Mathematics for Computer Science

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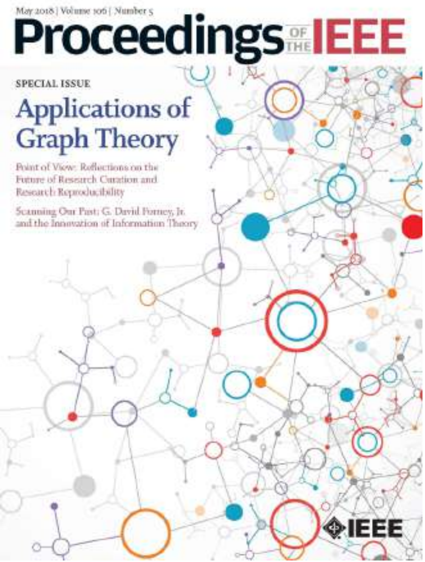
Lecture 10

Pictures of the Day



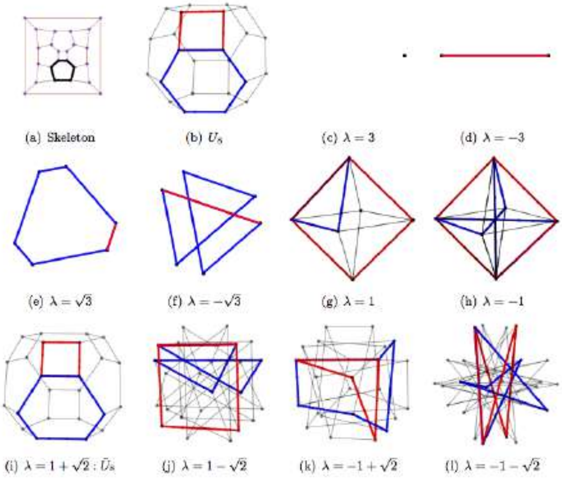
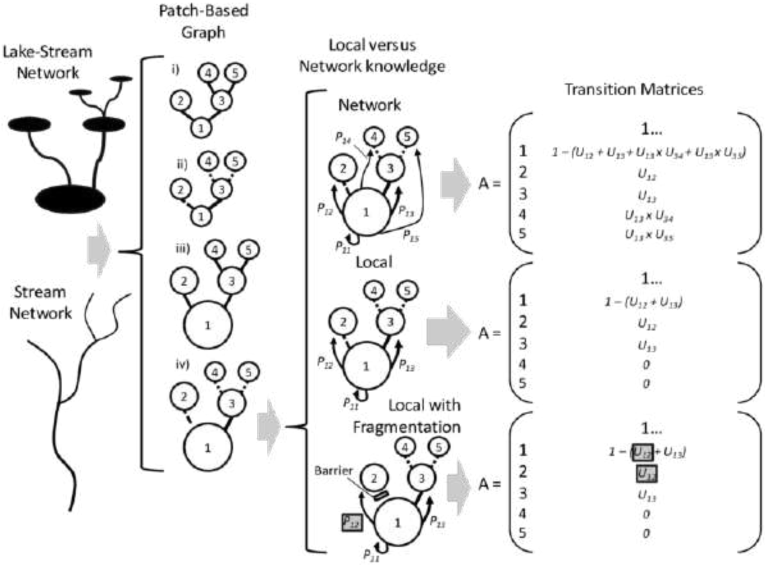
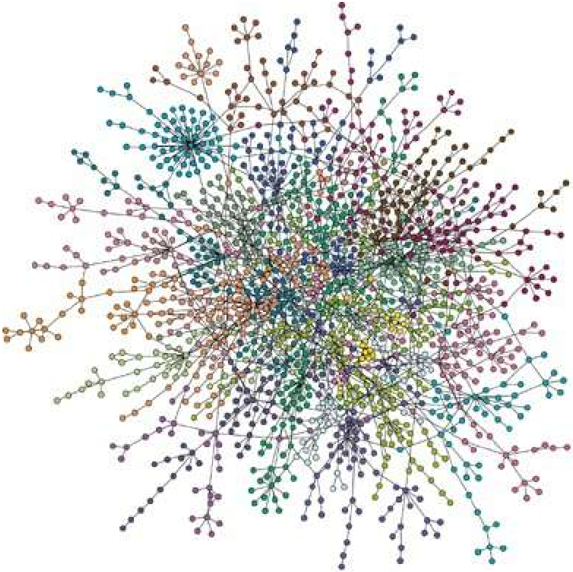


FIGURE 12. The U_8 skeleton (a), its classical realization (b), and its low-dimensional spectral realizations. (The $(1+\sqrt{2})$-realization is only pseudo-classical: the hexagonal faces are not regular.)

Pictures of the Day





Leonard Euler, 1707–1783

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Later, Vandermonde studied the knot problem using methods developed by Leibniz.

Euler also introduced his famous formula relating the number of edges, vertices, and faces of a convex polyhedron, generalized later by Cauchy.

Seven Bridges of Konigsberg

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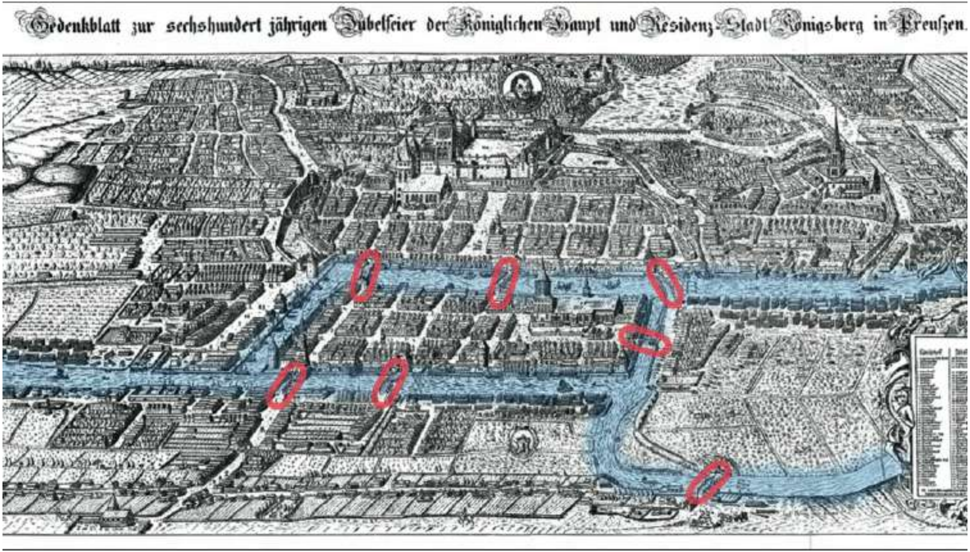

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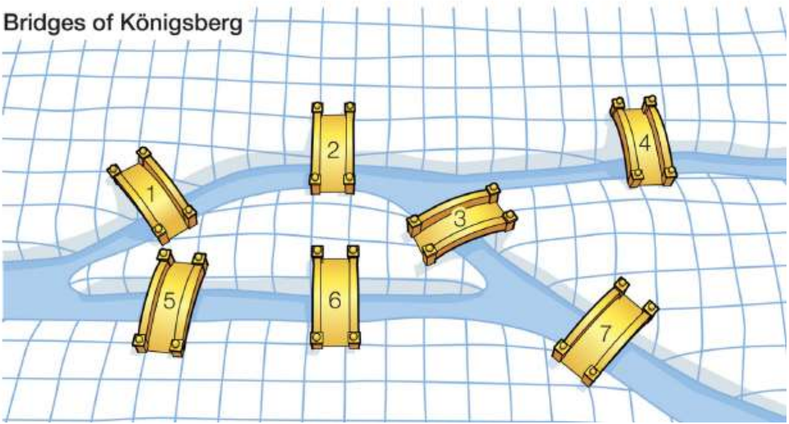

Euler's Question

Is it possible to find a walk through the city that would cross each bridge once and only once?


ICELAND
FINLAND
RUSSIA
UKRAINE
FRANCE
SPAIN
ITALY
GREECE
ALGERIA


UCRAINA
ROMÂNIA

The 4 Color Conjecture was first proposed on October 23, 1852 by Francis Guthrie, while trying to color the map of counties of England, noticed that only four different colors were needed.

Theorem (4 Color Theorem, Intuitive Formulation)

Given any separation of a plane into contiguous regions, called a map, the regions can be colored using at most four colors so that no two adjacent regions have the same color.

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In 1891 Tait's proof was shown incorrect by J. Petersen.

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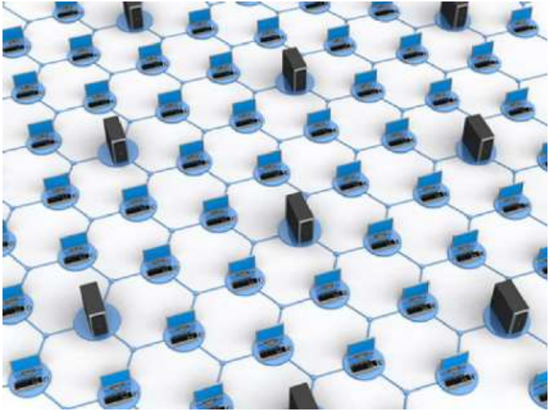
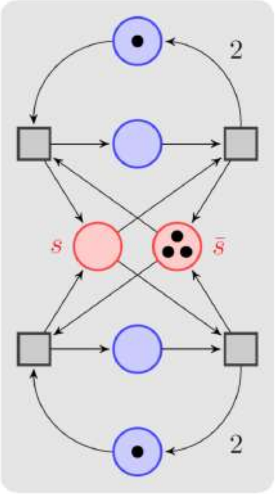
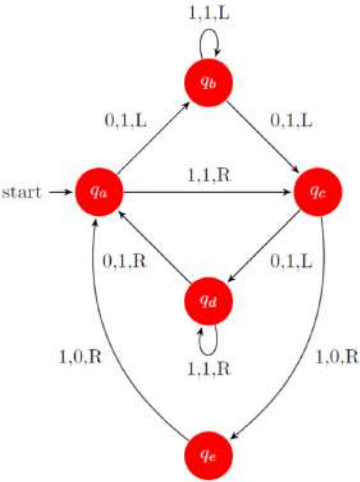
A simpler proof using the same ideas and still relying on computers was published in 1997 by Robertson, Sanders, Seymour, and Thomas.

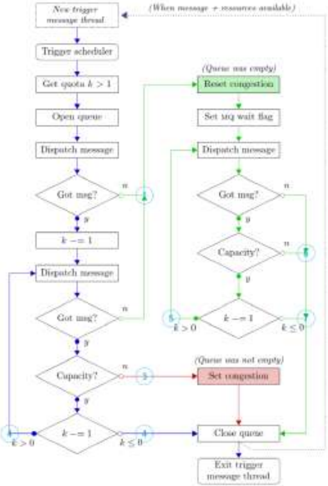
Graph Theory: Computer Networks



Graphs in Computer Science







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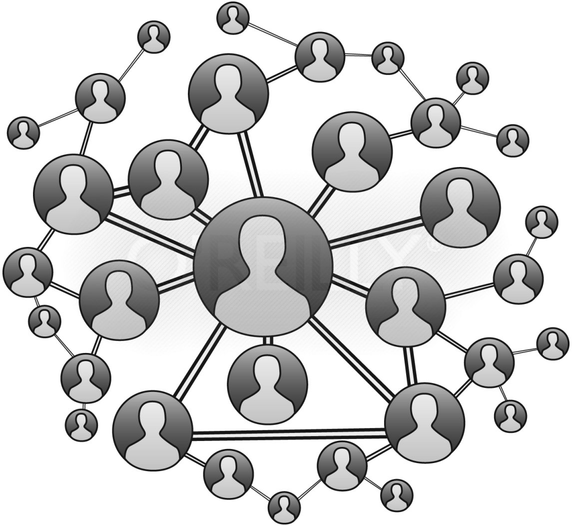

Question

How many handshakes (at most) are between, How many handshakes (at most) are between, say, ...

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Donald Trump and

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Donald Trump and **Toader de la Stâna din Vale**?

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How many handshakes are between Queen Elizabeth and a zulu warrior?

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Six degrees of separation is the theory that everyone and everything is six or fewer steps away, by way of introduction, from any other person in the world.

6 Degree of separation conjecture

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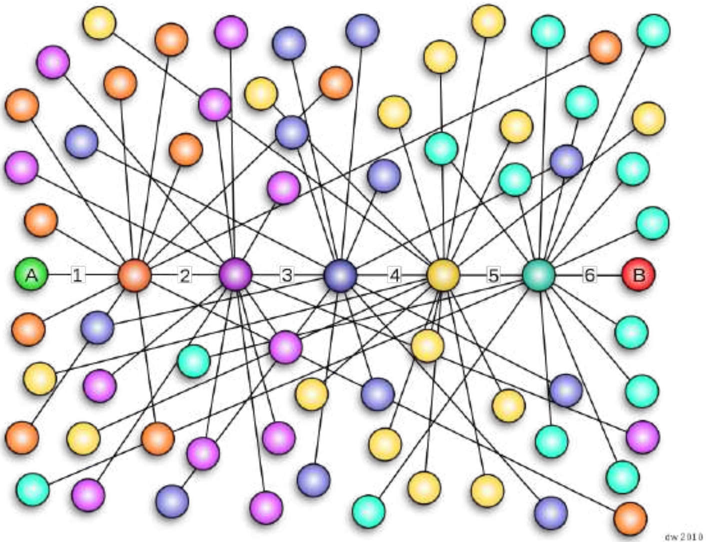
> *A fascinating game grew out of this discussion. One of us suggested performing the following experiment to prove that the population of the Earth is closer together now than they have ever been before. We should select any person from the 1.5 billion inhabitants of the Earth – anyone, anywhere at all. He bet us that, using no more than five individuals, one of whom is a personal acquaintance, he could contact the selected individual using nothing except the network of personal acquaintances.*

6 Degree of separation conjecture



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Google made it possible to search for any given actor's Bacon Number through their search engine.

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> *We found that six degrees actually overstates the number of links between typical pairs of users: While 99.6% of all pairs of users are connected by paths with 5 degrees (6 hops), 92% are connected by only four degrees (5 hops)*

6 Degree of separation on Facebook 2016

Each person in the world (at least among the 1.59 billion people active on Facebook) is connected to every other person by an average of (3.57) three and a half other people.

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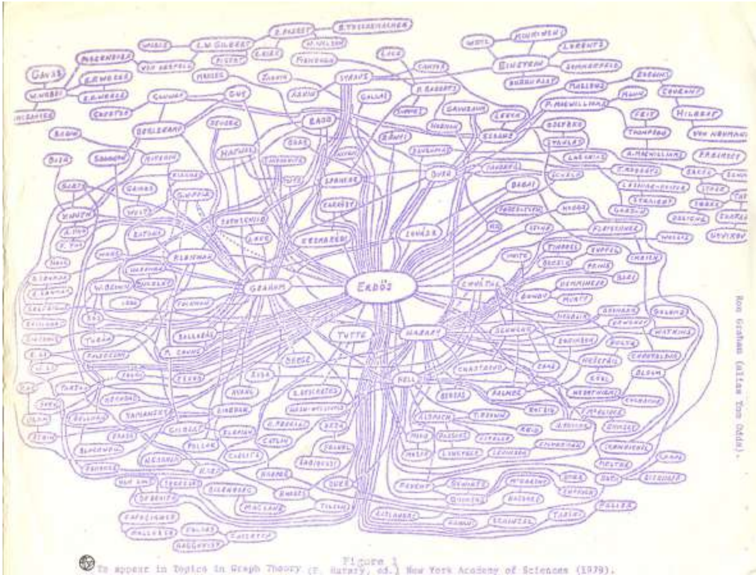
Some famous names have low Erdős numbers— Bill Gates has an Erdős number of 4, John von Neumann is 3, and Albert Einstein is 2.

Erdős number (2017)

Erdős number	
1	511
2	9267
3	33605
4	83642
5	87760
6	40014
7	11591
8	3146
9	819
10	244
11	68
12	23
13	5

Thus the median Erdős number is 5;
the mean is 4.65,
and the standard deviation is 1.21.

Erdős number



My Erdos number is:

Erdős number

My Erdos number is: 4

Erdős number

My Erdos number is: 4

Here is the proof:

4. Viorel Bostan coauthored with Weimin Han
3. Weimin Han coauthored with Michael J. McAsey
2. Michael J. McAsey coauthored with Lee A. Rubel
1. Lee A. Rubel coauthored with Paul Erdős

You can check your Erdos number on the website:
http://www.ams.org/mathscinet/collaborationDistance.html

Graph Theory: Planarity

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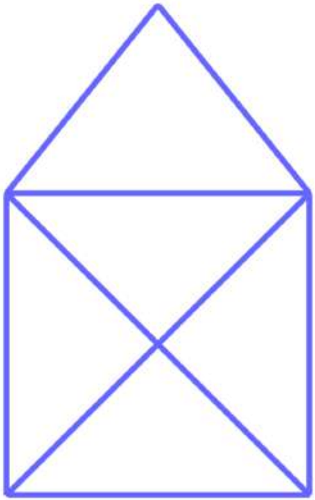

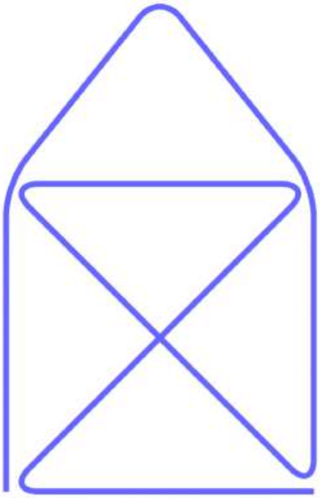

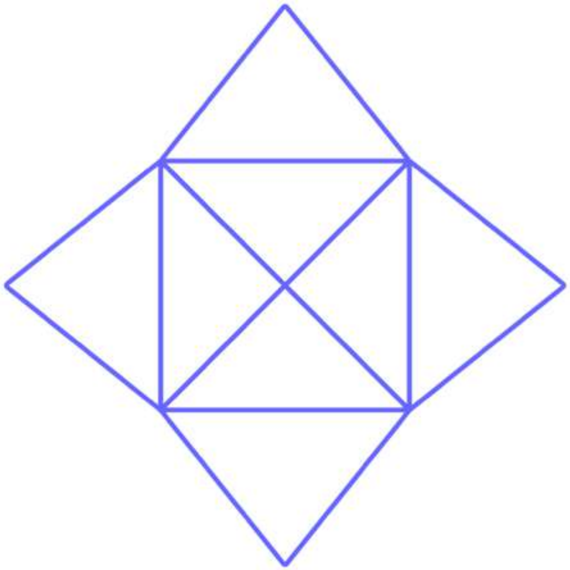

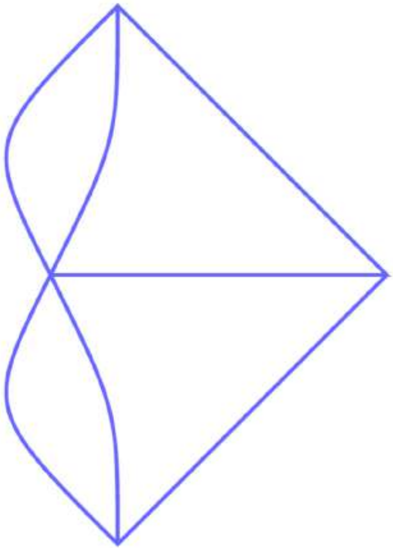

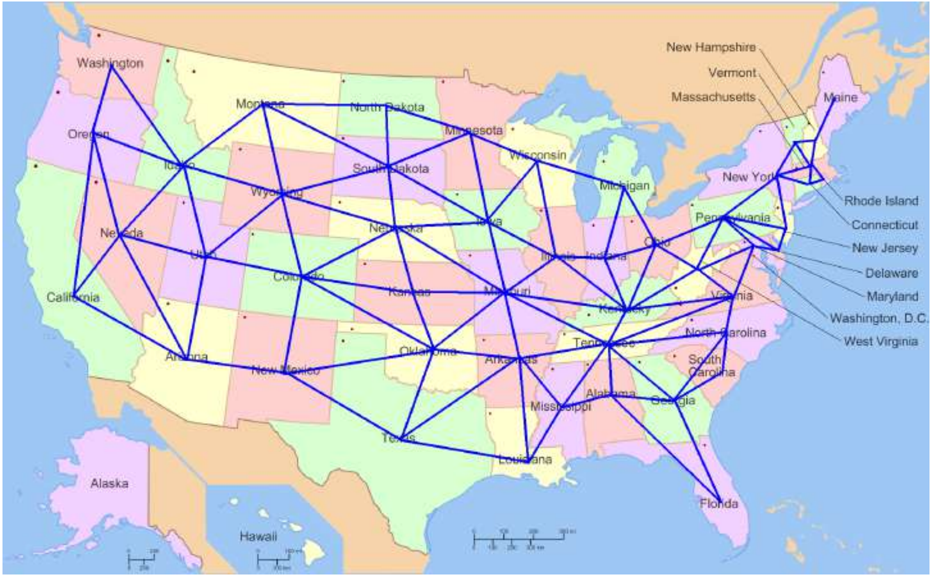
New Hampshire
Vermont
Massachusetts
Maine
Washington
Montana
North Dakota
Minnesota
Oregon
Idaho
South Dakota
Wisconsin
Michigan
New York
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Connecticut
Pennsylvania
Nebraska
Iowa
Nevada
Utah
Illinois
Indiana
Ohio
New Jersey
Delaware
Colorado
California
Kansas
Missouri
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Washington, D.C.
West Virginia
North Carolina
Tennessee
Oklahoma
Arizona
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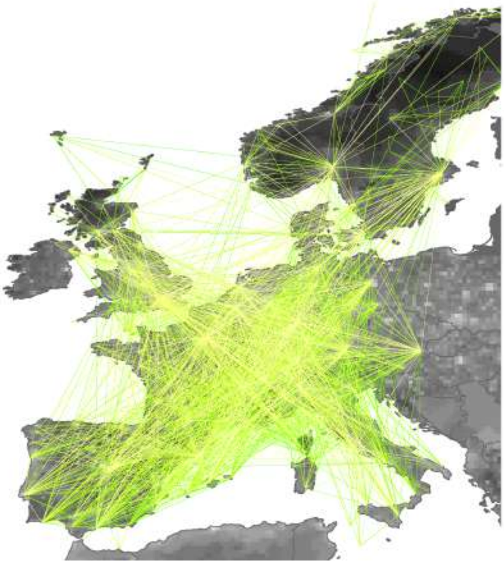
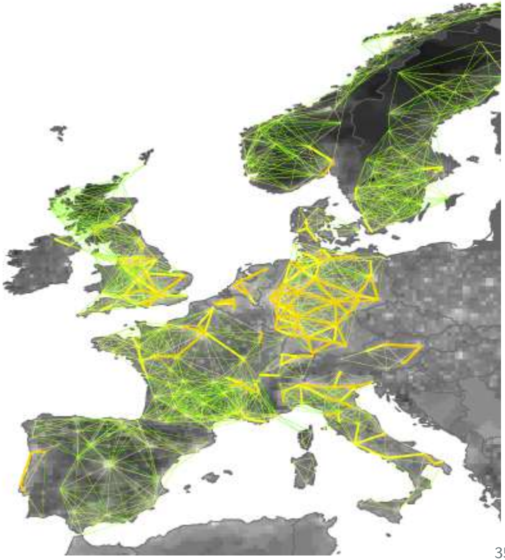
Hawaii

Google Maps Fastest Roundtrip Solver

Airline routes in Europe





Transportation Map

Definition

Let V be an arbitrary set. A **graph** is a pair of sets $G = (V, E)$ satisfying $E \subset V^2 \equiv V \times V$.

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The elements of V are called **vertices** (or nodes, or points) of the graph G, the elements of E are called its **edges** (or lines, or arcs).

Graph Theory. Definition of Graph

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The elements of V are called **vertices** (or nodes, or points) of the graph G, the elements of E are called its **edges** (or lines, or arcs).

The usual way to picture a graph is by drawing a dot for each vertex and joining two of these dots by a line, if the corresponding two vertices form an edge.

Graph definition. Example

$G = (V, E)$, $V = \{0, 1, \ldots, 6\}$, $E = \{(0, 2), (0, 4), (2, 4), (1, 3), (4, 5)\}$.

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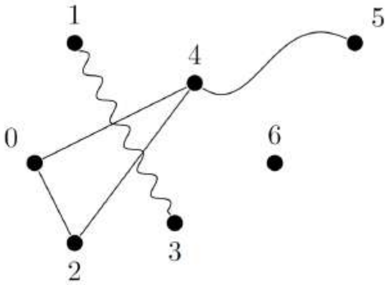

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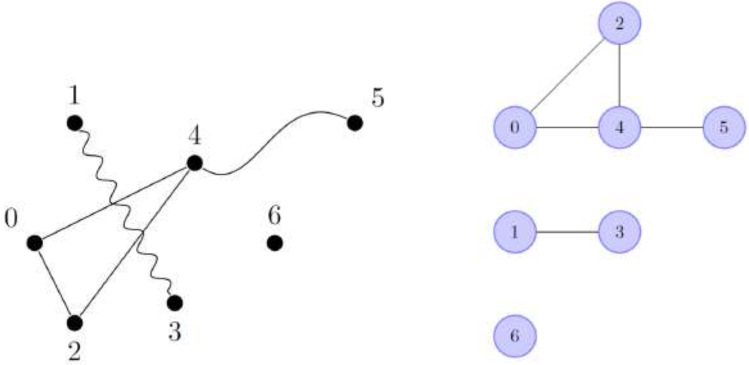


Important!

Just how these dots and lines are drawn is considered irrelevant:
All that matters is the information which pairs of vertices form edges!

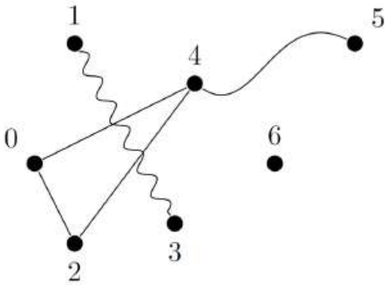
Order of a graph

Definition

A graph with vertex set V is said to be a graph on V . The **vertex set** of a graph G is referred to as $V(G)$, its **edge set** as $E(G)$.

Definition

The number of vertices of a graph G is called its **order**, written as $|G|$; its number of edges is denoted by $\|G\|$.



Finite and trivial graphs. Vertex incident with an edge

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Graphs are **finite** or **infinite** according to their order; unless otherwise stated, the graphs we consider are all finite.

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For the empty graph $(\varnothing, \varnothing)$ we simply write $\varnothing$.

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A graph of order 0 or 1 is called **trivial**.

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A vertex v is **incident** with an edge e if $v \in e$; then e is an edge at v. The two vertices incident with an edge are its **endvertices** or ends, and an edge joins its ends. An edge $\{x, y\}$ is usually written as xy (or yx).

Definition

The set of all the edges in E at a vertex v is denoted by $E(v)$.

Definition

Two vertices x and y of G are **adjacent**, or neighbours, if xy is an edge of G. Two edges e and f are **adjacent** if they have an endvertex in common.

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Adjacent vertices and edges. Complete graphs

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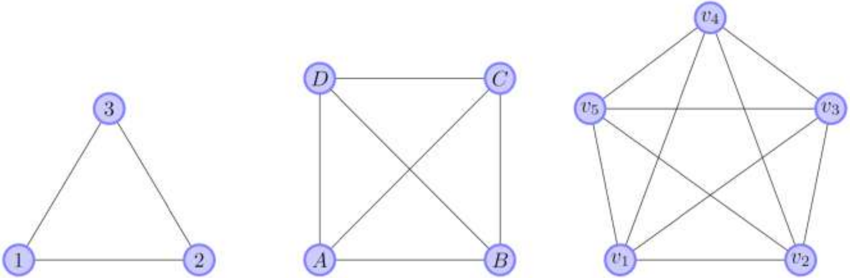
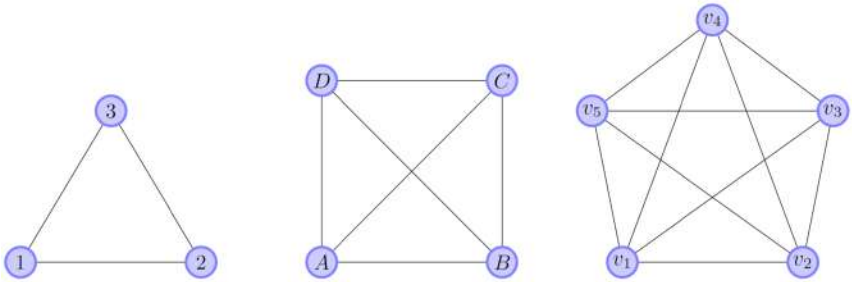
Complete graphs



Complete graphs K^3 (triangle), K^4 and K^5

Complete graphs



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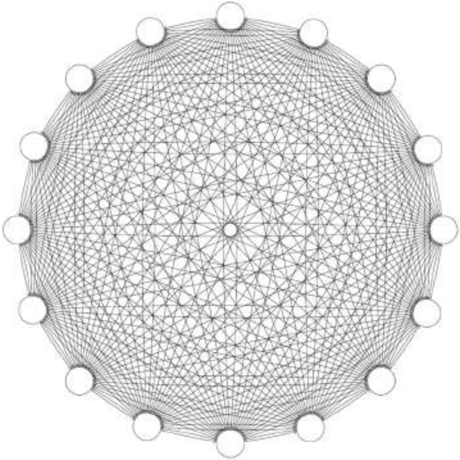
Problem

Count the number of edges in complete graph K^n.

Complete graphs



Graph K^{16}

Definition

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We call G and G' **isomorphic**, and write $G \simeq G'$, if there exists a bijection $\varphi : V \to V'$ with $xy \in E \iff \varphi(x)\varphi(y) \in E' \forall x, y \in V$. Such a map φ is called an **isomorphism**; if $G = G'$, it is called an **automorphism**.

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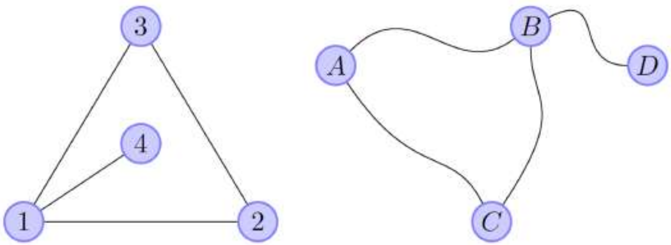
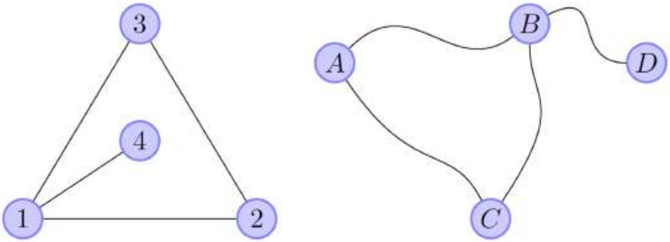


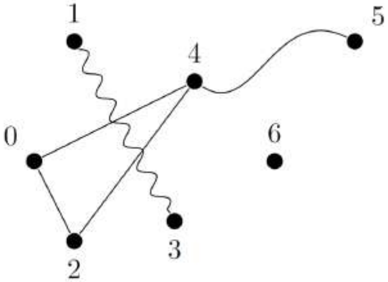
Isomorphic graphs

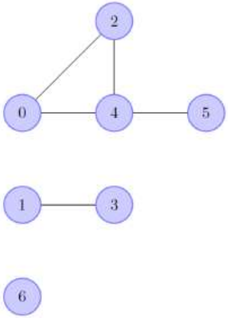
Definition

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We call G and G' **isomorphic**, and write $G \simeq G'$, if there exists a bijection $\varphi : V \to V'$ with $xy \in E \iff \varphi(x)\varphi(y) \in E' \forall x, y \in V$. Such a map φ is called an **isomorphism**; if $G = G'$, it is called an **automorphism**.

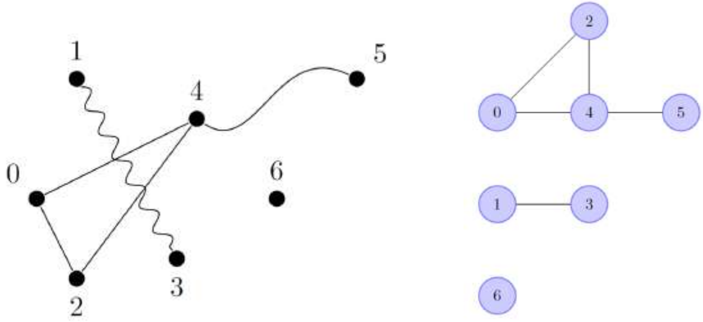


$$\varphi(1) = B,\ \varphi(2) = A,\ \varphi(3) = C,\ \varphi(4) = D$$





We do not distinguish between isomorphic graphs, and write $G = G'$ rather than $G \simeq G'$.



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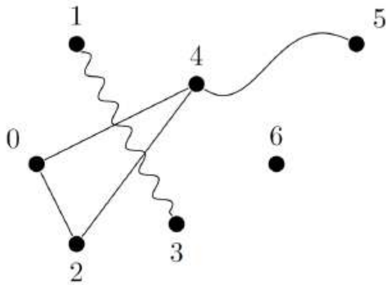
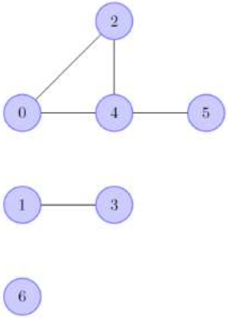
Definition

A map taking graphs as arguments is called a **graph invariant** if it assigns equal values to isomorphic graphs.

Isomorphic graphs. Graph invariant





We do not distinguish between isomorphic graphs, and write $G = G'$ rather than $G \simeq G'$.

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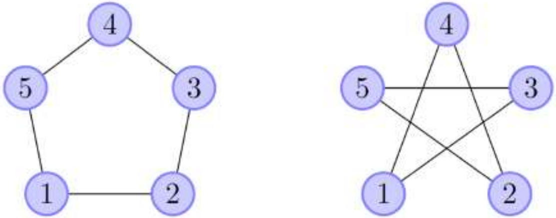
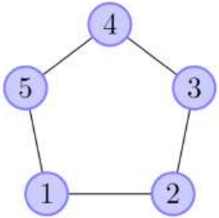
A map taking graphs as arguments is called a **graph invariant** if it assigns equal values to isomorphic graphs.

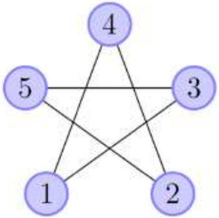
The number of vertices and number of edges of a graph are 2 simple graph invariant.

Isomorphic graphs

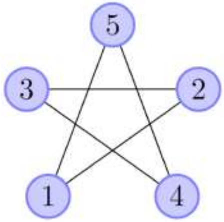


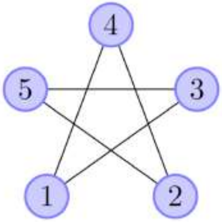
Isomorphic graphs





$$\varphi(1) = 1, \quad \varphi(2) = 3, \quad \varphi(3) = 5, \quad \varphi(4) = 2, \quad \varphi(5) = 4$$





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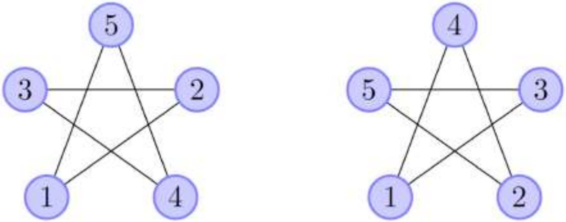
Isomorphic graphs



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Remark

No one has yet found a general procedure for determining whether two graphs are isomorphic that is guaranteed to run in polynomial time in $|V|$.

Subgraphs. Induced subgraph. Spanning subgraph

Definition

If $V' \subset V$ and $E' \subset E$, then G' is called a **subgraph** of G, written as $G' \subset G$.

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Thus if $U \subset V$ is any set of vertices, then $G[U]$ denotes the graph on U whose edges are precisely the edges of G with both ends in U.

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If H is a subgraph of G, not necessarily induced, we abbreviate $G[V(H)]$ to $G[H]$.

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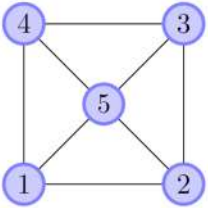
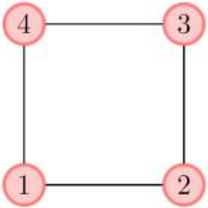
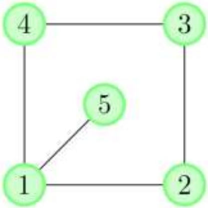
If H is a subgraph of G, not necessarily induced, we abbreviate $G[V(H)]$ to $G[H]$.

Definition

Let $G' \subset G$. Then G' is a **spanning subgraph** of G if V' spans all of G, i.e. if $V' = V$.
In other words subgraph G' contains all the vertices of G.



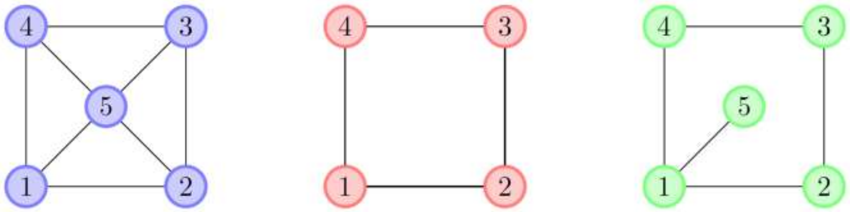




G (blue), G_1 (red), G_2 (green)

- G_1 and G_2 are subgraphs of G.

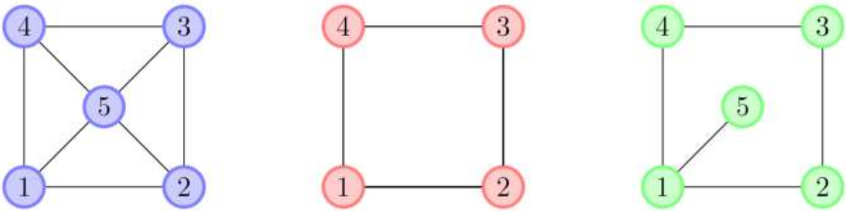
Induced subgraph. Spanning subgraph.



G (blue), G_1 (red), G_2 (green)

- G_1 and G_2 are subgraphs of G.
- G_1 is an induced subgraph of G, while G_2 is not.

Induced subgraph. Spanning subgraph.



G (blue), G_1 (red), G_2 (green)

- G_1 and G_2 are subgraphs of G.
- G_1 is an induced subgraph of G, while G_2 is not.
- G_2 is a spanning subgraph of G, while G_1 is not.

Union, intersection and difference of two graphs

Definition (Graph union and intersection)

We set $G \cup G' = (V \cup V', E \cup E')$ and $G \cap G' = (V \cap V', E \cap E')$.
If $G \cap G' = \varnothing$, then G and G' are called **disjoint**.

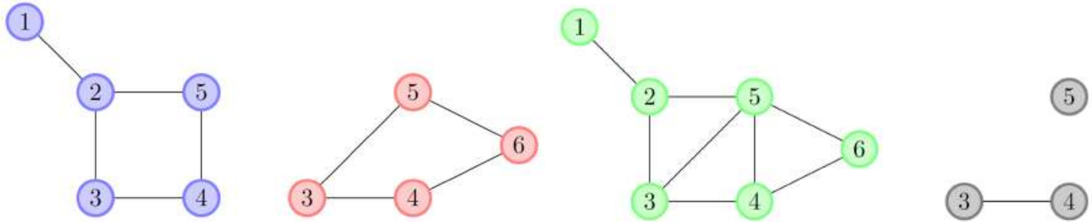
Definition (Graph difference)

If U is any set of vertices (usually of G), we write $G - U$ for $G[V - U]$. In other words, $G - U$ is obtained from G by deleting all the vertices in $U - V$ and their incident edges.

Definition

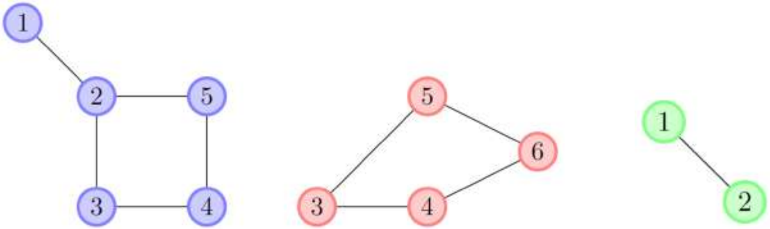
If $U = \{v\}$ is a singleton, we write $G - v$ rather than $G - \{v\}$. Instead of $G - V(G')$ we simply write $G - G'$.
For a subset F of V^2 we write $G - F = (V; E - F)$ and $G + F = (V; E \cup F)$;
as above, $G - \{e\}$ and $G + \{e\}$ are abbreviated to $G - e$ and $G + e$.

Union and intersection of two graphs



G_1 (blue), G_2 (red), $G_1 \cup G_2$ (green), $G_1 \cap G_2$ (grey)

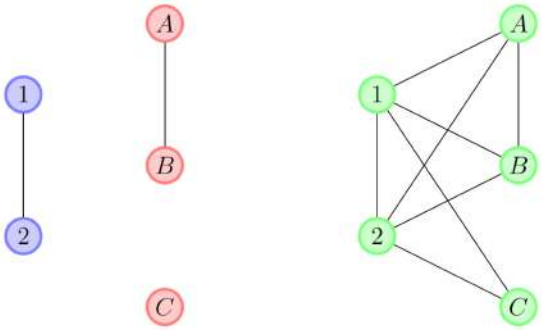
Difference of two graphs



G_1 (blue), G_2 (red), $G_1 - G_2$ (green)

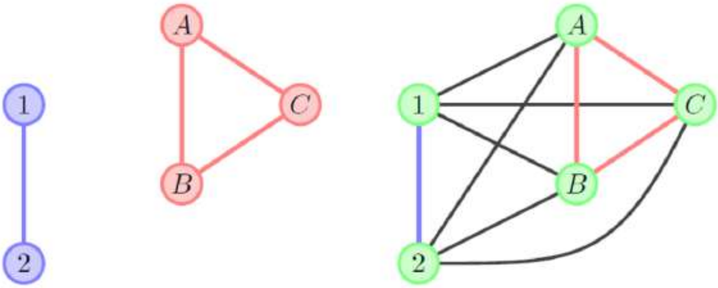
Definition (Star operation on graphs)

If G_1 and G_2 are disjoint, we denote by $G_1 * G_2$ the graph obtained from $G_1 \cup G_2$ by joining all the vertices of G_1 to all the vertices of G_2.



G_1 (blue), G_2 (red), $G_1 * G_2$ (green)

For example, $K^2 * K^3 = K^5$.



K^2 (blue), K^3 (red), $K^2 * K^3 = K^5$ (green).

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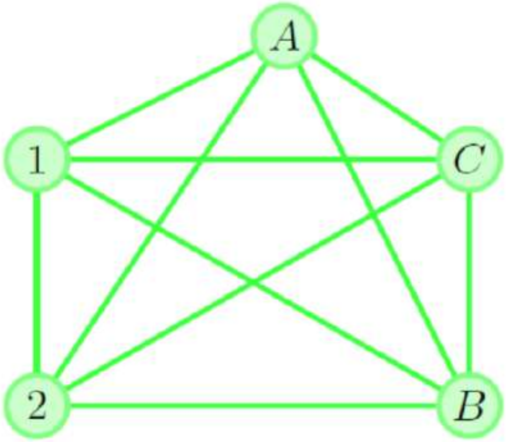

$K^2 * K^3 = K^5$ (green).

Complement of a graph

Definition (Graph complement)

The **complement** $\overline{G}$ of G is the graph on V with edge set $V^2 - E$. In other words, eliminate all the edges from G and add all the edges not present in G.

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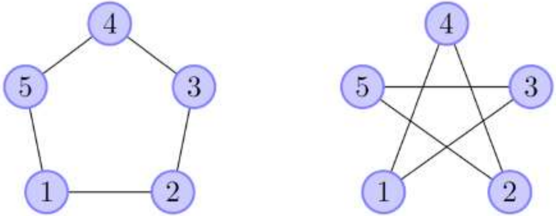

A graph isomorphic to its complement.

Neighbors of a vertex. Vertex degree.

Definition

The set of **neighbors** of a vertex v in G is denoted by $N_G(v)$, or $N(v)$. For $U \subset V$, the neighbors in $V - U$ of vertices in U are called **neighbors** of U, $N(U)$.

Neighbors of a vertex. Vertex degree.

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Definition

Degree (or valency) $d_G(v) = d(v)$ of v is the number of edges at v, that is the number of neighbors of v.
A vertex of degree 0 is called **isolated** vertex.

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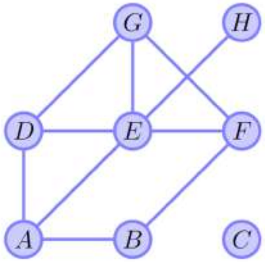
Definition

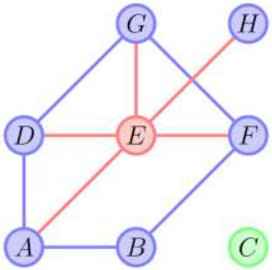
Minimum degree of G and **maximum degree** of G are numbers:

$$\delta(G) = \min_{v \in V} \{d_G(v)\}, \quad \Delta(G) = \max_{v \in V} \{d_G(v)\}$$

Vertex degree. Minimum and maximum degrees.

Consider the graph $G = (V, E)$:

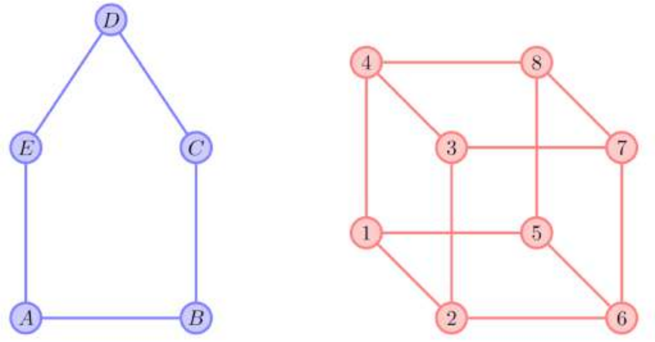




$$d_G(E) = d(E) = 5,\ d(C) = 0,\ \delta(G) = 0,\ \Delta(G) = 5$$

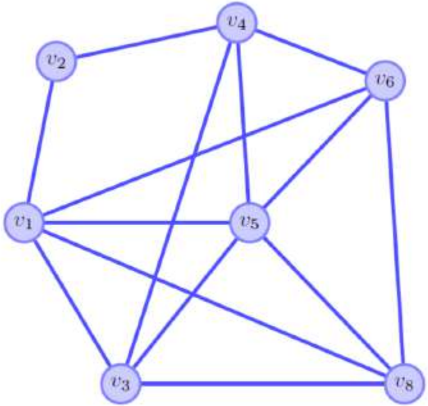
Definition (Regular graph)

If all the vertices of a graph G have the same degree k, then G is called $k-$ **regular**, or simply **regular**. A $3-$regular graph is called **cubic**.



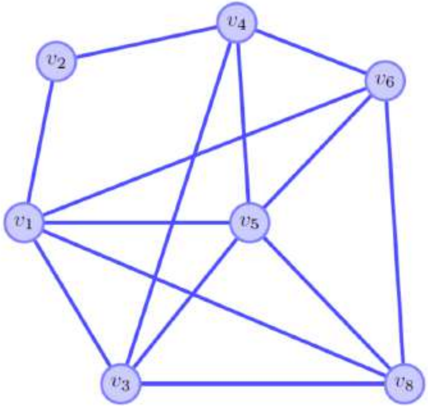
A $2-$regular graph (blue) and a $3-$regular (cubic) graph (red).

Adjacency Matrix



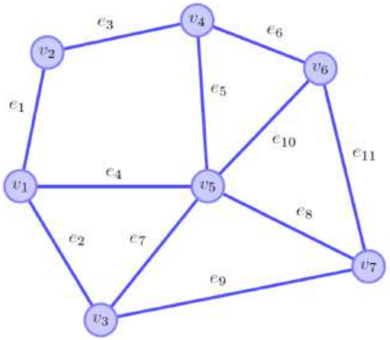
$$a_{ij} = \begin{cases} 1, \text{ if } (v_i, v_j) \in E \\ 0, \text{ otherwise} \end{cases}$$

Adjacency Matrix

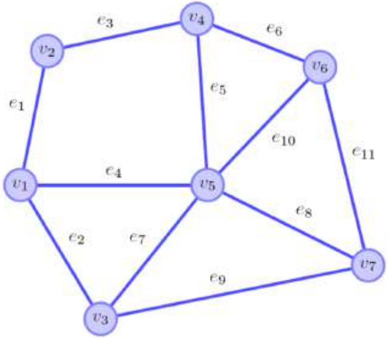


$$a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E \\ 0, & \text{otherwise} \end{cases}$$

	1	**2**	**3**	**4**	**5**	**6**	**8**
1	0	1	1	0	1	1	1
2	1	0	0	1	0	0	0
3	1	0	0	1	1	0	1
4	0	1	1	0	1	1	0
5	1	0	1	1	0	1	1
6	1	0	0	1	1	0	1
8	1	0	1	0	1	1	0



$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } e_j \text{ are incident} \\ 0, & \text{otherwise} \end{cases}$$



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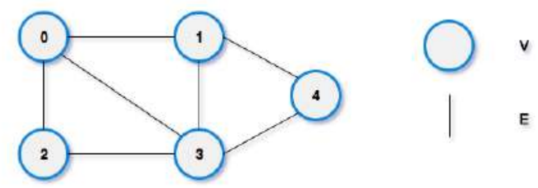
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	1	1	0	1	0	0	0	0	0	0	0
2	1	0	1	0	0	0	0	0	0	0	0
3	0	1	0	0	0	0	1	0	1	0	0
4	0	0	1	0	1	1	0	0	0	0	0
5	0	0	0	1	1	0	1	0	0	1	0
6	0	0	0	0	0	1	0	0	0	1	1
7	0	0	0	0	0	0	0	1	1	0	1

Adjacency List (Simple Graph)



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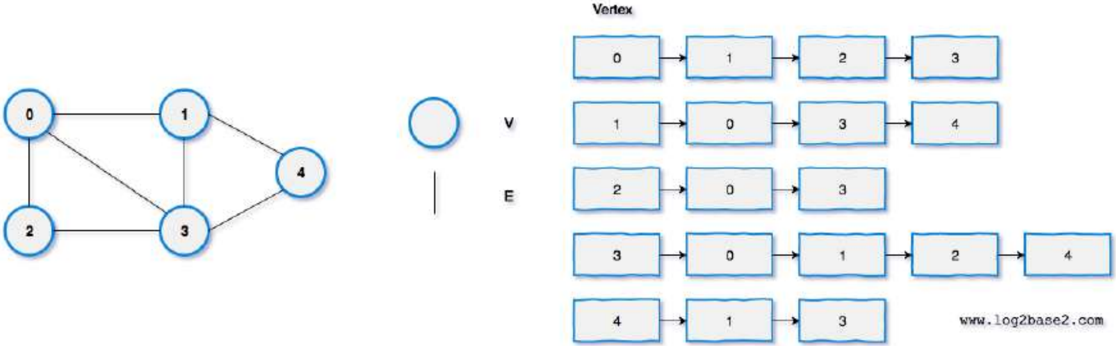
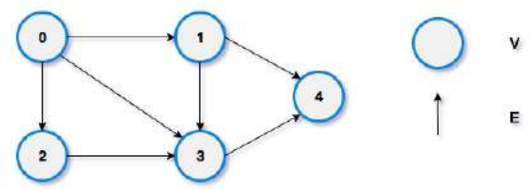


Adjacency List (Directed Graph)



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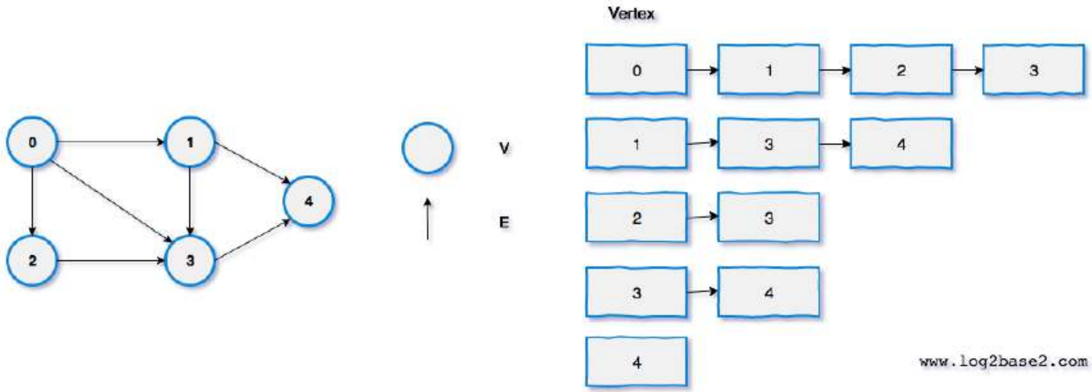

Average Degree of a Graph

Definition

The number

$$d(G) = \frac{1}{|G|} \sum_{v \in V} d_G(v)$$

is called the **average degree** of G.

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Clearly, average degree
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The average degree quantifies globally what is measured locally by the vertex degrees: the number of edges of G per vertex.

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The average degree quantifies globally what is measured locally by the vertex degrees: the number of edges of G per vertex.

Sometimes it will be convenient to express this ratio directly, as

$$\varepsilon(G) = \frac{\|G\|}{|G|}.$$

Handshaking Lemma

The quantities d and ε are related. Indeed,if we sum up all the vertex degrees in G, we count every edge exactly twice: once from each of its ends.

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Thus,

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Corollary

The number of vertices of odd degree in a graph is always even.

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Corollary

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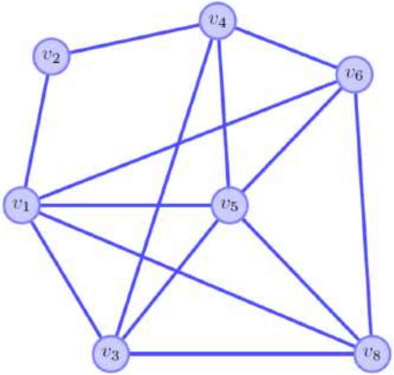
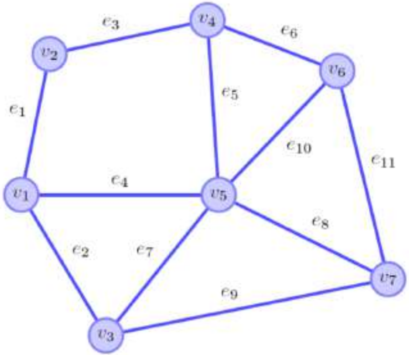
Lemma (Handshaking lemma)

For any graph the sum of the degrees of vertices equals twice the number of edges.

Average Degree





G_1 (left), G_2 (right). $d(G_1) = 4$, $\varepsilon(G_1) = 2$, $d(G_2) = \frac{22}{7} = 3.14$, $\varepsilon(G_2) = \frac{11}{7} = 1.57$

Lecture summary

- Graph theory history;
- Six degrees of separation;
- Graph applications;
- Graph definition, Graph representations;
- Complete graph, Isomorphic graphs;
- Subgraphs, Induced and spanning graph;
- Operations on graphs;
- Adjacency and Incident matrices;
- Vertex degree, Average degree, Regular graph;
- Handshaking lemma.