Mathematics for Computer Science

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Lecture 1. Part 2

Last Lecture Outline

- Naive Set Theory;
- Functions (bijections, injections, surjections);
- Cardinal number of a set;
- Mapping Rule:

Theorem (Mapping Rule)

If X and Y are sets then:

1. $f : X \to Y$ *is surjection (denoted by* $X \operatorname{surj} Y$*), if and only if* $|X| \geqslant |Y|$.
2. $f : X \to Y$ *is injection (denoted by* $X \operatorname{inj} Y$*), if and only if* $|X| \leqslant |Y|$.
3. $f : X \to Y$ *is bijection (denoted by* $X \operatorname{bij} Y$*), if and only if* $|X| = |Y|$.

- Infinite sets;

Last Lecture Outline

- Countable sets (finite or infinite)

Definition

A set X is called **infinitely countable** if and only if there is a bijection between $\mathbb{N}$ and X.

- $$\aleph_0 = |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|.$$
- Finite union of countable sets
$$\text{If } |A_1| = |A_2| = |A_3| = \aleph_0, \text{ then } |A_1 \cup A_2 \cup A_3| = \aleph_0.$$
- Union of countable sets
$$\text{If } |A_i| = \aleph_0, \text{ then } \left|\bigcup_{i=1}^{\infty} A_i\right| = |A_1 \cup A_2 \cup \cdots \cup A_n \cup \ldots| = \aleph_0.$$
- Cartesian product of countable sets
$$\aleph_0 = |\mathbb{N} \times \mathbb{N}| = |\mathbb{Q} \times \mathbb{Q}| = |\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}|.$$

Cardinality of real numbers

How about cardinality of $|\mathbb{R}|$?

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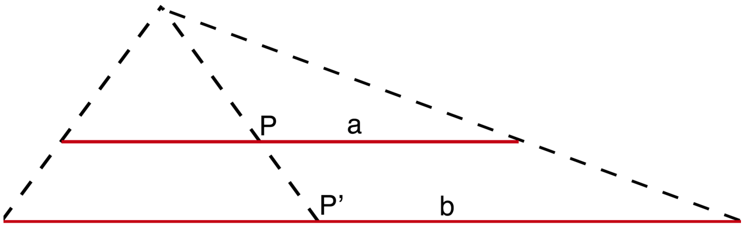

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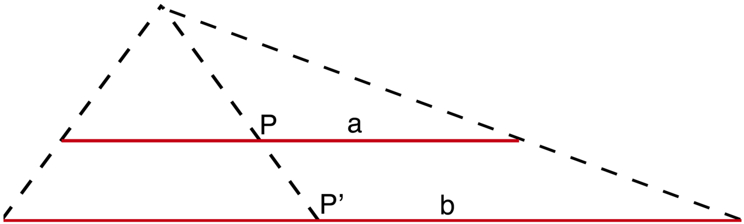

Clearly, there is a bijection between points in both intervals.

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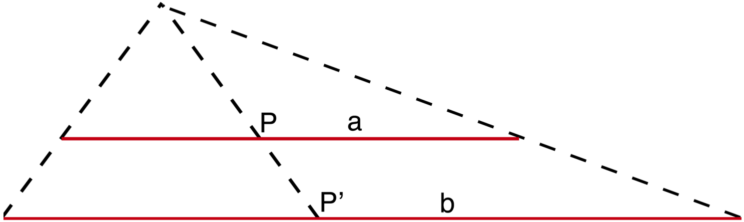

Clearly, there is a bijection between points in both intervals.

$\Big|[a,b]\Big| = \Big|[c,d]\Big|$ for any $a,b,c,d \in \mathbb{R}$.

Cardinality of real numbers

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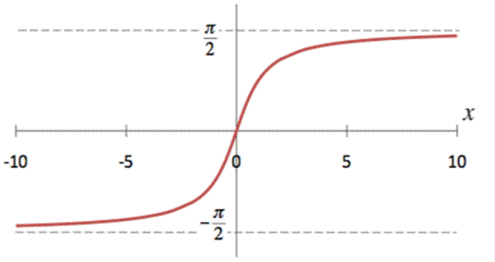
Consider function $f : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ defined by $f(x) = \arctan x$.



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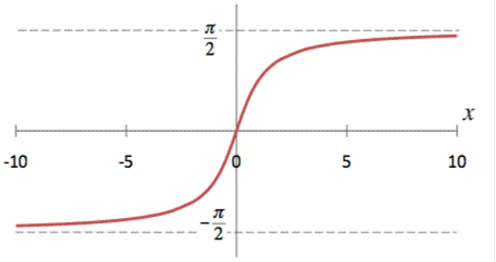

It is a bijection, and therefore,

$$|\mathbb{R}| = \left|\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right| = |(0,1)|$$

Cardinality of power set

Proposition

If $|X| = n$, then $\left|2^X\right| = 2^n$.

Notation

The collection of all n-bit sequences will be denoted by $\{0,1\}^n$.
The collection of all finite-bit sequences will be denoted by $\{0,1\}^*$.
The collection of all infinit-bit sequences will be denoted by $\{0,1\}^\omega$.

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Example

1. $\{0,1\}^2 = \{00, 01, 10, 11\}$;
2. $\{0,1\}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$;
3. $\{0,1\}^* = \left\{b_1 b_2 b_3 b_4 \dots b_m \mid b_i \in \{0,1\},\ 1 \leqslant i \leqslant m,\ m \in \mathbb{N}\right\}$.
4. $\{0,1\}^\omega = \left\{b_1 b_2 b_3 b_4 \dots, \mid b_i \in \{0,1\}, \quad i \in \mathbb{N}\right\}$.

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Proof.

Let X be a set and $|X| = n$. There is a natural bijection from power set 2^X to $\{0, 1\}^n$:

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Let $x_1, x_2, \ldots x_n \in X$.

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Let $x_1, x_2, \ldots x_n \in X$. Then, a subset of X maps to the sequence $(b_1, b_2, \ldots, b_n)$, where $b_i = 1$ if and only if $x_i \in$ that subset.

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For example, if $n = 10$, then the subset $\{x_3, x_4, x_8, x_{10}\}$ maps to a $10-$bit sequence as follows

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$$\begin{array}{cccccccccccc} \{ & & & x_3, & x_4, & & & & x_8, & & x_{10} & \} \\ (& 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 &) \end{array}$$

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Let $x_1, x_2, \dots x_n \in X$. Then, a subset of X maps to the sequence $(b_1, b_2, \dots, b_n)$, where $b_i = 1$ if and only if $x_i \in$ that subset.
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and vice versa, sequence (1011101010) corresponds to the subset $\{x_1, x_3, x_4, x_5, x_7, x_9\}$.
Clearly, this is a bijection.

Cardinality of power set

Proposition

If $|X| = n$, then $\left|2^X\right| = 2^n$.

Proof Contd.

Since,

$$2^X \text{ bij } \{0,1\}^n,$$

it follows that

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it follows that

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Show that

$$\left|\{0,1\}^n\right| = 2^n$$

and we are done. $\square$

Cardinality of power set

Cantor's discovery

Not all infinite sets are the same size!

Cardinality of power set

Cantor's discovery

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Theorem (Cantor)

For any set X,

$$|X| < \left|2^X\right|.$$

Cardinality of power set

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Theorem (Cantor)

For any set X,

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Let's prove a particular case of this theorem:

Theorem

$$|\mathbb{N}| < \left|2^{\mathbb{N}}\right|.$$

In other words, the set of all subsets of natural numbers is **uncountable**.

Cardinality of power set

Proof.

Establish a natural bijection between all subsets of $\mathbb{N}$ and $\{0,1\}^\omega$ (infinite-bit sequences).

$$\left|2^{\mathbb{N}}\right| = \left|\{0,1\}^\omega\right|.$$

Then, we will prove that

$$\aleph_0 < \left|\{0,1\}^\omega\right|.$$

Proof by contradiction.

Cardinality of power set

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Proof by contradiction.

Suppose that $\{0,1\}^\omega$ is a countable set. It means that its elements can be listed:

$$s_1, s_2, s_3, s_4, s_5, s_6, s_7, \ldots$$

Proof Contd.

$$
\begin{array}{rcccccccccccc}
s_1 = & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
s_2 = & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & \cdots \\
s_3 = & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & \cdots \\
s_4 = & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & \cdots \\
s_5 = & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \cdots \\
s_6 = & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & \cdots \\
s_7 = & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & \cdots \\
s_8 = & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & \cdots \\
s_9 = & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & \cdots \\
s_{10} = & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Cardinality of power set

Proof Contd.

$$
\begin{array}{rcccccccccccc}
s_1 = & & \color{red}{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
s_2 = & & 0 & \color{red}{0} & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & \cdots \\
s_3 = & & 1 & 1 & \color{red}{1} & 0 & 1 & 0 & 1 & 1 & 0 & 0 & \cdots \\
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\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
x = & & \color{blue}{1} & \color{blue}{1} & \color{blue}{0} & \color{blue}{1} & \color{blue}{1} & \color{blue}{0} & \color{blue}{0} & \color{blue}{0} & \color{blue}{0} & \color{blue}{1} & \cdots
\end{array}
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Cardinality of power set

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s_1 = & & \color{red}{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
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\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
x = & & \color{blue}{1} & \color{blue}{1} & \color{blue}{0} & \color{blue}{1} & \color{blue}{1} & \color{blue}{0} & \color{blue}{0} & \color{blue}{0} & \color{blue}{0} & \color{blue}{1} & \cdots
\end{array}$$

Is x listed above?

Cardinality of power set

Proof Contd.

$$
\begin{array}{lcccccccccccc}
s_1 = & & {\color{red}0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
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\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
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\end{array}
$$

Is x listed above? **NO**.

Cardinality of power set

Proof Contd.

$$
\begin{array}{rcccccccccccc}
s_1 = & \color{red}{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
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\end{array}
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Is x listed above? **NO**. Means that $x \notin \{0,1\}^{\omega}$.

Cardinality of power set

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\end{array}
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Is x listed above? **NO**. Means that $x \notin \{0,1\}^\omega$. Contradiction!

Cardinality of power set

Proof Contd.

Thus, initial supposition that we can list (count) all elements of $\{0, 1\}^\omega$ is **wrong**.

Proof Contd.

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Therefore, $\{0,1\}^\omega$ is uncountable, which means

$$\left|\{0,1\}^\omega\right| > \aleph_0.$$

We have proved that

$$|\mathbb{N}| < \left|2^{\mathbb{N}}\right|.$$

$\square$

Definition

Procedure used in the above proof is called **Cantor's diagonal argument**.

Corollary

The set of real numbers $\mathbb{R}$ is uncountable.

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Proof.

Proof by contradiction and using Cantor's diagonal argument.

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The set of real numbers $\mathbb{R}$ from interval $(0, 1)$ is uncountable.

Proof.

Proof by contradiction and using Cantor's diagonal argument.

Suppose that the set $(0, 1)$ is countable. Consider real numbers from $(0, 1)$, written in binary decimal form. Since $(0, 1)$ is countable, we can list all numbers from $(0, 1)$ in a sequence:

Proof Contd.

$$
\begin{array}{rcccccccccccc}
\alpha_1 = & 0. & \color{red}{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\alpha_2 = & 0. & 0 & \color{red}{0} & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & \cdots \\
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\alpha_5 = & 0. & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \cdots \\
\alpha_6 = & 0. & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & \cdots \\
\alpha_7 = & 0. & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & \cdots \\
\alpha_8 = & 0. & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & \cdots \\
\alpha_9 = & 0. & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & \cdots \\
\alpha_{10} = & 0. & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\alpha = & 0. & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & \cdots
\end{array}
$$

Proof Contd.

$$
\begin{array}{rcllllllllllll}
\alpha_1 & = & 0. & \color{red}{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\alpha_2 & = & 0. & 0 & \color{red}{0} & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & \cdots \\
\alpha_3 & = & 0. & 1 & 1 & \color{red}{1} & 0 & 1 & 0 & 1 & 1 & 0 & 0 & \cdots \\
\alpha_4 & = & 0. & 1 & 0 & 0 & \color{red}{0} & 1 & 1 & 0 & 1 & 0 & 1 & \cdots \\
\alpha_5 & = & 0. & 0 & 1 & 0 & 1 & \color{red}{0} & 1 & 1 & 0 & 1 & 1 & \cdots \\
\alpha_6 & = & 0. & 0 & 0 & 0 & 1 & 0 & \color{red}{1} & 0 & 1 & 1 & 0 & \cdots \\
\alpha_7 & = & 0. & 1 & 0 & 0 & 0 & 0 & 0 & \color{red}{1} & 0 & 1 & 1 & \cdots \\
\alpha_8 & = & 0. & 1 & 1 & 1 & 0 & 1 & 1 & 0 & \color{red}{1} & 0 & 1 & \cdots \\
\alpha_9 & = & 0. & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \color{red}{1} & 0 & \cdots \\
\alpha_{10} & = & 0. & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & \color{red}{0} & \cdots \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\alpha & = & \color{blue}{0.} & \color{blue}{1} & \color{blue}{1} & \color{blue}{0} & \color{blue}{1} & \color{blue}{1} & \color{blue}{0} & \color{blue}{0} & \color{blue}{0} & \color{blue}{0} & \color{blue}{1} & \cdots
\end{array}
$$

Is real number $\alpha \in (0,1)$ listed above?

Cardinality of real numbers

Proof Contd.

$$
\begin{array}{rcccccccccccc}
\alpha_1 = & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\alpha_2 = & 0. & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & \cdots \\
\alpha_3 = & 0. & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & \cdots \\
\alpha_4 = & 0. & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & \cdots \\
\alpha_5 = & 0. & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \cdots \\
\alpha_6 = & 0. & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & \cdots \\
\alpha_7 = & 0. & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & \cdots \\
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\alpha_{10} = & 0. & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\alpha = & 0. & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & \cdots
\end{array}
$$

Is real number $\alpha \in (0,1)$ listed above? **NO**.

Cardinality of real numbers

Proof Contd.

$$
\begin{array}{rcccccccccccc}
\alpha_1 = & 0. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\alpha_2 = & 0. & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & \cdots \\
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\alpha = & 0. & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & \cdots
\end{array}
$$

Is real number $\alpha \in (0,1)$ listed above? **NO**. Means that $\alpha \notin (0,1)$.

Proof Contd.

$$
\begin{array}{rcccccccccccc}
\alpha_1 = & 0. & \color{red}{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\alpha_2 = & 0. & 0 & \color{red}{0} & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & \cdots \\
\alpha_3 = & 0. & 1 & 1 & \color{red}{1} & 0 & 1 & 0 & 1 & 1 & 0 & 0 & \cdots \\
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\alpha_5 = & 0. & 0 & 1 & 0 & 1 & \color{red}{0} & 1 & 1 & 0 & 1 & 1 & \cdots \\
\alpha_6 = & 0. & 0 & 0 & 0 & 1 & 0 & \color{red}{1} & 0 & 1 & 1 & 0 & \cdots \\
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\alpha_9 = & 0. & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \color{red}{1} & 0 & \cdots \\
\alpha_{10} = & 0. & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & \color{red}{0} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\alpha = & \color{blue}{0.} & \color{blue}{1} & \color{blue}{1} & \color{blue}{0} & \color{blue}{1} & \color{blue}{1} & \color{blue}{0} & \color{blue}{0} & \color{blue}{0} & \color{blue}{0} & \color{blue}{1} & \cdots
\end{array}
$$

Is real number $\alpha \in (0,1)$ listed above? **NO**. Means that $\alpha \notin (0,1)$. **Contradiction!** $\square$

Transfinite numbers

Definition

The cardinal number of set $\mathbb{R}$ is called **continuum**: $\left|\mathbb{R}\right| = c$.

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Remark

It follows from Sroder-Bernstein Theorem that:

$$2^{\aleph_0} = c.$$

So far, we know that

$$\aleph_0 < c.$$

G. Cantor in 1878 formulated the question whether there is no set whose cardinality is strictly between that of the integers and the real numbers. D. Hilbert in his set of biggest problems of mathematics called it the 1st problem.

Continuum hypothesis

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It is known as:

Continuum Hypothesis (CH)

$$c = 2^{\aleph_0} \stackrel{?}{=} \aleph_1$$

Kurt Godel advanced in the solution of Hilbert's 1st problem, and Paul Cohen showed in 1963 that CH cannot be proven from ZFC axioms, in other words, CH is independent of ZFC. For his work, P. Cohen was awarded the Fields Medal (1966).