Linear Algebra and Analytic Geometry

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An Example of a Straight Line

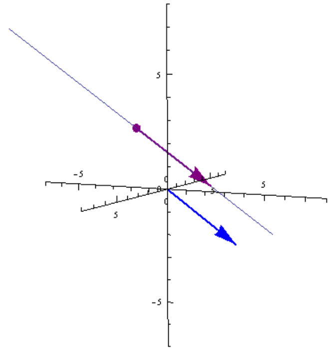
- Find an equation for the line through $P_0 = (5, 1, 3)$ and parallel to the vector $\boldsymbol{v} = \boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$.

Its vector equation is

$$\boldsymbol{r} = \langle 5, 1, 3\rangle + t\langle 1, 4, -2\rangle.$$

The corresponding system of parametric equations is

$$x = 5 + t, \quad y = 1 + 4t, \quad z = 3 - 2t.$$



Different Parameterizations of the Same Line

- Show that $\boldsymbol{r}_1(t) = \langle 1,1,0\rangle + t\langle -2,1,3\rangle$ and $\boldsymbol{r}_2(t) = \langle -3,3,6\rangle + t\langle 4,-2,-6\rangle$ parametrize the same line.

  The line $\boldsymbol{r}_1$ has direction vector $\boldsymbol{v} = \langle -2,1,3\rangle$. The line $\boldsymbol{r}_2$ has direction vector $\boldsymbol{w} = \langle 4,-2,-6\rangle$. These vectors are parallel because $\boldsymbol{w} = -2\boldsymbol{v}$. Therefore, the lines described by $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are parallel.

  We must check that they have a point in common. Choose any point on $\boldsymbol{r}_1$, say $P = (1,1,0)$ (corresponding to $t = 0$). This point lies on $\boldsymbol{r}_2$ if there is a t such that $\langle 1,1,0\rangle = \langle -3,3,6\rangle + t\langle 4,-2,-6\rangle$. This yields three equations $\left\{\begin{array}{l} 1 = -3+4t \\ 1 = 3-2t \\ 0 = 6-6t \end{array}\right\}$. All three are satisfied when $t = 1$. Therefore P also lies on $\boldsymbol{r}_2$.

  We conclude that $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ parametrize the same line.

Intersection of Two Lines

- Determine whether the following two lines $\boldsymbol{r}_1(t) = \langle 1, 0, 1\rangle + t\langle 3, 3, 5\rangle$, $\boldsymbol{r}_2(t) = \langle 3, 6, 1\rangle + t\langle 4, -2, 7\rangle$ intersect.

  The two lines intersect if there exist parameter values t_1 and t_2 such that $\boldsymbol{r}_1(t_1) = \boldsymbol{r}_2(t_2)$. This gives

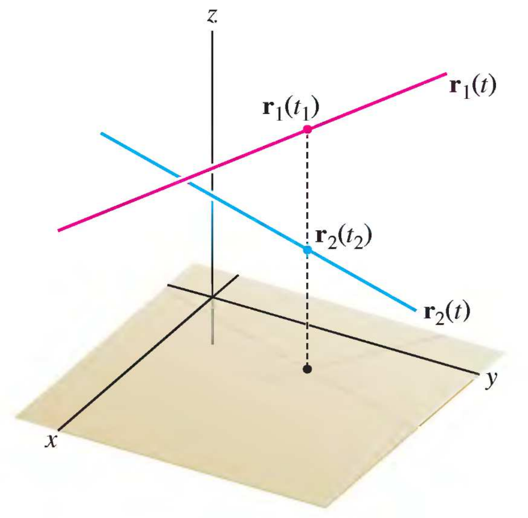
$$\langle 1, 0, 1\rangle + t_1\langle 3, 3, 5\rangle = \langle 3, 6, 1\rangle + t_2\langle 4, -2, 7\rangle.$$

  This is equivalent to three equations for the components:

$$x = 1 + 3t_1 = 3 + 4t_2,\ y = 3t_1 = 6 - 2t_2,\ z = 1 + 5t_1 = 1 + 7t_2.$$

  Solve the first two equations for t_1 and t_2. We get $t_1 = \frac{14}{9},\ t_2 = \frac{2}{3}$. These values satisfy the first two equations. However, t_1 and t_2 do not satisfy the third equation $1 + 5 \cdot \frac{14}{9} \neq 1 + 7 \cdot \frac{2}{3}$. Therefore, the lines do not intersect.

The Two Non-Intersecting Lines



Skew Lines

- Two straight lines in space are called **skew lines** if they do not intersect and are not parallel.

  Example: Show that the lines with parametric equations

  $$x = 1 + t, \quad y = -2 + 3t, \quad z = 4 - t;$$
  $$x = 2s, \quad y = 3 + s, \quad z = -3 + 4s$$

  are skew lines.

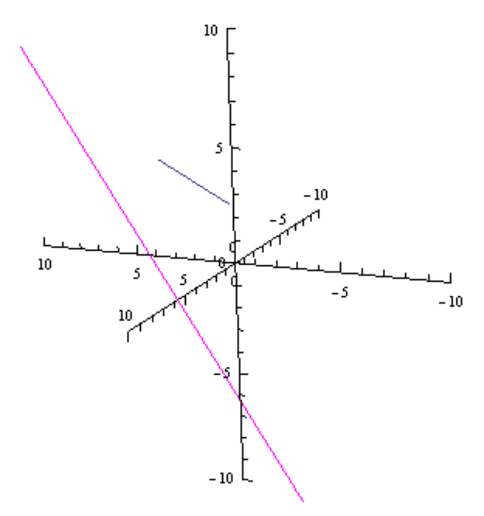
  To see that the lines do not intersect, we try to find a point of intersection by setting

  $$1 + t = 2s, \quad -2 + 3t = 3 + s, \quad 4 - t = -3 + 4s.$$

  The first two taken together give $(t, s) = (\frac{11}{5}, \frac{8}{5})$. These values do not satisfy the third equation! So there is no point of intersection.

  To see that they are not parallel, look at the direction vectors. The first has direction vector $\langle 1, 3, -1 \rangle$ and the second $\langle 2, 1, 4 \rangle$. These are not parallel vectors (Why?).

Plots of the Skew Lines



Line Segment Between Two Points

- Let $P_1 = (a_1, b_1, c_1)$ with position vector $\boldsymbol{r}_1 = \langle a_1, b_1, c_1 \rangle$ and $P_2 = (a_2, b_2, c_2)$ with position vector $\boldsymbol{r}_2 = \langle a_2, b_2, c_2 \rangle$.
  The vector equation of the line segment joining P_1 and P_2 is

$$\boldsymbol{r} = (1 - t)\boldsymbol{r}_1 + t\boldsymbol{r}_2, \quad 0 \leq t \leq 1.$$

Example: Find the vector equation and the parametric equations for the line segment joining $P_1 = (10, 3, 1)$ with $P_2 = (5, 6, -3)$.
For the vector equation, we have

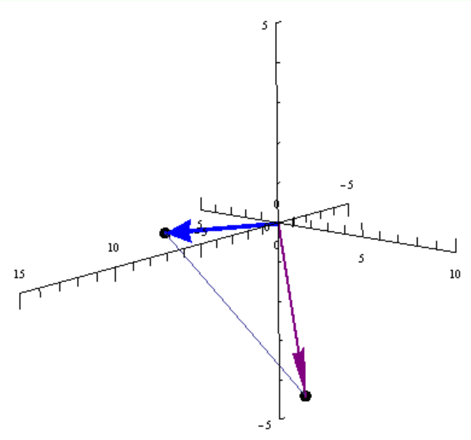
$$\boldsymbol{r} = (1 - t)\langle 10, 3, 1 \rangle + t\langle 5, 6, -3 \rangle, \quad 0 \leq t \leq 1.$$

For the parametric equations, we have

$$x = 10 - 5t, \quad y = 3 + 3t, \quad z = 1 - 4t, \quad 0 \leq t \leq 1.$$

The Line Segment

- The line segment joining $(10, 3, 1)$ with $(5, 6, -3)$.



Example

- Parametrize the segment $\overrightarrow{PQ}$ where $P = (1,0,4)$ and $Q = (3,2,1)$. Find the midpoint of the segment.

  The line through $P = (1,0,4)$ and $Q = (3,2,1)$ has the parametrization

  $$\boldsymbol{r}(t) = (1-t)\langle 1,0,4\rangle + t\langle 3,2,1\rangle = \langle 1+2t, 2t, 4-3t\rangle.$$

  The segment $\overline{PQ}$ is traced for $0 \leq t \leq 1$.

  The midpoint of $\overline{PQ}$ is the terminal point of the vector

  $$\boldsymbol{r}\left(\frac{1}{2}\right) = \frac{1}{2}\langle 1,0,4\rangle + \frac{1}{2}\langle 3,2,1\rangle = \langle 2,1,\frac{5}{2}\rangle.$$

  In other words, the midpoint is $(2,1,\frac{5}{2})$.

Symmetric Equations

- Given a point $P_0 = (x_0, y_0, z_0)$ in space, with position vector $\boldsymbol{r}_0 = \langle x_0, y_0, z_0 \rangle$, and a direction vector $\boldsymbol{v} = \langle a, b, c \rangle$, the **symmetric equations** for the straight line in space through P_0 with direction $\boldsymbol{v}$ are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Example: Find the symmetric equations for the line passing through $A = (2, 4, -3)$ and $B = (3, -1, 1)$ and the point at which this line intersects the xy-plane.

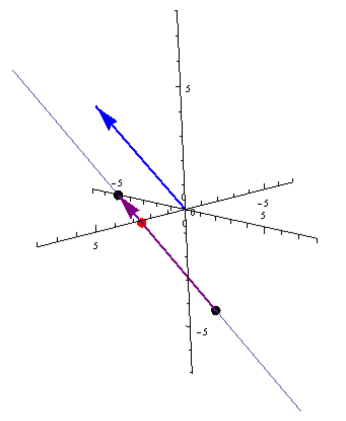
The line has the direction of the vector $\overrightarrow{AB} = \langle 1, -5, 4 \rangle$. Therefore, the symmetric equations are:

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{z + 3}{4}.$$

It intersects the xy-plane when $z = 0$. Therefore, we get $x - 2 = \frac{3}{4}$ and $-\frac{1}{5}(y - 4) = \frac{3}{4}$, which yield $(x, y, z) = (\frac{11}{4}, \frac{1}{4}, 0)$.

Picture of the Line

- The line passing through $A = (2, 4, -3)$ and $B = (3, -1, 1)$ and the point $(\frac{11}{4}, \frac{1}{4}, 0)$ at which this line intersects the xy-plane.



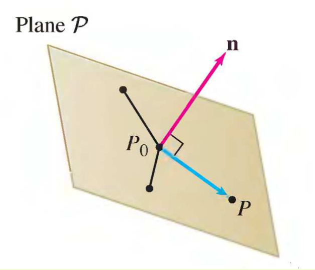
Subsection 5

Planes in Three-Space

Stewart, p816-823
12.5

Equation of a Plane

- Consider a plane $\mathcal{P}$ that passes through a point $P_0 = (x_0, y_0, z_0)$ and is orthogonal to a nonzero vector $\boldsymbol{n} = \langle a, b, c\rangle$, called a **normal vector**.
  A point $P = (x, y, z)$ lies on $\mathcal{P}$ precisely when $\overrightarrow{P_0P}$ is orthogonal to $\boldsymbol{n}$.



Therefore, P lies on the plane if $\boldsymbol{n} \cdot \overrightarrow{P_0P} = 0$. In components, $\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0\rangle$. So we get the **vector equation**

$\vec{n} \cdot \overrightarrow{P_0P} = 0$

$$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0.$$

This gives us the following **scalar equation**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Alternative Form

- The plane $\mathcal{P}$ that passes through a point $P_0 = (x_0, y_0, z_0)$ and is orthogonal to a nonzero vector $\boldsymbol{n} = \langle a, b, c \rangle$ has equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

This can also be written

$$ax + by + cz = ax_0 + by_0 + cz_0$$

or

$$\boldsymbol{n} \cdot \overrightarrow{OP} = \boldsymbol{n} \cdot \overrightarrow{OP_0}.$$

When we set $d = ax_0 + by_0 + cz_0 = \boldsymbol{n} \cdot \overrightarrow{OP_0}$, the equations become

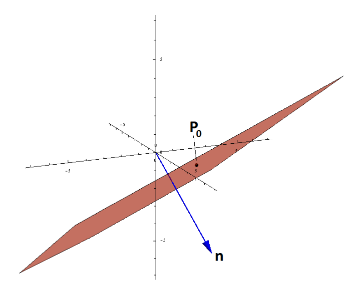
$$\boldsymbol{n} \cdot \langle x, y, z \rangle = d \quad \text{or} \quad ax + by + cz = d.$$

Example

- Find an equation of the plane through $P_0 = (3, 1, 0)$ with normal vector $\boldsymbol{n} = \langle 3, 2, -5 \rangle$.

  We have

  $$3(x-3) + 2(y-1) - 5(z-0) = 0.$$

  We may also compute $d = \boldsymbol{n} \cdot \overrightarrow{OP_0} = \langle 3, 2, -5 \rangle \cdot \langle 3, 1, 0 \rangle = 11.$

  Then we have

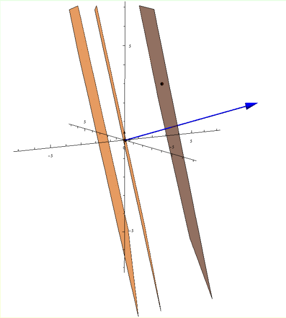
  $$\langle 3, 2, -5 \rangle \cdot \langle x, y, z \rangle = 11 \quad \text{or} \quad 3x + 2y - 5z = 11.$$

Parallel Planes

- Let $\mathcal{P}$ have equation $7x - 4y + 2z = -10$. Find equations of the plane parallel to $\mathcal{P}$ passing through (a) the origin and (b) $Q = (2, -1, 3)$.

  The given plane has normal

  $$\boldsymbol{n} = \langle 7, -4, 2 \rangle.$$

  All parallel planes have the same normal. For the first plane, we get

  a) $$7x - 4y + 2z = 0.$$

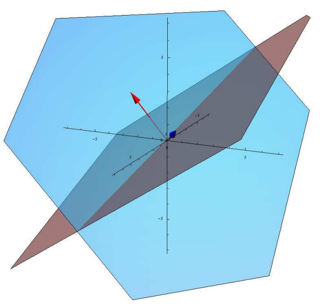
  For the second plane, we get $7(x-2) - 4(y+1) + 2(z-3) = 0$ or equivalently

  b) $$7x - 4y + 2z = 24.$$

Angle Between Planes

- To find the angle between two planes:
  - Find normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ of the planes;
  - Compute the angle between $\boldsymbol{n}_1, \boldsymbol{n}_2$ using dot product.

Example: Find the angle between $x + y + z = 1$ and $x - 2y + 3z = 1$.



The normals are

$$\begin{aligned} \boldsymbol{n}_1 &= \langle 1, 1, 1 \rangle; \\ \boldsymbol{n}_2 &= \langle 1, -2, 3 \rangle. \end{aligned}$$

Therefore the angle θ between the normals has

$$\cos\theta = \frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{\|\boldsymbol{n}_1\|\|\boldsymbol{n}_2\|} = \frac{1 - 2 + 3}{\sqrt{3}\sqrt{14}} = \frac{2}{\sqrt{42}}.$$

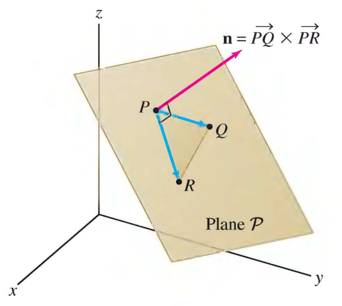
This gives $\theta = \cos^{-1} \frac{2}{\sqrt{42}} \approx 72^\circ$.

arccos

Plane Determined By Three Points

- Find an equation of the plane $\mathcal{P}$ determined by the points $P = (1, 0, -1),\ Q = (2, 2, 1),\ R = (4, 1, 2)$.

  Find a normal vector: The vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ lie in the plane $\mathcal{P}$. So their cross product is normal to $\mathcal{P}$. We have



$$\begin{aligned} \boldsymbol{n} &= \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 2 \\ 3 & 1 & 3 \end{vmatrix} \\ &= 4\boldsymbol{i} + 3\boldsymbol{j} - 5\boldsymbol{k}. \end{aligned}$$

  Now set up the equation of the plane using any of the three given points:

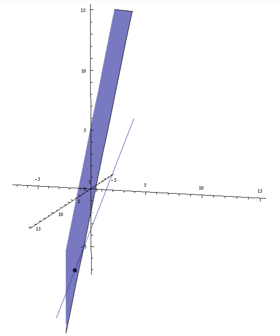
$$4(x-1) + 3y - 5(z+1) = 0 \quad \text{or} \quad 4x + 3y - 5z = 9.$$

Intersection of a Plane and a Line

- Find the point P where the plane $3x - 9y + 2z = 7$ and the line $\boldsymbol{r}(t) = \langle 1, 2, 1\rangle + t\langle -2, 0, 1\rangle$ intersect.

  The line has parametric equations

$$x = 1 - 2t, \; y = 2, \; z = 1 + t.$$



Substitute in the equation of the plane and solve for t: $3x - 9y + 2z = 3(1 - 2t) - 9(2) + 2(1 + t) = 7$. Simplification yields $-4t - 13 = 7$ or $t = -5$.

Therefore, P has coordinates

$$x = 1 - 2(-5) = 11, \; y = 2, \; z = 1 + (-5) = -4.$$

The plane and line intersect at the point $P = (11, 2, -4)$.

Equation of Line of Intersection of Two Planes

- To find a set of symmetric equations for the line of intersection between two planes, we need
  - a point on the line;
  - a vector in the direction of the line.

Stewart,p.821 example 7

Example: Find symmetric equations for the line of intersection of $x + y + z = 1$ and $x - 2y + 3z = 1$.

Set $z = 0$ and solve for x, y to find a point on the line. This gives $(x, y, z) = (1, 0, 0)$. Since the line of intersection lies in both planes, it has a direction perpendicular to both normals $\boldsymbol{n}_1 = \langle 1, 1, 1\rangle$ and $\boldsymbol{n}_2 = \langle 1, -2, 3\rangle$. Such a vector is given by the cross-product $\boldsymbol{n}_1 \times \boldsymbol{n}_2$. So we compute

$$\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}.$$

direction vector

Thus, the symmetric equations are $\dfrac{x-1}{5} = \dfrac{y}{-2} = \dfrac{z}{-3}$.

$$\begin{cases} x+y+z=1 \\ x-2y+3z=1 \end{cases}$$

2 equations, 3 variables

$z=0$

$$\begin{cases} x+y=1 \\ x-2y=1 \end{cases}$$

$3y=0$

$x=1$

$y=0$

$(1,0,0)$

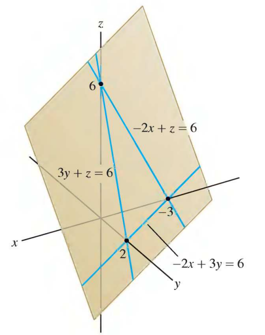
The Traces of a Plane

- The intersection of a plane $\mathcal{P}$ with a coordinate plane or a plane parallel to a coordinate plane is called a **trace**.

Example: Find the traces of the plane $-2x + 3y + z = 6$ in the coordinate planes.

We obtain the trace in the xy-plane by setting $z = 0$ in the equation of the plane. Thus, the trace is the line $-2x + 3y = 6$ in the xy-plane.

Similarly, the trace in the xz-plane is obtained by setting $y = 0$, which gives the line $-2x + z = 6$ in the xz-plane.

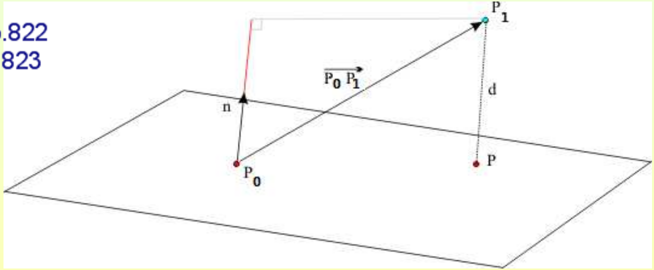


Finally, the trace in the yz-plane is $3y + z = 6$.

Distance of Point from a Plane

- To calculate the distance of a point $P_1 = (x_1, y_1, z_1)$ from a plane with linear equation $ax + by + cz = d$:
  - Take any point $P_0 = (x_0, y_0, z_0)$ on the plane;
  - Consider the vector $\overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$ from P_0 to P_1;
  - Calculate the length $\|\overrightarrow{P_0P_1}_{\|}\|$ of the projection of $\overrightarrow{P_0P_1}$ onto the normal $\boldsymbol{n} = \langle a, b, c \rangle$ of the plane using the dot product.

example8, p.822
example9,p.823

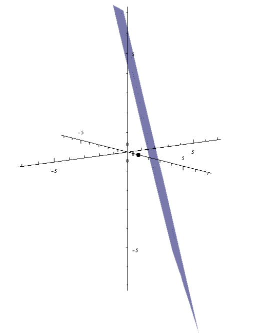


Example

- Find the distance of $P_1 = (1,0,0)$ from the plane $2x + 3y + z - 5 = 0$.

Consider $P_0 = (0,0,5)$ on the plane. Then, $\overrightarrow{P_0P_1} = \langle 1, 0, -5\rangle$. Thus, since $\mathbf{n} = \langle 2,3,1\rangle$, we have

$$
\begin{aligned}
|\overrightarrow{P_0P_1}_{\|}| &= \frac{|\overrightarrow{P_0P_1} \cdot \boldsymbol{n}|}{\|\boldsymbol{n}\|} \\
&= \frac{|\langle 1,0,-5\rangle \cdot \langle 2,3,1\rangle|}{\|\langle 2,3,1\rangle\|} \\
&= \frac{|2-5|}{\sqrt{4+9+1}} \\
&= \frac{3}{\sqrt{14}}.
\end{aligned}
$$



Example 10, p.823

Stewart