Calculus I

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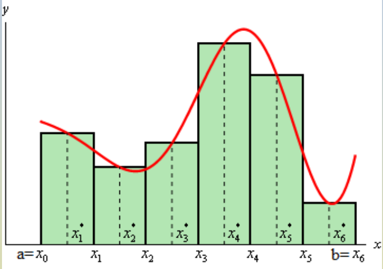
1. The Integral
   - Approximating and Computing Area
   - The Definite Integral
   - The Fundamental Theorem of Calculus, Part I
   - The Fundamental Theorem of Calculus, Part II
   - Net Change as the Integral of a Rate
   - Substitution Method
   - Further Transcendental Functions
   - Exponential Growth and Decay

Subsection 1

Approximating and Computing Area

Approximating Area by Rectangles

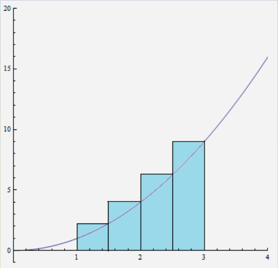
- Suppose, we want to approximate the area under the graph of $y = f(x)$ from $x = a$ to $x = b$;



- We may cut the interval $[a, b]$ into N subintervals of equal length; The common length will be equal to $\Delta x = \frac{b-a}{N}$;
- Suppose that in the first subinterval $[a, x_1]$, we take a point x_1^*, in the second $[x_1, x_2]$ a point x_2^*, etc.; Thus, in interval $[x_{i-1}, x_i]$, we will have a point x_i^*;
- Then we calculate the area of each rectangle by $\Delta A_i = f(x_i^*)\Delta x$;
- Finally, we sum all the elementary rectangular areas: $A \approx \Delta x[f(x_1^*) + f(x_2^*) + \cdots + f(x_N^*)]$;

Approximating Area Under $y = x^2$

- We use the method to approximate the area under $f(x) = x^2$ from $x = 1$ to $x = 3$ using $N = 4$ subintervals and taking as x_i^* the right endpoint of the corresponding interval:



- Since $\Delta x = \frac{3-1}{4} = \frac{1}{2}$, we get

$$\begin{aligned} A &\approx \tfrac{1}{2}[f(\tfrac{3}{2}) + f(2) + f(\tfrac{5}{2}) + f(3)] \\ &= \tfrac{1}{2}[\tfrac{9}{4} + 4 + \tfrac{25}{4} + 9] \\ &= \tfrac{1}{2}\tfrac{86}{4} = \tfrac{43}{4}. \end{aligned}$$

Summation ($\sum$) Notation

- We use the notation

$$\sum_{i=m}^{n} a_i := a_m + a_{m+1} + \cdots + a_{n-1} + a_n.$$

- Example:

$$\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55;$$

- Example: Compute

$$\begin{aligned} \sum_{k=4}^{6} (k^2 - 2k) &= (4^2 - 2 \cdot 4) + (5^2 - 2 \cdot 5) + (6^2 - 2 \cdot 6) \\ &= 8 + 15 + 24 = 47; \end{aligned}$$

Example: $\sum_{m=7}^{11} 1 = 1 + 1 + 1 + 1 + 1 = 5;$

Linearity Properties of Summation

Linearity of Summation

- $\displaystyle\sum_{i=m}^{n}(a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i;$
- $\displaystyle\sum_{i=m}^{n} Ca_i = C\sum_{i=m}^{n} a_i;$
- $\displaystyle\sum_{i=1}^{n} k = nk \quad \text{and} \quad \sum_{i=m}^{n} k = (n-m+1)k;$

- Example: $\displaystyle\sum_{i=3}^{5}(i^2 + i) = (3^2+3) + (4^2+4) + (5^2+5) =$

  $\displaystyle(3^2+4^2+5^2) + (3+4+5) = \sum_{i=3}^{5} i^2 + \sum_{i=3}^{5} i;$

Two More Examples

- Example:
$$\sum_{i=0}^{50}(3i^2 - 7i + 8) = \sum_{i=0}^{50} 3i^2 - \sum_{i=0}^{50} 7i + \sum_{i=0}^{50} 8 = 3\sum_{i=0}^{50} i^2 - 7\sum_{i=0}^{50} i + 8\sum_{i=0}^{50} 1;$$
- Example: The sum of the rectangle areas that approximate the area under the curve $y = f(x)$ on $[a, b]$ can be written very succinctly using summation notation

$$\begin{aligned} A &\approx \Delta x[f(x_1^*) + f(x_2^*) + \cdots + f(x_{N-1}^*) + f(x_N^*)] \\ &= \frac{b-a}{N}\sum_{i=1}^{N} f(x_i^*). \end{aligned}$$

Approximating Area Under $y = \frac{1}{x}$

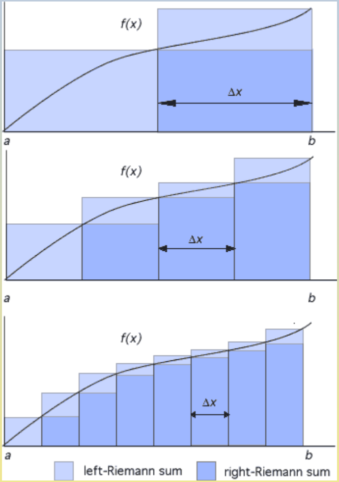
- Let us approximate the area under the graph of $f(x) = \frac{1}{x}$ on $[2,4]$ using $N = 6$ and mid-points as the x_i^*'s;

$$
\begin{aligned}
A &\approx \frac{4-2}{6}\sum_{i=1}^{6} f(2 + (i - \tfrac{1}{2})\tfrac{1}{3}) \\
&= \tfrac{1}{3}\sum_{i=1}^{6} f(\tfrac{11+2i}{6}) \\
&= \tfrac{1}{3}[f(\tfrac{13}{6}) + f(\tfrac{15}{6}) + f(\tfrac{17}{6}) + f(\tfrac{19}{6}) + f(\tfrac{21}{6}) + f(\tfrac{23}{6})] \\
&= \tfrac{1}{3}[\tfrac{6}{13} + \tfrac{6}{15} + \tfrac{6}{17} + \tfrac{6}{19} + \tfrac{6}{21} + \tfrac{6}{23}] \\
&= 2[\tfrac{1}{13} + \tfrac{1}{15} + \tfrac{1}{17} + \tfrac{1}{19} + \tfrac{1}{21} + \tfrac{1}{23}] \\
&\approx 2 \cdot 0.346 = 0.692.
\end{aligned}
$$

Exact Area as the Limit of Approximations

- When the number of rectangles N approaches infinity, then the area enclosed by the approximating rectangles tends to the exact amount of area under the curve;
- Thus

$$A = \lim_{N \to \infty} \frac{b-a}{N} \sum_{i=1}^{N} f(x_i^*).$$



- To use the limit of the approximating sums to compute areas, we need some summation formulas;

Sums of Powers

Power Sums

- $$\sum_{i=1}^{N} i = 1 + 2 + \cdots + N = \frac{N(N+1)}{2};$$
- $$\sum_{i=1}^{N} i^2 = 1^2 + 2^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6};$$
- $$\sum_{i=1}^{N} i^3 = 1^3 + 2^3 + \cdots + N^3 = \frac{N^2(N+1)^2}{4};$$

- Consider the function $f(x) = \frac{1}{2}x$. The area of the triangle under the graph of $y = f(x)$ from $x = 0$ to $x = 4$ can be computed using the familiar formula $A = \frac{1}{2}\text{base} \cdot \text{height}$; It is equal to $A = \frac{1}{2}4 \cdot 2 = 4$;
- We are going to compute this area using the limit of the approximating sums method in the next slide;

Using Limits of Approximating Sums

- We write an expression using the summation notation for the approximating sum of the area of the triangle under $y = \frac{1}{2}x$ on $[0, 4]$ using N rectangles and right endpoints as the x_i^*'s:

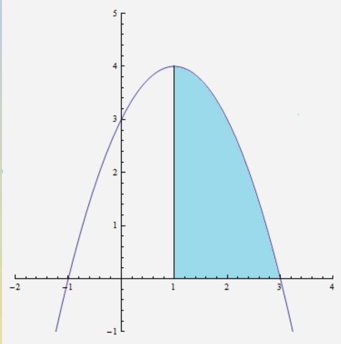
$$\begin{aligned} A &\approx \frac{4-0}{N}\sum_{i=1}^{N} f(\tfrac{4i}{N}) = \frac{4}{N}\sum_{i=1}^{N}\tfrac{1}{2}\cdot\tfrac{4i}{N} = \frac{4}{N}\sum_{i=1}^{N}\tfrac{2}{N}i \\ &= \frac{4}{N}\sum_{i=1}^{N}\tfrac{2}{N}i = \frac{8}{N^2}\sum_{i=1}^{N} i = \frac{8}{N^2}\cdot\frac{N(N+1)}{2} \\ &= \frac{8N(N+1)}{2N^2} = \frac{4N^2+4N}{N^2}. \end{aligned}$$

Therefore, the exact area is given by

$$A = \lim_{N\to\infty}\frac{4N^2+4N}{N^2} = 4.$$

Finding Area Under Curve

- Find the exact area under $f(x) = -x^2 + 2x + 3$ from $x = 1$ to $x = 3$;



The approximation sum for N subintervals using right endpoints for the x_i^*'s is

$$A \approx \frac{3-1}{N}\sum_{i=1}^{N} f(1+\frac{2i}{N})$$

$$= \frac{2}{N}\sum_{i=1}^{N}[-(1+\tfrac{2i}{N})^2 + 2(1+\tfrac{2i}{N}) + 3]$$

$$= \frac{2}{N}\sum_{i=1}^{N}[-1 - \tfrac{4i}{N} - \tfrac{4i^2}{N^2} + 2 + \tfrac{4i}{N} + 3]$$

Example (Cont'd)

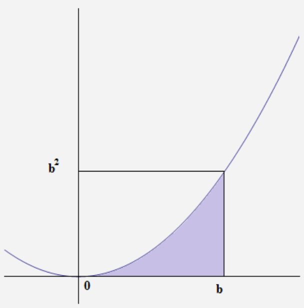
$$\begin{aligned} A &\approx \frac{2}{N}\sum_{i=1}^{N}[4 - \tfrac{4i^2}{N^2}] \\ &= \frac{2}{N}[\sum_{i=1}^{N}4 - \tfrac{4}{N^2}\sum_{i=1}^{N}i^2] \\ &= \frac{2}{N}[4N - \frac{4N(N+1)(2N+1)}{6N^2}] \\ &= 8 - \frac{4(N+1)(2N+1)}{3N^2}; \end{aligned}$$

Therefore

$$A = \lim_{N\to\infty}(8 - \frac{4(N+1)(2N+1)}{3N^2}) = 8 - \frac{8}{3} = \frac{16}{3}.$$

Area up to a Variable Endpoint

- Find the exact area under $f(x) = x^2$ from $x = 0$ to $x = b$ (a fixed constant);



The approximation sum for N subintervals using right endpoints for the x_i^*'s is

$$A \approx \frac{b-0}{N}\sum_{i=1}^{N} f(0+\frac{bi}{N})$$

$$= \frac{b}{N}\sum_{i=1}^{N}(\frac{bi}{N})^2 = \frac{b}{N}\frac{b^2}{N^2}\sum_{i=1}^{N} i^2 = \frac{b^3}{N^3}\frac{N(N+1)(2N+1)}{6};$$

Therefore,

$$A = \lim_{N\to\infty}\frac{b^3}{N^3}\frac{N(N+1)(2N+1)}{6} = \frac{1}{3}b^3.$$

Subsection 2

The Definite Integral

Riemann Sums and Definite Integrals

- Consider a function $f(x)$ on $[a, b]$;
- Choose a **partition** P of $[a, b]$ of size N, i.e.,

$$P : a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

- Choose **sample points** $C = \{c_1, \ldots, c_N\}$, with $c_i \in [x_{i-1}, x_i]$, for all i;
- Denoting $\Delta x_i = x_i - x_{i-1}$, we obtain the **Riemann sum**

$$R(f, P, C) = \sum_{i=1}^{N} f(c_i)\Delta x_i;$$

Definite Integral

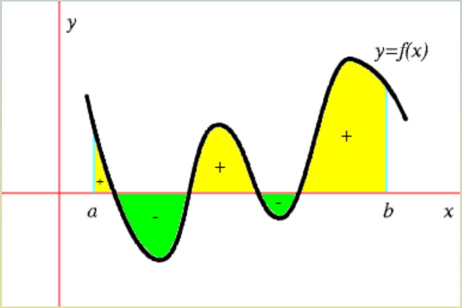
The **definite integral** of $f(x)$ over $[a, b]$ is the limit of the Riemann sums as the maximum length $\|P\|$ of the partition subintervals approaches zero:

$$\int_a^b f(x)dx = \lim_{\|P\| \to 0} R(f, P, C) = \lim_{\|P\| \to 0} \sum_{i=1}^{N} f(c_i)\Delta x_i.$$

If the limit exists $f(x)$ is called **integrable** over $[a, b]$;

Signed Areas

- Signed Area $=$ (Area Above x-Axis) $-$ (Area Below x-Axis);



- That is exactly the geometric interpretation of the definite integral:

$$\int_a^b f(x)dx = \text{Signed Area Between Graph and } x\text{-Axis over } [a, b];$$

Interpretation into Signed Area

- Compute $\int_0^5 (3-x)dx$

According to the previous interpretation, we have

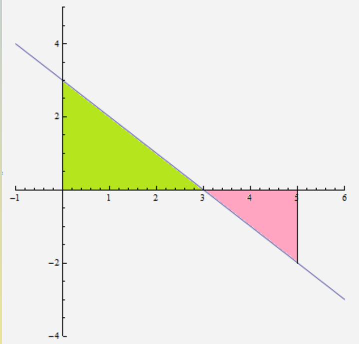
$$\int_0^5 (3-x)dx$$

$$= \text{(Area Above)} - \text{(Area Below)}$$

$$= \tfrac{1}{2}3\cdot 3 - \tfrac{1}{2}2\cdot 2$$

$$= \tfrac{9}{2} - 2$$

$$= \tfrac{5}{2};$$



Constant Functions and Linearity

Integral of a Constant

$$\int_a^b C dx = C(b-a).$$

Linearity of the Definite Integral

If f, g are integrable over $[a, b]$, then $f \pm g$ and Cf are also integrable over $[a, b]$ and:

- $\displaystyle\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx;$
- $\displaystyle\int_a^b Cf(x) dx = C \int_a^b f(x) dx.$

Example: Recall that $\int_0^b x^2 dx = \frac{1}{3} b^3$; Therefore, we have $\int_0^3 (2x^2 - 5) dx = \int_0^3 2x^2 dx - \int_0^3 5 dx = 2 \int_0^3 x^2 dx - \int_0^3 5 dx = 2\frac{3^3}{3} - 5(3-0) = 3;$

Reversing the Limits and Adding Over Intervals

Reversing the Limits of Integration

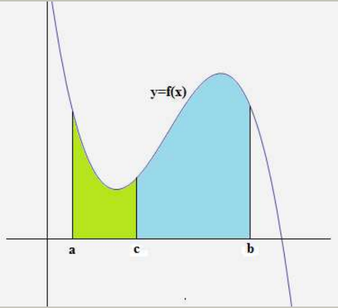
If $a < b$, then

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Additivity over Adjacent Intervals

If $a \leq b \leq c$ and $f(x)$ is integrable, then:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

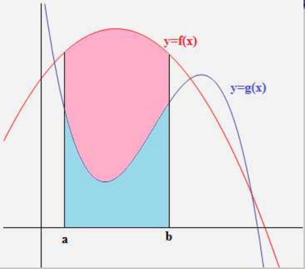


Comparison Theorem

Comparison Theorem

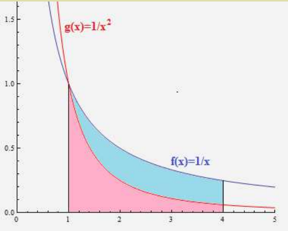
If f and g are integrable and $g(x) \leq f(x)$ for all $x \in [a, b]$, then

$$\int_a^b g(x)dx \leq \int_a^b f(x)dx.$$



Example: If $x \geq 1$, $x^2 \geq x$ and, hence, $\frac{1}{x^2} \leq \frac{1}{x}$. Therefore,

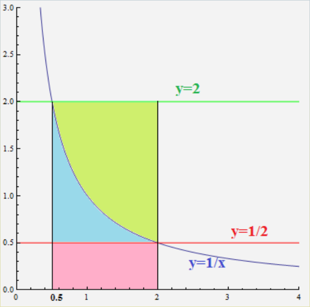
$$\int_1^4 \frac{1}{x^2}dx \leq \int_1^4 \frac{1}{x}dx;$$



Establishing Bounds

Consider the function $f(x) = \frac{1}{x}$ on $[\frac{1}{2}, 2]$; Clearly, if $\frac{1}{2} \le x \le 2$, $\frac{1}{2} \le \frac{1}{x} \le 2$; Therefore, by the Comparison Theorem,

$$\int_{1/2}^{2} \frac{1}{2} dx \le \int_{1/2}^{2} \frac{1}{x} dx \le \int_{1/2}^{2} 2dx;$$



This yields

$$\frac{3}{2} \cdot \frac{1}{2} \le \int_{1/2}^{2} \frac{1}{x} dx \le \frac{3}{2} \cdot 2; \quad \text{i.e.,} \quad \frac{3}{4} \le \int_{1/2}^{2} \frac{1}{x} dx \le 3;$$

Subsection 3

The Fundamental Theorem of Calculus, Part I

The Fundamental Theorem of Calculus, Part I

The Fundamental Theorem of Calculus, Part I

If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is an antiderivative of $f(x)$ on $[a, b]$, then

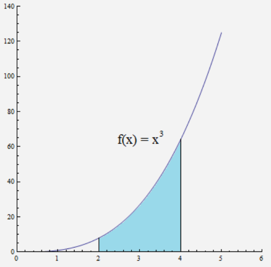
$$\int_a^b f(x)dx = F(b) - F(a).$$

- The difference $F(b) - F(a)$ is denoted $F(x) \mid_a^b$. Using this notation, we get

$$\int_a^b f(x)dx = F(x)|_a^b .$$

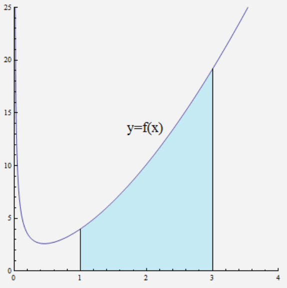
Example: Calculate the area under $f(x) = x^3$ over $[2, 4]$;

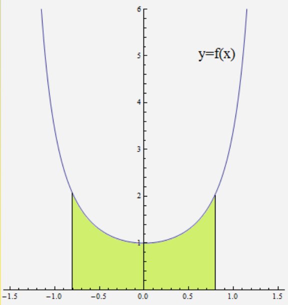
$$\begin{aligned} A &= \int_2^4 x^3 dx = \tfrac{1}{4}x^4 \mid_2^4 \\ &= \tfrac{1}{4}(4^4 - 2^4) = 60. \end{aligned}$$



More Examples

Example: Calculate the area under $f(x) = x^{-3/4} + 3x^{5/3}$ over $[1, 3]$;
$A = \int_1^3 (x^{-3/4} + 3x^{5/3})dx$
$= (4x^{1/4} + \frac{9}{8}x^{8/3}) \big|_1^3$
$= (4 \cdot 3^{1/4} + \frac{9}{8} \cdot 3^{8/3}) - (4 + \frac{9}{8}) \approx 21.2.$

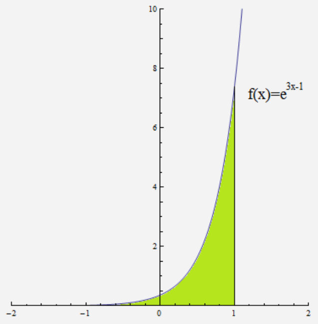


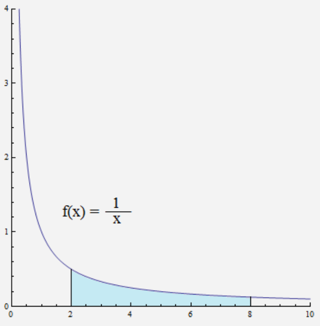


Example: Calculate the area under $f(x) = \sec^2 x$ over $[-\frac{\pi}{4}, \frac{\pi}{4}]$;
$A = \int_{-\pi/4}^{\pi/4} \sec^2 x dx = \tan x \big|_{-\pi/4}^{\pi/4} = \tan\frac{\pi}{4} - \tan(-\frac{\pi}{4}) = 2.$

Additional Examples

Example: Calculate the area under $f(x) = e^{3x-1}$ over $[-1, 1]$;
$A = \int_{-1}^{1} e^{3x-1} dx = \frac{1}{3} e^{3x-1} \Big|_{-1}^{1} = \frac{1}{3}(e^2 - e^{-4}) \approx 2.457$





Example: Calculate the area under $f(x) = \frac{1}{x}$ over $[2, 8]$;
$A = \int_{2}^{8} \frac{1}{x} dx = \ln x \Big|_{2}^{8}$
$= \ln 8 - \ln 2 \approx 1.386.$

Subsection 4

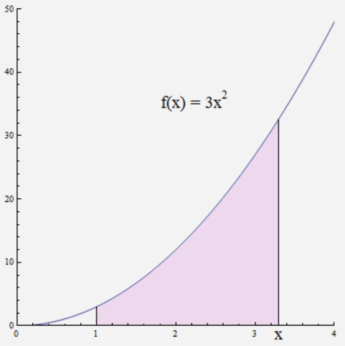
The Fundamental Theorem of Calculus, Part II

Illustration of Main Concept

- Consider $f(x) = 3x^2$;

  The area $A(x)$ under $y = f(x)$ over $[1, x]$ is given by

$$\begin{aligned} A(x) &= \int_1^x 3t^2 dt \\ &= t^3 \Big|_1^x \\ &= x^3 - 1; \end{aligned}$$



- Now, note that $A'(x) = (x^3 - 1)' = 3x^2 = f(x)$;

Fundamental Theorem of Calculus, Part II

Fundamental Theorem of Calculus, Part II

If $f(x)$ is continuous on an open interval I and $a \in I$, then the area function

$$A(x) = \int_a^x f(t)dt$$

is an antiderivative of $f(x)$ on I, i.e., $A'(x) = f(x)$; Equivalently,

$$\frac{d}{dx}\int_a^x f(t)dt = f(x);$$

Note that this antiderivative satisfies the initial condition $A(a) = 0$.

Examples

- Suppose $F(x)$ is a particular antiderivative of $f(x) = \sin(x^2)$ satisfying $F(-\sqrt{\pi}) = 0$. Express $F(x)$ as an integral.

  According to the Part II of the Fundamental Theorem, we have

$$F(x) = \int_{-\sqrt{\pi}}^{x} f(t)dx = \int_{-\sqrt{\pi}}^{x} \sin(t^2)dt.$$

- Find the derivative of $A(x) = \int_{2}^{x} \sqrt{1+t^3}dt$;

  By Part II of the Fundamental Theorem,

$$\frac{dA}{dx} = \frac{d}{dx}\int_{2}^{x} \sqrt{1+t^3}dt = \sqrt{1+x^3}.$$

Fundamental Theorem of Calculus and the Chain Rule

- Let us find the derivative of $G(x) = \int_{-2}^{x^2} \sin t dt$;

  It is important to realize that $G(x) = A(x^2)$, where
  $A(x) = \int_{-2}^{x} \sin t dt$;
  Thus, $G(x)$ is a composite function and, as such, the Chain Rule must be used to compute its derivative:

$$\begin{aligned} \frac{d}{dx} G(x) &= \frac{d}{dx} A(x^2) \underbrace{=}_{u=x^2} \frac{d}{du} A(u) \frac{du}{dx} \\ &= f(u) \cdot 2x = \sin u \cdot 2x \\ &= 2x \sin(x^2). \end{aligned}$$

Subsection 5

Net Change as the Integral of a Rate

Net Change as Integral of Rate of Change

- The **net change** in $s(t)$ over an interval $[t_1, t_2]$ is the integral

$$\int_{t_1}^{t_2} s'(t)dt = s(t_2) - s(t_1);$$

Example: If water leaks from a bucket at a rate of $2+5t$ lt/hr, where t is number of hours after 7 AM, how much water is lost between 9 and 11 AM?

We have

$$\begin{aligned} s(4) - s(2) &= \int_2^4 -(2+5t)dt = \left(-2t - \tfrac{5}{2}t^2\right)\Big|_2^4 \\ &= (-48) - (-14) = -34 \text{ lts.} \end{aligned}$$

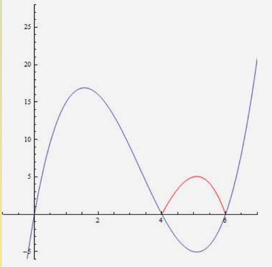
The Integral of Velocity

- For an object in linear motion with velocity $v(t)$,
  - Displacement during $[t_1, t_2] = \int_{t_1}^{t_2} v(t)dt$;
  - Distance traveled during $[t_1, t_2] = \int_{t_1}^{t_2} |v(t)|dt$;

Example: If $v(t) = t^3 - 10t^2 + 24t$ m/sec, compute both the displacement and the total distance over $[0, 6]$;

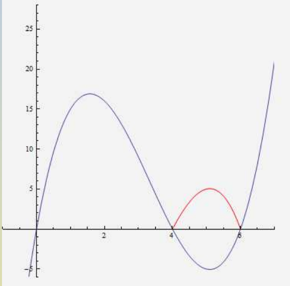
Thus, we have

$$\begin{aligned} &\int_0^6 v(t)dt \\ &= \int_0^6 (t^3 - 10t^2 + 24t)dt \\ &= \left(\tfrac{1}{4}t^4 - \tfrac{10}{3}t^3 + 12t^2\right)\Big|_0^6 \\ &= 36 \text{ meters;} \end{aligned}$$



The Integral of Velocity: Example (Cont'd)

Note that $|v(t)| =$
$$\begin{cases} t^3 - 10t^2 + 24t, & \text{if } 0 \le t \le 4 \\ -(t^3 - 10t^2 + 24t), & \text{if } 4 \le t \le 6 \end{cases}$$



Thus, we have

$$\begin{aligned}
&\int_0^6 |v(t)|dt \\
&= \int_0^4 (t^3 - 10t^2 + 24t)dt + \int_4^6 -(t^3 - 10t^2 + 24t)dt \\
&= \left(\tfrac{1}{4}t^4 - \tfrac{10}{3}t^3 + 12t^2\right)\Big|_0^4 + \left(-\tfrac{1}{4}t^4 + \tfrac{10}{3}t^3 - 12t^2\right)\Big|_4^6 \\
&= \frac{128}{3} + \frac{20}{3} = \frac{148}{3} \text{ meters.}
\end{aligned}$$

Total Versus Marginal Cost

- Let $C(x)$ be cost for producing x units of a product or a commodity;
- The derivative $C'(x)$ is called the **marginal cost**;
- The cost of increasing production from a to b is

$$C[a,b] = \int_a^b C'(x)dx;$$

Example: Suppose that the marginal cost for producing x computer chips (x in thousands) is $C'(x) = 300x^2 - 4000x + 40,000$ dollars per thousand chips;

- Determine the cost of increasing production from 10,000 to 15,000 chips.

$$\begin{aligned} C[10,15] &= \int_{10}^{15} C'(x)dx \\ &= \int_{10}^{15} (300x^2 - 4000x + 40,000)dx \\ &= (100x^3 - 2000x^2 + 40,000x)\Big|_{10}^{15} \\ &= \$187,500. \end{aligned}$$

Total Versus Marginal Cost: Example (Cont'd)

- The marginal cost for producing x computer chips (x in thousands) is $C'(x) = 300x^2 - 4000x + 40,000$ dollars per thousand chips;
  - Determine the total production cost for 15,000 chips assuming that the company incurs a cost of \$ 30,000 for setting up the manufacturing run, i.e., that $C(0) = 30,000$;

$$\begin{aligned} C(x) &= \int C'(x)dx \\ &= \int (300x^2 - 4000x + 40,000)dx \\ &= 100x^3 - 2000x^2 + 40,000x + C. \end{aligned}$$

Since $C(0) = 30,000$, we get $C = 30,000$; Hence,

$$C(x) = 100x^3 - 2000x^2 + 40,000x + 30,000.$$

Therefore,

$$C(15) = 100 \cdot 15^3 - 2000 \cdot 15^2 + 40,000 \cdot 15 + 30,000 = \$517,500;$$

Subsection 6

Substitution Method

The Substitution Method

- Recall the **Chain Rule** for computing derivatives:

$$\frac{d}{dx}F(u(x)) = F'(u(x))u'(x) = f(u(x))u'(x),$$

  where, of course $F(x)$ is an antiderivative of $f(x)$;
- This rule yields the **Substitution Rule** for computing indefinite integrals:

$$\boxed{\int f(u(x))u'(x)dx = F(u(x)) + C;}$$

- Usually, the Substitution Rule is applied in the form of the **Substitution** or **Change of Variable Method**:
  - We want to compute $\int f(u(x))u'(x)dx$;
  - Note that since $\frac{du}{dx} = u'(x)$, one gets $du = u'(x)dx$;
  - Therefore $\int f(u(x))u'(x)dx = \int f(u)du = F(u) + C$;

Example I

- Evaluate $\int 3x^2 \sin(x^3) dx$;
- Method 1 (Substitution Rule):

$$\begin{aligned}\int 3x^2 \sin(x^3) dx &= \int (x^3)' \sin(x^3) dx \\ &= -\cos(x^3) + C;\end{aligned}$$

- Method 2 (Substitution Method):
  Let $u = x^3$; Then $\frac{du}{dx} = 3x^2$; Therefore, $du = 3x^2 dx$;
  So we have

$$\begin{aligned}\int 3x^2 \sin(x^3) dx &= \int \sin u \; du \\ &= -\cos u + C \\ &= -\cos(x^3) + C;\end{aligned}$$

Example II

- Evaluate $\int x(x^2+9)^5 dx$;
- Method 1 (Substitution Rule):

$$\begin{aligned}\int x(x^2+9)^5 dx &= \frac{1}{2}\int 2x(x^2+9)^5 dx \\ &= \frac{1}{2}\int (x^2+9)'(x^2+9)^5 dx \\ &= \frac{1}{2}\cdot\frac{1}{6}(x^2+9)^6 + C;\end{aligned}$$

- Method 2 (Substitution Method):
  Let $u = x^2+9$; Then $\frac{du}{dx} = 2x$; Therefore, $\frac{1}{2}du = xdx$;
  So we have

$$\begin{aligned}\int x(x^2+9)^5 dx &= \frac{1}{2}\int u^5 du \\ &= \frac{1}{2}\cdot\frac{1}{6}u^6 + C \\ &= \frac{1}{12}(x^2+9)^6 + C;\end{aligned}$$

Example III

- Evaluate $\int \frac{x^2+2x}{(x^3+3x^2+12)^6} dx$;
  Let $u = x^3 + 3x^2 + 12$; Then $\frac{du}{dx} = 3x^2 + 6x = 3(x^2 + 2x)$;
  Therefore, $\frac{1}{3} du = (x^2 + 2x) dx$;
  So we have

$$\begin{aligned} \int \frac{x^2+2x}{(x^3+3x^2+12)^6} dx &= \frac{1}{3} \int \frac{1}{u^6} du \\ &= \frac{1}{3} \cdot \frac{1}{-5} u^{-5} + C \\ &= -\frac{1}{15u^5} + C \\ &= -\frac{1}{15(x^3+3x^2+12)^5} + C; \end{aligned}$$

More Examples

- Evaluate $\int \sin(7\theta + 5)d\theta$;

  Let $u = 7\theta + 5$; Then $\frac{du}{d\theta} = 7$; Therefore, $\frac{1}{7}du = d\theta$;
  So we have
  $$\begin{aligned} \int \sin(7\theta + 5)d\theta &= \frac{1}{7}\int \sin u du \\ &= \frac{1}{7}(-\cos u) + C \\ &= -\frac{1}{7}\cos(7\theta + 5) + C; \end{aligned}$$

- Evaluate $\int e^{-9t}dt$;

  Let $u = -9t$; Then $\frac{du}{dt} = -9$; Therefore, $-\frac{1}{9}du = dt$;
  So we have
  $$\begin{aligned} \int e^{-9t}dt &= -\frac{1}{9}\int e^{u}du \\ &= -\frac{1}{9}e^{u} + C \\ &= -\frac{1}{9}e^{-9t} + C; \end{aligned}$$

Additional Examples

- Evaluate $\displaystyle\int \tan\theta d\theta$;

  Rewrite $\int \tan\theta d\theta = \int \frac{\sin\theta}{\cos\theta} d\theta$;
  Let $u = \cos\theta$; Then $\frac{du}{d\theta} = -\sin\theta$; Therefore, $-du = \sin\theta d\theta$; Thus,

$$\begin{aligned} \int \tan\theta d\theta &= \int \frac{\sin\theta}{\cos\theta} d\theta = -\int \frac{1}{u} du \\ &= -\ln|u| + C = -\ln|\cos\theta| + C; \end{aligned}$$

- Evaluate $\displaystyle\int x\sqrt{5x+1}dx$;

  Let $u = 5x+1$; Then, $x = \frac{1}{5}u - \frac{1}{5}$; Also, $\frac{du}{dx} = 5$; So, $\frac{1}{5}du = dx$;
  We now have

$$\begin{aligned} \int x\sqrt{5x+1}dx &= \frac{1}{5}\int \left(\frac{1}{5}u - \frac{1}{5}\right)\sqrt{u}du = \frac{1}{25}\int (u^{3/2} - u^{1/2})du \\ &= \frac{1}{25}\left(\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C = \frac{2}{125}u^{5/2} + \frac{2}{75}u^{3/2} + C \\ &= \frac{2}{125}(5x+1)^{5/2} + \frac{2}{75}(5x+1)^{3/2} + C \end{aligned}$$

Substitution for Definite Integration

$$\boxed{\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du;}$$

- Example: Evaluate $\int_0^2 x^2\sqrt{x^3+1}dx$;
  Let $u = x^3 + 1$; Then, $\frac{du}{dx} = 3x^2$; So, $\frac{1}{3}du = x^2dx$; Also, for $x = 0$, $u = 1$ and for $x = 2$, $u = 9$;
  We now have

$$\begin{aligned} \int_0^2 x^2\sqrt{x^3+1}dx &= \frac{1}{3}\int_1^9 \sqrt{u}du = \frac{1}{3}\ \frac{2}{3}\sqrt{u^3}\Big|_1^9 \\ &= \frac{2}{9}(27-1) = \frac{52}{9}; \end{aligned}$$

Two More Examples

- Evaluate $\int_0^{\pi/4} \tan^3\theta \sec^2\theta d\theta$;

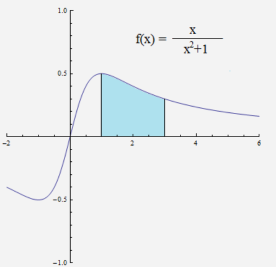
  Let $u = \tan\theta$; Then, $\frac{du}{d\theta} = \sec^2\theta$; So, $du = \sec^2\theta d\theta$; Also, for $\theta = 0$, $u = 0$ and for $\theta = \frac{\pi}{4}$, $u = 1$; We now have

$$\int_0^{\pi/4} \tan^3\theta \sec^2\theta d\theta = \int_0^1 u^3 du = \frac{1}{4}u^4\Big|_0^1 = \frac{1}{4};$$

- Evaluate $\int_1^3 \frac{x}{x^2+1} dx$;

  Let $u = x^2 + 1$; Then, $\frac{du}{dx} = 2x$; So, $\frac{1}{2}du = xdx$; Also, for $x = 1$, $u = 2$ and for $x = 3$, $u = 10$;

$$\begin{aligned} \int_1^3 \frac{x}{x^2+1} dx &= \frac{1}{2}\int_2^{10} \frac{1}{u} du = \\ \frac{1}{2}\ln u\Big|_2^{10} &= \frac{1}{2}(\ln 10 - \ln 2); \end{aligned}$$



Subsection 7

Further Transcendental Functions

Transcendental Functions Using Substitution

- Evaluate $\int_0^1 \frac{1}{x^2+1}dx$;
  We have
  $$\int_0^1 \frac{1}{x^2+1}dx = \tan^{-1} x\Big|_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \tfrac{\pi}{4};$$
- Evaluate $\int_{1/\sqrt{2}}^1 \frac{1}{x\sqrt{4x^2-1}}dx$;
  Let $u = 2x$; Then, $\frac{du}{dx} = 2$; So, $\frac{1}{2}du = dx$; Also, for $x = \frac{1}{\sqrt{2}}$, $u = \sqrt{2}$ and, for $x = 1$, $u = 2$; We now have
  $$\begin{aligned} \int_{1/\sqrt{2}}^1 \frac{1}{x\sqrt{4x^2-1}}dx &= \int_{\sqrt{2}}^2 \frac{\frac{1}{2}}{\frac{1}{2}u\sqrt{u^2-1}}du = \int_{\sqrt{2}}^2 \frac{1}{u\sqrt{u^2-1}}du \\ &= \sec^{-1} u\Big|_{\sqrt{2}}^2 = \sec^{-1} 2 - \sec^{-1} \sqrt{2} \\ &= \tfrac{\pi}{3} - \tfrac{\pi}{4} = \tfrac{\pi}{12}; \end{aligned}$$

Two More Examples

- Evaluate $\int_0^{3/4} \frac{1}{\sqrt{9-16x^2}} dx$;
  Rewrite $\frac{1}{\sqrt{9-16x^2}} = \frac{1}{3\sqrt{1-\frac{16}{9}x^2}} = \frac{1}{3\sqrt{1-(\frac{4x}{3})^2}}$; Set $u = \frac{4x}{3}$; Thus, $\frac{du}{dx} = \frac{4}{3}$; So, $\frac{3}{4}du = dx$; For $x = 0$, $u = 0$; and for $x = \frac{3}{4}$, $u = 1$;
  $$\begin{aligned}\int_0^{3/4} \frac{1}{\sqrt{9-16x^2}} dx &= \int_0^{3/4} \frac{1}{3\sqrt{1-(\frac{4x}{3})^2}} dx = \int_0^1 \frac{1}{4} \frac{1}{\sqrt{1-u^2}} du \\ &= \frac{1}{4} \sin^{-1} u \Big|_0^1 = \frac{1}{4} \cdot \frac{\pi}{2};\end{aligned}$$
- Evaluate $\int_0^{\pi/2} (\cos\theta) 10^{\sin\theta} d\theta$;
  Let $u = \sin\theta$; Then, $\frac{du}{d\theta} = \cos\theta$; So, $du = \cos\theta d\theta$; Also, for $\theta = 0$, $u = 0$ and, for $\theta = \frac{\pi}{2}$, $u = 1$; We now have
  $$\int_0^{\pi/2} (\cos\theta) 10^{\sin\theta} d\theta = \int_0^1 10^u du = \frac{1}{\ln 10} 10^u \Big|_0^1 = \frac{9}{\ln 10};$$

Subsection 8

Exponential Growth and Decay

Exponential Growth and Decay

- The quantity $P(t)$ depends **exponentially** on time t, if it varies according to
$$P(t) = P_0 e^{kt};$$
  - If $k > 0$, then $P(t)$ **grows exponentially** and k is the **growth constant**;
  - If $k < 0$, then $P(t)$ **decays exponentially** and k is the **decay constant**;

Example: If an E-coli culture grows exponentially with growth constant $k = 0.41$ hours^{-1} and there are 1000 bacteria at time $t = 0$, what is the population $P(t)$ at time t? When will the population reach the level of $10,000$?
We have $P(t) = 1000e^{0.41t}$;
Therefore, the population will reach 10,000 when
$1000e^{0.41t} = 10,000$; This yields $e^{0.41t} = 10$, or $t = \frac{1}{0.41} \ln 10$;

Differential Equations with Exponential Solutions

Theorem (Solutions of $y' = ky$)

If $y(t)$ obeys the differential equation $y' = ky$, then

$$y(t) = P_0 e^{ky},$$

where $P_0 = y(0)$.

Example: What are the general solutions of $y' = 3y$? Which one satisfies the initial condition $y(0) = 9$?
According to the Theorem,

$$y(t) = P_0 e^{3t};$$

Moreover, if $y(0) = 9$, then $P_0 = 9$, whence $y(t) = 9e^{3t}$;

Administering a Drug

- Suppose that a drug leaves the bloodstream at a rate proportional to the amount present.
  - Write a differential equation expressing this statement;
  - If 50 mg of the drug remain in the blood 7 hours after an injection of 450 mg, what is the decay constant?
  - At what time, will there be 200 mg present in the blood?
- We work as follows:
  - If y is the amount present, then $y' = -ky$;
  - The general solution of this equation is $y = P_0 e^{-kt}$; Under hypotheses, $50 = 450e^{-7k}$; Therefore, $-7k = \ln \frac{1}{9} = -\ln 9$, i.e., $k = \frac{\ln 9}{7}$;
  - We must solve $200 = 450e^{-\frac{\ln 9}{7}t}$; So $e^{-\frac{\ln 9}{7}t} = \frac{4}{9}$, i.e., $-\frac{\ln 9}{7}t = \ln \frac{4}{9}$; Thus, we get $t = -\frac{7\ln(4/9)}{\ln 9}$;

Doubling Time and Half-Life

- If $P(t) = P_0 e^{kt}$, with $k > 0$, then the **doubling time** of P is
$$\text{Doubling Time} = \frac{\ln 2}{k};$$
- If $P(t) = P_0 e^{-kt}$, with $k > 0$, then the **half-life** of P is
$$\text{Half-Life} = \frac{\ln 2}{k};$$
- The formulas above are very easy to establish; They need not be memorized!
Set $P(t) = 2P_0$; Then $2P_0 = P_0 e^{kt}$; Now solve for t: $2 = e^{kt}$, whence $kt = \ln 2$, and, therefore, $t = \dfrac{\ln 2}{k}$;

Compound Interest

- If P_0 dollars are invested in an account earning interest at annual rate r, compounded M times yearly, then the future amount $P(t)$ after t years is

$$P(t) = P_0 \left(1 + \frac{r}{M}\right)^{Mt};$$

Theorem (Limit Formulas for e and e^x)

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n.$$

- If P_0 dollars are invested in an account earning interest at annual rate r, compounded continuously, then the future amount $P(t)$ after t years is

$$P(t) = P_0 e^{rt};$$

Present Value of Future Amount

Present Value

The **present value** PV of P dollars to be received t years in the future under continuous compounding at an annual rate r, is given by

$$\text{PV} = Pe^{-rt};$$

Example: If the annual interest rate is $r = 0.03$, is it better to receive \$ 2000 today or \$ 2200 in two years?
The present value of \$ 2200 received two years from now is $\text{PV} = Pe^{-rt}$ i.e., $\text{PV} = 2200e^{-0.03 \cdot 2} \approx 2,071.88$; Therefore, it is better to receive \$ 2,200 two years from now;

Present value of an Income Stream

PV of an Income Stream

If the annual interest rate is r, the present value of an income stream paying out $R(t)$ dollars per year continuously for T years is

$$\mathrm{PV} = \int_0^T R(t)e^{-rt}dt;$$

Example: An investment pays ¥100,000 per year continuously for 10 years. What is the investment's present value for $r = 0.06$?

$$\begin{aligned} \mathrm{PV} &= \int_0^{10} 100,000e^{-0.06t}dt = \frac{100,000}{-0.06}e^{-0.06}\Big|_0^{10} \\ &\approx 1,666,666.67(e^{-0.6}-1) \approx ¥751,980.61; \end{aligned}$$

Example: An investment pays €50,000 per year continuously for 5 years. What is the investment's present value for $r = 0.02$?

$$\begin{aligned} \mathrm{PV} &= \int_0^{5} 50,000e^{-0.02t}dt = \frac{50,000}{-0.02}e^{-0.02}\Big|_0^{5} \\ &\approx 2,500,500(e^{-0.2}-1) \approx €453,173.12; \end{aligned}$$