Optimization Techniques

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2013 Lecture 7

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$$\begin{aligned} g(x) &\equiv \nabla f(x) = \left(\frac{\partial f}{\partial x_i} \right)_{i=1}^{n} \\ H(x) &= \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1}^{n} \end{aligned}$$

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Also, all vectors $x \in \mathbb{R}^n$ are column vectors, $x = (x_1, x_2, \ldots, x_n)^T$

Generic Linesearch Method

Definition

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1. Given an initial guess x_0, let $k = 0$
2. Until convergence:
    1. Find a descent direction p_k at x_k.
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In the steepest descent direction:$p_k = -g_k$. But convergence is slow, and numerical convergence sometimes does not occur at all. Therefore, steepest-descent is rarely used in most cases. Instead other linesearch methods are being used.

Newton and Newton-like methods

Lemma

Let B_k be a symmetric, positive definite matrix. Then p_k such that $B_k p_k = -g_k$ is a search direction.

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Suppose B_k is positive definite matrix and $B_k p_k = -g_k$. Therefore for any direction p_k:

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So p_k is a search direction. □

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If H_k is positive definite, then the descent search direction is

$$-H_k^{-1} g_k$$

Newton and Newton-like methods

Theorem

Suppose that $f \in C^1$ and that g is Lipschitz continuous on $\mathbb{R}^n$. Then, for the iterates generated by the Generic Linesearch Method using the Newton or Newton-like direction, either

$$g_l = 0 \quad \textit{for some } l$$

or

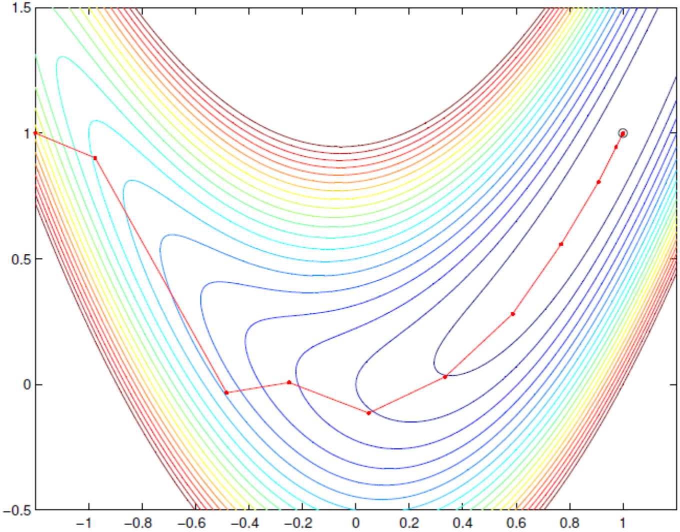
$$\lim_{k \to \infty} f_k = -\infty$$

or

$$\lim_{k \to \infty} g_k = 0$$

provided that the eigenvalues of B_k are uniformly bounded and bounded away from zero.

Newton and Newton-like methods



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It can be shown theoretically, that convergence of Newton method under suitable conditions will be quadratic.
In other words, there exists a positive constant C such that

$$\lim_{k \to \infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|^2} = C$$

Modified Newton methods

Away from a local minimizer there is no reason to believe that H_k will be positive definite, so precautions need to be taken to ensure that Newton and Newton-like linesearch methods, for which B_k is (or is close to) H_k, satisfy the assumptions of the global convergence

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If H_k is indefinite, it is usual to solve instead

$$(\underbrace{H_k + M_k}_{\equiv B_k})pk \equiv B_k p_k = -g_k;$$

where Mk is chosen so that $B_k = H_k + M_k$ is "sufficiently" positive definite and $M_k = 0$ when H_k is itself "sufficiently" positive definite.

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This may be achieved in a number of ways. There several choices for choosing M_k : using spectral decomposition or Cholesky decomposition.

Quasi-Newton methods

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There are broadly two classes of what may be called quasi-Newton methods: by finite differencies or by secant approximations.