Mathematical Analysis II

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Stewart, p.621 ex. 23

Stewart, p.600 ex. 22

Adv. Eng. Math., p.17

Reduction to Separable Form (homogeneous ODE)

ex. 7,8,9,10,16, 17, p.18

Refresh from the previous seminar

Ex.23, p.621 Stewart

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

nonlinear DE, I

Bernoulli DE

$$y(x) \rightsquigarrow u(x)$$

$$u = y^{1-n}$$

$$u' = (1-n)y^{-n} \cdot y'$$

$$(1-n)y^{-n}y' + (1-n)P(x)y^{1-n} = (1-n)Q(x)$$

$$u' + (1-n)P(x)u = (1-n)Q(x)$$

linear DE, I

Refresh from the previous seminar

Ex.25, p.621 Stewart

$$y' + \frac{2}{x}y = \frac{y^3}{x^2}$$

Bernoulli DE

$$u = y^{-2}$$

$$u' = -\frac{2}{y^3}y'$$

$$-2y^{-3}y' - \frac{4}{x}y^{-2} = -2\cdot\frac{1}{x^2}$$

$$u' - \frac{4}{x}u = -\frac{2}{x^2}$$

Linear DE ...

Refresh from the previous seminar

Adv. Eng. Math.
Ex.4, p.18

$$y' \sin 2\pi x = \pi y \cos 2\pi x$$

Separable ODE, I

$$\frac{dy}{dx} \sin 2\pi x = \pi y \cos 2\pi x$$

$$\frac{1}{y} dy = \pi \frac{\cos 2\pi x}{\sin 2\pi x} dx$$

CONT.

$$\frac{dy}{y} = \pi \operatorname{ctg} 2\pi x \, dx$$

$$\ln|y| = \frac{1}{2} \ln|\sin 2\pi x| + C$$

the general solution:

$$y = C\sqrt{|\sin 2\pi x|}, \quad C \in \mathbb{R}$$

Refresh from the previous seminar

Adv. Eng. Math. Ex.6, p.18

$$y' = e^{2x-1} y^2$$

Separable DE, I

$$y^{-2} dy = e^{2x-1} dx$$

$$-y^{-1} = \frac{1}{2} e^{2x-1} + c$$

$$y = -\frac{2}{e^{2x-1} + C}$$

Adv. Eng. Math.
Ex.7, p.18

$y' = f\left(\frac{y}{x}\right)$

$$xy' = y + 2x^3 \sin^2 \frac{y}{x}$$

homogeneous
ODE, I

$t = \frac{y}{x}$

$y = tx$

$y' = t'x + t$

$y(x) \rightsquigarrow t(x)$

$$x(t'x + t) = tx + 2x^3 \sin^2 t$$

$$x^2 t' + \cancel{tx} = \cancel{tx} + 2x^3 \sin^2 t$$

CONT.

Separable DE, I

$$x^2 t' = 2x^3 \sin^2 t$$

$$\frac{dt}{\sin^2 t} = 2x\,dx$$

$$-\operatorname{ctg} t = x^2 + C$$

$$\operatorname{ctg}\frac{y}{x} = -x^2 + C$$

$$y = x\operatorname{arcctg}(-x^2 + C)$$

Adv. Eng. Math.
Ex.17, p.18

$$xy' = y + 3x^4\cos^2\left(\frac{y}{x}\right), \quad y(1) = 0$$

$$y = tx$$
$$y' = t'x + t$$

$$x(t'x + t) = tx + 3x^4\cos^2 t$$

$$x^2t' + \cancel{tx} = \cancel{tx} + 3x^4\cos^2 t$$

$$x^2t' = 3x^4\cos^2 t$$

$$x^2 t' = 3x^4 \cos^2 t$$

$$\frac{dt}{\cos^2 t} = 3x^2\,dx$$

$$\operatorname{tg} t = x^3 + C$$

$$\operatorname{tg}\frac{y}{x} = x^3 + C \qquad y(1) = 0$$

the general solution:

$$y = x \operatorname{arctg}(x^3 + C) \qquad \operatorname{tg} 0 = 1^3 + C$$

the particular solution:

$$y = x \operatorname{arctg}(x^3 - 1) \qquad C = -1$$

Homogeneous DE, I

H1)

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

H2)

$$y' = \frac{y}{x} - e^{-\frac{y}{x}}$$

H3)

$$\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$$

H4*)

$$y' = \frac{x+2y-4}{2x-y-3}$$

Exercise.

$$(x^2+xy)\,dx-(xy+y^2)\,dy=0$$

$M(x,y)\,dx + N(x,y)\,dy = 0$

$$M(\alpha x,\alpha y)=(\alpha x)^2+\alpha x\cdot\alpha y=\alpha^2(x^2+xy)=$$
$$=\alpha^2 M(x,y)$$

$$N(\alpha x,\alpha y)=-(\alpha x)(\alpha y)-(\alpha y)^2=\alpha^2[-xy-y^2]=$$
$$=\alpha^2 N(x,y)$$

The given equation is a homogeneous DE

$$y=tx$$

Exercise. CONT.

$$(x^2+xy)\,dx-(xy+y^2)\,dy=0$$

The given equation is a homogeneous DE

The substitution: $y=t\,x$ or $y(x)=x\,t(x)$

$dy=x\,dt+t\,dx$

$$(x^2+x\,tx)\,dx-(x\,tx+(tx)^2)(x\,dt+t\,dx)=0$$

$$\left[(x^2(1+t))-tx^2t(1+t)\right]dx-x\cdot x^2t(1+t)\,dt=0$$

$$x^2(1+t)(1-t^2)\,dx=x^3t(1+t)\,dt$$

Exercise. CONT.

$$x^2(1+t)(1-t^2)\,dx = x^3 t(1+t)\,dt$$

$$\frac{dx}{x} = \frac{t\,dt}{1-t^2}$$

$$\ln|x| = -\frac{1}{2}\ln|1-t^2| + C$$

$$\ln|1-t^2| = -2\ln|x| + C \Rightarrow \ln|1-t^2| = \ln|x|^{-2}\cdot e^{C} \Rightarrow$$

$$\Rightarrow\ 1-t^2 = \frac{C}{x^2}$$

The substitution: $y = t\,x$

$$1-\frac{y^2}{x^2} = \frac{C}{x^2}$$

the general solution:

$$x^2 - y^2 = C$$

Adv. Eng. Math.
Ex.8, p.18

$$y' = (y + 4x)^2$$ ODE, I

$$y(x) \to v(x)$$

$$v = y + 4x$$

$$v' = y' + 4$$

Separable ODE, I

$$v' - 4 = v^2$$

$$v' = v^2 + 4$$

$$\frac{dv}{dx} = v^2 + 4$$

. . .

$y = f(x) \quad dy = f'(x)\,dx$

Exercise.

$$\frac{dy}{dx} = \frac{x+y-3}{x-y-1}$$

ODE, I , it is reducible to a homogeneous DE

$$\begin{cases} x+y = 3 \\ x-y = 1 \end{cases} \qquad \begin{cases} x_0 = 2 \\ y_0 = 1 \end{cases}$$

Substitution:

$$\begin{cases} x = 2+u \Rightarrow dx = du \\ y = 1+v \Rightarrow dy = dv \end{cases}$$

$y(x) \rightsquigarrow v(u)$

$$\frac{dv}{du} = \frac{u+v}{u-v}$$ is a homogeneous DE, I

$v(u) \rightsquigarrow t(u)$

$v = tu$

$dv = u\,dt + t\,du$

Exercise.

$$\frac{u\,dt + t\,du}{du} = \frac{u + tu}{u - tu}$$

$$u\frac{dt}{du} + t = \frac{u + tu}{u - tu}$$

$$u\frac{dt}{du} = \frac{1+t}{1-t} - t$$

$$u\frac{dt}{du} = \frac{1+t^2}{1-t}$$

Separable ODE, I

$$\frac{1-t}{1+t^2}\,dt = \frac{du}{u}$$

$$\int \frac{dt}{1+t^2} - \int \frac{t\,dt}{1+t^2} = \int \frac{du}{u}$$

$$\operatorname{arctg} t - \frac{1}{2}\ln(1+t^2) = \ln|u| + C$$

$$\operatorname{arctg} t = \ln\left|cu\sqrt{1+t^2}\right|$$

$$cu\sqrt{1+t^2} = e^{\operatorname{arctg} t}$$

$$cu\sqrt{1+\frac{v^2}{u^2}} = e^{\operatorname{arctg}\frac{v}{u}}$$

$$c\sqrt{u^2+v^2} = e^{\operatorname{arctg}\frac{v}{u}}$$

the general solution:

$$c\sqrt{(x-2)^2+(y-1)^2} = e^{\operatorname{arctg}\frac{y-1}{x-2}}$$

Exercise.

$$(x^2+2y^2)dy = xy\,dx$$

$$y = tx$$

$$dy = x\,dt + t\,dx$$

$$(x^2+2t^2x^2)(x\,dt+t\,dx) = x^2t\,dx$$

$$x^3(1+2t^2)\,dt = (\cancel{-x^2t} - 2t^3x^2 + \cancel{x^2t})dx$$

$$x^3(1+2t^2)\,dt = -2t^3x^2dx$$

$$\frac{1+2t^2}{t^3}\,dt = -2\frac{x^2}{x^3}\,dx$$

$$\int t^{-3}\,dt + 2\int \frac{dt}{t} = -2\int \frac{dx}{x}$$

$$-\frac{1}{2t^2} + 2\ln|t| = -2\ln|x| + C$$

$$\ln|cxt| = \frac{1}{4}\cdot\frac{1}{t^2}$$

the general solution:

$$y = c\cdot e^{x^2/4y^2}$$

Exercise.

$$(3x^2y+y^3)\,dy-(x^3+3xy^2)\,dx=0$$

$$y=tx\;;\quad dy=x\,dt+t\,dx$$

$$(3x^2tx+(tx)^3)(x\,dt+t\,dx)-(x^3+3x(tx)^2)\,dx=0$$

$$x(3x^3t+x^3t^3)\,dt+(\cancel{3x^3t^2}+x^3t^4-x^3-\cancel{3x^3t^2})\,dx=0$$

$$x^4t(3+t^2)\,dt=x^3(1-t^4)\,dx$$

$$\frac{t(3+t^2)}{1-t^4}\,dt=\frac{dx}{x}$$

$$\dots$$