton difference formula

$$P_n(x) = P_{n-1}(x) + \underbrace{f[x_0, x_1, x_2, \ldots, x_n]}_{\text{Divided difference}}(x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$

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Never use Lagrange formula for practical reasons!

So, we have essentially solved the polynomial interpolation problem of discrete data perfectly.

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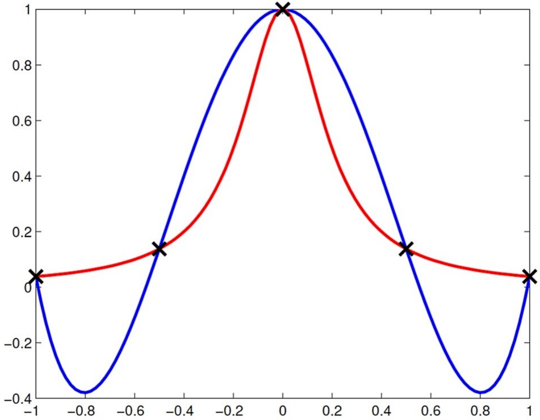
Another approach is by interpolation:

Consider the partition of $[a, b]$: $\quad a \leqslant x_0 < x_1 < x_2 < \cdots < x_n \leqslant b$

and the data points $\quad \{(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))\}$.

Construct the interpolation polynomial passing these data points to get $P_n(x)$.

Interpolation for Function Approximation



Function $f(x) = \frac{1}{1+25x^2}$ (red) and polynomial $P_4(x)$ (blue) passing 5 data points (black)

Polynomial approximation clearly depends on the interpolation partition considered.

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Error for polynomial interpolation is given by:

$$f(x) - P_n(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_n)}{(n+1)!} f^{(n+1)}(\xi) = \frac{\Psi_n(x)}{(n+1)!} f^{(n+1)}(\xi),$$

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where ξ is some point on $[a, b]$.

$$\left| f(x) - P_n(x) \right| \leqslant \frac{\left| \Psi_n(x) \right|}{(n+1)!} \cdot \max_{x \in [a,b]} \left| f^{(n+1)}(x) \right|.$$

Behavior (distribution and magnitude) of error depends on the shape of function $\Psi_n(x)$ (see Lecture 3 Reading)

Interpolation for Function Approximation

Generally, if we have more interpolation points, and hence the degree of the interpolation polynomial is higher, then the error will be smaller:

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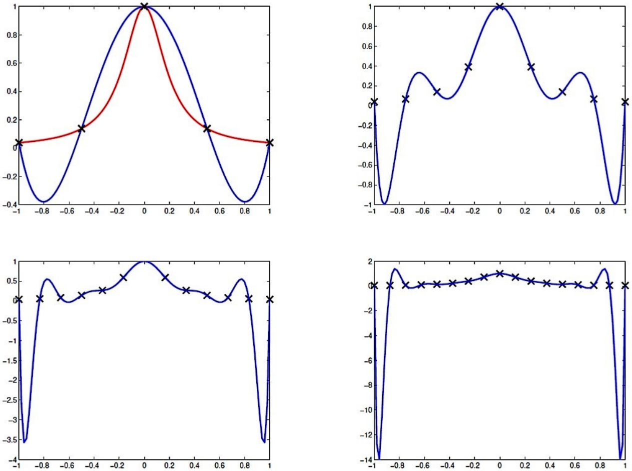
But ... interpolation points should be chosen appropriately.

If the interpolation partition is uniform (equally spaced),
then pathological Runge's example might happen (see Lecture 3 Reading).

Another Runge's Example



$f(x) = \frac{1}{1+25x^2}$ (red) and its interpolants on evenly spaced points P_4, P_8, P_{12} and P_{16}

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Best means the smallest error measure:

$$\rho_n(f) = \min_{P_n} \|f(x) - P_n(x)\|_\infty = \min_{P_n} \max_{x\in[a,b]} |f(x) - P_n(x)|.$$

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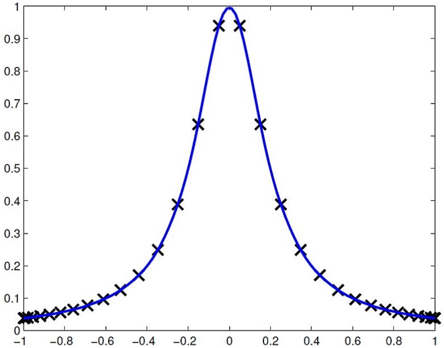
Answer is YES!
(Best approximation, Lecture 4 Reading), but practical implementation is cumbersome.

Near-Minimax Approximation

A practical alternative for best approximation is the near-minimax approximation (known as Chebyshev approximation) based on Chebyshev polynomials (Lecture 4).



Chebyshev approximation of the function $f(x) = \frac{1}{1+25x^2}$

Notice that, using Chebyshev points, we overcame the pathology of Runge's Example.

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Something that would measure the quality of interpolation points. This is provided by so-called **Lebesgue constant**.

Lebesgue constant

Definition

Let $\mathcal{P}$ denote a set of interpolation points:

$$\mathcal{P} = \{x_0, x_1, \ldots, x_n\} \subset [a, b].$$

The Lebesgue constant of $\mathcal{P}$ is defined as,

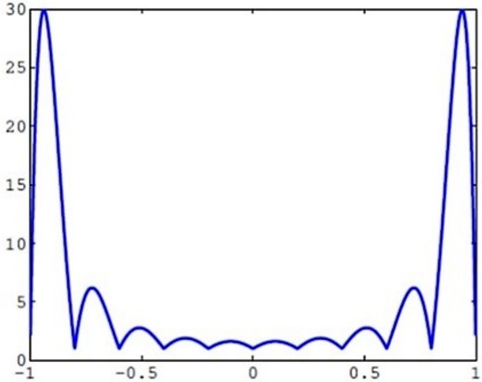
$$\Lambda_n(\mathcal{P}) = \max_{x \in [a,b]} \sum_{k=0}^{n} |L_k(x)|,$$

where $L_k(x)$ are the Lagrange basis functions associated with interpolation set $\mathcal{P}$.

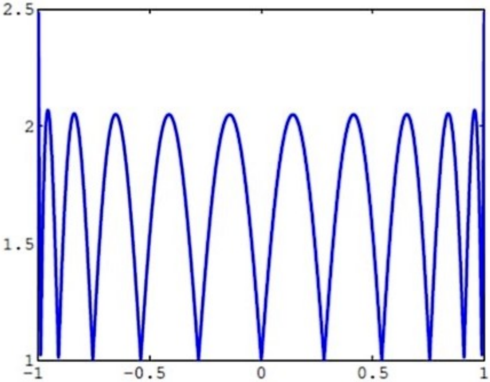
Small Lebesgue constant means that our interpolation can't be much worse that the best polynomial approximation!

Lebesgue constant

Plot of $\sum_{k=0}^{10} |L_k(x)|$ for $\mathcal{P}_{unif}$ and $\mathcal{P}_{cheb}$ with 11 data points in $[-1, 1]$.



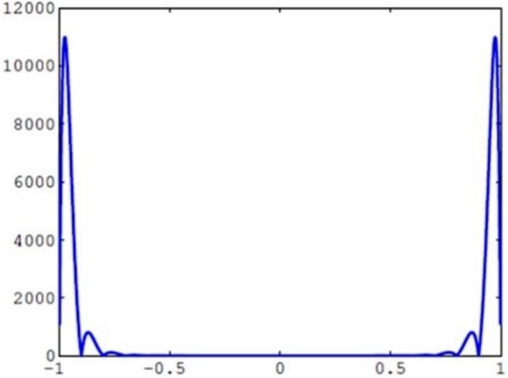
$\Lambda_{10}(\mathcal{P}_{unif}) \approx 29.9$



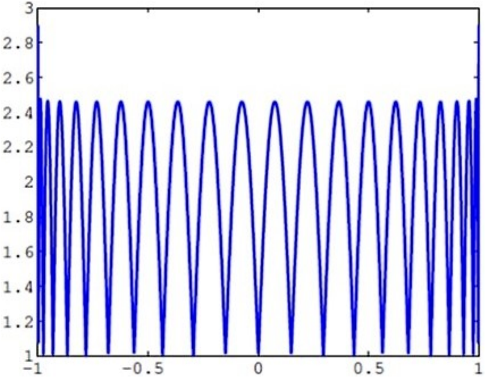
$\Lambda_{10}(\mathcal{P}_{cheb}) \approx 2.49$

Lebesgue constant

Plot of $\sum_{k=0}^{20} |L_k(x)|$ for $\mathcal{P}_{unif}$ and $\mathcal{P}_{cheb}$ with 21 data points in $[-1, 1]$.



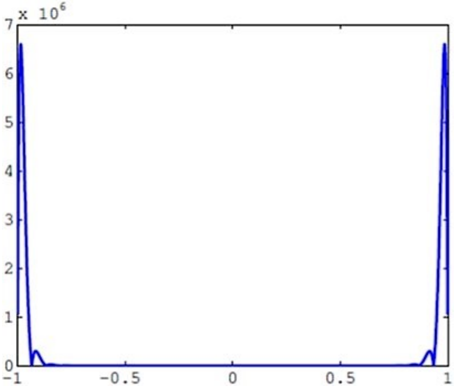
$\Lambda_{20}(\mathcal{P}_{unif}) \approx 10\,987$

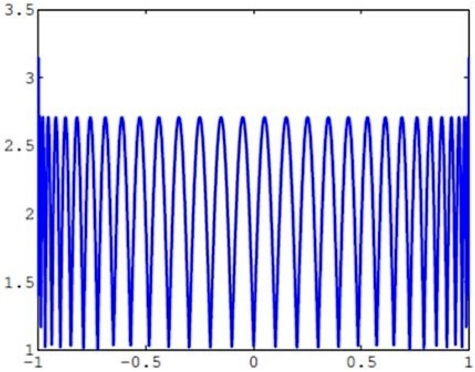


$\Lambda_{20}(\mathcal{P}_{cheb}) \approx 2.9$

Lebesgue constant

Plot of $\sum_{k=0}^{30} |L_k(x)|$ for $\mathcal{P}_{unif}$ and $\mathcal{P}_{cheb}$ with 31 data points in $[-1, 1]$.





$\Lambda_{30}(\mathcal{P}_{unif}) \approx 6\,600\,000$

$\Lambda_{30}(\mathcal{P}_{cheb}) \approx 3.15$

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$$\Lambda_n(\mathcal{P}_{unif}) \approx \frac{2^n}{e\, n \log n}, \qquad \text{BAD!}$$

whereas

$$\Lambda_n(\mathcal{P}_{cheb}) < \frac{2}{\pi} \log(n+1) + 1. \qquad \text{GOOD!}$$

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2. **Polynomial interpolation purpose 2 (approximating functions)**
   - For a given set of interpolation points, use point 1 methodology to build the approximation $P_n(x)$.
   - Interpolation points play an important role on the size of error $\|f(x) - P_n(x)\|_\infty$ (keep in mind Runge's example).

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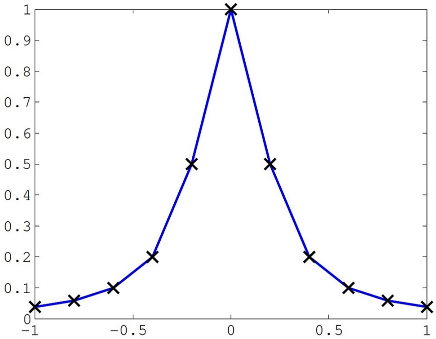

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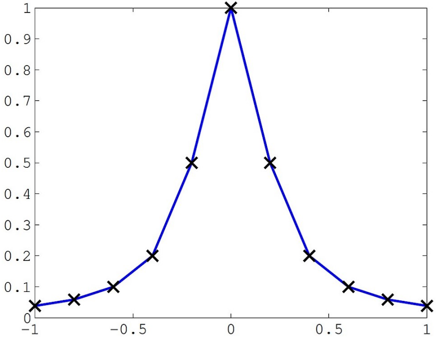

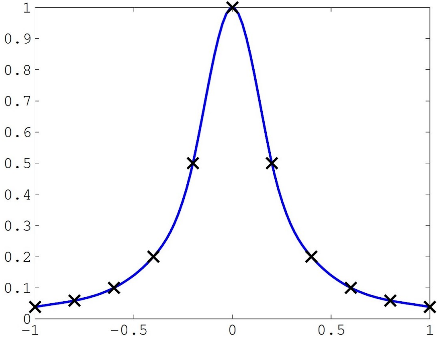

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