Mathematics for Computer Science

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Lecture 9

Picture of the day

Picture of the day

Picture of the day

A **“relation”** is a fundamental mathematical term expressing a relationship between elements of sets.

Definition

A $k-$**place relation** R on sets $S_1, S_2, \ldots, S_k$ is $R \subseteq S_1 \times S_2 \times \ldots \times S_k$.

Definition

A $k-$**place relation** R on set M is $R \subseteq M \times M \times \ldots \times M = M^k$.

Relations

Definition

A **binary relation** from a set A to a set B is a subset $R \subseteq A \times B$.

Definition

A **binary relation** on set A is a subset $R \subseteq A \times A \equiv A^2$.

A binary relation R is a set of ordered pairs $(a, b) \in R$.

Write $a \sim_R b$ or aRb, or Rab (for logic languages, prefix notation) to mean that $(a, b) \in R$.

Function $y = f(x)$ can be written as $x \sim_f y$ or xfy, or fxy.

Functions are a particular case of relation!

Relations

1. The relation "is taking class" as a subset of
$$\{\text{students at UTM}\} \times \{\text{classes at UTM}\}$$
   A relation from students set to classes set.
2. The relation "has lecture in" as a subset of
$$\{\text{classes at UTM}\} \times \{\text{rooms at UTM}\}$$
   A relation from classes set to rooms set.
3. The relation "is living in the same room" as a subset of
$$\{\text{students at UTM}\} \times \{\text{students at UTM}\}$$
   A relation on students set.
4. The relation "can drive from Chisinau to Cahul". (Not necessarily directly—just some way, on some roads.)
5. Relation on computers, "are connected (directly) by a wire"

6. "meet one another on a given day"
7. "likes"
8. Let $A = \mathbb{N}$ and define $a \sim b$ iff $a \leqslant b$.
9. Let $A = 2^{\mathbb{N}}$ and define $a \sim b$ iff $a \cap b$ is finite.
10. Let $A = \mathbb{R}^2$ and define $a \sim b$ iff $dist(a, b) = 1$.
11. Let $A = 2^{\{1,\ldots,n\}}$ and define $a \sim b$ iff $a \subseteq b$.
12. Let $A = \mathbb{N}$ and define $a \sim b$ iff $a \equiv b(\mod p)$ (i.e. if a and b have the same remainder after division by p).
13. Let A be the set of all triangles in $\mathbb{R}^2$ and define $a \sim b$ iff a is similar to b.
14. Let A be the set of all lines in $\mathbb{R}^2$ and define $a \sim b$ iff a and b have the same slope.
15. Let A be the set of all humans and define $a \sim b$ iff a is a child of b.

Properties of Relations

For a relation defined on a set A, there are several standard properties that occur commonly. Use these properties to classify different types of relations.

Definition

A binary relation R on A (in other words $R \subset A \times A$) is called:

$$
\begin{aligned}
(1) \quad & \textbf{reflexive} \ \text{ if } \forall a \in A,\ a \sim a. \\
(2) \quad & \textbf{symmetric} \ \text{ if } \forall a, b \in A,\ a \sim b \ \text{ implies } b \sim a. \\
(3) \quad & \textbf{antisymmetric} \ \text{ if } \forall a, b \in A, a \sim b \text{ and } b \sim a \ \text{ implies } a = b. \\
(4) \quad & \textbf{asymmetric} \text{ if } \forall a, b \in A,\ a \sim b \text{ implies } (b \nsim a). \\
(5) \quad & \textbf{transitive} \ \text{ if } \forall a, b, c \in A,\ a \sim b \text{ and } b \sim c \text{ implies } a \sim c.
\end{aligned}
$$

The difference between **antisymmetric** and **asymmetric** relations is that antisymmetric relations may contain pairs (a, a), i.e., elements can be in relations with themselves, while in an asymmetric relation this is not allowed.

Properties of Relations

$$
\begin{array}{rll}
(1) & \textbf{reflexive} & \text{if } \forall a \in A,\ a \sim a. \\
(2) & \textbf{symmetric} & \text{if } \forall a, b \in A,\ a \sim b \ \text{ implies } b \sim a. \\
(3) & \textbf{antisymmetric} & \text{if } \forall a, b \in A, a \sim b \text{ and } b \sim a \ \text{ implies } a = b. \\
(4) & \textbf{asymmetric} & \text{if } \forall a, b \in A,\ a \sim b \text{ implies } (b \nsim a). \\
(5) & \textbf{transitive} & \text{if } \forall a, b, c \in A,\ a \sim b \text{ and } b \sim c \text{ implies } a \sim c.
\end{array}
$$

3. The relation "is living in the same room" as a subset of $\{\text{students at UTM}\} \times \{\text{students at UTM}\}$.

Relation 3 is reflexive, symmetric, transitive.

4. The relation "can drive from Chisinau to Cahul".

Relation 4 is reflexive, transitive. Not necessarily symmetric, since roads could be one-way (consider one-way streets). But definitely not antisymmetric.

5. Relation on computers, "are connected (directly) by a wire"

Relation 5 is symmetric, but not transitive. Whether it is reflexive is open to interpretation.

$$\begin{array}{rrl} (1) & \textbf{reflexive} & \text{if } \forall a \in A,\ a \sim a. \\ (2) & \textbf{symmetric} & \text{if } \forall a, b \in A,\ a \sim b \ \text{ implies } b \sim a. \\ (3) & \textbf{antisymmetric} & \text{if } \forall a, b \in A, a \sim b \text{ and } b \sim a \ \text{ implies } a = b. \\ (4) & \textbf{asymmetric} & \text{if } \forall a, b \in A,\ a \sim b \text{ implies } (b \nsim a). \\ (5) & \textbf{transitive} & \text{if } \forall a, b, c \in A,\ a \sim b \text{ and } b \sim c \text{ implies } a \sim c. \end{array}$$

6. “meet one another on a given day”

Relation 6 is reflexive (obviously), symmetric, but not transitive.

7. “likes”

Relation 7 is not symmetric. Not antisymmetric. Not transitive. Not even reflexive!

8. Let $A = \mathbb{N}$ and define $a \sim_R b$ iff $a \leqslant b$.

Relation 8 is reflexive, antisymmetric, transitive.

9. Let $A = 2^{\mathbb{N}}$ and define $a \sim_R b$ iff $a \cap b$ is finite.

Relation 9 is not reflexive. It is symmetric. It is not transitive.
$\{$even naturals$\} \cap \{$odd naturals$\}$ is finite (empty),
but not $\{$even naturals$\} \cap \{$even naturals$\}$.

$$
\begin{aligned}
(1) \quad & \textbf{reflexive} && \text{if } \forall a \in A,\ a \sim a. \\
(2) \quad & \textbf{symmetric} && \text{if } \forall a, b \in A,\ a \sim b \ \text{ implies } b \sim a. \\
(3) \quad & \textbf{antisymmetric} && \text{if } \forall a, b \in A, a \sim b \text{ and } b \sim a \ \text{ implies } a = b. \\
(4) \quad & \textbf{asymmetric} && \text{if } \forall a, b \in A,\ a \sim b \text{ implies } (b \nsim a). \\
(5) \quad & \textbf{transitive} && \text{if } \forall a, b, c \in A,\ a \sim b \text{ and } b \sim c \text{ implies } a \sim c.
\end{aligned}
$$

10. Let $A = \mathbb{R}^2$ and define $a \sim_R b$ iff $d(a, b) = 1$.

Relation 10 is only symmetric.

11. Let $A = 2^{\{1,\ldots,n\}}$ and define $a \sim_R b$ iff $a \subseteq b$.

Relation 11 is reflexive, antisymmetric and transitive.

12. Let $A = \mathbb{N}$ and define $a \sim b$ iff $a \equiv b(\mod p)$ (i.e. if a and b have the same remainder after division by p).

Relation 12 is reflexive, symmetric and transitive.

Representations of Relations: Lists

There are many different ways of representing relations. One way is to describe them by properties.

For finite sets, can explicitly enumerate all elements of the relation. Some alternatives are *lists*, *matrices* and *graphs*.

Different representations may be more efficient for encoding different problems and also tend to highlight different properties of the relation.

A relation from set A to set B can be represented by a **list** of all the pairs.

Example (1)

The relation from $\{0, 1, 2, 3\}$ to $\{a, b, c\}$ defined by the list:

$$\{(0, a), (0, c), (1, c), (2, b), (1, a)\}.$$

Example (2)

The relation from $A = \{a, b, c, d, e\}$ to A:

$$\{(a, b), (a, d), (b, b), (c, b), (c, d), (d, c), (d, e), (e, b), (e, e)\}.$$

Representations of Relations: Lists

Example (3)

The divisibility relation on natural numbers $\{1, \ldots, 12\}$ is represented by the list:

$$\begin{aligned}\{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(1,9),\\&(1,10),(1,11),(1,12),(2,2),(2,4),(2,6),(2,8),(2,10),(2,12),\\&(3,3),(3,6),(3,9),(3,12),(4,4),(4,8),(4,12),(5,5),(5,10),(6,6),\\&(6,12),(7,7),(8,8),(9,9),(10,10),(11,11),(12,12)\}.\end{aligned}$$

Can recognize certain properties by examining this representation:

Reflexivity: It contains all pairs (a, a).

Symmetry: If it contains (a,b), then it contains (b, a).

Transitivity: If it contains (a, b) and (b, c) then it contains (a, c).

Representations of Relations: Boolean Matrices

Boolean matrices are a convenient representation for representing relations in computer programs.

The rows are for elements of A, columns for B, and for every entry there is a 1, if the pair is in the relation, and 0 otherwise.

Example

The relation from Example (1):

$$\{(0, a), (0, c), (1, c), (2, b), (1, a)\}.$$

is represented by the matrix

	a	b	c
0	1	0	1
1	1	0	1
2	0	1	0
3	0	0	0

Representations of Relations: Boolean Matrices

Example

The relation from Example (2):

$$\{(a,b),(a,d),(b,b),(c,b),(c,d),(d,c),(d,e),(e,b),(e,e)\}.$$

	a	b	c	d	e
a	0	1	0	1	0
b	0	1	0	0	0
c	0	1	0	1	0
d	0	0	1	0	1
e	0	1	0	0	1

Again, properties can by recognized by examining the representation:
Reflexivity: the major diagonal is all 1.
Symmetry: the matrix is clearly not symmetric across the major diagonal.
Transitivity: not so obvious ...

Representations of Relations: Boolean Matrices

Example

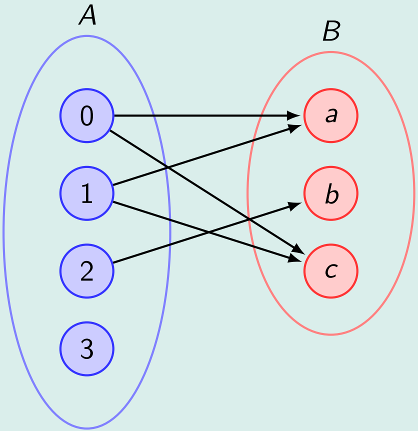
The divisibility relation over $\{1, 2, \ldots, 12\}$ is represented by:

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	0	1	0	1	0	1	0	1	0	1
3	0	0	1	0	0	1	0	0	1	0	0	1
4	0	0	0	1	0	0	0	1	0	0	0	1
5	0	0	0	0	1	0	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	0	0	0	1
7	0	0	0	0	0	0	1	0	0	0	0	0
8	0	0	0	0	0	0	0	1	0	0	0	0
9	0	0	0	0	0	0	0	0	1	0	0	0
10	0	0	0	0	0	0	0	0	0	1	0	0
11	0	0	0	0	0	0	0	0	0	0	1	0
12	0	0	0	0	0	0	0	0	0	0	0	1

Can represent a relation $R \subseteq A \times B$ by digraphs (directed graphs).

Example (1)

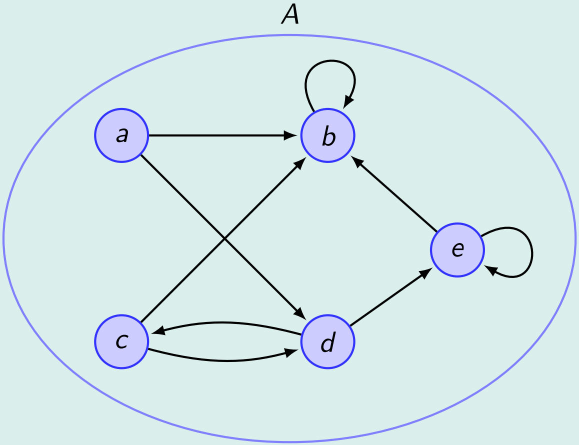
Relation $\{(0, a), (0, c), (1, c), (2, b), (1, a)\}$ is represented by:



Note that generally a binary relation on $A \times B$ is represented by a **bipartite graph**.

Example (2)

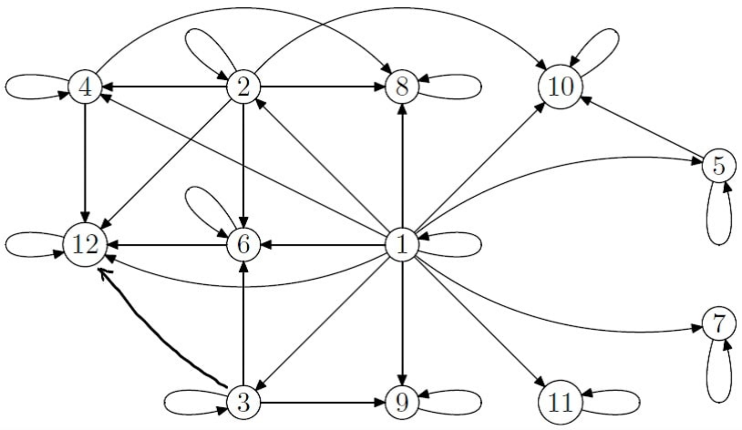
Relation $\{(a, b), (a, d), (b, b), (c, b), (c, d), (d, c), (d, e), (e, b), (e, e)\}$ is represented by:



Loops are for nodes in relation with themselves.

Example (3)

The divisibility relation is represented by:



How to detect properties from graph representation?

Reflexivity: All nodes have self-loops.

Symmetry: All edges are bidirectional.

Transitivity: Short-circuits—for any sequence of consecutive arrows, there is a single arrow from the first to the last node.

Operations on Relations: Inverse

Definition

If R is a relation on $A \times B$, then **inverse relation** of R, denoted by R^{-1} is a relation on $B \times A$ given by $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.

It's just the relation R turned backwards.

Example

Inverse of "is taking class" is the relation "has as a student" on the set $\{$classes at UTM$\} \times \{$students at UTM$\}$; a relation from classes to students.

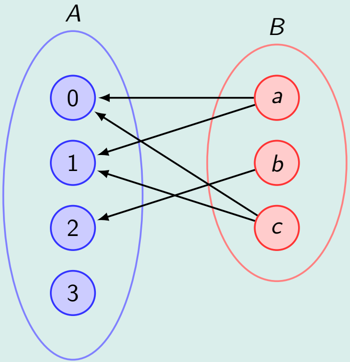
Given the matrix for R, we can get the matrix for R^{-1} by transposing the matrix for R.

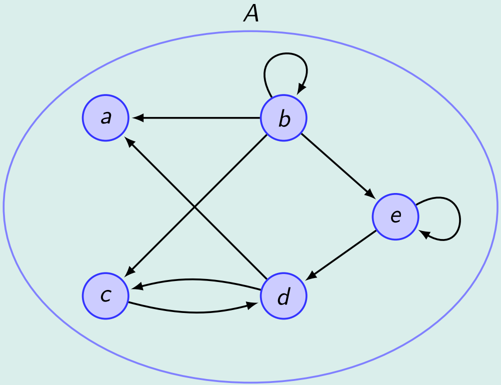
Note that the inverse of a relation is not the same thing as the inverse of the matrix representation.

Operations on Relations: Inverse

Given a digraph for R, we get the graph for R^{-1} by reversing every edge in the original digraph.

Example (1, 2)





Operations on Relations: Composition

Definition

The **composition** of relations $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$ is the relation

$$R_2 \circ R_1 = \{(a, c) \mid \exists b \quad ((a, b) \in R_1) \wedge ((b, c) \in R_2)\}.$$

In words, the pair (a, c) is in $R_2 \circ R_1$, if there exists an element b such that the pair (a, b) is in R_1 and the pair (b, c) is in R_2.

Another way of thinking about this is that a "path" exists from element a to c via some element b in the set B.

Similar to composition of functions.

Operations on Relations: Composition

Example

The composition of the relation "is taking class" with the relation "has lecture in" is the relation "should go to lecture in", a relation from {students at UTM} to {rooms at UTM}.

Example

Composition of the parent-of relation with itself gives grandparent-of. Composition of the child-of relation with the parent-of relation gives the sibling-of relation. Does composition of parent-of with child-of give married-to/domestic partners? No, because that misses childless couples.

Example

Let B be the set of boys, G be the set of girls, $R_1 \subseteq B \times G$ consists of all pairs (b, g) such that b is madly in love with g, and $R_2 \subseteq G \times B$ consist of all pairs (g, b) such that g is madly in love with b. What are the relations $R_2 \circ R_1$ and $R_1 \circ R_2$, respectively?

Operations on Relations: Closure

A closure "extends" a relation to satisfy some property. But extends it as little as possible.

Definition

The closure of relation R with respect to property P is the relation S that contains R, has property P, and it is contained in any relation satisfying the first two.

That is, S is the "smallest" relation with property P and containing R.

There are two possibilities to construct a closure of R with respect to property P:

1. Can start with R and add as few pairs as possible until the new relation has property P;
2. Can start with the largest possible relation (which is $A \times A$ for a relation on A) and then remove as many not-in-R pairs as possible while preserving the property P.

Example

Consider relation R on set $A = \{a, b, c, d, e\}$:

$$R = \{(a, b), (a, d), (b, b), (c, b), (c, d), (d, c), (d, e), (e, b), (e, e)\}.$$

then the **reflexive closure** of R is

$$\begin{aligned} R_1 = \{&(a, b), (a, d), (b, b), (c, b), (c, d), (d, c), (d, e), (e, b), (e, e), \\ &\color{red}{(a, a), (c, c), (d, d)}\} \end{aligned}$$

and the **symmetric closure** of R is

$$\begin{aligned} R_2 = \{&(a, b), (a, d), (b, b), (c, b), (c, d), (d, c), (d, e), (e, b), (e, e), \\ &\color{red}{(b, a), (d, a), (b, c), (e, d), (b, e)}\}. \end{aligned}$$

Equivalence Relations and Partitions

Use properties of relations to classify them into different types.
Will consider two important types of relations: **equivalence relations** and **partial orders**.

Definition

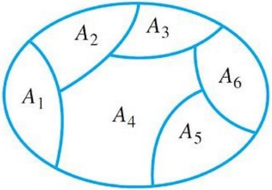
An **equivalence relation** is a relation that is reflexive, symmetric and transitive.

For example, the "roommates" relation is an equivalence relation.
So is "same size as", and "on same Ethernet hub".
A trivial example is the $=$ relation on natural numbers.

The hint of equivalence relation is the word same.
It provides a way to hide unimportant differences. Using an equivalence relation we can actually partition the universe into subsets of things that are the "same", in a natural way.

Definition

A **partition** of a set A is a collection of subsets $\{A_1, \dots, A_k\}$ such that any two of them are disjoint (for any $i \neq j$, $A_i \cap A_j = \varnothing$) and such that their union is A.



Let R be an equivalence relation on set A. For an element $a \in A$, let

$$[a] = \{b \in A, \text{ such that } a \sim_R b\}.$$

This set is called the **equivalence class** of a under R and a is called **representative** of $[a]$.

Lemma

The sets $[a]$ for $a \in A$ constitute a partition of A. That is, for every $a, b \in A$, either $[a] = [b]$ or $[a] \cap [b] = \varnothing$.

Partial Orders

Partial orders are another type of binary relation that is very important in Computer Science.

They have applications to task scheduling, database concurrency control, and logical time in distributed computing.

Definition

A binary relation $R \subseteq A \times A$ is a **partial order** if it is reflexive, transitive, and antisymmetric.

Recall that antisymmetric mean

$$aRb \text{ and } bRa \quad \text{implies} \quad a = b,$$

or

$$\forall a \neq b, aRb \quad \text{implies} \quad \neg(bRa).$$

In other words, this relation is never symmetric!

This single property is what distinguishes it from an equivalence relation.

The reflexivity, antisymmetry and transitivity properties are abstract properties that generally describe “ordering” relationships.

For a partial order relation we often write an ordering-style symbol like $\preccurlyeq$, instead of just a letter like R.

This lets us use notation similar to $\leqslant$.

For example, we write $a \preccurlyeq b$ if $a \preccurlyeq b$ and $a \neq b$.

Similarly, we write $b \succcurlyeq a$ as equivalent to $a \preccurlyeq b$.

A partial order is always defined on some set A.

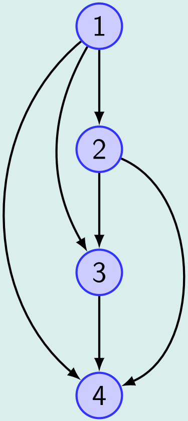
The set together with the partial order is called a “poset”:

Definition

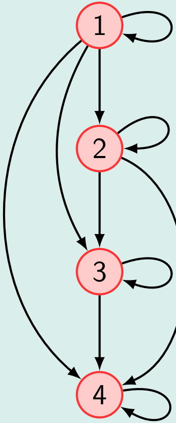
A set A together with a partial order $\preccurlyeq$ is called a **poset** $(A, \preccurlyeq)$.

Example

Consider the relation "$<$" on the set $A = \{1, 2, 3, 4\}$ (left) and the poset defined by the relation "$\leqslant$" on same set (right).



"$<$"



"$\leqslant$"

Directed Acyclic Graphs

Definition

A cycle in a graph (or digraph) is a path that ends where it started (i.e., the last vertex equals the first).

Definition

A directed acyclic graph (DAG) is a directed graph with no cycles.

Lemma

Any partial order is a DAG.

But not all DAG are partial orders!

Lemma

The transitive reflexive closure of a DAG is a partial order

Theorem

A poset has no directed cycles other than self-loops.

Total order

A partial order is "partial" because there can be two elements with no relation between them. In general, we say that two elements a and b are **incomparable**, if neither $a \preccurlyeq b$ nor $b \preccurlyeq a$. Otherwise, if $a \preccurlyeq b$ or $b \preccurlyeq a$, then we say that a and b are **comparable**.

Definition

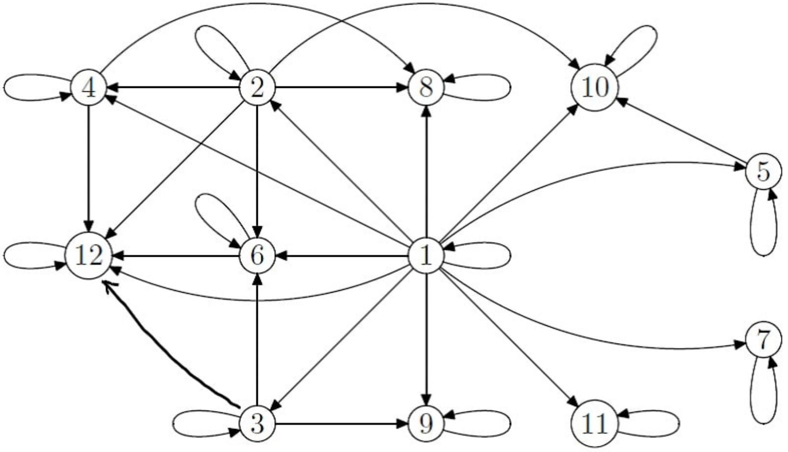
A **total order** is a partial order in which every pair of distinct elements is comparable.

Example

1. $\leqslant$ relation on $\mathbb{N}$ is a total order, since $\forall a, b \in \mathbb{N}$ we will have either $a \leqslant b$ or $b \leqslant a$.
2. Let $A = 2^{\{0,1\}} = \{\varnothing, \{0\}, \{1\}, \{0,1\}\}$ and partial order $\subseteq$. It is not a total order, since neither $\{0\}$ and $\{1\}$ are incomparable.
3. Let $B = \{1, 2, 3, 6\}$. Elements x and y are in relation $\sim$ if x divides y.
   So, for example $2 \sim 6$, and $3 \sim 3$. It is a partial order. But it is not a total order, since 2 and 3 are incomparable.

Hasse diagramms

One problem with viewing a poset as a digraph is that there tend to be lots of edges due to the transitivity property, and every vertex with a self-loop due to reflexivity.



We could choose not to draw any edge which would be implied by the transitivity property, knowing that it is really there by implication.

And, we could choose to skip all the self-loops, due to redundancy.

In general, following these considerations we will get the Hasse diagrams:

Definition

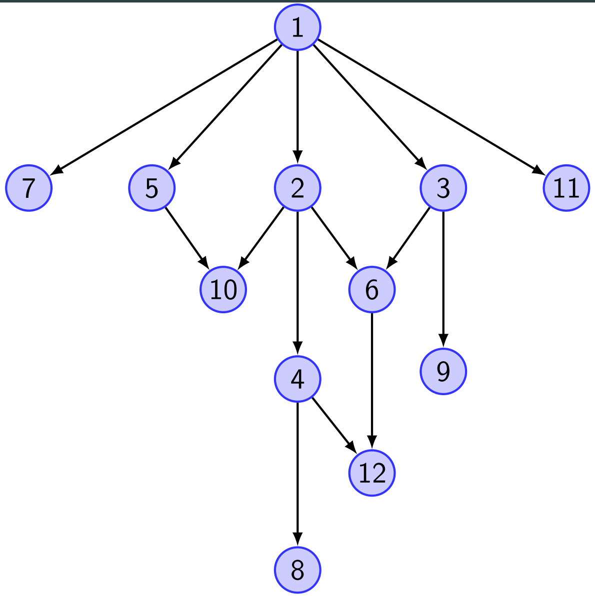
A **Hasse diagramm** for a poset $\{A, \preccurlyeq\}$ is a digraph with vertex set A and edge set minus all self-loops and edges implied by transitivity.

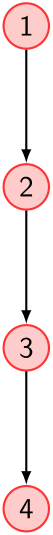
Hasse diagram will simplify greatly the digraph of a partial order relation.

Example

Hasse diagram of divisibility relation on set $B = \{1, 2, 3, ..., 12\}$ and
Hasse diagram of "less than" relation on set $A = \{1, 2, 3, 4\}$
are shown on next page.

Hasse diagramms





Directed Acyclic Graphs

A common source of partial orders in Computer Science is in so-called **task graphs**.

You have a set of tasks A, and a relation R on A in which aRb means " task b cannot be done until task a is finished".

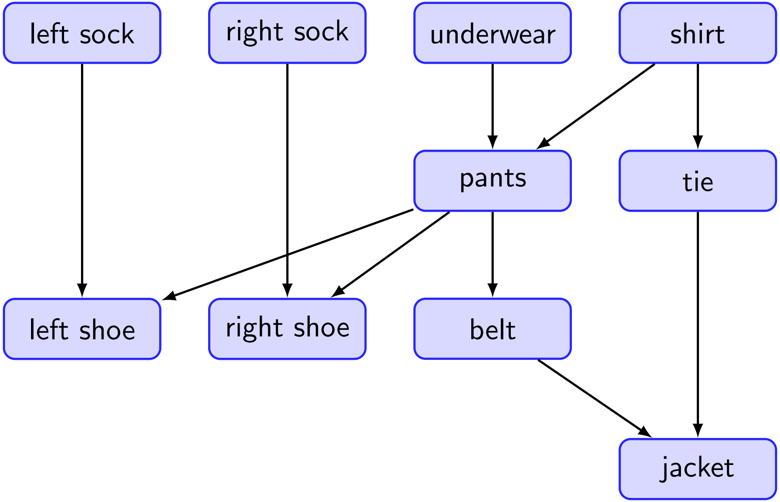
In other words, "task a precedes task b".

Implicitly, "if all the things that point at b are done, I can do b."

This can be nicely drawn as a graph, called **task graph**. We draw an arrow from a to b if aRb.

Let's consider the example that describes the order in which one would put on clothes in the morning. The set is of clothes, and the edges say what should be put on before what.

Directed Acyclic Graphs



Topological sort

Recall that, in a partial ordered set (poset) there might exist elements that are not comparable (that's why the relation is called partial order).

Also, if in a poset all elements are comparable, then the relation is called total order.

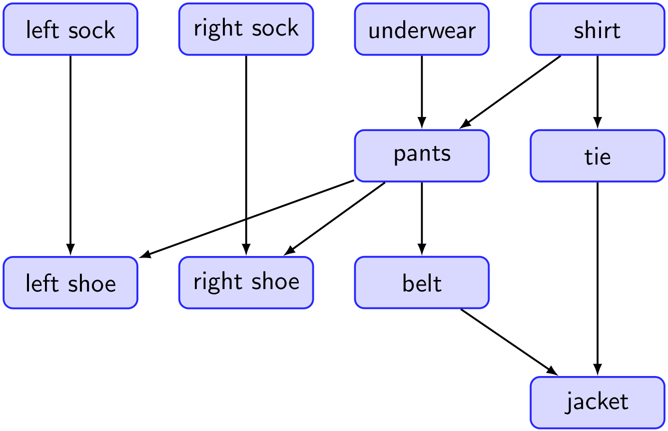
A total order that is consistent with a partial order is called a topological sort.

Definition

A **topological sort** of a poset $(A, \preccurlyeq)$ is a total order $(A, \preccurlyeq_T)$ such that

$$x \preccurlyeq y \qquad \text{implies} \qquad x \preccurlyeq_T y.$$

Topological sort



Clearly,

$$\textit{left sock} \preccurlyeq \textit{left shoe}$$
$$\textit{pants} \preccurlyeq \textit{belt}$$
$$\textit{shirt} \preccurlyeq \textit{belt}$$

On the other hand *left sock* and *right sock* are incomparable.

Also, *right shoe* and *belt* are incomparable too.

There are several total orders (topological sorts) that are consistent with this partial order.

Topological sort

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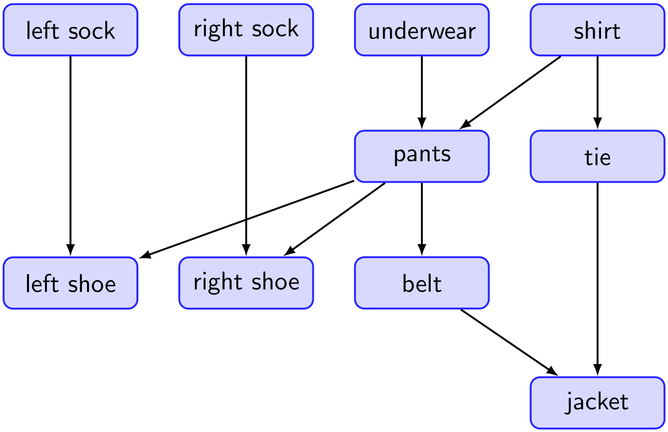
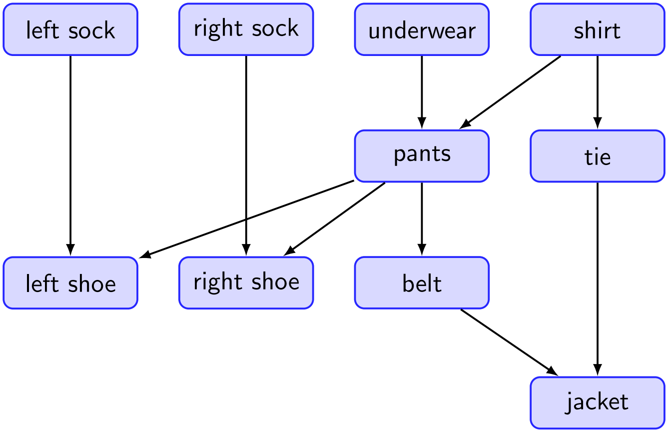


Example 1:

underwear
shirt
pants
belt
tie
jacket
left sock
right sock
left shoe
right shoe

There are several total orders (topological sorts) that are consistent with this partial order.



Example 2:

left sock
shirt
tie
underwear
right sock
pants
right shoe
belt
jacket
left shoe

Topological sort

Theorem

Every **finite** *poset has a topological sort.*

Proof by induction on the number of elements in poset. Idea is as follows: *Take the* **"smallest"** *element out, find a topological sort for the remaining, and then add this smallest element making it smaller than all the others.*
The first difficulty is that **"smallest"** is not such a simple concept in a set that is only partially ordered. Need to define it precisely!

Definition

In a poset $(A, \preccurlyeq)$ an element $x \in A$ is called **minimal**, if there is no other element $y \in A$ such that $y \preccurlyeq x$.

Example

There are four minimal elements in the getting-dressed poset:

$$\textit{left sock}, \quad \textit{right sock}, \quad \textit{underwear}, \quad \textit{shirt}.$$

Topological sort

Definition

An element $x \preccurlyeq A$ is **maximal**, if there is no other element $y \in A$ such that $x \preccurlyeq y$.

Proving that every poset has a minimal element is extremely difficult, because it is not true!

Example

The poset $(\mathbb{Z}, \leqslant)$ has no minimal element.

However, there is at least one minimal element in every **finite** poset.

Lemma

Every **finite** *poset has a minimal element.*

Suppose that the elements of a poset are tasks that need to be done such that partial order is a precedence constraint.
In that case, the topological sorting provides a way to execute the tasks **sequentially** without violating the precedence constraints.

What if there is a possibility to execute more than one task at the same time?

In other words the tasks can be executed in **parallel?**

Suppose that the tasks are programs and the partial order indicates data dependence.

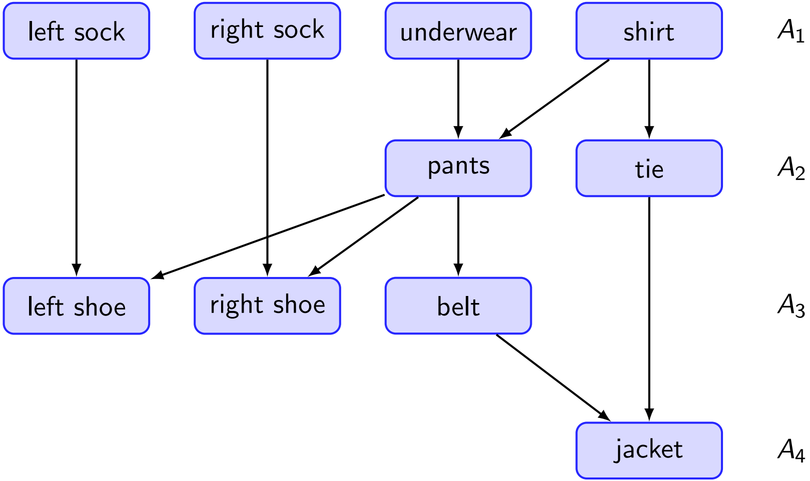
Consider a parallel machine with lots of processors instead of a sequential machine with only one processor.

How should we schedule the tasks so as to minimize the total time used?

For simplicity, assume all tasks take 1 unit of time and we have an unlimited number of identical processors.

For example, the getting dress procedure can be executed in 4 units of time (see next page).

Parallel task scheduling



Definition

A chain is a sequence $a_1 a_2 \dots a_n$, where $a_i \neq a_j$ for all $i \neq j$, such that each item is comparable to the next in the chain, and it is smaller with respect to $\preccurlyeq$. The length of the chain is n, the number of elements in the chain.

Theorem

Given any finite poset $(A, \preccurlyeq)$ for which the longest chain has length n, it is possible to partition A into n subsets $A_1, A_2, ..., A_n$ such that for all $i \in \{1, 2, \dots, n\}$ and for all $a_i \in A$, we have that all $b \preccurlyeq a$ appear in the set $A_1 \cup A_2 \cup \dots \cup A_{i-1}$.

Corollary

The total amount of parallel time needed to complete the tasks is the same as the length of the longest chain.

Dilworth's lemma

Definition

An antichain in a poset is a set of elements such that any two elements in the set are incomparable.

Lemma (Dilworth)

For all $k > 0$, every partially ordered set with n elements must have either a chain of size greater than k or an antichain of size at least $\frac{n}{k}$.

Corollary

Every partially ordered set with n elements has a chain of size greater than $\sqrt{n}$ or an antichain of size at least $\sqrt{n}$.

Summary

- Relations. Examples of relations
  - k-place relation, binary relation, binary relation on a set
- Properties of relations
  - reflexive, symmetric, antisymmetric, asymmetric, transitive
- Representations of relations
  - lists, boolean matrices, digraphs
- Operations on relations
  - inverse, composition closure
- Equivalence relation
- Partial order relation. Poset
- DAG
- Total order
- Hasse diagram
- Topological sort
- Dilworth's Lemma