Systems of Ordinary Differential Equations

Prof.univ.dr.hab. Viorel Bostan

Spring 2015

Predator-Prey Systems

We can consider a variety of models for the growth of a single species that lives alone in an environment.

Predator-Prey Systems

We can consider a variety of models for the growth of a single species that lives alone in an environment. For example

$$\frac{dR}{dt} = kR$$

where k is a positive growth coefficient.

Predator-Prey Systems

We can consider a variety of models for the growth of a single species that lives alone in an environment. For example

$$\frac{dR}{dt} = kR$$

where k is a positive growth coefficient.
Solution is

$$R(t) = Ce^{kt}$$

Predator-Prey Systems

We can consider a variety of models for the growth of a single species that lives alone in an environment. For example

$$\frac{dR}{dt} = kR$$

where k is a positive growth coefficient.
Solution is

$$R(t) = Ce^{kt}$$

It can be coupled with initial condition

$$R(t_0) = R_0$$

Predator-Prey Systems

We can consider a variety of models for the growth of a single species that lives alone in an environment. For example

$$\frac{dR}{dt} = kR$$

where k is a positive growth coefficient.
Solution is

$$R(t) = Ce^{kt}$$

It can be coupled with initial condition

$$R(t_0) = R_0$$

and in this case solution is

$$R(t) = R_0 e^{kt}$$

Predator-Prey Systems

We can consider a variety of models for the growth of a single species that lives alone in an environment. For example

$$\frac{dR}{dt} = kR$$

where k is a positive growth coefficient.
Solution is

$$R(t) = Ce^{kt}$$

It can be coupled with initial condition

$$R(t_0) = R_0$$

and in this case solution is

$$R(t) = R_0 e^{kt}$$

Also, we can consider a negative growth (i.e. decrease)

$$\frac{dR}{dt} = -kR$$

Predator-Prey Systems

Consider more realistic models that take into account the interaction of two species in the same habitat.

Predator-Prey Systems

Consider more realistic models that take into account the interaction of two species in the same habitat.

These models take the form of a pair of linked differential equations.

Predator-Prey Systems

Consider more realistic models that take into account the interaction of two species in the same habitat.

These models take the form of a pair of linked differential equations.

Consider the situation in which one species, called the **prey**, has an ample food supply

Predator-Prey Systems

Consider more realistic models that take into account the interaction of two species in the same habitat.

These models take the form of a pair of linked differential equations.

Consider the situation in which one species, called the **prey**, has an ample food supply and the second species, called the **predator**, feeds on the prey.

Predator-Prey Systems

Consider more realistic models that take into account the interaction of two species in the same habitat.

These models take the form of a pair of linked differential equations.

Consider the situation in which one species, called the **prey**, has an ample food supply and the second species, called the **predator**, feeds on the prey.

Examples of prey and predators include:

Predator-Prey Systems

Consider more realistic models that take into account the interaction of two species in the same habitat.

These models take the form of a pair of linked differential equations.

Consider the situation in which one species, called the **prey**, has an ample food supply and the second species, called the **predator**, feeds on the prey.

Examples of prey and predators include:

- rabbits and wolves in an isolated forest
- food fish and sharks
- aphids and ladybugs
- bacteria and amoebas.

Predator-Prey Systems

Let $R(t)$ be the number of prey (rabbits)

Predator-Prey Systems

Let $R(t)$ be the number of prey (rabbits) and $W(t)$ be the number of predators (wolves) at time t.

Predator-Prey Systems

Let $R(t)$ be the number of prey (rabbits) and $W(t)$ be the number of predators (wolves) at time t.

In the absence of predators, the ample food supply would support exponential growth of the prey:

Predator-Prey Systems

Let $R(t)$ be the number of prey (rabbits) and $W(t)$ be the number of predators (wolves) at time t.

In the absence of predators, the ample food supply would support exponential growth of the prey:

$$\frac{dR}{dt} = kR$$

Predator-Prey Systems

Let $R(t)$ be the number of prey (rabbits) and $W(t)$ be the number of predators (wolves) at time t.

In the absence of predators, the ample food supply would support exponential growth of the prey:

$$\frac{dR}{dt} = kR$$

In the absence of prey, assume that the predator population would decline at a rate proportional to itself

Predator-Prey Systems

Let $R(t)$ be the number of prey (rabbits) and $W(t)$ be the number of predators (wolves) at time t.

In the absence of predators, the ample food supply would support exponential growth of the prey:

$$\frac{dR}{dt} = kR$$

In the absence of prey, assume that the predator population would decline at a rate proportional to itself, that is, proportional to itself:

Predator-Prey Systems

Let $R(t)$ be the number of prey (rabbits) and $W(t)$ be the number of predators (wolves) at time t.

In the absence of predators, the ample food supply would support exponential growth of the prey:

$$\frac{dR}{dt} = kR$$

In the absence of prey, assume that the predator population would decline at a rate proportional to itself, that is, proportional to itself:

$$\frac{dW}{dt} = -rW$$

Predator-Prey Systems

Let $R(t)$ be the number of prey (rabbits) and $W(t)$ be the number of predators (wolves) at time t.

In the absence of predators, the ample food supply would support exponential growth of the prey:

$$\frac{dR}{dt} = kR$$

In the absence of prey, assume that the predator population would decline at a rate proportional to itself, that is, proportional to itself:

$$\frac{dW}{dt} = -rW$$

With both species present, assume that the principal cause of death among the prey is being eaten by a predator

Predator-Prey Systems

Let $R(t)$ be the number of prey (rabbits) and $W(t)$ be the number of predators (wolves) at time t.

In the absence of predators, the ample food supply would support exponential growth of the prey:

$$\frac{dR}{dt} = kR$$

In the absence of prey, assume that the predator population would decline at a rate proportional to itself, that is, proportional to itself:

$$\frac{dW}{dt} = -rW$$

With both species present, assume that the principal cause of death among the prey is being eaten by a predator , and the birth and survival rates of the predators depend on their available food supply, i.e., the prey.

Predator-Prey Systems

Assume that the two species encounter each other at a rate that is proportional to both populations

Predator-Prey Systems

Assume that the two species encounter each other at a rate that is proportional to both populationsand is therefore proportional to the product $R \cdot W$.

Predator-Prey Systems

Assume that the two species encounter each other at a rate that is proportional to both populationsand is therefore proportional to the product $R \cdot W$.(The more there are of either population, the more encounters there are likely to be.)

Predator-Prey Systems

Assume that the two species encounter each other at a rate that is proportional to both populationsand is therefore proportional to the product $R \cdot W$.(The more there are of either population, the more encounters there are likely to be.)

A system of two differential equations that incorporates these assumptions is as follows:

Assume that the two species encounter each other at a rate that is proportional to both populationsand is therefore proportional to the product $R \cdot W$.(The more there are of either population, the more encounters there are likely to be.)

A system of two differential equations that incorporates these assumptions is as follows:

$$\begin{aligned} \frac{dR}{dt} &= kR - aRW \\ \frac{dW}{dt} &= -rW + bRW \end{aligned}$$

where k, r, a abd b are positive constants

Predator-Prey Systems

Assume that the two species encounter each other at a rate that is proportional to both populationsand is therefore proportional to the product $R \cdot W$.(The more there are of either population, the more encounters there are likely to be.)

A system of two differential equations that incorporates these assumptions is as follows:

$$\begin{aligned} \frac{dR}{dt} &= kR - aRW \\ \frac{dW}{dt} &= -rW + bRW \end{aligned}$$

where k, r, a abd b are positive constants
Notice that the term aRW decreases the natural growth rate of the prey

Predator-Prey Systems

Assume that the two species encounter each other at a rate that is proportional to both populationsand is therefore proportional to the product $R \cdot W$.(The more there are of either population, the more encounters there are likely to be.)

A system of two differential equations that incorporates these assumptions is as follows:

$$\begin{aligned} \frac{dR}{dt} &= kR - aRW \\ \frac{dW}{dt} &= -rW + bRW \end{aligned}$$

where k, r, a abd b are positive constants
Notice that the term aRW decreases the natural growth rate of the prey and the term bRW increases the natural growth rate of the predators.

Predator-Prey Systems

The equations are known as the predator-prey equations, or the **Lotka-Volterra** equations:

$$\begin{aligned}\frac{dR}{dt} &= kR - aRW \\ \frac{dW}{dt} &= -rW + bRW\end{aligned}$$

Predator-Prey Systems

The equations are known as the predator-prey equations, or the **Lotka-Volterra** equations:

$$\begin{aligned} \frac{dR}{dt} &= kR - aRW \\ \frac{dW}{dt} &= -rW + bRW \end{aligned}$$

A **solution** of this system of equations is a pair of functions $R(t)$ and $W(t)$ that describe the populations of prey and predator as functions of time t.

Predator-Prey Systems

The equations are known as the predator-prey equations, or the **Lotka-Volterra** equations:

$$\begin{aligned}\frac{dR}{dt} &= kR - aRW \\ \frac{dW}{dt} &= -rW + bRW\end{aligned}$$

A **solution** of this system of equations is a pair of functions $R(t)$ and $W(t)$ that describe the populations of prey and predator as functions of time t.

Because the system is coupled (R and W occur in both equations), we can't solve one equation and then the other; we have to solve them simultaneously.

Predator-Prey Systems

The equations are known as the predator-prey equations, or the **Lotka-Volterra** equations:

$$\begin{aligned}\frac{dR}{dt} &= kR - aRW \\ \frac{dW}{dt} &= -rW + bRW\end{aligned}$$

A **solution** of this system of equations is a pair of functions $R(t)$ and $W(t)$ that describe the populations of prey and predator as functions of time t.

Because the system is coupled (R and W occur in both equations), we can't solve one equation and then the other; we have to solve them simultaneously. Unfortunately, it is usually impossible to find explicit formulas for R and W as functions of t.

Predator-Prey Systems

The equations are known as the predator-prey equations, or the **Lotka-Volterra** equations:

$$\begin{aligned}\frac{dR}{dt} &= kR - aRW \\ \frac{dW}{dt} &= -rW + bRW\end{aligned}$$

A **solution** of this system of equations is a pair of functions $R(t)$ and $W(t)$ that describe the populations of prey and predator as functions of time t.

Because the system is coupled (R and W occur in both equations), we can't solve one equation and then the other; we have to solve them simultaneously. Unfortunately, it is usually impossible to find explicit formulas for R and W as functions of t. We can, however, use graphical methods to analyze the equations.

Predator-Prey System Example

Suppose that populations of rabbits R and wolves W are described by the Lotka-Volterra equations with $k = 0.08$, $a = 0.001$, $r = 0.02$, and $b = 0.00002$.

Predator-Prey System Example

Suppose that populations of rabbits R and wolves W are described by the Lotka-Volterra equations with $k = 0.08$, $a = 0.001$, $r = 0.02$, and $b = 0.00002$.

The time t is measured in months.

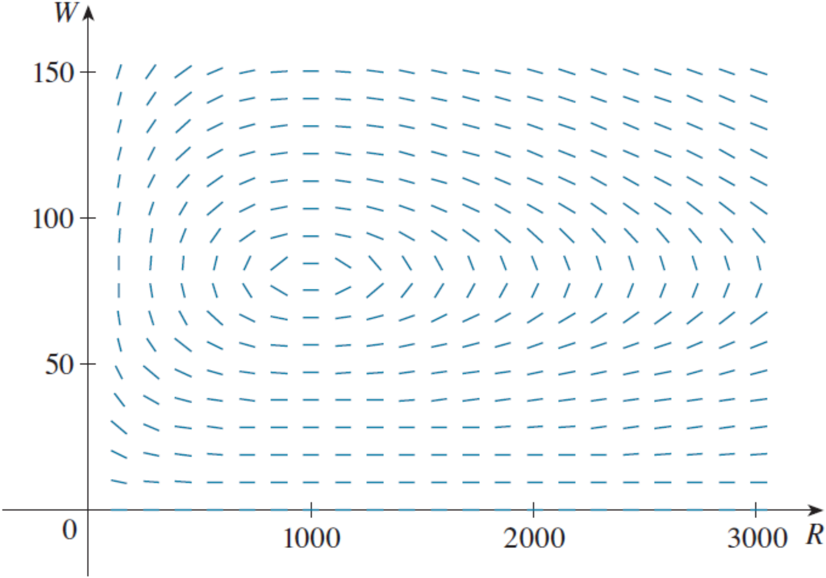
Predator-Prey System Example

Suppose that populations of rabbits R and wolves W are described by the Lotka-Volterra equations with $k = 0.08$, $a = 0.001$, $r = 0.02$, and $b = 0.00002$.

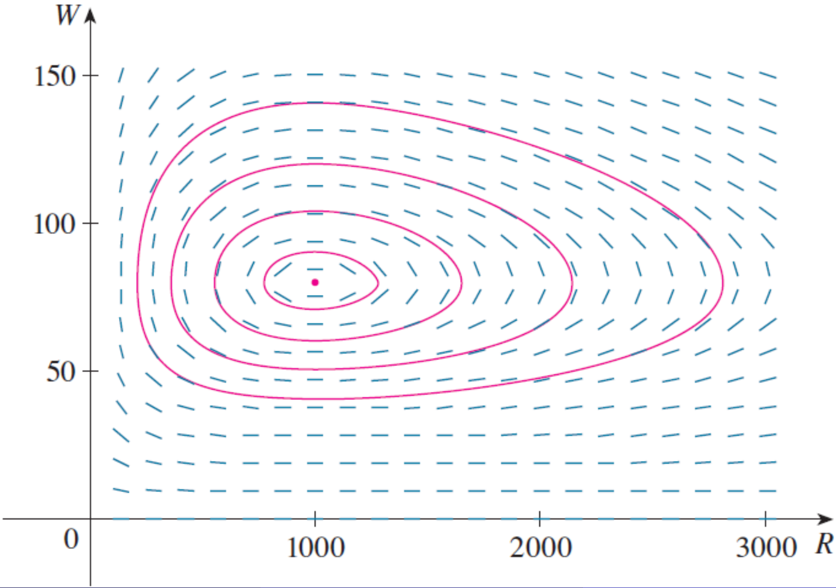
The time t is measured in months.

$$\begin{aligned} \frac{dR}{dt} &= 0.08R - 0.001RW \\ \frac{dW}{dt} &= -0.02W + 0.00002RW \end{aligned}$$

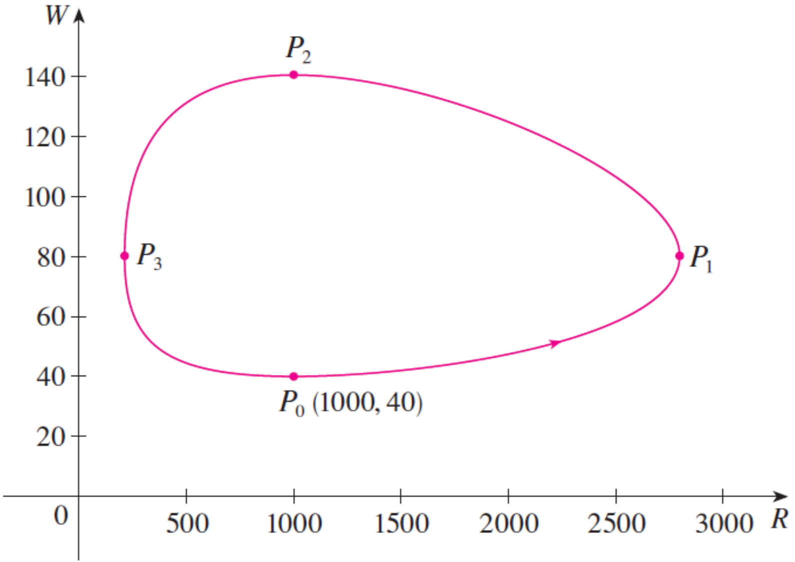
Predator-Prey Example



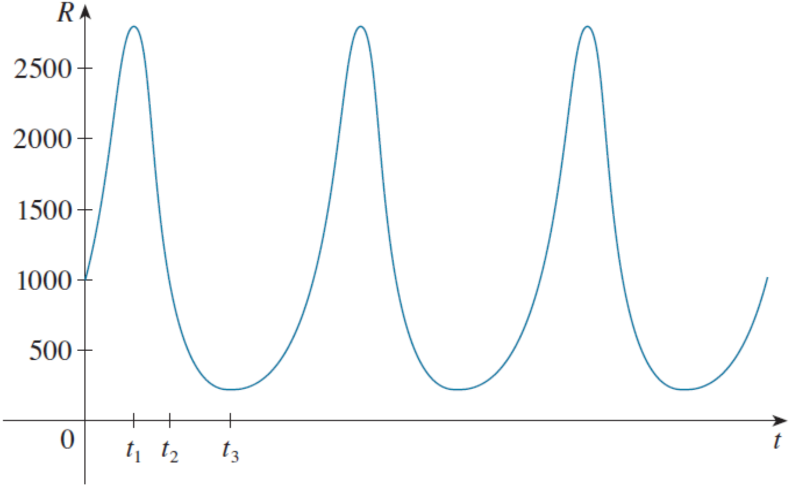
Predator-Prey Example



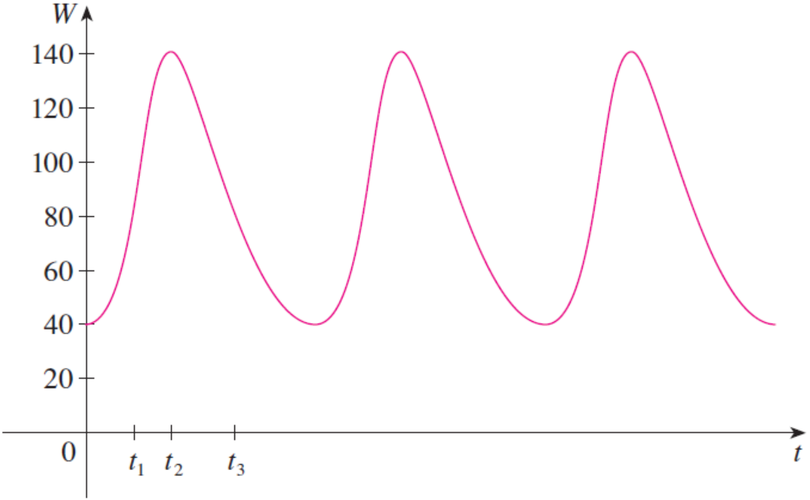
Predator-Prey Example



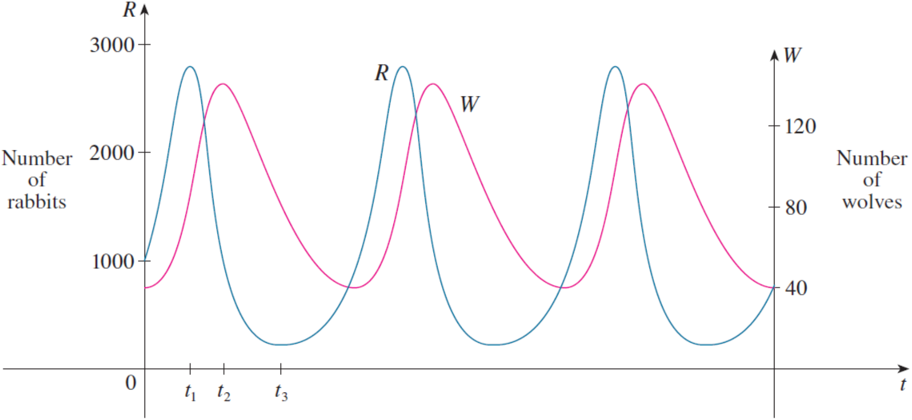
Predator-Prey Example



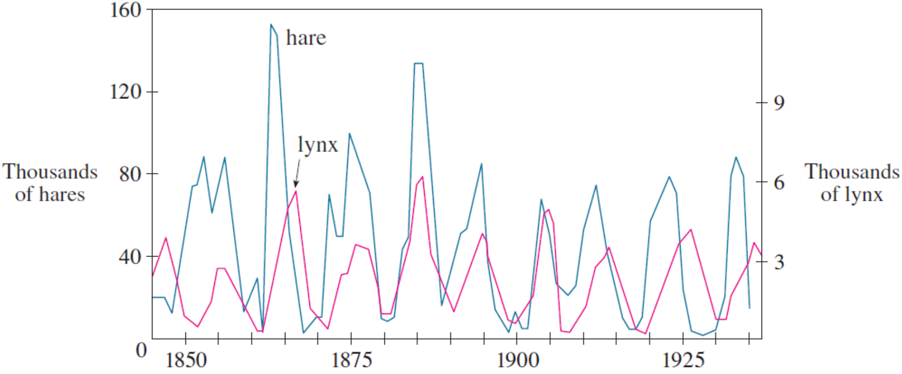
Predator-Prey Example



Predator-Prey Example



Predator-Prey Example



LAURA AND PETRARCH: AN INTRIGUING CASE OF CYCLICAL LOVE DYNAMICS
by SERGIO RINALDI

Francesco Petrarca (July 20, 1304 – July 19, 1374)

Love dynamics

Francis Petrarch, arguably the most lovesick poet of all time, is the author of the Canzoniere, a collection of 366 poems (sonnets, songs, sestinas, ballads, and madrigals).

Love dynamics

Francis Petrarch, arguably the most lovesick poet of all time, is the author of the Canzoniere, a collection of 366 poems (sonnets, songs, sestinas, ballads, and madrigals).

In Avignon, at the age of 23, he met Laura, a beautiful but married lady.

Love dynamics

Francis Petrarch, arguably the most lovesick poet of all time, is the author of the Canzoniere, a collection of 366 poems (sonnets, songs, sestinas, ballads, and madrigals).

In Avignon, at the age of 23, he met Laura, a beautiful but married lady.

He immediately fell in love with her and, although his love was not reciprocated, he addressed more than 200 poems to her over the next 21 years.

Love dynamics

Francis Petrarch, arguably the most lovesick poet of all time, is the author of the Canzoniere, a collection of 366 poems (sonnets, songs, sestinas, ballads, and madrigals).

In Avignon, at the age of 23, he met Laura, a beautiful but married lady.

He immediately fell in love with her and, although his love was not reciprocated, he addressed more than 200 poems to her over the next 21 years.

The poems express bouts of ardor and despair, snubs and reconciliations, and they mark the birth of modern love poetry.

Love dynamics

Francis Petrarch, arguably the most lovesick poet of all time, is the author of the Canzoniere, a collection of 366 poems (sonnets, songs, sestinas, ballads, and madrigals).

In Avignon, at the age of 23, he met Laura, a beautiful but married lady.

He immediately fell in love with her and, although his love was not reciprocated, he addressed more than 200 poems to her over the next 21 years.

The poems express bouts of ardor and despair, snubs and reconciliations, and they mark the birth of modern love poetry.

The verse has influenced countless poets, including Shakespeare.

Love dynamics

Unfortunately, only a few lyrics of the Canzoniere are dated.

Love dynamics

Unfortunately, only a few lyrics of the Canzoniere are dated.

The rest are collected in a bafflingly obscure order. The knowledge of the correct chronological order of the poems is a prerequisite for studying the lyrical, psychological, and stylistical development of any poet.

Love dynamics

Unfortunately, only a few lyrics of the Canzoniere are dated.

The rest are collected in a bafflingly obscure order. The knowledge of the correct chronological order of the poems is a prerequisite for studying the lyrical, psychological, and stylistical development of any poet.

This fact is particularly relevant for Petrarch, who somehow represents or, at least, interprets the spectacular transition from the Middle Ages to Humanism.

Love dynamics

Unfortunately, only a few lyrics of the Canzoniere are dated.

The rest are collected in a bafflingly obscure order. The knowledge of the correct chronological order of the poems is a prerequisite for studying the lyrical, psychological, and stylistical development of any poet.

This fact is particularly relevant for Petrarch, who somehow represents or, at least, interprets the spectacular transition from the Middle Ages to Humanism.

For this reason, the identification of the chronological order of the poems of the Canzoniere has been for centuries a problem of major concern for scholars.

Love dynamics

Amor con sue promesse lunsingando
mi ricondusse alla prigione antica

[Love's promises so softly flattering me
have led me back to my old prison's thrall]

Amor con sue promesse lunsingando
mi ricondusse alla prigione antica

[Love's promises so softly flattering me
have led me back to my old prison's thrall]

Quale mio destin, qual forza o qual inganno
mi riconduce disarmato al campo
l a 've sempre son vinto?

[What fate, what power or what insidiousness
still guides me back, disarmed, to that same field
wherein I'm always crushed?]

Love dynamics

Di tempo in tempo mi si fa men dura
l'angelica gura e'l dolce riso,
et l'aria del bel viso
e degli occhi leggiadri meno oscura

[From time to time less reproachful seem to me
her heavenly figure, and her charming face,
and sweet smile's airy grace,
while her dancing eyes grow far less dark I see]

Love dynamics

The emotions of Laura and Petrarch are now modeled by means of three ordinary differential equations.

Love dynamics

The emotions of Laura and Petrarch are now modeled by means of three ordinary differential equations.

Laura is described by a single variable $L(t)$, representing her love for the poet at time t.

Love dynamics

The emotions of Laura and Petrarch are now modeled by means of three ordinary differential equations.

Laura is described by a single variable $L(t)$, representing her love for the poet at time t.

Positive and high values of L mean warm friendship, while negative values should be associated with coldness and antagonism.

Love dynamics

The emotions of Laura and Petrarch are now modeled by means of three ordinary differential equations.

Laura is described by a single variable $L(t)$, representing her love for the poet at time t.

Positive and high values of L mean warm friendship, while negative values should be associated with coldness and antagonism.

The personality of Petrarch is more complex; its description requires two variables: $P(t)$, love for Laura, and $Z(t)$, poetic inspiration.

Love dynamics

The emotions of Laura and Petrarch are now modeled by means of three ordinary differential equations.

Laura is described by a single variable $L(t)$, representing her love for the poet at time t.

Positive and high values of L mean warm friendship, while negative values should be associated with coldness and antagonism.

The personality of Petrarch is more complex; its description requires two variables: $P(t)$, love for Laura, and $Z(t)$, poetic inspiration.

High values of P indicate ecstatic love, while negative values stand for despair.

Love dynamics

$$
\begin{aligned}
\frac{dL(t)}{dt} &= -\alpha_1 L(t) + R_L(P(t)) + \beta_1 A_P,\\
\frac{dP(t)}{dt} &= -\alpha_2 P(t) + R_P(L(t)) + \beta_2 \frac{A_L}{1+\delta Z(t)},\\
\frac{dZ(t)}{dt} &= -\alpha_3 Z(t) + \beta_3 P(t),
\end{aligned}
$$

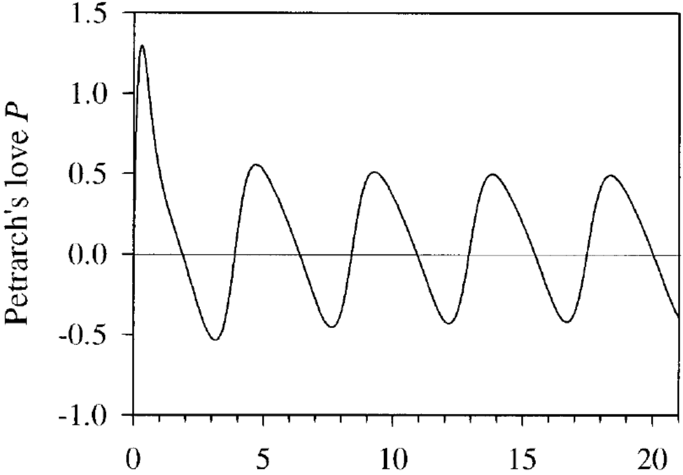
Love dynamics

$$
\begin{aligned}
\frac{dL(t)}{dt} &= -\alpha_1 L(t) + R_L(P(t)) + \beta_1 A_P, \\
\frac{dP(t)}{dt} &= -\alpha_2 P(t) + R_P(L(t)) + \beta_2 \frac{A_L}{1+\delta Z(t)}, \\
\frac{dZ(t)}{dt} &= -\alpha_3 Z(t) + \beta_3 P(t),
\end{aligned}
$$

where RL and RP are reaction functions speci ed below, $AP[AL]$ is the appeal (physical, as well as social and intellectual) of Petrarch [Laura], and all greek letters are positive constant parameters (this means that variations in the personalities of Laura and Petrarch due to aging or other external factors are not considered).

Love dynamics

Love dynamics



Love dynamics