Numerical Analysis

Prof.dr.hab. Bostan Viorel

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Furthermore, we can carry out only a finite number of such operations.

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How to evaluate other functions such as

$$e^x, \quad \cos x, \quad \log x, \quad \tan x, \quad \text{etc?}$$

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and we get $P_1(x) = 1 + x$.

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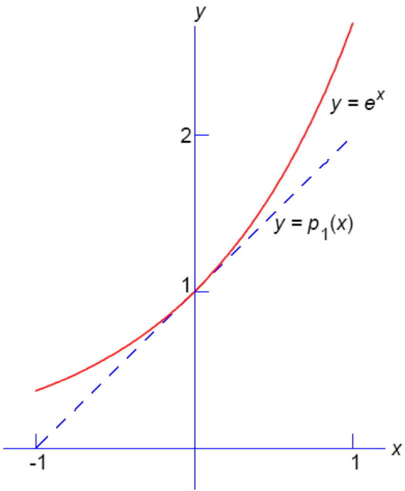

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This yields the approximation

$$P_2(x) = 1 + x + \frac{1}{2}x^2.$$

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Moreover, as $n \to \infty$ it can be shown that $P_n(x) \to e^x$

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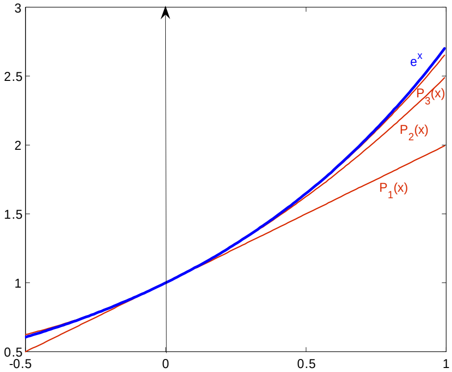

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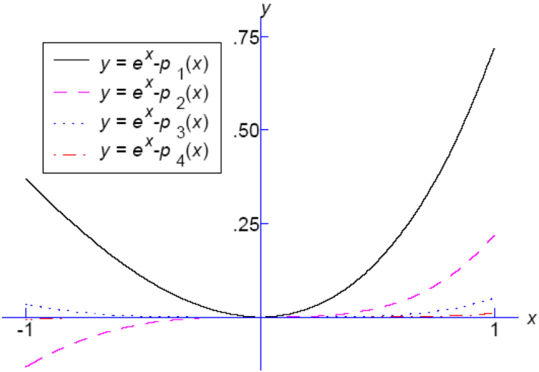

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This last term is also the final term in $P_{n+1}(x)$, and thus

$$e^x - P_n(x) \approx P_{n+1}(x) - P_n(x)$$

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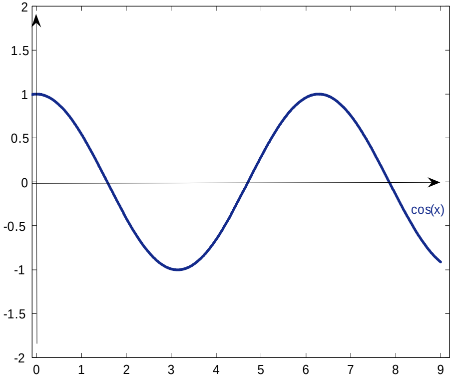

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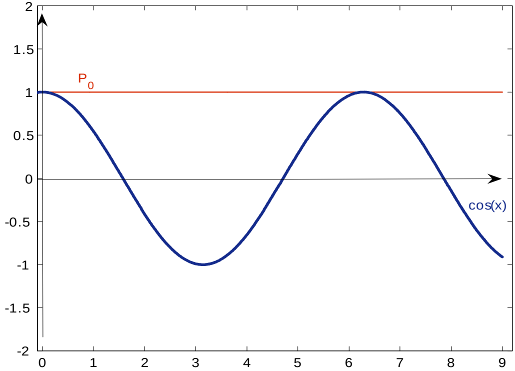

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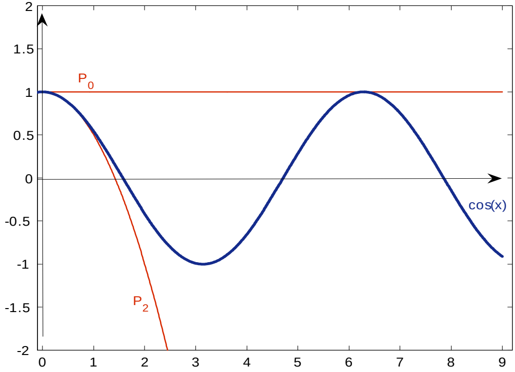

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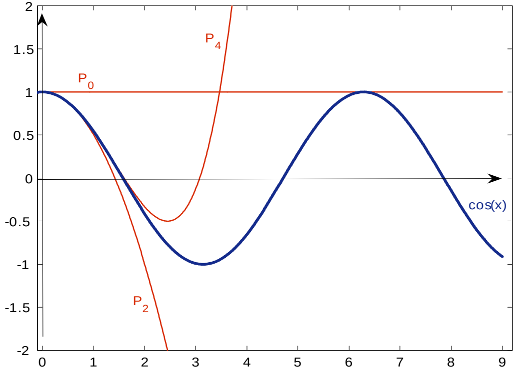

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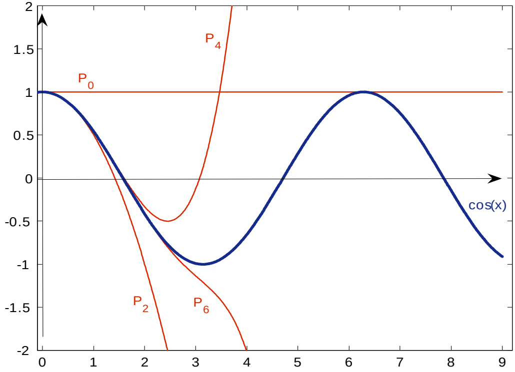

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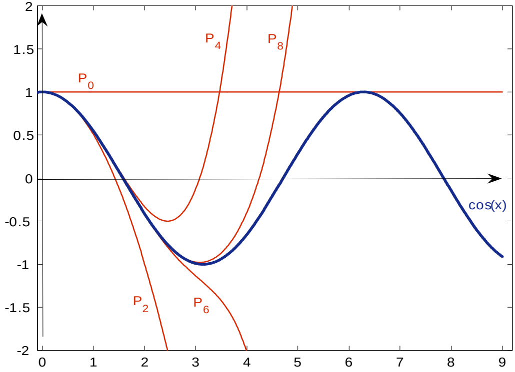

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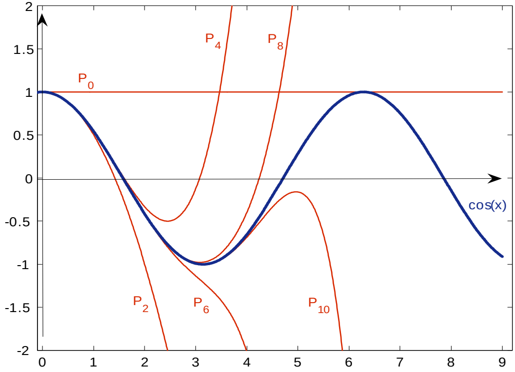

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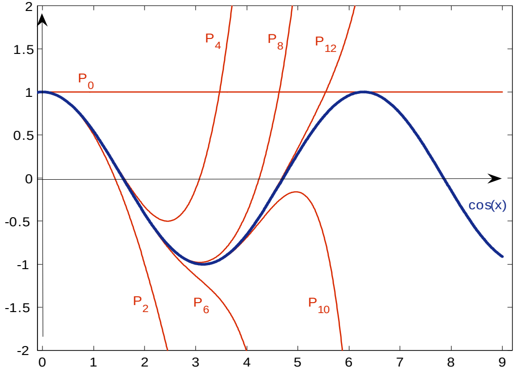

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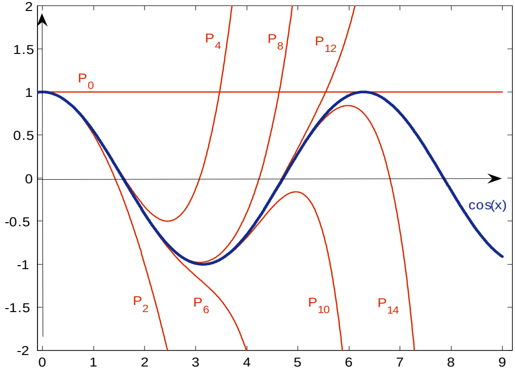

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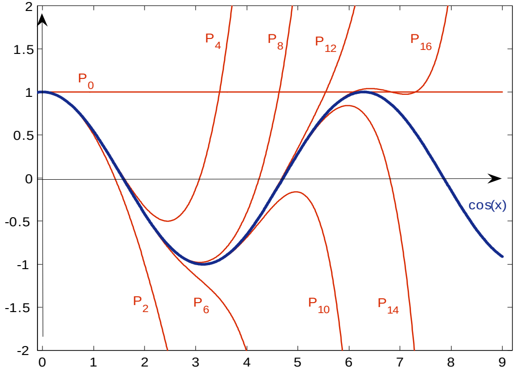

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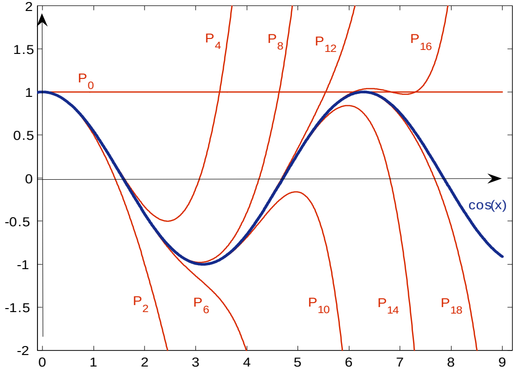
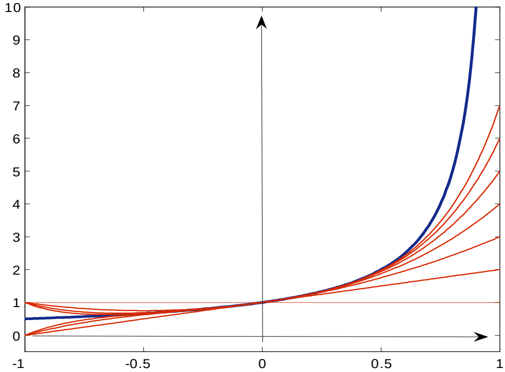


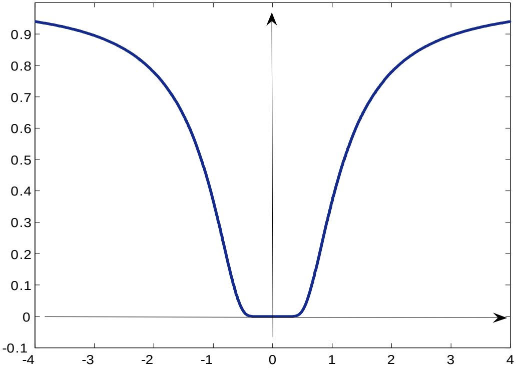
Taylor polynomial approximations

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \ldots, \qquad |x| < 1$$



Function whose Taylor series doesn't converge to the function itself

$$f(x) = \begin{cases} 0, & \textit{if} \quad x = 0 \\ e^{-\frac{1}{x^2}}, & \textit{if} \quad x \neq 0 \end{cases}$$



Computing approximation to e

Coming back to earlier approximation

$$e^x \approx P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots + \frac{1}{n!}x^n, \quad x \in \mathbb{R}$$

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Then calculate $P_8(1)$:

$$P_8(1) = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{8!} = 2.71827877,$$

Taylor polynomial approximations

In fact,

$$e - P_8(1) \leq \frac{e}{9!} \approx 7.5 \cdot 10^{-6}$$

Then calculate $P_8(1)$:

$$P_8(1) = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{8!} = 2.71827877,$$

and true error is

$$e - P_8(1) \approx 3.06 \cdot 10^{-6}$$

Evaluation of polynomials

Consider having a polynomial

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This may seem a strange question, but the answer is not as obvious as you might think.

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Above, the counts are as follows:

$$\begin{aligned} \textit{additions} &: \quad n \\ \textit{multiplications} &: \quad 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \end{aligned}$$

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The total operations cost is

$$\begin{aligned} \textit{additions} &: \quad n \\ \textit{multiplications} &: \quad n + n - 1 = 2n - 1 \end{aligned}$$

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For example, with $n = 20$,

the first method has 210 multiplications,

whereas the second has 39 multiplications.

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As examples of particular degrees, write

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n &= 3 : p(x) = a_0 + x(a_1 + x(a_2 + a_3 x)) \\
n &= 4 : p(x) = a_0 + x(a_1 + x(a_2 + x(a_3 + a_4 x)))
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The first method will need $3, 6$, and 10 multiplications.

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With all three methods, the number of additions is n; but the number of multiplications can be dramatically different for large values of n.

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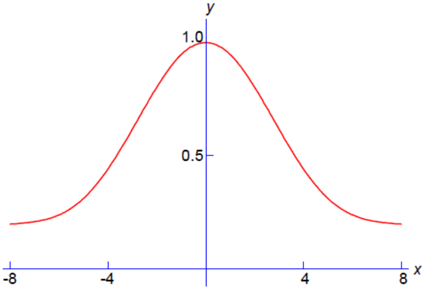

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As an example, begin with the degree 3 Taylor approximation to $\sin t$, expanded about $t = 0$:

$$\sin t = t - \frac{1}{6}t^3 + \frac{1}{120}t^5 \cos \xi, \qquad 0 \le \xi \le t$$

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\int_0^x \frac{\sin t}{t}\,dt &= \int_0^x 1\,dt - \frac{1}{6}\int_0^x t^2\,dt + \frac{1}{120}\int_0^x t^4 \cos\xi\,dt \\
\int_0^x \frac{\sin t}{t}\,dt &= x - \frac{1}{18}x^3 + \frac{1}{120}\int_0^x t^4 \cos\xi\,dt
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How large is the error in approximation

$$SF(x) \approx 1 - \frac{1}{18}x^2$$

on interval $x \in [-1, 1]$?

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To obtain a more accurate approximation, we can proceed exactly as above, but simply use a higher degree approximation to $\sin t$.