Linear Algebra and Analytic Geometry

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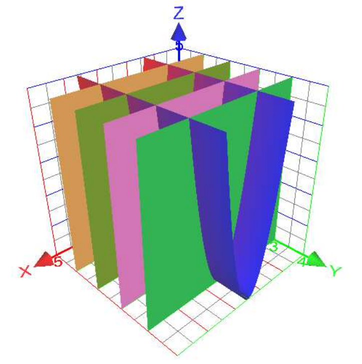
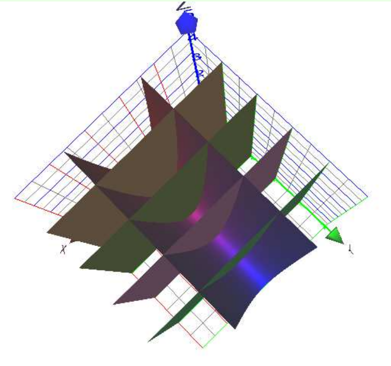
Subsection 6

A Survey of Quadratic Surfaces

Traces or Cross-Sections

- The curves of intersection of a given surface with planes parallel to the coordinate planes are called **traces** or **cross-sections** of the surface.

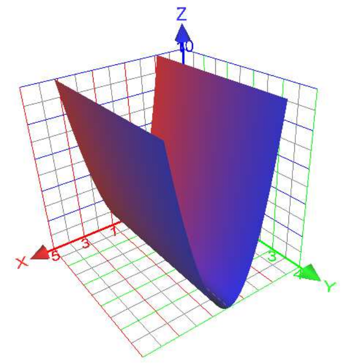


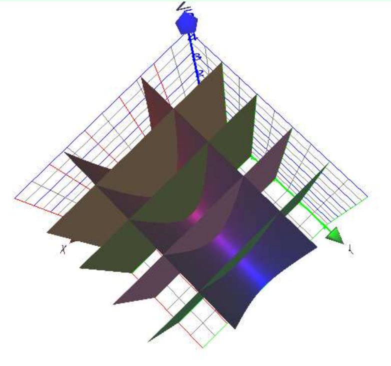


- Traces are very useful in sketching the graph of a 3-dimensional surface.

Parabolic Cylinders

- Consider the surface $z = x^2$.

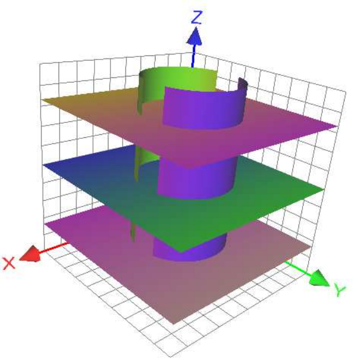




- For planes $y = k$ parallel to the coordinate xz-plane the traces are all curves with equations $z = x^2$, i.e., parabolas with vertex at the xz-origin and opening up.
- The surface $z = x^2$ is called a **parabolic cylinder**.

Cylinders

- Consider the surface $x^2 + y^2 = 1$.



- For planes $z = k$ parallel to the coordinate xy-plane the traces are all curves with equations $x^2 + y^2 = 1$, i.e., circles with center the xy-origin and radius 1.
- The surface $x^2 + y^2 = 1$ is called a **cylinder**.

Quadric Surfaces

- Quadric surfaces are the three dimensional analogs of the two dimensional conic sections, i.e., of parabolas, ellipses and hyperbolas.
- The general equation of a **quadric surface** is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0;$$

- If one translates and rotates the quadric surface, then its equation may be simplified to one of the forms

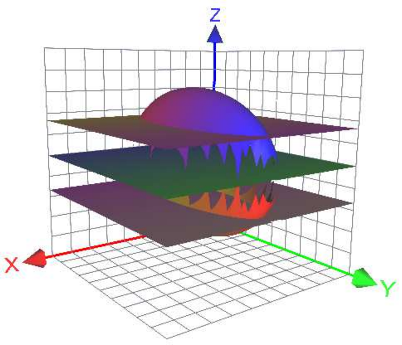
$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0.$$

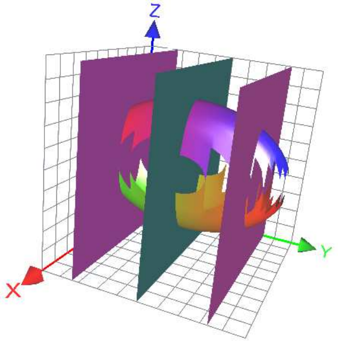
Example: What are the traces of the quadric $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$ parallel to the coordinate planes?

On plane $z = k$, the trace is $x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4}$, which is the equation of an ellipse.

The quadric $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$ (Cont'd)

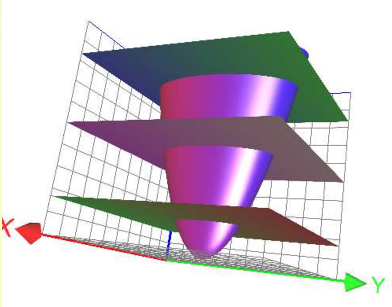
- On plane $y = k$, the trace is $x^2 + \dfrac{z^2}{4} = 1 - \dfrac{k^2}{9}$, which is the equation of an ellipse. On plane $x = k$, the trace is $\dfrac{y^2}{9} + \dfrac{z^2}{4} = 1 - k^2$, which is also the equation of an ellipse. Since all traces are ellipses, this surface is called an **ellipsoid**.

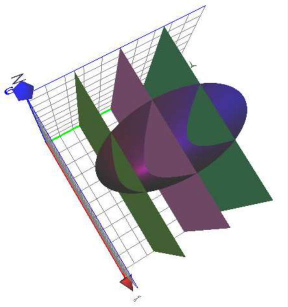




The Quadric Surface $z = 4x^2 + y^2$

- What are the traces of the quadric $z = 4x^2 + y^2$ parallel to the coordinate planes?
  On plane $z = k$, the trace is $x^2 + \frac{y^2}{4} = \frac{k}{4}$, which is the equation of an ellipse. On plane $y = k$, the trace is $z = 4x^2 + k^2$, which is the equation of a parabola. On plane $x = k$, the trace is $z = y^2 + 4k^2$, which is also the equation of a parabola. This surface is called an **elliptic paraboloid**.

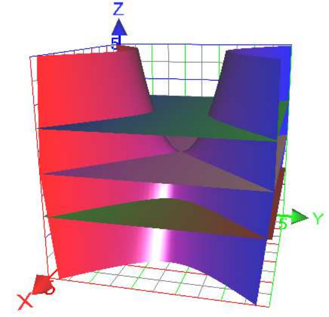


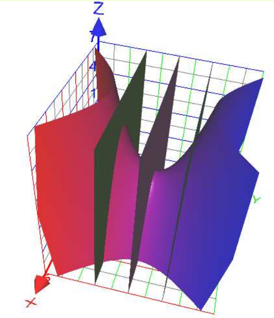


The Quadric Surface $z = y^2 - x^2$

- What are the traces of the quadric $z = y^2 - x^2$ parallel to the coordinate planes?

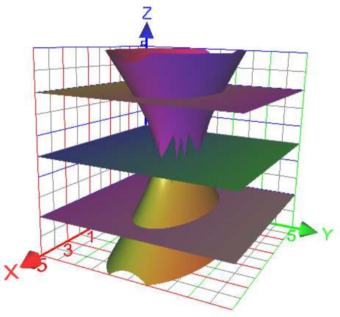
  On plane $z = k$, the trace is $y^2 - x^2 = k$, which is the equation of a hyperbola. On plane $y = k$, the trace is $z = -x^2 + k^2$, which is the equation of a parabola. On plane $x = k$, the trace is $z = y^2 - k^2$, which is also the equation of a parabola; This surface is called an **hyperbolic paraboloid**.

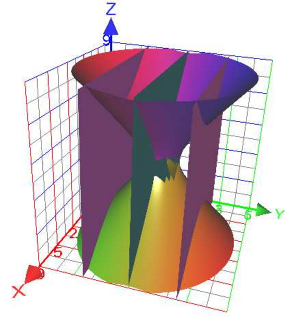




The Quadric Surface $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$

- What are the traces of the quadric $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$ parallel to the coordinate planes?
  On plane $z = k$, the trace is $\frac{x^2}{4} + y^2 = 1 + \frac{k^2}{4}$, which is the equation of an ellipse. On plane $y = k$, the trace is $\frac{x^2}{4} - \frac{z^2}{4} = 1 - k^2$, which is the equation of a hyperbola. On plane $x = k$, the trace is $y^2 - \frac{z^2}{4} = 1 - \frac{k^2}{4}$, which is also the equation of a hyperbola; This surface is called an **hyperboloid of one sheet**.



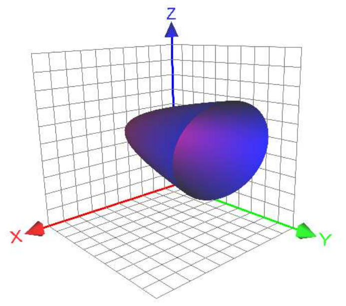


Types of Quadric Surfaces

- **Ellipsoids** with equations $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.
- **Elliptic Paraboloids** with equations $\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$.
- **Hyperbolic Paraboloids** with equations $\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$.
- **Cones** with equations $\dfrac{z^2}{c^2} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$.
- **Hyperboloids of One Sheet** with equations $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$.
- **Hyperboloid of Two Sheets** with equations $-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

Identifying a Quadric Surface

- Classify the quadric surface $x^2 + 2z^2 - 6x - y + 10 = 0$.
  Rewrite $x^2 - 6x - y + 2z^2 = -10$. Complete x-square
  $(x-3)^2 - y + 2z^2 = -1$. Separate square terms from linear terms
  $y - 1 = (x-3)^2 + 2z^2$. Divide by 2 and put in standard form
  $\dfrac{y-1}{(\sqrt{2})^2} = \dfrac{(x-3)^2}{(\sqrt{2})^2} + z^2$. This has form of an elliptic Paraboloid with vertex $(3, 1, 0)$ opening in the positive y-direction.



Subsection 7

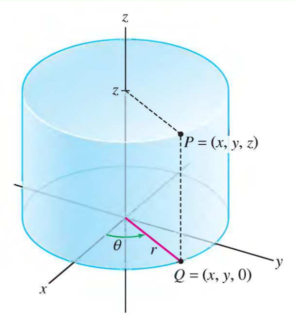
Cylindrical and Spherical Coordinates

Cylindrical Coordinates

- In cylindrical coordinates, we replace the x- and y-coordinates of a point $P = (x, y, z)$ by polar coordinates.
- The **cylindrical coordinates** of $P = (x, y, z)$ are

$$(r, \theta, z),$$

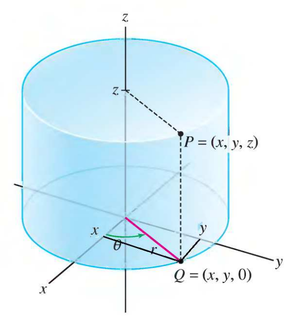
where (r, θ) are polar coordinates of the projection $Q = (x, y, 0)$ of P onto the xy-plane. We usually assume $r \geq 0$.



- Note that the points at fixed distance r from the z-axis make up a cylinder, hence the name "cylindrical coordinates".

Cylindrical and Rectangular

- We convert between rectangular and cylindrical coordinates using the familiar rectangular-polar formulas and we usually assume $r > 0$.



Cylindrical to Rectangular

$$\begin{aligned} x &= r\cos\theta; \\ y &= r\sin\theta; \\ z &= z. \end{aligned}$$

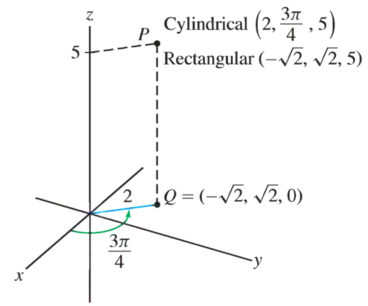
Rectangular to Cylindrical

$$\begin{aligned} r &= \sqrt{x^2+y^2}; \\ \tan\theta &= \frac{y}{x}; \\ z &= z. \end{aligned}$$

From Cylindrical to Rectangular

- Find the rectangular coordinates of the point P with cylindrical coordinates $(r, \theta, z) = (2, \frac{3\pi}{4}, 5)$.

$$\begin{aligned} x &= r\cos\theta = 2\cos\frac{3\pi}{4} \\ &= 2(-\frac{\sqrt{2}}{2}) = -\sqrt{2}; \\ y &= r\sin\theta = 2\sin\frac{3\pi}{4} \\ &= 2(\frac{\sqrt{2}}{2}) = \sqrt{2}. \end{aligned}$$



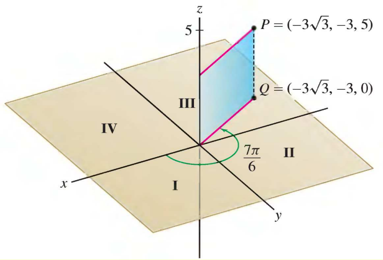
The z-coordinate is unchanged. So $(x, y, z) = (-\sqrt{2}, \sqrt{2}, 5)$.

From Rectangular to Cylindrical

- Find cylindrical coordinates for the point with rectangular coordinates $(x, y, z) = (-3\sqrt{3}, -3, 5)$.

  We have

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-3\sqrt{3})^2 + (-3)^2} = 6. \end{aligned}$$



The angle θ satisfies $\tan\theta = \frac{y}{x} = \frac{-3}{-3\sqrt{3}} = \frac{1}{\sqrt{3}}$. So $\theta = \frac{\pi}{6}$ or $\frac{7\pi}{6}$. The correct choice is $\theta = \frac{7\pi}{6}$ because the projection $Q = (-3\sqrt{3}, -3, 0)$ lies in the third quadrant.

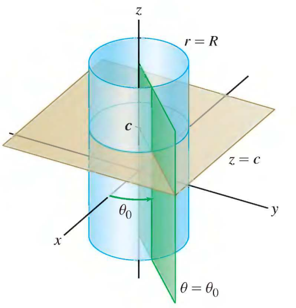
The cylindrical coordinates are $(r, \theta, z) = (6, \frac{7\pi}{6}, 5)$.

Level Surfaces of Cylindrical Coordinates

- The level surfaces of a coordinate system are the surfaces obtained by setting one of the coordinates equal to a constant.
  - In rectangular coordinates, the level surfaces are the planes $x = x_0$, $y = y_0$, and $z = z_0$.
  - In cylindrical coordinates, the level surfaces come in three types.

Level Surfaces in Cylindrical Coordinates:

- $r = R$: Cylinder of radius R with the z-axis as axis of symmetry;
- $\theta = \theta_0$: Half-plane through the z-axis making an angle θ_0 with the xz-plane;
- $z = c$: Horizontal plane at height c.



Equations in Cylindrical Coordinates

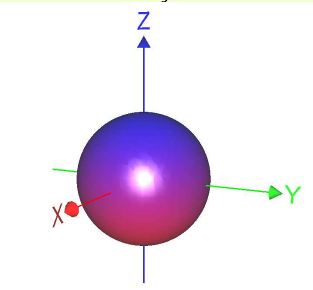
- Find an equation of the form $z = f(r, \theta)$ for the surfaces

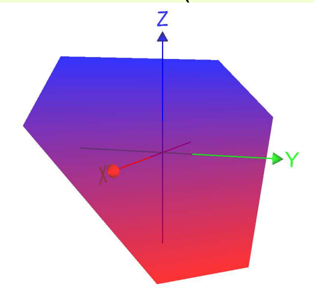
$$(a)\ x^2 + y^2 + z^2 = 9; \quad (b)\ x + y + z = 1.$$

We use the formulas $x^2 + y^2 = r^2$, $x = r\cos\theta$, $y = r\sin\theta$.

(a) The equation $x^2 + y^2 + z^2 = 9$ becomes $r^2 + z^2 = 9$, or $z = \pm\sqrt{9 - r^2}$. This is a sphere of radius 3.

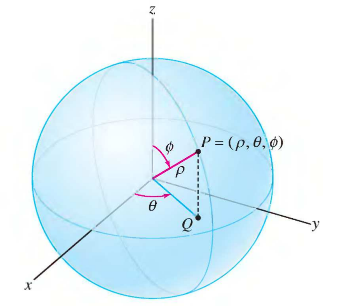
(b) The plane $x + y + z = 1$ becomes $z = 1 - x - y = 1 - r\cos\theta - r\sin\theta$ or $z = 1 - r(\cos\theta + \sin\theta)$.





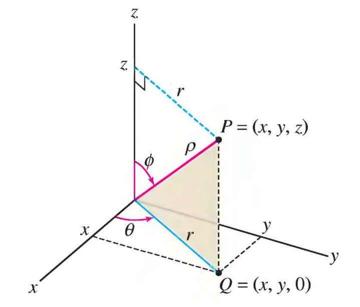
Spherical Coordinates

- Spherical coordinates make use of the fact that a point P on a sphere of radius ρ is determined by two angular coordinates θ and ϕ:
  - θ is the polar angle of the projection Q of P onto the xy-plane;
  - ϕ is the **angle of declination**, which measures how much the ray through P declines from the vertical.



Thus P is determined by the triple (ρ, θ, ϕ), which are called **spherical coordinates**.

Spherical and Rectangular



Spherical to Rectangular

$$\begin{aligned} x &= r\cos\theta = \rho\sin\phi\cos\theta; \\ y &= r\sin\theta = \rho\sin\phi\sin\theta; \\ z &= \rho\cos\phi. \end{aligned}$$

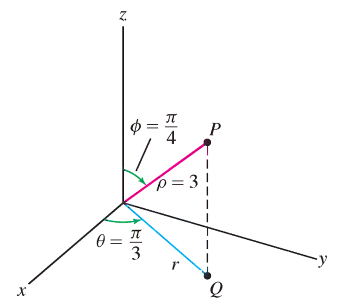
Rectangular to Spherical

$$\begin{aligned} \rho &= \sqrt{x^2+y^2+z^2}; \\ \tan\theta &= \frac{y}{x}; \\ \cos\phi &= \frac{z}{\rho}. \end{aligned}$$

From Spherical to Rectangular

- Find the rectangular coordinates of $P = (\rho, \theta, \phi) = (3, \frac{\pi}{3}, \frac{\pi}{4})$, and find the radial coordinate r of its projection Q onto the xy-plane.

$$\begin{aligned}
x &= \rho \sin\phi \cos\theta \\
&= 3 \sin\tfrac{\pi}{4} \cos\tfrac{\pi}{3} \\
&= 3\tfrac{\sqrt{2}}{2}\tfrac{1}{2} = \tfrac{3\sqrt{2}}{4}. \\
y &= \rho \sin\phi \sin\theta \\
&= 3 \sin\tfrac{\pi}{4} \sin\tfrac{\pi}{3} \\
&= 3\tfrac{\sqrt{2}}{2}\tfrac{\sqrt{3}}{2} = \tfrac{3\sqrt{6}}{4}. \\
z &= \rho \cos\phi \\
&= 3\cos\tfrac{\pi}{4} = \tfrac{3\sqrt{2}}{2}.
\end{aligned}$$

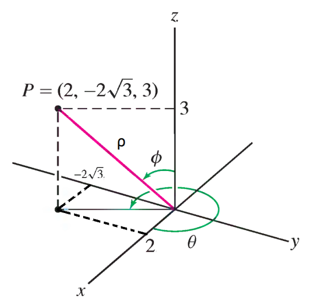


Now consider the projection $Q = (x, y, 0) = (\frac{3\sqrt{2}}{4}, \frac{3\sqrt{6}}{4}, 0)$. The radial coordinate r of Q is $r = \rho \sin\phi = 3\sin\frac{\pi}{4} = \frac{3\sqrt{2}}{2}$.

From Rectangular to Spherical

- Find the spherical coordinates of the point $P = (x, y, z) = (2, -2\sqrt{3}, 3)$.

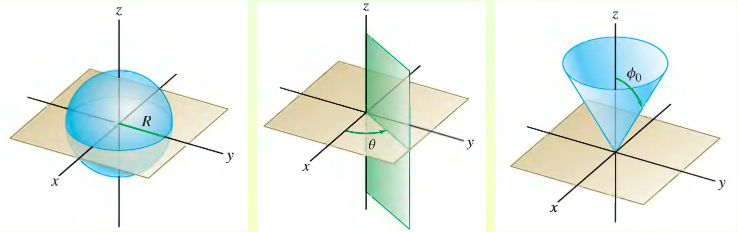
  The radial coordinate is $\rho = \sqrt{2^2 + (-2\sqrt{3})^2 + 3^2} = \sqrt{25} = 5$.
  The angular coordinate θ satisfies $\tan\theta = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$. Thus, $\theta = \frac{2\pi}{3}$ or $\frac{5\pi}{3}$. Since the point $(x, y) = (2, -2\sqrt{3})$ lies in the fourth quadrant, the correct choice is $\theta = \frac{5\pi}{3}$.



  Finally, $\cos\phi = \frac{z}{\rho} = \frac{3}{5}$. Thus, $\phi = \cos^{-1}\frac{3}{5}$.
  Therefore, P has spherical coordinates $(5, \frac{5\pi}{3}, \cos^{-1}\frac{3}{5})$.

Level Surfaces of Cylindrical Coordinates

- There are three types of level surfaces in spherical coordinates.
  - $\rho = R$: Sphere of radius R;
  - $\theta = \theta_0$: Vertical half-plane at angle θ_0 from x-axis;
  - If $\phi \neq 0, \frac{\pi}{2}, \pi$, $\phi = \phi_0$ is the right circular cone consisting of points P such that $\overline{OP}$ makes an angle ϕ_0 with the z-axis.



There are three exceptional cases:
- $\phi = \frac{\pi}{2}$ defines the xy-plane;
- $\phi = 0$ is the positive z-axis;
- $\phi = \pi$ is the negative z-axis.

Equations in Spherical

- Find an equation of the form $\rho = f(\theta, \phi)$ for the following surfaces:

$$(a)\ x^2 + y^2 + z^2 = 9 \quad (b)\ z = x^2 - y^2.$$

(a) The equation $x^2 + y^2 + z^2 = 9$ defines the sphere of radius 3 centered at the origin. We know $\rho^2 = x^2 + y^2 + z^2$. So the equation in spherical coordinates is $\rho = 3$.

(b) To convert $z = x^2 - y^2$ to spherical coordinates, we substitute the formulas for x, y, and z in terms of ρ, θ, and ϕ:

$$\begin{aligned} &\rho \cos\phi = (\rho \sin\phi \cos\theta)^2 - (\rho \sin\phi \sin\theta)^2 \\ &\Rightarrow\ \cos\phi = \rho \sin^2\phi (\cos^2\theta - \sin^2\theta) \\ &\Rightarrow\ \cos\phi = \rho \sin^2\phi \cos 2\theta. \end{aligned}$$

Solving for ρ, we obtain

$$\rho = \frac{\cos\phi}{\sin^2\phi \cos 2\theta}.$$