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Linear Algebra and Analytic Geometry

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1 Vector Geometry
- Vectors in the Plane
- Vectors in Three Dimensions
- Dot Product and Angle Between Vectors
- The Cross Product
- Planes in Three-Space
- A Survey of Quadratic Surfaces
- Cylindrical and Spherical Coordinates

Subsection 3

Dot Product and Angle Between Vectors

The Dot Product

- Recall that the scalar product $\lambda \boldsymbol{v}$ of a real number λ times a vector $\boldsymbol{v}$ is a vector with length $|\lambda|\|\boldsymbol{v}\|$ and direction
  - the same as $\boldsymbol{v}$, if $\lambda > 0$;
  - opposite of $\boldsymbol{v}$, if $\lambda < 0$.
- In contrast, the **dot product** $\boldsymbol{v} \cdot \boldsymbol{w}$ of two vectors $\boldsymbol{v} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{w} = \langle b_1, b_2, b_3 \rangle$ is a real number, defined by

$$\boldsymbol{v} \cdot \boldsymbol{w} = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Examples:

$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2 \cdot 3 + 4 \cdot (-1) = 2.$

$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\frac{1}{2} \rangle = (-1) \cdot 6 + 7 \cdot 2 + 4 \cdot (-\frac{1}{2}) = -6 + 14 - 2 = 6.$

$(\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}) \cdot (2\boldsymbol{i} - \boldsymbol{k}) = 1 \cdot 2 + 2 \cdot 0 + (-3) \cdot (-1) = 2 + 3 = 5.$

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Properties of the Dot Product

- **Zero Property:** $\mathbf{0} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \mathbf{0} = 0$. $\vec{v} \cdot \vec{v} = 0$
- **Commutativity:** $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$.
- **Pulling out Scalars:** $(\lambda \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{v} \cdot (\lambda \boldsymbol{w}) = \lambda(\boldsymbol{v} \cdot \boldsymbol{w})$.
- **Distributive Law:** $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$.

$$\begin{aligned}
\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) &= \langle a_1, a_2, a_3 \rangle \cdot (\langle b_1, b_2, b_3 \rangle + \langle c_1, c_2, c_3 \rangle) \\
&= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle \\
&= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\
&= (a_1 b_1 + a_1 c_1) + (a_2 b_2 + a_2 c_2) + (a_3 b_3 + a_3 c_3) \\
&= (a_1 b_1 + a_2 b_2 + a_3 b_3) + (a_1 c_1 + a_2 c_2 + a_3 c_3) \\
&= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle + \langle a_1, a_2, a_3 \rangle \cdot \langle c_1, c_2, c_3 \rangle \\
&= \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}.
\end{aligned}$$

- **Relation with Length:** $\boldsymbol{v} \cdot \boldsymbol{v} = \|\boldsymbol{v}\|^2$.

$$\boldsymbol{v} \cdot \boldsymbol{v} = \langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle = a_1^2 + a_2^2 + a_3^2$$
$$= \left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)^2 = \|\boldsymbol{v}\|^2.$$

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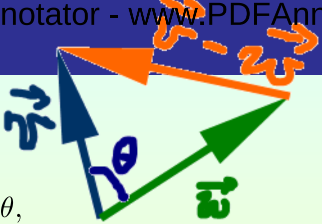
The Cosine Identity



- For vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, we have

$$\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|\|\boldsymbol{w}\| \cos\theta,$$

  where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

- Law of cosines: $\|\boldsymbol{v} - \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 - 2\|\boldsymbol{v}\|\|\boldsymbol{w}\| \cos\theta$.

  We get

$$\begin{aligned}
(\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) &= \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 - 2\|\boldsymbol{v}\|\|\boldsymbol{w}\| \cos\theta \\
\boldsymbol{v} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w} &= \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 - 2\|\boldsymbol{v}\|\|\boldsymbol{w}\| \cos\theta \\
\|\boldsymbol{v}\|^2 - 2(\boldsymbol{v} \cdot \boldsymbol{w}) + \|\boldsymbol{w}\|^2 &= \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 - 2\|\boldsymbol{v}\|\|\boldsymbol{w}\| \cos\theta \\
-2(\boldsymbol{v} \cdot \boldsymbol{w}) &= -2\|\boldsymbol{v}\|\|\boldsymbol{w}\| \cos\theta \\
\boldsymbol{v} \cdot \boldsymbol{w} &= \|\boldsymbol{v}\|\|\boldsymbol{w}\| \cos\theta.
\end{aligned}$$

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Computing Angle Between Two Vectors

- One of the most useful applications of the Cosine Formula is finding the angle θ between the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$:

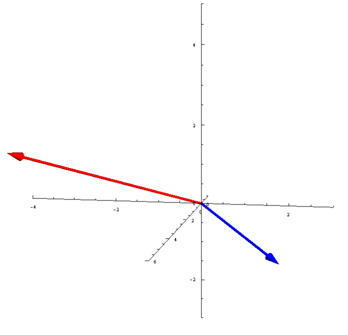
$$\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|}.$$

Example: Suppose $\boldsymbol{v} = \langle 2, 2, -1\rangle$ and $\boldsymbol{w} = \langle 5, -3, 2\rangle$.

$\cos\theta = \dfrac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \dfrac{2}{3\sqrt{38}}$. So

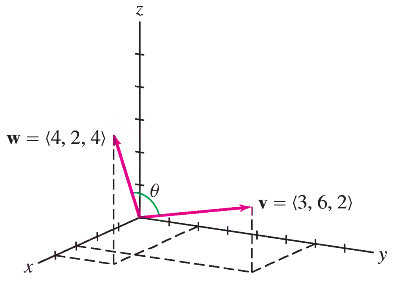
$$\theta = \cos^{-1}\frac{2}{3\sqrt{38}} \approx 1.46 \text{ rads.}$$

arc cos



A Second Example

- Find the angle θ between $\boldsymbol{v} = \langle 3, 6, 2 \rangle$ and $\boldsymbol{w} = \langle 4, 2, 4 \rangle$.



$$\|\boldsymbol{v}\| = \sqrt{3^2 + 6^2 + 2^2} = \sqrt{49} = 7. \ \|\boldsymbol{w}\| = \sqrt{4^2 + 2^2 + 4^2} = \sqrt{36} = 6.$$

$$\cos\theta = \frac{\langle 3, 6, 2 \rangle \cdot \langle 4, 2, 4 \rangle}{7 \cdot 6} = \frac{3 \cdot 4 + 6 \cdot 2 + 2 \cdot 4}{42} = \frac{16}{21}.$$

The angle is $\theta = \cos^{-1}\left(\frac{16}{21}\right) \approx 0.705$ rad.

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Orthogonality

- Two nonzero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are called **perpendicular** or **orthogonal** if the angle between them is $\frac{\pi}{2}$.
  In this case we write $\boldsymbol{v} \perp \boldsymbol{w}$.
- We can use the dot product to test whether $\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal.
  Because an angle between 0 and π satisfies $\cos\theta = 0$ if and only if $\theta = \frac{\pi}{2}$, we see that

$$\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|\|\boldsymbol{w}\| \cos\theta = 0 \iff \theta = \frac{\pi}{2}.$$

We conclude that $\boldsymbol{v} \perp \boldsymbol{w}$ if and only if $\boldsymbol{v} \cdot \boldsymbol{w} = 0$.

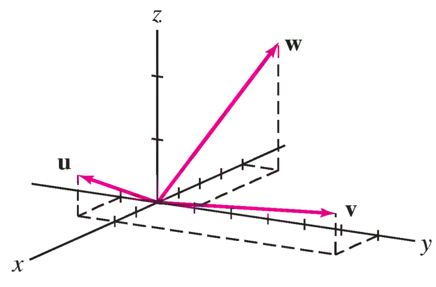
Example: The standard basis vectors are mutually orthogonal and have length 1.

In terms of dot products, because $\boldsymbol{i} = \langle 1, 0, 0\rangle$, $\boldsymbol{j} = \langle 0, 1, 0\rangle$, and $\boldsymbol{k} = \langle 0, 0, 1\rangle$,

$$\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{j} \cdot \boldsymbol{k} = 0, \quad \boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1.$$

Testing for Orthogonality

- Determine whether $\boldsymbol{v} = \langle 2, 6, 1\rangle$ is orthogonal to $\boldsymbol{u} = \langle 2, -1, 1\rangle$ or $\boldsymbol{w} = \langle -4, 1, 2\rangle$.



We test for orthogonality by computing the dot products:

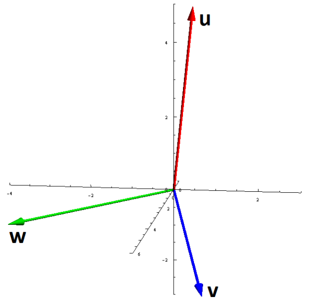
$$\boldsymbol{v} \cdot \boldsymbol{u} = \langle 2, 6, 1\rangle \cdot \langle 2, -1, 1\rangle = 2(2) + 6(-1) + 1(1) = -1;$$
$$\boldsymbol{v} \cdot \boldsymbol{w} = \langle 2, 6, 1\rangle \cdot \langle -4, 1, 2\rangle = 2(-4) + 6(1) + 1(2) = 0.$$

We conclude $\boldsymbol{v} \not\perp \boldsymbol{u}$, but $\boldsymbol{v} \perp \boldsymbol{w}$.

Testing for Obtuseness

- Determine whether the angles between the vector $\boldsymbol{v} = \langle 3, 1, -2 \rangle$ and the vectors $\boldsymbol{u} = \langle \frac{1}{2}, \frac{1}{2}, 5 \rangle$ and $\boldsymbol{w} = \langle 4, -3, 0 \rangle$ are obtuse.

The angle θ between $\boldsymbol{v}$ and $\boldsymbol{u}$ is obtuse if $\frac{\pi}{2} < 0 \leq \pi$, and this is the case if $\cos\theta < 0$. Since $\boldsymbol{v} \cdot \boldsymbol{u} = \|\boldsymbol{v}\| \|\boldsymbol{u}\| \cos\theta$ and the lengths $\|\boldsymbol{v}\|$ and $\|\boldsymbol{u}\|$ are positive, we see that $\cos\theta$ is negative if and only if $\boldsymbol{v} \cdot \boldsymbol{u}$ is negative.



We have

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{u} &= \langle 3, 1, -2 \rangle \cdot \langle \tfrac{1}{2}, \tfrac{1}{2}, 5 \rangle = \tfrac{3}{2} + \tfrac{1}{2} - 10 = -8 < 0; \\ \boldsymbol{v} \cdot \boldsymbol{w} &= \langle 3, 1, -2 \rangle \cdot \langle 4, -3, 0 \rangle = 12 - 3 + 0 = 9 > 0. \end{aligned}$$

Thus, the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$ is obtuse, whereas the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ is acute.

Using the Distributive Law

- Calculate the dot product $\boldsymbol{v} \cdot \boldsymbol{w}$, where $\boldsymbol{v} = 4\boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{w} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$.

  Use the Distributive Law and the orthogonality of $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$:

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{w} &= (4\boldsymbol{i} - 3\boldsymbol{j}) \cdot (\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) \\ &= 4\boldsymbol{i} \cdot (\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) - 3\boldsymbol{j} \cdot (\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) \\ &= 4\boldsymbol{i} \cdot \boldsymbol{i} - 3\boldsymbol{j} \cdot (2\boldsymbol{j}) \\ &= 4 - 6 = -2. \end{aligned}$$

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Projection

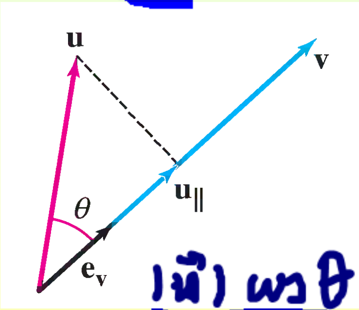
$\vec{e_v} = \frac{\vec{v}}{|\vec{v}|}$

- Assume $\boldsymbol{v} \neq \mathbf{0}$. The **projection of $\boldsymbol{u}$ along $\boldsymbol{v}$** is the vector

$\text{proj}_{\vec{v}}\vec{u}$

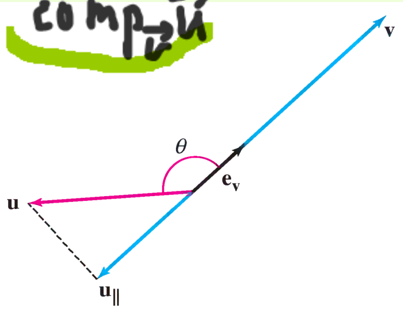
$$\boldsymbol{u}_{\parallel} = (\boldsymbol{u} \cdot \boldsymbol{e}_{\boldsymbol{v}})\boldsymbol{e}_{\boldsymbol{v}} \quad \text{or} \quad \boldsymbol{u}_{\parallel} = \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}}\right)\boldsymbol{v}.$$

The scalar $\boldsymbol{u} \cdot \boldsymbol{e}_{\boldsymbol{v}}$ is called the **component of $\boldsymbol{u}$ along $\boldsymbol{v}$**.



$|\vec{u}| \cos\theta$

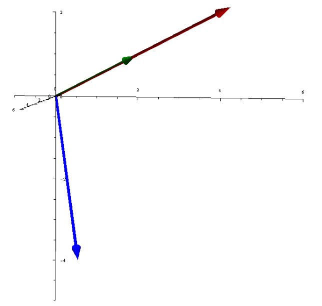
$\text{comp}_{\vec{v}}\vec{u}$



Example

- Find the projection of $\boldsymbol{u} = \langle 5, 1, -3 \rangle$ along $\boldsymbol{v} = \langle 4, 4, 2 \rangle$.

  We use the second formula:



$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} &= \langle 5, 1, -3 \rangle \cdot \langle 4, 4, 2 \rangle \\ &= 20 + 4 - 6 = 18; \\ \boldsymbol{v} \cdot \boldsymbol{v} &= 4^2 + 4^2 + 2^2 = 36. \end{aligned}$$

  Therefore,

$$\boldsymbol{u}_{\|} = \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}}\right)\boldsymbol{v} = \left(\frac{18}{36}\right)\langle 4, 4, 2 \rangle = \langle 2, 2, 1 \rangle.$$

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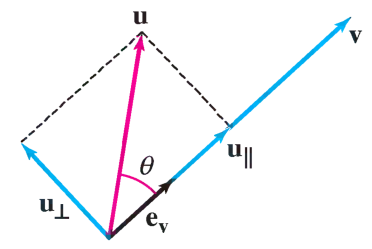
Decomposition of $\boldsymbol{u}$ with respect to $\boldsymbol{v}$

OPTIONAL

- If $\boldsymbol{v} \neq \mathbf{0}$, then every vector $\boldsymbol{u}$ can be written as the sum of the projection $\boldsymbol{u}_{\parallel}$ and a vector $\boldsymbol{u}_{\perp}$ that is orthogonal to $\boldsymbol{v}$.



  If we set $\boldsymbol{u}_{\perp} = \boldsymbol{u} - \boldsymbol{u}_{\parallel}$, then we have

$$\boldsymbol{u} = \boldsymbol{u}_{\parallel} + \boldsymbol{u}_{\perp}.$$

  This equation is called the **decomposition of $\boldsymbol{u}$ with respect to $\boldsymbol{v}$**.

- We verify that $\boldsymbol{u}_{\perp}$ is orthogonal to $\boldsymbol{v}$:

$$\boldsymbol{u}_{\perp} \cdot \boldsymbol{v} = (\boldsymbol{u} - \boldsymbol{u}_{\parallel}) \cdot \boldsymbol{v} = (\boldsymbol{u} - (\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}})\boldsymbol{v}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v} - (\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}})(\boldsymbol{v} \cdot \boldsymbol{v}) = 0.$$

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Example

OPTIONAL

- Find the decomposition of $\boldsymbol{u} = \langle 5, 1, -3 \rangle$ with respect to $\boldsymbol{v} = \langle 4, 4, 2 \rangle$.

  We showed that $\boldsymbol{u}_{\|} = \langle 2, 2, 1 \rangle$. The orthogonal vector is

$$\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{u}_{\|} = \langle 5, 1, -3 \rangle - \langle 2, 2, 1 \rangle = \langle 3, -1, -4 \rangle.$$

The decomposition of $\boldsymbol{u}$ with respect to $\boldsymbol{v}$ is

$$\boldsymbol{u} = \langle 5, 1, -3 \rangle = \boldsymbol{u}_{\|} + \boldsymbol{u}_{\perp} = \underbrace{\langle 2, 2, 1 \rangle}_{\text{Projection along } \boldsymbol{v}} + \underbrace{\langle 3, -1, -4 \rangle}_{\text{Orthogonal to } \boldsymbol{v}}.$$

Subsection 4

The Cross Product

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The Cross-Product

The Vector product

- Recall that the dot product $\boldsymbol{v} \cdot \boldsymbol{w}$ of two vectors is a scalar quantity, not a vector.
- The **cross-product** $\boldsymbol{v} \times \boldsymbol{w}$ of two vectors $\boldsymbol{v} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{w} = \langle b_1, b_2, b_3 \rangle$, on the other hand, is a vector defined by

$$\boldsymbol{v} \times \boldsymbol{w} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

- Matrices are useful when dealing with cross-products. Recall that, given a 2×2-matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ its **determinant** is computed by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12} a_{21}.$$

$\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}$

- Using this matrix notation, we have

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \boldsymbol{k}.$$

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Stewart Calculus p.808 12.4

? $\vec{u}$, $\vec{u} \perp \vec{v}$, $\vec{u} \perp \vec{w}$

$\vec{u} = (c_1, c_2, c_3)$, $\begin{cases} \vec{u} \cdot \vec{v} = 0 \\ \vec{u} \cdot \vec{w} = 0 \end{cases}$

Independent Reading

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Cross-Products Using 3×3-Determinants

- In fact, there is a similar formula for determinants of 3×3-matrices:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

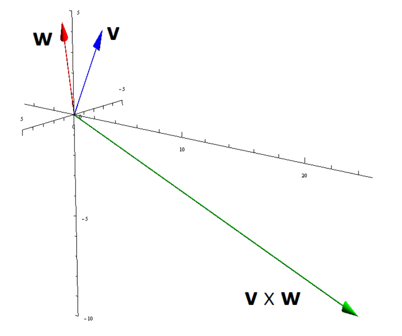
- In that notation, the cross-product of $\boldsymbol{v} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{w} = \langle b_1, b_2, b_3 \rangle$ can be written more succinctly as

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \boldsymbol{k}$$

Example

- Suppose $\boldsymbol{v} = \langle -2, 1, 4\rangle$, $\boldsymbol{w} = \langle 3, 2, 5\rangle$.
  Then

$$\begin{aligned}\boldsymbol{v} \times \boldsymbol{w} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2 & 1 & 4 \\ 3 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} -2 & 4 \\ 3 & 5 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix} \boldsymbol{k} \\ &= -3\boldsymbol{i} + 22\boldsymbol{j} - 7\boldsymbol{k}.\end{aligned}$$



Direction of the Cross-Product

Orthogonality

For all vectors $\boldsymbol{v} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{w} = \langle b_1, b_2, b_3 \rangle$, the vector $\boldsymbol{v} \times \boldsymbol{w}$ is orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$. Moreover, $\{\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{v} \times \boldsymbol{w}\}$ forms a right-handed system.

- To see this compute the dot product:

$$\begin{aligned}
\boldsymbol{v} \cdot (\boldsymbol{v} \times \boldsymbol{w}) &= \langle a_1, a_2, a_3 \rangle \cdot \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle \\
&= a_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - a_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + a_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= 0.
\end{aligned}$$

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Length of the Cross-Product

The length of $\boldsymbol{v} \times \boldsymbol{w}$

For all vectors $\boldsymbol{v}, \boldsymbol{w}$, $\|\boldsymbol{v} \times \boldsymbol{w}\| = \|\boldsymbol{v}\|\|\boldsymbol{w}\| \sin\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$, with $0 \leq \theta \leq \pi$.

- We have, for $\boldsymbol{v} = \langle a_1, a_2, a_3\rangle$ and $\boldsymbol{w} = \langle b_1, b_2, b_3\rangle$,

$$\begin{aligned}
\|\boldsymbol{v} \times \boldsymbol{w}\|^2 &= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\
&= \|\boldsymbol{v}\|^2\|\boldsymbol{w}\|^2 - (\boldsymbol{v} \cdot \boldsymbol{w})^2 \\
&= \|\boldsymbol{v}\|^2\|\boldsymbol{w}\|^2 - \|\boldsymbol{v}\|^2\|\boldsymbol{w}\|^2 \cos^2\theta \\
&= \|\boldsymbol{v}\|^2\|\boldsymbol{w}\|^2(1 - \cos^2\theta) \\
&= \|\boldsymbol{v}\|^2\|\boldsymbol{w}\|^2 \sin^2\theta.
\end{aligned}$$

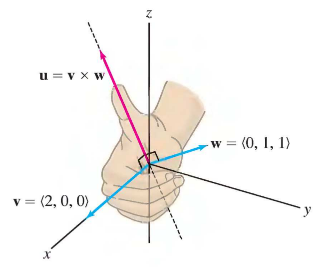
p.810, Th9

- Thus, if $\boldsymbol{v} \| \boldsymbol{w}$, then $\boldsymbol{v} \times \boldsymbol{w} = \boldsymbol{0}$.

Using the Geometric Properties

- Let $\boldsymbol{v} = \langle 2, 0, 0 \rangle$ and $\boldsymbol{w} = \langle 0, 1, 1 \rangle$. Determine $\boldsymbol{u} = \boldsymbol{v} \times \boldsymbol{w}$ using the geometric properties of the cross product rather than its definition.
  First, $\boldsymbol{u} = \boldsymbol{v} \times \boldsymbol{w}$ is orthogonal to $\boldsymbol{v}$ and $\boldsymbol{w}$. Since $\boldsymbol{v}$ lies along the x-axis, $\boldsymbol{u}$ must lie in the yz-plane, i.e., $\boldsymbol{u} = \langle 0, b, c \rangle$. But $\boldsymbol{u}$ is also orthogonal to $\boldsymbol{w} = \langle 0, 1, 1 \rangle$, so $\boldsymbol{u} \cdot \boldsymbol{w} = b + c = 0$. Thus, $\boldsymbol{u} = \langle 0, b, -b \rangle$.
  Next, we compute $\|\boldsymbol{v}\| = 2$ and $\|\boldsymbol{w}\| = \sqrt{2}$. Furthermore, the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ is $\theta = \frac{\pi}{2}$ since $\boldsymbol{v} \cdot \boldsymbol{w} = 0$.

  Thus, $\|\boldsymbol{u}\| = \|\boldsymbol{v} \times \boldsymbol{w}\|$ yields $|b|\sqrt{2} = \|\boldsymbol{v}\|\|\boldsymbol{w}\| \sin \frac{\pi}{2} = 2\sqrt{2}$. So $|b| = 2$, i.e., $b = \pm 2$.
  By the right hand rule $\boldsymbol{u}$ points in the positive z-direction. So $b = -2$. We get $\boldsymbol{u} = \langle 0, -2, 2 \rangle$.



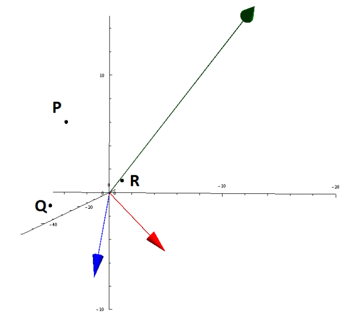
Determining A Vector Perpendicular to a Plane

- Determine a vector that is perpendicular to the plane passing through the points $P = (1, 4, 6)$, $Q = (-2, 5, -1)$ and $R = (1, -1, 1)$.

  Note that since P, Q, R are on the plane, the vectors $\overrightarrow{PQ} = \langle -3, 1, -7 \rangle$ and $\overrightarrow{PR} = \langle 0, -5, -5 \rangle$ are also on the plane.

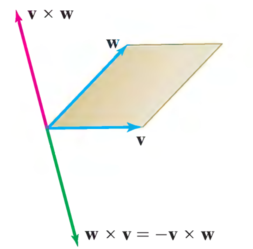
  Therefore, a vector perpendicular to the plane is given by the cross-product

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} \\ &= \langle -40, -15, 15 \rangle. \end{aligned}$$



Anticommutativity

- The cross product is **anticommutative**, i.e., reversing the order changes the sign:
$$\boldsymbol{w} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{w}.$$
- To verify this using the definition, note that when we interchange $\boldsymbol{v}$ and $\boldsymbol{w}$, each of the 2×2 determinants changes sign
$$\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = a_2 b_1 - a_1 b_2 = -(a_1 b_2 - a_2 b_1) = -\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$
- Anticommutativity also follows from the geometric description of the cross product.
    - $\boldsymbol{v} \times \boldsymbol{w}$ and $\boldsymbol{w} \times \boldsymbol{v}$ are both orthogonal to $\boldsymbol{v}$ and $\boldsymbol{w}$ and have the same length.
    - However, $\boldsymbol{v} \times \boldsymbol{w}$ and $\boldsymbol{w} \times \boldsymbol{v}$ point in opposite directions by the right-hand rule, whence $\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$.



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Basic Properties of the Cross Product

(i) $\boldsymbol{w} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{w}$;

(ii) $\boldsymbol{v} \times \boldsymbol{v} = \boldsymbol{0}$;

(iii) $\boldsymbol{v} \times \boldsymbol{w} = \boldsymbol{0}$ if and only if $\boldsymbol{w} = \lambda \boldsymbol{v}$, for some scalar λ, or $\boldsymbol{v} = \boldsymbol{0}$; Or w =0

(iv) $(\lambda \boldsymbol{v}) \times \boldsymbol{w} = \boldsymbol{v} \times (\lambda \boldsymbol{w}) = \lambda(\boldsymbol{v} \times \boldsymbol{w})$;

(v) $(\boldsymbol{u} + \boldsymbol{v}) \times \boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{w} + \boldsymbol{v} \times \boldsymbol{w}$;
$\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$.

- As a special case, we obtain that the cross product of any two of the standard basis vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ is equal to the third, possibly with a minus sign.

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}, \quad \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}, \quad \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j},$$
$$\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{0}.$$

Using the $\boldsymbol{ijk}$ Relations

- Compute $(2\boldsymbol{i}+\boldsymbol{k})\times(3\boldsymbol{j}+5\boldsymbol{k})$.

  We use the Distributive Law for cross products:

$$\begin{aligned}
&(2\boldsymbol{i}+\boldsymbol{k})\times(3\boldsymbol{j}+5\boldsymbol{k})\\
&=(2\boldsymbol{i})\times(3\boldsymbol{j})+(2\boldsymbol{i})\times(5\boldsymbol{k})+\boldsymbol{k}\times(3\boldsymbol{j})+\boldsymbol{k}\times(5\boldsymbol{k})\\
&=6(\boldsymbol{i}\times\boldsymbol{j})+10(\boldsymbol{i}\times\boldsymbol{k})+3(\boldsymbol{k}\times\boldsymbol{j})+5(\boldsymbol{k}\times\boldsymbol{k})\\
&=6\boldsymbol{k}-10\boldsymbol{j}-3\boldsymbol{i}+5(\boldsymbol{0})\\
&=-3\boldsymbol{i}-10\boldsymbol{j}+6\boldsymbol{k}.
\end{aligned}$$

Area of a Parallelogram

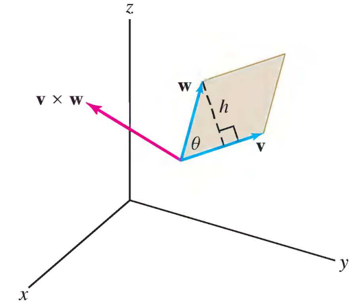
- Consider the parallelogram $\mathcal{P}$ spanned by nonzero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ with a common basepoint.

  $\mathcal{P}$ has:

  - base $b = \|\boldsymbol{v}\|$;
  - height $h = \|\boldsymbol{w}\| \sin\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

  Therefore, $\mathcal{P}$ has area

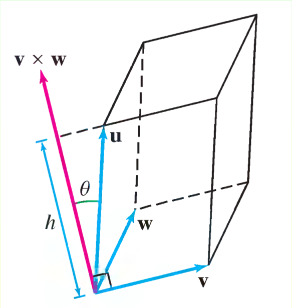
  $$A = bh = \|\boldsymbol{v}\|\|\boldsymbol{w}\|\sin\theta = \|\boldsymbol{v} \times \boldsymbol{w}\|.$$



Volume of a Parallelepiped

- Consider the parallelepiped $\boldsymbol{P}$ spanned by three nonzero vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

  The base of $\boldsymbol{P}$ is the parallelogram spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$. So the area of the base is $\|\boldsymbol{v} \times \boldsymbol{w}\|$.
  The height is $h = \|\boldsymbol{u}\| \cdot |\cos\theta|$, where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v} \times \boldsymbol{w}$.



  Therefore,

$$\text{Volume of } \boldsymbol{P} = (\text{area of base})(\text{height}) = \|\boldsymbol{v} \times \boldsymbol{w}\| \cdot \|\boldsymbol{u}\| \cdot |\cos\theta|.$$

  Thus,

$$\text{Volume of } \boldsymbol{P} = |\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})|.$$

  The quantity $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$ is called the **vector triple product**.

The Vector Triple Product

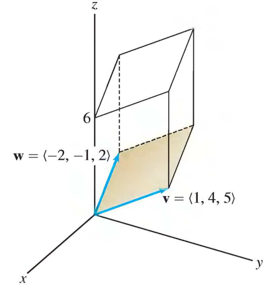
- The vector triple product $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$ can be expressed as a determinant.

  Suppose $\boldsymbol{u} = \langle a_1, a_2, a_3 \rangle$, $\boldsymbol{v} = \langle b_1, b_2, b_3 \rangle$ and $\boldsymbol{w} = \langle c_1, c_2, c_3 \rangle$. Then we have:

$$
\begin{aligned}
\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) &= \boldsymbol{u} \cdot \left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \boldsymbol{k} \right) \\
&= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= \det \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix}.
\end{aligned}
$$

Example

- Let $\boldsymbol{v} = \langle 1, 4, 5\rangle$ and $\boldsymbol{w} = \langle -2, -1, 2\rangle$. Calculate:
  (a) The area A of the parallelogram spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$.
  (b) The volume V of the parallelepiped in the figure.



Both the area and the volume require computing the cross product

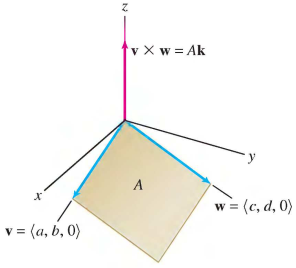
$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} 4 & 5 \\ -1 & 2 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} 1 & 5 \\ -2 & 2 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} 1 & 4 \\ -2 & -1 \end{vmatrix} \boldsymbol{k} = \langle 13, -12, 7\rangle.$$

(a) The area of the parallelogram spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$ is $A = \|\boldsymbol{v} \times \boldsymbol{w}\| = \sqrt{13^2 + (-12)^2 + 7^2} = \sqrt{362}.$

(b) The vertical leg of the parallelepiped is the vector $6\boldsymbol{k}$. So $V = |(6\boldsymbol{k}) \cdot (\boldsymbol{v} \times \boldsymbol{w})| = |\langle 0, 0, 6\rangle \cdot \langle 13, -12, 7\rangle| = 6(7) = 42.$

Parallelograms on the Plane

- We can compute the area A of the parallelogram spanned by vectors $\boldsymbol{v} = \langle a, b \rangle$ and $\boldsymbol{w} = \langle c, d \rangle$ by regarding $\boldsymbol{v}$ and $\boldsymbol{w}$ as vectors in space with zero component in the z-direction.



We write $\boldsymbol{v} = \langle a, b, 0 \rangle$ and $\boldsymbol{w} = \langle c, d, 0 \rangle$. The cross product $\boldsymbol{v} \times \boldsymbol{w}$ is a vector pointing in the z-direction:

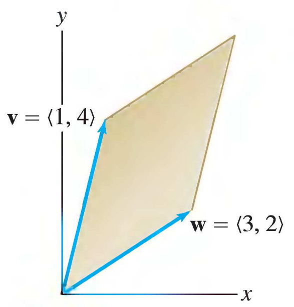
$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a & b & 0 \\ c & d & 0 \end{vmatrix} = \begin{vmatrix} b & 0 \\ d & 0 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a & b \\ c & d \end{vmatrix} \boldsymbol{k} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \boldsymbol{k}.$$

Thus, the parallelogram spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$ has area

$$A = \|\boldsymbol{v} \times \boldsymbol{w}\| = \left| \det \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix} \right|.$$

Example

- Compute the area A of the parallelogram in the figure



We have

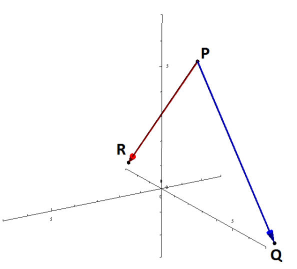
$$A = \left|\det\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix}\right| = \left|\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix}\right| = |1 \cdot 2 - 3 \cdot 4| = |-10| = 10.$$

Area of Triangle

- Find the area of a triangle with vertices

$$P = (1,4,6), Q = (-2,5,-1), R = (1,-1,1).$$

This triangle has sides $\overrightarrow{PQ} = \langle -3, 1, -7\rangle$ and $\overrightarrow{PR} = \langle 0, -5, -5\rangle$.



Its area is

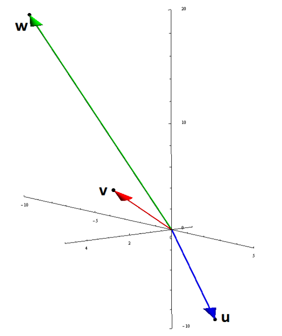
$$A = \frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \frac{1}{2}\|\langle -40, -15, 15\rangle\| = \frac{1}{2}5\sqrt{82}.$$

An Example of Co-Planar Vectors

- Show that the vectors $\boldsymbol{u} = \langle 1, 4, -7 \rangle$, $\boldsymbol{v} = \langle 2, -1, 4 \rangle$ and $\boldsymbol{w} = \langle 0, -9, 18 \rangle$ are co-planar.

  We show that the vector triple product

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = 0.$$



$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} + (-7) \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix} = 0.$$