# Mathematical Analysis II

**conf. univ., dr. Elena COJUHARI**

*elena.cojuhari@mate.utm.md*

Technical University of Moldova

2022

# Elementary Differential Equations

# General Framework

- We deal with first-order differential equations $\dfrac{dy}{dt} = f(t, y)$, where $f$ is a given function of two variables;
- Any differentiable function $y = \phi(t)$ that satisfies this equation for all t in some interval is called a **solution**;
- We want to determine whether such functions exist and, if so, to develop methods for finding them;
- For an arbitrary function $f$, there is no general method for solving the equation in terms of elementary functions;
- So we focus on special types of first order equations:
  - Linear Equations;
  - Separable Equations;
  - Exact Equations;

Bernoulli eq. (see Stewart, ex.23, p.621 )

homogeneous dif.eq.

## Cauchy's Theorem $(\exists \ \& \ \exists! \text{ of the sol. of } y' = f(x,y))$

If the function f(x,y) and its partial derivative $f'_y(x,y)$ are defined and continuous in a domain D of the xy-plane, then $\forall (x_0, y_0) \in D \ \exists! \text{ sol.}$ in some vecinity of this interior point, and this solution satisfies the conditions

$$y = y_0 \text{ for } x = x_0 .$$

the initial condition

$$y(x_0) = y_0 \quad \text{or} \quad y\big|_{x=x_0} = y_0$$

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

Cauchy's problem

the initial-value problem

Subsection 1

## Linear Equations; Method of Integrating Factors

# Linear Equations

- If the function $f$ in $\frac{dy}{dt} = f(t, y)$ depends linearly on the dependent variable $y$, then the equation is called a **first order linear equation**;
- A typical example is
$$\frac{dy}{dt} = -ay + b,$$
where $a, b$ are constants;
- We consider a more general first order linear equation, obtained by replacing the coefficients $a$ and $b$ by arbitrary functions of $t$;
- The **general first order linear equation in the standard form** is
$$\frac{dy}{dt} + p(t)y = g(t),$$
where $p$ and $g$ are given functions of the independent variable $t$;

# Solving $\frac{dy}{dt} = -ay + b$ by Integrating

We work as follows:

$$\frac{dy}{dt} = -ay + b \quad \overset{a\neq 0}{\Rightarrow} \quad \frac{dy}{dt} = -a\left(y - \frac{b}{a}\right)$$

$$\overset{y\neq \frac{b}{a}}{\Rightarrow} \quad \frac{dy}{y - \frac{b}{a}} = -adt \quad \Rightarrow \quad \int \frac{dy}{y - \frac{b}{a}} = \int -adt$$

$$\Rightarrow \quad \ln\left|y - \frac{b}{a}\right| = -at + C \quad \Rightarrow \quad \left|y - \frac{b}{a}\right| = e^C e^{-at}$$

$$\overset{y > \frac{b}{a}}{\Rightarrow} y = \frac{b}{a} + ce^{-at};$$

# Leibniz's Integrating Factor Method: An Example

- Solve the differential equation $\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}$;
  Multiply both sides by a function $\mu(t)$, as yet undetermined:

$$\mu(t)\frac{dy}{dt} + \frac{1}{2}\mu(t)y = \frac{1}{2}\mu(t)e^{t/3};$$

  Can we choose $\mu(t)$ so that the left side is recognizable as the derivative of some particular expression?
  Note that, by the product rule

$$\frac{d}{dt}[\mu(t)y] = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y;$$

  Thus, we need to choose

$$\frac{d\mu(t)}{dt} = \frac{1}{2}\mu(t);$$

# Example (Cont'd)

- We want to solve the differential equation $\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}$; We multiplied by $\mu(t)$: $\mu(t)\frac{dy}{dt} + \frac{1}{2}\mu(t)y = \frac{1}{2}\mu(t)e^{t/3}$; We found $\frac{d\mu(t)}{dt} = \frac{1}{2}\mu(t)$;

$$\begin{aligned} \frac{\frac{d\mu(t)}{dt}}{\mu(t)} = \frac{1}{2} \quad &\Rightarrow \quad \frac{d}{dt}\ln|\mu(t)| = \frac{1}{2} \\ &\Rightarrow \quad \ln|\mu(t)| = \frac{1}{2}t + C \quad \Rightarrow \quad \mu(t) = ce^{t/2}; \end{aligned}$$

Now, with $c = 1$, we obtain

$$\begin{aligned} e^{t/2}\frac{dy}{dt} + \frac{1}{2}e^{t/2}y = \frac{1}{2}e^{5t/6} \quad &\Rightarrow \quad \frac{d}{dt}(e^{t/2}y) = \frac{1}{2}e^{5t/6} \\ \Rightarrow \quad e^{t/2}y = \frac{1}{2}\frac{6}{5}e^{5t/6} + c \quad &\Rightarrow \quad y = \frac{3}{5}e^{t/3} + ce^{-t/2}; \end{aligned}$$

# The Integrating Factor Method

$$\frac{dy}{dt} + ay = g(t)$$

Multiply by the integrating factor $\mu(t) = e^{at}$:

$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}g(t)$$

$$\frac{d}{dt}[e^{at}y] = e^{at}g(t)$$

$$e^{at}y = \int e^{at}g(t)dt + c$$

$$y = e^{-at}\int e^{at}g(t)dt + ce^{-at};$$

or, if not possible to integrate explicitly,

$$y = e^{-at}\int_{t_0}^{t} e^{as}g(s)ds + ce^{-at}.$$

# Applying the Method to An Example

- Solve the differential equation $\frac{dy}{dt} - 2y = 4 - t$;
  First a reminder:

$$\int te^{-2t}dt = \int t(-\tfrac{1}{2}e^{-2t})'dt =$$
$$-\tfrac{1}{2}te^{-2t} - \int -\tfrac{1}{2}e^{-2t}dt = -\tfrac{1}{2}te^{-2t} - \tfrac{1}{4}e^{-2t} + c;$$

Now we start the main work:

$(e^{-2t})'$

$\times e^{-2t}$

$$\begin{aligned}
\frac{dy}{dt} - 2y = 4 - t \quad &\Rightarrow \quad e^{-2t}\frac{dy}{dt} - 2e^{-2t}y = (4-t)e^{-2t} \\
&\Rightarrow \quad \frac{d}{dt}[e^{-2t}y] = (4-t)e^{-2t} \\
&\Rightarrow \quad e^{-2t}y = \int 4e^{-2t}dt - \int te^{-2t}dt \\
&\Rightarrow \quad e^{-2t}y = -2e^{-2t} + \tfrac{1}{2}te^{-2t} + \tfrac{1}{4}e^{-2t} + c \\
&\Rightarrow \quad e^{-2t}y = \tfrac{1}{2}te^{-2t} - \tfrac{7}{4}e^{-2t} + c \\
&\Rightarrow \quad y = \tfrac{1}{2}t - \tfrac{7}{4} + ce^{2t};
\end{aligned}$$

# Integrating Factor Method: The General Case

$I'(t) = \left(e^{\int p(t)dt}\right)' = p(t)e^{\int p(t)dt}$

$$\frac{dy}{dt} + p(t)y = g(t) \qquad I'(t)$$

$$I(t)\; e^{\int p(t)dt}\frac{dy}{dt} + p(t)e^{\int p(t)dt}y = e^{\int p(t)dt}g(t)$$

$$\frac{d}{dt}[e^{\int p(t)dt}y] = e^{\int p(t)dt}g(t)$$

$$e^{\int p(t)dt}y = \int e^{\int p(t)dt}g(t)dt + c$$

$$y = e^{-\int p(t)dt}\left[\int e^{\int p(t)dt}g(t)dt + c\right]$$

or, if not possible to integrate explicitly,

$$y = e^{-\int p(t)dt}\left[\int_{t_0}^{t} e^{\int p(s)ds}g(s)ds + c\right];$$

# Example I

- Solve the initial value problem

$$t\frac{dy}{dt} + 2y = 4t^2, \quad y(1) = 2, \quad \text{for } t > 0;$$

$t\frac{dy}{dt} + 2y = 4t^2 \Rightarrow \frac{dy}{dt} + \frac{2}{t}y = 4t;$ We compute the integrating factor: $\mu(t) = e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln(t^2)} = t^2;$ We start work on the equation:

$$\begin{aligned} \frac{dy}{dt} + \frac{2}{t}y = 4t \quad &\Rightarrow \quad t^2\frac{dy}{dt} + 2ty = 4t^3 \\ \Rightarrow \quad \frac{d}{dt}[t^2 y] = 4t^3 \quad &\Rightarrow \quad t^2 y = t^4 + c \\ \Rightarrow \quad y = t^2 + \frac{c}{t^2}; & \end{aligned}$$

Finally we find the particular solution based on the given initial condition:

$$y(1) = 2 \quad \Rightarrow \quad 1 + c = 2 \quad \Rightarrow \quad c = 1;$$

So the particular solution is $y = t^2 + \frac{1}{t^2}, t > 0;$ ✓

# Example II

- Solve the initial value problem

$$2y' + ty = 2, \quad y(0) = 1;$$

$2y' + ty = 2 \Rightarrow y' + \frac{t}{2}y = 1;$ We compute the integrating factor: $\mu(t) = e^{\int \frac{1}{2} t dt} = e^{\frac{1}{4}t^2};$ We start work on the equation:

$$\begin{aligned} y' + \tfrac{t}{2}y = 1 \quad &\Rightarrow \quad e^{\frac{1}{4}t^2}y' + \tfrac{t}{2}e^{\frac{1}{4}t^2}y = e^{\frac{1}{4}t^2} \\ &\Rightarrow \quad \tfrac{d}{dt}[e^{\frac{1}{4}t^2}y] = e^{\frac{1}{4}t^2} \quad \Rightarrow \quad e^{\frac{1}{4}t^2}y = \int e^{\frac{1}{4}t^2}dt + c \\ &\Rightarrow \quad y = e^{-\frac{1}{4}t^2}\left[\int_0^t e^{\frac{1}{4}s^2}ds + c\right]; \end{aligned}$$

Finally we find the particular solution based on the given initial condition: $y(0) = 1 \quad \Rightarrow \quad c = 1;$ So the particular solution is $y = e^{-\frac{1}{4}t^2}\left[\int_0^t e^{\frac{1}{4}s^2}ds + 1\right]$

# The Spread of an Infectious Disease

https://en.wikipedia.org/wiki/COVID-19_pandemic_on_Diamond_Princess

Suppose we have a constant population of N individuals [Diamond Princess, winter 2020]
and at time t the number of infected members is P(t)
uninfected N-P(t)

A reasonable assumption is that the rate of spread of the disease at time t is propotional to the product of noninfected and infected individuals:

$$(*) \quad \frac{dP}{dt} = k(N-P)P, \qquad \text{where k is the constant of proportionality}$$

Ex.

the initial-value problem : $P(0)=P_0$

$(*)$ has two equilibrium solutions:

P=0 If noone is infected
P=N if everyone is infected

---

at time t=0, when 100 000 members of a population of 500 000 are known to be infected, medical authorities intervene with medical treatment.
As a consequence of intervention, the rate factor k is no longer constant: k(t) $= 2e^{-t} - 1$,
where time t is measured in month and k(t) represents the rate of infection per month per 100 000 individiuals.

Subsection 2

## Separable Equations

$$y' = f(x)\,g(y) \quad \text{or} \quad M(x)N(y)\,dx + P(x)Q(y)\,dy = 0$$

where

f, M, P are some cont. f. of x
g, N, Q are some cont. f. of y

# Separable Equations

$$\frac{dy}{dx} = f(x,y) \quad \text{is a special case of} \quad M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

Just take $M(x,y) = -f(x,y),\ N(x,y) = 1;$

Assume that $M(x,y) = M(x)$ is a function of $x$ only and that $N(x,y) = N(y)$ is a function of $y$ only; Then

$$M(x) + N(y)\frac{dy}{dx} = 0$$

$$N(y)\frac{dy}{dx} = -M(x)$$

$$N(y)dy = -M(x)dx;$$

Because $x, y$ can be separated in either side of the equation $M(x) + N(y)\dfrac{dy}{dx} = 0$ is called a **separable differential equation**;

# Solving a Separable Equation: Example

- Find the general solution of the separable differential equation

$$\frac{dy}{dx} = \frac{x^2}{1-y^2};$$

$$\begin{aligned} \frac{dy}{dx} = \frac{x^2}{1-y^2} \quad & \Rightarrow \quad (1-y^2)dy = x^2 dx \\ & \Rightarrow \quad \int (1-y^2)dy = \int x^2 dx \\ & \Rightarrow \quad y - \frac{1}{3}y^3 = \frac{1}{3}x^3 + C; \end{aligned}$$

# The General Separable Equation

Consider the separable differential equation $M(x) + N(y)\dfrac{dy}{dx} = 0$;
Assume that we are able to find $H_1(x)$ and $H_2(y)$, such that

$$\int M(x)dx = H_1(x), \qquad \int N(y)dy = H_2(y);$$

Then, we get

$$N(y)dy = -M(x)dx$$

which yields

$$\int N(y)dy = -\int M(x)dx$$

and, therefore,

$$H_2(y) = -H_1(y) + c, \quad \text{for some constant } c;$$

# Solving a Separable Equation I

- Solve the separable equation $\dfrac{dy}{dx} = \dfrac{3x^2 + 4x + 2}{2(y-1)}$, subject to the initial condition $y(0) = -1$;

$$\begin{aligned} \frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)} & \\ \Rightarrow \quad & (2y-2)dy = (3x^2 + 4x + 2)dx \\ \Rightarrow \quad & \int (2y-2)dy = \int (3x^2 + 4x + 2)dx \\ \Rightarrow \quad & y^2 - 2y = x^3 + 2x^2 + 2x + c; \qquad \textbf{the general solution} \end{aligned}$$

For the particular solution:
$y(0) = -1 \Rightarrow (-1)^2 - 2(-1) = 0 + c \Rightarrow c = 3;$ Therefore, we obtain

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3; \qquad \textbf{the particular sol.}$$

# Solving a Separable Equation II

- Solve the separable equation $\dfrac{dy}{dx} = \dfrac{4x - x^3}{4 + y^3}$; Find the solution curve passing through the point $(0, 1)$;

$$\begin{aligned} \frac{dy}{dx} = \frac{4x - x^3}{4 + y^3} \\ \Rightarrow \quad & (4 + y^3)dy = (4x - x^3)dx \\ \Rightarrow \quad & \int (4 + y^3)dy = \int (4x - x^3)dx \\ \Rightarrow \quad & 4y + \frac{1}{4}y^4 = 2x^2 - \frac{1}{4}x^4 + c; \end{aligned}$$

For the particular solution:

$y(0) = 1 \Rightarrow 4 \cdot 1 + \dfrac{1}{4} \cdot 1^4 = 0 + c \Rightarrow c = \frac{17}{4}$; Therefore, we obtain

$$4y + \frac{1}{4}y^4 = 2x^2 - \frac{1}{4}x^4 + \frac{17}{4} \quad \Rightarrow \quad y^4 + 16y + x^4 - 8x^2 = 17;$$

Ex. $x(y^2-4)\,dx + y\,dy = 0$ $\quad : y^2-4 \neq 0$

$$x\,dx + \frac{y}{y^2-4}\,dy = 0$$

$$x^2 + \ln|y^2-4| = C$$

$$\ln|y^2-4| = C - x^2$$

$$|y^2-4| = e^{c-x^2}$$

$$y^2-4 = \pm e^{c} \cdot e^{-x^2}$$

$y = \pm 2$

$$y^2-4 = C \cdot e^{-x^2}, \quad C \in \mathbb{R}$$

Ex. $^{88}Ra$

$t=0 \quad m_0$

$k>0$

$$\frac{dm}{dt} = -km$$

$$\frac{dm}{m} = -k\,dt$$

$$\ln m = -kt + C$$

$$m = C \cdot e^{-kt}$$

$$m = m_0 e^{-kt}$$

$k$ experimental observation

$Ra \quad k = 0.000436$

$$T = \frac{\ln 2}{k}$$

1600 years

Subsection 3 **Ind. reading**

# Modeling with First Order Equations

# Application: Mixing

At time $t = 0$ a tank contains $Q_0$ lb of salt dissolved in 100 gallons of water; Water containing $\frac{1}{4}$ lb of salt/gal is entering the tank at a rate of $r$ gal/min and the mixture is draining from the tank at the same rate; Set up the initial value problem that describes this flow process and find the amount of salt $Q(t)$ in the tank at time $t$;


$$\frac{dQ}{dt} = \text{rate in} - \text{rate out} \quad \Rightarrow \quad \frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100} \quad \text{and} \quad Q(0) = Q_0;$$

$$\frac{dQ}{dt} + \frac{r}{100}Q = \frac{r}{4} \quad \Rightarrow \quad e^{\frac{r}{100}t}\frac{dQ}{dt} + \frac{r}{100}e^{\frac{r}{100}t}Q = \frac{r}{4}e^{\frac{r}{100}t}$$

$$\Rightarrow \quad \frac{d}{dt}(e^{\frac{r}{100}t}Q) = \frac{r}{4}e^{\frac{r}{100}t} \quad \Rightarrow \quad e^{\frac{r}{100}t}Q = 25e^{\frac{r}{100}t} + c$$

$$\Rightarrow \quad Q = 25 + ce^{-\frac{r}{100}t};$$

Now $Q(0) = Q_0 \Rightarrow 25 + c = Q_0 \Rightarrow c = Q_0 - 25$; Therefore $Q(t) = 25 + (Q_0 - 25)e^{-\frac{r}{100}t}$;

# Application: Compound Interest

Suppose that a sum of money $S_0$ is deposited in an account that pays interest at an annual rate $r$; Assume that compounding takes place continuously; Set up a simple initial value problem that describes the value $S(t)$ of the investment at time $t$.

$$\begin{aligned}
&\tfrac{dS}{dt} = rS \quad \text{and} \quad S(0) = S_0; \\
&\tfrac{dS}{dt} - rS = 0 \quad \Rightarrow \quad e^{-rt}\tfrac{dS}{dt} - re^{-rt}S = 0 \\
&\quad \Rightarrow \quad \tfrac{d}{dt}(e^{-rt}S) = 0 \quad \Rightarrow \quad e^{-rt}S = c \\
&\quad \Rightarrow \quad S = ce^{rt};
\end{aligned}$$

Now $S(0) = S_0 \Rightarrow c = S_0$; Therefore, $S(t) = S_0 e^{rt}$;

# Reviewing By-Parts Integration

Compute the integral $\int e^{t/2} \sin 2t dt$;

$$\begin{aligned}
\int e^{t/2} \sin 2t dt &= \int (2e^{t/2})' \sin 2t dt \\
&= 2e^{t/2} \sin 2t - \int 4e^{t/2} \cos 2t dt \\
&= 2e^{t/2} \sin 2t - \int (8e^{t/2})' \cos 2t dt \\
&= 2e^{t/2} \sin 2t - 8e^{t/2} \cos 2t - \int 16e^{t/2} \sin 2t dt;
\end{aligned}$$

Therefore,

$$17 \int e^{t/2} \sin 2t dt = 2e^{t/2} \sin 2t - 8e^{t/2} \cos 2t$$

$$\int e^{t/2} \sin 2t dt = \tfrac{2}{17} e^{t/2} \sin 2t - \tfrac{8}{17} e^{t/2} \cos 2t + C;$$

# Application: Chemicals in a Pond

Consider a pond that initially contains 10 million gal of fresh water; Water containing a chemical flows into the pond at the rate of 5 million gal/year, and the mixture in the pond flows out at the same rate; The concentration $\gamma(t)$ of chemical in the incoming water varies periodically with time according to the expression $\gamma(t) = 2 + \sin 2t$ grams/gal; Construct a mathematical model of this flow process and determine the amount $Q(t)$ of chemical in the pond at time $t$;

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out} = 5 \cdot 10^6 (2 + \sin 2t) - 5 \cdot 10^6 \frac{Q}{10^7}$$

$$= 10^7 + 5 \cdot 10^6 \sin 2t - \frac{1}{2} Q \quad \text{and} \quad Q_0 = 0;$$

$$\frac{dQ}{dt} + \frac{1}{2} Q = 10^7 + 5 \cdot 10^6 \sin 2t;$$

# Chemicals in a Pond (Cont'd)

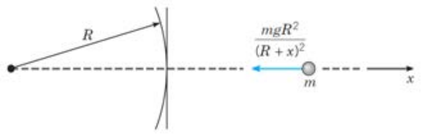
$$\frac{dQ}{dt} + \frac{1}{2}Q = 10^7 + 5\cdot 10^6 \sin 2t$$
$$\Rightarrow \quad e^{t/2}\frac{dQ}{dt} + \frac{1}{2}e^{t/2}Q = (10^7 + 5\cdot 10^6 \sin 2t)e^{t/2}$$
$$\Rightarrow \quad \frac{d}{dt}(e^{t/2}Q) = 10^7 e^{t/2} + 5\cdot 10^6 e^{t/2}\sin 2t$$
$$\Rightarrow \quad e^{t/2}Q = 2\cdot 10^7 e^{t/2} + \frac{10^7}{17}e^{t/2}\sin 2t - \frac{4\cdot 10^7}{17}e^{t/2}\cos 2t + c$$
$$\Rightarrow \quad Q = 2\cdot 10^7 + \frac{10^7}{17}\sin 2t - \frac{4\cdot 10^7}{17}\cos 2t + ce^{-t/2};$$

Now $Q(0) = 0 \Rightarrow 2\cdot 10^7 - \frac{4\cdot 10^7}{17} + c = 0 \Rightarrow c = -\frac{30}{17}\cdot 10^7$; Therefore,

$$Q(t) = 2\cdot 10^7 + \frac{10^7}{17}\sin 2t - \frac{4\cdot 10^7}{17}\cos 2t - \frac{30}{17}\cdot 10^7 e^{-t/2};$$

# Application: Velocity and Gravitation

A body of constant mass $m$ is projected away from the earth in a direction perpendicular to the earth's surface with an initial velocity $v_0$; Assuming that there is no air resistance, but taking into account the variation of the earths gravitational field with distance, find an expression for the velocity during the ensuing motion;




The weight is inversely proportional to the square of the distance $R + x$ of the object from the center of the earth $w(x) = -\frac{k}{(R+x)^2}$;
Since on the surface of the earth, $w(0) = -mg$, we get that
$-\frac{k}{R^2} = -mg \Rightarrow k = mgR^2$; Therefore, $w(x) = -\frac{mgR^2}{(R+x)^2}$;
An application of Newton's Law Force = Mass × Acceleration, gives

$$m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}, \quad v(0) = v_0;$$

# Velocity and Gravitation (Cont'd)

$$
\begin{aligned}
& m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2} \quad \Rightarrow \quad \frac{dv}{dx}\frac{dx}{dt} = -\frac{gR^2}{(R+x)^2} \\
& \Rightarrow \quad v\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2} \quad \Rightarrow \quad \int vdv = \int -\frac{gR^2}{(R+x)^2}dx \\
& \Rightarrow \quad \frac{v^2}{2} = \frac{gR^2}{R+x} + c;
\end{aligned}
$$

Now, $v(0) = v_0$ and $x(0) = 0$ yield $\frac{v_0^2}{2} = \frac{gR^2}{R} + c \Rightarrow c = \frac{v_0^2}{2} - gR$;
Therefore, $\frac{v^2}{2} = \frac{gR^2}{R+x} + \frac{v_0^2}{2} - gR$ and, thus,

$$
v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}};
$$

Subsection 4

# Exact Equations and Integrating Factors

# Example of Solving an Exact Equation

- Solve the differential equation $2x + y^2 + 2xyy' = 0$;

  The function $\psi(x, y) = x^2 + xy^2$ is such that

  $$\frac{\partial \psi}{\partial x} = 2x + y^2 \quad \text{and} \quad \frac{\partial \psi}{\partial y} = 2xy;$$

  Therefore the differential equation can be written as

  $$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0;$$

  Assuming that $y$ is a function of $x$ and considering the chain rule, we obtain $\dfrac{d\psi}{dx} = \dfrac{d}{dx}(x^2 + xy^2) = 0$;
  Thus, $\psi(x, y) = x^2 + xy^2 = c$, where $c$ is an arbitrary constant, is an equation that defines the solutions of the given differential equation implicitly.

# General Form of Exact Equations

- Consider the differential equation

$$M(x,y) + N(x,y)y' = 0;$$

- Suppose that we can identify a function $\psi$, such that $\dfrac{\partial\psi}{\partial x}(x,y) = M(x,y)$, $\dfrac{\partial\psi}{\partial y}(x,y) = N(x,y)$ and $\psi(x,y) = c$ defines $y = \phi(x)$ implicitly as a differentiable function of $x$;
- Then

$$M(x,y) + N(x,y)y' = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}[\psi(x,\phi(x))]$$

- So the differential equation becomes $\dfrac{d}{dx}[\psi(x,\phi(x))] = 0$;
- In this case the equation is called an **exact differential equation**;
- Its solutions are given implicitly by $\psi(x,y) = c$, where c is an arbitrary constant;

# A Recognition Theorem for Exact Differential Equations

- For some equations it may not be possible to detect that they are exact very easily;
- The following theorem provides a systematic way of doing this:

**Theorem (Detection of Exactness)**

Let the functions $M, N, M_y$, and $N_x$, where subscripts denote partial derivatives, be continuous in the rectangular region $R: \alpha < x < \beta, \gamma < y < \delta$; Then $M(x, y) + N(x, y)y' = 0$ is an exact differential equation in $R$ if and only if $M_y(x, y) = N_x(x, y)$ at each point of $R$; That is, there exists a function $\psi$ satisfying

$$\psi_x(x, y) = M(x, y) \quad \text{and} \quad \psi_y(x, y) = N(x, y),$$

if and only if M and N satisfy $M_y(x, y) = N_x(x, y)$;

$\vec{F} = M(x,y)\vec{i} + N(x,y)\vec{j}$ should be conservative

# Example I

- Solve the differential equation
  $(y\cos x + 2xe^y) + (\sin x + x^2 e^y - 1)y' = 0$;

  Calculate $M_y$ and $N_x$: $M_y(x,y) = \cos x + 2xe^y$;
  $N_x(x,y) = \cos x + 2xe^y$; Therefore, $M_y(x,y) = N_x(x,y)$, i.e., the given equation is exact;
  Thus there exists a $\psi(x,y)$ such that

  $$\psi_x(x,y) = y\cos x + 2xe^y \quad \text{and} \quad \psi_y(x,y) = \sin x + x^2 e^y - 1;$$

  Integrating the first, we obtain $\psi(x,y) = y\sin x + x^2 e^y + h(y)$;
  Setting $\psi_y = N$ gives
  $\psi_y(x,y) = \sin x + x^2 e^y + h'(y) = \sin x + x^2 e^y - 1$; Thus $h'(y) = -1$ and $h(y) = -y$; (The constant of integration can be omitted;)
  Substituting for $h(y)$ gives $\psi(x,y) = y\sin x + x^2 e^y - y$;
  Hence the solutions are given implicitly by $y\sin x + x^2 e^y - y = c$;

# Example II

- Solve the differential equation $(3xy + y^2) + (x^2 + xy)y' = 0$;

  We get $M_y(x, y) = 3x + 2y$; $N_x(x, y) = 2x + y$; Since $M_y \neq N_x$, the given equation is not exact;

  To see that it cannot be solved by the procedure described above, let us seek a function $\psi$, such that $\psi_x(x, y) = 3xy + y^2$ and $\psi_y(x, y) = x^2 + xy$;

  Integrating the first gives $\psi(x, y) = \frac{3}{2}x^2 y + xy^2 + h(y)$, where $h$ is an arbitrary function of y only; To try to satisfy the second, we compute $\psi_y$ and set it equal to $N$, obtaining $\frac{3}{2}x^2 + 2xy + h'(y) = x^2 + xy$ or $h'(y) = -\frac{1}{2}x^2 - xy$;

  Since the right side depends on $x$ as well as $y$, it is impossible to solve for $h(y)$; There is no $\psi(x, y)$ satisfying both partial derivative equations $\psi_x(x, y) = 3xy + y^2$ and $\psi_y(x, y) = x^2 + xy$;

# Integrating Factors: From Non-exact to Exact Equations

- Consider the equation $M(x, y)dx + N(x, y)dy = 0$;
- Multiply by a function $\mu$ and try to choose $\mu$ so that the resulting equation $\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$ be exact;
- For this to be exact, we need $(\mu M)_y = (\mu N)_x$;
- Thus, the integrating factor $\mu$ must satisfy the first order partial differential equation $M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$;
- If such a function $\mu$ can be found, then the original equation will be exact;
- The derived partial differential equation may have more than one solution; If this is the case, any such solution may be used as an integrating factor of the original equation;

# Case Where Simple Integrating Factors Exist

- Let us determine necessary conditions on $M$ and $N$ so that $M(x,y)dx + N(x,y)dy = 0$ has an integrating factor $\mu$ that depends on $x$ only;
- Assuming that $\mu$ is a function of x only, we have $(\mu M)_y = \mu M_y, (\mu N)_x = \mu N_x + N\frac{d\mu}{dx}$;
- Thus, for $(\mu M)_y = (\mu N)_x$, it is necessary that $\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$;
- If $\frac{M_y - N_x}{N}$ is a function of $x$ only, then there is an integrating factor $\mu$ that also depends only on $x$; Further, $\mu(x)$ can be found by solving $\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$, which is both linear and separable;
- A similar procedure can be used to determine a condition under which $M(x,y)dx + N(x,y)dy = 0$ has an integrating factor $\mu$ that depends on $y$ only;

# Example of Conversion into an Exact Equation

- Find an integrating factor for the equation $(3xy + y^2) + (x^2 + xy)y' = 0$ and then solve the equation;

  We have shown that this equation is not exact;

  Let us determine whether it has an integrating factor that depends on $x$ only;

  On computing the quantity $\frac{M_y - N_x}{N}$, we find that

$$\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{1}{x};$$

  Thus there is an integrating factor $\mu$ that is a function of $x$ only, and it satisfies the differential equation $\frac{d\mu}{dx} = \frac{\mu}{x}$; Hence $\mu(x) = x$;

  Multiplying the original by this integrating factor, we obtain

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0;$$

# Example of Conversion into an Exact Equation (Cont'd)

- We obtained

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0;$$

This equation is exact, since

$$\frac{\partial M}{\partial y} = \frac{\partial(3x^2y + xy^2)}{\partial y} = 3x^2 + 2xy = \frac{\partial(x^3 + x^2y)}{\partial x} = \frac{\partial N}{\partial x};$$

Moreover,

$$\psi(x, y) = \int (3x^2y + xy^2)dx = x^3y + \frac{1}{2}x^2y^2 + h(y);$$

Setting

$$x^3 + x^2y + h'(y) = x^3 + x^2y$$

we get $y'(y) = 0$; So we can take $h(y) = 0$;
Thus $h(x, y) = x^3y + \frac{1}{2}x^2y^2$;
So the solutions are given implicitly by $x^3y + \frac{1}{2}x^2y^2 = c$;

Homogeneous dif. eq. I $\quad y' = g\left(\frac{y}{x}\right)$ $\quad (1)$

g is a function of one variable

or $\quad M(x,y)\,dx + N(x,y)\,dy = 0$, where $M(x,y), N(x,y)$ are homogeneous fuction of the same degree

$y = f(x,y)$ is a homogeneous function of degree k in variables x and y, if $f(\alpha x, \alpha y) = \alpha^k f(x,y)$

for any $\alpha$.

1) $f(x,y) = x^2 - xy$

$f(\alpha x, \alpha y) = (\alpha x)^2 - \alpha x \cdot \alpha y = \alpha^2 (x^2 - xy) = \alpha^2 f(x,y)$

$k=2$

2) $f(x,y) = \frac{2x-3y}{x+y}$

$k=0$

3) $f(x,y) = xy + 1$ $\quad$ not hom.

by the substitution y = tx, the equation (1) will be reduced to the separable equation

$$y = tx \quad \text{or} \quad t = \frac{y}{x} \qquad y(x) \rightsquigarrow t(x)$$

$$y' = t'x + t$$

dy= xdt +tdx

## Equations reducible to homogeneous dif. eq.

$$y' = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right) \quad (2)$$

1) $c_1 = c_2 = 0$ (2) is H.D.E

2) $c_1^2 + c_2^2 \neq 0$ $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

the substitution $\begin{cases} x = x_0 + u \\ y = y_0 + v \end{cases}$, where $(x_0, y_0)$ is a sol. $\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$

(2) will be reduced to the homogeneous equation

3) $c_1^2 + c_2^2 \neq 0$ $\Delta = 0$ $z(x) = a_1x + b_1y$

(2) will be reduced to the separable equation