Practice set 4

ANSWERS

Problem 4.1

It follows directly:

$$\int_a^b f(x)\,dx \approx \int_a^b f\big(\tfrac{a+b}{2}\big)\,dx = f\big(\tfrac{a+b}{2}\big)\int_a^b dx = (b-a)f\big(\tfrac{a+b}{2}\big).$$

Now, consider an evenly spaced partition of $[a,b]$: $\quad a = z_0 < z_1 < z_2 < \cdots < z_{N-1} < z_N = b$ with $h = \frac{b-a}{N}$.

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_{z_0}^{z_1} f(x)\,dx + \int_{z_1}^{z_2} f(x)\,dx + \ldots + \int_{z_{N-1}}^{z_N} f(x)\,dx \\ &\approx hf\big(\tfrac{z_0+z_1}{2}\big) + hf\big(\tfrac{z_1+z_2}{2}\big) + \ldots + hf\big(\tfrac{z_{N-1}+z_N}{2}\big) \\ &= h\big(f(x_1) + f(x_2) + \ldots + f(x_N)\big),\end{aligned}$$

where $x_j = \frac{z_{j-1}+z_j}{2} = a + (j-\frac{1}{2})h, \quad j = 1, 2, \ldots, N.$

Problem 4.2

Let $h = \frac{b-a}{4}$.
Consider points $x_0 = a$, $x_1 = a+h$, $x_2 = a+2h$, $x_3 = a+3h$, $x_4 = b$.
Let $P_4(x)$ be the degree 4 interpolating polynomial at the points $\big(x_j, f(x_j)\big)$, $j = 0,1,2,3,4$:

$$P_4(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x) + f(x_3)L_3(x) + f(x_4)L_4(x),$$

where $L_j(x)$ are the Lagrange basis functions associated to points x_0, x_1, x_2, x_3, x_4. Then

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \int_a^b P_4(x)\,dx \\ &= \int_a^b \Big(f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x) + f(x_3)L_3(x) + f(x_4)L_4(x)\Big) \\ &= f(x_0)\int_a^b L_0(x)\,dx + f(x_1)\int_a^b L_1(x)\,dx + f(x_2)\int_a^b L_2(x)\,dx + f(x_3)\int_a^b L_3(x)\,dx + f(x_4)\int_a^b L_4(x)\,dx \quad (1)\end{aligned}$$

Compute each integral $\int_a^b L_j(x)\,dx$. For example (using substitution $x - a - 2h = y$),

$$\begin{aligned}\int_a^b L_0(x)\,dx &= \int_a^b \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}\,dx \\ &= \int_a^b \frac{(x-a-h)(x-a-2h)(x-a-3h)(x-a-4h)}{h\cdot 2h\cdot 3h\cdot 4h}\,dx \\ &= \frac{1}{24h^4}\int_{-2h}^{2h}(y+h)y(y-h)(y-2h)\,dy \\ &= \frac{1}{24h^4}\int_{-2h}^{2h}(y^2-h^2)(y^2-2hy)\,dy \\ &= \frac{1}{24h^4}\int_{-2h}^{2h}(y^4-2hy^3-h^2y^2+2h^3y)\,dy \\ &= \frac{1}{24h^4}\left(\frac{y^5}{5} - 2h\frac{y^4}{4} - h^2\frac{y^3}{3} + 2h^3\frac{y^2}{2}\right)\Bigg|_{-2h}^{2h} \\ &= \frac{1}{24h^4}\left(\frac{64h^5}{5} - \frac{16h^5}{3}\right) \\ &= \frac{14}{45}h.\end{aligned}$$

Similarly,

$$\int_a^b L_1(x)\,dx = \frac{64}{45}h = \int_a^b L_3(x)\,dx, \quad \int_a^b L_2(x)\,dx = \frac{24}{45}h, \quad \int_a^b L_4(x)\,dx = \frac{14}{45}h$$

Substituting the last integrals in (1) we get the Boole's formula.

Problem 4.3

We have (see Lecture 13)

$$E_n^T(f) = -\frac{h^2(b-a)}{12} f''(\theta)$$

for some $\theta \in [a,b]$. Introducing in the last identity $a=0$, $b=1$, $h=\frac{b-a}{n}=\frac{1}{n}$ and $f(x)=\frac{e^x+e^{-x}}{2}$ we get

$$\begin{aligned}
E_n^T(f) &= -\frac{e^\theta + e^{-\theta}}{24\,n^2} \quad \text{for some } \theta \in [a,b] \\
\left|E_n^T(f)\right| &= \frac{e^\theta + e^{-\theta}}{24\,n^2} \quad \text{for some } \theta \in [a,b], \\
&\le \frac{\max_{[0,1]} |e^x + e^{-x}|}{24\,n^2} \\
&= \frac{e+e^{-1}}{24\,n^2} \\
&\approx \frac{3.0862}{24\,n^2}.
\end{aligned}$$

Then, in order to have $\left|E_n^T(f)\right| < 10^{-8}$, we ask

$$\begin{aligned}
\frac{3.0862}{24\,n^2} &< 10^{-8} \\
n^2 &> \frac{3.0862 \cdot 10^8}{24} \\
n &\ge 3586.
\end{aligned}$$

Therefore, for composite trapezoidal rule we need at least 3586 integration points.

Similarly, for composite Simpson's rule, we obtain

$$\begin{aligned}
E_n^S(f) &= -\frac{e^\theta + e^{-\theta}}{360\,n^4} \quad \text{for some } \theta \in [a,b] \\
\left|E_n^T(f)\right| &\le \frac{3.0862}{360\,n^4}.
\end{aligned}$$

And consequently, in order to have $\left|E_n^S(f)\right| < 10^{-8}$, we need

$$\begin{aligned}
\frac{3.0862}{360\,n^4} &< 10^{-8} \\
n^4 &> \frac{3.0862 \cdot 10^8}{360} \\
n &\ge 31
\end{aligned}$$

Thus, for composite Simpson's rule we will need at least 31 integration points.

Problem 4.4

Similar to the above problem.

Problem 4.5

According to the error formula for trapezoidal rule, indeed we should have $I - T_n \approx Ch^2$. But, recall the integration example $I^{(3)}$ from Lecture 13, pages 26-27, which shows that the convergence is much faster. Explanation lies with Remark 2 (from page 7, lecture 14) according to which if the integrand is periodic, then the quadrature error will have an exponential decay. so the answer is **False**.

Problem 4.6

Error for trapezoidal rule:

$$E_n^T(f) = -\frac{h^2(b-a)}{12} f''(\theta)$$

for some $\theta \in [a, b]$. Here $a = 1$, $b = 3$, $h = \frac{2}{n}$, $f(x) = \ln x$, $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2}$. Therefore,

$$\left|E_n^T(f)\right| \leq \frac{2}{3\,n^2} \max_{x\in[1,3]} \left|\frac{1}{x^2}\right| = \frac{2}{3\,n^2}$$

since $\max_{x\in[1,3]} \left|\frac{1}{x^2}\right| = 1$.
In order to ensure $\left|E_n^T(f)\right| < 10^{-5}$ we ask

$$\begin{aligned} \frac{2}{3\,n^2} &< 10^{-5} \\ n^2 &> \frac{3\cdot 10^5}{2} \\ n &> 387.3 \end{aligned}$$

Therefore, $n = 388$ will suffice.

Problem 4.7

Compute the column ratio in the given tabel

n	I_n	$I_n - I_{\frac{1}{2}n}$	*Ratio*
2	0.402368927062		
4	0.400431916045	$-1.937E-3$	
8	0.400077249447	$-3.547E-4$	5.46
16	0.400013713469	$-6.354E-5$	5.58
32	0.400002427846	$-1.129E-5$	5.63
64	0.400000429413	$-1.998E-6$	5.65
128	0.400000075924	$-3.535E-7$	5.65
256	0.400000013423	$-6.250E-8$	5.66
512	0.400000002373	$-1.105E-8$	5.66

Since the ratio converges to 5.66 it seems that the error is consistent with

$$I - I_n \approx \frac{c}{n^p}.$$

In order to find p and c use the approach from "Richardson formula and Aitken extrapolation for integration" file posted on course web page. According to formula

$$\frac{I_{2n} - I_n}{I_{4n} - I_{2n}} \approx 2^p$$

Thus

$$\begin{aligned} 2^p &\approx 5.66 \\ p &\approx \log_2 5.66 \approx 2.5 \end{aligned}$$

In order to find c, consider Richardson estimate formula (formula (5)):

$$I - I_{2n} \approx \frac{I_{2n} - I_n}{2^p - 1}$$

On the other hand

$$I - I_{2n} \approx \frac{c}{2^p n^p}$$

Threrefore,

$$\begin{aligned} c &\approx \frac{2^p n^p (I_{2n} - I_n)}{2^p - 1} \\ &\approx \frac{5.66 \cdot 256^{2.5} \cdot (I_{512} - I_{256})}{5.66 - 1} \\ &\approx \frac{5.66 \cdot 256^{2.5} \cdot (-1.105E-8)}{4.66} \\ &\approx -0.014 \end{aligned}$$

In order to estimate $I - I_{512}$ use Richardson estimate formula to get

$$I - I_{512} \approx \frac{I_{512} - I_{256}}{5.66 - 1} \approx \frac{-1.105E-8}{4.66} \approx 2.371E-9$$

In order to ensure that error is to be less than 10^{-10} we ask

$$\begin{aligned} |I - I_n| \approx \frac{|c|}{n^p} &< 10^{-10} \\ n^p &> |c| 10^{10} \\ n &> \left(|c| 10^{10}\right)^{1/p} \\ &> \left(0.014 \cdot 10^{10}\right)^{1/2.5} \\ &> 1813.2 \end{aligned}$$

Therefore, (since we are working in powers of 2 subdivisions) we will need 2056 subdivisions.

Problem 4.8

Compute the corresponding ratio from the table

n	*Error*	*Ratio*
2	$2.860E-2$	
4	$1.012E-2$	2.83
8	$3.587E-3$	2.82
16	$1.268E-3$	2.83
32	$4.485E-4$	2.83

Ratio converges to 2.83 and thus the error is consistent with formula

$$I - I_n \approx \frac{c}{n^p}.$$

In order to find p use formula

$$\begin{aligned} 2^p &\approx 2.83 \\ p &\approx \log_2 2.83 \approx 1.5 \end{aligned}$$

It can be obserdved that even if theoretically for Simpson's rule we must have $ratio \approx 16$ and correspondingly $p = 4$, in our case we have $ratio = 2.83$ and $p = 1.5$. This is explained probably by the fact that the integrand function $f(x)$ is not smooth enough on $[a, b]$ (at least $C^4[a, b]$ for Simpson's). Similar behaviour was observed for $\int_0^1 \sqrt{x}\, dx$.

Problem 4.9

According to Lecture 14, it is necessary to check if the integration formula is exact for polynomials $1,\ x,\ x^2,\ \ldots,\ x^r$, and it is not exact for polynomial x^{r+1}.

$$f(x)=1, \qquad \int_0^{2h} 1\,dx = 2h, \qquad I_h=\frac{3}{4}h\cdot 1+3\cdot 1=\frac{3}{4}h+3$$

It can be seen that the given integration formula is not exact even for polynomials of degree 0. Therefore, it does not have any degree of precission.

Problem 4.10

Consider the quadrature

$$\int_0^2 f(x)dx \approx w_1 f(x_1)+w_2 f(2)$$

We need to find 3 unknowns: w_1, w_2 and x_1. Thus, we will need three conditions. Let consequently $f(x)$ be $f(x)=1$, $f(x)=x$ and $f(x)=x^2$. Then, we get

$$\int_0^2 1\,dx = w_1+w_2$$
$$\int_0^2 x\,dx = w_1x_1+2w_2$$
$$\int_0^2 x^2\,dx = w_1x_1^2+4w_2$$

So we obtain the following system of equations

$$\begin{cases} 2 & = w_1+w_2 \\ 2 & = w_1x_1+2w_2 \\ \frac{8}{3} & = w_1x_1^2+4w_2 \end{cases}$$

Solve this system by elimination of variables and find solution:

$$w_1=\frac{3}{2},\ w_2=\frac{1}{2},\ x_1=\frac{2}{3}.$$

Therefore, we obtained the following quadrature

$$\int_0^2 f(x)\,dx \approx \frac{3}{2}\,f\left(\frac{2}{3}\right)+\frac{1}{2}\,f(2).$$

This quadrature is exact for ploynomials for degree up to 2.
Let's check if it is exact for polynomials of degree 3:

$$\int_0^2 x^3\,dx = 4 \neq 4\frac{4}{9}=\frac{3}{2}\cdot\left(\frac{2}{3}\right)^3+\frac{1}{2}\cdot 2^3.$$

Thus, the above quadrature has degree of precission 2.

Problem 4.11

By forward difference formula we have

$$f'(0.5)\approx\frac{f(x+h)-f(x)}{h}=\frac{f(0.6)-f(0.5)}{0.1}=\frac{7.6141-7.5383}{0.1}=0.758$$

We could have used $h=0.2$:

$$f'(0.5)\approx\frac{f(x+h)-f(x)}{h}=\frac{f(0.7)-f(0.5)}{0.2}=\frac{7.6906-7.5383}{0.2}=0.7615,$$

but it is less accurate since h is bigger and we know that error for forward difference is $O(h)$.
By backward difference formula we get

$$f'(0.5) \approx \frac{f(x) - f(x-h)}{h} = \frac{f(0.5) - f(0.4)}{0.1} = \frac{7.5383 - 7.4633}{0.1} \approx 0.750$$

and by centered difference formula we obtain

$$f'(0.5) \approx \frac{f(x+h) - f(x-h)}{2h} = \frac{f(0.6) - f(0.4)}{0.2} = \frac{7.6141 - 7.4633}{0.2} \approx 0.754$$

Since, error in centered difference formula is $O(h^2)$, the last result is most accurate. Thus, $f'(0.5) \approx 0.754$.
In order to estimate $f''(0.5)$, let's use second order centered difference formula

$$f''(0.5) = \frac{f(0.6) - 2f(0.5) + f(0.4)}{0.1^2} = \frac{7.6141 - 2 \cdot 7.5383 + 7.6141}{0.01} \approx 15.160$$