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Introduction to Complex Analysis

Overview of Consequences of the Residue Theorem

- The residue theory can be used to evaluate real integrals of the forms
  - $\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$;
  - $\int_{-\infty}^{\infty} f(x)dx$;
  - $\int_{-\infty}^{\infty} f(x)\cos\alpha x dx$;
  - $\int_{-\infty}^{\infty} f(x)\sin\alpha x dx$.

  Here F and f are rational functions of the form $f(x) = \frac{p(x)}{q(x)}$ in which the polynomials p and q are assumed to have no common factors.
- Residues can be used to evaluate real improper integrals that require integration along a branch cut.
- A relationship exists between the residue theory and the zeros of an analytic function.
- Residues can, in certain cases, be used to find the sum of an infinite series.

Subsection 1

Evaluation of Real Trigonometric Integrals

Integrals of the Form $\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$

- The basic idea is to convert those into a complex integral, where the contour C is the unit circle $|z| = 1$ centered at the origin.
- To do this we parametrize this contour by $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. We write $dz = ie^{i\theta}d\theta,\ \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2},\ \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$. Since $dz = ie^{i\theta}d\theta = izd\theta$ and $z^{-1} = \frac{1}{z} = e^{-i\theta}$, these three quantities are equivalent to $d\theta = \frac{dz}{iz},\ \cos\theta = \frac{1}{2}(z + z^{-1}),\ \sin\theta = \frac{1}{2i}(z - z^{-1})$. The conversion of the given integral into a contour integral is

$$\oint_C F\left(\frac{1}{2}(z+z^{-1}), \frac{1}{2i}(z-z^{-1})\right)\frac{dz}{iz},$$

where C is the unit circle $|z| = 1$.

A Real Trigonometric Integral

- Evaluate $\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2} d\theta$.
- We use the substitutions: $\oint_C \frac{1}{(2+\frac{1}{2}(z+z^{-1}))^2} \frac{dz}{iz} = \oint_C \frac{1}{(2+\frac{z^2+1}{2z})^2} \frac{dz}{iz}$.

  Simplifying, $\frac{4}{i} \oint_C \frac{z}{(z^2+4z+1)^2} dz$. Factoring the denominator $z^2+4z+1 = (z-z_1)(z-z_2)$, where $z_1 = -2-\sqrt{3}$ and $z_2 = -2+\sqrt{3}$. Thus, $\frac{z}{(z^2+4z+1)^2} = \frac{z}{(z-z_1)^2(z-z_2)^2}$. Only z_2 is inside the unit circle C. Thus, we have $\oint_C \frac{z}{(z^2+4z+1)^2} dz = 2\pi i \mathrm{Res}(f(z), z_2)$.

  To calculate the residue, note that z_2 is a pole of order 2:

  $\mathrm{Res}(f(z), z_2) = \lim\limits_{z\to z_2} \frac{d}{dz}(z-z_2)^2 f(z) = \lim\limits_{z\to z_2} \frac{d}{dz} \frac{z}{(z-z_1)^2} =$

  $\lim\limits_{z\to z_2} \frac{-z-z_1}{(z-z_1)^3} = \frac{1}{6\sqrt{3}}$.

  Hence, $\frac{4}{i}\oint_C \frac{z}{(z^2+4z+1)^2} dz = \frac{4}{i} \cdot 2\pi i \mathrm{Res}(f(z), z_1) = \frac{4}{i} \cdot 2\pi i \cdot \frac{1}{6\sqrt{3}}$

  and, finally, $\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2} d\theta = \frac{4\pi}{3\sqrt{3}}$.

Subsection 2

Evaluation of Real Improper Integrals

Integrals of the Form $\int_{-\infty}^{\infty} f(x)dx$

- Suppose $y = f(x)$ is a real function that is defined and continuous on the interval $[0, \infty)$.
- In elementary calculus the improper integral $I_1 = \int_0^\infty f(x)dx$ is defined as the limit $I_1 = \int_0^\infty f(x)dx = \lim_{R\to\infty} \int_0^R f(x)dx$. If the limit exists, the integral I_1 is said to be **convergent**; otherwise, it is **divergent**.
- The improper integral $I_2 = \int_{-\infty}^0 f(x)dx$ is defined similarly: $I_2 = \int_{-\infty}^0 f(x)dx = \lim_{R\to\infty} \int_{-R}^0 f(x)dx$.
- Finally, if f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = I_1 + I_2,$$

provided both integrals I_1 and I_2 are convergent. If either one, I_1 or I_2, is divergent, then $\int_{-\infty}^{\infty} f(x)dx$ is divergent.

Cauchy Principal Value of $\int_{-\infty}^{\infty} f(x)dx$

- It is important to remember that $\lim_{R\to\infty}\int_{-R}^{0} f(x)dx + \lim_{R\to\infty}\int_{0}^{R} f(x)dx$ is not the same as $\lim_{R\to\infty}(\int_{-R}^{0} f(x)dx + \int_{0}^{R} f(x)dx) = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$.
- For the integral $\int_{-\infty}^{\infty} f(x)dx$ to be convergent, the limits $\lim_{R\to\infty}\int_{-R}^{0} f(x)dx$ and $\lim_{R\to\infty}\int_{0}^{R} f(x)dx$ must exist independently of one another.
- In the event that we know (a priori) that an improper integral $\int_{-\infty}^{\infty} f(x)dx$ converges, we can then evaluate it by means of the single limiting process $\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$.
- On the other hand, the symmetric limit may exist even though the improper integral $\int_{-\infty}^{\infty} f(x)dx$ is divergent.
- The limit $\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$, if it exists, is called the **Cauchy principal value** (**P.V.**) of the integral and is written P.V. $\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$.

Principal Value and Integrals of Even Functions

- Suppose $f(x)$ is continuous on $(-\infty, \infty)$ and is an even function, i.e., $f(-x) = f(x)$. Then its graph is symmetric with respect to the y-axis. As a consequence, $\int_{-R}^{0} f(x)dx = \int_{0}^{R} f(x)dx$. Therefore, $\int_{-R}^{R} f(x)dx = \int_{-R}^{0} f(x)dx + \int_{0}^{R} f(x)dx = 2\int_{0}^{R} f(x)dx$.
  If the Cauchy principal value exists, $\int_{0}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} f(x)dx$ converge. The values of the integrals are

$$\int_{0}^{\infty} f(x)dx = \frac{1}{2}\text{P.V.} \int_{-\infty}^{\infty} f(x)dx$$

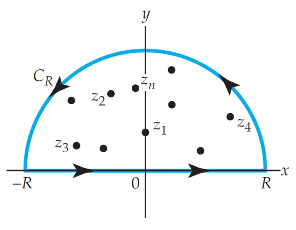
  and

$$\int_{-\infty}^{\infty} f(x)dx = \text{P.V.} \int_{-\infty}^{\infty} f(x)dx.$$

Evaluation of Integral $\int_{-\infty}^{\infty} f(x)dx$

- To evaluate $\int_{-\infty}^{\infty} f(x)dx$, where the rational function $f(x) = \frac{p(x)}{q(x)}$ is continuous on $(-\infty, \infty)$,

  we replace x by the complex variable z and integrate the complex function f over a closed contour C that consists of the interval $[-R, R]$ on the real axis and a semicircle C_R of radius large enough to enclose all the poles of $f(z) = \frac{p(z)}{q(z)}$ in the upper half-plane $\mathrm{Im}(z) > 0$.



  Then,

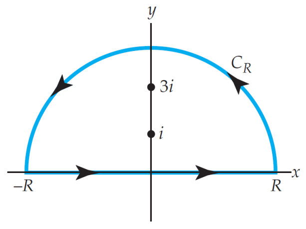
  $\oint_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$,
  where z_k, $k = 1, 2, \ldots, n$ denotes poles in the upper half-plane. If we can show that the $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$, then we have

  $\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$.

Cauchy P.V. of an Improper Integral

- Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\,dx$.

  Let $f(z) = \frac{1}{(z^2+1)(z^2+9)}$.

  Since $(z^2+1)(z^2+9) = (z-i)(z+i)(z-3i)(z+3i)$, we take C be the closed contour consisting of the interval $[-R, R]$ on the x-axis and the semicircle C_R of radius $R > 3$.

  Then,
  $\oint_C \frac{1}{(z^2+1)(z^2+9)}\,dz = \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+9)}\,dx + \int_{C_R} \frac{1}{(z^2+1)(z^2+9)}\,dz = I_1 + I_2$ and $I_1 + I_2 = 2\pi i[\mathrm{Res}(f(z), i) + \mathrm{Res}(f(z), 3i)]$. At the simple poles $z = i$ and $z = 3i$ we find $\mathrm{Res}(f(z), i) = \frac{1}{16i}$ and $\mathrm{Res}(f(z), 3i) = -\frac{1}{48i}$, whence $I_1 + I_2 = 2\pi i[\frac{1}{16i} + (-\frac{1}{48i})] = \frac{\pi}{12}$.

Letting $R \to \infty$

- $\oint_C \frac{1}{(z^2+1)(z^2+9)} dz = \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+9)} dx + \int_{C_R} \frac{1}{(z^2+1)(z^2+9)} dz = \frac{\pi}{12}$.
  Before letting $R \to \infty$, note that $|(z^2+1)(z^2+9)| = |z^2+1| \cdot |z^2+9| \geq ||z^2|-1| \cdot ||z^2|-9| = (R^2-1)(R^2-9)$. Since the length L of the semicircle is πR, it follows, by the ML-inequality, $|I_2| = \left| \int_{C_R} \frac{1}{(z^2+1)(z^2+9)} dz \right| \leq \frac{\pi R}{(R^2-1)(R^2-9)}$. Hence, $|I_2| \to 0$ as $R \to \infty$, and we conclude that $\lim_{R\to\infty} I_2 = 0$. It follows that $\lim_{R\to\infty} I_1 = \frac{\pi}{12}$. I.e., $\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+9)} dx = \frac{\pi}{12}$ or P.V. $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)} dx = \frac{\pi}{12}$.

Behavior of Integral as $R \to \infty$

- To show that the contour integral along C_R approaches zero as $R \to \infty$ the following sufficient conditions are useful:

Theorem (Behavior of Integral as $R \to \infty$)

Suppose $f(z) = \frac{p(z)}{q(z)}$ is a rational function, where the degree of $p(z)$ is n and the degree of $q(z)$ is $m \geq n + 2$. If C_R is a semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, then $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$.

- In other words, the integral along C_R approaches zero as $R \to \infty$ when the denominator of f is of a power at least 2 more than its numerator.
- The proof of this fact follows as in the preceding example, in which degree of $p(z) = 1$ is 0 and degree of $q(z) = (z^2 + 1)(z^2 + 9)$ is 4.

Another Cauchy P.V. of an Improper Integral

- Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx$.

  The conditions given in the preceding theorem are satisfied. Moreover, $f(z) = \frac{1}{z^4+1}$ has simple poles in the upper half-plane at $z_1 = e^{\pi i/4}$ and $z_2 = e^{3\pi i/4}$. We have seen the residues at these poles are

$$\text{Res}(f(z), z_1) = -\frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i \quad \text{and} \quad \text{Res}(f(z), z_2) = \frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i.$$

  Thus,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = 2\pi i [\text{Res}(f(z), z_1) + \text{Res}(f(z), z_2)] = \frac{\pi}{\sqrt{2}}.$$

  Since the integrand is an even function, the original integral converges to $\frac{\pi}{\sqrt{2}}$.

Integrals $\int_{-\infty}^{\infty} f(x)\sin\alpha x dx$ and $\int_{-\infty}^{\infty} f(x)\cos\alpha x dx$

- Integrals of Form $\int_{-\infty}^{\infty} f(x)\sin\alpha x dx$ and $\int_{-\infty}^{\infty} f(x)\cos\alpha x dx$ are referred to as **Fourier integrals**.
- They appear as the real and imaginary parts of $\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx$.
- Suppose $f(x) = \frac{p(x)}{q(x)}$ is a rational function continuous on $(-\infty, \infty)$. Then both Fourier integrals can be evaluated by considering the complex integral $\oint_C f(z)e^{i\alpha z}dz$, where $\alpha > 0$, and the contour C consists of $[-R, R]$ and a semicircular contour C_R with radius large enough to enclose the poles of $f(z)$ in the upper-half plane.
- Sufficient conditions under which the contour integral along C_R approaches zero as $R \to \infty$ are given by

Theorem (Behavior of Integral as $R \to \infty$)

Suppose $f(z) = \frac{p(z)}{q(z)}$ is a rational function, where the degree of $p(z)$ is n and the degree of $q(z)$ is $m \geq n+2$. If C_R is a semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, and $\alpha > 0$, then $\int_{C_R} f(z)e^{i\alpha z}dz \to 0$ as $R \to \infty$.

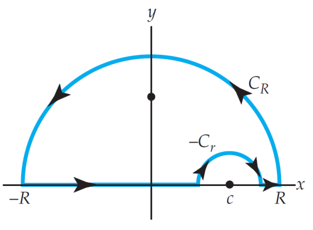
Evaluating a Fourier Integral

- Evaluate the Cauchy principal value of $\int_0^\infty \frac{x \sin x}{x^2+9} dx$.

  First note that the limits of integration in the given integral are not from $-\infty$ to ∞ as required by the method just described. Since the integrand is an even function of x, $\int_0^\infty \frac{x \sin x}{x^2+9} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x \sin x}{x^2+9} dx$. We now form the contour integral $\oint_C \frac{z}{z^2+9} e^{iz} dz$, where C is the contour described before, with $R > 3$. We have
  $\int_{C_R} \frac{z}{z^2+9} e^{iz} dz + \int_{-R}^R \frac{x}{x^2+9} e^{ix} dx = 2\pi i \text{Res}(f(z)e^{iz}, 3i)$, where
  $f(z) = \frac{z}{z^2+9}$, and $\text{Res}(f(z)e^{iz}, 3i) = \left. \frac{ze^{iz}}{z+3i} \right|_{z=3i} = \frac{e^{-3}}{2}$. Since, by the theorem, $\int_{C_R} f(z)e^{iz} dz \to 0$ as $R \to \infty$, we get
  P.V. $\int_{-\infty}^\infty \frac{x}{x^2+9} e^{ix} dx = 2\pi i (\frac{e^{-3}}{2}) = \frac{\pi}{e^3} i$. Note that $\int_{-\infty}^\infty \frac{x}{x^2+9} e^{ix} dx = \int_{-\infty}^\infty \frac{x \cos x}{x^2+9} dx + i \int_{-\infty}^\infty \frac{x \sin x}{x^2+9} dx = \frac{\pi}{e^3} i$. Equating real and imaginary parts: P.V. $\int_{-\infty}^\infty \frac{x \cos x}{x^2+9} dx = 0$ and P.V. $\int_{-\infty}^\infty \frac{x \sin x}{x^2+9} dx = \frac{\pi}{e^3}$. This implies that $\int_0^\infty \frac{x \sin x}{x^2+9} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x \sin x}{x^2+9} dx = \frac{\pi}{2e^3}$.

Indented Contours

- Up to this point we considered improper integrals of functions continuous on the interval $(-\infty, \infty)$, i.e., the complex function $f(z) = \frac{p(z)}{q(z)}$ did not have poles on the real axis.
- Suppose we want to evaluate $\int_{-\infty}^{\infty} f(x)dx$ by residues when $f(z)$ has a pole at $z = c$, where c is a real number. Then we use an indented contour: The symbol C_r denotes a semicircular contour centered at $z = c$ and oriented in the positive direction.



Theorem (Behavior of Integral as $r \to 0$)

Suppose f has a simple pole $z = c$ on the real axis. If C_r is the contour defined by $z = c + re^{i\theta}$, $0 \le \theta \le \pi$, then

$$\lim_{r \to 0} \int_{C_r} f(z)dz = \pi i \operatorname{Res}(f(z), c).$$

Proof of the Theorem

- Since f has a simple pole at $z = c$, its Laurent series is
$$f(z) = \frac{a_{-1}}{z - c} + g(z),$$
where $a_{-1} = \mathrm{Res}(f(z), c)$ and g is analytic at the point c. Using the Laurent series and the parametrization of C_r, we have
$$\int_{C_r} f(z)dz = a_{-1} \int_0^\pi \frac{ire^{i\theta}}{re^{i\theta}} d\theta + ir \int_0^\pi g(c + re^{i\theta})e^{i\theta} d\theta = I_1 + I_2.$$
  - $I_1 = a_{-1} \int_0^\pi \frac{ire^{i\theta}}{re^{i\theta}} d\theta = a_{-1} \int_0^\pi i d\theta = \pi i a_{-1} = \pi i \mathrm{Res}(f(z), c)$.
  - Since g is analytic at c, it is continuous at this point and bounded in a neighborhood of the point. I.e., there exists an $M > 0$ for which $|g(c + re^{i\theta})| \leq M$. Hence,
$$|I_2| = |ir \int_0^\pi g(c + re^{i\theta})d\theta| \leq r \int_0^\pi M d\theta = \pi r M.$$
It follows that $\lim_{r\to 0} |I_2| = 0$ and, consequently, $\lim_{r\to 0} I_2 = 0$.

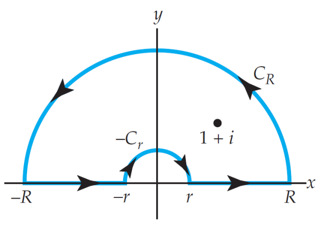
By taking the limit of the sum as $r \to 0$, we get the conclusion.

Using an Indented Contour

- Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2-2x+2)} dx$.

  We consider $\oint_C \frac{e^{iz}}{z(z^2-2z+2)} dz$.

  $f(z) = \frac{1}{z(z^2-2z+2)}$ has a pole at $z = 0$ and at $z = 1 + i$ in the upper half-plane. The contour C, is indented at the origin. We have $\oint_C = \int_{C_R} + \int_{-R}^{-r} + \int_{-C_r} + \int_r^R = 2\pi i \text{Res}(f(z)e^{iz}, 1+i)$, $\int_{-C_r} = -\int_{C_r}$.



  If we take the limits as $R \to \infty$ and as $r \to 0$,

  P.V. $\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2-2x+2)} dx - \pi i \text{Res}(f(z)e^{iz}, 0) = 2\pi i \text{Res}(f(z)e^{iz}, 1+i)$.

  Now, $\text{Res}(f(z)e^{iz}, 0) = \frac{1}{2}$ and $\text{Res}(f(z)e^{iz}, 1+i) = -\frac{e^{-1+i}}{4}(1+i)$.

  Therefore, P.V. $\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2-2x+2)} dx = \pi i \frac{1}{2} + 2\pi i(-\frac{e^{-1+i}}{4}(1+i))$. Using $e^{-1+i} = e^{-1}(\cos 1 + i \sin 1)$ and equating real and imaginary parts:

  P.V. $\int_{-\infty}^{\infty} \frac{\cos x}{x(x^2-2x+2)} dx = \frac{\pi}{2} e^{-1}(\sin 1 + \cos 1)$,

  P.V. $\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2-2x+2)} dx = \frac{\pi}{2}[1 + e^{-1}(\sin 1 - \cos 1)]$.

Subsection 3

Integration along a Branch Cut

Branch Point at $z = 0$

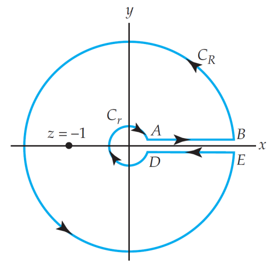
- Suppose that, if $f(x)$ is converted to a complex function, $f(z)$ has, in addition to poles, a nonisolated singularity at $z = 0$.
- In that case, computing $\int_0^\infty f(x)dx$ requires a special type of contour.
- Example: Consider the real integral $\int_0^\infty \frac{x^{\alpha-1}}{x+1}dx$, (21) where α is a real constant restricted to the interval $0 < \alpha < 1$. When $\alpha = \frac{1}{2}$ and x is replaced by z, the integrand becomes the multiple-valued function $\frac{1}{z^{1/2}(z+1)}$. The origin is a branch point because $z^{1/2}$ has two values for any $z \neq 0$. Traveling in a complete circle around the origin $z = 0$, starting from a point $z = re^{i\theta}$, $r > 0$, we return to the same starting point z, but θ has increased by 2π. Thus, the value of $z^{1/2}$ changes from $z^{1/2} = \sqrt{r}e^{i\theta/2}$ to a different value or different branch:
  $z^{1/2} = \sqrt{r}e^{i(\theta+2\pi)/2} = \sqrt{r}e^{i\theta/2}e^{i\pi} = -\sqrt{r}e^{i\pi/2}$.
  Recall, we can force $z^{1/2}$ to be single valued by restricting θ to some interval of length 2π. E.g., by restricting θ to $0 < \theta < 2\pi$, we guarantee that $z^{1/2} = \sqrt{r}e^{i\theta/2}$ is single valued.

Integration along a Branch Cut

- Evaluate $\int_0^\infty \frac{1}{\sqrt{x}(x+1)}dx$.

  The real integral is improper for two reasons:
  - There is an infinite discontinuity at $x = 0$;
  - The limit of integration is infinite.

  We form the integral $\int_C \frac{1}{z^{1/2}(z+1)}dz$, where C is the contour shown, which consists of
  - C_r and C_R, which are portions of circles;
  - AB and ED, which are parallel horizontal line segments running along opposite sides of the branch cut.



  The integrand $f(z)$ of the contour integral is single valued and analytic on and within C, except for the simple pole at $z = -1 = e^{\pi i}$. Hence, we can write $\oint_C \frac{1}{z^{1/2}(z+1)}dz = 2\pi i \text{Res}(f(z), -1)$ or

  $\int_{C_R} + \int_{ED} + \int_{C_r} + \int_{AB} = 2\pi i \text{Res}(f(z), -1)$.

Integration along a Branch Cut (Cont'd)

- We think of AB as coinciding with the upper side of the positive real axis for which $\theta = 0$ and of ED with the lower side of the positive real axis for which $\theta = 2\pi$.

  On AB, $z = xe^{0i}$;

  On ED, $z = xe^{(0+2\pi)i} = xe^{2\pi i}$; Thus,
  $\int_{ED} = \int_R^r \frac{(xe^{2\pi i})^{-1/2}}{xe^{2\pi i}+1}(e^{2\pi i}dx) = -\int_R^r \frac{x^{-1/2}}{x+1}dx = \int_r^R \frac{x^{-1/2}}{x+1}dx$ and
  $\int_{AB} = \int_r^R \frac{(xe^{0i})^{-1/2}}{xe^{0i}+1}(e^{0i}dx) = \int_r^R \frac{x^{-1/2}}{x+1}dx$.
  Now with $z = re^{i\theta}$ and $z = Re^{i\theta}$ on C_r and C_R, respectively, it can be shown that $\int_{C_r} \to 0$ as $r \to 0$ and $\int_{C_R} \to 0$ as $R \to \infty$. Thus,
  $\lim_{\substack{r\to 0\\ R\to\infty}} [\int_{C_R} + \int_{ED} + \int_{C_r} + \int_{AB} = 2\pi i \text{Res}(f(z), -1)]$ is the same as
  $2\int_0^\infty \frac{1}{\sqrt{x}(x+1)}dx = 2\pi i \text{Res}(f(z), -1)$. Since
  $\text{Res}(f(z), -1) = z^{-1/2}\big|_{z=e^{\pi i}} = e^{-\pi i/2} = -i$, $\int_0^\infty \frac{1}{\sqrt{x}(x+1)}dx = \pi$.

Subsection 4

The Argument Principle and Rouché's Theorem

Number of Zeros and Poles

- We apply residue theory to the location of zeros of an analytic function.
- In the first theorem we need to count the number of zeros and poles of a function f that are located within a simple closed contour C, taking into account the order or multiplicity of each zero and pole.
- Example: If $f(z) = \frac{(z-1)(z-9)^4(z+i)^2}{(z^2-2z+2)^2(z-i)^6(z+6i)^7}$ and C is taken to be the circle $|z| = 2$, then:
  - Inspection of the numerator of f reveals that the zeros inside C are $z = 1$ (a simple zero) and $z = -i$ (a zero of order or multiplicity 2). Therefore, the number N_0 of zeros inside C is taken to be $N_0 = 1 + 2 = 3$.
  - Similarly , inspection of the denominator of f shows, after factoring $z^2 - 2z + 2 = (z-1-i)(z-1+i)$, that the poles inside C are $z = 1 - i$ (pole of order 2), $z = 1 + i$ (pole of order 2), and $z = i$ (pole of order 6). The number N_p of poles inside C is taken to be $N_p = 2 + 2 + 6 = 10$.

Argument Principle

Theorem (Argument Principle)

Let C be a simple closed contour lying entirely within a domain D. Suppose f is analytic in D except at a finite number of poles inside C, and that $f(z) \neq 0$ on C. Then $\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = N_0 - N_p$, where N_0 is the total number of zeros of f inside C and N_p is the total number of poles of f inside C, counting their order or multiplicities.

- The integrand $\frac{f'(z)}{f(z)}$ is analytic in and on the contour C except at the points in the interior of C where f has a zero or a pole. If z_0 is a zero of order n of f inside C, then we can write $f(z) = (z - z_0)^n \phi(z)$, where ϕ is analytic at z_0 and $\phi(z_0) \neq 0$. We differentiate f by the product rule, $f'(z) = (z - z_0)^n \phi'(z) + n(z - z_0)^{n-1}\phi(z)$, and divide this expression by f. In some punctured disk centered at z_0, we have $\frac{f'(z)}{f(z)} = \frac{(z-z_0)^n\phi'(z)+n(z-z_0)^{n-1}\phi(z)}{(z-z_0)^n\phi(z)} = \frac{\phi'(z)}{\phi(z)} + \frac{n}{z-z_0}$. Thus, the integrand $\frac{f'(z)}{f(z)}$ has a simple pole at z_0.

Proof of the Argument Principle

- We found $\frac{f'(z)}{f(z)} = \frac{\phi'(z)}{\phi(z)} + \frac{n}{z-z_0}$. The residue at z_0 is $\mathrm{Res}(\frac{f'(z)}{f(z)}, z_0) = \lim_{z\to z_0}(z-z_0)\left(\frac{\phi'(z)}{\phi(z)} + \frac{n}{z-z_0}\right) = \lim_{z\to z_0}\left(\frac{(z-z_0)\phi'(z)}{\phi(z)} + n\right) = 0 + n = n$, which is the order of the zero z_0.

  Now if z_p is a pole of order m of f within C, then $f(z) = \frac{g(z)}{(z-z_p)^m}$, where g is analytic at z_p and $g(z_p) \neq 0$. By differentiating, $f'(z) = (z-z_p)^{-m}g'(z) - m(z-z_p)^{-m-1}g(z)$. Therefore, in some punctured disk centered at z_p, $\frac{f'(z)}{f(z)} = \frac{(z-z_p)^{-m}g'(z) - m(z-z_p)^{-m-1}g(z)}{(z-z_p)^{-m}g(z)} = \frac{g'(z)}{g(z)} + \frac{-m}{z-z_p}$. Thus, $\frac{f'(z)}{f(z)}$ has a simple pole at z_p. We also see that the residue at z_p is equal to $-m$, which is the negative of the order of the pole of f.

Proof of the Argument Principle (Cont'd)

- Finally, suppose that z_{0_1}, z_{0_2}, $\ldots$, z_{0_r} and z_{p_1}, z_{p_2}, $\ldots$, z_{p_s} are the zeros and poles of f within C and that the order of the zeros are $n_1, n_2, \ldots, n_r$ and that order of the poles are $m_1, m_2, \ldots, m_s$. Then each of these points is a simple pole of the integrand $\frac{f'(z)}{f(z)}$ with corresponding residues n_1, n_2, $\ldots$, n_r and $-m_1, -m_2, \ldots, -m_s$. It follows from the residue theorem that $\oint_C \frac{f'(z)}{f(z)} dz$ is equal to $2\pi i$ times the sum of the residues at the poles: $\oint_C \frac{f'(z)}{f(z)} dz = 2\pi i \left[\sum_{k=1}^r \mathrm{Res}(\frac{f'(z)}{f(z)}, z_{0_k}) + \sum_{k=1}^s \mathrm{Res}(\frac{f'(z)}{f(z)}, z_{p_k})\right] = 2\pi i \left(\sum_{k=1}^r n_k + \sum_{k=1}^s (-m_k)\right) = 2\pi i [N_0 - N_p]$.

Illustrating the Argument Principle

- Suppose the simple closed contour is $|z| = 2$ and the function
$$f(z) = \frac{(z-1)(z-9)^4(z+i)^2}{(z^2-2z+2)^2(z-i)^6(z+6i)^7}.$$
In the evaluation of $\oint_C \frac{f'(z)}{f(z)} dz$, each zero of f within C contributes $2\pi i$ times the order of multiplicity of the zero and each pole contributes $2\pi i$ times the negative of the order of the pole:
$$\oint_C \frac{f'(z)}{f(z)} dz$$
$$= [2\pi i(1) + 2\pi i(2)] + [2\pi i(-2) + 2\pi i(-2) + 2\pi i(-6)] = -14\pi i.$$
- The name "argument principle" originates from a relation between the number $N_0 - N_p$ and $\arg(f(z))$: We have
$$\begin{aligned} N_0 - N_p &= \frac{1}{2\pi}[\text{change in } \arg(f(z)) \text{ as } z \text{ traverses } C \\ &\qquad \text{once in the positive direction}]. \end{aligned}$$

Rouché's Theorem

- The following theorem is helpful in determining the number of zeros of an analytic function.

Theorem (Rouché's Theorem)

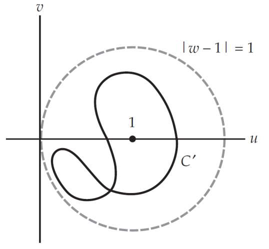
Let C be a simple closed contour lying entirely within a domain D. Suppose f and g are analytic in D. If the strict inequality $|f(z) - g(z)| < |f(z)|$ holds for all z on C, then f and g have the same number of zeros, counting their order or multiplicities, inside C.

- The hypothesis that $|f(z) - g(z)| < |f(z)|$ holds, for all z on C, indicates that both f and g have no zeros on the contour C. From $|f(z) - g(z)| = |g(z) - f(z)|$, we see that, by dividing the inequality by $|f(z)|$. we have, for all z on C, $|F(z) - 1| < 1$, where $F(z) = \frac{g(z)}{f(z)}$.

Proof of Rouché's Theorem

- We have $|F(z)-1|<1$, where $F(z)=\frac{g(z)}{f(z)}$.

  This inequality shows that the image C' in the w-plane of the curve C under the mapping $w=F(z)$ is a closed path and must lie within the unit open disk $|w-1|<1$ centered at $w=1$.



  As a consequence, the curve C' does not enclose $w=0$, and therefore $\frac{1}{w}$ is analytic in and on C'. By the Cauchy-Goursat Theorem, $\oint_{C'}\frac{1}{w}dw=0$. Since $w=F(z)$ and $dw=F'(z)dz$, $\oint_C\frac{F'(z)}{F(z)}dz=0$. From the quotient rule, $F'(z)=\frac{f(z)g'(z)-g(z)f'(z)}{[f(z)]^2}$, we get $\frac{F'(z)}{F(z)}=\frac{g'(z)}{g(z)}-\frac{f'(z)}{f(z)}$. Therefore, $\oint_C(\frac{g'(z)}{g(z)}-\frac{f'(z)}{f(z)})dz=0$ or $\oint_C\frac{g'(z)}{g(z)}dz=\oint_C\frac{f'(z)}{f(z)}dz$. By the argument principle, the number of zeros of g inside C is the same as the number of zeros of f inside C.

Location of Zeros

- Locate the zeros of the polynomial function $g(z) = z^9 - 8z^2 + 5$.

  We begin by choosing $f(z) = z^9$ because it has the same number of zeros as g. Since f has a zero of order 9 at $z = 0$, we search for the zeros of g by examining circles centered at $z = 0$. If we can establish that $|f(z) - g(z)| < |f(z)|$, for all z on some circle $|z| = R$, then Rouché's Theorem asserts that f and g have the same number of zeros within $|z| < R$.

  By the triangle inequality, $|f(z) - g(z)| = |z^9 - (z^9 - 8z^2 + 5)| = |8z^2 - 5| \leq 8|z|^2 + 5$. Also, $|f(z)| = |z|^9$.
  Since $|f(z) - g(z)| < |f(z)|$ or $8|z|^2 + 5 < |z|^9$ is not true for all z on $|z| = 1$, we can draw no conclusion.
  By expanding the search to the larger circle $|z| = \frac{3}{2}$, we see $|f(z) - g(z)| \leq 8|z|^2 + 5 = 8 \cdot (\frac{3}{2})^2 + 5 = 23 < (\frac{3}{2})^9 = |f(z)|$. Thus, since f has a zero of order 9 within $|z| < \frac{3}{2}$, all nine zeros of g lie within the same disk.

Revisiting the Zeros of g I

- By more refined reasoning, we can show that $g(z) = z^9 - 8z^2 + 5$ has some zeros inside $|z| < 1$.

  To see this suppose we choose $f(z) = -8z^2 + 5$. Then, for all z on $|z| = 1$,
  $|f(z) - g(z)| = |(-8z^2 + 5) - (z^9 - 8z^2 + 5)| = |-z^9| = |z|^9 = (1)^9 = 1.$
  For all z on $|z| = 1$,
  $|f(z)| = |-f(z)| = |8z^2 - 5| \geq |8|z|^2 - |-5|| = |8 - 5| = 3.$
  Therefore, for all z on $|z| = 1$, $|f(z) - g(z)| < |f(z)|$.

  Because f has two zeros within $|z| < 1$ (namely, $\pm\sqrt{\frac{5}{8}}$), we can conclude, by Rouché's Theorem, that two zeros of g also lie within this disk.

Revisiting the Zeros of g II

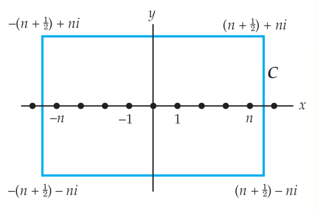
- Continuing to reason about the zeros of $g(z) = z^9 - 8z^2 + 5$, suppose we choose $f(z) = 5$ and $|z| = \frac{1}{2}$. Then, for all z on $|z| = \frac{1}{2}$, $|f(z) - g(z)| = |5 - (z^9 - 8z^2 + 5)| = |-z^9 + 8z^2| \leq |z|^9 + 8|z|^2 = (\frac{1}{2})^9 + 2 \approx 2.002$.
  We now have $|f(z) - g(z)| < |f(z)| = 5$, for all z on $|z| = \frac{1}{2}$. Since f has no zeros within the disk $|z| < \frac{1}{2}$, neither does g.
  At this point we are able to conclude that all nine zeros of $g(z) = z^9 - 8z^2 + 5$ lie within the annular region $\frac{1}{2} < |z| < \frac{3}{2}$. Moreover, two of these zeros lie within $\frac{1}{2} < |z| < 1$.

Subsection 5

Summing Infinite Series

Using $\cot \pi z$

- The residues at the simple poles of $\cot \pi z$ can help find the sum of an infinite series.
- The zeros of $\sin z$ are the reals $z = k\pi$, $k = 0, \pm 1, \pm 2, \ldots$. Thus, $\cot \pi z$ has simple poles at $\pi z = k\pi$ or $z = k$, $k = 0, \pm 1, \pm 2, \ldots$.
- If a polynomial function $p(z)$ has (i) real coefficients; (ii) degree $n \geq 2$, and (iii) no integer zeros, then the function $f(z) = \frac{\pi \cot \pi z}{p(z)}$ has an infinite number of simple poles $z = 0, \pm 1, \pm 2, \ldots$ from $\cot \pi z$ and a finite number of poles $z_{p_1}, z_{p_2}, \ldots, z_{p_r}$ from the zeros of $p(z)$.
- The closed rectangular contour is C, where n is taken large enough so that C encloses the simple poles $z = 0$, $\pm 1, \pm 2, \ldots, \pm n$ and all of the poles $z_{p_1}, z_{p_2}, \ldots, z_{p_r}$. By the residue theorem,



$$\oint_C \frac{\pi \cot \pi z}{p(z)} dz = 2\pi i \left(\sum_{k=-n}^{n} \operatorname{Res}\left(\frac{\pi \cot \pi z}{p(z)}, k\right) + \sum_{j=1}^{r} \operatorname{Res}\left(\frac{\pi \cot \pi z}{p(z)}, z_{p_j}\right)\right).$$

Using $\cot \pi z$ (Cont'd)

- Since it can be shown that $\oint_C \frac{\pi \cot \pi z}{p(z)} dz \to 0$ as $n \to \infty$, we get $0 = \sum_k$ residues $+ \sum_j$ residues. That is,

$$\sum_{k=-\infty}^{\infty} \operatorname{Res}\left(\frac{\pi \cot \pi z}{p(z)}, k\right) = -\sum_{j=1}^{r} \operatorname{Res}\left(\frac{\pi \cot \pi z}{p(z)}, z_{p_j}\right).$$

- If a function f can be written as a quotient $f(z) = \frac{g(z)}{h(z)}$, where g and h are analytic at $z = z_0$, $g(z_0) \neq 0$ and h has a zero of order 1 at z_0, then f has a simple pole at $z = z_0$ and $\operatorname{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}$.
- Hence, with $g(z) = \frac{\pi \cos \pi z}{p(z)}$ and $h(z) = \sin \pi z$, we get $\operatorname{Res}(\frac{\pi \cot \pi z}{p(z)}, k) = \frac{\frac{\pi \cos k\pi}{p(k)}}{\pi \cos k\pi} = \frac{1}{p(k)}$.
- Therefore, we arrive at

$$\sum_{k=-\infty}^{\infty} \frac{1}{p(k)} = -\sum_{j=1}^{r} \operatorname{Res}(\frac{\pi \cot \pi z}{p(z)}, z_{p_j}).$$

Using $\csc \pi z$

- If $p(z)$ is a polynomial function satisfying the same assumptions, i.e.,
  (i) has real coefficients;
  (ii) has degree $n \geq 2$, and
  (iii) no integer zeros,

  then the function $f(z) = \frac{\pi \csc \pi z}{p(z)}$ has an infinite number of simple poles $z = 0, \pm 1, \pm 2, \ldots$ from $\csc \pi z$ and a finite number of poles $z_{p_1}, z_{p_2}, \ldots, z_{p_r}$ from the zeros of $p(z)$.
- In this case it can be shown that

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{p(k)} = -\sum_{j=1}^{r} \operatorname{Res}\left(\frac{\pi \csc \pi z}{p(z)}, z_{p_j}\right).$$

Summing an Infinite Series

- Find the sum of the series $\sum_{k=0}^{\infty} \frac{1}{k^2+4}$.

  If we identify $p(z) = z^2 + 4$, then the three assumptions (i)-(iii) are satisfied. The zeros of $p(z)$ are $\pm 2i$ and correspond to simple poles of $f(z) = \frac{\pi \cot \pi z}{z^2+4}$. According to the formula $\sum_{k=-\infty}^{\infty} \frac{1}{k^2+4} = -\left(\operatorname{Res}(\frac{\pi \cot \pi z}{z^2+4}, -2i) + \operatorname{Res}(\frac{\pi \cot \pi z}{z^2+4}, 2i)\right)$. Since $\operatorname{Res}(\frac{\pi \cot \pi z}{z^2+4}, -2i) = \frac{\pi \cot 2\pi i}{4i}$ and $\operatorname{Res}(\frac{\pi \cot \pi z}{z^2+4}, 2i) = \frac{\pi \cot 2\pi i}{4i}$, the sum of the residues is $\frac{\pi}{2i} \cot 2\pi i$. This sum is a real quantity because $\frac{\pi}{2i} \cot 2\pi i = \frac{\pi}{2i} \frac{\cosh(-2\pi)}{(-i \sinh(-2\pi))} = -\frac{\pi}{2} \coth 2\pi$. Hence,

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+4} = \frac{\pi}{2} \coth 2\pi.$$

Summing an Infinite Series (Cont'd)

- To get the desired sum, we must manipulate the summation $\sum_{-\infty}^{\infty}$ in order to put it in the form $\sum_{k=0}^{\infty}$.

  We have

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\frac{1}{k^2+4} &= \sum_{k=-\infty}^{-1}\frac{1}{k^2+4}+\frac{1}{4}+\sum_{k=1}^{\infty}\frac{1}{k^2+4}\\ &= \sum_{k=1}^{\infty}\frac{1}{(-k)^2+4}+\frac{1}{4}+\sum_{k=1}^{\infty}\frac{1}{k^2+4}\\ &= 2\sum_{k=1}^{\infty}\frac{1}{k^2+4}+\frac{1}{4}=2\sum_{k=0}^{\infty}\frac{1}{k^2+4}-\frac{1}{4}.\end{aligned}$$

  Finally, since $\sum_{k=-\infty}^{\infty}\frac{1}{k^2+4}=2\sum_{k=0}^{\infty}\frac{1}{k^2+4}-\frac{1}{4}=\frac{\pi}{2}\coth 2\pi$, we obtain

$$\sum_{k=0}^{\infty}\frac{1}{k^2+4}=\frac{1}{8}+\frac{\pi}{4}\coth 2\pi.$$

Subsection 6

Laplace and Fourier Transforms

Laplace and Inverse Laplace Transforms

- The **Laplace transform** of a real function f is defined, for $t \geq 0$, by $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$.
  - (i) The direct problem: Given a function $f(t)$ satisfying certain conditions, find its Laplace transform. When the integral converges, the result is a function of s. The relationship between a function and its transform is exhibited by using a lowercase letter to denote the function and the corresponding uppercase letter to denote its Laplace transform, e.g., $\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\{y(t)\} = Y(s)$, and so on.
  - (ii) The inverse problem: Find the function $f(t)$ that has a given transform $F(s)$. The function $f(t)$ is called the **inverse Laplace transform** and is denoted by $\mathcal{L}^{-1}\{F(s)\}$.
- We will see that the inverse Laplace transform is not merely a symbol but actually another integral transform, actually a special type of complex contour integral.

Integral Transforms

- Suppose $f(x, y)$ is a real-valued function of two real variables.
- A definite integral of f with respect to one of the variables leads to a function of the other variable.

  Example: If we hold y constant, integration with respect to the real variable x gives $\int_1^2 4xy^2 dx = 2x^2y^2\Big|_1^2 = 8y^2 - 2y^2 = 6y^2$.
- Thus, a definite integral such as $F(\alpha) = \int_a^b f(x)K(\alpha, x)dx$ transforms a function f of the variable x into a function F of the variable α.
- We say that $F(\alpha) = \int_a^b f(x)K(\alpha, x)dx$ is an **integral transform** of the function f.
- Integral transforms appear in **transform pairs**, meaning that the original function f can be recovered by another integral transform $f(x) = \int_c^d F(\alpha)H(\alpha, x)d\alpha$, called the **inverse transform**.
- The functions $K(\alpha, x)$ and $H(\alpha, x)$ are the **kernels** of the transforms.
- If α represents a complex variable, then the second definite integral is replaced by a contour integral.

The Laplace Transform

- Suppose that, in $F(\alpha) = \int_a^b f(x)K(\alpha, x)dx$, α is replaced by the symbol s, and that f represents a real function that is defined on the unbounded interval $[0, \infty)$.
- Then $F(s) = \int_0^\infty f(t)K(s, t)dt$ is an improper integral, defined by

$$\int_0^\infty K(s,t)f(t)dt = \lim_{b \to \infty} \int_0^b K(s,t)f(t)dt.$$

- If the limit exists, we say that the integral **exists** or is **convergent**; otherwise, the integral **does not exist** and is said to be **divergent**.
- The choice $K(s, t) = e^{-st}$, where s is a complex variable, gives the **Laplace transform** $\mathcal{L}\{f(t)\}$ defined previously.
- The integral that defines the Laplace transform may not converge for certain kinds of functions f.

  Example: Neither $\mathcal{L}\{e^{t^2}\}$ nor $\mathcal{L}\{\frac{1}{t}\}$ exists.
- Also, the limit may exist for only certain values of the variable s.

Existence of a Laplace Transform

- The Laplace transform of $f(t) = 1$, $t \geq 0$, is

$$\begin{aligned}
\mathcal{L}\{1\} &= \int_0^\infty e^{-st}(1)dt \\
&= \lim_{b\to\infty} \int_0^b e^{-st} dt \\
&= \lim_{b\to\infty} -\frac{e^{-st}}{s}\Big|_0^b \\
&= \lim_{b\to\infty} [\tfrac{1-e^{-sb}}{s}].
\end{aligned}$$

If $s = x + iy$, then $e^{-sb} = e^{-bx}(\cos by - i \sin by)$. Thus, $e^{-sb} \to 0$ as $b \to \infty$, if $x > 0$. In other words,

$$\mathcal{L}\{1\} = \frac{1}{s}, \text{ provided } \operatorname{Re}(s) > 0.$$

Existence of $\mathcal{L}\{f(t)\}$

- Conditions that are sufficient to guarantee the existence of $\mathcal{L}\{f(t)\}$ are that f be piecewise continuous on $[0, \infty)$ and that f be of exponential order.
  - **Piecewise continuity** on $[0, \infty)$ means that, on any interval, there are at most a finite number of points t_k, $k = 1, 2, \ldots, n$, $t_{k-1} < t_k$, at which f has finite discontinuities and is continuous on each open interval $t_{k-1} < t < t_k$.
  - A function f is said to be **of exponential order** c if there exist constants $c, M > 0$, and $T > 0$, so that $|f(t)| \leq Me^{ct}$, for $t > T$. The condition $|f(t)| \leq Me^{ct}$, for $t > T$, states that the graph of f on the interval (T, ∞) does not grow faster than the graph of the exponential function Me^{ct}.
    Alternatively, $e^{-ct}|f(t)|$ is bounded, i.e., $e^{-ct}|f(t)| \leq M$, for $t > T$.
- All bounded functions are necessarily of exponential order $c = 0$.

Existence Theorem for $\mathcal{L}\{f(t)\}$

Theorem (Sufficient Conditions for Existence)

Suppose f is piecewise continuous on $[0, \infty)$ and of exponential order c for $t > T$. Then $\mathcal{L}\{f(t)\}$ exists for $\mathrm{Re}(s) > c$.

- We have $\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt = I_1 + I_2$.
  - The integral I_1 exists since it can be written as a sum of integrals over intervals on which $e^{-st} f(t)$ is continuous.
  - To prove the existence of I_2, let $s = x + iy$. Then $|e^{-st}| = |e^{-xt}(\cos yt - i \sin yt)| = e^{-xt}$. Further, by the definition of exponential order, $|f(t)| \le Me^{ct}$, $t > T$. Hence, $|I_2| \le \int_T^\infty |e^{-st} f(t)|dt$ $\le M \int_T^\infty e^{-xt} e^{ct} dt = M \int_T^\infty e^{-(x-c)t} dt = -M \left. \frac{e^{-(x-c)t}}{x-c} \right|_T^\infty =$ $M \frac{e^{-(x-c)T}}{x-c}$, for $x = \mathrm{Re}(s) > c$. Since $\int_T^\infty Me^{-(x-c)t} dt$ converges, $\int_T^\infty |e^{-st} f(t)|dt$ converges by the comparison test. This, in turn, implies that I_2 exists for $\mathrm{Re}(s) > c$.

The existence of I_1 and I_2 implies that $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$ exists for $\mathrm{Re}(s) > c$.

Analyticity of the Laplace Transform

- The following theorem is stated without proof:

Theorem (Analyticity of the Laplace Transform)

Suppose f is piecewise continuous on $[0, \infty)$ and of exponential order c for $t \geq 0$. Then the Laplace transform of f,

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

is an analytic function in the right half-plane defined by $\mathrm{Re}(s) > c$.

- Although the complex function $F(s)$ is analytic to the right of the line $x = c$ in the complex plane, $F(s)$ will, in general, have singularities to the left of that line.

The Inverse Laplace Transform

Theorem (Inverse Laplace Transform)

If f and f' are piecewise continuous on $[0, \infty)$ and f is of exponential order c for $t \geq 0$, and $F(s)$ is a Laplace transform, then the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$ is

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\gamma - iR}^{\gamma + iR} e^{st} F(s) ds,$$

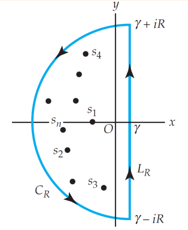
where $\gamma > c$.

- We write $f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s) ds$, where the limits of integration indicate that the integration is along the infinitely long vertical-line contour $\text{Re}(s) = x = \gamma$.
- γ is a positive real constant greater than c and greater than all the real parts of the singularities in the left half-plane.
- This integral is called a **Bromwich contour integral**.
- The kernel of the inverse transform is $H(s, t) = \frac{e^{st}}{2\pi i}$.

Evaluating the Inverse Laplace Transform

- The Bromwich contour integral
  $f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)ds.$
- The fact that $F(s)$ has singularities $s_1, s_2, \ldots, s_n$ to the left of the line $x = \gamma$ makes it possible to evaluate the integral by using an appropriate closed contour encircling the singularities.

  A closed contour C that is commonly used consists of a semicircle C_R of radius R centered at $(\gamma, 0)$ and a vertical line segment L_R parallel to the y-axis passing through the point $(\gamma, 0)$ and extending from $y = \gamma - iR$ to $y = \gamma + iR$. R is larger than the largest number in $\{|s_1|, |s_2|, \ldots, |s_n|\}$.



  With the contour C chosen in this manner, the integral can often be evaluated using Cauchy's residue theorem. If we allow the radius R of the semicircle to approach ∞, the vertical part of the contour approaches the infinite vertical line of the Bromwich integral.

Inverse Laplace Transform Theorem

Theorem (Inverse Laplace Transform)

Suppose $F(s)$ is a Laplace transform that has a finite number of poles $s_1, s_2, \ldots, s_n$ to the left of the vertical line $\mathrm{Re}(s) = \gamma$ and that C is the contour on the preceding slide. If $sF(s)$ is bounded as $R \to \infty$, then $\mathcal{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} \mathrm{Res}(e^{st}F(s), s_k)$.

- By Cauchy's residue theorem, we have $\int_{C_R} e^{st}F(s)ds + \int_{L_R} e^{st}F(s)ds = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(e^{st}F(s), s_k)$ or $\frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR} e^{st}F(s)ds = \sum_{k=1}^{n} \mathrm{Res}(e^{st}F(s), s_k) - \frac{1}{2\pi i}\int_{C_R} e^{st}F(s)ds$. We let $R \to \infty$ and show that $\lim_{R\to\infty}\int_{C_R} e^{st}F(s)ds = 0$.

  If the semicircle C_R is parametrized by $s = \gamma + Re^{i\theta}$, $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$, then $ds = Rie^{i\theta}d\theta = (s-\gamma)id\theta$, and so, $\frac{1}{2\pi i}\int_{C_R} e^{st}F(s)ds = \frac{1}{2\pi i}\int_{\pi/2}^{3\pi/2} e^{\gamma t + Rte^{i\theta}} F(\gamma + Re^{i\theta})Rie^{i\theta}d\theta$, whence $\frac{1}{2\pi}\left|\int_{C_R} e^{st}F(s)ds\right| \le \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}\left|e^{\gamma t + Rte^{i\theta}}\right|\left|F(\gamma + Re^{i\theta})\right|\left|Rie^{i\theta}\right|d\theta$.

Proof of the Inverse Laplace Transform Theorem

- We examine the three moduli involved:
  - $\left|e^{\gamma t+Rte^{i\theta}}\right| = \left|e^{\gamma t}e^{Rt(\cos\theta+i\sin\theta)}\right| = e^{\gamma t}e^{Rt\cos\theta}$.
  - For $|s|$ sufficiently large, we can write $\left|Rie^{i\theta}\right| = |s-\gamma||i| \leq |s|+|\gamma| < |s|+|s| = 2|s|$.
  - Finally, by hypothesis, $|sF(s)| < M$.

Thus, we get $\frac{1}{2\pi}\left|\int_{C_R} e^{st}F(s)ds\right| \leq$ $\frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}\left|e^{\gamma t+Rte^{i\theta}}\right|\left|F(\gamma+Re^{i\theta})\right|\left|Rie^{i\theta}\right|d\theta \leq \frac{M}{\pi}e^{\gamma t}\int_{\pi/2}^{3\pi/2}e^{Rt\cos\theta}d\theta$. Let $\theta = \phi+\frac{\pi}{2}$ and notice that the integral becomes $\int_0^{\pi}e^{-Rt\sin\phi}d\phi = 2\int_0^{\pi/2}e^{-Rt\sin\phi}d\phi$. We have $\sin\phi \geq \frac{2\phi}{\pi}$, whence $2\int_0^{\pi/2}e^{-Rt\sin\phi}d\phi \leq 2\int_0^{\pi/2}e^{-2Rt\phi/\pi}d\phi = -\frac{\pi}{Rt}e^{-2Rt\phi/\pi}\Big|_0^{\pi/2} =$ $\frac{\pi}{Rt}[1-e^{-Rt}]$. We conclude that $\frac{1}{2\pi}\left|\int_{C_R}e^{st}F(s)ds\right| \leq \frac{Me^{\gamma t}}{Rt}[1-e^{-Rt}]$. The right-hand side approaches zero as $R\to\infty$ for $t>0$, whence $\lim_{R\to\infty}\int_{C_R}e^{st}F(s)ds = 0$.

An Inverse Laplace Transform

- Evaluate $\mathcal{L}^{-1}\{\frac{1}{s^3}\}$, $\text{Re}(s) > 0$.

  The function $F(s) = \frac{1}{s^3}$ has a pole of order 3 at $s = 0$. Thus, by the theorem,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{\tfrac{1}{s^3}\} \\ &= \text{Res}(e^{st}\tfrac{1}{s^3}, 0) \\ &= \tfrac{1}{2}\lim_{s\to 0}\tfrac{d^2}{ds^2}(s-0)^3\tfrac{e^{st}}{s^3} \\ &= \tfrac{1}{2}\lim_{s\to 0}\tfrac{d^2}{ds^2}e^{st} \\ &= \tfrac{1}{2}\lim_{s\to 0}t^2e^{st} \\ &= \tfrac{1}{2}t^2. \end{aligned}$$

Fourier Transform

- Suppose now that $f(x)$ is a real function defined on the interval $(-\infty, \infty)$.
- Another important transform pair consists of
  - the **Fourier transform**
    $$\mathfrak{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx = F(\alpha).$$
  - the **inverse Fourier transform**
    $$\mathfrak{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha = f(x).$$
- The kernel of the Fourier transform is $K(\alpha, x) = e^{i\alpha x}$, whereas the kernel of the inverse transform is $H(\alpha, x) = \frac{e^{-i\alpha x}}{2\pi}$.
- We assume that α is a real variable.
- In contrast to the Laplace case, the inverse Fourier transform is not a contour integral.

Computing a Fourier Transform

- Find the Fourier transform of $f(x) = e^{-|x|}$.

We have $f(x) = \begin{cases} e^{x}, & \text{if } x < 0 \\ e^{-x}, & \text{if } x \geq 0 \end{cases}$. The Fourier transform of f is $\mathfrak{F}\{f(x)\} = \int_{-\infty}^{0} e^{x} e^{i\alpha x} dx + \int_{0}^{\infty} e^{-x} e^{i\alpha x} dx = I_1 + I_2$.

  - For I_2, we have $I_2 = \lim_{b \to \infty} \int_0^b e^{-x} e^{i\alpha x} dx = \lim_{b \to \infty} \int_0^b e^{-x(1-\alpha i)} dx = \lim_{b \to \infty} \frac{e^{-x(1-\alpha i)}}{\alpha i - 1}\Big|_0^b = \lim_{b \to \infty} \frac{e^{-b(1-\alpha i)} - 1}{\alpha i - 1} = \frac{1}{\alpha i - 1} \lim_{b \to \infty} (e^{-b} \cos b\alpha + i e^{-b} \sin b\alpha - 1) = \frac{1}{1 - \alpha i}$.
  - The integral I_1 can be evaluate similarly to obtain $I_1 = \frac{1}{1 + \alpha i}$.

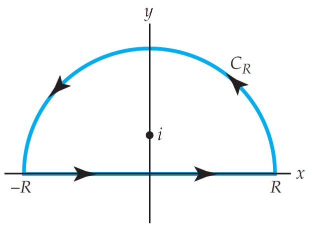
Adding I_1 and I_2 gives the value of the Fourier transform:

$$\mathcal{F}\{f(x)\} = \frac{1}{1 - \alpha i} + \frac{1}{1 + \alpha i} = \frac{2}{1 + \alpha^2}.$$

Computing an Inverse Fourier Transform

- Find the inverse Fourier transform of $F(\alpha) = \frac{2}{1+\alpha^2}$.

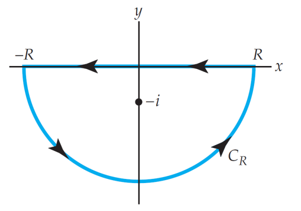
  The idea here is to recover the function f of the preceding example. We have $\mathfrak{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{1+\alpha^2} e^{-i\alpha x} d\alpha = f(x)$.



Let z be a complex variable and introduce the contour integral $\oint_C \frac{1}{\pi(1+z^2)} e^{-izx} dz$. The integrand has simple poles at $z = \pm i$. The contour C is shown in the figure.

We get $\oint_C \frac{1}{\pi(1+z^2)} e^{-izx} dz = 2\pi i \mathrm{Res}(\frac{1}{\pi(1+z^2)} e^{-izx}, i) = e^x$. The contour integral along C_R approaches zero as $R \to \infty$ only if we assume that $x < 0$. Thus, the answer is $e^x, x < 0$.

Computing an Inverse Fourier Transform (Cont'd)

- 

If we consider $\oint_C \frac{1}{\pi(1+z^2)} e^{-izx} dz$, where C is the contour on the left, it can be shown that the integral along C_R now approaches zero as $R \to \infty$ when x is assumed to be positive. Hence, $\oint_C \frac{1}{\pi(1+z^2)} e^{-izx} dz = -2\pi i \text{Res}(\frac{1}{\pi(1+z^2)} e^{-izx}, -i) = e^{-x}$, $x > 0$. The extra minus sign appearing in front of the factor $2\pi i$ comes from the fact that on C, $\int_C = \int_{C_R} + \int_R^{-R} = \int_{C_R} - \int_{-R}^R = 2\pi i \text{Res}(z = -i)$. As $R \to \infty$, $\int_{C_R} \to 0$, for $x > 0$, whence $-\lim_{R\to\infty} \int_{-R}^R = 2\pi i \text{Res}(z = -i)$ or $\lim_{R\to\infty} \int_{-R}^R = -2\pi i \text{Res}(z = -i)$.
- By combining the findings, we get

$$\mathfrak{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1+\alpha^2} e^{-i\alpha x} d\alpha = \begin{cases} e^x, & \text{if } x < 0 \\ e^{-x}, & \text{if } x > 0 \end{cases}.$$