Calculus I

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1 Applications of the Integral

- Area Between Two Curves
- Setting Up Integrals: Volume, Density, Average Value
- Volumes of Revolution
- The Method of Cylindrical Shells
- Work and Energy

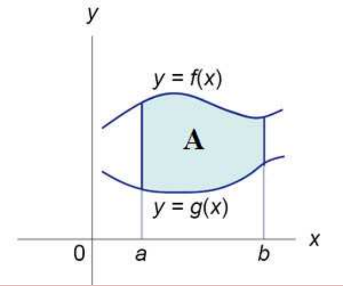
Subsection 1

Area Between Two Curves

Area Between Two Curves

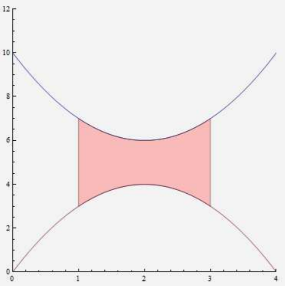
Consider two functions $y = f(x)$ and $y = g(x)$, such that $f(x) \geq g(x)$, for all x in $[a, b]$;
Then the area of the region between the two graphs is given by

$$\begin{aligned} A &= \int_a^b f(x)dx - \int_a^b g(x)dx \\ &= \int_a^b (f(x) - g(x))dx; \end{aligned}$$



Computing Area Between Two Curves

- Example: Find the area of the region bounded by the graphs of $f(x) = x^2 - 4x + 10$ and $g(x) = 4x - x^2$, for $1 \leq x \leq 3$;



By setting $f(x) = g(x)$, we find that the two graphs do not intersect. To discover which graph is the top graph, we compute $f(2) = 6 > 4 = g(2)$; Therefore f's graph is higher;
We get

$$A = \int_1^3 (f(x) - g(x))dx =$$

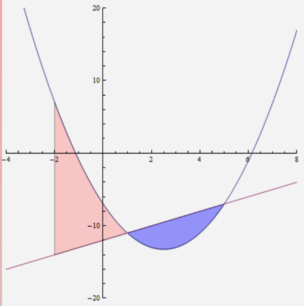
$$\int_1^3 ((x^2 - 4x + 10) - (4x - x^2))dx = \int_1^3 (2x^2 - 8x + 10)dx =$$

$$(\tfrac{2}{3}x^3 - 4x^2 + 10x)\Big|_1^3 = 12 - \frac{20}{3} = \frac{16}{3};$$

Intersecting Curves

- Find the area between the graphs of $f(x) = x^2 - 5x - 7$ and $g(x) = x - 12$ over $[-2, 5]$;

  To discover the points of intersection in $[-2, 5]$, set $f(x) = g(x) \Rightarrow x^2 - 5x - 7 = x - 12 \Rightarrow x^2 - 6x + 5 = 0 \Rightarrow (x-1)(x-5) = 0 \Rightarrow x = 1$ or $x = 5$; Thus, the two curves intersect at $x = 1$; Since $g(0) < f(0)$, f is higher on $[-2, 1]$ and, since $f(2) < g(2)$, g is higher on $[1, 5]$;



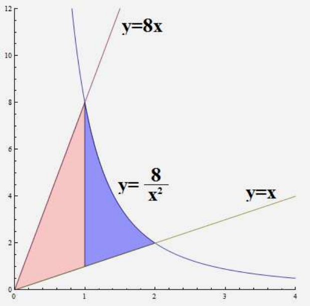
$$A = \int_{-2}^{1} ((x^2 - 5x - 7) - (x - 12))dx + \int_{1}^{5} ((x - 12) - (x^2 - 5x - 7))dx$$
$$= \int_{-2}^{1} (x^2 - 6x + 5)dx + \int_{1}^{5} (-x^2 + 6x - 5)dx = \left(\tfrac{1}{3}x^3 - 3x^2 + 5x\right)\Big|_{-2}^{1} +$$
$$\left(-\tfrac{1}{3}x^3 + 3x^2 - 5x\right)\Big|_{1}^{5} = \left(\tfrac{7}{3} - \tfrac{-74}{3}\right) + \left(\tfrac{25}{3} - \tfrac{-7}{3}\right) = \tfrac{113}{3};$$

Dividing the Region in Pieces

- Find the area of the region bounded by the graphs of $y = \dfrac{8}{x^2}, y = 8x$ and $y = x$;

  To discover the points of intersection, we set

  1. $\dfrac{8}{x^2} = 8x \Rightarrow x^3 = 1 \Rightarrow x = 1;$
  2. $\dfrac{8}{x^2} = x \Rightarrow x^3 = 8 \Rightarrow x = 2;$
  3. $8x = x \Rightarrow 7x = 0 \Rightarrow x = 0;$

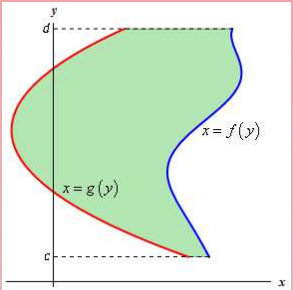


$$A = \int_0^1 (8x - x)dx + \int_1^2 \left(\frac{8}{x^2} - x\right)dx$$
$$= \int_0^1 7x dx + \int_1^2 (8x^{-2} - x)dx = \left(\tfrac{7}{2}x^2\right)\Big|_0^1 + \left(-\tfrac{8}{x} - \tfrac{1}{2}x^2\right)\Big|_1^2$$
$$= \left(\tfrac{7}{2} - 0\right) + \left(-6 - \left(-\tfrac{17}{2}\right)\right) = 6;$$

Area Between Two Curves Along the y-Axis

Consider two functions $x = f(y)$ and $x = g(y)$, such that $f(y) \geq g(y)$, for all y in $[c, d]$;
Then the area of the region between the two graphs is given by

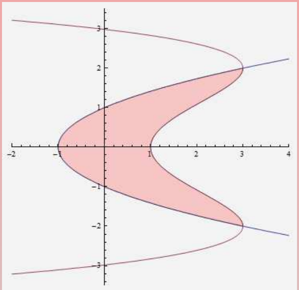
$$\begin{aligned} A &= \int_c^d f(y)dy - \int_c^d g(y)dy \\ &= \int_c^d (f(y) - g(y))dy; \end{aligned}$$



Computing an Area Along y-Axis

- Calculate the area of the region enclosed by $h(y) = y^2 - 1$ and $g(y) = y^2 - \frac{1}{8}y^4 + 1$;

  To discover points of intersection, set $g(y) = h(y) \Rightarrow y^2 - 1 = y^2 - \frac{1}{8}y^4 + 1 \Rightarrow \frac{1}{8}y^4 = 2 \Rightarrow y^4 = 16 \Rightarrow y = \pm 2$; Since $g(0) > h(0)$, g is to the right: We get



$$A = \int_{-2}^{2} (g(y) - h(y))dy =$$

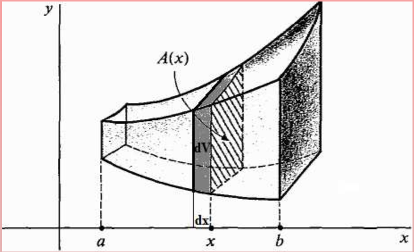
$$\int_{-2}^{2} (y^2 - \frac{1}{8}y^4 + 1) - (y^2 - 1))dy = \int_{-2}^{2} (2 - \frac{1}{8}y^4)dy =$$
$$(2y - \frac{1}{40}y^5)\Big|_{-2}^{2} = \frac{16}{5} - (-\frac{16}{5}) = \frac{32}{5};$$

Subsection 2

Setting Up Integrals: Volume, Density, Average Value

Volume

- Suppose that we want to compute the volume of the solid shown in the figure on the right:
- At a given x in $[a, b]$ the area of the cross section is known to be $A(x)$;

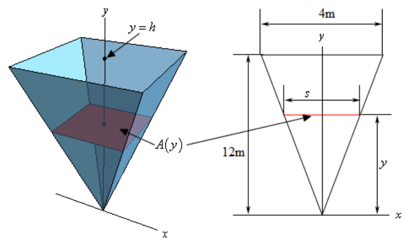


- We start by computing the volume of a thin slice of thickness dx;
- Assuming that the area of the side is almost constant and equal to $A(x)$, the volume of the slice is $dV = A(x)dx$;
- To compute the entire volume of the solid we integrate (i.e., take the sum of the volumes of all slices and, then, the limit as the max thickness of each slice approaches 0):

$$V = \int_a^b dV = \int_a^b A(x)dx;$$

Volume: Pyramid

- Calculate the volume V of a pyramid of height 12 meters whose base is a square of side 4 meters;
  Taking into account similar triangles, we obtain
  $\frac{s}{y} = \frac{4}{12} \Rightarrow s = \frac{1}{3}y;$



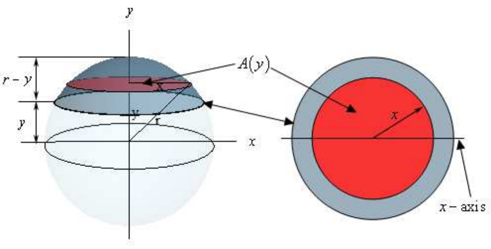
Therefore, the area of a cross section at height y is
$A(y) = s^2 = (\frac{1}{3}y)^2 = \frac{1}{9}y^2;$
Now we set up the volume integral:

$$V = \int_0^{12} A(y)dy = \int_0^{12} \frac{y^2}{9}dy = \frac{1}{9}\int_0^{12} y^2 dy = \frac{1}{9}\left(\frac{1}{3}y^3\right)\Big|_0^{12} = 64\text{m}^3;$$

Volume: Sphere

- Calculate the volume V of a sphere of radius r; Note that

$$x^2 + y^2 = r^2$$
$$\Rightarrow x^2 = r^2 - y^2;$$

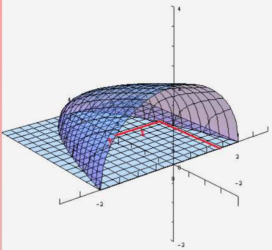


Therefore, the area of a cross section at height y is
$A(y) = \pi x^2 = \pi(r^2 - y^2) = \pi r^2 - \pi y^2;$
Now we set up the volume integral:

$$V = \int_{-r}^{r} A(y)dy = \int_{-r}^{r} (\pi r^2 - \pi y^2)dy = \left(\pi r^2 y - \frac{1}{3}\pi y^3\right)\Big|_{-r}^{r} =$$
$$\pi r^3 - \frac{1}{3}\pi r^3 - \left(-\pi r^3 + \frac{1}{3}\pi r^3\right) = \frac{4}{3}\pi r^3;$$

Volume: Paraboloid

- Calculate the volume V of the solid whose base is the region between the inverted parabola $y = 4 - x^2$ and the x-axis and whose vertical cross sections perpendicular to the y-axis are semicircles;

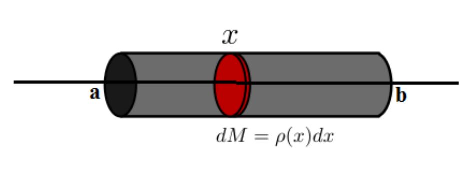


Note that, for fixed $0 \leq y \leq 4$, the radius of the semicircle is $x = \sqrt{4-y}$;
Thus, $A(y) = \frac{1}{2}\pi x^2 = \frac{1}{2}\pi(\sqrt{4-y})^2 = \pi(2 - \dfrac{1}{2}y)$; Now we set up the volume integral:

$$V = \int_0^4 A(y)dy = \int_0^4 \pi(2 - \frac{1}{2}y)dy =$$
$$\pi \left(2y - \frac{1}{4}y^2)\right|_0^4 = \pi(8-4) = 4\pi;$$

Linear Density

- Consider a rod extending along the x-axis from $x = a$ to $x = b$;



- Assume that at point x, the rod has linear density $\rho(x)$;
- Then, the total mass M of the rod is given by

$$M = \int_a^b \rho(x)dx;$$

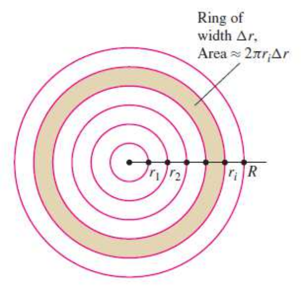
Mass of a Rod

- What is the mass M of a 2 meter long rod of linear density $\rho(x) = 1 + x(2 - x)$ kg/m, where x is the distance from one end of the rod;

$$\begin{aligned} M &= \int_0^2 \rho(x)dx \\ &= \int_0^2 (1 + x(2 - x))dx \\ &= \int_0^2 (-x^2 + 2x + 1)dx \\ &= (-\frac{1}{3}x^3 + x^2 + x)\Big|_0^2 \\ &= -\frac{8}{3} + 4 + 2 \\ &= \frac{10}{3}\ \text{Kg}; \end{aligned}$$

Radial Density

- Consider a disk centered at the origin with radius R;
- Assume that, at any point on the disk, the density $\rho(r)$ depends only on the distance r of the point from the center;



- The area dA of a thin slice with radius r (considered almost constant) and thickness dr is $dA = 2\pi r dr$;
- The mass dM of that thin slice is $dM = \rho(r) dA = 2\pi r \rho(r) dr$;
- Therefore, the total mass M of the disk is given by

$$M = 2\pi \int_0^R r\rho(r) dr;$$

Computing Total Population

- The population in a city has radial density $\rho(r) = 15(1+r^2)^{-1/2}$, where r is the distance from the city center in kilometers and ρ is in thousands of people per square kilometer; How many people live in the ring between 10 and 30 kilometers from the city center?

$$P = 2\pi \int_{10}^{30} r\rho(r)dr = 2\pi \int_{10}^{30} 15r(1+r^2)^{-1/2}dr;$$

Now use substitution:

$$u = 1 + r^2 \quad \Rightarrow \quad du = 2rdr; \quad u(10) = 101; \quad u(30) = 901;$$

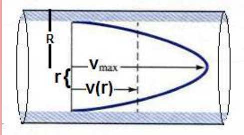
Therefore,

$$P = 2\pi \int_{101}^{901} \frac{15}{2} u^{-1/2} du = 15\pi \left(2u^{1/2}\right)\Big|_{101}^{901} =$$

$$30\pi(\sqrt{901} - \sqrt{101}) \approx 1881 \text{ thousand};$$

Flow Rate

- Consider a tube or blood vessel in which a fluid is flowing;



- The **flow rate** Q is defined as the volume of the liquid flowing per unit of time;
- If all the particles of the fluid travel with the same velocity v and the tube has radius R, then the flow rate is $Q = \pi R^2 v$;
- If, however, the velocity $v(r)$ of a particle depends on the distance r of the particle from the center of the tube, then

$$Q = 2\pi \int_0^R rv(r)dr;$$

Poiseuille's Law for Blood Flow

- **Poiseuille's Law** gives the velocity of blood flowing in a vessel of radius R cm as

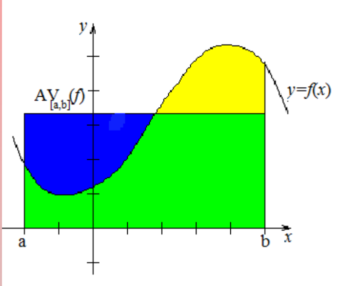
$$v(r) = k(R^2 - r^2),$$

where r is the distance from the center of the vessel in cm and k is a constant; What is the flow rate Q in terms of R?

$$\begin{aligned} Q &= 2\pi \int_0^R rv(r)dr = 2\pi \int_0^R rk(R^2 - r^2)dr \\ &= 2\pi k \int_0^R (R^2 r - r^3)dr = 2\pi k \left(\frac{1}{2}R^2 r^2 - \frac{1}{4}r^4)\right|_0^R \\ &= 2\pi k(\frac{1}{2}R^4 - \frac{1}{4}R^4) = 2\pi k \frac{1}{4}R^4 = \frac{1}{2}\pi k R^4 \text{ cm}^3/\text{sec}; \end{aligned}$$

Average Value of a Function

- Consider a function $f(x)$ continuous on $[a, b]$;
- The **average value** $\mathrm{AV}_{[a,b]}(f)$ of f on $[a, b]$ is the height of a rectangle with base $[a, b]$ that has the same area as the area under the curve from a to b;



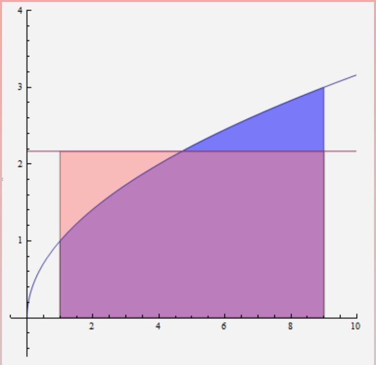
- Since the area under the curve is $\int_a^b f(x)dx$ and the area of the rectangle is $(b-a)\mathrm{AV}_{[a,b]}(f)$, and these are equal: $\int_a^b f(x)dx = (b-a)\mathrm{AV}_{[a,b]}(f)$, we get

$$\mathrm{AV}_{[a,b]}(f) = \frac{1}{b-a}\int_a^b f(x)dx;$$

Computing Average I

- Find the average value of $f(x) = \sqrt{x}$ from $x = 1$ to $x = 9$;

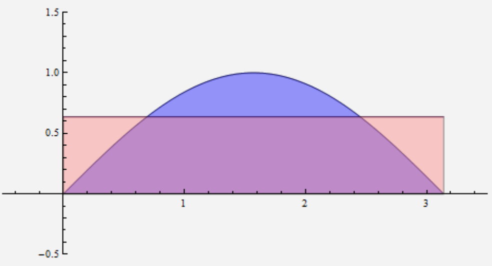
$$\begin{aligned} \mathrm{AV}_{[1,9]}(f) &= \frac{1}{9-1}\int_1^9 \sqrt{x}\,dx \\ &= \frac{1}{8}\cdot\frac{2}{3}x^{3/2}\Big|_1^9 \\ &= \frac{1}{12}\sqrt{x}^3\Big|_1^9 \\ &= \frac{1}{12}(27-1) \\ &= \frac{13}{6}; \end{aligned}$$



Computing Average II

- Find the average value of $f(x) = \sin x$ on $[0, \pi]$;

$$\begin{aligned} AV_{[0,\pi]}(f) &= \frac{1}{\pi - 0}\int_0^\pi \sin x dx = \frac{1}{\pi} \cdot (-\cos x)\big|_0^\pi \\ &= \frac{1}{\pi}(-\cos\pi + \cos 0) = \frac{1}{\pi}(-(-1)+1) = \frac{2}{\pi}; \end{aligned}$$



Computing Average: An Application

- The bushbaby has a remarkable vertical jumping ability; Find the average speed during his jump if its initial velocity is $v_0 = 600$ cm/sec and its height is given by $h(t) = v_0 t - \frac{1}{2} g t^2$ in centimeters, where $g = 980$ cm/sec^2;

  First, we find when its jump ends: $h = 0 \Rightarrow 600t - \frac{1}{2} \cdot 980 t^2 = 0 \Rightarrow t(600 - 490t) = 0 \Rightarrow t = \frac{60}{49}$ seconds;

  Its velocity at time t is $v(t) = \dfrac{dh}{dt} = v_0 - gt = 600 - 980t$; Therefore, its average speed during the jump is

$$\mathrm{AV}_{[0,\frac{60}{49}]}(|v|) = \frac{49}{60}\int_0^{60/49} |600 - 980t| dt = 2\frac{49}{60}\left(600t - 490t^2\right)\Big|_0^{60/98}$$
$$= \frac{98}{60}[600 \cdot \frac{60}{98} - 490 \cdot (\frac{60}{98})^2] = \frac{98}{60}(\frac{36000}{98} - \frac{18000}{98}) = 300 \text{ cm/sec};$$

Mean Value Theorem for Integrals

Mean Value Theorem for Integrals

If $f(x)$ is continuous on $[a, b]$, then there exists a value $c \in [a, b]$, such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

Example: Let M be the average value of $f(x) = x^3$ on $[0, A]$, where $A > 0$. Which theorem guarantees that $f(c) = M$ has a solution c in $[0, A]$? Find c.

Since $f(x) = x^3$ is continuous on $[0, A]$, the Intermediate Value Theorem for Integrals guarantees that there exists $c \in [0, A]$, such that $f(c) = M = \frac{1}{A}\int_0^A x^3 dx$. We have:

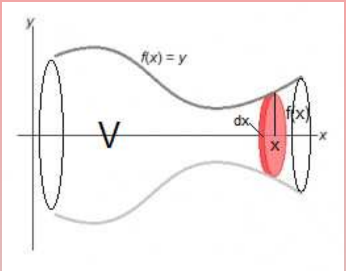
$$c^3 = \frac{1}{A}\left.\frac{x^4}{4}\right|_0^A = \frac{1}{A}\frac{A^4}{4} \;\Rightarrow\; c^3 = \frac{A^3}{4} \Rightarrow\; c = \frac{A}{\sqrt[3]{4}}.$$

Subsection 3

Volumes of Revolution

Volume of Revolution: Disk Method

- A **solid of revolution** is obtained by rotating a region in the xy-plane about an axis;
- Consider $y = f(x)$ for $a \leq x \leq b$; All vertical cross-sections of the solid obtained by rotating the region around the x-axis are circles; The area of such a cross-section is $A(x) = \pi f(x)^2$;



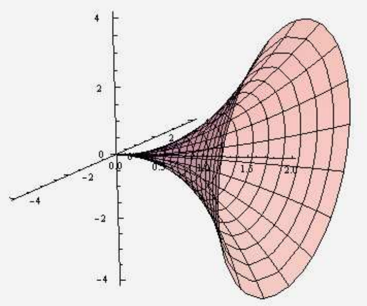
- Thus, a small volume of a thin slice of thickness dx is $dV = A(x)dx = \pi f(x)^2 dx$;
- The volume of the entire solid is then given by

$$V = \int_a^b \pi f(x)^2 dx;$$

Example of Computing a Volume of Revolution

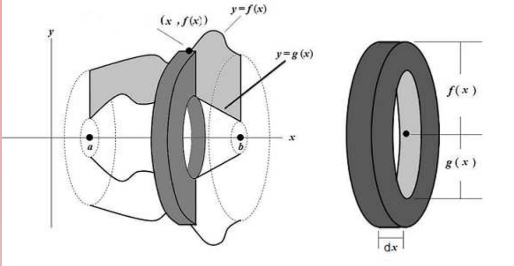
- Calculate the volume V of the solid obtained by rotating the region under $y = x^2$ around the x-axis for $0 \leq x \leq 2$;

$$V = \int_0^2 \pi f(x)^2 dx =$$
$$\pi \int_0^2 x^4 dx =$$
$$\pi \left(\frac{1}{5}x^5\right)\Big|_0^2 =$$
$$\frac{32\pi}{5};$$



Region Between Two Curves

- Consider the region between two curves $y = f(x)$ and $y = g(x)$;
- Rotation about the x axis results in a solid whose volume can be seen as the sum of elementary volumes dV of **washers** of thickness dx;



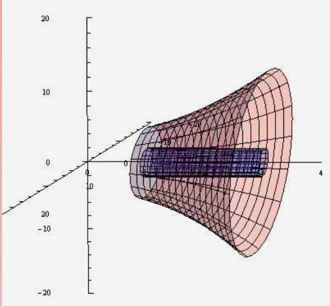
- We have $dV = \pi f(x)^2 dx - \pi g(x)^2 dx = \pi[f(x)^2 - g(x)^2]dx$;
- So the volume of the entire solid is

$$V = \int_a^b \pi[f(x)^2 - g(x)^2]dx;$$

Example of Region Between Two Curves

- Calculate the volume V of the solid obtained by revolving the region between $y = x^2 + 4$ and $y = 2$ about the x-axis for $1 \leq x \leq 3$;

$$V = \int_1^3 \pi[f(x)^2 - g(x)^2]dx =$$
$$\pi \int_1^3 ((x^2+4)^2 - 2^2)dx =$$
$$\pi \int_1^3 (x^4 + 8x^2 + 12)dx =$$
$$\pi \left(\frac{1}{5}x^5 + \frac{8}{3}x^3 + 12x\right)\Big|_1^3 =$$

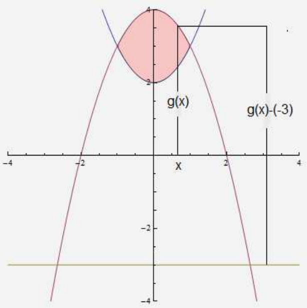


$$\pi \left[\frac{243}{5} + 72 + 36 - \left(\frac{1}{5} + \frac{8}{3} + 12\right)\right] = \frac{2126\pi}{15};$$

Revolving About a Horizontal Axis I

- Find the volume V of the "wedding band" obtained by rotating the region between the graphs of $f(x) = x^2 + 2$ and $g(x) = 4 - x^2$ about the horizontal line $y = -3$;

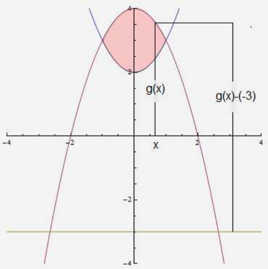
The key is to realize that, for given x, the upper radius is $g(x)+3$ and the lower radius is $f(x)+3$ and not simply $g(x)$ and $f(x)$, as in previous examples; To find the endpoints, we solve $f(x) = g(x) \Rightarrow x^2+2 = 4-x^2 \Rightarrow 2x^2-2 = 0 \Rightarrow 2(x^2-1) = 0 \Rightarrow 2(x+1)(x-1) = 0 \Rightarrow x = -1$ or $x = 1$;



Revolving About a Horizontal Axis I (Cont'd)

$$f(x) = x^2 + 2$$

$$g(x) = 4 - x^2$$



$$V = \int_{-1}^{1} \pi[(g(x)+3)^2 - (f(x)+3)^2]dx =$$

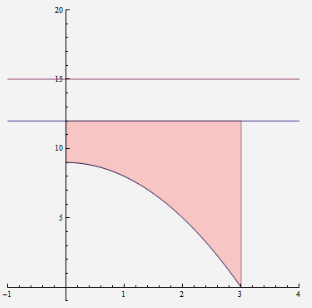
$$\pi \int_{-1}^{1} [(7-x^2)^2 - (x^2+5)^2]dx =$$

$$\pi \int_{-1}^{1} [(49 - 14x^2 + x^4) - (x^4 + 10x^2 + 25)]dx =$$

$$\pi \int_{-1}^{1} (24 - 24x^2)dx = \pi\ (24x - 8x^3)\Big|_{-1}^{1} = 32\pi;$$

Revolving About a Horizontal Axis II

- Find the volume V of the solid obtained by rotating the region between the graphs of $f(x) = 9 - x^2$ and $g(x) = 12$ about the horizontal line $y = 15$;
  The key, again, is to realize that, for given x, the upper radius is $15 - f(x)$ and the lower radius is $15 - 12 = 3$;



$$V = \int_0^3 \pi[(15 - f(x))^2 - 3^2]dx =$$

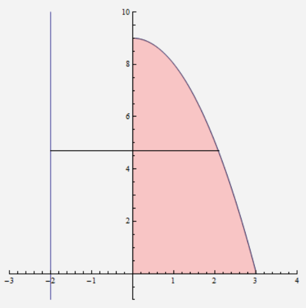
$$\pi \int_0^3 [(x^2 + 6)^2 - 9]dx =$$

$$\pi \int_0^3 (x^4 + 12x^2 + 27)dx =$$

$$\pi \left(\frac{1}{5}x^5 + 4x^3 + 27x\right)\Big|_0^3 = \frac{1188\pi}{5};$$

Revolving About a Vertical Axis

- Find the volume V of the solid obtained by rotating the region under the graph of $f(x) = 9 - x^2$ for $0 \leq x \leq 3$ about the vertical line $x = -2$;
  For given y, the outer radius is $2 + \sqrt{9 - y}$ and the inner radius is 2;



$$V = \int_0^9 \pi[(2 + \sqrt{9 - y})^2 - 2^2]dy =$$

$$\pi \int_0^9 [4 + 4\sqrt{9 - y} + (\sqrt{9 - y})^2 - 4]dy =$$

$$\pi \int_0^9 (9 - y + 4\sqrt{9 - y})dy =$$

$$\pi \left(9y - \frac{1}{2}y^2 - \frac{8}{3}(\sqrt{9 - y})^3\right)\Big|_0^9 = \pi[(81 - \frac{81}{2}) - (-\frac{8}{3} \cdot 27)] = \frac{225\pi}{2};$$

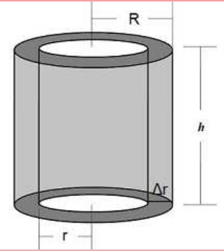
Subsection 4

The Method of Cylindrical Shells

Volume of Revolution: The Shell Method

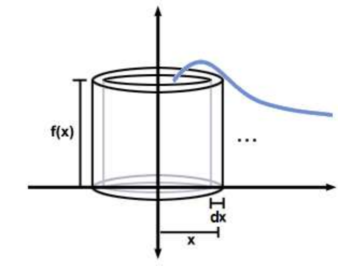
Consider the cylindrical shell shown in the picture; Its volume is given by

$$V = \pi R^2 h - \pi r^2 h = \pi h(R^2 - r^2) = \pi h(R+r)(R-r) = 2\pi h \frac{R+r}{2} \Delta r \approx 2\pi h R \Delta r;$$



To compute the volume of the solid of revolution of $f(x)$ around the y-axis, we estimate the volume dV of a cylindrical shell: $dV = 2\pi x f(x)dx$ and then integrate from a to b

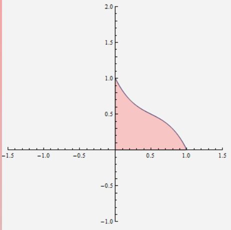
$$V = \int_a^b 2\pi x f(x)dx;$$

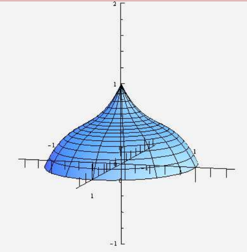


Example of the Shell Method

- Find the volume V of the solid obtained by rotating the region under the graph of $f(x) = 1 - 2x + 3x^2 - 2x^3$ over $[0, 1]$ about the y-axis;

$$V = 2\pi \int_0^1 x f(x) dx =$$
$$2\pi \int_0^1 x(1 - 2x + 3x^2 - 2x^3) dx =$$
$$2\pi \int_0^1 (x - 2x^2 + 3x^3 - 2x^4) dx =$$
$$2\pi \left(\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{3}{4}x^4 - \frac{2}{5}x^5\right)\Big|_0^1 =$$
$$\frac{11\pi}{30};$$



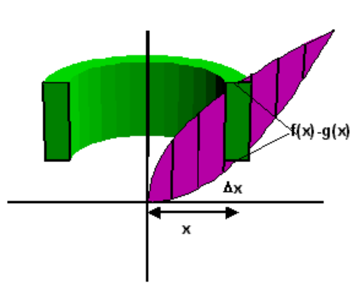


Region Between Two Curves

In case we rotate the region between two curves $f(x)$ and $g(x)$ around the y-axis, we get

$$V = 2\pi \int_a^b (\text{Radius})(\text{Height})dx$$

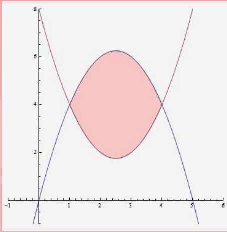
$$= 2\pi \int_a^b x(f(x) - g(x))dx;$$



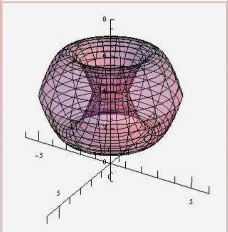
Example With Two Curves

- Find the volume V obtained by rotating the region enclosed by $f(x) = x(5-x)$ and $g(x) = 8 - x(5-x)$ about the y-axis;

  Find points of intersection: $x(5-x) = 8 - x(5-x) \Rightarrow 2x^2 - 10x + 8 = 0 \Rightarrow x^2 - 5x + 4 = 0 \Rightarrow (x-1)(x-4) = 0 \Rightarrow x = 1$ or $x = 4$;



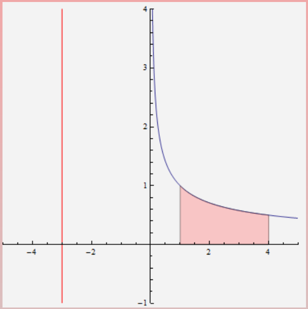
$$V = 2\pi \int_1^4 x(f(x) - g(x))dx =$$
$$2\pi \int_1^4 (-2x^3 + 10x^2 - 8x)dx =$$
$$2\pi\,(-\frac{1}{2}x^4 + \frac{10}{3}x^3 - 4x^2)\Big|_1^4 = 45\pi;$$



Rotating About a Vertical Axis

- Calculate the volume V obtained by rotating the region under the graph of $f(x) = x^{-1/2}$ over $[1, 4]$ about $x = -3$;

  Note that the radius of revolution is $x + 3$ and the height of the shell is $f(x)$: Thus, we get

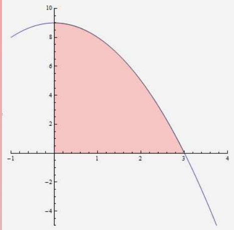


$$V = 2\pi \int_1^4 (x+3)x^{-1/2}dx =$$
$$2\pi \int_1^4 (x^{1/2} + 3x^{-1/2})dx =$$
$$2\pi \left(\frac{2}{3}x^{3/2} + 6x^{1/2}\right)\Big|_1^4 =$$
$$2\pi\left(\frac{16}{3} + 12 - \left(\frac{2}{3} + 6\right)\right) = \frac{64\pi}{3};$$

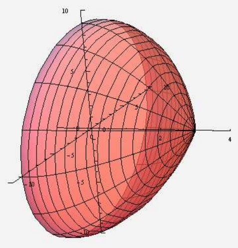
Rotating About the x-Axis

- Calculate, using the shell method, the volume V obtained by rotating the region under $y = 9 - x^2$ over $[0, 3]$ about the x-axis;

  Note that the radius of revolution is y and the height of the shell is $x = \sqrt{9-y}$: Thus, we get



$$V = 2\pi \int_0^9 y\sqrt{9-y}dx \overset{u=9-y}{=} -2\pi \int_9^0 (9-u)u^{1/2}du = 2\pi \int_0^9 (9u^{1/2} - u^{3/2})du = 2\pi \left(6u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_0^9 = 2\pi(6 \cdot 27 - \frac{2}{5} \cdot 243) = \frac{648\pi}{5};$$



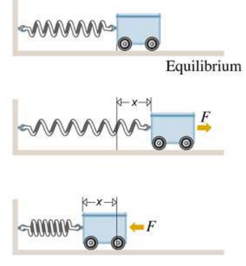
Subsection 5

Work and Energy

Work and Hooke's Law

- The energy expended when a force is applied to move an object is called **work**;
- If a constant force F is applied to move an object a distance d in the direction of the force, then the work W is defined by $W = F \cdot d$;
- If a varying force $F(x)$ is applied along the x-axis to move an object along the axes from a point a to a point b, then the work expended is

$$W = \int_a^b F(x)dx;$$



Hooke's Law: When a spring is stretched or compressed by distance x from equilibrium, it exerts restoring force of magnitude $-kx$, where k is **spring constant**;

To stretch or compress the spring we must apply force $F = kx$ to counteract the restoring force of the string;

Example on Hooke's Law

- Suppose that a spring has spring constant $k = 400$ N/m;
  - Find the work required to stretch the spring 10 cm beyond equilibrium;

$$W = \int_0^{0.1} F(x)dx = \int_0^{0.1} kxdx = 400 \left. \frac{1}{2}x^2 \right|_0^{0.1} =$$
$$200(0.1^2 - 0^2) = 200 \cdot 0.01 = 2 \text{ J};$$

  - Find the work required to compress the string 2 cm more when it is already compressed 3 cm;

$$W = \int_{-0.03}^{-0.05} F(x)dx = \int_{-0.03}^{-0.05} kxdx = 400 \left. \frac{1}{2}x^2 \right|_{-0.03}^{-0.05} =$$
$$200((-0.05)^2 - (-0.03)^2) = 200 \cdot 0.0016 = 0.32 \text{ J};$$

Building a Cement Column

- Compute the work against gravity required to build a cement column of height 5 meters and square base of side 2 meters, assuming that cement has density 1500 $\mathrm{Kg/m^3}$;
  Consider a thin layer of cement of thickness dy at height y, having volume dV, mass dM and requiring force dF to be lifted to height y;
  Then, we have

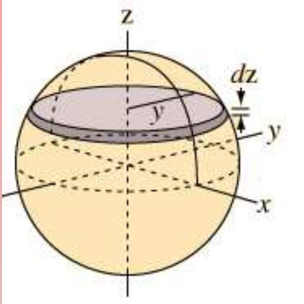
$$\begin{aligned} dV &= l \cdot w \cdot h = 2 \cdot 2 \cdot dy = 4dy; \\ dM &= \rho \cdot dV = 4\rho dy; \\ dF &= g \cdot dM = 4\rho g dy = 4 \cdot 1500 \cdot 9.8dy = 58800dy; \end{aligned}$$

Thus, we get

$$W = \int_0^5 ydF = \int_0^5 58800ydy = (29400y^2)\Big|_0^5 = 29400(25 - 0) = 735,000 \text{ J};$$

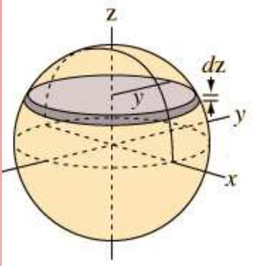
Pumping Water Out of a Tank

A spherical tank of radius R meters is filled with water; Calculate the work needed against gravity in pumping out the water through a small hole at the top, assuming the density of water is 1000 Kg/m^3;
Consider a thin layer of water of thickness dz at depth $R - z$, having volume dV, mass dM and requiring force dF to be lifted by a height $h = R - z$;
Then, we have



$$\begin{aligned} dV &= \pi y^2 dz = \pi(R^2 - z^2)dz; \\ dM &= \rho \cdot dV = 1000\pi(R^2 - z^2)dz; \\ dF &= g \cdot dM = 1000 \cdot 9.8\pi(R^2 - z^2)dz = 9800\pi(R^2 - z^2)dz; \end{aligned}$$

Pumping Water Out of a Tank (Cont'd)



$$dF = 9800\pi(R^2 - z^2)dz;$$

Thus, we get

$$W = \int_{-R}^{R}(R-z)dF =$$

$$\int_{-R}^{R} 9800\pi(R^2 - z^2)(R-z)dz =$$

$$9800\pi\int_{-R}^{R}(z^3 - Rz^2 - R^2 z + R^3)dz =$$

$$9800\pi\ (\frac{1}{4}z^4 - \frac{1}{3}Rz^3 - \frac{1}{2}R^2z^2 + R^3 z)\Big|_{-R}^{R} =$$

$$9800\pi[(\frac{R^4}{4} - \frac{R^4}{3} - \frac{R^4}{2} + R^4) - (\frac{R^4}{4} + \frac{R^4}{3} - \frac{R^4}{2} - R^4)] =$$

$$9800\pi\frac{4R^4}{3} = \frac{39200\pi R^4}{3}\ \text{J};$$