Probability theory

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Post Lecture 5

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The assistant can also choose which four cards to reveal in $\binom{5}{4} = 5$ different ways.Still not sufficient since the magician does not know which of these 5 possibilities the assistant selected because of the secret card.

Trick:

Magic trick revealed

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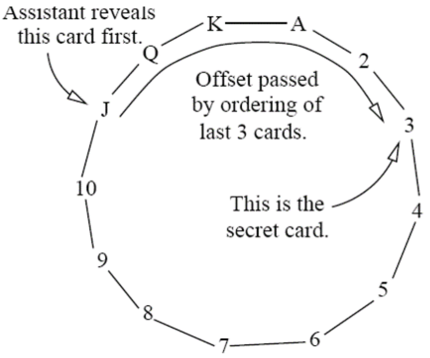
The value of the secret card is offset from the value of the first card announced by the value of the remaining three cards:



Offset 5 has to be communicated.

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$$A\clubsuit, 2\clubsuit, 3\clubsuit, \ldots, K\clubsuit, A\diamondsuit, 2\diamondsuit, \ldots, K\diamondsuit, \ldots K\spadesuit$$

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The offset is encoded by the order of these three cards,

$$\begin{aligned} SML &= 1, \quad SLM = 2, \quad MSL = 3 \\ MLS &= 4, \quad LSM = 5, \quad LMS = 6 \end{aligned}$$

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The remaining cards shown are $A\spadesuit, 6\diamondsuit, 8\diamondsuit$. Meaning order $LSM = 5$.

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Then,

$$|X| = \binom{52}{4} = 270\,725, \qquad |Y| = 52 \cdot 51 \cdot 50 = 132\,600.$$

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In other words, if magician hears a set of three cards, then there are at least two possibilities for the fourth card which he cannot distinguish.

Data compression

Stars and bars help us to solve a data compression problem:

Problem

How many bits are needed to store a **multiset** *of n arbitrary integers in the range $[0, 2n]$?*

For example, how much disk space is necessary to store one million numbers in the range from zero to two million?

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The biggest binary number with m digits is

$$\underbrace{(111 \ldots 1)}_{m \text{ digits}} = 1 \cdot 2^{m-1} + 1 \cdot 2^{m-2} + \ldots + 1 \cdot 2^1 + 1 \cdot 2^0 = 2^m - 1.$$

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Could store 10^6 numbers in the range $\left[0, 2 \cdot 10^6\right]$ in only $3 \cdot 10^6$ bits.

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This inequality is satisfied if $k \geq \log_2 \binom{3n}{n}$. Let's estimate it using Stirling's formula.

Data compression

$$\binom{3n}{n} = \frac{(3n)!}{(2n)!\, n!} \sim \frac{\sqrt{2\pi 3n}\left(\frac{3n}{e}\right)^{3n}}{\sqrt{2\pi 2n}\left(\frac{2n}{e}\right)^{2n}\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}}$$
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Therefore, the lower bound is

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$$= n \log_2 \tfrac{27}{4} - \tfrac{1}{2}\log_2 \tfrac{4\pi n}{3} \approx 2.755n.$$

Slightly better than $3n$.

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The index is a number from 0 to $\binom{3n}{n} - 1$, which can be stored in exactly $\log_2 \binom{3n}{n}$ bits.