

Practice set 4

Problem 4.1

Another approximation to $I(f) = \int_a^b f(x) dx$ is obtained if we replace $f(x)$ by the constant $f\left(\frac{a+b}{2}\right)$ on the entire interval $[a, b]$. Show that this leads to the numerical integration formula, called **midpoint rule**:

$$M_1(f) = (b - a) f\left(\frac{a + b}{2}\right)$$

In analogy to the derivation of the composite trapezoidal and Simpson's rules, generalize to the numerical integration formula

$$M_n(f) = h [f(x_1) + f(x_2) + \dots + f(x_{n-1}) + f(x_n)]$$

where $h = \frac{b-a}{2}$ and $x_j = a + (j - \frac{1}{2})h$. This rule is called **composite midpoint rule**.

Problem 4.2

As another approximation to $I(f) = \int_a^b f(x) dx$ replace $f(x)$ by the degree 4 interpolation polynomial at five evenly spaced points $x_j = a + jh, j = 0, 1, 2, 3, 4$. Show that this leads to the approximation formula

$$B_4(f) = \frac{2h}{45} [7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)]$$

This formula is called the **Boole's rule**.

Problem 4.3

Consider the numerical integration of

$$I = \int_0^1 \frac{e^x + e^{-x}}{2} dx$$

using the trapezoidal rule $T_n(f)$ and Simpson's rule $S_n(f)$. Give an exact error formulas $E_n^T(f)$ and $E_n^S(f)$. Using the formulas for E_n^T and E_n^S , determine how large n should be chosen in order to have $|I - T_n| \leq 10^{-8}$ and $|I - S_n| \leq 10^{-8}$.

Problem 4.4

Repeat previous problem for

$$I = \int_{-1}^2 e^{3x-1} dx.$$

Problem 4.5

Consider using the trapezoidal rule T_n to approximate

$$I = \int_0^\pi \frac{dx}{2 + \cos(x)}$$

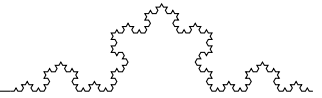
Then $I - T_n \approx Ch^2$. Answer True or False. Give appropriate explanations for your answer.

Problem 4.6

Using error formula for trapezoidal rule estimate the number n of subdivisions to evaluate the following integral

$$I = \int_1^3 \ln x dx$$

to the accuracy of 10^{-5} .



Problem 4.7

Following is a table of numerical integrals I_n approximating an integral I , where and also are given the successive differences $I_n - I_{\frac{1}{2}n}$. Show that the table values seem to be consistent with an error of the form

$$I - I_n \approx \frac{c}{n^p}$$

for some constant c and some p . Estimate p and c . Then estimate $I - I_{512}$. How large n should be chosen if the error is to be less than 10^{-10} ?

n	I_n	$I_n - I_{\frac{1}{2}n}$	<i>Ratio</i>
2	0.402368927062		
4	0.400431916045	$-1.937E - 3$	
8	0.400077249447	$-3.547E - 4$	
16	0.400013713469	$-6.354E - 5$	
32	0.400002427846	$-1.129E - 5$	
64	0.400000429413	$-1.998E - 6$	
128	0.400000075924	$-3.535E - 7$	
256	0.400000013423	$-6.250E - 8$	
512	0.400000002373	$-1.105E - 8$	

Problem 4.8

Following is a table of numerical integrals I_n approximating an integral $I = \int_0^1 f(x) dx$ with Simpson's rule. Show that the table values seem to be consistent with an error of the form

$$I - I_n \approx \frac{c}{n^p}.$$

Find value of p and comment on anything unusual in the above numerical results.

n	<i>Error</i>	<i>Ratio</i>
2	$2.860E - 2$	
4	$1.012E - 2$	
8	$3.587E - 3$	
16	$1.268E - 3$	
32	$4.485E - 4$	

Problem 4.9

Let

$$I_h(f) = \frac{3}{4}h f(0) + 3f(2h).$$

Find the degree of precision of the approximation

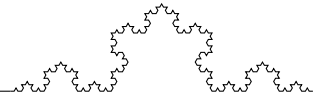
$$I_h(f) \approx \int_0^{2h} f(x) dx$$

Problem 4.10

For the formula

$$\int_0^2 f(x) dx \approx w_1 f(x_1) + w_2 f(2)$$

determine the weights w_1, w_2 and x_2 such that the quadrature formula is exact for all polynomials of as large degree as possible.

**Problem 4.11**

Using the table below estimate the value of $f'(0.5)$ using forward, backward and centered difference formulas. What is the best estimate and argue your answer. Also, estimate $f''(0.5)$.

x	$f(x)$
0.3	7.3891
0.4	7.4633
0.5	7.5383
0.6	7.6141
0.7	7.6906