Calculus I

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1. Techniques of Integration
    - Integration by Parts
    - Trigonometric Integrals
    - Trigonometric Substitution
    - Hyperbolic and Inverse Hyperbolic Functions
    - The Method of Partial Fractions
    - Improper Integrals
    - Numerical Integration

Subsection 1

Integration by Parts

Integration By Parts

- Recall the Product Rule for Derivatives:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x);$$

Integrate both sides with respect to x:

$$\int (f(x)g(x))' dx = \int [f'(x)g(x) + f(x)g'(x)]dx;$$

Since integration is the reverse operation of differentiation, we have

$$\int (f(x)g(x))' dx = f(x)g(x);$$

Moreover, because of the sum rule for integrals:

$$\int [f'(x)g(x) + f(x)g'(x)]dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx;$$

Putting all these together:

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx;$$

Finally, subtract to get the **Integration By Parts Formula**:

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

Alternative Form

- We came up with the formula

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

- Use two new variables u and v as follows: Set

$$\begin{aligned} u &= g(x) & \qquad du &= g'(x)dx \\ v &= f(x) & \qquad dv &= f'(x)dx \end{aligned}$$

- Now substitute into the formula above to get the uv-form of the **By Parts Rule**:

$$\int u dv = uv - \int v du;$$

Example I (fg-form)

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

$$\begin{aligned}
\int x\cos x dx &= \int x(\sin x)' dx \\
&= x\sin x - \int (x)' \sin x dx \\
&= x\sin x - \int \sin x dx \\
&= x\sin x - (-\cos x) + C; \\
&= x\sin x + \cos x + C;
\end{aligned}$$

Example I (uv-form)

$$\int u dv = uv - \int v du;$$

We want to compute $\int x \cos x dx$;

Set $u = x$ and $dv = \cos x dx$; Then $\frac{du}{dx} = 1 \Rightarrow du = dx$; Moreover $\frac{dv}{dx} = \cos x \Rightarrow v = \sin x$;

$$\begin{aligned} \int x \cos x dx &= \int u dv \\ &= uv - \int v du \\ &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x + C; \end{aligned}$$

Example II (fg-form)

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

$$\begin{aligned}
\int xe^x dx &= \int x(e^x)' dx \\
&= xe^x - \int (x)' e^x dx \\
&= xe^x - \int e^x dx \\
&= xe^x - e^x + C;
\end{aligned}$$

Example II (*uv*-form)

$$\int u\,dv = uv - \int v\,du;$$

We want to compute $\int xe^x\,dx$;

Set $u = x$ and $dv = e^x\,dx$; Then $\frac{du}{dx} = 1 \Rightarrow du = dx$; Moreover $\frac{dv}{dx} = e^x \Rightarrow v = e^x$;

$$\begin{aligned} \int xe^x\,dx &= \int u\,dv \\ &= uv - \int v\,du \\ &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C; \end{aligned}$$

Example III (*fg*-form)

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

$$\begin{aligned}
\int x^7 \ln x dx &= \int (\frac{1}{8}x^8)' \ln x dx \\
&= \frac{1}{8}x^8 \ln x - \int \frac{1}{8}x^8 (\ln x)' dx \\
&= \frac{1}{8}x^8 \ln x - \frac{1}{8}\int x^8 \cdot \tfrac{1}{x} dx \\
&= \frac{1}{8}x^8 \ln x - \frac{1}{8}\int x^7 dx \\
&= \frac{1}{8}x^8 \ln x - \frac{1}{8} \cdot \frac{1}{8}x^8 + C \\
&= \frac{1}{8}x^8 \ln x - \frac{1}{64}x^8 + C;
\end{aligned}$$

Example III (uv-form)

$$\int udv = uv - \int vdu;$$

We want to compute $\int x^7 \ln x dx$;

Set $u = \ln x$ and $dv = x^7 dx$; Then $\frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x}dx$; Moreover $\frac{dv}{dx} = x^7 \Rightarrow v = \frac{1}{8}x^8$;

$$\begin{aligned}
\int x^7 \ln x dx &= \int udv = uv - \int vdu \\
&= \frac{1}{8}x^8 \ln x - \int \frac{1}{8}x^8 \cdot \frac{1}{x}dx = \frac{1}{8}x^8 \ln x - \frac{1}{8}\int x^7 dx \\
&= \frac{1}{8}x^8 \ln x - \frac{1}{64}x^8 + C;
\end{aligned}$$

Example IV: Applying By Parts Twice

$$
\begin{aligned}
\int x^2 \cos x dx &= \int x^2 (\sin x)' dx \\
&= x^2 \sin x - \int (x^2)' \sin x dx \\
&= x^2 \sin x - \int 2x \sin x dx \\
&= x^2 \sin x - \int 2x(-\cos x)' dx \\
&= x^2 \sin x - [-2x \cos x - \int (2x)'(-\cos x) dx] \\
&= x^2 \sin x + 2x \cos x - \int 2 \cos x dx \\
&= x^2 \sin x + 2x \cos x - 2 \sin x + C;
\end{aligned}
$$

Example V: Applying By Parts Twice

$$
\begin{aligned}
\int x^2 e^{3x}\,dx &= \int x^2 \left(\tfrac{1}{3}e^{3x}\right)' dx \\
&= x^2 \tfrac{1}{3}e^{3x} - \int (x^2)' \tfrac{1}{3}e^{3x}\,dx \\
&= x^2 \tfrac{1}{3}e^{3x} - \int 2x \tfrac{1}{3}e^{3x}\,dx \\
&= x^2 \tfrac{1}{3}e^{3x} - \int 2x \left(\tfrac{1}{9}e^{3x}\right)' dx \\
&= x^2 \tfrac{1}{3}e^{3x} - \left[2x \tfrac{1}{9}e^{3x} - \int (2x)' \tfrac{1}{9}e^{3x}\,dx\right] \\
&= x^2 \tfrac{1}{3}e^{3x} - 2x \tfrac{1}{9}e^{3x} + \int 2 \tfrac{1}{9}e^{3x}\,dx \\
&= \tfrac{1}{3}x^2 e^{3x} - \tfrac{2}{9}x e^{3x} + \tfrac{2}{27}e^{3x} + C;
\end{aligned}
$$

Example VI: Integral of $\ln x$

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

$$\begin{aligned}
\int \ln x dx &= \int (x)' \ln x dx \\
&= x \ln x - \int x (\ln x)' dx \\
&= x \ln x - \int x \cdot \frac{1}{x} dx \\
&= x \ln x - \int dx \\
&= x \ln x - x + C;
\end{aligned}$$

Example VII: Returning to the Original Form

$$\begin{aligned}
\int e^x \cos x dx &= \int (e^x)' \cos x dx \\
&= e^x \cos x - \int e^x (\cos x)' dx \\
&= e^x \cos x + \int e^x \sin x dx \\
&= e^x \cos x + \int (e^x)' \sin x dx \\
&= e^x \cos x + e^x \sin x - \int e^x (\sin x)' dx \\
&= e^x \cos x + e^x \sin x - \int e^x \cos x dx
\end{aligned}$$

Thus, $\int e^x \cos x dx = e^x(\cos x + \sin x) - \int e^x \cos x dx$, and, hence,

$2\int e^x \cos x dx = e^x(\cos x + \sin x) \Rightarrow \int e^x \cos x dx = \frac{1}{2} e^x(\cos x + \sin x) + C;$

Example VIII

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

$$\begin{aligned}
\int (x-2)(x+4)^8 dx &= \int (x-2)[\frac{1}{9}(x+4)^9]' dx \\
&= \frac{1}{9}(x-2)(x+4)^9 - \int (x-2)' \frac{1}{9}(x+4)^9 dx \\
&= \frac{1}{9}(x-2)(x+4)^9 - \frac{1}{9}\int (x+4)^9 dx \\
&= \frac{1}{9}(x-2)(x+4)^9 - \frac{1}{9}\cdot\frac{1}{10}(x+4)^{10} + C \\
&= \frac{1}{9}(x-2)(x+4)^9 - \frac{1}{90}(x+4)^{10} + C;
\end{aligned}$$

Example IX: A Definite Integral By Parts

$$
\begin{aligned}
\int_0^{\pi/4} x \sin 2x dx &= \int_0^{\pi/4} x(-\tfrac{1}{2}\cos 2x)' dx \\
&= -\tfrac{1}{2} x \cos 2x \Big|_0^{\pi/4} - \int_0^{\pi/4} (x)'(-\tfrac{1}{2}\cos 2x) dx \\
&= -\tfrac{1}{2} x \cos 2x \Big|_0^{\pi/4} + \int_0^{\pi/4} \tfrac{1}{2}\cos 2x dx \\
&= -\tfrac{1}{2} x \cos 2x \Big|_0^{\pi/4} + \tfrac{1}{4}\sin 2x \Big|_0^{\pi/4} \\
&= (0-0) + (\tfrac{1}{4} - 0) \\
&= \tfrac{1}{4};
\end{aligned}
$$

Subsection 2

Trigonometric Integrals

Odd Powers of $\sin x$

$$
\begin{aligned}
\int \sin^3 x dx &= \int \sin^2 x \sin x dx \\
&= \int (1-\cos^2 x)\sin x dx \\
&\overset{u=\cos x}{=} \int (1-u^2)(-du) \\
&= \int (u^2-1) du \\
&= \frac{1}{3}u^3 - u + C \\
&= \frac{1}{3}\cos^3 x - \cos x + C;
\end{aligned}
$$

Odd Power of $\sin x$ or $\cos x$

$$
\begin{aligned}
\int \sin^4 x \cos^5 x dx &= \int \sin^4 x \cos^4 x \cos x dx \\
&= \int \sin^4 x (\cos^2 x)^2 \cos x dx \\
&= \int \sin^4 x (1 - \sin^2 x)^2 \cos x dx \\
&\overset{u=\sin x}{=} \int u^4 (1-u^2)^2 du \\
&= \int u^4 (u^4 - 2u^2 + 1) du \\
&= \int (u^8 - 2u^6 + u^4) du \\
&= \frac{1}{9} u^9 - \frac{2}{7} u^7 + \frac{1}{5} u^5 + C \\
&= \frac{1}{9} \sin^9 x - \frac{2}{7} \sin^7 x + \frac{1}{5} \sin^5 x + C
\end{aligned}
$$

Double-Angle Identities

$$\cos^2 x = \frac{1+\cos 2x}{2}, \qquad \sin^2 x = \frac{1-\cos 2x}{2};$$

$$\begin{aligned}
\int \sin^4 x dx &= \int (\sin^2 x)^2 dx = \int (\frac{1-\cos 2x}{2})^2 dx \\
&= \int (\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x) dx \\
&= \int (\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}(\frac{1+\cos 4x}{2})) dx \\
&= \int (\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x) dx \\
&= \int (\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x) dx \\
&= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C;
\end{aligned}$$

Integrals of Tangent and Secant

$$\begin{aligned}
\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\
&\overset{u=\cos x}{=} \int \frac{1}{u}(-du) \\
&= -\int \frac{1}{u} du \\
&= -\ln|u| + c \\
&= -\ln|\cos x| + C \\
&= \ln\left|\frac{1}{\cos x}\right| + C \\
&= \ln|\sec x| + C;
\end{aligned}$$

$$\begin{aligned}
\int \sec x dx &= \int \frac{\sec x(\tan x + \sec x)}{\tan x + \sec x} dx \\
&= \int \frac{\sec^2 x + \tan x \sec x}{\tan x + \sec x} dx \\
&\overset{u=\tan x+\sec x}{=} \int \frac{1}{u} du \\
&= \ln|u| + C \\
&= \ln|\tan x + \sec x| + C;
\end{aligned}$$

Tangent and Secant I

Recall the identity $1 + \tan^2 x = \sec^2 x$;

$$\begin{aligned}
\int \tan^3 x \sec^4 x dx &= \int \tan^3 x \sec^2 x \sec^2 x dx \\
&= \int \tan^3 x (1 + \tan^2 x) \sec^2 x dx \\
&\overset{u=\tan x}{=} \int u^3 (1 + u^2) du \\
&= \int (u^5 + u^3) du \\
&= \frac{1}{6} u^6 + \frac{1}{4} u^4 + C \\
&= \frac{1}{6} \tan^6 x + \frac{1}{4} \tan^4 x + C;
\end{aligned}$$

Tangent and Secant II

Again, we will use $1 + \tan^2 x = \sec^2 x$;

$$\begin{aligned}
\int \tan^3 x \sec^3 x dx &= \int \tan^2 x \sec^2 x \tan x \sec x dx \\
&= \int (\sec^2 x - 1) \sec^2 x \tan x \sec x dx \\
&\overset{u=\sec x}{=} \int (u^2 - 1) u^2 du \\
&= \int (u^4 - u^2) du \\
&= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C \\
&= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C;
\end{aligned}$$

Subsection 3

Trigonometric Substitution

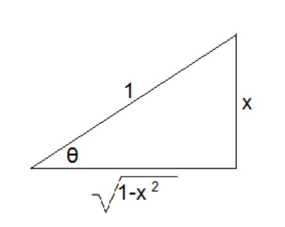
Integrals Involving $\sqrt{a^2 - x^2}$

Integrals Involving $\sqrt{a^2 - x^2}$

$$x = a\sin\theta, \qquad dx = a\cos\theta d\theta, \qquad \sqrt{a^2 - x^2} = a\cos\theta;$$

- Example: Evaluate $\int \sqrt{1 - x^2}dx$;

  Set $x = \sin\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$; Then $dx = \cos\theta d\theta$; Moreover, $\sqrt{1 - x^2} = \sqrt{1 - \sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$; Note, also, that $\theta = \sin^{-1}x$ and $\cos\theta = \sqrt{1 - x^2}$;



$$\int \sqrt{1 - x^2}dx = \int \cos\theta\cos\theta d\theta = \int \frac{1 + \cos 2\theta}{2}d\theta =$$
$$\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C =$$
$$\frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1 - x^2} + C;$$

A Trigonometric Identity

- Show that $\sin\theta\tan\theta+\cos\theta=\dfrac{1}{\cos\theta}$;

$$\begin{aligned}
\sin\theta\tan\theta+\cos\theta &= \sin\theta\frac{\sin\theta}{\cos\theta}+\cos\theta\\
&= \frac{\sin^2\theta}{\cos\theta}+\cos\theta\\
&= \frac{\sin^2\theta}{\cos\theta}+\frac{\cos^2\theta}{\cos\theta}\\
&= \frac{\sin^2\theta+\cos^2\theta}{\cos\theta}\\
&= \frac{1}{\cos\theta};
\end{aligned}$$

Integral of $\tan^2\theta$

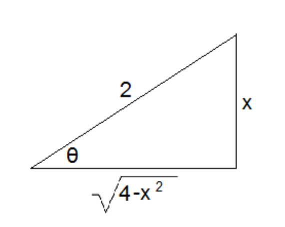
- Evaluate $\int \tan^2\theta d\theta$;

$$\int \tan^2\theta d\theta = \int \frac{\sin^2\theta}{\cos^2\theta} d\theta = \int \sin^2\theta \sec^2\theta d\theta =$$
$$\int \sin^2\theta (\tan\theta)' d\theta = \sin^2\theta \tan\theta - \int (\sin^2\theta)' \tan\theta d\theta =$$
$$\sin^2\theta \tan\theta - \int 2\sin\theta\cos\theta \frac{\sin\theta}{\cos\theta} d\theta =$$
$$\sin^2\theta \tan\theta - \int 2\sin^2\theta d\theta = \sin^2\theta\tan\theta - \int (1-\cos 2\theta) d\theta =$$
$$\sin^2\theta\tan\theta - (\theta - \frac{1}{2}\sin 2\theta) + C =$$
$$\sin^2\theta\tan\theta - \theta + \sin\theta\cos\theta + C =$$
$$\sin^2\theta\tan\theta + \sin\theta\cos\theta - \theta + C =$$
$$\sin\theta[\sin\theta\tan\theta + \cos\theta] - \theta + C \overset{\text{preceding slide}}{=}$$
$$\sin\theta \cdot \frac{1}{\cos\theta} - \theta + C = \tan\theta - \theta + C;$$

Integrals Involving $(a^2 - x^2)^{n/2}$

- Example: Evaluate $\int \frac{x^2}{(4-x^2)^{3/2}}dx$;

  Set $x = 2\sin\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$; Then $dx = 2\cos\theta d\theta$; Moreover, $(4-x^2)^{3/2} = (4-4\sin^2\theta)^{3/2} = (4\cos^2\theta)^{3/2} = 8\cos^3\theta$; Note, also, that $\theta = \sin^{-1}\left(\frac{x}{2}\right)$ and $\tan\theta = \frac{x}{\sqrt{4-x^2}}$;



$$\int \frac{x^2}{(4-x^2)^{3/2}}dx = \int \frac{4\sin^2\theta}{8\cos^3\theta}2\cos\theta d\theta = \int \frac{\sin^2\theta}{\cos^2\theta}d\theta =$$
$$\int \tan^2\theta d\theta \overset{\text{preceding slide}}{=} \tan\theta - \theta + C = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C;$$

Integral of $\sec^3\theta$

- Recall that $\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$; Evaluate $\int \sec^3\theta d\theta$;

  $\int \sec^3\theta d\theta = \int \sec\theta \sec^2\theta d\theta = \int \sec\theta(\tan\theta)' d\theta =$

  $\sec\theta\tan\theta - \int (\sec\theta)'\tan\theta d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta d\theta =$

  $\sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)d\theta =$

  $\sec\theta\tan\theta - \int (\sec^3\theta - \sec\theta)d\theta =$

  $\sec\theta\tan\theta - \int \sec^3\theta d\theta + \int \sec\theta d\theta$;

  Therefore, $2\int \sec^3\theta d\theta = \sec\theta\tan\theta + \int \sec\theta d\theta \Rightarrow 2\int \sec^3\theta d\theta =$ $\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C$, i.e.,

$$\int \sec^3\theta d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C;$$

Integrals Involving $\sqrt{x^2+a^2}$

Integrals Involving $\sqrt{x^2+a^2}$

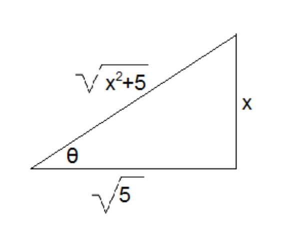
$$x = a\tan\theta, \qquad dx = a\sec^2\theta d\theta, \qquad \sqrt{x^2+a^2} = a\sec\theta;$$

- Example: Evaluate $\int \sqrt{4x^2+20}dx$;

  Note $\int \sqrt{4x^2+20}dx = \int \sqrt{4(x^2+5)}dx = 2\int \sqrt{x^2+5}dx$;

  Set $x = \sqrt{5}\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$; Then $dx = \sqrt{5}\sec^2\theta d\theta$; Moreover, $\sqrt{x^2+5} = \sqrt{5\tan^2\theta+5} = \sqrt{5\sec^2\theta} = \sqrt{5}\sec\theta$; Note, also, that $\theta = \tan^{-1}\left(\frac{x}{\sqrt{5}}\right)$ and

  $$\sec\theta = \frac{1}{\cos\theta} = \frac{\sqrt{x^2+5}}{\sqrt{5}};$$



Integrals Involving $\sqrt{x^2 + a^2}$ (Cont'd)

Recall

$$\sqrt{x^2+5} = \sqrt{5}\sec\theta, \qquad dx = \sqrt{5}\sec^2\theta d\theta$$

$$\theta = \tan^1\left(\frac{x}{\sqrt{5}}\right), \qquad \sec\theta = \frac{\sqrt{x^2+5}}{\sqrt{5}};$$

$$2\int\sqrt{x^2+5}dx = 2\int(\sqrt{5}\sec\theta)\sqrt{5}\sec^2\theta d\theta = 10\int\sec^3\theta d\theta \overset{\text{preceding problem}}{=}$$

$$10\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left(\sec\theta+\tan\theta\right)\right] + C =$$

$$5\frac{x}{\sqrt{5}}\cdot\frac{\sqrt{x^2+5}}{\sqrt{5}} + 5\ln\left(\frac{\sqrt{x^2+5}}{\sqrt{5}} + \frac{x}{\sqrt{5}}\right) + C =$$

$$x\sqrt{x^2+5} + 5\ln\left(\frac{\sqrt{x^2+5}+x}{\sqrt{5}}\right) + C;$$

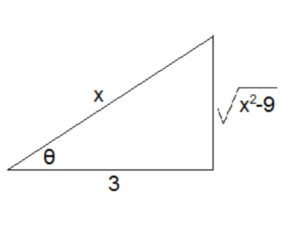
Integrals Involving $\sqrt{x^2 - a^2}$

Integrals Involving $\sqrt{x^2 - a^2}$

$$x = a\sec\theta, \qquad dx = a\sec\theta\tan\theta d\theta, \qquad \sqrt{x^2 - a^2} = a\tan\theta;$$

- Example: Evaluate $\int \frac{1}{x^2\sqrt{x^2-9}}dx$;

  Set $x = 3\sec\theta$; Then $dx = 3\sec\theta\tan\theta d\theta$; Moreover, $\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = \sqrt{9\tan^2\theta} = 3\tan\theta$; Note, also, that $\theta = \sec^{-1}\left(\frac{x}{3}\right)$ and $\sin\theta = \frac{\sqrt{x^2-9}}{x}$;



$$\int \frac{1}{x^2\sqrt{x^2-9}}dx = \int \frac{1}{9\sec^2\theta 3\tan\theta} 3\sec\theta\tan\theta d\theta = \int \frac{1}{9\sec\theta}d\theta = \frac{1}{9}\int \cos\theta d\theta = \frac{1}{9}\sin\theta + C = \frac{\sqrt{x^2-9}}{9x} + C;$$

Completing the Square

- Example: Evaluate $\displaystyle\int \frac{1}{(x^2-6x+11)^2}dx$;

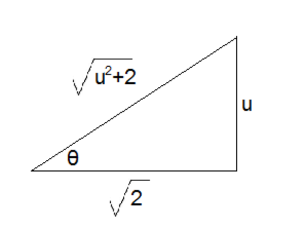
Note $\displaystyle\int \frac{1}{(x^2-6x+11)^2}dx = \int \frac{1}{[(x^2-6x+9)+2]^2}dx =$
$\displaystyle\int \frac{1}{[(x-3)^2+2]^2}dx \overset{u=x-3}{=} \int \frac{1}{(u^2+2)^2}du$;

Set $u=\sqrt{2}\tan\theta$; Then $du=\sqrt{2}\sec^2\theta d\theta$;

Moreover, $u^2+2=2\tan^2\theta+2=2\sec^2\theta$;

Note, also, that $\theta = \tan^{-1}\left(\frac{u}{\sqrt{2}}\right)$ and

$\sin\theta = \frac{u}{\sqrt{u^2+2}}$, $\cos\theta = \frac{\sqrt{2}}{\sqrt{u^2+2}}$;



Completing the Square (Cont'd)

Recall

$$u = x - 3 \qquad u^2 + 2 = 2\sec^2\theta, \qquad du = \sqrt{2}\sec^2\theta d\theta$$
$$\sin\theta = \tfrac{u}{\sqrt{u^2+2}}, \qquad \cos\theta = \tfrac{\sqrt{2}}{\sqrt{u^2+2}};$$

$$\int \frac{1}{(u^2+2)^2} du = \int \frac{\sqrt{2}\sec^2\theta}{(2\sec^2\theta)^2} d\theta = \int \frac{\sqrt{2}\sec^2\theta}{4\sec^4\theta} d\theta =$$
$$\frac{\sqrt{2}}{4}\int \cos^2\theta d\theta = \frac{\sqrt{2}}{4}\int \frac{1}{2}(1+\cos 2\theta) d\theta =$$
$$\frac{\sqrt{2}}{8}(\theta + \frac{1}{2}\sin 2\theta) + C = \frac{\sqrt{2}}{8}\theta + \frac{\sqrt{2}}{8}\sin\theta\cos\theta + C =$$
$$\frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + \frac{\sqrt{2}}{8}\frac{u}{\sqrt{u^2+2}}\frac{\sqrt{2}}{\sqrt{u^2+2}} + C =$$
$$\frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x-3}{\sqrt{2}}\right) + \frac{x-3}{4(x^2-6x+11)} + C;$$

Subsection 4

Hyperbolic and Inverse Hyperbolic Functions

Hyperbolic Functions and Derivatives

Definition of Hyperbolics

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \tanh x = \frac{\sinh x}{\cosh x} \quad \mathrm{sech} x = \frac{1}{\cosh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \coth x = \frac{\cosh x}{\sinh x} \quad \mathrm{csch} x = \frac{1}{\sinh x}$$

Basic Derivatives

$$(\sinh x)' = \cosh x \quad (\tanh x)' = \mathrm{sech}^2 x \quad (\mathrm{sech} x)' = -\mathrm{sech} x \tanh x$$

$$(\cosh x)' = \sinh x \quad (\coth x)' = -\mathrm{csch}^2 x \quad (\mathrm{csch} x)' = -\mathrm{csch} x \coth x$$

Basic Integral Formulas

Basic Integrals

$$\int \sinh x dx = \cosh x + C \qquad \int \cosh x dx = \sinh x + C$$

$$\int \text{sech}^2 x dx = \tanh x + C \qquad \int \text{csch}^2 x dx = -\coth x + C$$

$$\int \text{sech} x \tanh x dx = -\text{sech} x + C \qquad \int \text{csch} x \coth x dx = -\text{csch} x + C$$

- Example: Calculate $\int x \cosh(x^2) dx$;

$$\int x \cosh(x^2) dx \overset{u=x^2}{=} \int \frac{1}{2} \cosh u du = \frac{1}{2} \sinh u + C = \frac{1}{2} \sinh(x^2) + C;$$

Powers of $\sinh x$ and $\cosh x$

- Calculate $\int \sinh^4 x \cosh^5 x dx$;

$$\int \sinh^4 x \cosh^5 x dx = \int \sinh^4 x (\cosh^2 x)^2 \cosh x dx =$$
$$\int \sinh^4 x (1+\sinh^2 x)^2 \cosh x dx \overset{u=\sinh x}{=} \int u^4 (1+u^2)^2 du =$$
$$\int u^4(u^4+2u^2+1)du = \int (u^8+2u^6+u^4)du =$$
$$\frac{1}{9}u^9 + \frac{2}{7}u^7 + \frac{1}{5}u^5 + C = \frac{1}{9}\sinh^9 x + \frac{2}{7}\sinh^7 x + \frac{1}{5}\sinh^5 x + C;$$

- Calculate $\int \cosh^2 x dx$;

$$\int \cosh^2 x dx = \int \frac{1}{2}(1+\cosh 2x)dx = \frac{1}{2}(x + \frac{1}{2}\sinh 2x) + C =$$
$$\frac{1}{2}x + \frac{1}{4}\sinh 2x + C;$$

Hyperbolic Substitutions (instead of Trig Substitutions)

- Instead of trigonometric substitutions, one may sometimes perform hyperbolic substitutions to calculate an integral:

The Method

- For expressions of the form $\sqrt{x^2+a^2}$, instead of $x = a\tan\theta$, we may use $x = a\sinh u$; In that case
  - $dx = a\cosh u du$;
  - $\sqrt{x^2+a^2} = a\cosh u$;
- For expressions of the form $\sqrt{x^2-a^2}$, instead of $x = a\sec\theta$, we may use $x = a\cosh u$; In that case
  - $dx = a\sinh u du$;
  - $\sqrt{x^2-a^2} = a\sinh u$;

Example of Hyperbolic Substitution

- Example: Calculate $\int \sqrt{x^2+16}dx$;
  We set $x = 4\sinh u$; Then $dx = 4\cosh u du$,
  $\sqrt{x^2+16} = \sqrt{16\sinh^2 u + 16} = 4\cosh u$; Moreover, $u = \sinh^{-1}\frac{x}{4}$,
  $\sinh u = \frac{x}{4}$ and $\cosh u = \sqrt{\sinh^2 u + 1} = \sqrt{\frac{x^2}{16}+1}$; Therefore,

$$\int \sqrt{x^2+16}dx = \int 4\cosh u 4\cosh u du = \int 16\cosh^2 u du = \int 8(1+\cosh 2u)du = 8u + 4\sinh 2u + C =$$

$$8u + 8\sinh u\cosh u + C = 8\sinh^{-1}\frac{x}{4} + 8\frac{x}{4}\sqrt{\frac{x^2}{16}+1} + C;$$

Integrals of Inverse Hyperbolic Functions

Integrals Involving Inverse Trigonometric Functions

- $\displaystyle\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x + C;$
- $\displaystyle\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C; \qquad (x > 1)$
- $\displaystyle\int \frac{dx}{1-x^2} = \tanh^{-1} x + C; \qquad (|x| < 1)$
- $\displaystyle\int \frac{dx}{1-x^2} = \coth^{-1} x + C; \qquad (|x| > 1)$
- $\displaystyle\int \frac{dx}{x\sqrt{1-x^2}} = -\operatorname{sech}^{-1} x + C; \qquad (0 < x < 1)$
- $\displaystyle\int \frac{dx}{|x|\sqrt{1+x^2}} = -\operatorname{csch}^{-1} x + C; \qquad (x \neq 0)$

Examples of Inverse Hyperbolic Integrals

- Evaluate the following integrals:
  - $\int_2^4 \frac{dx}{\sqrt{x^2-1}}$;

$$\int_2^4 \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x\Big|_2^4 = \cosh^{-1} 4 - \cosh^{-1} 2;$$

  - $\int_{0.2}^{0.6} \frac{xdx}{1-x^4}$;

$$\int_{0.2}^{0.6} \frac{xdx}{1-x^4} \overset{u=x^2}{=} \int_{0.04}^{0.36} \frac{\frac{1}{2}du}{1-u^2} =$$

$$\frac{1}{2}\tanh^{-1} u\Big|_{0.04}^{0.36} = \frac{1}{2}(\tanh^{-1} 0.36 - \tanh^{-1} 0.04);$$

Subsection 5

The Method of Partial Fractions

Outline of Partial Fractions Method

- To integrate a rational function $f(x) = \frac{P(x)}{Q(x)}$, we write it as a sum of simpler rational functions that can be integrated directly;
- For example, to integrate $\int \frac{1}{x^2-1}dx$:
  1. We decompose the fraction into **partial fractions**:
  $$\frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1};$$
  2. Then, work as follows
  $$\begin{aligned} \int \frac{1}{x^2-1}dx &= \int \left[\frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\right] dx \\ &= \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{2}\int \frac{1}{x+1}dx \\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C; \end{aligned}$$

Distinct Linear Factors I

- Evaluate $\int \frac{1}{x^2-7x+10}dx$;
  Factor the denominator: $x^2-7x+10=(x-2)(x-5)$;
  Decompose into partial fractions:
  $\frac{1}{x^2-7x+10}=\frac{A}{x-2}+\frac{B}{x-5} \quad \Rightarrow \quad \frac{(x-2)(x-5)}{x^2-7x+10}=$
  $\frac{A(x-2)(x-5)}{x-2}+\frac{B(x-2)(x-5)}{x-5} \quad \Rightarrow \quad 1=$
  $A(x-5)+B(x-2) \quad \Rightarrow \quad 1=(A+B)x+(-5A-2B) \quad \Rightarrow$
  $\left\{\begin{array}{rcl} A+B & = & 0 \\ -5A-2B & = & 1 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{rcl} A & = & -B \\ 5B-2B & = & 1 \end{array}\right\} \quad \Rightarrow$
  $\left\{\begin{array}{rcl} A & = & -\frac{1}{3} \\ B & = & \frac{1}{3} \end{array}\right\}$; So we get
  $$\frac{1}{x^2-7x+10}=\frac{-\frac{1}{3}}{x-2}+\frac{\frac{1}{3}}{x-5};$$

Distinct Linear Factors I (Cont'd)

We obtained

$$\frac{1}{x^2-7x+10} = \frac{-\frac{1}{3}}{x-2} + \frac{\frac{1}{3}}{x-5};$$

So, we have

$$\begin{aligned}
\int \frac{1}{x^2-7x+10}dx &= \int \left[\frac{-\frac{1}{3}}{x-2} + \frac{\frac{1}{3}}{x-5}\right] dx \\
&= -\frac{1}{3}\int \frac{1}{x-2}dx + \frac{1}{3}\int \frac{1}{x-5}dx \\
&= -\frac{1}{3}\ln|x-2| + \frac{1}{3}\ln|x-5| + C;
\end{aligned}$$

Distinct Linear Factors II

- Evaluate $\int \frac{x^2+2}{(x-1)(2x-8)(x+2)}dx$;

$$\frac{x^2+2}{(x-1)(2x-8)(x+2)} = \frac{A}{x-1} + \frac{B}{2x-8} + \frac{C}{x+2} \quad \Rightarrow$$

$$\frac{(x-1)(2x-8)(x+2)(x^2+2)}{(x-1)(2x-8)(x+2)} = \frac{A(x-1)(2x-8)(x+2)}{x-1} +$$

$$\frac{B(x-1)(2x-8)(x+2)}{2x-8} + \frac{C(x-1)(2x-8)(x+2)}{x+2} \quad \Rightarrow$$

$x^2+2 = A(2x-8)(x+2) + B(x-1)(x+2) + C(x-1)(2x-8)$;

Now, we get:

  - $x = 1 \Rightarrow 3 = A \cdot (-6) \cdot 3 \Rightarrow A = -\frac{1}{6}$;
  - $x = 4 \Rightarrow 18 = B \cdot 3 \cdot 6 \Rightarrow B = 1$;
  - $x = -2 \Rightarrow 6 = C \cdot (-3) \cdot (-12) \Rightarrow C = \frac{1}{6}$;

Therefore, we obtain

$$\frac{x^2+2}{(x-1)(2x-8)(x+2)} = \frac{-\frac{1}{6}}{x-1} + \frac{1}{2x-8} + \frac{\frac{1}{6}}{x+2};$$

Distinct Linear Factors II (Cont'd)

We obtained

$$\frac{x^2+2}{(x-1)(2x-8)(x+2)} = \frac{-\frac{1}{6}}{x-1} + \frac{1}{2x-8} + \frac{\frac{1}{6}}{x+2};$$

Now, we integrate $\displaystyle\int \frac{x^2+2}{(x-1)(2x-8)(x+2)} dx$

$$\begin{aligned}
&= \int \left[\frac{-\frac{1}{6}}{x-1} + \frac{1}{2x-8} + \frac{\frac{1}{6}}{x+2} \right] dx \\
&= -\frac{1}{6} \int \frac{1}{x-1} dx + \int \frac{1}{2x-8} dx + + \frac{1}{6} \int \frac{1}{x+2} dx \\
&= -\frac{1}{6} \ln|x-1| + \frac{1}{2} \ln|2x-8| + \frac{1}{6} \ln|x+2| + C;
\end{aligned}$$

Long Division First...

- Evaluate $\int \frac{x^3+1}{x^2-4} dx$;
  Numerator has higher degree than denominator!
  Start by performing the long division $(x^3+1) \div (x^2-4)$;

$$
\begin{array}{r|l}
 & \quad x \\
\cline{2-2}
x^2-4 & x^3 \qquad\qquad\quad +1 \\
 & \underline{x^3 \quad -4x \quad} \\
 & \qquad\quad 4x \quad +1
\end{array}
$$

  It has quotient x and remainder $4x+1$; Thus,
  $\frac{x^3+1}{x^2-4} = x + \frac{4x+1}{x^2-4} = x + \frac{4x+1}{(x-2)(x+2)}$;

...Breaking Into Partial Fractions Next...

- We found $\frac{x^3+1}{x^2-4} = x + \frac{4x+1}{(x-2)(x+2)}$.

  Decompose the second fraction:
  $\frac{4x+1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \quad \Rightarrow \quad \frac{(4x+1)(x-2)(x+2)}{(x-2)(x+2)} = \frac{A(x-2)(x+2)}{x-2} + \frac{B(x-2)(x+2)}{x+2} \quad \Rightarrow \quad 4x+1 = A(x+2) + B(x-2)$;
  - $x = 2 \Rightarrow 9 = 4A \Rightarrow A = \frac{9}{4}$;
  - $x = -2 \Rightarrow -7 = -4B \Rightarrow B = \frac{7}{4}$;

  This gives $\frac{x^3+1}{x^2-4} = x + \frac{\frac{9}{4}}{x-2} + \frac{\frac{7}{4}}{x+2}$;

...and Integrating

We got

$$\frac{x^3+1}{x^2-4} = x + \frac{\frac{9}{4}}{x-2} + \frac{\frac{7}{4}}{x+2};$$

Hence, we have

$$\begin{aligned}
\int \frac{x^3+1}{x^2-4}dx &= \int \left[x + \frac{\frac{9}{4}}{x-2} + \frac{\frac{7}{4}}{x+2}\right] dx \\
&= \int x dx + \frac{9}{4}\int \frac{1}{x-2}dx + \frac{7}{4}\int \frac{1}{x+2}dx \\
&= \frac{1}{2}x^2 + \frac{9}{4}\ln|x-2| + \frac{7}{4}\ln|x+2| + C;
\end{aligned}$$

Repeated Linear Factors

- Evaluate $\int \frac{3x-9}{(x-1)(x+2)^2}dx$;

  Decompose into partial fractions:

  $$\frac{3x-9}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \quad \Rightarrow$$

  $$\frac{(3x-9)(x-1)(x+2)^2}{(x-1)(x+2)^2} =$$

  $$\frac{A(x-1)(x+2)^2}{x-1} + \frac{B(x-1)(x+2)^2}{x+2} + \frac{C(x-1)(x+2)^2}{(x+2)^2} \quad \Rightarrow$$

  $3x-9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$;
  - $x=1 \Rightarrow -6 = 9A \Rightarrow A = -\frac{2}{3}$;
  - $x=-2 \Rightarrow -15 = -3C \Rightarrow C = 5$;
  - $x=0 \Rightarrow -9 = 4A - 2B - C \Rightarrow B = \frac{4A-C+9}{2} = \frac{2}{3}$;

  So we get

  $$\frac{3x-9}{(x-1)(x+2)^2} = \frac{-\frac{2}{3}}{x-1} + \frac{\frac{2}{3}}{x+2} + \frac{5}{(x+2)^2};$$

Repeated Linear Factors (Cont'd)

We just got

$$\frac{3x-9}{(x-1)(x+2)^2} = \frac{-\frac{2}{3}}{x-1} + \frac{\frac{2}{3}}{x+2} + \frac{5}{(x+2)^2};$$

Now, we integrate $\displaystyle\int \frac{3x-9}{(x-1)(x+2)^2}dx$

$$\begin{aligned}
&= \int \left[\frac{-\frac{2}{3}}{x-1} + \frac{\frac{2}{3}}{x+2} + \frac{5}{(x+2)^2} \right] dx \\
&= -\frac{2}{3}\int \frac{1}{x-1}dx + \frac{2}{3}\int \frac{1}{x+2}dx + 5\int \frac{1}{(x+2)^2}dx \\
&= -\frac{2}{3}\ln|x-1| + \frac{2}{3}\ln|x+2| - \frac{5}{x+2} + C;
\end{aligned}$$

Irreducible Quadratic Factors

- Evaluate $\int \frac{18}{(x+3)(x^2+9)} dx$;
  Decompose the fraction
  $\frac{18}{(x+3)(x^2+9)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+9} \quad \Rightarrow \quad \frac{18(x+3)(x^2+9)}{(x+3)(x^2+9)} = \frac{A(x+3)(x^2+9)}{x+3} + \frac{(Bx+C)(x+3)(x^2+9)}{x^2+9} \quad \Rightarrow \quad 18 = A(x^2+9) + (Bx+C)(x+3)$;
  - $x = -3 \Rightarrow 18 = 18A \Rightarrow A = 1$;
  - $x = 0 \Rightarrow 18 = 9 + 3C \Rightarrow 3C = 9 \Rightarrow C = 3$;
  - $x = 1 \Rightarrow 18 = 10 + (B+3) \cdot 4 \Rightarrow 8 = 4B + 12 \Rightarrow B = -1$;

  Therefore $\frac{18}{(x+3)(x^2+9)} = \frac{1}{x+3} + \frac{-x+3}{x^2+9}$;

Irreducible Quadratic Factors (Cont'd)

We found that

$$\frac{18}{(x+3)(x^2+9)} = \frac{1}{x+3} + \frac{-x+3}{x^2+9};$$

Now we integrate: $\displaystyle\int \frac{18}{(x+3)(x^2+9)}dx$

$$\begin{aligned}
&= \int \left[\frac{1}{x+3} + \frac{-x+3}{x^2+9}\right] dx \\
&= \int \frac{1}{x+3}dx + \int \frac{-x+3}{x^2+9}dx \\
&= \int \frac{1}{x+3}dx - \int \frac{x}{x^2+9}dx + 3\int \frac{1}{x^2+9}dx \\
&= \ln|x+3| - \frac{1}{2}\ln(x^2+9) + 3\cdot\frac{1}{3}\tan^{-1}\frac{x}{3} + C \\
&= \ln|x+3| - \frac{1}{2}\ln(x^2+9) + \tan^{-1}\frac{x}{3} + C;
\end{aligned}$$

Repeated Quadratic Factors

- Evaluate $\int \frac{4-x}{x(x^2+2)^2}dx$;

Decompose the fraction $\frac{4-x}{x(x^2+2)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2} \quad \Rightarrow$

$A = 1, B = -1, C = 0, D = -2, E = -1$; So, we get

$\frac{4-x}{x(x^2+2)^2} = \frac{1}{x} + \frac{-x}{x^2+2} + \frac{-2x-1}{(x^2+2)^2}$; Integrating, we get

$$\begin{aligned}
\int \frac{4-x}{x(x^2+2)^2}dx \\
&= \int \left[\frac{1}{x} + \frac{-x}{x^2+2} + \frac{-2x-1}{(x^2+2)^2}\right] dx \\
&= \int \frac{1}{x}dx - \int \frac{x}{x^2+2}dx - \int \frac{2x}{(x^2+2)^2}dx - \int \frac{1}{(x^2+2)^2}dx \\
&= \ln|x| - \frac{1}{2}\ln(x^2+2) - \frac{-1}{x^2+2} - \int \frac{1}{(x^2+2)^2}dx;
\end{aligned}$$

The Integral $\int \frac{1}{(x^2+2)^2} dx$

Set $x = \sqrt{2}\tan\theta$; Then $dx = \sqrt{2}\sec^2\theta d\theta,\ x^2+2 = 2\tan^2\theta + 2 = 2\sec^2\theta,\ \theta = \tan^{-1}\frac{x}{\sqrt{2}},\ \sin\theta = \frac{x}{\sqrt{x^2+2}},\ \cos\theta = \frac{\sqrt{2}}{\sqrt{x^2+2}}$;

$$
\begin{aligned}
&\int \frac{1}{(x^2+2)^2} dx = \int \frac{1}{4\sec^4\theta}\sqrt{2}\sec^2\theta d\theta = \\
&\frac{\sqrt{2}}{4}\int \cos^2\theta d\theta = \frac{\sqrt{2}}{8}\int (1+\cos 2\theta) d\theta = \\
&\frac{\sqrt{2}}{8}(\theta + \frac{1}{2}\sin 2\theta) + C = \frac{\sqrt{2}}{8}\theta + \frac{\sqrt{2}}{8}\sin\theta\cos\theta + C = \\
&\frac{\sqrt{2}}{8}\tan^{-1}\frac{x}{\sqrt{2}} + \frac{\sqrt{2}}{8}\frac{x}{\sqrt{x^2+2}}\frac{\sqrt{2}}{\sqrt{x^2+2}} + C = \\
&\frac{\sqrt{2}}{8}\tan^{-1}\frac{x}{\sqrt{2}} + \frac{x}{4(x^2+2)} + C;
\end{aligned}
$$