Mathematical analysis I

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Subsection 7

Taylor Series

Taylor Series

- Assume that a function $f(x)$ is represented by a power series centered at $x = c$ on $(c - R, c + R)$ with $R > 0$, i.e.,

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots ;$$

- Then, for the derivatives of f on $(c - R, c + R)$, we have

$$\begin{aligned}
f(x) &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots ; \\
f'(x) &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdots ; \\
f''(x) &= 2a_2 + 2 \cdot 3a_3(x-c) + 3 \cdot 4a_4(x-c)^2 + 4 \cdot 5(x-c)^3 \cdots ; \\
f'''(x) &= 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4(x-c) + 3 \cdot 4 \cdot 5(x-c)^2 + \cdots ;
\end{aligned}$$

- Plug in $x = c$ to get

$$f(c) = a_0, f'(c) = a_1, f''(c) = 2!a_2, f'''(c) = 3!a_3, f^{(4)}(c) = 4!a_4, \ldots ;$$

- In general, we get $a_n = \dfrac{f^{(n)}(c)}{n!}$;

Taylor and Maclaurin Series

Stewart, p.754

Taylor Series Expansion

If f is represented as a power series centered at $x = c$ in an interval $|x - c| < R, R > 0$, then the power series is the **Taylor series**:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n;$$

Maclaurin Series

The special case of the Taylor series for $c = 0$ is the **Maclaurin series**:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots;$$

Finding a Taylor Series

- Find the Taylor series for $f(x) = x^{-3}$ centered at $c = 1$;

$$f(x) = x^{-3}, \quad f(1) = 1;$$
$$f'(x) = (-3)x^{-4}, \quad f'(1) = -3;$$
$$f''(x) = (-3)(-4)x^{-5}, \quad f''(1) = +3 \cdot 4;$$
$$f'''(x) = (-3)(-4)(-5)x^{-6}, \quad f'''(1) = -3 \cdot 4 \cdot 5;$$
$$\vdots$$
$$f^{(n)}(x) = (-3)(-4) \cdot \dots \cdot (-n-2)x^{-n-3},$$
$$f^{(n)}(1) = (-1)^n \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot (n+2) = \frac{(-1)^n}{2}(n+2)!;$$

Now we get by the Taylor series formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)!}{2 \cdot n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)(n+2)}{2}(x-1)^n;$$

Convergence Issues

Stewart, p.756

- We know that if $f(x)$ can be represented by a power series centered at $x = c$, then that power series will be the Taylor series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n;$$

- However, there is no guarantee that $T(x)$ converges; Moreover, there is no guarantee that, even if it converges, it will converge to $f(x)$!
- Let

$$T_k(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(c-x)^2 + \cdots + \frac{f^{(k)}(c)}{k!}(x-c)^k;$$

Define the **remainder**

$$R_k(x) = f(x) - T_k(x);$$

The Taylor series converges to $f(x)$ if and only if $\lim_{k\to\infty} R_k(x) = 0$;

Convergence Theorem

Taylor's Inequality, Th9, p.756 [Stewart]

Theorem

Let $I = (c - R, c + R)$, $R > 0$; If there exists a $K > 0$, such that all derivatives of f are bounded by K on I, i.e.,

$$|f^{(k)}(x)| \leq K, \text{ for all } k \geq 0, x \in I,$$

then, for all $x \in I$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x - c)^n.$$

Formulas for the Taylor Remainder Term, p.756

Sine and Cosine

- Show that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Let $f(x) = \sin x$;

$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$	$\cdots$
$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cdots$
0	1	0	-1	0	$\cdots$

Note, also that for all x, $|f^{(k)}(x)| \leq 1$; Therefore, we have convergence of the Taylor series of f centered at $x = 0$ to $f(x) = \sin x$ everywhere and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots ;$$

One either works similarly from scratch for $g(x) = \cos x$ or notices that $\cos x = (\sin x)'$ and appeals to term-by-term differentiation of the series for $\sin x$;

- for small values of x
  $\sin x \approx x$

$$\cos x \approx 1-\frac{x^2}{2}$$

If $x>0$

$$x-\frac{x^3}{3!} < \sin x < x$$

Example

$$x = 1^\circ = \frac{\pi}{180} \approx 0{,}01\ 7453292\ldots$$

$$\frac{x^2}{2!} = 0{,}000152309$$

$$\frac{x^3}{3!} = 0{,}000000886\ldots$$

$$\frac{x^4}{4!} = 0{,}000000004\ldots$$

$$\frac{x^5}{5!} = 0{,}000000000\ldots$$

$$\sin 1^\circ = 0{,}017452406 \ ; \qquad \cos 1^\circ = 0{,}999847695\ldots$$

Infinite Series for e^x

Stewart, p.754 example 1

- The Maclaurin series for $f(x) = e^x$ is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!};$$

R=∞

- Example: Find a Maclaurin series for $f(x) = x^2 e^x$;

$$\begin{aligned} f(x) &= x^2 e^x = x^2 \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right] \\ &= x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \frac{x^6}{4!} + \cdots = \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!}; \end{aligned}$$

- Example: Find the Maclaurin series for $f(x) = e^{-x^2}$;

$$\begin{aligned} f(x) &= e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \cdots \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}; \end{aligned}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ for all x

put x=1

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

Stewart, pp.754-757, example 1, 2, 3

Using Integration

- Find the Maclaurin series for $f(x) = \ln(1+x)$;

$$
\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \cdots \qquad R=1 \\
\frac{1}{1+x} &= \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \cdots; \\
\ln(1+x) &= \int \frac{1}{1+x} dx \\
&= \int (1 - x + x^2 - x^3 + x^4 - \cdots) dx \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}; \qquad R=1,\ (-1,1]
\end{aligned}
$$

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Binomial Coefficients

- For any number a (integer or not) and any integer $n \geq 0$, we define the **binomial coefficient**

$$C_a^n = \binom{a}{n} = \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}, \quad \binom{a}{0} = 1;$$

- Example:

$$\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3!} = 20;$$

$$\binom{\frac{4}{3}}{3} = \frac{\frac{4}{3} \cdot \frac{1}{3} \cdot (-\frac{2}{3})}{3!} = \frac{-\frac{8}{27}}{6} = -\frac{4}{81};$$

$$C_n^k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}, \quad n, k \in \mathbb{N}$$

Binomial Series

Stewart, p.760, example 8; p.761

The Binomial Series

For any exponent a and for $|x| < 1$,

$$(1+x)^a = 1 + \frac{a}{1!}x + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots + \binom{a}{n}x^n + \cdots ;$$

$a \in \mathbb{N}$

Newton's binomial

- Example: Find the terms through degree four of the Maclaurin expansion of $f(x) = (1+x)^{4/3}$;

$$\begin{aligned} T_4(x) &= 1 + \frac{a}{1!}x + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \frac{a(a-1)(a-2)(a-3)}{4!}x^4 \\ &= 1 + \frac{\frac{4}{3}}{1!}x + \frac{\frac{4}{3}\cdot\frac{1}{3}}{2!}x^2 + \frac{\frac{4}{3}\cdot\frac{1}{3}\cdot(-\frac{2}{3})}{3!}x^3 + \frac{\frac{4}{3}\cdot\frac{1}{3}\cdot(-\frac{2}{3})\cdot(-\frac{5}{3})}{4!}x^4 \\ &= 1 + \frac{4}{3}x + \frac{2}{9}x^2 - \frac{4}{81}x^3 + \frac{5}{243}x^4; \end{aligned}$$

Applying the Binomial Series Expansion

- Find the Maclaurin series for $f(x) = \frac{1}{\sqrt{1-x^2}}$; Recall that $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$; Hence, for $a = -\frac{1}{2}$, we get

$$(1+x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n;$$

Therefore, we obtain

$$f(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} = (1+(-x^2))^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-x^2)^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2}) \cdots (-\frac{1}{2} - n + 1)}{n!} (-1)^n x^{2n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!} (-1)^n x^{2n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!} x^{2n};$$

Example. $\sqrt[7]{150}$ – ?

$150 = 128+22 = 2^7\left(1+\frac{11}{64}\right)$ $\qquad 0<\frac{11}{64}<1$

$$\sqrt[7]{150} = 2\left(1+\frac{11}{64}\right)^{1/7} =$$

$$\simeq 2\left(1+\frac{1}{7}\cdot\frac{11}{64}+\frac{1}{2}\,\frac{1}{7}\left(\frac{1}{7}-1\right)\cdot\left(\frac{11}{64}\right)^2+\dots\right) =$$

$$\approx 2{,}05$$

example 9,10,p.761-762
Table 1

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \qquad \left(\begin{matrix}\text{B.5}\\ a = -1\end{matrix}\right)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \qquad R = 1$$

$|x| < 1$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots \qquad x^2 < 1$$

$|x| < 1$

integrate term-by-term

$R = 1$

$$\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

[-1,1]

x=1

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

- power series to limits, definite integrals, improper integrals

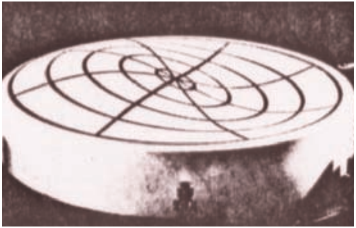
Stewart, p762,763 example 11, 12

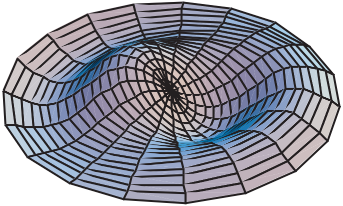
example 12, p.763

$$\lim_{x \to 0} \frac{\sin x - \operatorname{arctg} x}{x^3} = \dots = -\frac{1}{6}$$

p.768 Applications to Physics

We will see that the main use of a power series is that it provides a way to represent some of the most important functions that arise in mathematics, physics, and chemistry. In particular, the sum of the power series in the next example is called a **Bessel function**, after the German astronomer Friedrich Bessel (1784–1846), and the function given in Exercise 35 is another example of a Bessel function. In fact, these functions first arose when Bessel solved Kepler's equation for describing planetary motion. Since that time, these functions have been applied in many different physical situations, including the temperature distribution in a circular plate and the shape of a vibrating drumhead.





Notice how closely the computer-generated model (which involves Bessel functions and cosine functions) matches the photograph of a vibrating rubber membrane.

EXAMPLE 3 Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

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$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

SOLUTION Let $a_n = (-1)^n x^{2n}/[2^{2n}(n!)^2]$. Then

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right| \\
&= \frac{x^{2n+2}}{2^{2n+2}(n+1)^2(n!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}} \\
&= \frac{x^2}{4(n+1)^2} \to 0 < 1 \qquad \text{for all } x
\end{aligned}$$

Thus, by the Ratio Test, the given series converges for all values of x. In other words, the domain of the Bessel function J_0 is $(-\infty, \infty) = \mathbb{R}$.

Recall that the sum of a series is equal to the limit of the sequence of partial sums. So when we define the Bessel function in Example 3 as the sum of a series we mean that, for every real number x,

$$J_0(x) = \lim_{n\to\infty} s_n(x) \qquad \text{where} \qquad s_n(x) = \sum_{i=0}^{n} \frac{(-1)^i x^{2i}}{2^{2i}(i!)^2}$$

The first few partial sums are

$$s_0(x) = 1 \qquad s_1(x) = 1 - \frac{x^2}{4} \qquad s_2(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64}$$

$$s_3(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} \qquad s_4(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147{,}456}$$

Figure 1 shows the graphs of these partial sums, which are polynomials. They are all approximations to the function J_0, but notice that the approximations become better when more terms are included. Figure 2 shows a more complete graph of the Bessel function.

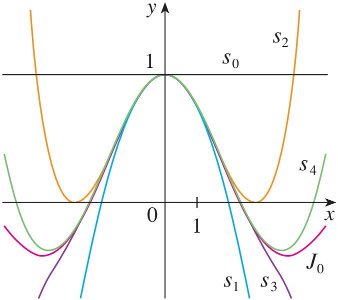


FIGURE 1

Partial sums of the Bessel function J_0

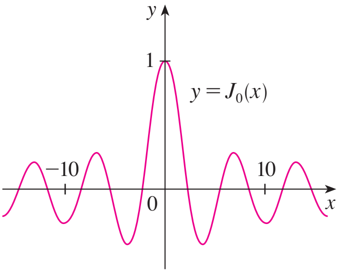


FIGURE 2

EXAMPLE 4 In Example 3 we saw that the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

is defined for all x. Thus, by Theorem 2, J_0 is differentiable for all x and its derivative is found by term-by-term differentiation as follows:

$$J_0'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2}$$

Exercise 35. (a) Show that J_0 (the Bessel function of order 0 given in Example 4) satisfies the differential equation

$$x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x) = 0$$

(b) Evaluate $\int_0^1 J_0(x)\,dx$ correct to three decimal places.

Exercise 36. The Bessel function of order 1 is defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$$

(a) Show that J_1 satisfies the differential equation

$$x^2 J_1''(x) + x J_1'(x) + (x^2 - 1) J_1(x) = 0$$

(b) Show that $J_0'(x) = -J_1(x)$.

```
s0 = Normal[Series[BesselJ[0, x], {x, 0, 1}]]
s1 = Normal[Series[BesselJ[0, x], {x, 0, 2}]]
s2 = Normal[Series[BesselJ[0, x], {x, 0, 4}]]
s3 = Normal[Series[BesselJ[0, x], {x, 0, 6}]]
s4 = Normal[Series[BesselJ[0, x], {x, 0, 8}]]
s5 = Normal[Series[BesselJ[0, x], {x, 0, 8}]]
s6 = BesselJ[0, x];
Plot[Evaluate@Tooltip[{s0, s1, s2, s3, s4, s5, s6}],
 {x, -20, 20}, PlotRange → {-2, 2} , ImageSize → 800]
```

1

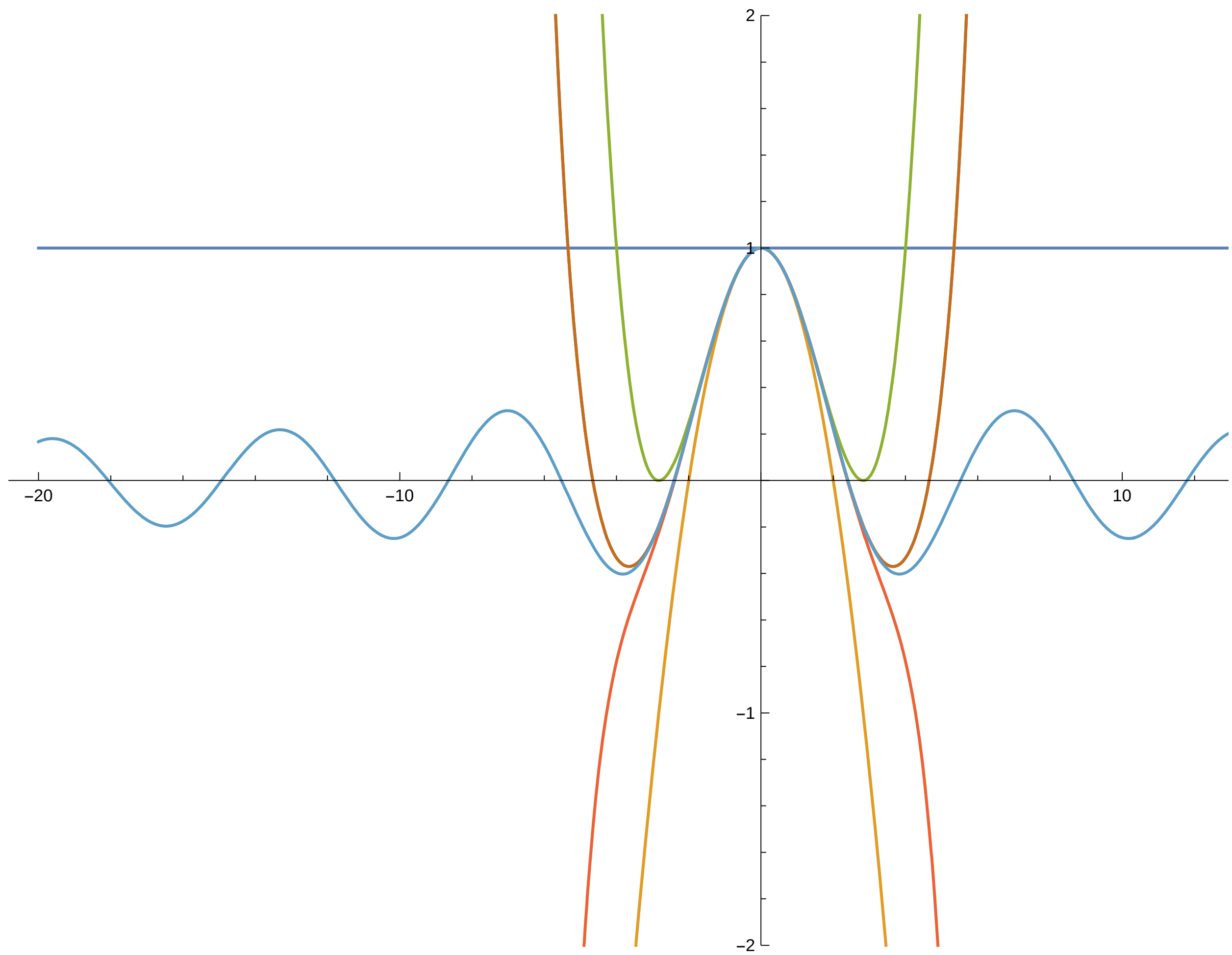
$$1-\frac{x^2}{4}$$

$$1-\frac{x^2}{4}+\frac{x^4}{64}$$

$$1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}$$

$$1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147\,456}$$

$$1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+\frac{x^8}{147\,456}$$



2
1
−1
−2
−20
−10
10

```
p1 = Normal[Series[1 / (1 - x), {x, 0, 1}]]
p2 = Normal[Series[1 / (1 - x), {x, 0, 2}]]
p5 = Normal[Series[1 / (1 - x), {x, 0, 5}]]
p8 = Normal[Series[1 / (1 - x), {x, 0, 8}]]
p11 = Normal[Series[1 / (1 - x), {x, 0, 11}]]
p = 1 / (1 - x)
Plot[Evaluate@Tooltip[{p1, p2, p5, p8, p11, p}], {x, -1.5, 1},
 PlotStyle → {Thin, Thin, Thin, Thin, Thin, Thick}, PlotRange → {-1, 10} , ImageSize → 800]
```

$1+x$

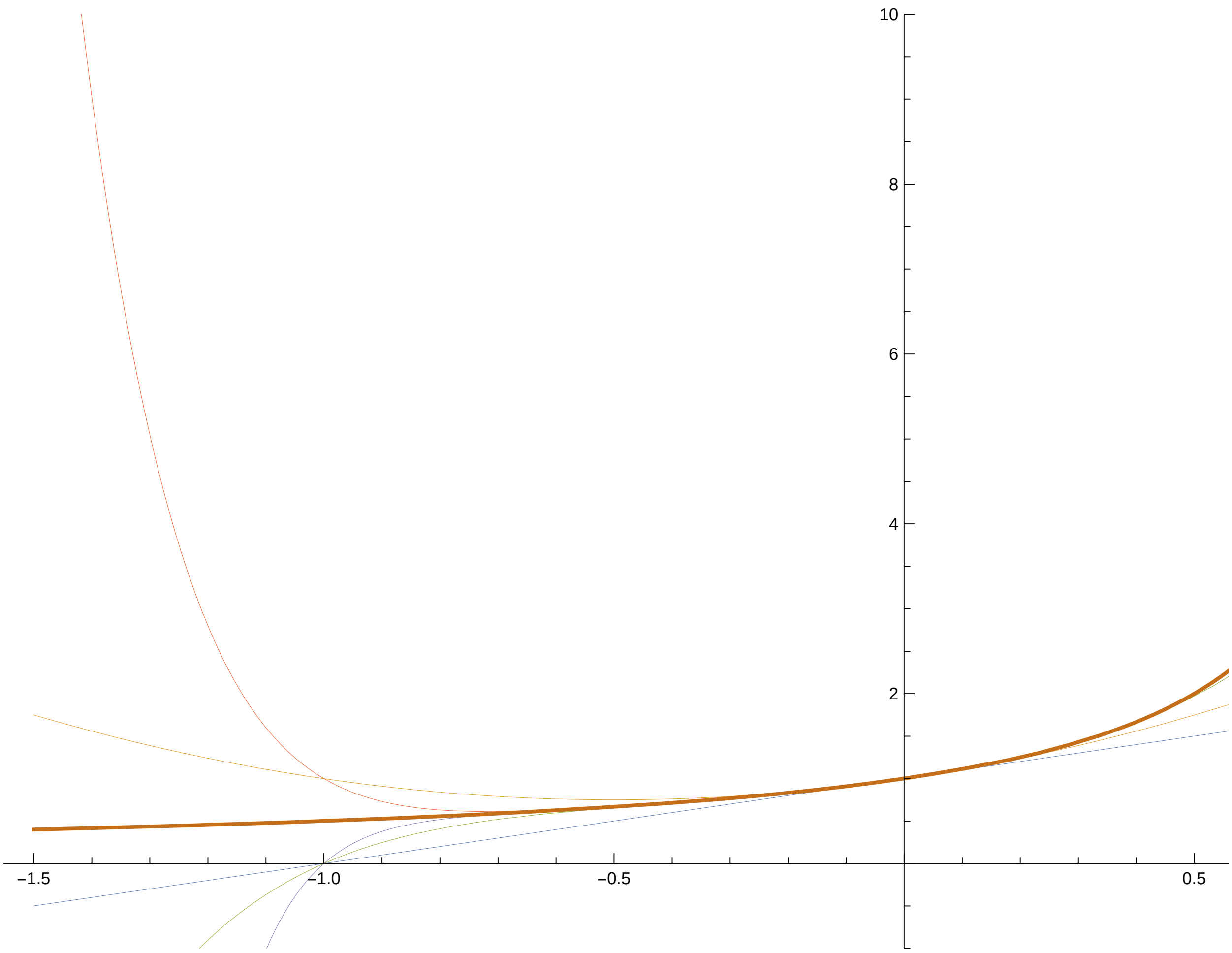
$1+x+x^2$

$1+x+x^2+x^3+x^4+x^5$

$1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8$

$1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}$

$\frac{1}{1-x}$

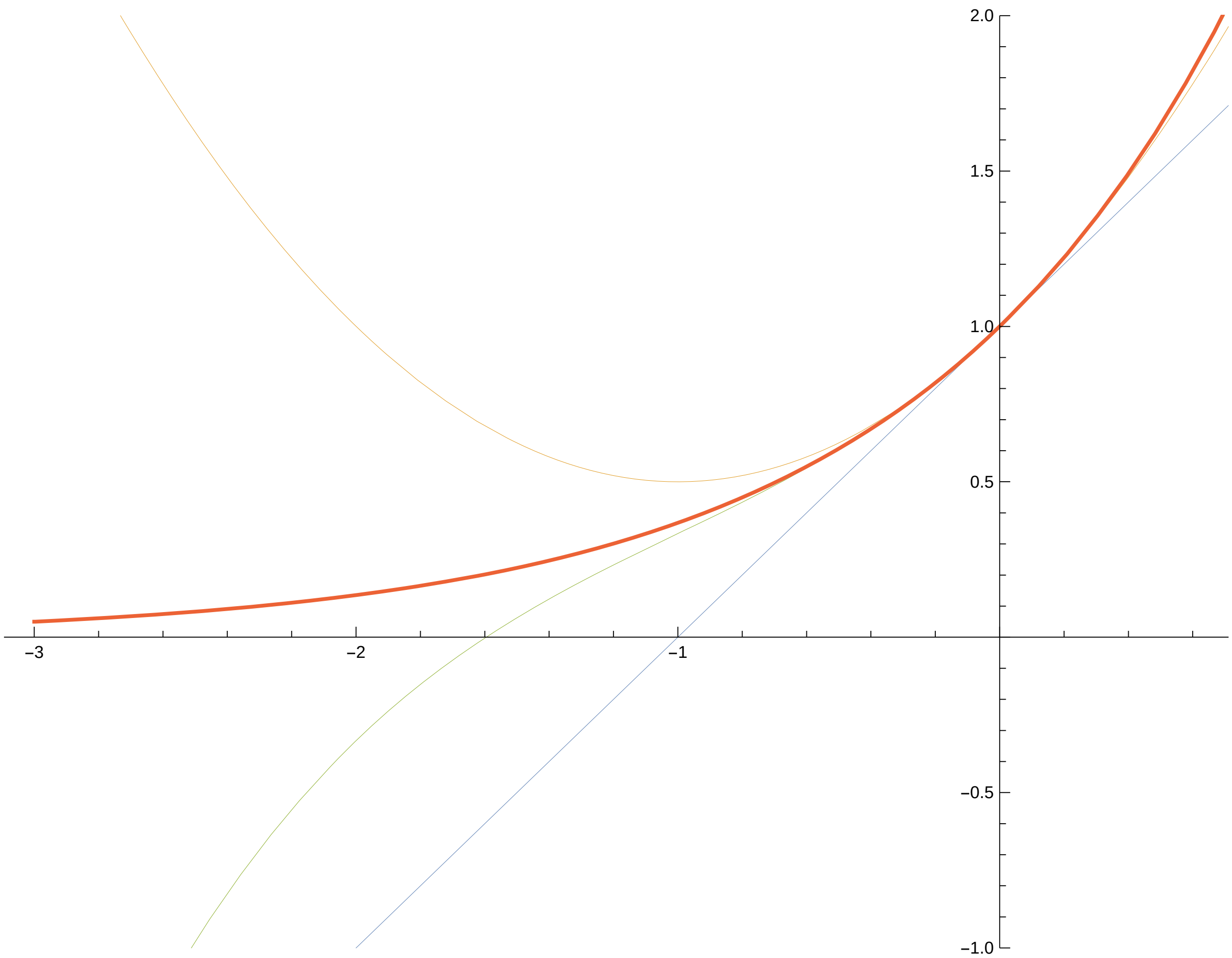


```
p1 = Normal[Series[Exp[x], {x, 0, 1}]]
p2 = Normal[Series[Exp[x], {x, 0, 2}]]
p3 = Normal[Series[Exp[x], {x, 0, 3}]]
Plot[Evaluate@Tooltip[{p1, p2, p3, Exp[x]}], {x, -3, 1.5},
 PlotStyle → {Thin, Thin, Thin, Thick}, PlotRange → {-1, 2} , ImageSize → 800]
```

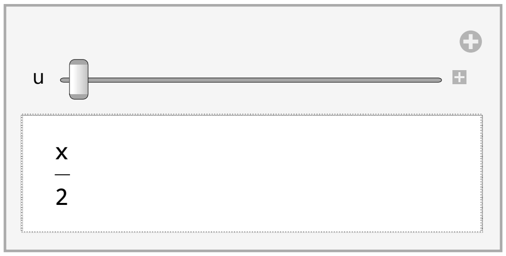
$1 + x$

$1 + x + \frac{x^2}{2}$

$1 + x + \frac{x^2}{2} + \frac{x^3}{6}$



```
Manipulate[Sum[x^n / (n + 1), {n, u}], {u, 1, 10, 1}]
```

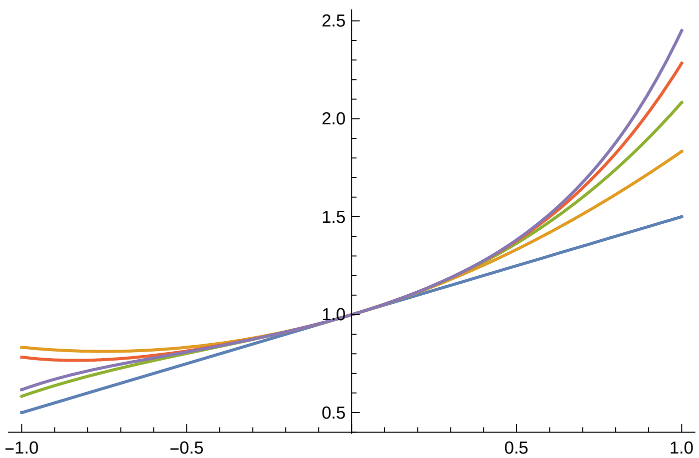



```
s[n_] := Sum[x^k / (k + 1), {k, 0, n}]
```

```
partialsums = Table[s[n], {n, 1, 5}]
```

```
Plot[Evaluate@Tooltip[partialsums], {x, -1, 1}]
```

$$\left\{1+\frac{x}{2},\ 1+\frac{x}{2}+\frac{x^2}{3},\ 1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4},\ 1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\frac{x^4}{5},\ 1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\frac{x^4}{5}+\frac{x^5}{6}\right\}$$



```
p1 = Normal[Series[Sin[x], {x, 0, 1}]]
p2 = Normal[Series[Sin[x], {x, 0, 3}]]
p3 = Normal[Series[Sin[x], {x, 0, 5}]]
p4 = Normal[Series[Sin[x], {x, 0, 7}]]
p5 = Normal[Series[Sin[x], {x, 0, 9}]]
Plot[Evaluate@Tooltip[{p1, p2, p3, p4, p5, Sin[x]}], {x, -2, 4},
 PlotRange → {-1.2, 1.2} , PlotStyle → {{Thin, Red}, {Thin, Blue},
   {Thin, Green}, {Thin, Cyan}, {Thin, Magenta}, {Thick, Black}}, ImageSize → 800 ]
```

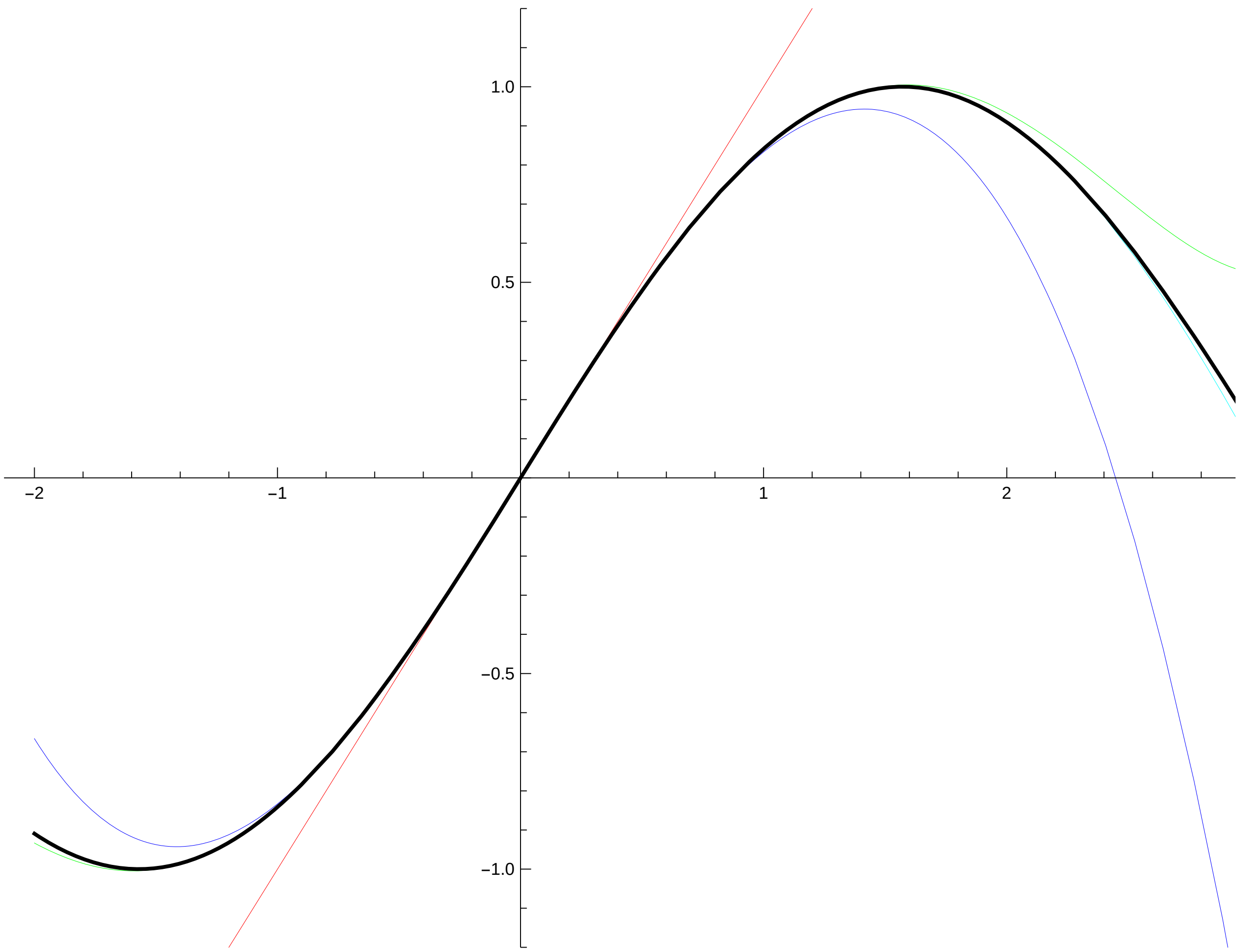
$$x$$

$$x - \frac{x^3}{6}$$

$$x - \frac{x^3}{6} + \frac{x^5}{120}$$

$$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$$

$$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362\,880}$$



```
p1 = Normal[Series[Sin[x], {x, Pi / 3, 1}]]
p2 = Normal[Series[Sin[x], {x, Pi / 3, 3}]]
p3 = Normal[Series[Sin[x], {x, Pi / 3, 5}]]
p4 = Normal[Series[Sin[x], {x, Pi / 3, 7}]]
p5 = Normal[Series[Sin[x], {x, Pi / 3, 9}]]
Plot[Evaluate@Tooltip[{p1, p2, p3, p4, p5, Sin[x]}], {x, -2.5, 4.5},
 PlotRange → {-1.2, 1.2} , PlotStyle → {{Thin, Red}, {Thin, Blue},
   {Thin, Green}, {Thin, Cyan}, {Thin, Magenta}, {Thick, Black}}, ImageSize → 800 ]
```

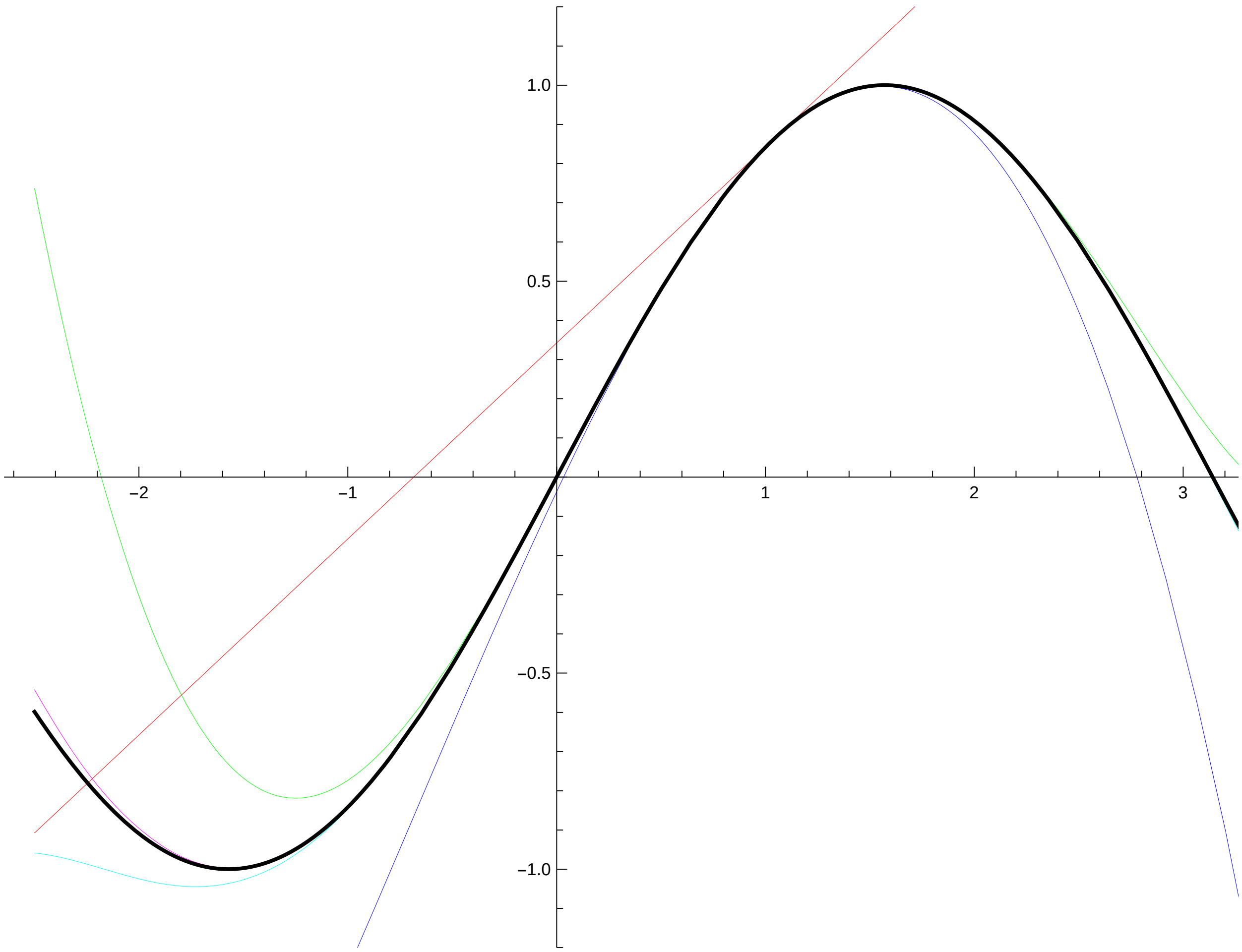
$$\frac{\sqrt{3}}{2}+\frac{1}{2}\left(-\frac{\pi}{3}+x\right)$$

$$\frac{\sqrt{3}}{2}+\frac{1}{2}\left(-\frac{\pi}{3}+x\right)-\frac{1}{4}\sqrt{3}\left(-\frac{\pi}{3}+x\right)^2-\frac{1}{12}\left(-\frac{\pi}{3}+x\right)^3$$

$$\frac{\sqrt{3}}{2}+\frac{1}{2}\left(-\frac{\pi}{3}+x\right)-\frac{1}{4}\sqrt{3}\left(-\frac{\pi}{3}+x\right)^2-\frac{1}{12}\left(-\frac{\pi}{3}+x\right)^3+\frac{\left(-\frac{\pi}{3}+x\right)^4}{16\sqrt{3}}+\frac{1}{240}\left(-\frac{\pi}{3}+x\right)^5$$

$$\frac{\sqrt{3}}{2}+\frac{1}{2}\left(-\frac{\pi}{3}+x\right)-\frac{1}{4}\sqrt{3}\left(-\frac{\pi}{3}+x\right)^2-\frac{1}{12}\left(-\frac{\pi}{3}+x\right)^3+$$
$$\frac{\left(-\frac{\pi}{3}+x\right)^4}{16\sqrt{3}}+\frac{1}{240}\left(-\frac{\pi}{3}+x\right)^5-\frac{\left(-\frac{\pi}{3}+x\right)^6}{480\sqrt{3}}-\frac{\left(-\frac{\pi}{3}+x\right)^7}{10\,080}$$

$$\frac{\sqrt{3}}{2}+\frac{1}{2}\left(-\frac{\pi}{3}+x\right)-\frac{1}{4}\sqrt{3}\left(-\frac{\pi}{3}+x\right)^2-\frac{1}{12}\left(-\frac{\pi}{3}+x\right)^3+\frac{\left(-\frac{\pi}{3}+x\right)^4}{16\sqrt{3}}+$$
$$\frac{1}{240}\left(-\frac{\pi}{3}+x\right)^5-\frac{\left(-\frac{\pi}{3}+x\right)^6}{480\sqrt{3}}-\frac{\left(-\frac{\pi}{3}+x\right)^7}{10\,080}+\frac{\left(-\frac{\pi}{3}+x\right)^8}{26\,880\sqrt{3}}+\frac{\left(-\frac{\pi}{3}+x\right)^9}{725\,760}$$



$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n =$$

$$1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} +$$

$$\cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} =$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} =$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} =$$

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n} =$$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n =$$

$$1 + \mathrm{kx} + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots \qquad R = 1$$