Mathematical analysis I

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Subsection 6

Power Series

$$1 + x + x^2 + \dots + x^n + \dots$$

$$c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots =$$

$$= \sum_{n=0}^{\infty} c_n x^n \qquad \text{power series} \qquad (1)$$

$$\sum_{n=0}^{\infty} c_n x_0^n \qquad \text{number series}$$

the set of all values x_0

Theorem. If the power series (1) is convergent for x=a ($a \neq 0$)

then the series is absolutely convergent for any x, |x|<|a|

If the power series (1) is divergent for x=b, then this series is divergent for any x, |x|>|b|

Proof. $\sum_{n=0}^{\infty} c_n x^n$, $\sum_{n=0}^{\infty} c_n a^n$ conv.

the the general term tens to zero, if $n \to \infty$, $c_n a^n \to 0$

$\exists M > 0 : |c_n a^n| < M$

$$\sum \left| c_n a^n \left(\frac{x}{a}\right)^n \right| \quad \sum M \left|\frac{x}{a}\right|^n$$

the geometric series with $q = \left|\frac{x}{a}\right|$, conv. $\left|\frac{x}{a}\right| < 1$

Power Series Centered at c

- A **power series with center** c is an infinite series

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} a_n (x-c)^n \\ &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots ; \end{aligned}$$

- Example: The following is a power series centered at $c = 2$:

$$F(x) = 1 + (x-2) + 2(x-2)^2 + 3(x-2)^3 + \cdots ;$$

- A power series may converge for some values of x and diverge for some other values of x;
- Take a look again at $F(x) = 1 + (x-2) + 2(x-2)^2 + 3(x-2)^3 + \cdots ;$
  - $F(\frac{5}{2}) = 1 + \frac{1}{2} + 2(\frac{1}{2})^2 + 3(\frac{1}{2})^3 + \cdots = \sum_{n=0}^{\infty} \frac{n}{2^n}$; This series converges by the Ratio Test!
  - $F(3) = 1 + 1 + 2 + 3 + 4 + \cdots ;$ This series diverges by the Divergence Test!

Radius and Interval of Convergence

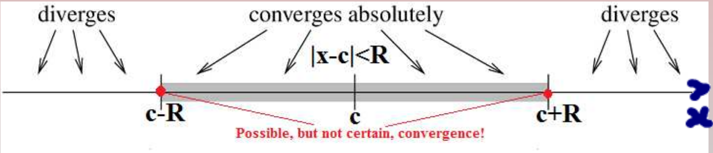
Th.3, Stewart, p.743

Theorem (Radius of Convergence)

Every power series $F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ has a **radius of convergence** R, which is either a nonnegative number ($R \geq 0$) or infinity ($R = \infty$).

- If R is finite, $F(x)$ converges absolutely when $|x-c| < R$ (i.e., in $(c-R, c+R)$) and diverges when $|x-c| > R$;
- If $R = \infty$, then $F(x)$ converges absolutely for all x.

- According to the Theorem, $F(x)$ converges in an **interval of convergence** consisting of the open $(c-R, c+R)$ and possibly one or both of the endpoints $c-R$ and $c+R$;



Using the Ratio Test I

- Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$;

  Let $a_n = \frac{x^n}{2^n}$ and compute the ratio ρ of the Ratio Test:

  $$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{2} = \frac{|x|}{2};$$

  Therefore, we get $\rho < 1 \Rightarrow \frac{|x|}{2} < 1 \Rightarrow |x| < 2$; This shows that, if $|x| < 2$ the series converges absolutely; If $|x| > 2$ the series diverges;
  - If $x = -2$, then $\sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n$, which diverges!
  - If $x = 2$, then $\sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1$, which also diverges!

  Thus, the interval of convergence is $(-2, 2)$;

Using the Ratio Test I An Even Power Series

- Find the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n}$;

Let $a_n = \dfrac{x^{n\cdot 2}}{2^n}$ and compute the ratio ρ of the Ratio Test:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{\overset{2n+2}{\cancel{n+1}}}}{2^{n+1}} \cdot \frac{2^n}{x^{2n}} \right| = \lim_{n\to\infty} \frac{|x|^2}{2} = \frac{|x|^2}{2};$$

Therefore, we get $\rho < 1 \Rightarrow \dfrac{|x|^2}{2} < 1 \Rightarrow |x|^2 < 2$; This shows that, if $|x| < \sqrt{2}$ the series converges absolutely; If $|x| > \sqrt{2}$ the series diverges;

- If $x = -\sqrt{2}$, then $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n} = \sum_{n=0}^{\infty} \frac{(+2)^n}{2^n} = \sum_{n=0}^{\infty} (+1)^n$, which diverges!
- If $x = \sqrt{2}$, then $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n} = \sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1$, which also diverges!

Thus, the interval of convergence is $(-\sqrt{2}, \sqrt{2})$;

Using the Ratio Test II

- Find the interval of convergence of $F(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n}(x-5)^n$;

  Let $a_n = \frac{(-1)^n}{4^n n}(x-5)^n$ and compute the ratio ρ of the Ratio Test:

  $$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-5)^{n+1}}{4^{n+1}(n+1)} \cdot \frac{4^n n}{(-1)^n(x-5)^n}\right| =$$
  $$|x-5| \lim_{n\to\infty}\left|\frac{n}{4(n+1)}\right| = \frac{1}{4}|x-5|;$$

  Therefore, we get $\rho < 1 \Rightarrow \frac{|x-5|}{4} < 1 \Rightarrow |x-5| < 4$; This shows that, if $|x-5| < 4$ the series converges absolutely; If $|x-5| > 4$ the series diverges;

  - If $x-5=-4$, then $F(1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n}(-4)^n = \sum_{n=0}^{\infty} \frac{1}{n}$, which diverges!
  - If $x-5=4$, then $F(9) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n}(4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$, which converges! Thus, interval of convergence is $(1,9]$;

$$\sum_{n=1}^{\infty} n!\,x^n$$

By Ratio Test, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!\,|x|^{n+1}}{n!\,|x|^n} =$$

$$= |x|\lim_{n\to\infty}(n+1) = \infty, \quad x \neq 0$$

$R = 0$,

{0} domain of convergence

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \ldots = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

$R = \infty$

$(-\infty, +\infty)$

An Even Power Series

- Where does $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ converge?

  Let $a_n = \dfrac{x^{2n}}{(2n)!}$ and compute the ratio ρ of the Ratio Test:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{x^{2n}} \right| =$$

$$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| = x^2 \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)} = 0;$$

  Therefore, we get $\rho < 1$, for all x; This shows that the series is absolutely convergent everywhere;

Geometric Power Series

- Recall that the geometric infinite series $S = a + ar + ar^2 + \cdots$ converges when $|r| < 1$ and has sum $S = \frac{a}{1-r}$;
- As a special case, when $a = 1$ and $r = x$, we get the geometric series with center 0: $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$; We have
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1;$$
- Example: Show that $\frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n$, for $|x| < \frac{1}{2}$;

  If $|x| < \frac{1}{2}$, then $2|x| < 1$ and, therefore $|2x| < 1$; Thus, the geometric series with ratio $2x$ converges; We have
$$\frac{1}{1-2x} \overset{\text{Geometric Sum}}{=} \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n;$$

Another Example of a Geometric Power Series

- Find a power series expansion with center $c = 0$ for $f(x) = \dfrac{1}{5+4x^2}$ and find the interval of convergence;

$$\frac{1}{5+4x^2} = \frac{1}{5}\cdot\frac{1}{1+\frac{4}{5}x^2} = \frac{1}{5}\cdot\frac{1}{1-(-\frac{4}{5}x^2)};$$

Therefore, if $|-\frac{4}{5}x^2| = \frac{4}{5}x^2 < 1 \Rightarrow x^2 \leq \frac{5}{4} \Rightarrow |x| \leq \frac{\sqrt{5}}{2}$, we have

$$\frac{1}{5+4x^2} = \frac{1}{5}\cdot\frac{1}{1-(-\frac{4}{5}x^2)} \overset{\text{Geometric}}{=} \frac{1}{5}\sum_{n=0}^{\infty}(-\frac{4}{5}x^2)^n =$$

$$\frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\frac{4^n}{5^n}x^{2n} = \sum_{n=0}^{\infty}(-1)^n\frac{4^n}{5^{n+1}}x^{2n};$$

Term-by-Term Differentiation and Integration

St.,p.748

Term-by-Term Differentiation and Integration

Assume that $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ has radius of convergence $R > 0$;

Then $F(x)$ is differentiable on $(c - R, c + R)$ (or for all x, if $R = \infty$);

Moreover, we can integrate and differentiate term-by-term, i.e.,

1. $F'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$;
2. $\int F(x)dx = A + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$;

Both series for $F'(x)$ and $\int F(x)dx$ have the same radius of convergence R as $F(x)$;

Example of Differentiation of Power Series

- Prove that for $-1 < x < 1$,
  $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$;
  We know that, for $|x| < 1$, we have
  $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots ;$$
  Therefore, by Term-by-Term Differentiation, we get, for $|x| < 1$:
  $$\begin{aligned} \frac{1}{(1-x)^2} &= \left(\frac{1}{1-x}\right)' \\ &= (1 + x + x^2 + x^3 + x^4 + x^5 + \cdots)' \\ &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots ; \end{aligned}$$

Example of Integration of Power Series

- Prove that for $|x| < 1$, we have
  $$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots;$$
  Since for $|x| < 1$, we have $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$, we obtain, also for $|x| < 1$,
  $$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots;$$
  Therefore, by Term-by-Term Integration we get
  $$\begin{aligned} \tan^{-1} x &= \int \frac{1}{1+x^2} dx \\ &= \int (1 - x^2 + x^4 - x^6 + x^8 - \cdots) dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots; \end{aligned}$$

Power Series Solution of Differential Equations

OPTIONAL for semester I

- Consider $y' = y$ and $y(0) = 1$;

  Assume that the power series $F(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ is a solution of the given initial value problem; Compute $F'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$; Since $F(x) = F'(x)$, we must have $a_0 = a_1, a_1 = 2a_2, a_2 = 3a_3, a_3 = 4a_4, \ldots$; Looking at these carefully, we obtain $a_n = \frac{a_{n-1}}{n}$, for all n; Thus,

$$a_n = \frac{1}{n} a_{n-1} = \frac{1}{n}\frac{1}{n-1} a_{n-2} = \frac{1}{n}\frac{1}{n-1}\frac{1}{n-2} a_{n-3} = \cdots = \frac{1}{n(n-1)(n-2)\cdot\cdots\cdot 1} a_0 = \frac{1}{n!} a_0;$$

Example I (Cont'd)

OPTIONAL for semester I

- We were solving $y' = y$ and $y(0) = 1$;

  We assumed $F(x) = \sum_{n=0}^{\infty} a_n x^n$ is a solution; We found $a_n = \frac{1}{n!}a_0$; This yields $F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = a_0 + a_0\frac{1}{1!}x + a_0\frac{1}{2!}x^2 + a_0\frac{1}{3!}x^3 + \cdots = a_0(1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots) = a_0\sum_{n=0}^{\infty}\frac{x^n}{n!}$; Since $F(0) = 1 = a_0$, we get $F(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}$;

- Since e^x is also a solution, we get

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots;$$

Example II

OPTIONAL for semester I

- Find a series solution to $x^2y'' + xy' + (x^2 - 1)y = 0$, with $y'(0) = 1$;

Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$; Then $y' = F'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$ and

$y'' = F''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$; Plug those in equation:

$$
\begin{aligned}
& x^2y'' + xy' + (x^2 - 1)y = \\
& x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + (x^2 - 1) \sum_{n=0}^{\infty} a_n x^n = \\
& \sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_{n+2} x^n = \\
& \sum_{n=0}^{\infty} (n^2 - 1) a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0;
\end{aligned}
$$

Thus,

$$
\sum_{n=0}^{\infty} (n^2 - 1) a_n x^n = -\sum_{n=2}^{\infty} a_{n-2} x^n \;\Rightarrow\; a_n = -\frac{a_{n-2}}{n^2 - 1};
$$

Example II (Cont'd)

OPTIONAL for semester I

- We were solving $x^2y'' + xy' + (x^2 - 1)y = 0$, with $y'(0) = 1$;
  We assumed $F(x) = \sum_{n=0}^{\infty} a_n x^n$ is a solution; We found $a_n = -\dfrac{a_{n-2}}{n^2 - 1}$;
  Now, note $a_0 = 0$; Thus, $a_2 = -\frac{a_0}{2^2-1} = 0$; Then $a_4 = -\frac{a_2}{4^2-1} = 0$;
  We see that $a_{2n} = 0$, for all n;
  Moreover, $a_1 = 1$; Thus, $a_3 = -\frac{a_1}{3^2-1} = -\frac{1}{2\cdot4}$; Then
  $a_5 = -\frac{a_3}{5^2-1} = +\frac{1}{2\cdot4\cdot4\cdot6}$; Also $a_7 = -\frac{a_5}{7^2-1} = -\frac{1}{2\cdot4\cdot4\cdot6\cdot6\cdot8}$; In general
  $a_{2n+1} = \frac{(-1)^n}{2\cdot4\cdot4\cdot6\cdot6\cdot\cdots\cdot(2n)(2n+2)} = \frac{(-1)^n}{2^n(1\cdot2\cdot3\cdot\cdots\cdot n)2^n(2\cdot3\cdot4\cdot\cdots\cdot(n+1))} = \frac{(-1)^n}{4^n n!(n+1)!}$;
  So we get $F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!(n+1)!} x^{2n+1}$;