Fourier Series

Conf.dr. Bostan Viorel

Fall 2017

Jean Baptiste Joseph Fourier (21/03/1768 – 16/05/1830)

Jean Baptiste Joseph Fourier (21/03/1768 – 16/05/1830)

- Publishing his initial results on Fourier series in 1807 *"Me'moire sur la propagation de la chaleur dans les corps solides"*

Jean Baptiste Joseph Fourier (21/03/1768 – 16/05/1830)

- Publishing his initial results on Fourier series in 1807 *"Me'moire sur la propagation de la chaleur dans les corps solides"*
- When Fourier submitted his paper in 1807, the committee (which included Lagrange, Laplace, Malus and Legendre, among others) concluded:

Jean Baptiste Joseph Fourier (21/03/1768 – 16/05/1830)

- Publishing his initial results on Fourier series in 1807 *"Me'moire sur la propagation de la chaleur dans les corps solides"*
- When Fourier submitted his paper in 1807, the committee (which included Lagrange, Laplace, Malus and Legendre, among others) concluded:
- ...the manner in which the author arrives at these equations is not exempt of difficulties and [...] his analysis to integrate them still leaves something to be desired on the score of generality and even rigour.

Fourier Series: Original Problem

- Consider a long thin insulated rod or wire of length L.

Fourier Series: Original Problem

- Consider a long thin insulated rod or wire of length L.
- Assume that the $x-$axis is aligned with the length L of the rod and that $0 < x < L$.

Fourier Series: Original Problem

- Consider a long thin insulated rod or wire of length L.
- Assume that the $x-$axis is aligned with the length L of the rod and that $0 < x < L$.
- The temperature $T(x, t)$ along the length of the rod generally varies with both position x and time t.

Fourier Series: Original Problem

- Consider a long thin insulated rod or wire of length L.
- Assume that the $x-$axis is aligned with the length L of the rod and that $0 < x < L$.
- The temperature $T(x, t)$ along the length of the rod generally varies with both position x and time t.
- The problem is to determine function $T(x, t)$ given the initial temperature $T(x, 0) = f(x)$ along the rod.

Fourier Series: Original Problem

- Consider a long thin insulated rod or wire of length L.
- Assume that the $x-$axis is aligned with the length L of the rod and that $0 < x < L$.
- The temperature $T(x,t)$ along the length of the rod generally varies with both position x and time t.
- The problem is to determine function $T(x,t)$ given the initial temperature $T(x,0) = f(x)$ along the rod.
- For example, it might be hot at one end and cooler at the other end, so heat will flow from the hot to the cool end and we might want to know what temperature will look like in for example, one hour time.

Fourier Series: Original Problem

- Consider a long thin insulated rod or wire of length L.
- Assume that the $x-$axis is aligned with the length L of the rod and that $0 < x < L$.
- The temperature $T(x,t)$ along the length of the rod generally varies with both position x and time t.
- The problem is to determine function $T(x,t)$ given the initial temperature $T(x,0) = f(x)$ along the rod.
- For example, it might be hot at one end and cooler at the other end, so heat will flow from the hot to the cool end and we might want to know what temperature will look like in for example, one hour time.
- The metnod proposed by Fourier requires the expansion

$$f(x) \quad = \quad b_1 \sin \frac{\pi x}{L} + b_2 \sin \frac{2\pi x}{L} + b_3 \sin \frac{3\pi x}{L} + \ldots$$

Fourier Series: Original Problem

- Consider a long thin insulated rod or wire of length L.
- Assume that the $x-$axis is aligned with the length L of the rod and that $0 < x < L$.
- The temperature $T(x,t)$ along the length of the rod generally varies with both position x and time t.
- The problem is to determine function $T(x,t)$ given the initial temperature $T(x,0) = f(x)$ along the rod.
- For example, it might be hot at one end and cooler at the other end, so heat will flow from the hot to the cool end and we might want to know what temperature will look like in for example, one hour time.
- The metnod proposed by Fourier requires the expansion

$$f(x) \quad = \quad b_1 \sin \frac{\pi x}{L} + b_2 \sin \frac{2\pi x}{L} + b_3 \sin \frac{3\pi x}{L} + \dots$$

Fourier Series: Original Problem

- Consider a long thin insulated rod or wire of length L.
- Assume that the $x-$axis is aligned with the length L of the rod and that $0 < x < L$.
- The temperature $T(x,t)$ along the length of the rod generally varies with both position x and time t.
- The problem is to determine function $T(x,t)$ given the initial temperature $T(x,0) = f(x)$ along the rod.
- For example, it might be hot at one end and cooler at the other end, so heat will flow from the hot to the cool end and we might want to know what temperature will look like in for example, one hour time.
- The metnod proposed by Fourier requires the expansion

$$\begin{aligned} f(x) &= b_1 \sin \frac{\pi x}{L} + b_2 \sin \frac{2\pi x}{L} + b_3 \sin \frac{3\pi x}{L} + \ldots \\ &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \end{aligned}$$

Fourier Series

Fourier series have a wide range of science and engineering applications like

- study of heat conduction,
- wave phenomena,
- concentrations of chemicals and pollutants,
- communications (radio, tv, mobile phones),
- image processing,
- sound and video recording

Fourier Series

Problem

If $f(x)$ *is defined on the interval* $-L < x < L$ *we need to know the coefficients* $a_0, a_n, b_n,\ n \geq 1$ *such that*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

Fourier Series

Problem

If $f(x)$ *is defined on the interval* $-L < x < L$ *we need to know the coefficients* $a_0, a_n, b_n,\ n \geq 1$ *such that*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{FS}$$

$$\begin{aligned} f(x) &= \frac{a_0}{2} + a_1 \cos \frac{\pi x}{L} + b_1 \sin \frac{\pi x}{L} \\ &\quad + a_2 \cos \frac{2\pi x}{L} + b_2 \sin \frac{2\pi x}{L} \\ &\quad + a_3 \cos \frac{3\pi x}{L} + b_3 \sin \frac{3\pi x}{L} + \dots \end{aligned}$$

Problem

If $f(x)$ is defined on the interval $-L < x < L$ we need to know the coefficients $a_0, a_n, b_n,\ n \geq 1$ such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{FS}$$

$$\begin{aligned} f(x) &= \frac{a_0}{2} + a_1 \cos \frac{\pi x}{L} + b_1 \sin \frac{\pi x}{L} \\ &\quad + a_2 \cos \frac{2\pi x}{L} + b_2 \sin \frac{2\pi x}{L} \\ &\quad + a_3 \cos \frac{3\pi x}{L} + b_3 \sin \frac{3\pi x}{L} + \ldots \end{aligned}$$

Note that the interval $-L < x < L$ is symmetric around origin. Equation (FS) is called a Fourier series for function f on interval $(-L, L)$.

Coefficients in a Fourier Series

Lemma

If $m, n \in \mathbb{N}$, *then*

$$1. \quad \int_{-L}^{L} \cos \frac{n\pi x}{L} dx = 0$$

$$2. \quad \int_{-L}^{L} \sin \frac{n\pi x}{L} dx = 0$$

$$3. \quad \int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \begin{cases} 0, & n \neq m \\ L, & n = m \end{cases}$$

$$4. \quad \int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \begin{cases} 0, & n \neq m \\ L, & n = m \end{cases}$$

$$5. \quad \int_{-L}^{L} \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = 0$$

Coefficients in a Fourier Series

- Assume that function $f(x)$ can be decomposed in a convergent series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \qquad \text{(FS)}$$

Coefficients in a Fourier Series

- Assume that function $f(x)$ can be decomposed in a convergent series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{FS}$$

- Integrate from $-L$ to L both sides

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} \frac{a_0}{2} dx + \int_{-L}^{L} \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) dx$$

Coefficients in a Fourier Series

- Assume that function $f(x)$ can be decomposed in a convergent series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \qquad \text{(FS)}$$

- Integrate from $-L$ to L both sides

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} \frac{a_0}{2} dx + \int_{-L}^{L} \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) dx$$

- $$\int_{-L}^{L} f(x)dx = \frac{a_0}{2} \int_{-L}^{L} dx +$$

Coefficients in a Fourier Series

- Assume that function $f(x)$ can be decomposed in a convergent series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{FS}$$

- Integrate from $-L$ to L both sides

$$\int_{-L}^{L} f(x) dx = \int_{-L}^{L} \frac{a_0}{2} dx + \int_{-L}^{L} \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) dx$$

- $$\int_{-L}^{L} f(x) dx = \frac{a_0}{2} \int_{-L}^{L} dx +$$

Coefficients in a Fourier Series

- Assume that function $f(x)$ can be decomposed in a convergent series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \qquad \text{(FS)}$$

- Integrate from $-L$ to L both sides

$$\int_{-L}^{L} f(x) dx = \int_{-L}^{L} \frac{a_0}{2} dx + \int_{-L}^{L} \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) dx$$

-

$$\int_{-L}^{L} f(x) dx = \frac{a_0}{2} \int_{-L}^{L} dx + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L} \cos \frac{n\pi x}{L} dx}_{=0} +$$

Coefficients in a Fourier Series

- Assume that function $f(x)$ can be decomposed in a convergent series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \qquad \text{(FS)}$$

- Integrate from $-L$ to L both sides

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} \frac{a_0}{2} dx + \int_{-L}^{L} \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) dx$$

-

$$\int_{-L}^{L} f(x)dx = \frac{a_0}{2} \int_{-L}^{L} dx + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L} \cos \frac{n\pi x}{L} dx}_{=0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L} \sin \frac{n\pi x}{L} dx}_{=0}$$

Coefficients in a Fourier Series

- Assume that function $f(x)$ can be decomposed in a convergent series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \qquad \text{(FS)}$$

- Integrate from $-L$ to L both sides

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} \frac{a_0}{2} dx + \int_{-L}^{L} \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) dx$$

-

$$\int_{-L}^{L} f(x)dx = \frac{a_0}{2} \int_{-L}^{L} dx + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L} \cos \frac{n\pi x}{L} dx}_{=0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L} \sin \frac{n\pi x}{L} dx}_{=0}$$

-

$$\int_{-L}^{L} f(x)dx = \frac{a_0}{2} \int_{-L}^{L} dx$$

Coefficients in a Fourier Series

- Assume that function $f(x)$ can be decomposed in a convergent series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \qquad \text{(FS)}$$

- Integrate from $-L$ to L both sides

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} \frac{a_0}{2} dx + \int_{-L}^{L} \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) dx$$

- $$\int_{-L}^{L} f(x)dx = \frac{a_0}{2} \int_{-L}^{L} dx + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L} \cos \frac{n\pi x}{L} dx}_{=0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L} \sin \frac{n\pi x}{L} dx}_{=0}$$

- $$\int_{-L}^{L} f(x)dx = \frac{a_0}{2} \int_{-L}^{L} dx$$

Coefficients in a Fourier Series

- Assume that function $f(x)$ can be decomposed in a convergent series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \qquad \text{(FS)}$$

- Integrate from $-L$ to L both sides

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} \frac{a_0}{2} dx + \int_{-L}^{L} \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) dx$$

- $$\int_{-L}^{L} f(x)dx = \frac{a_0}{2} \int_{-L}^{L} dx + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L} \cos \frac{n\pi x}{L} dx}_{=0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L} \sin \frac{n\pi x}{L} dx}_{=0}$$

- $$\int_{-L}^{L} f(x)dx = \frac{a_0}{2} \int_{-L}^{L} dx = La_0$$

Coefficients in a Fourier Series

- Assume that function $f(x)$ can be decomposed in a convergent series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \qquad \text{(FS)}$$

- Integrate from $-L$ to L both sides

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} \frac{a_0}{2}dx + \int_{-L}^{L} \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) dx$$

-

$$\int_{-L}^{L} f(x)dx = \frac{a_0}{2}\int_{-L}^{L} dx + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L} \cos\frac{n\pi x}{L}dx}_{=0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L} \sin\frac{n\pi x}{L}dx}_{=0}$$

-

$$\int_{-L}^{L} f(x)dx = \frac{a_0}{2}\int_{-L}^{L} dx = La_0 \quad \Rightarrow \quad a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

Multiply both sides of (FS) with $\cos \frac{m\pi x}{L}$ and integrate from $-L$ to L

Multiply both sides of (FS) with $\cos\frac{m\pi x}{L}$ and integrate from $-L$ to L

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx = \frac{a_0}{2}\underbrace{\int_{-L}^{L}\cos\frac{m\pi x}{L}dx}_{=0}$$

Multiply both sides of (FS) with $\cos\frac{m\pi x}{L}$ and integrate from $-L$ to L

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx = \frac{a_0}{2}\underbrace{\int_{-L}^{L}\cos\frac{m\pi x}{L}dx}_{=0} + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx}_{=0,\text{ if } n\neq m}$$

Multiply both sides of (FS) with $\cos\frac{m\pi x}{L}$ and integrate from $-L$ to L

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx &= \frac{a_0}{2}\underbrace{\int_{-L}^{L}\cos\frac{m\pi x}{L}dx}_{=0} \\ &+ \sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx}_{=0,\text{ if } n\neq m} \\ &+ \sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx}_{=0}\end{aligned}$$

Multiply both sides of (FS) with $\cos\frac{m\pi x}{L}$ and integrate from $-L$ to L

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx = \frac{a_0}{2}\underbrace{\int_{-L}^{L}\cos\frac{m\pi x}{L}dx}_{=0}$$

$$+\sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx}_{=0,\text{ if } n\neq m}$$

$$+\sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx}_{=0}$$

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx = a_m\underbrace{\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{m\pi x}{L}dx}_{=L} = a_m L$$

Multiply both sides of (FS) with $\cos \frac{m\pi x}{L}$

Multiply both sides of (FS) with $\cos\frac{m\pi x}{L}$ and integrate from $-L$ to L

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx &= \frac{a_0}{2}\underbrace{\int_{-L}^{L}\cos\frac{m\pi x}{L}dx}_{=0} \\ &+ \sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx}_{=0,\text{ if } n\neq m} \\ &+ \sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx}_{=0}\end{aligned}$$

Multiply both sides of (FS) with $\cos\frac{m\pi x}{L}$ and integrate from $-L$ to L

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx = \frac{a_0}{2}\underbrace{\int_{-L}^{L}\cos\frac{m\pi x}{L}dx}_{=0}$$

$$+\sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx}_{=0,\text{ if } n\neq m}$$

$$+\sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx}_{=0}$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx$$

Coefficients in a Fourier Series

To compute coefficient b_m, multiply both sides of (FS) with $\sin \frac{m\pi x}{L}$ and integrate from $-L$ to L.

Coefficients in a Fourier Series

To compute coefficient b_m, multiply both sides of (FS) with $\sin \frac{m\pi x}{L}$ and integrate from $-L$ to L.In a similar manner, using formulas from Lemma we get

Coefficients in a Fourier Series

To compute coefficient b_m, multiply both sides of (FS) with $\sin \frac{m\pi x}{L}$ and integrate from $-L$ to L.In a similar manner, using formulas from Lemma we get

$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} dx$$

Fourier Series

Definition

The trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

whose coefficients $a_0, a_n, b_n, n \geq 1$ are determined by

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^{L} f(x) dx \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx \end{aligned}$$

is called the **Fourier series** of the function f over interval $(-L, L)$

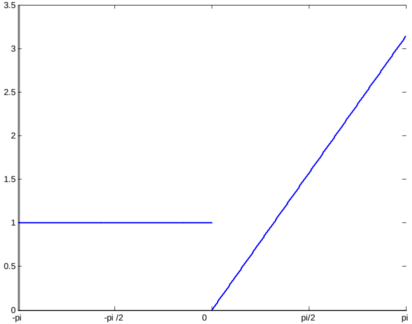
Finding Fourier Series

Example. Find the Fourier series of the function

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Example. Find the Fourier series of the function

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$



Finding Fourier Series

Note that $L = \pi$. Then

Finding Fourier Series

Note that $L = \pi$. Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} dx + \frac{1}{\pi}\int_{-\pi}^{0} xdx = 1 + \frac{\pi}{2}$$

Note that $L = \pi$. Then

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} dx + \frac{1}{\pi}\int_{-\pi}^{0} x\,dx = 1 + \frac{\pi}{2} \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \end{aligned}$$

Finding Fourier Series

Note that $L = \pi$. Then

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} dx + \frac{1}{\pi}\int_{-\pi}^{0} xdx = 1 + \frac{\pi}{2} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx
\end{aligned}
$$

Finding Fourier Series

Note that $L = \pi$. Then

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} dx + \frac{1}{\pi}\int_{-\pi}^{0} xdx = 1 + \frac{\pi}{2} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx \\
&= \frac{1}{n\pi}\sin(nx)\Big|_{-\pi}^{0} + \frac{1}{\pi}\,\frac{x}{n}\sin(nx)\Big|_{0}^{\pi} - \frac{1}{n\pi}\int_{0}^{\pi} \sin(nx)\,dx
\end{aligned}$$

Finding Fourier Series

Note that $L = \pi$. Then

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} dx + \frac{1}{\pi}\int_{-\pi}^{0} xdx = 1 + \frac{\pi}{2} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx \\
&= \frac{1}{n\pi}\sin(nx)\Big|_{-\pi}^{0} + \frac{1}{\pi}\,\frac{x}{n}\sin(nx)\Big|_{0}^{\pi} - \frac{1}{n\pi}\int_{0}^{\pi}\sin(nx)\,dx \\
&= \frac{1}{\pi n^2}\cos(nx)\Big|_{0}^{\pi} = \frac{1}{\pi n^2}(\cos(n\pi) - 1) = \frac{(-1)^n - 1}{\pi n^2}
\end{aligned}$$

Finding Fourier Series

In a similar manner

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

Finding Fourier Series

In a similar manner

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx
\end{aligned}
$$

Finding Fourier Series

In a similar manner

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx \\
&= \frac{(-1)^n(1-\pi)-1}{\pi n^2}
\end{aligned}
$$

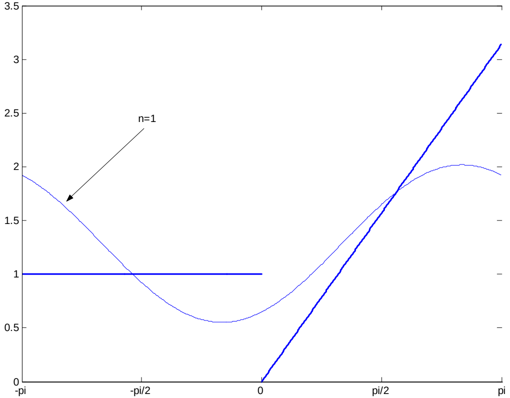
In a similar manner

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx \\
&= \frac{(-1)^n(1-\pi)-1}{\pi n^2}
\end{aligned}$$

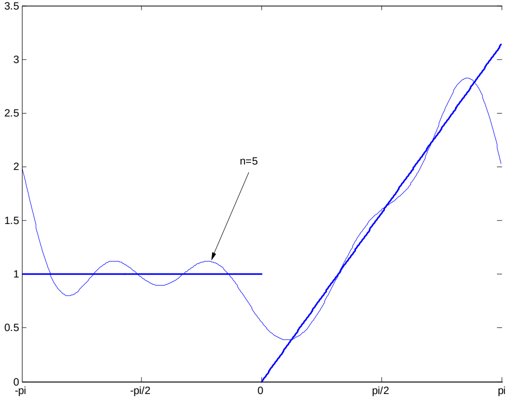
Therefore, for given function, the Fourier series expansion is

$$\frac{1}{2}+\frac{\pi}{4}+\sum_{n=1}^{\infty}\frac{(-1)^n-1}{\pi n^2}\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty}\frac{(-1)^n(1-\pi)-1}{\pi n^2}\sin\frac{n\pi x}{L}$$

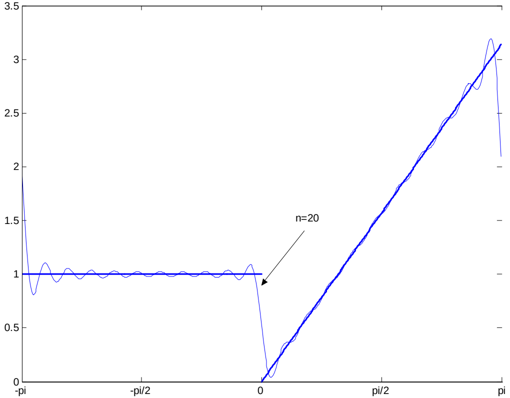
Finding Fourier Series



Finding Fourier Series



Finding Fourier Series



- In computing the coefficients $a_0, a_n, b_n, n \geq 1$ we assumed that function f was integrable on the interval $(-L, L)$.

Fourier Series

- In computing the coefficients $a_0, a_n, b_n, n \geq 1$ we assumed that function f was integrable on the interval $(-L, L)$.
- We also assumed that the trigonometric series as well as the series obtained when we multiply it by $\cos \frac{m\pi x}{L}$ or $\sin \frac{m\pi x}{L}$ converge in a way to allow us the term by term integration.

- In computing the coefficients $a_0, a_n, b_n, n \geq 1$ we assumed that function f was integrable on the interval $(-L, L)$.
- We also assumed that the trigonometric series as well as the series obtained when we multiply it by $\cos \frac{m\pi x}{L}$ or $\sin \frac{m\pi x}{L}$ converge in a way to allow us the term by term integration.
- These convergence issues are studied in advanced calculus, as is the question of when the Fourier series actually equals $f(x)$ for $-L < x < L$.

- In computing the coefficients $a_0, a_n, b_n, n \geq 1$ we assumed that function f was integrable on the interval $(-L, L)$.
- We also assumed that the trigonometric series as well as the series obtained when we multiply it by $\cos \frac{m\pi x}{L}$ or $\sin \frac{m\pi x}{L}$ converge in a way to allow us the term by term integration.
- These convergence issues are studied in advanced calculus, as is the question of when the Fourier series actually equals $f(x)$ for $-L < x < L$.
- Most of the functions we will encounter in applications guarantee both convergence of the Fourier series and its equality with f.

Fourier Series

- In computing the coefficients $a_0, a_n, b_n, n \geq 1$ we assumed that function f was integrable on the interval $(-L, L)$.
- We also assumed that the trigonometric series as well as the series obtained when we multiply it by $\cos \frac{m\pi x}{L}$ or $\sin \frac{m\pi x}{L}$ converge in a way to allow us the term by term integration.
- These convergence issues are studied in advanced calculus, as is the question of when the Fourier series actually equals $f(x)$ for $-L < x < L$.
- Most of the functions we will encounter in applications guarantee both convergence of the Fourier series and its equality with f.
- We now state without proof the resulty concerning the convergence of the Foruier series expansion for a wide class of functions commonly encountered in engineeering and science applications.

Fourier Series

First recall some definitions:

Definition

A function f is called piecewise continuous over an interval I if both limits

$$\lim_{x \to c+} f(x) = f(c+) \quad \text{and} \quad \lim_{x \to c-} f(x) = f(c-)$$

exist at every interior point $c \in I$ and at endpoints the appropriate one-sded limits exist, and moreover function f has at most finitely many discontinuites in I.

Fourier Series

First recall some definitions:

Definition

A function f is called piecewise continuous over an interval I if both limits

$$\lim_{x \to c+} f(x) = f(c+) \quad \text{and} \quad \lim_{x \to c-} f(x) = f(c-)$$

exist at every interior point $c \in I$ and at endpoints the appropriate one-sded limits exist, and moreover function f has at most finitely many discontinuites in I.

Notice that a piecewise contnuous function is bounded on its domain I, in other words it is not tending toward infinity.

Theorem (Convergence of Fourier series)

If the function f *and its derivetive* f' *are piecewise continuous over the interval* $-L < x < L$*, then* f *equals its Fourier series at all points of continuity. At a point* c *where the jump discontinuity occurs in* f*, the fourier series convreges to the average*

$$\frac{f(c+) + f(c-)}{2}$$

where $f(c+)$ *and* $f(c-)$ *denote the right and left limits of* f *at* c*.*

Theorem (Convergence of Fourier series)

If the function f and its derivetive f' are piecewise continuous over the interval $-L < x < L$, then f equals its Fourier series at all points of continuity. At a point c where the jump discontinuity occurs in f, the fourier series convreges to the average

$$\frac{f(c+)+f(c-)}{2}$$

where $f(c+)$ and $f(c-)$ denote the right and left limits of f at c.

In the example considered previously, the function satiesfies the conditions of the Theorem. Also, for every $x \neq 0$ in the interval $(-\pi, \pi)$ the Fourier series converges to $f(x)$. At $x = 0$ where jump discotinuity happens, the Fourier series converges to

$$\frac{f(0+)+f(0-)}{2} = \frac{0+1}{2} = \frac{1}{2}$$

Fourier Series

The trigonometric terms $\cos \frac{n\pi x}{L}$ and $\sin \frac{n\pi x}{L}$ in the Fourier series are periodic with period $2L$. Recall

Fourier Series

The trigonometric terms $\cos \frac{n\pi x}{L}$ and $\sin \frac{n\pi x}{L}$ in the Fourier series are periodic with period $2L$. Recall

Definition

Function f is periodic function with period T, if

$$f(x+T) = f(x), \quad \forall x$$

Fourier Series

The trigonometric terms $\cos\frac{n\pi x}{L}$ and $\sin\frac{n\pi x}{L}$ in the Fourier series are periodic with period $2L$. Recall

Definition

Function f is periodic function with period T, if

$$f(x+T) = f(x), \quad \forall x$$

Indeed, we have

$$\sin\frac{n\pi(x+2L)}{L} \quad = \quad \sin\left(\frac{n\pi x}{L} + 2n\pi\right)$$

Fourier Series

The trigonometric terms $\cos\frac{n\pi x}{L}$ and $\sin\frac{n\pi x}{L}$ in the Fourier series are periodic with period $2L$. Recall

Definition

Function f is periodic function with period T, if

$$f(x+T)=f(x), \quad \forall x$$

Indeed, we have

$$\sin\frac{n\pi(x+2L)}{L} \quad = \quad \sin\left(\frac{n\pi x}{L}+2n\pi\right)=\sin\frac{n\pi x}{L}$$

Fourier Series

The trigonometric terms $\cos \frac{n\pi x}{L}$ and $\sin \frac{n\pi x}{L}$ in the Fourier series are periodic with period $2L$. Recall

Definition

Function f is periodic function with period T, if

$$f(x+T) = f(x), \quad \forall x$$

Indeed, we have

$$\begin{aligned} \sin \frac{n\pi(x+2L)}{L} &= \sin\left(\frac{n\pi x}{L} + 2n\pi\right) = \sin \frac{n\pi x}{L} \\ \cos \frac{n\pi(x+2L)}{L} &= \cos\left(\frac{n\pi x}{L} + 2n\pi\right) \end{aligned}$$

Fourier Series

The trigonometric terms $\cos\frac{n\pi x}{L}$ and $\sin\frac{n\pi x}{L}$ in the Fourier series are periodic with period $2L$. Recall

Definition

Function f is periodic function with period T, if

$$f(x+T) = f(x), \quad \forall x$$

Indeed, we have

$$\begin{aligned} \sin\frac{n\pi(x+2L)}{L} &= \sin\left(\frac{n\pi x}{L} + 2n\pi\right) = \sin\frac{n\pi x}{L} \\ \cos\frac{n\pi(x+2L)}{L} &= \cos\left(\frac{n\pi x}{L} + 2n\pi\right) = \cos\frac{n\pi x}{L} \end{aligned}$$

Fourier Series

The trigonometric terms $\cos\frac{n\pi x}{L}$ and $\sin\frac{n\pi x}{L}$ in the Fourier series are periodic with period $2L$. Recall

Definition

Function f is periodic function with period T, if

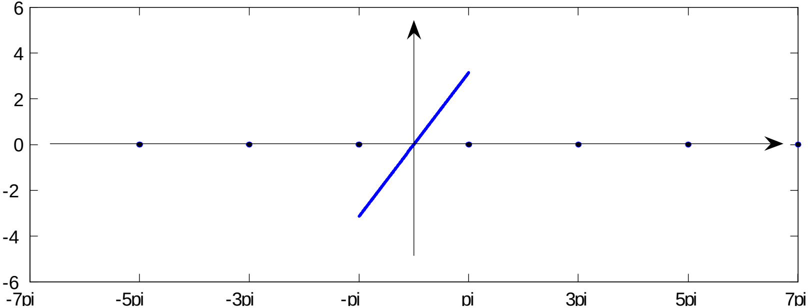
$$f(x+T) = f(x), \quad \forall x$$

Indeed, we have

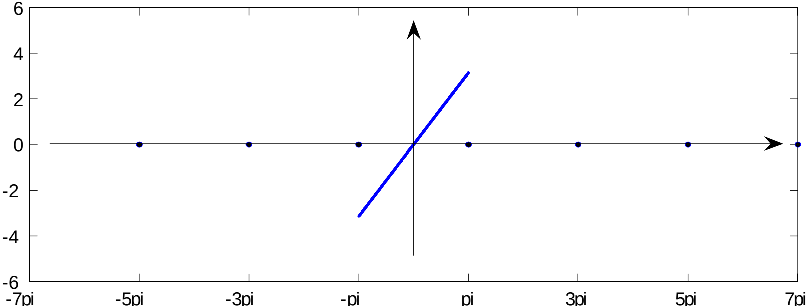
$$\begin{aligned}
\sin\frac{n\pi(x+2L)}{L} &= \sin\left(\frac{n\pi x}{L} + 2n\pi\right) = \sin\frac{n\pi x}{L} \\
\cos\frac{n\pi(x+2L)}{L} &= \cos\left(\frac{n\pi x}{L} + 2n\pi\right) = \cos\frac{n\pi x}{L}
\end{aligned}$$

It follows, that the Fourier series is also periodic with period $2L$. Therefore, the Fourier series not only represents the function f over the interval $-L < x < L$, but it also produces the periodic extension of f over the entire real number line.

Consider function $f(x) = x$ on interval $(-\pi, \pi)$:



Consider function $f(x) = x$ on interval $(-\pi, \pi)$:

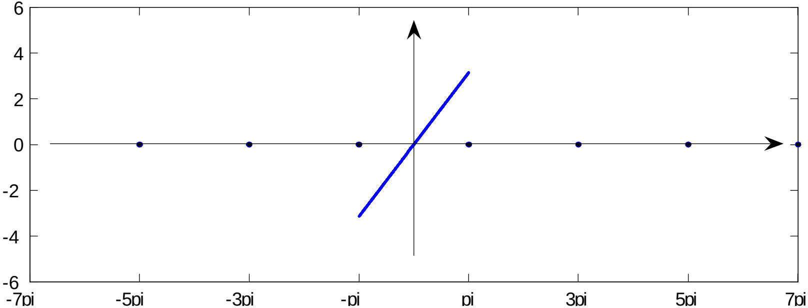


The Fourier series for $f(x) = x$ on interval $(-\pi, \pi)$ is

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$$

Fourier Series

Consider function $f(x) = x$ on interval $(-\pi, \pi)$:



The Fourier series for $f(x) = x$ on interval $(-\pi, \pi)$ is

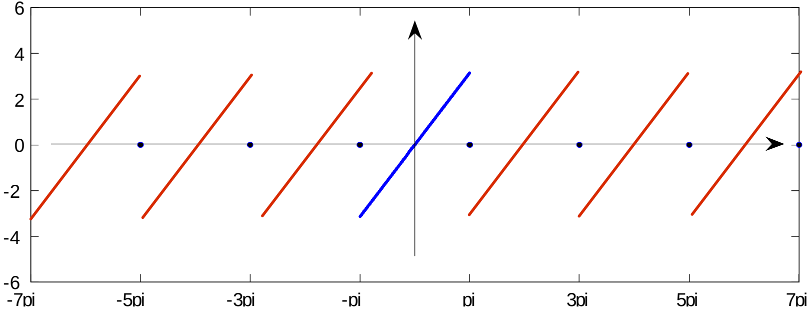
$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$$

$$f(x) = \frac{2}{1}\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x + \frac{2}{5}\sin 5x - \dots$$

Fourier Series

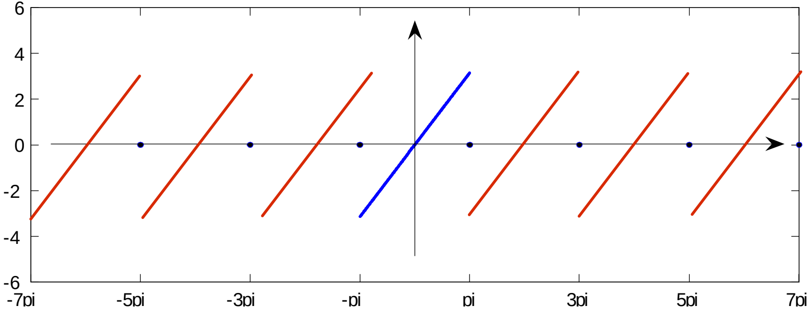
- The Fourier series for $f(x) = x$ on interval $(-\pi, \pi)$ is periodic with period 2π and converges to the periodic extension of $f(x) = x$ over entire $x-$axis.

- The Fourier series for $f(x) = x$ on interval $(-\pi, \pi)$ is periodic with period 2π and converges to the periodic extension of $f(x) = x$ over entire $x-$axis.



-

- The Fourier series for $f(x) = x$ on interval $(-\pi, \pi)$ is periodic with period 2π and converges to the periodic extension of $f(x) = x$ over entire $x-$axis.



- The solid dots represent the values

$$\frac{f(\pi+) + f(\pi-)}{2} = \frac{\pi + (-\pi)}{2} = 0$$

Series converges to 0 at the endpoints $\pm\pi, \pm 3\pi, \pm 5\pi, \pm 7\pi, \ldots$

Fourier Cosine and Fourier Sine Series

- How to compute the Fourier series if the function f is not defined over a symmetric interval?

Fourier Cosine and Fourier Sine Series

- How to compute the Fourier series if the function f is not defined over a symmetric interval?
- For example, the function f is defined over $(0, L)$.

Fourier Cosine and Fourier Sine Series

- How to compute the Fourier series if the function f is not defined over a symmetric interval?
- For example, the function f is defined over $(0, L)$.
- To do so, we extend our function over the symmetric interval $-L < x < L$.

Fourier Cosine and Fourier Sine Series

- How to compute the Fourier series if the function f is not defined over a symmetric interval?
- For example, the function f is defined over $(0, L)$.
- To do so, we extend our function over the symmetric interval $-L < x < L$.
- How we should extend it?

Fourier Cosine and Fourier Sine Series

- How to compute the Fourier series if the function f is not defined over a symmetric interval?
- For example, the function f is defined over $(0, L)$.
- To do so, we extend our function over the symmetric interval $-L < x < L$.
- How we should extend it?
- We can define the extension to be any function over $-L < x < 0$ as long as the extension and its derivative are piecewise continuous functions.

Fourier Cosine and Fourier Sine Series

- How to compute the Fourier series if the function f is not defined over a symmetric interval?
- For example, the function f is defined over $(0, L)$.
- To do so, we extend our function over the symmetric interval $-L < x < L$.
- How we should extend it?
- We can define the extension to be any function over $-L < x < 0$ as long as the extension and its derivative are piecewise continuous functions.
- No matter what piecewise extension we choose the resulting Fourier series is guaranteed to to equal $f(x)$ at all points of continuity over the original domain $0 < x < L$.

Fourier Cosine and Fourier Sine Series

- How to compute the Fourier series if the function f is not defined over a symmetric interval?
- For example, the function f is defined over $(0, L)$.
- To do so, we extend our function over the symmetric interval $-L < x < L$.
- How we should extend it?
- We can define the extension to be any function over $-L < x < 0$ as long as the extension and its derivative are piecewise continuous functions.
- No matter what piecewise extension we choose the resulting Fourier series is guaranteed to to equal $f(x)$ at all points of continuity over the original domain $0 < x < L$.
- There are two special extensions that are useful and whose Fourier coefficients are easy to compute: the **odd and even extensions.**

Lemma

If g is an odd function then

$$\int_{-L}^{L} g(x)dx = 0$$

and if g is an even function then

$$\int_{-L}^{L} g(x)dx = 2\int_{0}^{L} g(x)dx$$

Fourier Cosine and Fourier Sine Series

Lemma

If g is an odd function then

$$\int_{-L}^{L} g(x)dx = 0$$

and if g is an even function then

$$\int_{-L}^{L} g(x)dx = 2\int_{0}^{L} g(x)dx$$

Also keep in mind that

1. The product of 2 even functions is even.

Fourier Cosine and Fourier Sine Series

Lemma

If g is an odd function then

$$\int_{-L}^{L} g(x)dx = 0$$

and if g is an even function then

$$\int_{-L}^{L} g(x)dx = 2\int_{0}^{L} g(x)dx$$

Also keep in mind that

1. The product of 2 even functions is even.
2. The product of 2 odd functions is even.

Fourier Cosine and Fourier Sine Series

Lemma

If g is an odd function then

$$\int_{-L}^{L} g(x)dx = 0$$

and if g is an even function then

$$\int_{-L}^{L} g(x)dx = 2\int_{0}^{L} g(x)dx$$

Also keep in mind that

1. The product of 2 even functions is even.
2. The product of 2 odd functions is even.
3. The product of an even and odd functions is odd.

Even Extension: Fourier Cosine Series

Suppose that f is defined on $(0, L)$. Define the even extension of f by

$$f(-x) = f(x) \text{ on } (-L, L)$$

Even Extension: Fourier Cosine Series

Suppose that f is defined on $(0, L)$. Define the even extension of f by

$$f(-x) = f(x) \text{ on } (-L, L)$$

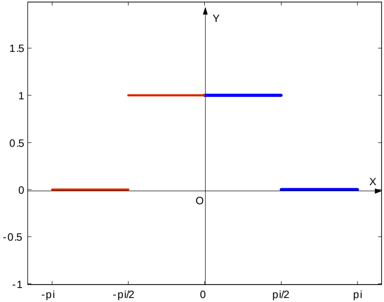
This is simply obtained by reflecting the function about $y-$axis.

Even Extension: Fourier Cosine Series

Suppose that f is defined on $(0, L)$. Define the even extension of f by

$$f(-x) = f(x) \text{ on } (-L, L)$$

This is simply obtained by reflecting the function about $y-$axis.



Even Extension: Fourier Cosine Series

If we use the even extension, then

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{2}{L}\int_{0}^{L} f(x)dx$$

Even Extension: Fourier Cosine Series

If we use the even extension, then

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\,dx \\
a_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\frac{n\pi x}{L}}_{even}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx
\end{aligned}
$$

Even Extension: Fourier Cosine Series

If we use the even extension, then

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\,dx \\
a_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\frac{n\pi x}{L}}_{even}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\
b_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\frac{n\pi x}{L}}_{odd}\,dx = 0
\end{aligned}
$$

Even Extension: Fourier Cosine Series

If we use the even extension, then

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\,dx \\
a_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\frac{n\pi x}{L}}_{even}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\
b_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\frac{n\pi x}{L}}_{odd}\,dx = 0
\end{aligned}
$$

and the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

which is called the **Fourier Cosine series** of function f.

Even Extension: Fourier Cosine Series

If we use the even extension, then

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{2}{L}\int_{0}^{L} f(x)dx \\
a_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\frac{n\pi x}{L}}_{even} dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}dx \\
b_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\frac{n\pi x}{L}}_{odd} dx = 0
\end{aligned}
$$

and the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

which is called the **Fourier Cosine series** of function f. It converges to the original function f over interval $(0, L)$ and to the even extension of f over interval $(-L, 0)$.

Even Extension: Fourier Cosine Series

We summarise this result

Even Extension: Fourier Cosine Series

We summarise this result

Theorem (Fourier Cosine Series)

The Fourier series of an even function on the interval (-L,L) is the cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

where

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)\,dx \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \end{aligned}$$

Even Extension: Fourier Cosine Series

Consider function

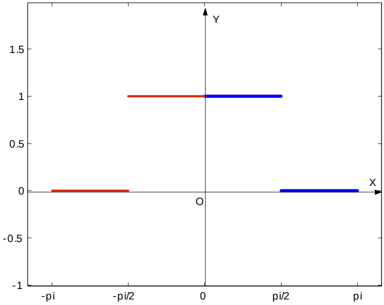
$$f(x) = \begin{cases} 1, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi \end{cases}$$

Even Extension: Fourier Cosine Series

Consider function

$$f(x) = \begin{cases} 1, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi \end{cases}$$

and its even extension over $(-\pi, \pi)$



Even Extension: Fourier Cosine Series

Compute Fourier coefficients to find

Even Extension: Fourier Cosine Series

Compute Fourier coefficients to find

$$a_0 = 1$$

Even Extension: Fourier Cosine Series

Compute Fourier coefficients to find

$$
\begin{aligned}
a_0 &= 1 \\
a_n &= \frac{2}{n\pi} \sin \frac{n\pi}{2}
\end{aligned}
$$

Even Extension: Fourier Cosine Series

Compute Fourier coefficients to find

$$
\begin{aligned}
a_0 &= 1 \\
a_n &= \frac{2}{n\pi} \sin \frac{n\pi}{2}
\end{aligned}
$$

The Fourier Cosine series is

$$
f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{2} \cos \frac{n\pi x}{L}
$$

Even Extension: Fourier Cosine Series

Compute Fourier coefficients to find

$$\begin{aligned} a_0 &= 1 \\ a_n &= \frac{2}{n\pi} \sin \frac{n\pi}{2} \end{aligned}$$

The Fourier Cosine series is

$$\begin{aligned} f(x) &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{2} \cos \frac{n\pi x}{L} \\ &= \frac{1}{2} + \frac{2}{1 \cdot \pi} \sin \frac{1 \cdot \pi}{2} \cos \frac{1 \cdot \pi x}{L} + \frac{1}{2\pi} \sin \frac{2\pi}{2} \cos \frac{2\pi x}{L} + \frac{2}{3\pi} \sin \frac{3\pi}{2} \text{co} \end{aligned}$$

Even Extension: Fourier Cosine Series

Compute Fourier coefficients to find

$$
\begin{aligned}
a_0 &= 1 \\
a_n &= \frac{2}{n\pi} \sin \frac{n\pi}{2}
\end{aligned}
$$

The Fourier Cosine series is

$$
\begin{aligned}
f(x) &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{2} \cos \frac{n\pi x}{L} \\
&= \frac{1}{2} + \frac{2}{1 \cdot \pi} \sin \frac{1 \cdot \pi}{2} \cos \frac{1 \cdot \pi x}{L} + \frac{1}{2\pi} \sin \frac{2\pi}{2} \cos \frac{2\pi x}{L} + \frac{2}{3\pi} \sin \frac{3\pi}{2} \text{co} \\
&= \frac{1}{2} + \frac{2}{\pi} \underbrace{\sin \frac{\pi}{2}}_{=1} \cos \frac{\pi x}{L} + \frac{1}{2\pi} \underbrace{\sin \pi}_{=0} \cos \frac{2\pi x}{L} + \frac{2}{3\pi} \underbrace{\sin \frac{3\pi}{2}}_{=-1} \cos \frac{3\pi x}{L} + \ldots
\end{aligned}
$$

Even Extension: Fourier Cosine Series

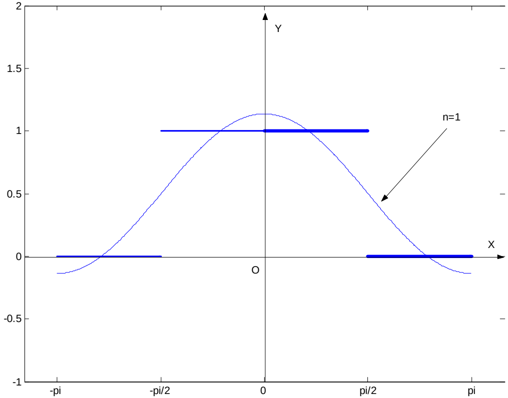
Compute Fourier coefficients to find

$$
\begin{aligned}
a_0 &= 1 \\
a_n &= \frac{2}{n\pi} \sin \frac{n\pi}{2}
\end{aligned}
$$

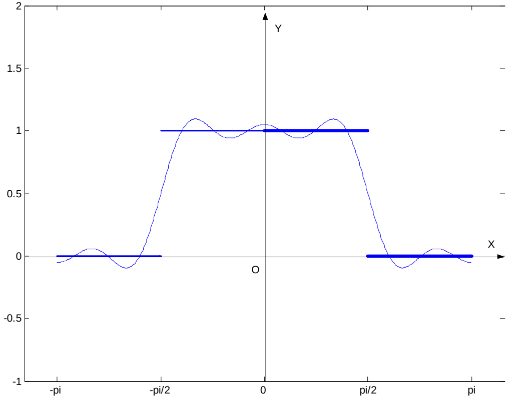
The Fourier Cosine series is

$$
\begin{aligned}
f(x) &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{2} \cos \frac{n\pi x}{L} \\
&= \frac{1}{2} + \frac{2}{1\cdot\pi} \sin \frac{1\cdot\pi}{2} \cos \frac{1\cdot\pi x}{L} + \frac{1}{2\pi} \sin \frac{2\pi}{2} \cos \frac{2\pi x}{L} + \frac{2}{3\pi} \sin \frac{3\pi}{2} \mathrm{co} \\
&= \frac{1}{2} + \frac{2}{\pi} \underbrace{\sin \frac{\pi}{2}}_{=1} \cos \frac{\pi x}{L} + \frac{1}{2\pi} \underbrace{\sin \pi}_{=0} \cos \frac{2\pi x}{L} + \frac{2}{3\pi} \underbrace{\sin \frac{3\pi}{2}}_{=-1} \cos \frac{3\pi x}{L} + \ldots \\
&= \frac{1}{2} + \frac{2}{\pi} \cos \frac{\pi x}{L} - \frac{2}{3\pi} \cos \frac{3\pi x}{L} + \ldots
\end{aligned}
$$

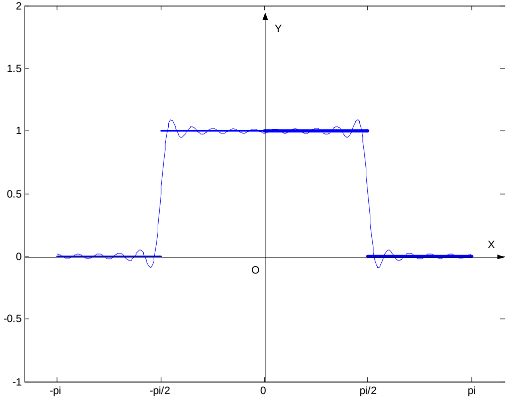
Even Extension: Fourier Cosine Series



Even Extension: Fourier Cosine Series



Even Extension: Fourier Cosine Series



Odd Extension: Fourier Sine Series

Suppose that f is defined on $(0, L)$. Define the odd extension of f by

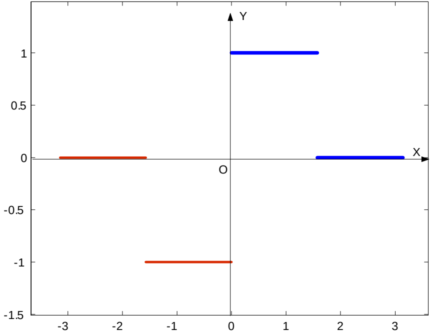
$$f(-x) = -f(x) \text{ on } (-L, L)$$

Odd Extension: Fourier Sine Series

Suppose that f is defined on $(0, L)$. Define the odd extension of f by

$$f(-x) = -f(x) \text{ on } (-L, L)$$

This is simply obtained by reflecting the function about origin.



Odd Extension: Fourier Sine Series

If we use the odd extension, then

$$a_0 = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{odd} dx = 0$$

Odd Extension: Fourier Sine Series

If we use the odd extension, then

$$a_0 = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{odd} dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\frac{n\pi x}{L}}_{odd} dx = 0$$

Odd Extension: Fourier Sine Series

If we use the odd extension, then

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{odd} dx = 0 \\
a_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\frac{n\pi x}{L}}_{odd} dx = 0 \\
b_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\frac{n\pi x}{L}}_{even} dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L} dx
\end{aligned}
$$

Odd Extension: Fourier Sine Series

If we use the odd extension, then

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{odd} dx = 0 \\
a_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\frac{n\pi x}{L}}_{odd} dx = 0 \\
b_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\frac{n\pi x}{L}}_{even} dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L} dx
\end{aligned}$$

and the Fourier series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Odd Extension: Fourier Sine Series

If we use the odd extension, then

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{odd} dx = 0 \\
a_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\frac{n\pi x}{L}}_{odd} dx = 0 \\
b_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\frac{n\pi x}{L}}_{even} dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L} dx
\end{aligned}
$$

and the Fourier series is

$$
f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}
$$

which is called the **Fourier Sine series** of function f.

Odd Extension: Fourier Sine Series

If we use the odd extension, then

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{odd} dx = 0 \\ a_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\frac{n\pi x}{L}}_{odd} dx = 0 \\ b_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\frac{n\pi x}{L}}_{even} dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L} dx \end{aligned}$$

and the Fourier series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

which is called the **Fourier Sine series** of function f.It converges to the original function f over interval $(0, L)$ and to the odd extension of f over interval $(-L, 0)$.

Odd Extension: Fourier Sine Series

We summarise this result

Theorem (Fourier Sine Series)

The Fourier series of an odd function on the interval $(-L, L)$ *is the sine series*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Odd Extension: Fourier Sine Series

Consider function

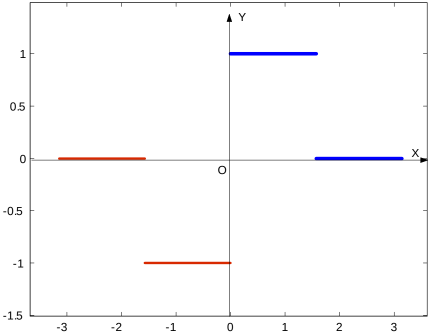
$$f(x) = \begin{cases} 1, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi \end{cases}$$

Odd Extension: Fourier Sine Series

Consider function

$$f(x) = \begin{cases} 1, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi \end{cases}$$

and its odd extension over $(-\pi, \pi)$



Odd Extension: Fourier Sine Series

Compute Fourier coefficients to find

$$b_n = \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)$$

Odd Extension: Fourier Sine Series

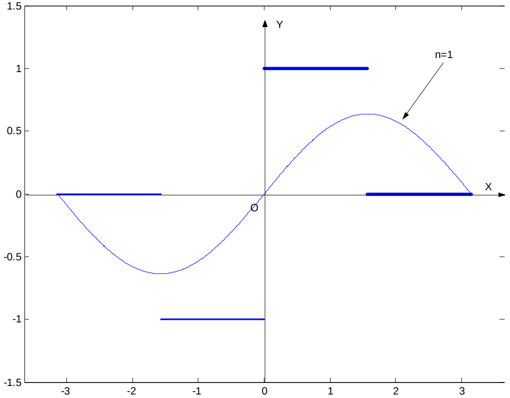
Compute Fourier coefficients to find

$$b_n = \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)$$

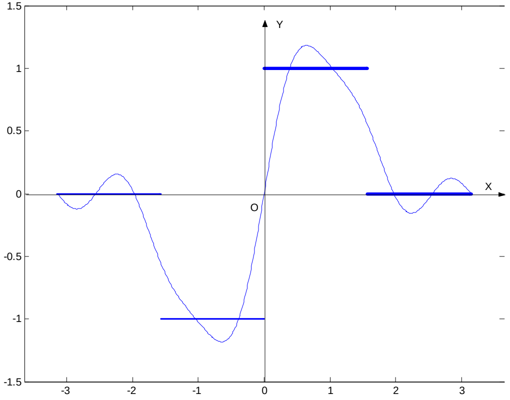
The Fourier Sine series is

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)\sin\frac{n\pi x}{L}$$

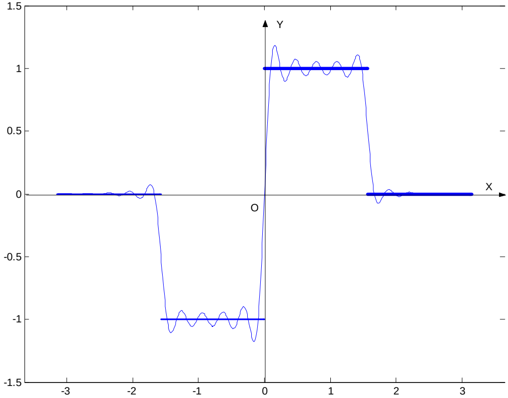
Odd Extension: Fourier Sine Series



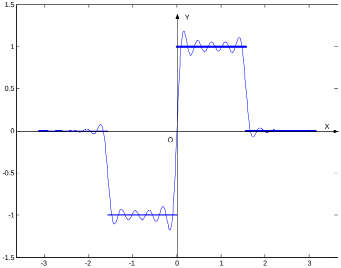
Odd Extension: Fourier Sine Series



Odd Extension: Fourier Sine Series

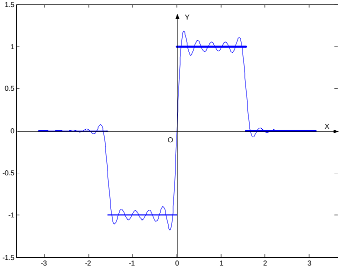


Gibbs Phenomenon



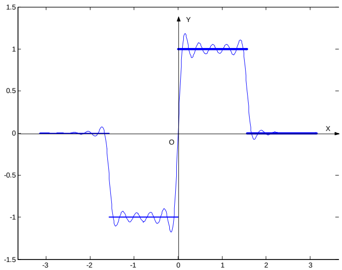
In previuos figures, like in this one, the overshoot at $x = \pi/2+$ and undershoot at $x = \pi/2-$ are characteristic of fourier series expansions near points of jump discontinuity.

Gibbs Phenomenon



In previuos figures, like in this one, the overshoot at $x = \pi/2+$ and undershoot at $x = \pi/2-$ are characteristic of fourier series expansions near points of jump discontinuity. Known as **Gibbs phenomenon**, after american mathematical physicist Josiah Willard Gibbs, this characteristics persists even when a large number of terms are summed.

Gibbs Phenomenon



In previuos figures, like in this one, the overshoot at $x = \pi/2+$ and undershoot at $x = \pi/2-$ are characteristic of fourier series expansions near points of jump discontinuity. Known as **Gibbs phenomenon**, after american mathematical physicist Josiah Willard Gibbs, this characteristics persists even when a large number of terms are summed. The combined overshoot and undershoot amount to approximately 18% of the difference between the functions values at the point of discontinuity.

Gibbs Phenomenon

Consider function

$$f(x) = \begin{cases} 1 & , \quad 0 < x < 1 \\ 0 & , \quad 1 < x < 2 \end{cases}$$

Gibbs Phenomenon

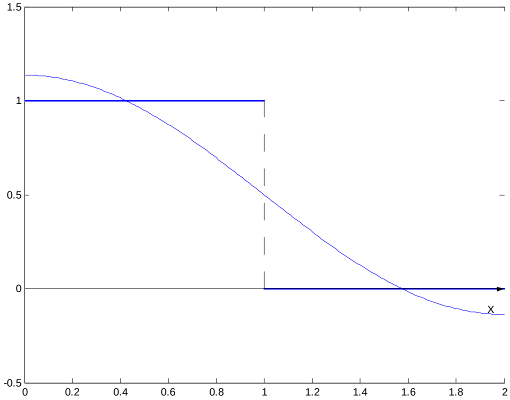


Figure: $n = 2$

Gibbs Phenomenon

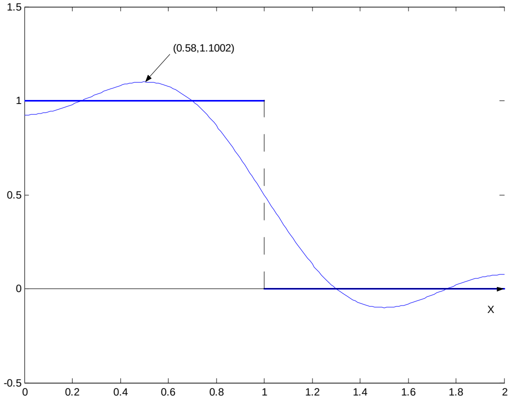


Figure: $n = 4$

Gibbs Phenomenon

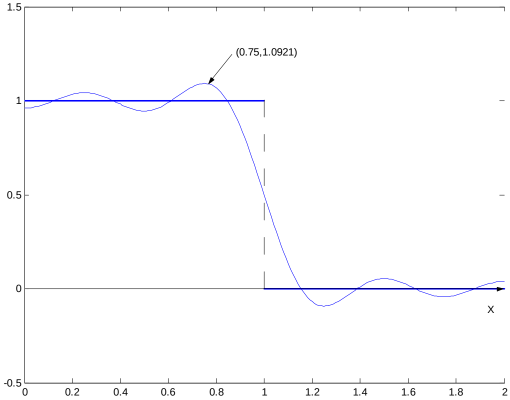


Figure: $n = 8$

Gibbs Phenomenon

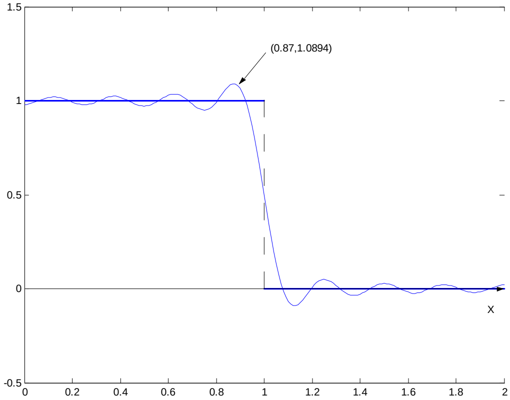


Figure: $n = 16$

Gibbs Phenomenon

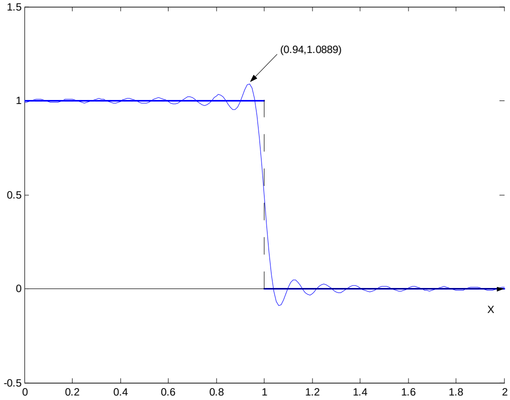


Figure: $n = 32$

Gibbs Phenomenon

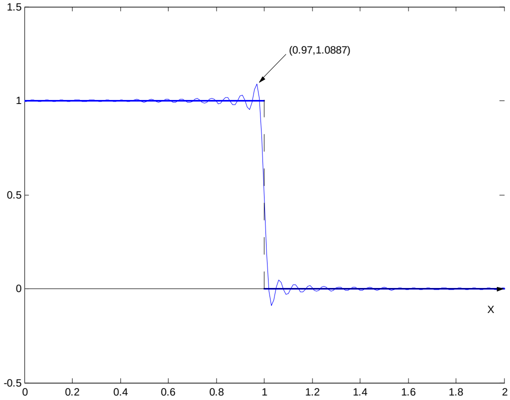


Figure: $n = 64$

Gibbs Phenomenon

The Gibbs phenomenon for $n = 2, 4, 8, 16, 32.$ and 64 terms in Fourier approximation presents the highest peak that moves from 0.58 to 0.75, then to 0.87, and so forth, getting closer to the discontinuity at $x = 1$.

Gibbs Phenomenon

The Gibbs phenomenon for $n = 2, 4, 8, 16, 32.$ and 64 terms in Fourier approximation presents the highest peak that moves from 0.58 to 0.75, then to 0.87, and so forth, getting closer to the discontinuity at $x = 1$.

The overshoot is always close to 1.09, or about 9% of the distance between $y = 0$ and $y = 1$ at the discontinuity point $x = 1$.