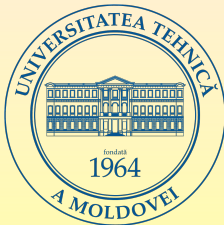


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Introduction to Complex Analysis

- 1 Complex Functions and Mappings
 - Functions
 - Complex Functions as Mappings
 - Linear Mappings
 - Special Power Functions
 - Reciprocal Function

Subsection 1

Functions

Real versus Complex Functions

- A **function** f **from** a set A **to** a set B is a rule of correspondence that assigns to **each** element in A **one and only one** element in B .
- We often think of a function as a rule or a machine that accepts **inputs** from the set A and returns **outputs** in the set B .
- In calculus we studied functions whose inputs and outputs were **real numbers**. Such functions are called **real-valued functions of a real variable**.
- Now we study functions whose inputs and outputs are **complex numbers**. We call these functions **complex functions of a complex variable**, or **complex functions** for short.
- Many interesting complex functions are simply generalizations of well-known functions from calculus.

Domain and Range

- Suppose that f is a function from the set A to the set B .
- If f assigns to a in A the element b in B , then we say that b is the **image** of a under f , or the **value** of f at a , and we write $b = f(a)$.
- The set A , the set of inputs, is called the **domain** of f and the set of images in B , the set of outputs, is called the **range** of f .
- We denote the domain of f by $\text{Dom}(f)$ and the range of f by $\text{Range}(f)$.

Example: Consider the “squaring” function $f(x) = x^2$ defined for the real variable x .

Since any real number can be squared, the domain of f is the set \mathbb{R} of all real numbers, i.e., $\text{Dom}(f) = A = \mathbb{R}$. The range of f consists of all real numbers x^2 , where x is a real number. Of course, $x^2 \geq 0$, for all real x , and one can see from the graph of f that $\text{Range}(f) = [0, \infty)$.

- The **range of f need not be the same as the set B** . For instance, because the interval $[0, \infty)$ is a subset of \mathbb{R} , f can be viewed as a function from $A = \mathbb{R}$ to $B = \mathbb{R}$, so the range of f is not equal to B .

Complex Functions

Definition (Complex Function)

A **complex function** is a function f whose domain and range are subsets of the set \mathbb{C} of complex numbers.

- A complex function is also called a **complex-valued function of a complex variable**.
- Ordinarily, the usual symbols f, g and h will denote complex functions.
- Inputs to a complex function f will typically be denoted by the variable z and outputs by the variable $w = f(z)$.
- When referring to a complex function we will use three notations interchangeably: E.g.,

$$f(z) = z - i, \quad w = z - i, \quad \text{or, simply, the function } z - i.$$

- The notation $w = f(z)$ will always denote a **complex function**; the notation $y = f(x)$ will represent a **real-valued function of a real variable** x .

Examples of Complex Functions

- (a) The expression $z^2 - (2 + i)z$ can be evaluated at any complex number z and always yields a single complex number, and so

$$f(z) = z^2 - (2 + i)z$$

defines a complex function.

Values of f are found by using the arithmetic operations for complex numbers. For instance, at the points $z = i$ and $z = 1 + i$ we have:

$$\begin{aligned}f(i) &= (i)^2 - (2 + i)(i) = -1 - 2i + 1 = -2i; \\f(1 + i) &= (1 + i)^2 - (2 + i)(1 + i) = 2i - 1 - 3i = -1 - i.\end{aligned}$$

- (b) The expression $g(z) = z + 2\operatorname{Re}(z)$ also defines a complex function. Some values of g are:

$$\begin{aligned}g(i) &= i + 2\operatorname{Re}(i) = i + 2(0) = i; \\g(2 - 3i) &= 2 - 3i + 2\operatorname{Re}(2 - 3i) = 2 - 3i + 2(2) = 6 - 3i.\end{aligned}$$

Natural Domains

- When the domain of a complex function is not explicitly stated, we assume the domain to be the **set of all complex numbers z for which $f(z)$ is defined**. This set is sometimes referred to as the **natural domain** of f .

- Example:** The functions

$$f(z) = z^2 - (2 + i)z \quad \text{and} \quad g(z) = z + 2\operatorname{Re}(z)$$

are defined for all complex numbers z , and so, $\operatorname{Dom}(f) = \mathbb{C}$ and $\operatorname{Dom}(g) = \mathbb{C}$. The complex function $h(z) = \frac{z}{z^2 + 1}$ is not defined at $z = i$ and $z = -i$ because the denominator $z^2 + 1$ is equal to 0 when $z = \pm i$. Therefore, $\operatorname{Dom}(h)$ is the set of all complex numbers except i and $-i$, written $\operatorname{Dom}(h) = \mathbb{C} - \{-i, i\}$.

- Since \mathbb{R} is a subset of \mathbb{C} , **every real-valued function of a real variable is also a complex function**. We will see that real-valued functions of **two real variables** x and y are also special types of complex functions.

Real and Imaginary Parts of a Complex Function

- If $w = f(z)$ is a complex function, then the image of a complex number $z = x + iy$ under f is a complex number $w = u + iv$. By simplifying the expression $f(x + iy)$, we can write the real variables u and v in terms of the real variables x and y .

Example: By replacing the symbol z with $x + iy$ in the complex function $w = z^2$, we obtain:

$$w = u + iv = (x + iy)^2 = x^2 - y^2 + 2xyi.$$

Thus, $u = x^2 - y^2$ and $v = 2xy$, respectively.

- If $w = u + iv = f(x + iy)$ is a complex function, then both u and v are real functions of the two real variables x and y , i.e., by setting $z = x + iy$, we can express any complex function $w = f(z)$ in terms of two real functions as:

$$f(z) = u(x, y) + iv(x, y).$$

- The functions $u(x, y)$ and $v(x, y)$ are called the **real** and **imaginary parts** of f , respectively.

Examples

- Find the real and imaginary parts of the functions:

(a) $f(z) = z^2 - (2 + i)z$;

(b) $g(z) = z + 2\operatorname{Re}(z)$.

In each case, we replace the symbol z by $x + iy$, then simplify.

(a) $f(z) = (x + iy)^2 - (2 + i)(x + iy) = x^2 - 2x + y - y^2 + (2xy - x - 2y)i$.

So,

$$u(x, y) = x^2 - 2x + y - y^2 \quad \text{and} \quad v(x, y) = 2xy - x - 2y.$$

(b) Since $g(z) = x + iy + 2\operatorname{Re}(x + iy) = 3x + iy$, we have

$$u(x, y) = 3x \quad \text{and} \quad v(x, y) = y.$$

Specifying w via u and v

- Every complex function is completely determined by the real functions $u(x, y)$ and $v(x, y)$.
- Thus, a complex function $w = f(z)$ can be defined by arbitrarily specifying two real functions $u(x, y)$ and $v(x, y)$, even though $w = u + iv$ may not be obtainable through familiar operations performed solely on the symbol z .

Example: If we take $u(x, y) = xy^2$ and $v(x, y) = x^2 - 4y^3$, then

$$f(z) = xy^2 + i(x^2 - 4y^3)$$

defines a complex function. In order to find the value of f at the point $z = 3 + 2i$, we substitute $x = 3$ and $y = 2$:

$$f(3 + 2i) = 3 \cdot 2^2 + i(3^2 - 4 \cdot 2^3) = 12 - 23i.$$

- Of course, complex functions defined in terms of $u(x, y)$ and $v(x, y)$ can always be expressed in terms of operations on the symbols z and \bar{z} .

Exponential Function

- The complex exponential function e^z is an example of a function defined by specifying its real and imaginary parts.

Definition (Complex Exponential Function)

The function e^z defined by

$$e^z = e^x \cos y + ie^x \sin y$$

is called the **complex exponential function**.

- The real and imaginary parts of the complex exponential function are

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y.$$

- Thus, values of the complex exponential function $w = e^z$ are found by expressing the point z as $z = x + iy$ and then substituting the values of x and y in $u(x, y)$ and $v(x, y)$.

Values of the Complex Exponential Function

- Find the values of the complex exponential function e^z at:

$$(a) \quad z = 0 \qquad (b) \quad z = i \qquad (c) \quad z = 2 + \pi i.$$

In each part we substitute $x = \operatorname{Re}(z)$ and $y = \operatorname{Im}(z)$ in $e^z = e^x \cos y + ie^x \sin y$ and then simplify:

- (a) For $z = 0$, we have $x = 0$ and $y = 0$, and so
$$e^0 = e^0 \cos 0 + ie^0 \sin 0 = 1 \cdot 1 + i1 \cdot 0 = 1.$$
- (b) For $z = i$, we have $x = 0$ and $y = 1$, and so:
$$e^i = e^0 \cos 1 + ie^0 \sin 1 = \cos 1 + i \sin 1.$$
- (c) For $z = 2 + \pi i$, we have $x = 2$ and $y = \pi$, and so
$$e^{2+\pi i} = e^2 \cos \pi + ie^2 \sin \pi = e^2 \cdot (-1) + ie^2 \cdot 0 = -e^2.$$

Exponential Form of a Complex Number

- The exponential function enables us to express the polar form of a nonzero complex number $z = r(\cos \theta + i \sin \theta)$ in a particularly convenient and compact form:

$$z = re^{i\theta}.$$

- This form is called the **exponential form** of the complex number z .
- **Example:** A polar form of the complex number $3i$ is $3(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$, whereas an exponential form of $3i$ is $3e^{i\pi/2}$.
- In the exponential form of a complex number, the value of $\theta = \arg(z)$ is not unique.

Example: All forms $\sqrt{2}e^{i\pi/4}$, $\sqrt{2}e^{i9\pi/4}$, and $\sqrt{2}e^{i17\pi/4}$ are all valid exponential forms of the complex number $1 + i$.

Some Additional Properties

- If z is a real number, that is, if $z = x + 0i$, then

$$e^z = e^x \cos 0 + ie^x \sin 0 = e^x.$$

Thus, the complex exponential function agrees with the usual real exponential function for real z .

- Many well-known properties of the real exponential function are also satisfied by the complex exponential function: If z_1 and z_2 are complex numbers, then:
 - $e^0 = 1$;
 - $e^{z_1} e^{z_2} = e^{z_1+z_2}$;
 - $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$;
 - $(e^{z_1})^n = e^{nz_1}$, for $n = 0, 1, 2, \dots$

Periodicity of e^z

- The most unexpected difference between the real and complex exponential functions is:

Proposition (Periodicity of e^z)

The complex exponential function is periodic; Indeed, we have

$$e^{z+2\pi i} = e^z, \text{ for all complex numbers } z.$$

$$\begin{aligned} e^{z+2\pi i} &= e^{x+iy+2\pi i} \\ &= e^{x+i(y+2\pi)} \\ &= e^x \cos(y+2\pi) + ie^x \sin(y+2\pi) \\ &= e^x \cos y + ie^x \sin y \\ &= e^{x+iy} = e^z. \end{aligned}$$

Corollary

The complex exponential function has a pure imaginary period $2\pi i$.

Polar Coordinates

- It is often more convenient to express the complex variable z using either the polar form $z = r(\cos \theta + i \sin \theta)$ or, equivalently, the exponential form $z = re^{i\theta}$.
- Given a complex function $w = f(z)$, if we replace the symbol z with $r(\cos \theta + i \sin \theta)$, then we can write this function as:

$$f(z) = u(r, \theta) + iv(r, \theta).$$

We still call the real functions $u(r, \theta)$ and $v(r, \theta)$ the **real** and **imaginary parts** of f , respectively.

- **Example:** Replacing z with $r(\cos \theta + i \sin \theta)$ in $f(z) = z^2$ yields

$$f(z) = (r(\cos \theta + i \sin \theta))^2 = r^2 \cos 2\theta + ir^2 \sin 2\theta.$$

Thus, the real and imaginary parts of $f(z) = z^2$ are

$$u(r, \theta) = r^2 \cos 2\theta \quad \text{and} \quad v(r, \theta) = r^2 \sin 2\theta.$$

Note that u and v are not the same as the functions u and v previously computed using $z = x + iy$.

Definition in Polar Coordinates

- A complex function can be defined by specifying its real and imaginary parts in polar coordinates.
- **Example:** The expression

$$f(z) = r^3 \cos \theta + (2r \sin \theta)i$$

defines a complex function.

To find the value of this function at, say, the point $z = 2i$, we first express $2i$ in polar form $2i = 2(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$. We then set $r = 2$ and $\theta = \frac{\pi}{2}$ in the expression for f :

$$f(2i) = (2)^3 \cos \frac{\pi}{2} + (2 \cdot 2 \sin \frac{\pi}{2})i = 8 \cdot 0 + (4 \cdot 1)i = 4i.$$

Remarks

- (i) The complex exponential function provides a good example of how complex functions can be similar to and, at the same time, different from their real counterparts.
- (ii) Every complex function can be defined in terms of two real functions $u(x, y)$ and $v(x, y)$ as $f(z) = u(x, y) + iv(x, y)$. Thus, the study of complex functions is closely related to the study of real multivariable functions of two real variables.
- (iii) **Real-valued functions of a real variable** and **real-valued functions of two real variables** are special types of complex functions. Other types include:
 - **Real-valued functions of a complex variable** are functions $y = f(z)$ where z is a complex number and y is a real number. The functions $x = \operatorname{Re}(z)$ and $r = |z|$ are both examples of this type of function.
 - **Complex-valued functions of a real variable** are functions $w = f(t)$ where t is a real number and w is a complex number. It is customary to express such functions in terms of two real-valued functions of the real variable t , $w(t) = x(t) + iy(t)$. An example is $w(t) = 3t + i \cos t$.

Subsection 2

Complex Functions as Mappings

Complex Mappings

- The graph of a complex function lies in four-dimensional space, and so we **cannot use graphs** to study complex functions.
- The concept of a **complex mapping** gives a geometric representation of a complex function:
 - The basic idea is that every complex function describes a correspondence between points in two copies of the complex plane.
 - The point z in the z -plane is associated with the unique point $w = f(z)$ in the w -plane.
- The alternative term **complex mapping** in place of “complex function” is used when considering the function as this correspondence between **points in the z -plane** and **points in the w -plane**.
- The geometric representation of a complex mapping $w = f(z)$ consists of two figures:
 - the first, a subset S of points in the **z -plane**;
 - the second, the set S' of the images of points in S under $w = f(z)$ in the **w -plane**.

Mappings

- If $y = f(x)$ is a real-valued function of a real variable x , then the **graph** of f is defined to be the set of all points $(x, f(x))$ in the two-dimensional Cartesian plane.
- If $w = f(z)$ is a complex function, then both z and w lie in a complex plane, whence the set of all points $(z, f(z))$ lies in **four-dimensional space**.

A subset of four-dimensional space cannot be easily illustrated and, thus, the graph of a complex function cannot be drawn.

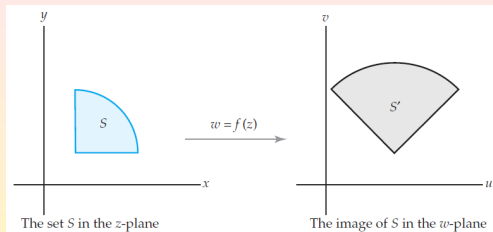
- The term **complex mapping** refers to the correspondence determined by a complex function $w = f(z)$ between **points in a z -plane** and **images in a w -plane**.
- If the point z_0 in the z -plane corresponds to the point $w_0 = f(z_0)$ in the w -plane, then we say that f **maps** z_0 onto w_0 or that z_0 is **mapped** onto w_0 by f .

Example (Physical Motion)

- Consider the real function $f(x) = x + 2$.
- The known representation of this function is a line of slope 1 and y-intercept $(0, 2)$.
- Another representation shows how one copy of the real line (the x -line) is mapped onto another copy of the real line (the y -line) by f : Each point on the x -line is mapped onto a point two units to the right on the y -line.
- You can visualize the action of this mapping by imagining the real line as an infinite rigid rod that is **physically moved** two units to the right.

Representing a Complex Mapping

- To create a geometric representation of a complex mapping, we begin with two copies of the complex plane, the **z-plane** and the **w-plane**.
- A complex mapping is represented by drawing a set S of **points in the z-plane** and the corresponding set of images of the points in S under f **in the w-plane**.

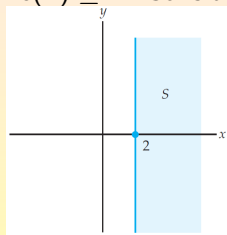


- If $w = f(z)$ is a complex mapping and if S is a set of points in the z -plane, then we call the set of images of the points in S under f the **image** of S under f , denoted S' .
- If S is a domain or a curve, we also use symbols such as D and D' or C and C' , in place of S and S' .
- Sometimes $f(C)$ is used to denote the image of C under $w = f(z)$.

Image of a Half-Plane under $w = iz$

- Find the image of the half-plane $\operatorname{Re}(z) \geq 2$ under the complex mapping $w = iz$ and represent the mapping graphically.

Let S be the half-plane consisting of all complex points z with $\operatorname{Re}(z) \geq 2$. Consider first the vertical boundary line $x = 2$ of S :

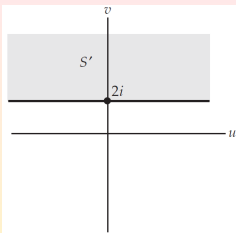
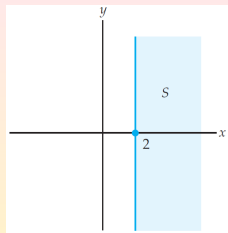


For any point z on this line we have $z = 2 + iy$, where $-\infty < y < \infty$. The value of $f(z) = iz$ at a point on this line is $w = f(2 + iy) = i(2 + iy) = -y + 2i$. The set of points $w = -y + 2i$, $-\infty < y < \infty$, is the line $v = 2$ in the w -plane.

Hence, the vertical line $x = 2$ in the z -plane is mapped onto the horizontal line $v = 2$ in the w -plane by the mapping $w = iz$.

Image of a Half-Plane under $w = iz$ (Cont'd)

- Therefore, the vertical line on the left is mapped onto the horizontal line shown on the right.



Now consider the entire half-plane S . This set can be described by the two simultaneous inequalities, $x \geq 2$ and $-\infty < y < \infty$. In order to describe the image of S :

- We express $w = iz$ in terms of its real and imaginary parts u and v .
- Then we use the bounds on x and y in the z -plane to determine bounds on u and v in the w -plane.

We have $w = i(x + iy) = -y + ix$. So the real and imaginary parts of $w = iz$ are $u(x, y) = -y$ and $v(x, y) = x$. We conclude that $v \geq 2$ and $-\infty < u < \infty$. That is, the set S' is the half-plane lying on or above the horizontal line $v = 2$.

Image of a Line under $w = z^2$

- Find the image of the vertical line $x = 1$ under the complex mapping $w = z^2$ and represent the mapping graphically.

Let C be the set of points on the vertical line $x = 1$, i.e., the set of points $z = 1 + iy$ with $-\infty < y < \infty$. The real and imaginary parts of $w = z^2 = (x + iy)^2$ are

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

For a point $z = 1 + iy$ in C , we have

$$u(1, y) = 1 - y^2 \quad \text{and} \quad v(1, y) = 2y.$$

Thus, the image of S is the set of points $w = u + iv$ satisfying $u = 1 - y^2$ and $v = 2y$, for $-\infty < y < \infty$.

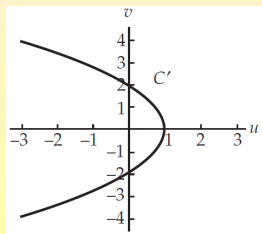
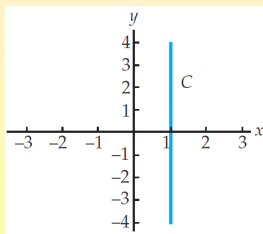
Image of a Line under $w = z^2$ (Cont'd)

- We found $w = u + iv$, with $u = 1 - y^2$, $v = 2y$, $-\infty < y < \infty$.

Note that these are parametric equations in the real parameter y , and they define a curve in the w -plane. By eliminating the parameter y , we find

$$u = 1 - \left(\frac{v}{2}\right)^2 = 1 - \frac{v^2}{4}.$$

Since y can take on any real value and since $v = 2y$, it follows that v can take on any real value. Consequently, C' is a parabola in the w -plane with vertex at $(1, 0)$ and u -intercepts at $(0, \pm 2)$:



Parametric Curves in the Complex Plane

- We can often gain a good understanding of a complex mapping by analyzing the images of curves (one-dimensional subsets of the complex plane).
- This process is facilitated by the use of parametric equations.
- If $x = x(t)$ and $y = y(t)$ are real-valued functions of a real variable t , then the set C of all points $(x(t), y(t))$, where $a \leq t \leq b$, is called a **parametric curve**.
- The equations $x = x(t)$, $y = y(t)$, and $a \leq t \leq b$ are called **parametric equations** of C .
- A parametric curve can be regarded as lying in the **complex plane** by letting x and y represent the real and imaginary parts of a point in the complex plane, i.e., if $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, are parametric equations of a curve C in the Cartesian plane, then the set of points $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, is a description of the curve C in the complex plane.

Definition of Parametric Curves in the Complex Plane

- **Example:** Consider the parametric equations

$$x = \cos t, y = \sin t, 0 \leq t \leq 2\pi,$$

of a curve C in the xy -plane. The set of points $z(t) = \cos t + i \sin t$, $0 \leq t \leq 2\pi$, describes the curve C in the complex plane (a circle of radius 1 centered at the origin). If, say, $t = 0$, then

- the point $(\cos 0, \sin 0) = (1, 0)$ is on the curve C in the Cartesian plane;
- the point $z(0) = \cos 0 + i \sin 0 = 1$ represents this point on C in the complex plane.

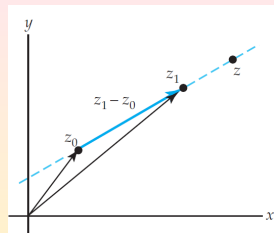
Definition (Parametric Curve in the Complex Plane)

If $x(t)$ and $y(t)$ are real-valued functions of a real variable t , then the set C consisting of all points $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, is called a **parametric curve** or a **complex parametric curve**.

The complex valued function of the real variable t , $z(t) = x(t) + iy(t)$, is called a **parametrization** of C .

Parametrization of a Line

- Suppose we want to find a parametrization of the line in the complex plane containing the points z_0 and z_1 .
- The difference $z_1 - z_0$ represents the vector originating at z_0 and terminating at z_1 : If z is any point on the line containing z_0 and z_1 , then the vector $z - z_0$ is a real multiple of the vector $z_1 - z_0$.



- Therefore, if z is on the line containing z_0 and z_1 , then there is a real number t such that $z - z_0 = t(z_1 - z_0)$.
- Solving this equation for z gives the parametrization

$$z(t) = z_0 + t(z_1 - z_0) = z_0(1 - t) + z_1 t, \quad -\infty < t < \infty.$$

- If we restrict the parameter t to the interval $[0, 1]$, then the points $z(t)$ range from z_0 to z_1 .

Common Parametric Curves in the Complex Plane

- **Line:** A parametrization of the line containing the points z_0 and z_1 is:

$$z(t) = z_0(1 - t) + z_1 t, \quad -\infty < t < \infty.$$

- **Line Segment:** A parametrization of the line segment from z_0 to z_1 is:

$$z(t) = z_0(1 - t) + z_1 t, \quad 0 \leq t \leq 1.$$

- **Ray:** A parametrization of the ray emanating from z_0 and containing z_1 is:

$$z(t) = z_0(1 - t) + z_1 t, \quad 0 \leq t < \infty.$$

- **Circle:** A parametrization of the circle centered at z_0 with radius r is:

$$z(t) = z_0 + r(\cos t + i \sin t), \quad 0 \leq t \leq 2\pi.$$

In exponential notation:

$$z(t) = z_0 + re^{it}, \quad 0 \leq t \leq 2\pi.$$

Images of Parametric Curves

- Parametric curves are important in the study of complex mappings because it is **easy to determine a parametrization of the image of a parametric curve**.
- **Example:** If $w = iz$ and C is the line $x = 2$ given by $z(t) = 2 + it$, $-\infty < t < \infty$, then the value of $f(z) = iz$ at a point on this line is

$$w = f(2 + it) = i(2 + it) = -t + 2i,$$

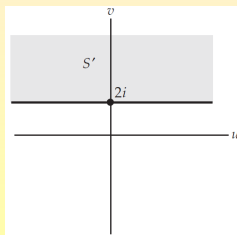
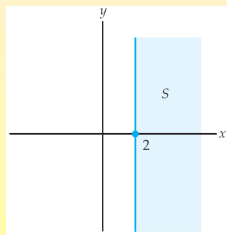
and so the image of $z(t)$ is $w(t) = -t + 2i$. Since $w(t) = -t + 2i$, $-\infty < t < \infty$, is a parametrization of the image C' , C' is the line $v = 2$.

Image of a Parametric Curve under a Complex Mapping

If $w = f(z)$ is a complex mapping and if C is a curve parametrized by $z(t)$, $a \leq t \leq b$, then $w(t) = f(z(t))$, $a \leq t \leq b$, is a parametrization of the image C' of C under $w = f(z)$.

Using a Single Copy of the Plane

- In some instances it is convenient to represent a complex mapping using a **single copy of the complex plane**.
- This is done by superimposing the w -plane on top of the z -plane, so that the real and imaginary axes in each copy of the plane coincide.
- Because such a figure simultaneously represents both the z and the w -planes, we omit all labels x, y, u and v from the axes.
- **Example:** If we plot the half-plane S and its image S' from the first example in the same copy of the complex plane,



then we see that the half-plane S' may be obtained by rotating the half-plane S through an angle $\frac{\pi}{2}$ radians counter-clockwise about the origin.

Example I: Image of a Parametric Curve

- Find the image of the line segment from 1 to i under the complex mapping $w = \bar{iz}$.

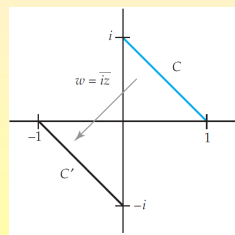
Let C denote the line segment from 1 to i and let C' denote its image under $f(z) = \bar{iz}$. A parametrization of C is

$$z(t) = (1 - t) + it, \quad 0 \leq t \leq 1.$$

The image C' is then given by

$$w(t) = f(z(t)) = \overline{i(1 - t + it)} = -i(1 - t) - t, \quad 0 \leq t \leq 1.$$

We see that $w(t)$ is a parametrization of the line segment from $-i$ to -1 . Therefore, C' is the line segment from $-i$ to -1 .



Example II: Image of a Parametric Curve

- Find the image of the upper semicircle centered at the origin with radius 2 under the complex mapping $w = z^2$.
Let C denote the given semicircle and C' its image under $f(z) = z^2$.
A parametrization of C is

$$z(t) = 2e^{it}, \quad 0 \leq t \leq \pi.$$

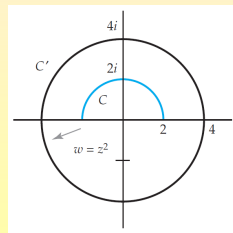
Thus, we get a parametrization of C' :

$$w(t) = f(z(t)) = (2e^{it})^2 = 4e^{2it}, \quad 0 \leq t \leq \pi.$$

If we set $t = \frac{1}{2}s$, then we obtain a new parametrization of C' :

$$W(s) = 4e^{is}, \quad 0 \leq s \leq 2\pi.$$

C' is a circle centered at 0 with radius 4:



Mappings and Parametric Curves

- (i) An important difference between real and complex analysis is that we cannot graph a complex function. Instead, we represent a complex function with two images:
- a subset S in the complex plane;
 - the image S' of the set S under a complex mapping.

A complete understanding of a complex mapping is obtained when we understand the relationship between any set S and its image S' .

- (ii) **Complex mappings** are closely related to **parametric curves** in the plane.
- This very important relationship will be used to help visualize the notions of **limit**, **continuity**, and **differentiability** of complex functions.
 - Parametric curves will also be of central importance in the study of **complex integrals**.

Subsection 3

Linear Mappings

Real and Complex Linear Functions

- A real function of the form $f(x) = ax + b$, where a and b are any real constants, is called a **linear function**.
- By analogy, we define a **complex linear function** to be a function of the form

$$f(z) = az + b,$$

where a and b are any complex constants.

- Just as real linear functions are the easiest types of real functions to graph, complex linear functions are the easiest types of complex functions to visualize as mappings of the complex plane.
- We will show that every nonconstant complex linear mapping can be described as a composition of **three basic types of motions**:
 - a translation,
 - a rotation, and
 - a magnification.

Translations

- We use the symbols T , R and M to represent mapping by translation, rotation, and magnification, respectively.

Definition (Translation)

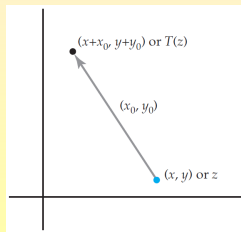
A complex linear function

$$T(z) = z + b, \quad b \neq 0,$$

is called a **translation**.

- If we set $z = x + iy$ and $b = x_0 + iy_0$, then we obtain:

$$T(z) = (x + iy) + (x_0 + iy_0) = x + x_0 + i(y + y_0).$$



The linear mapping $T(z) = z + b$ can be visualized in a single copy of the complex plane as the process of translating the point z along the vector representation (x_0, y_0) of b to the point $T(z)$.

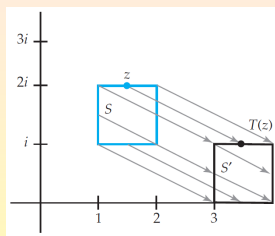
Image of a Square under Translation

- Find the image S' of the square S with vertices at $1 + i$, $2 + i$, $2 + 2i$ and $1 + 2i$ under the linear mapping $T(z) = z + 2 - i$.

We will represent S and S' in the same copy of the complex plane.

The mapping T is a translation. Identify $b = x_0 + iy_0 = 2 + i(-1)$.

Plot the vector $(2, -1)$ originating at each point in S .



The set of terminal points of these vectors is S' . S' is a square with vertices at: $T(1 + i) = 3$, $T(2 + i) = 4$, $T(2 + 2i) = 4 + i$, $T(1 + 2i) = 3 + i$. Therefore, the blue square S is mapped onto the black square S' by the translation $T(z) = z + 2 - i$.

Rotations

- A **translation** does not change the shape or size of a figure in the complex plane, i.e., the image of a line, circle, or triangle under a translation will also be a line, circle, or triangle, respectively. A mapping with this property is sometimes called a **rigid motion**.

Definition (Rotation)

A complex linear function

$$R(z) = az, \quad |a| = 1,$$

is called a **rotation**.

- If α is any nonzero complex number, then $a = \frac{\alpha}{|\alpha|}$ is a complex number for which $|a| = 1$.
- So, for any nonzero complex number α , we have that $R(z) = \frac{\alpha}{|\alpha|}z$ is a rotation.

Description of Rotations

- Consider the rotation $R = az$ and assume that $\text{Arg}(a) > 0$. Since $|a| = 1$ and $\text{Arg}(a) > 0$, we can write a in exponential form as $a = e^{i\theta}$, with $0 < \theta \leq \pi$. If we set $a = e^{i\theta}$ and $z = re^{i\phi}$, then we obtain the following description of R :

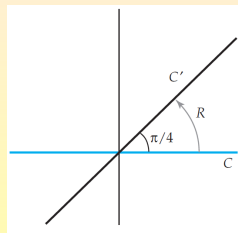
$$R(z) = e^{i\theta} re^{i\phi} = re^{i(\theta+\phi)}.$$

- The modulus of $R(z)$ is r , which is the same as the modulus of z . Therefore, if z and $R(z)$ are plotted in the same copy of the complex plane, then both points lie on a circle centered at 0 with radius r .
- An argument of $R(z)$ is $\theta + \phi$, which is θ radians greater than an argument of z . Therefore, $R(z) = az$ rotates z counterclockwise through an angle of θ radians about the origin to $R(z)$.
- If $\text{Arg}(a) < 0$, then the linear mapping $R(z) = az$ can be visualized in a single copy of the complex plane as the process of rotating points clockwise through an angle of θ radians about the origin.
- The angle $\theta = \text{Arg}(a)$ is called an **angle of rotation** of R .

Image of a Line under Rotation

- Find the image of the real axis $y = 0$ under the linear mapping $R(z) = (\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i)z$.
- Let C denote the real axis $y = 0$ and let C' denote the image of C under R . Since $|\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i| = 1$, the complex mapping $R(z)$ is a rotation. In order to determine the angle of rotation, we write $\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i$ in exponential form $\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i = e^{i\pi/4}$.

If z and $R(z)$ are plotted in the same copy of the complex plane, then the point z is rotated counterclockwise through $\frac{\pi}{4}$ radians about the origin to the point $R(z)$. The image C' is, therefore, the line $v = u$, which contains the origin and makes an angle of $\frac{\pi}{4}$ radians with the real axis.



Magnifications

- Rotations will not change the shape or size of a figure in the complex plane either.

Definition (Magnification)

A complex linear function

$$M(z) = az, \quad a > 0,$$

is called a **magnification**.

- It is implicit in $a > 0$ that the symbol a represents a **real number**.
- If $z = x + iy$, then $M(z) = az = ax + iay$. So the image of the point (x, y) is the point (ax, ay) . If $z = re^{i\theta}$, $M(z) = a(re^{i\theta}) = (ar)e^{i\theta}$, so that the magnitude of $M(z)$ is ar .
 - If $a > 1$, then the complex points z and $M(z)$ have the same argument θ , but different moduli $r \neq ar$. $M(z)$ is the unique point on the ray emanating from 0 and containing z whose distance from 0 is a times further than z . a is called the **magnification factor** of M .
 - If $0 < a < 1$, then the point $M(z)$ is a times closer to the origin than the point z . This case of a magnification is a **contraction**.

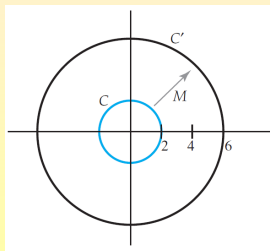
Image of a Circle under Magnification

- Find the image of the circle C given by $|z| = 2$ under the linear mapping $M(z) = 3z$.

Since M is a magnification with magnification factor of 3, each point on the circle $|z| = 2$ will be mapped onto a point with the same argument but with modulus magnified by 3.

- Thus, each point in the image will have modulus $3 \cdot 2 = 6$.
- The image points can have any argument since the points z in the circle $|z| = 2$ can have any argument.

Therefore, the image C' is the circle $|w| = 6$, centered at the origin with radius 6.



Composition and Mappings

- A magnification mapping will **change the size** of a figure in the complex plane, but it will **not change its basic shape**.
- We will now show that a general linear mapping $f(z) = az + b$ is a **composition of a rotation, a magnification, and a translation**.
- Recall that if f and g are two functions, then the **composition of f and g** is the function $f \circ g$ defined by

$$f \circ g(z) = f(g(z)).$$

- The value $w = f \circ g(z)$ is determined by
 - first evaluating the function g at z ;
 - and, then, evaluating the function f at $g(z)$.
- In a similar manner, the image, S'' , of set S under a composition $w = f \circ g(z)$ is determined by
 - first finding the image S' of S under g ;
 - and, then, finding the image S'' of S' under f .

Linear Mappings

- Suppose that $f(z) = az + b$ is a complex linear function, with $a \neq 0$ (if $f(z) = b$, every point is mapped onto the single point b).
- We can express f as:

$$f(z) = az + b = |a| \cdot \frac{a}{|a|}z + b.$$

- Consider a point z_0 :
 - First, z_0 is multiplied by the complex number $\frac{a}{|a|}$. Since $\left| \frac{a}{|a|} \right| = \frac{|a|}{|a|} = 1$, the complex mapping $w = \frac{a}{|a|}z$ is a rotation that rotates the point z_0 through an angle of $\theta = \text{Arg}\left(\frac{a}{|a|}\right)$ radians about the origin. The angle of rotation can also be written as $\theta = \text{Arg}(a)$, since $\frac{1}{|a|}$ is a real number. Let z_1 be the image of z_0 under this rotation by $\text{Arg}(a)$.
 - Then z_1 is multiplied by $|a|$. Because $|a| > 0$ is a real number, the complex mapping $w = |a|z$ is a magnification with a magnification factor $|a|$. Let z_2 be the image of z_1 under magnification by $|a|$.
 - The last step is to add b to z_2 . The complex mapping $w = z + b$ translates z_2 by b onto the point $w_0 = f(z_0)$.

Image of a Point under a Linear Mapping

- Let $f(z) = az + b$ be a linear mapping with $a \neq 0$ and let z_0 be a point in the complex plane.
- If the point $w_0 = f(z_0)$ is plotted in the same copy of the complex plane as z_0 , then w_0 is the point obtained by
 - (i) rotating z_0 through an angle of $\text{Arg}(a)$ about the origin;
 - (ii) magnifying the result by $|a|$, and
 - (iii) translating the result by b .
- The image S' of a set S under $f(z) = az + b$ is the set of points obtained by
 - rotating S through $\text{Arg}(a)$,
 - magnifying by $|a|$,
 - and, then, translating by b .
- Thus, every nonconstant complex linear mapping is a composition of **at most one** rotation, one magnification, and one translation.

Example and Remarks

- **Example:** The linear mapping $f(z) = 3z + i$ involves
 - a magnification by 3,
 - and a translation by i .
- If $a \neq 0$ is a complex number, and if
 - $R(z)$ is a rotation through $\text{Arg}(a)$,
 - $M(z)$ is a magnification by $|a|$, and
 - $T(z)$ is a translation by b ,

then the composition $f(z) = T \circ M \circ R(z) = T(M(R(z)))$ is a complex linear function.

- Since the composition of any finite number of linear functions is again a linear function, it follows that **the composition of finitely many rotations, magnifications, and translations is a linear mapping.**

Preservation of Shapes and Order of Composition

- Since translations, rotations, and magnifications all preserve the basic shape of a figure in the complex plane, a **linear mapping will also preserve the basic shape of a figure in the complex plane.**
- A complex linear mapping $w = az + b$, with $a \neq 0$, can distort the size of a figure, but it cannot alter the basic shape of the figure.
- When writing a linear function as a composition of a rotation, a magnification and a translation, the **order is important.**

Example: The mapping $f(z) = 2z + i$ magnifies by 2, then translates by i . Thus, 0 maps onto i . If we reverse the order of composition, i.e., translate by i , then magnify by 2, the effect is 0 maps onto $2i$.

- A complex linear mapping can always be **represented as a composition in more than one way.**

Example: $f(z) = 2z + i$ can also be expressed as $f(z) = 2(z + i/2)$. Therefore, a magnification by 2 followed by translation by i is the same mapping as translation by $i/2$ followed by magnification by 2.

Image of a Rectangle under a Linear Mapping

- Find the image of the rectangle with vertices $-1 + i$, $1 + i$, $1 + 2i$, and $-1 + 2i$ under the linear mapping $f(z) = 4iz + 2 + 3i$.

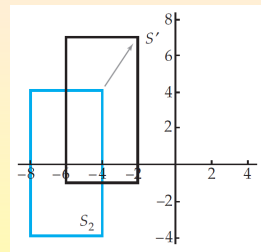
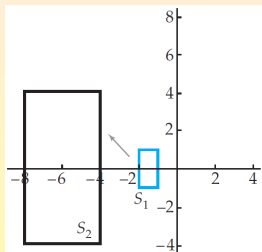
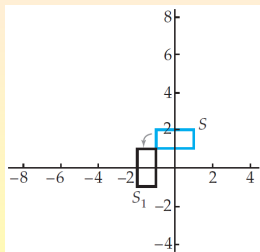
Let S be the rectangle with the given vertices and let S' denote the image of S under f . Because f is a linear mapping, S' has the same shape as S , i.e., it is a rectangle. Thus, in order to determine S' , we need only find its vertices, which are the images of the vertices of S under f :

$$\begin{aligned}f(-1 + i) &= -2 - i & f(1 + i) &= -2 + 7i \\f(1 + 2i) &= -6 + 7i & f(-1 + 2i) &= -6 - i.\end{aligned}$$

Therefore, S' is the rectangle with vertices $-2 - i$, $-2 + 7i$, $-6 + 7i$ and $-6 - i$.

Alternative Point of View

- The linear mapping $f(z) = 4iz + 2 + 3i$ is a composition of
 - a rotation through $\text{Arg}(4i) = \frac{\pi}{2}$ radians;
 - a magnification by $|4i| = 4$ and
 - a translation by $2 + 3i$.

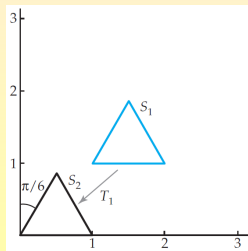


A Linear Mapping of a Triangle I

- Find a complex linear function that maps the equilateral triangle with vertices $1 + i$, $2 + i$ and $\frac{3}{2} + (1 + \frac{1}{2}\sqrt{3})i$ onto the equilateral triangle with vertices i , $\sqrt{3} + 2i$ and $3i$.

Let S_1 denote the triangle with vertices $1 + i$, $2 + i$ and $\frac{3}{2} + (1 + \frac{1}{2}\sqrt{3})i$ and let S' represent the triangle with vertices i , $3i$ and $\sqrt{3} + 2i$.

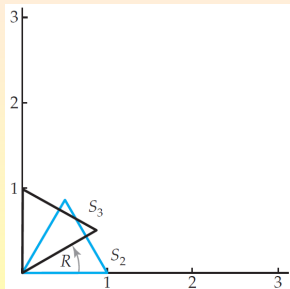
- We first translate S_1 to have one of its vertices at the origin. If $1 + i$ should be mapped onto 0 , then this is accomplished by the translation $T_1(z) = z - (1 + i)$. Let S_2 be the image of S_1 under T_1 .



Note that the angle between the imaginary axis and the edge of S_2 containing the vertices 0 and $\frac{1}{2} + \frac{1}{2}\sqrt{3}i$ is $\frac{\pi}{6}$.

A Linear Mapping of a Triangle II

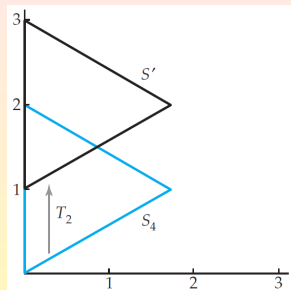
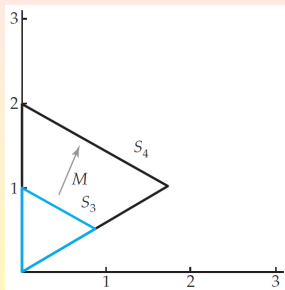
- A rotation through an angle of $\frac{\pi}{6}$ radians counterclockwise about the origin will map S_2 onto a triangle with two vertices on the imaginary axis. This rotation is given by $R(z) = (e^{i\pi/6})z = (\frac{1}{2}\sqrt{3} + \frac{1}{2}i)z$. The image of S_2 under R is the triangle S_3 with vertices at 0 , $\frac{1}{2}\sqrt{3} + \frac{1}{2}i$ and i :



It is easy to verify that each side of the triangle S_3 has length 1.

A Linear Mapping of a Triangle III

- Because each side of the desired triangle S' has length 2, we next magnify S_3 by a factor of 2. The magnification $M(z) = 2z$ maps the triangle S_3 onto the triangle S_4 with vertices 0 , $\sqrt{3} + i$, and $2i$:



- Finally, we translate S_4 by i using the mapping $T_2(z) = z + i$. This maps triangle S_4 onto the triangle S' with vertices i , $\sqrt{3} + 2i$, and $3i$.
- Thus, the linear mapping: $f(z) = T_2 \circ M \circ R \circ T_1(z) = (\sqrt{3} + i)z + 1 - \sqrt{3} + \sqrt{3}i$ maps the triangle S_1 onto the triangle S' .

Linear Approximations

- The study of differential calculus is based on the principle that **real linear functions are the easiest types of functions to understand**. One of the many uses of the derivative is to find a **linear function that approximates f in a neighborhood of a point x_0** .
- The **linear approximation** of a differentiable function $f(x)$ at $x = x_0$ is the linear function $\ell(x) = f(x_0) + f'(x_0)(x - x_0)$. Geometrically, the graph of $\ell(x)$ is the tangent line to the graph of f at $(x_0, f(x_0))$.
- The linear approximation formula can be applied to complex functions once an appropriate definition of the derivative of complex function is given. If $f'(z_0)$ represents the derivative of the complex function $f(z)$ at z_0 , then the **linear approximation of f in a neighborhood of z_0** is the complex linear function $\ell(z) = f(z_0) + f'(z_0)(z - z_0)$. Geometrically, $\ell(z)$ approximates how $f(z)$ acts as a complex mapping near the point z_0 .

An Example

- The derivative of the complex function $f(z) = z^2$ is $f'(z) = 2z$.
- Therefore, the linear approximation of $f(z) = z^2$ at $z_0 = 1 + i$ is

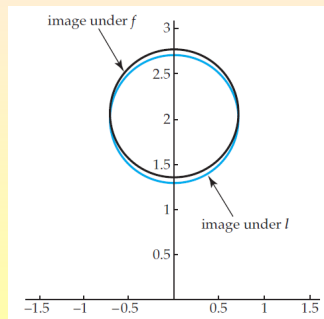
$$\ell(z) = 2i + 2(1 + i)(z - 1 - i) = 2\sqrt{2}(e^{i\pi/4}z) - 2i.$$

Near the point $z_0 = 1 + i$ the mapping $w = z^2$ can be approximated by the linear mapping consisting of the composition of:

- rotation through $\frac{\pi}{4}$,
- magnification by $2\sqrt{2}$,
- and translation by $-2i$.



The image of the circle $|z - (1 + i)| = 0.25$ under both f and ℓ are shown on the right.



Subsection 4

Special Power Functions

Complex Polynomial and Principal Root Functions

- A **complex polynomial function** is a function of the form $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where n is a positive integer and $a_n, a_{n-1}, \dots, a_1, a_0$ are complex constants.
- In general, a complex polynomial mapping can be quite complicated, but in many special cases the action of the mapping is easily understood.
- We now study complex polynomials of the form $f(z) = z^n, n \geq 2$.
- Unlike the linear mappings, the mappings $w = z^n, n \geq 2$, **do not preserve the basic shape** of every figure in the complex plane.
- Associated to the function $z^n, n \geq 2$, we also have the *principal n th root function* $z^{1/n}$.

The principal n th root functions are inverse functions of the functions z^n defined on a sufficiently restricted domain.

Power Functions

- A real function of the form $f(x) = x^a$, where a is a real constant, is called a **power function**.
- We form a **complex power function** by allowing the input or the exponent a to be a complex number.
- A **complex power function** is a function of the form

$$f(z) = z^\alpha, \quad \alpha \text{ a complex constant.}$$

- If α is an integer, then the power function z^α can be evaluated using the algebraic operations on complex numbers seen earlier:
Example: $z^2 = z \cdot z$ and $z^{-3} = \frac{1}{z \cdot z \cdot z}$.
- We can also use the formulas for taking roots of complex numbers to define power functions with fractional exponents of the form $\frac{1}{n}$.
- We restrict attention to special complex power functions of the form z^n and $z^{1/n}$, where $n \geq 2$ and n is an integer.
- More complicated complex power functions such as $z^{\sqrt{2}-i}$, will be discussed after the introduction of the complex logarithmic function.

The Power Function z^n

- We consider complex power functions of the form z^n , $n \geq 2$.
- We begin with the simplest of these functions, the **complex squaring function** z^2 .
- Values of the complex power function $f(z) = z^2$ are easily found using complex multiplication.

Example: At $z = 2 - i$, we have

$$f(2 - i) = (2 - i)^2 = (2 - i) \cdot (2 - i) = 3 - 4i.$$

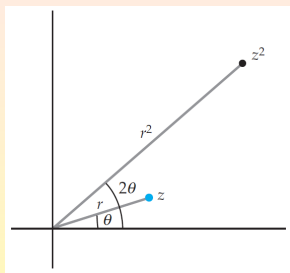
- We express $w = z^2$ in exponential notation by replacing z with $re^{i\theta}$:

$$w = z^2 = (re^{i\theta})^2 = r^2 e^{i2\theta}.$$

- The modulus r^2 of the point w is the square of the modulus r of the point z ;
- The argument 2θ of w is twice the argument θ of z .

The Complex Squaring Function z^2

- If we plot both z and w in the same copy of the complex plane, then w is obtained by magnifying z by a factor of r and then by rotating the result through the angle θ about the origin.



- The figure shows z and $w = z^2$, when $r > 1$ and $\theta > 0$.
- If $0 < r < 1$, then z is contracted by a factor of r , and if $\theta < 0$, then the rotation is clockwise.

Magnification and Rotation in Complex Squaring

- The magnification factor and the rotation angle associated to $w = f(z) = z^2$ depend on where z is located in the complex plane.

Example: Since $f(2) = 4$ and $f(\frac{i}{2}) = -\frac{1}{4}$, the point $z = 2$ is magnified by 2 but not rotated, whereas the point $z = \frac{i}{2}$ is contracted by $\frac{1}{2}$ and rotated through $\frac{\pi}{2}$.

- The function z^2 does not magnify the modulus of points on the unit circle $|z| = 1$ and it does not rotate points on the positive real axis.
- Consider a ray emanating from the origin and making an angle of ϕ with the positive real axis.
 - The images of all points have an argument of 2ϕ . Thus, they lie on a ray emanating from the origin and making an angle of 2ϕ with the positive real axis.
 - The modulus ρ of a point on the ray can be any value in $[0, \infty]$. So the modulus ρ^2 of a point in the image can also be any value in $[0, \infty]$.

Hence, the ray is mapped onto a ray emanating from the origin making an angle 2ϕ with the positive real axis.

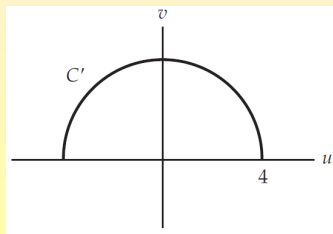
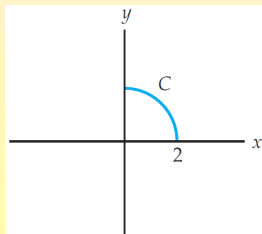
Image of a Circular Arc under $w = z^2$

- Find the image of the circular arc defined by $|z| = 2$, $0 \leq \arg(z) \leq \frac{\pi}{2}$, under the mapping $w = z^2$.

Let C be the circular arc defined by $|z| = 2$, $0 \leq \arg(z) \leq \frac{\pi}{2}$, and let C' denote the image of C under $w = z^2$.

- Since each point in C has modulus 2, each point in C' has modulus $2^2 = 4$. Thus, the image C' must be contained in the circle $|w| = 4$.
- Since the arguments of the points in C take on every value in $[0, \frac{\pi}{2}]$, the points in C' have arguments that take on every value in $[0, \pi]$.

So C' is the semicircle defined by $|w| = 4$, $0 \leq \arg(w) \leq \pi$.



Alternative Solution

- An alternative way to find the image of the circular arc defined by $|z| = 2$, $0 \leq \arg(z) \leq \frac{\pi}{2}$, under the mapping $w = z^2$ is to use a parametrization.

The circular arc C can be parametrized by $z(t) = 2e^{it}$, $0 \leq t \leq \frac{\pi}{2}$. Its image C' is given by $w(t) = f(z(t)) = 4e^{i2t}$, $0 \leq t \leq \frac{\pi}{2}$. By replacing the parameter t with $s = 2t$, we obtain $W(s) = 4e^{is}$, $0 \leq s \leq \pi$. This is a parametrization of the semicircle $|w| = 4$, $0 \leq \arg(w) \leq \pi$.

- Similarly, the squaring function maps a semicircle

$$|z| = r, -\frac{\pi}{2} \leq \arg(z) \leq \frac{\pi}{2},$$

onto a circle $|w| = r^2$.

Mapping of a Half-Plane onto the Entire Plane

- Since the right half-plane $\operatorname{Re}(z) \geq 0$ consists of the collection of semicircles $|z| = r$, $-\frac{\pi}{2} \leq \arg(z) \leq \frac{\pi}{2}$, where r takes on every value in the interval $[0, \infty)$, the image of this half-plane consists of the collection of circles $|w| = r^2$ where r takes on any value in $[0, \infty)$. This implies that $w = z^2$ maps the right half-plane $\operatorname{Re}(z) \geq 0$ onto the entire complex plane.

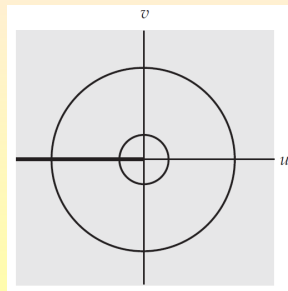
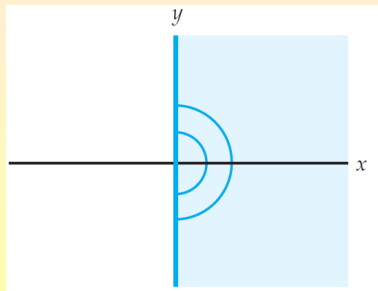


Image of a Vertical Line under $w = z^2$

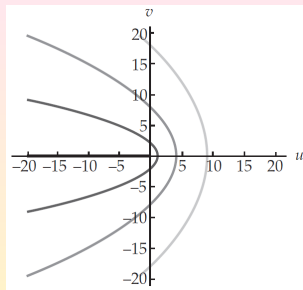
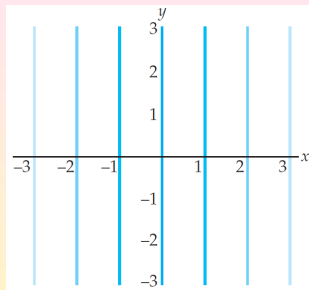
- Find the image of the vertical line $x = k$ under the mapping $w = z^2$. In this example it is convenient to work with real and imaginary parts of $w = z^2$ which are

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy.$$

Since the vertical line $x = k$ consists of the points $z = k + iy, -\infty < y < \infty$, it follows that the image of this line consists of all points $w = u + iv$, where $u = k^2 - y^2, v = 2ky$. If $k \neq 0$, we get $y = \frac{v}{2k}$ and then $u = k^2 - \frac{v^2}{4k^2}, -\infty < v < \infty$. Thus, the image of the line $x = k$ (with $k \neq 0$) under $w = z^2$ is a parabola that opens in the direction of the negative u -axis, has its vertex at $(k^2, 0)$, and has v -intercepts at $(0, \pm 2k^2)$. Since the image is unchanged if k is replaced by $-k$, if $k \neq 0$, the pair of vertical lines $x = k$ and $x = -k$ are both mapped onto the parabola $u = k^2 - \frac{v^2}{4k^2}$.

Image of a Vertical Line under $w = z^2$ (Cont'd)

- The action of the mapping $w = z^2$ on vertical lines is depicted below:



- The lines $x = 3$ and $x = -3$ are mapped onto the parabola with vertex at $(9, 0)$.
- Similarly, the lines $x = \pm 2$ are mapped onto the parabola with vertex at $(4, 0)$, and the lines $x = \pm 1$ onto the parabola with vertex at $(1, 0)$.
- In the case when $k = 0$, the image of the line $x = 0$ (the imaginary axis) is given by: $u = -y^2, v = 0, -\infty < y < \infty$. Therefore, the imaginary axis is mapped onto the negative real axis.

Image of a Horizontal Line under $w = z^2$

- The same method can be used to show that a horizontal line $y = k$, $k \neq 0$, is mapped by $w = z^2$ onto the parabola

$$u = \frac{v^2}{4k^2} - k^2.$$

- The image is unchanged if k is replaced by $-k$. So the pair $y = k$ and $y = -k$, $k \neq 0$, are both mapped onto the same parabola.
- If $k = 0$, then the horizontal line $y = 0$ (the real axis) is mapped onto the positive real axis.

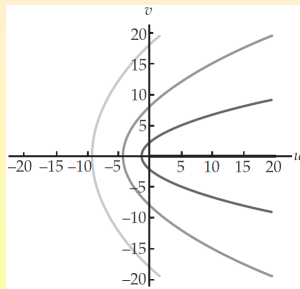
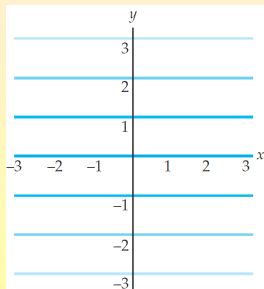


Image of a Triangle under $w = z^2$

- Find the image of the triangle with vertices 0 , $1 + i$ and $1 - i$ under the mapping $w = z^2$.

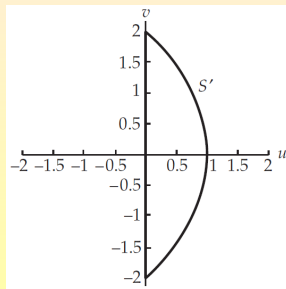
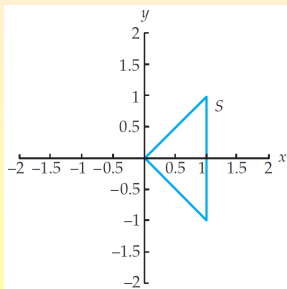
Let S denote the triangle with vertices at 0 , $1 + i$ and $1 - i$, and let S' denote its image under $w = z^2$.

- The side of S containing the vertices 0 and $1 + i$ lies on a ray emanating from the origin and making an angle of $\frac{\pi}{4}$ radians with the positive x -axis. The image of this segment must lie on a ray making an angle of $2\frac{\pi}{4} = \frac{\pi}{2}$ radians with the positive u -axis. Since the moduli of the points on the edge containing 0 and $1 + i$ vary from 0 to $\sqrt{2}$, the moduli of the images of these points vary from 0 to 2 . Thus, the image of this side is a vertical line segment from 0 to $2i$ contained in the v -axis.
- In a similar manner, we find that the image of the side of S containing the vertices 0 and $1 - i$ is a vertical line segment from 0 to $-2i$ contained in the v -axis.

Image of a Triangle under $w = z^2$ (Cont'd)

- We continue with the image of the triangle with vertices $0, 1 + i$ and $1 - i$ under the mapping $w = z^2$:
 - The remaining side of S contains the vertices $1 - i$ and $1 + i$. This side consists of the set of points $z = 1 + iy$, $-1 \leq y \leq 1$. Because this side is contained in the vertical line $x = 1$, its image is a parabolic segment given by: $u = 1 - \frac{v^2}{4}$, $-2 \leq v \leq 2$.

Thus, we have shown that the image of triangle S is the figure S' shown below.



The Function z^n , $n > 2$

- An analysis similar to that used for the mapping $w = z^2$ can be applied to the mapping $w = z^n$, $n > 2$.
- By replacing the symbol z with $re^{i\theta}$ we obtain:

$$w = z^n = r^n e^{in\theta}.$$

- Consequently, if z and $w = z^n$ are plotted in the same copy of the complex plane, then this mapping can be visualized as the process of
 - magnifying or contracting the modulus r of z to the modulus r^n of w ;
 - rotating z about the origin to increase an argument θ of z to an argument $n\theta$ of w .
- **Example:** A ray emanating from the origin and making an angle of ϕ radians with the positive x -axis is mapped onto a ray emanating from the origin and making an angle of $n\phi$ radians with the positive u -axis.

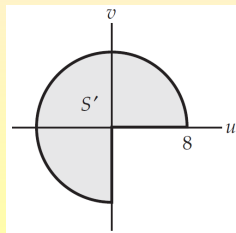
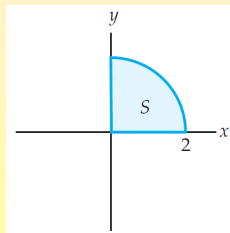
Image of a Circular Wedge under $w = z^3$

- Determine the image of the quarter disk defined by the inequalities $|z| \leq 2$, $0 \leq \arg(z) \leq \frac{\pi}{2}$, under the mapping $w = z^3$.

Let S denote the quarter disk and let S' denote its image under $w = z^3$.

- Since the moduli of the points in S vary from 0 to 2 the moduli of the points in S' vary from 0 to 8.
- In addition, because the arguments of the points in S vary from 0 to $\frac{\pi}{2}$, the arguments of the points in S' vary from 0 to $\frac{3\pi}{2}$.

Therefore, S' is given by the inequalities $|w| \leq 8$, $0 \leq \arg(w) \leq \frac{3\pi}{2}$:



The Power Function $z^{1/n}$

- We now investigate complex power functions of the form $z^{1/n}$, where n is an integer and $n \geq 2$. We begin with $n = 2$.
- We have seen that the n n -th roots of a nonzero complex number $z = r(\cos \theta + i \sin \theta) = re^{i\theta}$ are given by:

$$\sqrt[n]{r} \left[\cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right] = \sqrt[n]{r} e^{i(\theta + 2k\pi)/n},$$

for $k = 0, \dots, n - 1$.

- For $n = 2$, we get

$$\sqrt{r} \left[\cos \frac{\theta + 2k\pi}{2} + i \sin \frac{\theta + 2k\pi}{2} \right] = \sqrt{r} e^{i(\theta + 2k\pi)/2}, \quad k = 0, 1.$$

- By setting $\theta = \text{Arg}(z)$ and $k = 0$, we can define a **function** that assigns to z the **unique principal square root**.

The Principal Square Root Function

Definition (The Principal Square Root Function)

The function $z^{1/2}$ defined by

$$z^{1/2} = \sqrt{|z|}e^{i\text{Arg}(z)/2}$$

is called the **principal square root function**.

- If we set $\theta = \text{Arg}(z)$ and replace z with $re^{i\theta}$, then we obtain an alternative description of the principal square root function for $|z| > 0$:

$$z^{1/2} = \sqrt{r}e^{i\theta/2}, \quad r = |z| \text{ and } \theta = \text{Arg}(z).$$

- Note that the symbol $z^{1/2}$, as used in the definition, represents something different from the same symbol as used previously.

Values of the Principal Square Root Function

- **Example:** Find the values of the principal square root function $z^{1/2}$ at the following points: (a) $z = 4$ (b) $z = -2i$ (c) $z = -1 + i$.
- (a) For $z = 4$, $|z| = |4| = 4$ and $\text{Arg}(z) = \text{Arg}(4) = 0$. Thus,
 $4^{1/2} = \sqrt{4}e^{i(0/2)} = 2e^{i(0)} = 2$.
- (b) For $z = -2i$, $|z| = |-2i| = 2$ and $\text{Arg}(z) = \text{Arg}(-2i) = -\frac{\pi}{2}$,
whence $(-2i)^{1/2} = \sqrt{2}e^{i(-\pi/2)/2} = \sqrt{2}e^{-i\pi/4} = 1 - i$.
- (c) For $z = -1 + i$, $|z| = |-1 + i| = \sqrt{2}$ and $\text{Arg}(z) = \text{Arg}(-1 + i) = \frac{3\pi}{4}$, and, hence, $(-1 + i)^{1/2} = \sqrt{(\sqrt{2})}e^{i(3\pi/4)/2} = \sqrt[4]{2}e^{i(3\pi/8)}$.

One-to-One Functions

- The principal square root function $z^{1/2}$ is an **inverse function** of the squaring function z^2 .
- A real function must be one-to-one in order to have an inverse function. The same is true for a complex function.
- A complex function f is **one-to-one** if each point w in the range of f is the image of a unique point z , called the **pre-image** of w , in the domain of f . That is, f is one-to-one if whenever $f(z_1) = f(z_2)$, then $z_1 = z_2$. Equivalently, if $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$.

Example: The function $f(z) = z^2$ is not one-to-one because $f(i) = f(-i) = -1$.

- If f is a one-to-one complex function, then for any point w in the range of f there is a unique pre-image in the z -plane, which we denote by $f^{-1}(w)$.
- This correspondence between a point w and its pre-image $f^{-1}(w)$ defines the **inverse function** of a one-to-one complex function.

Inverse Functions

Definition (Inverse Function)

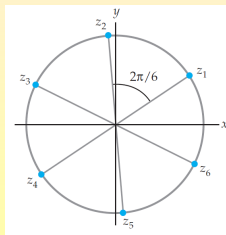
If f is a **one-to-one** complex function with domain A and range B , then the **inverse function** of f , denoted by f^{-1} , is the function with domain B and range A defined by

$$f^{-1}(z) = w \quad \text{if} \quad f(w) = z.$$

- If a set S is mapped onto a set S' by a one-to-one function f , then f^{-1} maps S' onto S .
- If f has an inverse function, then $f(f^{-1}(z)) = z$ and $f^{-1}(f(z)) = z$.
I.e., the two compositions $f \circ f^{-1}$ and $f^{-1} \circ f$ are the identities.
- **Example:** Show that the complex function $f(z) = z + 3i$ is one-to-one on the entire complex plane and find a formula for its inverse function.
 $f(z_1) = f(z_2)$ implies $z_1 + 3i = z_2 + 3i$ which implies $z_1 = z_2$.
The inverse function of f can often be found algebraically by solving the equation $z = f(w)$ for the symbol w : $z = w + 3i$ implies $w = z - 3i$. Therefore, $f^{-1}(z) = z - 3i$.

Functions of z^n , $n \geq 2$, Not One-to-One

- The function $f(z) = z^n$, $n \geq 2$, is not one-to-one: Consider the points $z_1 = re^{i\theta}$ and $z_2 = re^{i(\theta+2\pi/n)}$ with $r \neq 0$. Because $n \geq 2$, the points z_1 and z_2 are distinct. Note $f(z_1) = r^n e^{in\theta}$ and $f(z_2) = r^n e^{i(n\theta+2\pi)} = r^n e^{in\theta} e^{i2\pi} = r^n e^{in\theta}$. Therefore, f is not one-to-one.
- In fact, the n distinct points $z_1 = re^{i\theta}$, $z_2 = re^{i(\theta+2\pi/n)}$, $z_3 = re^{i(\theta+4\pi/n)}$, \dots , $z_n = re^{i(\theta+2(n-1)\pi/n)}$ are all mapped onto the single point $w = r^n e^{in\theta}$ by $f(z) = z^n$.
- This fact is illustrated for $n = 6$:



Restricting the Domain

- Recall that even though the real functions $f(x) = x^2$ and $g(x) = \sin x$ are not one-to-one and, thus, appear not to have inverses, yet we still have the inverse functions $f^{-1}(x) = \sqrt{x}$ and $g^{-1}(x) = \arcsin x$.
- The key is to appropriately restrict the domains of $f(x) = x^2$ and $g(x) = \sin x$ to sets on which the functions are one-to-one.

Example: Whereas $f(x) = x^2$ defined on $(-\infty, \infty)$ is not one-to-one, the same function defined on $[0, \infty)$ is one-to-one.

Similarly, $g(x) = \sin x$ is not one-to-one on $(-\infty, \infty)$, but it is one-to-one on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

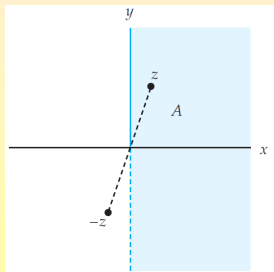
The function $f^{-1}(x) = \sqrt{x}$ is the inverse of $f(x) = x^2$ defined on the interval $[0, \infty)$. Since $\text{Dom}(f) = [0, \infty)$ and $\text{Range}(f) = [0, \infty)$, the domain and range of $f^{-1}(x) = \sqrt{x}$ are both $[0, \infty)$ as well.

Similarly, $g^{-1}(x) = \arcsin x$ is the inverse function of the function $g(x) = \sin x$ defined on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The domain and range of g^{-1} are $[-1, 1]$ and $[-\frac{\pi}{2}, \frac{\pi}{2}]$, respectively.

A Restricted Domain for $f(z) = z^2$

- Show that $f(z) = z^2$ is a one-to-one function on the set A defined by $-\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$

We show that f is one-to-one by demonstrating that if z_1 and z_2 are in A and if $f(z_1) = f(z_2)$, then $z_1 = z_2$. If $f(z_1) = f(z_2)$, then $z_1^2 = z_2^2$, or, equivalently, $z_1^2 - z_2^2 = 0$. By factoring this expression, we obtain $(z_1 - z_2)(z_1 + z_2) = 0$. It follows that either $z_1 = z_2$ or $z_1 = -z_2$. By definition of the set A , both z_1 and z_2 are nonzero. The complex points z and $-z$ are symmetric about the origin.



Inspection shows that if z_2 is in A , then $-z_2$ is not in A . This implies that $z_1 \neq -z_2$, since z_1 is in A . Therefore, we conclude that $z_1 = z_2$, and this proves that f is a one-to-one function on A .

An Alternative Approach

- The preceding technique does not extend to the function z^n , $n > 2$.
- We present an alternative approach.
- We prove that $f(z) = z^2$ is one-to-one on A by showing that if $f(z_1) = f(z_2)$ for two complex numbers z_1 and z_2 in A , then $z_1 = z_2$.
Suppose that z_1 and z_2 are in A . Then we may write $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ with $-\frac{\pi}{2} < \theta_1 \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} < \theta_2 \leq \frac{\pi}{2}$. If $f(z_1) = f(z_2)$, then it follows $r_1^2 e^{i2\theta_1} = r_2^2 e^{i2\theta_2}$. We conclude that the complex numbers $r_1^2 e^{i2\theta_1}$ and $r_2^2 e^{i2\theta_2}$ have the same modulus and principal argument: $r_1^2 = r_2^2$ and $\text{Arg}(r_1^2 e^{i2\theta_1}) = \text{Arg}(r_2^2 e^{i2\theta_2})$. Because both r_1 and r_2 are positive, we get $r_1 = r_2$. Moreover, since $-\frac{\pi}{2} < \theta_1 \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} < \theta_2 \leq \frac{\pi}{2}$, it follows that $-\pi < 2\theta_1 \leq \pi$ and $-\pi < 2\theta_2 \leq \pi$. This means that $\text{Arg}(r_1^2 e^{i2\theta_1}) = 2\theta_1$ and $\text{Arg}(r_2^2 e^{i2\theta_2}) = 2\theta_2$. This fact combined with the second equation implies that $2\theta_1 = 2\theta_2$, or $\theta_1 = \theta_2$. Therefore, z_1 and z_2 are equal because they have the same modulus and principal argument.

An Inverse of $f(z) = z^2$

- The squaring function z^2 is one-to-one on the set A defined by $-\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$. Thus, this function has a well-defined inverse function f^{-1} . We show this **inverse function is the principal square root function $z^{1/2}$** .

Let $z = re^{i\theta}$ and $w = \rho e^{i\phi}$, where θ and ϕ are the principal arguments of z and w , respectively. Suppose that $w = f^{-1}(z)$. Since the range of f^{-1} is the domain of f , the principal argument ϕ of w must satisfy: $-\frac{\pi}{2} < \phi \leq \frac{\pi}{2}$. On the other hand, $f(w) = w^2 = z$.

Hence, w is one of the two square roots of z , i.e., either $w = \sqrt{r}e^{i\theta/2}$ or $w = \sqrt{r}e^{i(\theta+2\pi)/2}$. Assume that w is the latter, i.e., assume that $w = \sqrt{r}e^{i(\theta+2\pi)/2}$. Because $\theta = \text{Arg}(z)$, we have $-\pi < \theta \leq \pi$, and so, $\frac{\pi}{2} < \frac{\theta+2\pi}{2} \leq \frac{3\pi}{2}$. We conclude that the principal argument ϕ of w must satisfy either $-\pi < \phi \leq -\frac{\pi}{2}$ or $\frac{\pi}{2} < \phi \leq \pi$. However, this cannot be true since $-\frac{\pi}{2} < \phi \leq \frac{\pi}{2}$. So $w = \sqrt{r}e^{i\pi/2}$, which is the value of the principal square root function $z^{1/2}$.

Domain and Range of $f^{-1}(z) = z^{1/2}$

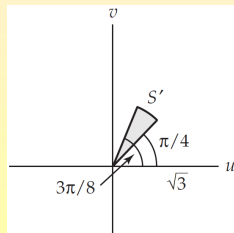
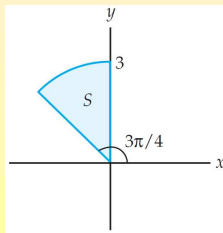
- Since $z^{1/2}$ is an inverse function of $f(z) = z^2$ defined on the set $-\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$, it follows that the domain and range of $z^{1/2}$ are the range and domain of f , respectively. In particular, $\text{Range}(z^{1/2}) = A$, that is, **the range of $z^{1/2}$ is the set of complex w satisfying $-\frac{\pi}{2} < \arg(w) \leq \frac{\pi}{2}$** . In order to find $\text{Dom}(z^{1/2})$ we need to find the range of f . We saw that $w = z^2$ maps the right half-plane $\text{Re}(z) \geq 0$ onto the entire complex plane. The set A is equal to the right half-plane $\text{Re}(z) \geq 0$ excluding the set of points on the ray emanating from the origin and containing the point $-i$. That is, A does not include the point $z = 0$ or the points satisfying $\arg(z) = -\frac{\pi}{2}$. However, we have seen that the image of the set $\arg(z) = \frac{\pi}{2}$, the positive imaginary axis, is the same as the image of the set $\arg(z) = -\frac{\pi}{2}$. Both sets are mapped onto the negative real axis. Since the set $\arg(z) = \frac{\pi}{2}$ is contained in A , it follows that the only difference between the image of the set A and the image of the right half-plane $\text{Re}(z) \geq 0$ is the image of the point $z = 0$, which is the point $w = 0$. Since A is mapped onto the entire complex plane excluding the point $w = 0$, **the domain of $f^{-1}(z) = z^{1/2}$ is the entire complex plane \mathbb{C} excluding 0**.

The Mapping $w = z^{1/2}$

- As a mapping, z^2 squares the modulus of z and doubles its argument.
- Thus, the mapping $w = z^{1/2}$ takes the square root of the modulus of a point and halves its principal argument, i.e., if $w = z^{1/2}$, then we have $|w| = \sqrt{|z|}$ and $\text{Arg}(w) = \frac{1}{2}\text{Arg}(z)$.
- Example** (Image of a Circular Sector under $w = z^{1/2}$): Find the image of the set S defined by $|z| \leq 3$, $\frac{\pi}{2} \leq \arg(z) \leq \frac{3\pi}{4}$, under $w = z^{1/2}$.

Let S' denote the image of S under $w = z^{1/2}$.

- Since $|z| \leq 3$ for points in S , we have that $|w| \leq \sqrt{3}$ for points w in S' .
- Since $\frac{\pi}{2} \leq \arg(z) \leq \frac{3\pi}{4}$ for points in S , $\frac{\pi}{4} \leq \arg(w) \leq \frac{3\pi}{8}$ for points w in S' .



Principal n -th Root Function

- The complex power function $f(z) = z^n$, $n > 2$, is one-to-one on the set defined by $-\frac{\pi}{n} < \arg(z) \leq \frac{\pi}{n}$.
- It can be seen that the image of this set under the mapping $w = z^n$ is the entire complex plane \mathbb{C} excluding $w = 0$.
- Therefore, there is a well-defined inverse function for f .
- Analogous to the case $n = 2$, this inverse function of z^n is called the **principal n -th root function** $z^{1/n}$.
- The domain of $z^{1/n}$ is the set of all nonzero complex numbers, and the range of $z^{1/n}$ is the set of w satisfying $-\frac{\pi}{n} < \arg(w) \leq \frac{\pi}{n}$.

Definition (Principal n -th Root Functions)

For $n \geq 2$, the function $z^{1/n}$ defined by

$$z^{1/n} = \sqrt[n]{|z|} e^{i \operatorname{Arg}(z)/n}$$

is called the **principal n -th root function**.

- By setting $z = re^{i\theta}$, with $\theta = \operatorname{Arg}(z)$, we have $z^{1/n} = \sqrt[n]{r} e^{i\theta/n}$.

Values of $z^{1/n}$

- Find the value of the given principal n th root function $z^{1/n}$ at the given point z : (a) $z^{1/3}$; $z = i$ (b) $z^{1/5}$; $z = 1 - \sqrt{3}i$.

(a) For $z = i$, $|z| = 1$ and $\text{Arg}(z) = \frac{\pi}{2}$. Thus, we obtain:

$$i^{1/3} = \sqrt[3]{1}e^{i(\pi/2)/3} = e^{i\pi/6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i.$$

(b) For $z = 1 - \sqrt{3}i$, we have $|z| = 2$ and $\text{Arg}(z) = -\frac{\pi}{3}$. Thus, we get

$$(1 - \sqrt{3}i)^{1/5} = \sqrt[5]{2}e^{i(-\pi/3)/5} = \sqrt[5]{2}e^{-i(\pi/15)}.$$

Multiple-Valued Functions

- A nonzero complex number z has n distinct n -th roots in the complex plane. Thus, the process of “taking the n -th root” of a complex number z does not define a complex function. We introduced the symbol $z^{1/n}$ to represent the set consisting of the n n -th roots of z .
- Similarly, $\arg(z)$ represents an infinite set of values.
- These types of operations on complex numbers are examples of **multiple-valued functions**.

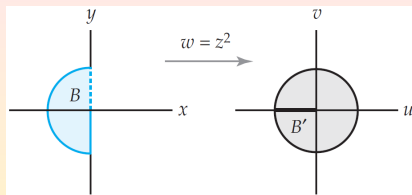
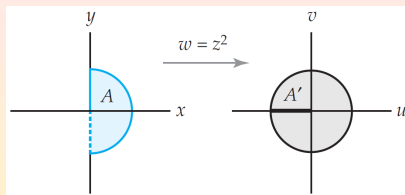
When representing multiple-valued functions with functional notation, we will use uppercase letters such as $F(z) = z^{1/2}$ or $G(z) = \arg(z)$.

Lowercase letters such as f and g will be reserved for functions.

- **Example:** $g(z) = z^{1/3}$ refers to the principal cube root function whereas $G(z) = z^{1/3}$ represents the multiple-valued function that assigns the three cube roots of z to the value of z . Thus, $g(i) = \frac{1}{2}\sqrt{3} + \frac{1}{2}i$ and $G(i) = \{\frac{1}{2}\sqrt{3} + \frac{1}{2}i, -\frac{1}{2}\sqrt{3} + \frac{1}{2}i, -i\}$.

Riemann Surface of $f(z) = z^2$

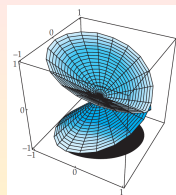
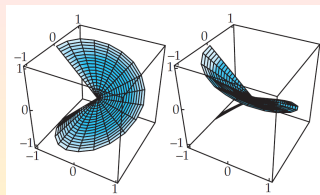
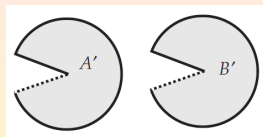
- $f(z) = z^2$ is not one-to-one. $f(z) = z^2$ is one-to-one on A defined by $|z| \leq 1, -\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$.



- $w = z^2$ is a one-to-one mapping of the set B defined by $|z| \leq 1, \frac{\pi}{2} < \arg(z) \leq \frac{3\pi}{2}$, onto the closed unit disk $|w| \leq 1$.
- Since the unit disk $|z| \leq 1$ is the union of the sets A and B , the image of the disk $|z| \leq 1$ under $w = z^2$ covers the disk $|w| \leq 1$ twice (once by A and once by B).
- We visualize this “covering” by considering two image disks for $w = z^2$.

Riemann Surface of $f(z) = z^2$ (Cont'd)

- Let A' denote the image of A under f and B' the image of B under f .
- Imagine the disks A' and B' cut open along the negative real axis:



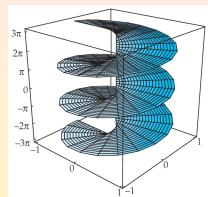
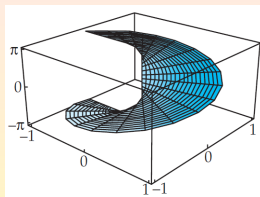
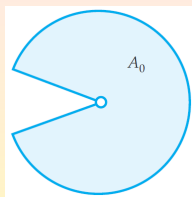
- We construct a Riemann surface for $f(z) = z^2$ by stacking the cut disks A' and B' one atop the other in xyz -space and attaching them by gluing together their edges.

After attaching in this manner we obtain the **Riemann surface**:

- Although $w = z^2$ is not a one-to-one mapping of the closed unit disk $|z| \leq 1$ onto the closed unit disk $|w| \leq 1$, it is a one-to-one mapping of the closed unit disk $|z| \leq 1$ onto the Riemann surface.

Riemann Surface of $G(z) = \arg(z)$

- Another interesting Riemann surface is one for the multiple valued function $G(z) = \arg(z)$ defined on $0 < |z| \leq 1$. We take a copy A_0 of the punctured disk $0 < |z| \leq 1$ and cut it open along the negative real axis. Let A_0 represent the points $re^{i\theta}$, $-\pi < \theta \leq \pi$.



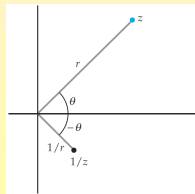
Take another copy A_1 and let it represent $re^{i\theta}$, $\pi < \theta \leq 3\pi$. Let A_{-1} represent the points $re^{i\theta}$, $-3\pi < \theta \leq -\pi$. We have an infinite set of cut disks $\dots, A_{-2}, A_{-1}, A_0, A_1, A_2, \dots$. Place A_n in xyz -space so that $re^{i\theta}$, with $(2n-1)\pi < \theta \leq (2n+1)\pi$, lies at height θ above the point $re^{i\theta}$ in the xy -plane. The collection of all the cut disks in xyz -space forms the Riemann surface for the multiple-valued function $G(z)$.

Subsection 5

Reciprocal Function

The Reciprocal Function

- Analogous to real functions, we define a **complex rational function** to be a function of the form $f(z) = \frac{p(z)}{q(z)}$ where both $p(z)$ and $q(z)$ are complex polynomial functions.
- The most basic complex rational function is the **reciprocal function**.
- The function $\frac{1}{z}$, whose domain is the set of all nonzero complex numbers, is called the **reciprocal function**.
- Given $z \neq 0$, if we set $z = re^{i\theta}$, we obtain: $w = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$.
 - The modulus of w is the reciprocal of the modulus of z ;
 - The argument of w is the negative of the argument of z .
- Therefore, the reciprocal function maps a point in the z -plane with polar coordinates (r, θ) onto a point in the w -plane with polar coordinates $(\frac{1}{r}, -\theta)$.

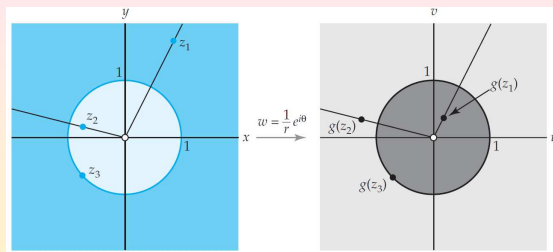


Inversion in the Unit Circle

- The function $g(z) = \frac{1}{\bar{z}}e^{i\theta}$, whose domain is the set of all nonzero complex numbers, is called **inversion in the unit circle**.
- We consider separately the images of points on the unit circle, points outside the unit circle, and points inside the unit circle.
 - Consider, first, a point z on the unit circle. Since $z = 1 \cdot e^{i\theta}$, $g(z) = \frac{1}{1}e^{i\theta} = z$. So each point on the unit circle is mapped onto itself by g .
 - If, on the other hand, z is a nonzero complex number that does not lie on the unit circle, then $z = re^{i\theta}$, with $r \neq 1$.
 - When $r > 1$ (z is outside of the unit circle), we have that $|g(z)| = |\frac{1}{r}e^{i\theta}| = \frac{1}{r} < 1$. So, the image under g of a point z outside the unit circle is a point inside the unit circle.
 - Conversely, if $r < 1$ (z is inside the unit circle), then $|g(z)| = \frac{1}{r} > 1$. Thus, if z is inside the unit circle, then its image under g is outside the unit circle.

Illustration of the Inversion in the Unit Circle

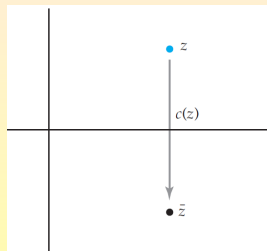
- The mapping $w = \frac{1}{z}$ is represented below:



- The arguments of z and $g(z)$ are equal. So, if $z_1 \neq 0$ is a point with modulus r in the z -plane, then $g(z_1)$ is the unique point in the w -plane with modulus $\frac{1}{r}$ lying on a ray emanating from the origin making an angle of $\arg(z_1)$ with the positive u -axis.
- The moduli of z and $g(z)$ are inversely proportional: the farther a point z is from 0 in the z -plane, the closer its image $g(z)$ is to 0 in the w -plane, and, the closer z is to 0, the farther $g(z)$ is from 0.

Complex Conjugation

- The second complex mapping that is helpful for describing the reciprocal mapping is a reflection across the real axis.
- Under this mapping the image of the point (x, y) is $(x, -y)$.
- This complex mapping is given by the function $c(z) = \bar{z}$, called the **complex conjugation function**.
- The relationship between z and its image $c(z)$ is shown below:



- If $z = re^{i\theta}$, then $c(z) = \overline{re^{i\theta}} = \bar{r}\bar{e^{i\theta}} = re^{-i\theta}$.

Reciprocal Mapping

- The reciprocal function $f(z) = \frac{1}{z}$ can be written as the **composition of inversion in the unit circle and complex conjugation**.
- Since $c(z) = re^{-i\theta}$ and $g(z) = \frac{1}{r}e^{i\theta}$, we get

$$c(g(z)) = c\left(\frac{1}{r}e^{i\theta}\right) = \frac{1}{r}e^{-i\theta} = \frac{1}{z}.$$

- Thus, as a mapping, the reciprocal function
 - first inverts in the unit circle,
 - then reflects across the real axis.
- In summary: Given z_0 a nonzero point in the complex plane the point $w_0 = f(z_0) = \frac{1}{z_0}$ is obtained by:
 - (i) inverting z_0 in the unit circle, then
 - (ii) reflecting the result across the real axis.

Image of a Semicircle under $w = \frac{1}{z}$

- Find the image of the semicircle $|z| = 2$, $0 \leq \arg(z) \leq \pi$, under the reciprocal mapping $w = \frac{1}{z}$.

Let C denote the semicircle and let C' denote its image under $w = \frac{1}{z}$. In order to find C' , we first invert C in the unit circle, then we reflect the result across the real axis.

- Under inversion in the unit circle, points with modulus 2 have images with modulus $\frac{1}{2}$. Moreover, inversion in the unit circle does not change arguments. The image is the semicircle $|w| = \frac{1}{2}$, $0 \leq \arg(w) \leq \pi$.
- Reflecting this set across the real axis negates the argument of a point but does not change its modulus. Hence, the image is the semicircle given by $|w| = \frac{1}{2}$, $-\pi \leq \arg(w) \leq 0$.

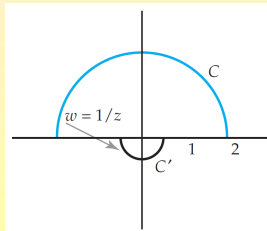


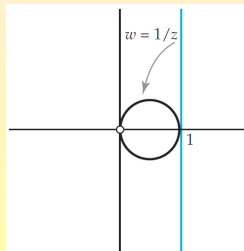
Image of a Line under $w = \frac{1}{z}$

- Find the image of the vertical line $x = 1$ under the mapping $w = \frac{1}{z}$.

The vertical line $x = 1$ consists of $z = 1 + iy$, $-\infty < y < \infty$. After replacing z with $1 + iy$ in $w = \frac{1}{z}$ and simplifying, we obtain:

$w = \frac{1}{1+iy} = \frac{1}{1+y^2} - \frac{y}{1+y^2}i$. It follows that the image of $x = 1$ under $w = \frac{1}{z}$ consists of all points $u + iv$ satisfying: $u = \frac{1}{1+y^2}$, $v = -\frac{y}{1+y^2}$, $-\infty < y < \infty$. We eliminate y : We have $v = -yu$. The first equation implies that $u \neq 0$, so we get $y = -\frac{v}{u}$. Thus, we obtain the quadratic equation $u^2 - u + v^2 = 0$.

Complete the square to get $(u - \frac{1}{2})^2 + v^2 = \frac{1}{4}$, $u \neq 0$. It defines a circle centered at $(\frac{1}{2}, 0)$ with radius $\frac{1}{2}$. However, because $u \neq 0$, the point $(0,0)$ is not in the image. Using the complex variable $w = u + iv$, we can describe this image by $|w - \frac{1}{2}| = \frac{1}{2}$, $w \neq 0$.



Reverting to the Extended Complex Number System

- The image of $x = 1$ is not the entire circle $|w - \frac{1}{2}| = \frac{1}{2}$ because points on the line $x = 1$ with extremely large modulus map onto points on the circle $|w - \frac{1}{2}| = \frac{1}{2}$ that are extremely close to 0, but there is no point on the line $x = 1$ that actually maps onto 0.
- To obtain the entire circle as the image, we must consider the **reciprocal function defined on the extended complex number system**.
- The extended complex number system consists of all the points in the complex plane adjoined with the ideal point ∞ .
- In the context of mappings this set of points is commonly referred to as the **extended complex plane**.
- The important property of the extended complex plane is the correspondence between points on the extended complex plane and the points on the complex plane.
 - In particular, points in the extended complex plane that are near the ideal point ∞ correspond to points with extremely large modulus in the complex plane.

Extending the Reciprocal Function

- We use this correspondence to extend the reciprocal function to a function whose domain and range are the extended complex plane.
- Since $w = \frac{1}{r}e^{-i\theta}$ already defines the reciprocal function for all points $z \neq 0$ or ∞ in the extended complex plane, we extend this function by specifying the images of 0 and ∞ .
 - If $z = re^{i\theta}$ is a point close to 0, then r is small, whence w is a point whose modulus $\frac{1}{r}$ is large. In the extended complex plane, if z is a point that is near 0, then $w = \frac{1}{z}$ is a point that is near the ideal point ∞ . So we define the reciprocal function $f(z) = \frac{1}{z}$ on the extended complex plane so that $f(0) = \infty$.
 - If z is a point that is near ∞ , in the extended complex plane, then $f(z)$ is a point that is near 0. Thus, we define the reciprocal function on the extended complex plane so that $f(\infty) = 0$.

The Reciprocal Function on the Extended Complex Plane

Definition (The Reciprocal Function on the Extended Complex Plane)

The **reciprocal function on the extended complex plane** is the function defined by

$$f(z) = \begin{cases} \frac{1}{z}, & \text{if } z \neq 0 \text{ or } \infty \\ \infty, & \text{if } z = 0 \\ 0, & \text{if } z = \infty \end{cases}$$

- We use the notation $\frac{1}{z}$ to represent both the reciprocal function and the reciprocal function on the extended complex plane.
- Whenever the ideal point ∞ is mentioned, it will be assumed that $\frac{1}{z}$ represents the reciprocal function defined on the extended complex plane.

Image of a Line under $w = \frac{1}{z}$

- Find the image of the vertical line $x = 1$ under the reciprocal function on the extended complex plane.

Since the line $x = 1$ is an unbounded set in the complex plane, the ideal point ∞ is on the line in the extended complex plane.

- We already saw that the image of the points $z \neq \infty$ on the line $x = 1$ is the circle $|w - \frac{1}{2}| = \frac{1}{2}$ excluding the point $w = 0$.
- We have that $f(\infty) = 0$, and so $w = 0$ is the image of the ideal point. This “fills in” the missing point in the circle $|w - \frac{1}{2}| = \frac{1}{2}$.

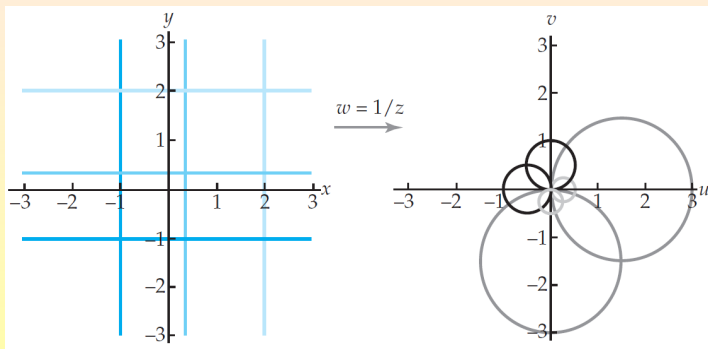
Therefore, the vertical line $x = 1$ is mapped onto the entire circle $|w - \frac{1}{2}| = \frac{1}{2}$ by the reciprocal mapping on the extended complex plane.

Mapping Lines to Circles with $w = \frac{1}{z}$

Mapping Lines to Circles with $w = \frac{1}{z}$

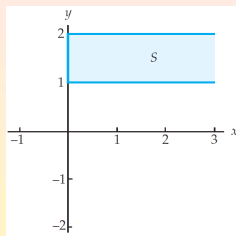
The reciprocal function on the extended complex plane maps:

- (i) The vertical line $x = k$ with $k \neq 0$ onto the circle $|w - \frac{1}{2k}| = |\frac{1}{2k}|$;
- (ii) The horizontal line $y = k$ with $k \neq 0$ onto the circle $|w + \frac{1}{2k}i| = |\frac{1}{2k}|$.



Mapping of a Semi-infinite Strip

- Find the image of the semi-infinite horizontal strip defined by $1 \leq y \leq 2$, $x \geq 0$, under $w = \frac{1}{z}$.



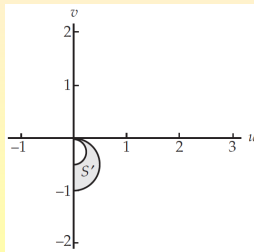
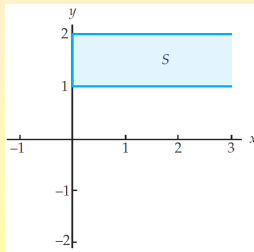
Let S denote the semi-infinite horizontal strip defined by $1 \leq y \leq 2$, $x \geq 0$. The boundary of S consists of the line segment $x = 0$, $1 \leq y \leq 2$, and the two half-lines $y = 1$ and $y = 2$, $0 \leq x < \infty$. We first determine the images of these boundary curves.

- The line segment $x = 0$, $1 \leq y \leq 2$, can also be described as the set $1 \leq |z| \leq 2$, $\arg(z) = \frac{\pi}{2}$. Since $w = \frac{1}{z}$, $\frac{1}{2} \leq |w| \leq 1$. In addition, we have that $\arg(w) = \arg(1/z) = -\arg(z)$, and so, $\arg(w) = -\frac{\pi}{2}$. Thus, the image of $x = 0$, $1 \leq y \leq 2$, is the line segment on the v -axis from $-\frac{1}{2}i$ to $-i$.

Mapping of a Semi-infinite Strip (Cont'd)

- Now consider $y = 1$, $0 \leq x < \infty$. The image is an arc in $|w + \frac{1}{2}i| = \frac{1}{2}$. The arguments satisfy $0 < \arg(z) \leq \frac{\pi}{2}$, so $-\frac{\pi}{2} \leq \arg(w) < 0$. Moreover, ∞ is on the half-line, and so $w = 0$ is in its image. Thus, the image of $y = 1$, $0 \leq x < \infty$, is $|w + \frac{1}{2}i| = \frac{1}{2}$, $-\frac{\pi}{2} \leq \arg(w) \leq 0$.
- Similarly, the image of $y = 2$, $0 \leq x < \infty$, is the circular arc $|w + \frac{1}{4}i| = \frac{1}{4}$, $-\frac{\pi}{2} \leq \arg(w) \leq 0$.

Every half-line $y = k$, $1 \leq k \leq 2$, between the boundary half-lines maps onto $|w + \frac{1}{2k}i| = \frac{1}{2k}$, $-\frac{\pi}{2} \leq \arg(w) \leq 0$, between these circular arcs:



The Inverse Mapping of $\frac{1}{z}$

- The reciprocal function $f(z) = \frac{1}{z}$ is one-to-one.
- Thus, f has a well-defined inverse function f^{-1} .
- Solving the equation $z = f(w)$ for w , we get $f^{-1}(z) = \frac{1}{z}$.
- This observation extends our understanding of the complex mapping $w = \frac{1}{z}$.
 - We have seen that the image of the line $x = 1$ under $\frac{1}{z}$ is the circle $|w - \frac{1}{2}| = \frac{1}{2}$. Since $f^{-1}(z) = \frac{1}{z} = f(z)$, the image of the circle $|z - \frac{1}{2}| = \frac{1}{2}$ under $\frac{1}{z}$ is the line $u = 1$.
 - Similarly, we see that the circles $|w - \frac{1}{2k}| = |\frac{1}{2k}|$ and $|w + \frac{1}{2k}i| = |\frac{1}{2k}|$ are mapped onto the lines $x = k$ and $y = k$, respectively.