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Introduction to Complex Analysis

1. Integration in the Complex Plane
   - Real Integrals
   - Complex Integrals
   - Cauchy-Goursat Theorem
   - Independence of Path
   - Cauchy's Integral Formulas
   - Consequences of the Integral Formulas

Subsection 1

Real Integrals

Definite Integrals

- If $F(x)$ is an antiderivative of a continuous function f, i.e., F is a function for which $F'(x) = f(x)$, then the definite integral of f on the interval $[a, b]$ is the number

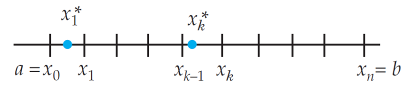
$$\int_a^b f(x)dx = F(x)\big|_a^b = F(b) - F(a).$$

- Example: $\int_{-1}^2 x^2 dx = \frac{1}{3}x^3\big|_{-1}^2 = \frac{8}{3} - \frac{-1}{3} = 3.$
- The fundamental theorem of calculus is a method of evaluating $\int_a^b f(x)dx$; it is not the definition of $\int_a^b f(x)dx$.
- We next define:
  - The definite (or Riemann) integral of a function f;
  - Line integrals in the Cartesian plane.

Both definitions rest on the limit concept.

Steps Leading to the Definition of the Definite Integral

1. Let f be a function of a single variable x defined at all points in a closed interval $[a, b]$.
2. Let P be a partition:
$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$
of $[a, b]$ into n subintervals $[x_{k-1}, x_k]$ of length $\Delta x_k = x_k - x_{k-1}$.
3. Let $\|P\|$ be the **norm** of the partition P of $[a, b]$, i.e., the length of the longest subinterval.
4. Choose a number x_k^* in each subinterval $[x_{k-1}, x_k]$ of $[a, b]$.



5. Form n products $f(x_k^*)\Delta x_k$, $k = 1, 2, \ldots, n$, and then sum these products:
$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

The Definition of the Definite Integral

Definition (Definite Integral)

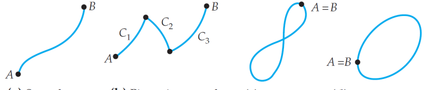
The **definite integral** of f on $[a, b]$ is

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

- Whenever the limit exists we say that f is **integrable** on the interval $[a, b]$ or that the definite integral of f **exists**.
- It can be proved that if f is continuous on $[a, b]$, then the integral exists.

Terminology About Curves

- Suppose a curve C in the plane is parametrized by a set of equations $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, where $x(t)$ and $y(t)$ are continuous real functions. Let the initial and terminal points of C $(x(a), y(a))$, $(x(b), y(b))$ be denoted by A, B. We say that:
  - (i) C is a **smooth curve** if x' and y' are continuous on the closed interval $[a, b]$ and not simultaneously zero on the open interval (a, b).
  - (ii) C is a **piecewise smooth curve** if it consists of a finite number of smooth curves $C_1, C_2, \ldots, C_n$ joined end to end, i.e., the terminal point of one curve C_k coinciding with the initial point of the next curve C_{k+1}.
  - (iii) C is a **simple curve** if the curve C does not cross itself except possibly at $t = a$ and $t = b$.
  - (iv) C is a **closed curve** if $A = B$.
  - (v) C is a **simple closed curve** if the curve C does not cross itself and $A = B$, i.e., C is simple and closed.



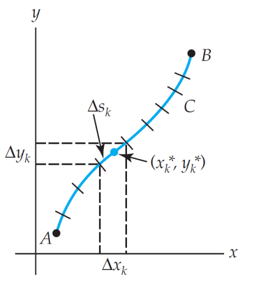
(a) Smooth curve and simple **(b)** Piecewise smooth curve and simple **(c)** Closed but not simple **(d)** Simple closed curve

Steps Leading to the Definition of Line Integrals

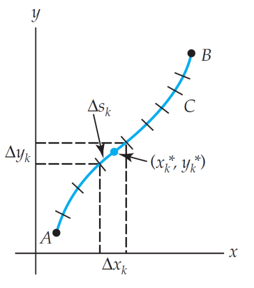
1. Let G be a function of two real variables x and y, defined at all points on a smooth curve C that lies in some region of the xy-plane. Let C be defined by the parametrization $x = x(t)$, $y = y(t)$, $a \leq t \leq b$.
2. Let P be a partition of the parameter interval $[a, b]$ into n subintervals $[t_{k-1}, t_k]$ of length $\Delta t_k = t_k - t_{k-1}$:

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

The partition P induces a partition of the curve C into n subarcs of length Δs_k. Let the projection of each subarc onto the x- and y-axes have lengths Δx_k and Δy_k, respectively.



Steps Leading to the Definition of Line Integrals (Cont'd)



3. Let $\|P\|$ be the **norm** of the partition P of $[a, b]$, that is, the length of the longest subinterval.
4. Choose a point (x_k^*, y_k^*) on each subarc of C.
5. Form n products $G(x_k^*, y_k^*)\Delta x_k$, $G(x_k^*, y_k^*)\Delta y_k$, $G(x_k^*, y_k^*)\Delta s_k$, $k = 1, 2, \ldots, n$, and then sum these products
$$\sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta x_k, \quad \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta y_k, \quad \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta s_k.$$

The Definition of Line Integrals

Definition (Line Integrals in the Plane)

(i) The **line integral of G along C with respect to x** is

$$\int_C G(x,y)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta x_k.$$

(ii) The **line integral of G along C with respect to y** is

$$\int_C G(x,y)dy = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta y_k.$$

(iii) The **line integral of G along C with respect to arc length s** is

$$\int_C G(x,y)ds = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta s_k.$$

- If G is continuous on C, then the three types of line integrals exist.
- The curve C is referred to as the **path of integration**.

Method of Evaluation: C Defined Parametrically

- Convert a line integral to a definite integral in a single variable.
- If C is a smooth curve parametrized by $x = x(t)$, $y = y(t)$, $a \le t \le b$, then replace
  - x and y in the integral by the functions $x(t)$ and $y(t)$;
  - the appropriate differential dx, dy, or ds by

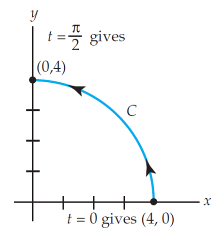
$$x'(t)dt, \quad y'(t)dt, \quad \sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$$

- The term $ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}dt$ is called the **differential of the arc length**.
- The line integrals become definite integrals in which the variable of integration is the parameter t:

$$\begin{aligned} \int_C G(x,y)dx &= \int_a^b G(x(t),y(t))x'(t)dt, \\ \int_C G(x,y)dy &= \int_a^b G(x(t),y(t))y'(t)dt, \\ \int_C G(x,y)ds &= \int_a^b G(x(t),y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}dt. \end{aligned}$$

Evaluation of a Line Integral I

- Evaluate $\int_C xy^2 dx$, where the path of integration C is the quarter circle defined by $x = 4\cos t, y = 4\sin t$, $0 \le t \le \frac{\pi}{2}$.



We have

$$dx = -4\sin t dt.$$

Thus,

$$\begin{aligned}\int_C xy^2 dx &= \int_0^{\pi/2} (4\cos t)(4\sin t)^2(-4\sin t dt)\\ &= -256\int_0^{\pi/2}\sin^3 t\cos t dt\\ &= -256[\tfrac{1}{4}\sin^4 t]_0^{\pi/2}\\ &= -64.\end{aligned}$$

Evaluation of a Line Integral II

- Evaluate $\int_C xy^2 dy$, where the path of integration C is the quarter circle defined by $x = 4\cos t$, $y = 4\sin t$, $0 \leq t \leq \frac{\pi}{2}$.

  We have

  $$dy = 4\cos t dt.$$

  Thus,

  $$\begin{aligned}
\int_C xy^2 dy &= \int_0^{\pi/2} (4\cos t)(4\sin t)^2 (4\cos t dt) \\
&= 256 \int_0^{\pi/2} \sin^2 t \cos^2 t dt \\
&= 256 \int_0^{\pi/2} \frac{1}{4} \sin^2 2t dt \\
&= 64 \int_0^{\pi/2} \frac{1}{2}(1 - \cos 4t) dt \\
&= 32\left[t - \frac{1}{4}\sin 4t\right]_0^{\pi/2} = 16\pi.
\end{aligned}$$

Evaluation of a Line Integral III

- Evaluate $\int_C xy^2 ds$, where the path of integration C is the quarter circle defined by $x = 4\cos t, y = 4\sin t$, $0 \leq t \leq \frac{\pi}{2}$.

  We have

$$ds = \sqrt{16(\sin^2 t + \cos^2 t)}dt = 4dt.$$

  Therefore,

$$\begin{aligned} \int_C xy^2 ds &= \int_0^{\pi/2} (4\cos t)(4\sin t)^2(4dt) \\ &= 256 \int_0^{\pi/2} \sin^2 t \cos t dt \\ &= 256[\tfrac{1}{3}\sin^3 t]_0^{\pi/2} \\ &= \tfrac{256}{3}. \end{aligned}$$

Method of Evaluation: C Defined by a Function

- If the path of integration C is the graph of an explicit function $y = f(x)$, $a \le x \le b$, then we can use x as a parameter:
- The differential of y is $dy = f'(x)dx$, and the differential of arc length is $ds = \sqrt{1 + [f'(x)]^2}dx$.
- We, thus, obtain the definite integrals:

$$\begin{array}{rcl} \int_C G(x,y)dx & = & \int_a^b G(x, f(x))dx, \\ \int_C G(x,y)dy & = & \int_a^b G(x, f(x))f'(x)dx, \\ \int_C G(x,y)ds & = & \int_a^b G(x, f(x))\sqrt{1 + [f'(x)]^2}dx. \end{array}$$

- A line integral along a piecewise smooth curve C is defined as the sum of the integrals over the various smooth pieces.
- Example: To evaluate $\int_C G(x,y)ds$ when C is composed of two smooth curves C_1 and C_2, we write

$$\int_C G(x,y)ds = \int_{C_1} G(x,y)ds + \int_{C_2} G(x,y)ds.$$

Notation for Line Integrals

- In many applications, line integrals appear as a sum

$$\int_C P(x,y)dx + \int_C Q(x,y)dy.$$

- It is common practice to write this sum as one integral without parentheses as

$$\int_C P(x,y)dx + Q(x,y)dy$$

or simply

$$\int_C Pdx + Qdy.$$

- A line integral along a closed curve C is usually denoted by

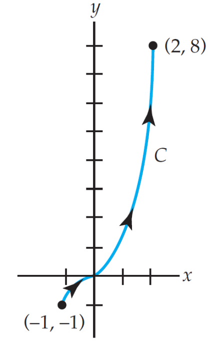
$$\oint_C Pdx + Qdy.$$

C Defined by an Explicit Function

- Evaluate $\int_C xydx + x^2dy$, where C is the graph of $y = x^3$, $-1 \leq x \leq 2$.

  We have $dy = 3x^2dx$. Therefore,

$$\begin{aligned}
\int_C xydx + x^2dy &= \int_{-1}^{2} xx^3dx + x^2 3x^2dx \\
&= \int_{-1}^{2} (x^4 + 3x^4)dx \\
&= \int_{-1}^{2} 4x^4dx \\
&= \tfrac{4}{5}x^5\Big|_{-1}^{2} \\
&= \tfrac{4}{5}(32 - (-1)) = \tfrac{132}{5}.
\end{aligned}$$



C a Closed Curve

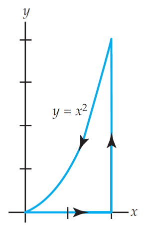
- Evaluate $\oint_C x dx$, where C is the circle defined by $x = \cos t, y = \sin t$, $0 \leq t \leq 2\pi$.

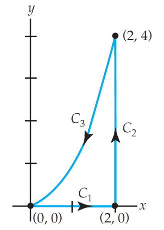
  We have $dx = -\sin t dt$, whence:

$$\begin{aligned} \oint_C x dx &= \int_0^{2\pi} \cos t (-\sin t dt) \\ &= \frac{1}{2} \cos^2 t \Big|_0^{2\pi} \\ &= \frac{1}{2}(1-1) \\ &= 0. \end{aligned}$$

C Another Closed Curve

- Evaluate $\oint_C y^2 dx - x^2 dy$, where C is the closed curve shown on the left.





C is piecewise smooth. So, the given integral is expressed as a sum of integrals, i.e., we write $\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$, with C_1, C_2, C_3 as shown on the right.

  - On C_1, with x as a parameter: $\int_{C_1} y^2 dx - x^2 dy = \int_0^2 0 dx - x^2(0) = 0$.
  - On C_2, with y as a parameter: $\int_{C_2} y^2 dx - x^2 dy = \int_0^4 y^2(0) - 4dy = -\int_0^4 4dy = -16$.
  - On C_3, we again use x as a parameter. From $y = x^2$, we get $dy = 2xdx$. Thus, $\int_{C_3} y^2 dx - x^2 dy = \int_2^0 (x^2)^2 dx - x^2(2xdx) = \int_2^0 (x^4 - 2x^3) dx = (\frac{1}{5}x^5 - \frac{1}{2}x^4)\Big|_2^0 = \frac{8}{5}$.

Hence, $\oint_C y^2 dx - x^2 dy = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + (-16) + \frac{8}{5} = -\frac{72}{5}$.

Orientation of a Curve

- If C is not a closed curve, then we say the **positive direction** on C, or that C has **positive orientation**, if we traverse C from its initial point A to its terminal point B, i.e., if $x = x(t), y = y(t)$, $a \le t \le b$, are parametric equations for C, then the positive direction on C corresponds to increasing values of the parameter t.
- If C is traversed in the sense opposite to that of the positive orientation, then C is said to have **negative orientation**.
- If C has an orientation (positive or negative), then the **opposite curve**, the curve with the opposite orientation, will be denoted $-C$.
- Then
$$\int_{-C} Pdx + Qdy = -\int_C Pdx + Qdy,$$
or, equivalently
$$\int_{-C} Pdx + Qdy + \int_C Pdx + Qdy = 0.$$
- A line integral is independent of the parametrization of C, provided C is given the same orientation.

Subsection 2

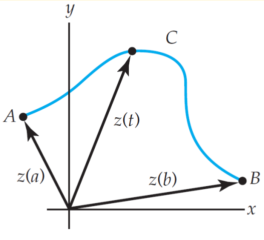
Complex Integrals

Curves Revisited

- Suppose the continuous real-valued functions $x = x(t), y = y(t)$, $a \leq t \leq b$, are parametric equations of a curve C in the complex plane.
- By considering $z = x + iy$, we can describe the points z on C by means of a complex-valued function of a real variable t, called a **parametrization** of C: $z(t) = x(t) + iy(t)$, $a \leq t \leq b$.

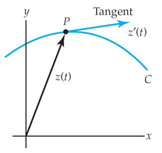
  Example: The parametric equations $x = \cos t, y = \sin t$, $0 \leq t \leq 2\pi$, describe a unit circle centered at the origin. A parametrization of this circle is $z(t) = \cos t + i \sin t$, or $z(t) = e^{it}$, $0 \leq t \leq 2\pi$.
- The point $z(a) = x(a) + iy(a)$ or $A = (x(a), y(a))$ is called the **initial point** of C. and $z(b) = x(b) + iy(b)$ or $B = (x(b), y(b))$ the **terminal point**.

  As t varies from $t = a$ to $t = b$, C is being traced out by the moving arrowhead of the vector corresponding to $z(t)$.



Smooth Curves and Contours

- Suppose the derivative of $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, is $z'(t) = x'(t) + iy'(t)$.
- We say C is **smooth** if $z'(t)$ is continuous and never zero in the interval $a \leq t \leq b$.

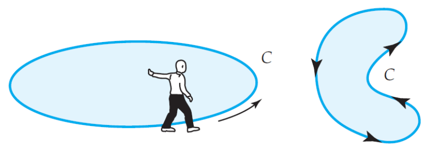


Since the vector $z'(t)$ is not zero at any point P on C, the vector $z'(t)$ is tangent to C at P. In other words, a smooth curve has a continuously turning tangent.

- A **piecewise smooth curve** C has a continuously turning tangent, except possibly at the points where the component smooth curves $C_1, C_2, \ldots, C_n$ are joined together.
- A curve C in the complex plane is **simple** if $z(t_1) \neq z(t_2)$, for $t_1 \neq t_2$, except possibly for $t = a$ and $t = b$.
- C is a **closed curve** if $z(a) = z(b)$.
- C is a **simple closed curve** if it is simple and closed.
- A piecewise smooth curve C is also called a **contour** or **path**.

Positive and Negative Directions

- We define the **positive direction** on a contour C to be the direction on the curve corresponding to increasing values of the parameter t. It is also said that the curve C has **positive orientation**.
- In the case of a *simple closed curve* C, the **positive direction** roughly corresponds to the counterclockwise direction or the direction that a person must walk on C in order to keep the interior of C to the left.



- The **negative direction** on a contour C is the direction opposite the positive direction.
- If C has an orientation, the **opposite curve**, that is, a curve with opposite orientation, is denoted by $-C$.
- On a *simple closed curve*, the **negative direction** corresponds to the clockwise direction.

Steps Leading to the Definition of the Complex Integral I

1. Let f be a function of a complex variable z defined at all points on a smooth curve C that lies in some region of the plane. Suppose C is defined by the parametrization $z(t) = x(t) + iy(t)$, $a \leq t \leq b$.
2. Let P be a partition of the parameter interval $[a, b]$ into n subintervals $[t_{k-1}, t_k]$ of length $\Delta t_k = t_k - t_{k-1}$:

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

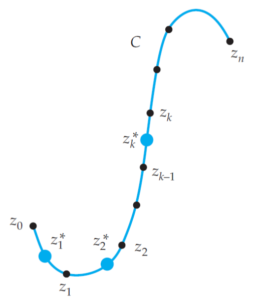
The partition P induces a partition of the curve C into n subarcs whose initial and terminal points are the pairs of numbers

$$\begin{aligned} z_0 &= x(t_0) + iy(t_0), & z_1 &= x(t_1) + iy(t_1), \\ z_1 &= x(t_1) + iy(t_1), & z_2 &= x(t_2) + iy(t_2), \\ &\vdots & &\vdots \\ z_{n-1} &= x(t_{n-1}) + iy(t_{n-1}), & z_n &= x(t_n) + iy(t_n). \end{aligned}$$

Let $\Delta z_k = z_k - z_{k-1}$, $k = 1, 2, \ldots, n$.

Steps Leading to the Definition of the Complex Integral II

3. Let $\|P\|$ be the **norm** of the partition P of $[a, b]$, i.e., the length of the longest subinterval.
4. Choose a point $z_k^* = x_k^* + iy_k^*$ on each subarc of C.



5. Form n products $f(z_k^*)\Delta z_k$, $k = 1, 2, \ldots, n$, and then sum these products: $\sum_{k=1}^{n} f(z_k^*)\Delta z_k$.

The Definition of the Complex Integral

Definition (Complex Integral)

The **complex integral** of f on C is

$$\int_C f(z)dz = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(z_k^*)\Delta z_k.$$

- If the limit exists, f is said to be **integrable** on C.
- The limit exists whenever f is continuous at all points on C and C is either smooth or piecewise smooth.
- Thus, we always assume that these conditions are fulfilled.
- By convention, we will use the notation $\oint_C f(z)dz$ to represent a complex integral around a *positively oriented closed curve* C.
- The notations $\ointctrclockwise_C f(z)dz$, $\ointclockwise_C f(z)dz$ denote more explicitly integration in the positive and negative directions, respectively.
- We shall refer to $\int_C f(z)dz$ as a **contour integral**.

Complex-Valued Function of a Real Variable

- Example: If t represents a real variable, then $f(t) = (2t + i)^2$ is a complex number. For $t = 2$, $f(2) = (4 + i)^2 = 16 + 8i + i^2 = 15 + 8i$.
- If f_1 and f_2 are real-valued functions of a real variable t, then $f(t) = f_1(t) + if_2(t)$ is a complex-valued function of a real variable t.
- We are interested in integration of a complex-valued function $f(t) = f_1(t) + if_2(t)$ of a real variable t carried out over a real interval.
- Example: On the interval $0 \le t \le 1$, it seems reasonable for $f(t) = (2t + i)^2$ to write

$$\int_0^1 (2t+i)^2 dt = \int_0^1 (4t^2 - 1 + 4ti)dt = \int_0^1 (4t^2 - 1)dt + i\int_0^1 4tdt.$$

The integrals $\int_0^1 (4t^2 - 1)dt$ and $\int_0^1 4tdt$ are real, and could be called the real and imaginary parts of $\int_0^1 (2t + i)^2 dt$. Each can be evaluated using the fundamental theorem of calculus to get:
$\int_0^1 (2t + i)^2 dt = \left(\frac{4}{3}t^3 - t\right)\Big|_0^1 + i\, 2t^2\Big|_0^1 = \frac{1}{3} + 2i.$

Integral of Complex Valued Function of a Real Variable

- If f_1 and f_2 are real-valued functions of a real variable t continuous on a common interval $a \leq t \leq b$, then we define the **integral** of the complex-valued function $f(t) = f_1(t) + if_2(t)$ on $a \leq t \leq b$ by

$$\int_a^b f(t)dt = \int_a^b f_1(t)dt + i\int_a^b f_2(t)dt.$$

- The continuity of f_1 and f_2 on $[a, b]$ guarantees that both integrals on the right exist.
- If $f(t) = f_1(t) + if_2(t)$ and $g(t) = g_1(t) + ig_2(t)$, are complex-valued functions of a real variable t continuous on $a \leq t \leq b$, then
  - $\int_a^b kf(t)dt = k\int_a^b f(t)dt$, k a complex constant;
  - $\int_a^b (f(t) + g(t))dt = \int_a^b f(t)dt + \int_a^b g(t)dt$;
  - $\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$, if $c \in [a, b]$;
  - $\int_b^a f(t)dt = -\int_a^b f(t)dt$.

Evaluation of Contour Integrals

- If we use $u + iv$ for f, $\Delta x + i\Delta y$ for Δz, lim for $\lim_{\|P\|\to 0}$ and $\sum$ for $\sum_{k=1}^{n}$, we get $\int_C f(z)dz = \lim \sum (u+iv)(\Delta x + i\Delta y) = \lim [\sum (u\Delta x - v\Delta y) + i \sum (v\Delta x + u\Delta y)]$.
- Thus, we have
$$\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udy.$$
- If $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, are parametric equations of C, then $dx = x'(t)dt$, $dy = y'(t)dt$.
- Now we obtain $\int_a^b [u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)]dt + i\int_a^b [v(x(t), y(t))x'(t) + u(x(t), y(t))y'(t)]dt$.
- This is the same as $\int_a^b f(z(t))z'(t)dt$ when the integrand $f(z(t))z'(t) = [u(x(t), y(t)) + iv(x(t), y(t))][x'(t) + iy'(t)]$ is multiplied out and $\int_a^b f(z(t))z'(t)dt$ is expressed in terms of its real and imaginary parts.

Evaluating of a Contour Integral

Theorem (Evaluation of a Contour Integral)

If f is continuous on a smooth curve C given by $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, then

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

- Example: Evaluate $\int_C \overline{z}dz$, where C is given by $x = 3t$, $y = t^2$, $-1 \leq t \leq 4$.

  A parametrization of the contour C is $z(t) = 3t + it^2$. Thus, since $f(z) = \overline{z}$, we have $f(z(t)) = \overline{3t + it^2} = 3t - it^2$. Also, $z'(t) = 3 + 2it$. Now, we have

$$\begin{aligned}
\int_C \overline{z}dz &= \int_{-1}^4 (3t - it^2)(3 + 2it)dt \\
&= \int_{-1}^4 (2t^3 + 9t)dt + i \int_{-1}^4 3t^2 dt \\
&= \left(\tfrac{1}{2}t^4 + \tfrac{9}{2}t^2\right)\Big|_{-1}^4 + i\ t^3\Big|_{-1}^4 = 195 + 65i.
\end{aligned}$$

Another Evaluation of a Contour Integral

- Evaluate $\oint_C \frac{1}{z} dz$, where C is the circle $x = \cos t, y = \sin t$, $0 \leq t \leq 2\pi$.

  In this case $z(t) = \cos t + i \sin t = e^{it},\ z'(t) = ie^{it},$ and $f(z(t)) = \frac{1}{z(t)} = e^{-it}$. Hence,

$$
\begin{aligned}
\oint_c \frac{1}{z} dz &= \int_0^{2\pi} (e^{-it}) ie^{it} dt \\
&= i \int_0^{2\pi} dt \\
&= 2\pi i.
\end{aligned}
$$

Using x as a Parameter

- For some curves the real variable x itself can be used as the parameter.
- Example: Evaluate $\int_C (8x^2 - iy)dz$ on the line segment $y = 5x$, $0 \leq x \leq 2$.

  We write $z = x + 5xi$, whence $dz = (1 + 5i)dx$. Therefore,

$$\begin{aligned}\int_C (8x^2 - iy)dz &= (1+5i)\int_0^2 (8x^2 - 5ix)dx \\ &= (1+5i)\,\frac{8}{3}x^3\Big|_0^2 - (1+5i)i\,\frac{5}{2}x^2\Big|_0^2 \\ &= \frac{214}{3} + \frac{290}{3}i.\end{aligned}$$

- If x and y are related by means of a continuous real function $y = f(x)$, then the corresponding curve C can be parametrized by $z(x) = x + if(x)$.

Properties of Contour Integrals

Theorem (Properties of Contour Integrals)

Suppose the functions f and g are continuous in a domain D, and C is a smooth curve lying entirely in D. Then:

(i) $\int_C kf(z)dz = k\int_C f(z)dz$, k a complex constant.

(ii) $\int_C [f(z)+g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$.

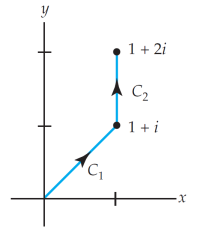
(iii) $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$, where C consists of the smooth curves C_1 and C_2 joined end to end.

(iv) $\int_{-C} f(z)dz = -\int_C f(z)dz$, where $-C$ denotes the curve having the opposite orientation of C.

- The four parts of the theorem also hold if C is a *piecewise smooth* curve in D.

C a Piecewise Smooth Curve

- Evaluate $\int_C (x^2 + iy^2)dz$, where C is the contour shown:



We write $\int_C (x^2 + iy^2)dz = \int_{C_1} (x^2 + iy^2)dz + \int_{C_2} (x^2 + iy^2)dz$.
Since the curve C_1 is defined by $y = x$, we use x as a parameter: $z(x) = x + ix$, $z'(x) = 1 + i$, $f(z) = x^2 + iy^2$, $f(z(x)) = x^2 + ix^2$,

whence, finally, $\int_{C_1} (x^2 + iy^2)dz = \int_0^1 (x^2 + ix^2)(i + 1)dx = (1 + i)^2 \int_0^1 x^2 dx = \frac{(1+i)^2}{3} = \frac{2}{3}i$.
The curve C_2 is defined by $x = 1$, $1 \leq y \leq 2$. If we use y as a parameter, then $z(y) = 1 + iy$, $z'(y) = i$, $f(z(y)) = 1 + iy^2$, and $\int_{C_2} (x^2 + iy^2)dz = \int_1^2 (1 + iy^2)idy = -\int_1^2 y^2 dy + i \int_1^2 dy = -\frac{7}{3} + i$.
Therefore $\int_C (x^2 + iy^2)dz = \frac{2}{3}i + (-\frac{7}{3} + i) = -\frac{7}{3} + \frac{5}{3}i$.

A Bounding Theorem

- We find an upper bound for the modulus of a contour integral.
- Recall the length of a plane curve $L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}dt$. If $z'(t) = x'(t) + iy'(t)$, then $|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$, whence $L = \int_a^b |z'(t)|dt$.

Theorem (A Bounding Theorem)

If f is continuous on a smooth curve C and if $|f(z)| \leq M$, for all z on C, then $|\int_C f(z)dz| \leq ML$, where L is the length of C.

- By triangle inequality, $|\sum_{k=1}^n f(z_k^*)\Delta z_k| \leq \sum_{k=1}^n |f(z_k^*)||\Delta z_k|$ $\leq M\sum_{k=1}^n |\Delta z_k|$. Because $|\Delta z_k| = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$, we can interpret $|\Delta z_k|$ as the length of the chord joining the points z_k and z_{k-1} on C. Moreover, since the sum of the lengths of the chords cannot be greater than L, we get $|\sum_{k=1}^n f(z_k^*)\Delta z_k| \leq ML$. Finally, the continuity of f guarantees that $\int_C f(z)dz$ exists. Thus, letting $\|P\| \to 0$, the last inequality yields $|\int_C f(z)dz| \leq ML$.

A Bound for a Contour Integral

- Find an upper bound for the absolute value of $\int_C \frac{e^z}{z+1} dz$ where C is the circle $|z| = 4$.

  First, the length L (circumference) of the circle of radius 4 is 8π. Next, for all points z on the circle, we have that $|z+1| \geq |z| - 1 = 4 - 1 = 3$. Thus, $\left|\dfrac{e^z}{z+1}\right| \leq \dfrac{|e^z|}{|z|-1} = \dfrac{|e^z|}{3}$. In addition, $|e^z| = |e^x(\cos y + i \sin y)| = e^x$. For points on the circle $|z| = 4$, the maximum that $x = \text{Re}(z)$ can be is 4, whence $\left|\dfrac{e^z}{z+1}\right| \leq \dfrac{e^4}{3}$. From the theorem, we have

$$\left|\int_C \frac{e^z}{z+1} dz\right| \leq \frac{8\pi e^4}{3}.$$

Single Contour: Many Parametrizations

- There is no unique parametrization for a contour C.
- Example: All of the following:

$$
\begin{aligned}
z(t) &= e^{it} = \cos t + i \sin t, \quad 0 \le t \le 2\pi, \\
z(t) &= e^{2\pi it} = \cos 2\pi t + i \sin 2\pi t, \quad 0 \le t \le 1, \\
z(t) &= e^{\pi it/2} = \cos \tfrac{\pi t}{2} + i \sin \tfrac{\pi t}{2}, \quad 0 \le t \le 4,
\end{aligned}
$$

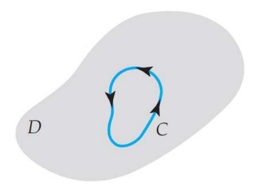
are all parametrizations, oriented in the positive direction, for the unit circle $|z| = 1$.

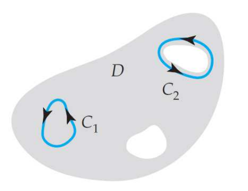
Subsection 3

Cauchy-Goursat Theorem

Simply and Multiply Connected Domains

- A **domain** is an open connected set in the complex plane.
- A domain D is **simply connected** if every simple closed contour C lying entirely in D can be shrunk to a point without leaving D.





Example: The entire complex plane is a simply connected domain. The annulus defined by $1 < |z| < 2$ is not simply connected.

- A domain that is not simply connected is called a **multiply connected domain**.
  - A domain with one "hole" is **doubly connected**;
  - A domain with two "holes" **triply connected**, and so on.

Example: The open disk $|z| < 2$ is a simply connected domain. The open circular annulus $1 < |z| < 2$ is doubly connected.

Cauchy's Theorem

Cauchy's Theorem (1825)

Suppose that a function f is analytic in a simply connected domain D and that f' is continuous in D. Then, for every simple closed contour C in D,

$$\oint_C f(z)dz = 0.$$

- We apply Green's theorem and the Cauchy-Riemann equations. Recall from calculus that, if C is a positively oriented, piecewise smooth, simple closed curve forming the boundary of a region R within D, and if the real-valued functions $P(x, y)$ and $Q(x, y)$ along with their first-order partial derivatives are continuous on a domain that contains C and R, then $\oint_C Pdx + Qdy = \iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA$. Since f' is continuous throughout D, the real and imaginary parts of $f(z) = u + iv$ and their first partial derivatives are continuous throughout D.

Proof of Cauchy's Theorem

- We have by Green's Theorem

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA.$$

By continuity of u, v and their first partial derivatives, $\oint_C f(z)dz = \oint_C u(x,y)dx - v(x,y)dy + i\oint_C v(x,y)dx + u(x,y)dy = \iint_R (-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dA + i\iint_R (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})dA$. f being analytic in D, u and v satisfy the Cauchy-Riemann equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Therefore,

$$\begin{aligned} \oint_C f(z)dz &= \iint_R \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right)dA + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right)dA \\ &= 0. \end{aligned}$$

The Cauchy-Goursat Theorem

- Edouard Goursat proved in 1883 that the assumption of continuity of f' is not necessary to reach the conclusion of Cauchy's theorem:

Cauchy-Goursat Theorem

Suppose that a function f is analytic in a simply connected domain D. Then, for every simple closed contour C in D,

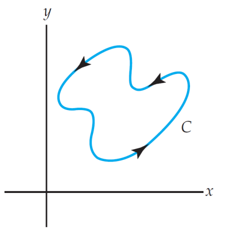
$$\oint_C f(z)dz = 0.$$

- Since the interior of a simple closed contour is a simply connected domain, the Cauchy-Goursat theorem can also be stated as:

If f is analytic at all points within and on a simple closed contour C, then $\oint_C f(z)dz = 0$.

Applying the Cauchy-Goursat Theorem I

- Evaluate $\oint_C e^z\,dz$, where the contour C is shown below.



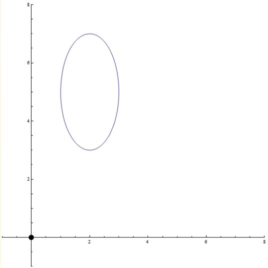
$f(z) = e^z$ is entire. Thus, it is analytic at all points within and on the simple closed contour C. It follows from the Cauchy-Goursat theorem that $\oint_C e^z\,dz = 0$.

- We have $\oint_C e^z\,dz = 0$, for any simple closed contour in the complex plane.
- Moreover, for any simple closed contour C and any entire function f, such as $f(z) = \sin z$, $f(z) = \cos z$, and $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, $n = 0, 1, 2, \ldots$, we also have

$$\oint_C \sin z dz = 0, \ \oint_C \cos z dz = 0, \ \oint_C p(z) dz = 0, \text{ etc.}$$

Applying the Cauchy-Goursat Theorem II

- Evaluate $\oint_C \frac{1}{z^2}dz$, where C is the ellipse $(x-2)^2 + \frac{1}{4}(y-5)^2 = 1$. The rational function $f(z) = \frac{1}{z^2}$ is analytic everywhere except at $z = 0$. But $z = 0$ is not a point interior to or on the simple closed elliptical contour C.

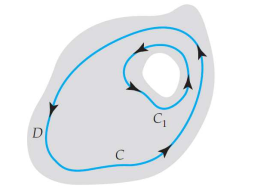


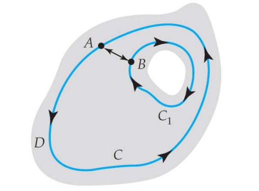
Thus, again by the Cauchy-Goursat Theorem, we get

$$\oint_C \frac{1}{z^2}dz = 0.$$

Cauchy-Goursat Theorem for Multiply Connected Domains

- If f is analytic in a multiply connected domain D, then we cannot conclude that $\oint_C f(z)dz = 0$, for every simple closed contour C in D.
- Suppose that D is a doubly connected domain and C and C_1 are simple closed contours placed as follows:





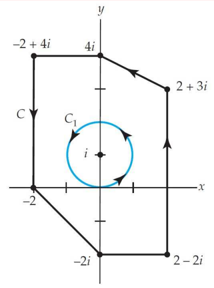
Suppose, also, that f is analytic on each contour and at each point interior to C but exterior to C_1.

By introducing the crosscut AB, the region bounded between the curves is now simply connected. So: $\oint_C f(z)dz + \int_{AB} f(z)dz + \oint_{-C_1} f(z)dz + \int_{-AB} f(z)dz = 0$ or $\oint_C f(z)dz = \oint_{C_1} f(z)dz$.

- This is sometimes called the **principle of deformation of contours**.
- It allows evaluation of an integral over a complicated simple closed contour C by replacing C with a more convenient contour C_1.

Applying Deformation of Contours

- Evaluate $\oint_C \frac{1}{z-i}\,dz$, where C is the black contour:



We choose the more convenient circular contour C_1 drawn in blue. By taking the radius of the circle to be $r = 1$, we are guaranteed that C_1 lies within C. C_1 is the circle $|z - i| = 1$.
It can be parametrized by

$$z = i + e^{it},\ 0 \le t \le 2\pi.$$

From $z - i = e^{it}$ and $dz = ie^{it}dt$, we get:

$$\begin{aligned}\oint_C \frac{1}{z-i}dz &= \oint_{C_1} \frac{1}{z-i}dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}dt \\ &= i\int_0^{2\pi} dt = 2\pi i.\end{aligned}$$

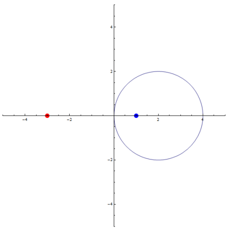
A Generalization

- This result can be generalized: If z_0 is any constant complex number interior to any simple closed contour C, and n an integer, we have
$$\oint_C \frac{1}{(z-z_0)^n} dz = \begin{cases} 2\pi i, & \text{if } n = 1 \\ 0, & \text{if } n \neq 1 \end{cases}.$$
- That the integral is zero when $n \neq 1$ follows only partially from the Cauchy-Goursat theorem.
  - When $n = 0$ or negative, $\frac{1}{(z-z_0)^n}$ is a polynomial and therefore entire. Then, clearly, $\oint_C \frac{1}{(z-z_0)^n} dz = 0$.
  - It is not very difficult to see that the integral is still zero when n is a positive integer different from 1.
- Analyticity of the function f at all points within and on a simple closed contour C is sufficient to guarantee that $\oint_C f(z) dz = 0$.
- This result emphasizes that analyticity is not necessary, i.e., it can happen that $\oint_C f(z) dz = 0$ without f being analytic within C.
Example: If C is the circle $|z| = 1$, then $\oint_C \frac{1}{z^2} dz = 0$, but $f(z) = \frac{1}{z^2}$ is not analytic at $z = 0$ within C.

Applying the Formula for the Integral of $1/(z-z_0)^n$

- Evaluate $\oint_C \frac{5z+7}{z^2+2z-3}dz$, where C is circle $|z-2|=2$.

The denominator factors as $z^2+2z-3=(z-1)(z+3)$. Thus, the integrand fails to be analytic at $z=1$ and $z=-3$.



Of these two points, only $z=1$ lies within the contour C, which is a circle centered at $z=2$ of radius $r=2$. By partial fractions

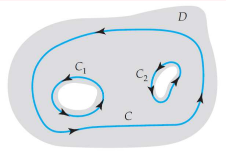
$$\frac{5z+7}{z^2+2z-3}=\frac{3}{z-1}+\frac{2}{z+3}.$$

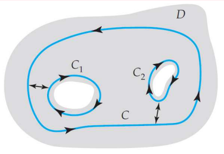
Hence, $\oint_C \frac{5z+7}{z^2+2z-3}dz = 3\oint_C \frac{1}{z-1}dz + 2\oint_C \frac{1}{z+3}dz$. The first integral has the value $2\pi i$, whereas the value of the second integral is 0 by the Cauchy-Goursat theorem. Hence,

$$\oint_C \frac{5z+7}{z^2+2z-3}dz = 3(2\pi i)+2(0)=6\pi i.$$

Cauchy-Goursat Theorem: Multiply Connnected Domains

- If C, C_1, and C_2 are simple closed contours as shown below





and f is analytic on each of the three contours as well as at each point interior to C but exterior to both C_1 and C_2, then by introducing crosscuts between C_1 and C and between C_2 and C, we get $\oint_C f(z)dz + \oint_{-C_1} f(z)dz + \oint_{-C_2} f(z)dz = 0$, whence $\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$.

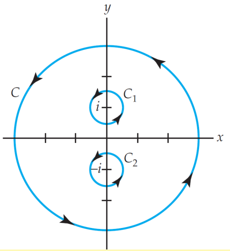
Cauchy-Goursat Theorem for Multiply Connnected Domains

Suppose $C, C_1, \ldots, C_n$ are simple closed curves with a positive orientation, such that $C_1, C_2, \ldots, C_n$ are interior to C, but the regions interior to each C_k, $k = 1, 2, \ldots, n$, have no points in common. If f is analytic on each contour and at each point interior to C but exterior to all the C_k, $k = 1, 2, \ldots, n$, then $\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz$.

Integrals in Multiply Connected Domains

- Evaluate $\oint_C \frac{1}{z^2+1}dz$, where C is the circle $|z| = 4$.

  The denominator of the integrand factors as $z^2 + 1 = (z - i)(z + i)$. So, the integrand $\frac{1}{z^2+1}$ is not analytic at $z = i$ and at $z = -i$. Both points lie within C. Using partial fractions, $\frac{1}{z^2+1} = \frac{1}{2i}\frac{1}{z-i} - \frac{1}{2i}\frac{1}{z+i}$. whence $\oint_C \frac{1}{z^2+1}dz = \frac{1}{2i}\oint_C (\frac{1}{z-i} - \frac{1}{z+i})dz$.

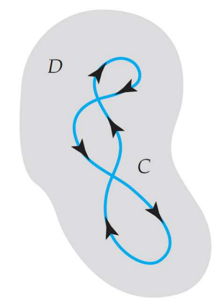
  Surround $z = i$ and $z = -i$ by circular contours C_1 and C_2, respectively, that lie entirely within C. The choice $|z - i| = \frac{1}{2}$ for C_1 and $|z + i| = \frac{1}{2}$ for C_2 will suffice.
  We have $\oint_C \frac{1}{z^2+1}dz =$

  $\frac{1}{2i}\oint_{C_1}(\frac{1}{z-i} - \frac{1}{z+i})dz + \frac{1}{2i}\oint_{C_2}(\frac{1}{z-i} - \frac{1}{z+i})dz = \frac{1}{2i}\oint_{C_1}\frac{1}{z-i}dz -$
  $\frac{1}{2i}\oint_{C_1}\frac{1}{z+i}dz + \frac{1}{2i}\oint_{C_2}\frac{1}{z-i}dz - \frac{1}{2i}\oint_{C_2}\frac{1}{z+i}dz = \frac{1}{2i}2\pi i - 0 + 0 - \frac{1}{2i}2\pi i = 0.$

Non-Simple Closed Contours

- Throughout the foregoing discussion we assumed that C was a simple closed contour, in other words, C did not intersect itself.
- It can be shown that the Cauchy-Goursat theorem is valid for any closed contour C in a simply connected domain D.
- For a contour C that is closed but not simple, if f is analytic in D, then

$$\oint_C f(z)dz = 0.$$



Subsection 4

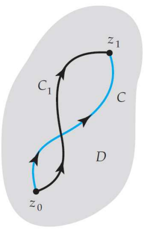
Independence of Path

Path Independence

Definition (Independence of the Path)

Let z_0 and z_1 be points in a domain D. A contour integral $\int_C f(z)dz$ is said to be **independent of the path** if its value is the same for all contours C in D with initial point z_0 and terminal point z_1.

- The Cauchy-Goursat theorem holds for closed contours, not just simple closed contours, in a simply connected domain D.
- Suppose that C and C_1 are two contours lying entirely in a simply connected domain D and both with initial point z_0 and terminal point z_1. C joined with $-C_1$ forms a closed contour. Thus, if f is analytic in D, $\int_C f(z)dz + \int_{-C_1} f(z)dz = 0$. Therefore, $\int_C f(z)dz = \int_{C_1} f(z)dz$.

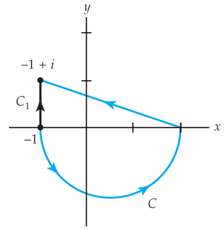


Theorem (Analyticity Implies Path Independence)

Suppose that a function f is analytic in a simply connected domain D and C is any contour in D. Then $\int_C f(z)dz$ is independent of the path C.

Choosing a Different Path

- Evaluate $\int_C 2z dz$, where C is the contour shown in blue.



The function $f(z) = 2z$ is entire. By the theorem, we can replace the piecewise smooth path C by any convenient contour C_1 joining $z_0 = -1$ and $z_1 = -1 + i$. We choose the contour C_1 to be the vertical line segment $x = -1, 0 \leq y \leq 1$.

Since $z = -1 + iy,\ dz = idy$. Therefore,

$$\begin{aligned} \int_C 2zdz &= \int_{C_1} 2zdz \\ &= \int_0^1 2(-1+iy)idy \\ &= \int_0^1 (-2i - 2y)dy \\ &= (-2iy - y^2)\Big|_0^1 \\ &= -1 - 2i. \end{aligned}$$

Antiderivatives

- A contour integral $\int_C f(z)dz$ that is independent of the path C is usually written $\int_{z_0}^{z_1} f(z)dz$, where z_0 and z_1 are the initial and terminal points of C.

Definition (Antiderivative)

Suppose that a function f is continuous on a domain D. If there exists a function F such that $F'(z) = f(z)$, for each z in D, then F is called an **antiderivative** of f.

Example: The function $F(z) = -\cos z$ is an antiderivative of $f(z) = \sin z$ since $F'(z) = \sin z$.

- The most general antiderivative, or **indefinite integral**, of a function $f(z)$ is written $\int f(z)dz = F(z) + C$, where $F'(z) = f(z)$ and C is some complex constant.
- Differentiability implies continuity, whence, since an antiderivative F of a function f has a derivative at each point in a domain D, it is necessarily analytic and hence continuous at each point in D.

Fundamental Theorem for Contour Integrals

Fundamental Theorem for Contour Integrals

Suppose that a function f is continuous on a domain D and F is an antiderivative of f in D. Then, for any contour C in D with initial point z_0 and terminal point z_1,

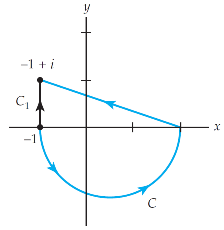
$$\int_C f(z)dz = F(z_1) - F(z_0).$$

- We prove the FTCI in the case when C is a smooth curve parametrized by $z = z(t)$, $a \leq t \leq b$. The initial and terminal points on C are $z(a) = z_0$ and $z(b) = z_1$. Since $F'(z) = f(z)$, for all z in D,

$$\begin{aligned} \int_C f(z)dz &= \int_a^b f(z(t))z'(t)dt = \int_a^b F'(z(t))z'(t)dt \\ &= \int_a^b \frac{d}{dt}F(z(t))dt = F(z(t))\Big|_a^b \\ &= F(z(b)) - F(z(a)) \\ &= F(z_1) - F(z_0). \end{aligned}$$

Applying the Fundamental Theorem I

- The integral $\int_C 2z dz$, where C is shown



is independent of the path. Since $f(z) = 2z$ is an entire function, it is continuous. Moreover, $F(z) = z^2$ is an antiderivative of f since $F'(z) = 2z = f(z)$. Hence, by the Fundamental Theorem, we have

$$\begin{aligned} \int_{-1}^{-1+i} 2z dz &= z^2\Big|_{-1}^{-1+i} \\ &= (-1+i)^2 - (-1)^2 \\ &= -1 - 2i. \end{aligned}$$

Applying the Fundamental Theorem II

- Evaluate $\int_C \cos z dz$, where C is any contour with initial point $z_0 = 0$ and terminal point $z_1 = 2 + i$.

  $F(z) = \sin z$ is an antiderivative of $f(z) = \cos z$, since $F'(z) = \cos z = f(z)$. Therefore, by the Fundamental Theorem, we have

$$\begin{aligned}\int_C \cos z dz &= \int_0^{2+i} \cos z dz \\ &= \sin z\Big|_0^{2+i} \\ &= \sin(2+i) - \sin 0 \\ &= \sin(2+i).\end{aligned}$$

Some Conclusions

- Observe that if the contour C is closed, then $z_0 = z_1$ and, consequently, $\oint_C f(z)dz = F(z_1) - F(z_0) = 0$.
- Since the value of $\int_C f(z)dz$ depends only on the points z_0 and z_1, this value is the same for any contour C in D connecting these points:

> If a continuous function f has an antiderivative F in D, then $\int_C f(z)dz$ is independent of the path.

- Moreover, we have a sufficient condition:

> If f is continuous and $\int_C f(z)dz$ is independent of the path C in a domain D, then f has an antiderivative everywhere in D.

- Assume f is continuous and $\int_C f(z)dz$ is independent of the path in a domain D and that F is a function defined by $F(z) = \int_{z_0}^{z} f(s)ds$, where s denotes a complex variable, z_0 is a fixed point in D, and z represents any point in D. We wish to show that $F'(z) = f(z)$, i.e., that $F(z) = \int_{z_0}^{z} f(s)ds$ is an antiderivative of f in D.

$F(z) = \int_{z_0}^{z} f(s)ds$ is an Antiderivative of f in D

- We have $F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(s)ds - \int_{z_0}^{z} f(s)ds = \int_{z}^{z+\Delta z} f(s)ds$. Because D is a domain, we can choose Δz so that $z+\Delta z$ is in D. Moreover, z and $z+\Delta z$ can be joined by a straight segment. With z fixed, we can write $f(z)\Delta z = f(z)\int_{z}^{z+\Delta z} ds = \int_{z}^{z+\Delta z} f(z)ds$ or $f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)ds$. Therefore, we have $\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z}[f(s)-f(z)]ds$. Since f is continuous at the point z, for any $\varepsilon > 0$, there exists a $\delta > 0$, so that $|f(s)-f(z)| < \epsilon$ whenever $|s-z| < \delta$. Consequently, if we choose Δz so that $|\Delta z| < \delta$, it follows from the ML-inequality, that $\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| = \left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z}[f(s)-f(z)]ds\right| =$ $\left|\frac{1}{\Delta z}\right|\left|\int_{z}^{z+\Delta z}[f(s)-f(z)]ds\right| \leq \left|\frac{1}{\Delta z}\right|\varepsilon|\Delta z| = \varepsilon$. Hence, $\lim_{\Delta z\to 0}\frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z)$ or $F'(z) = f(z)$.

Existence of Antiderivative

- If f is an analytic function in a simply connected domain D, it is continuous throughout D. This implies, by the Path Independence Theorem, that path independence holds for f in D. Therefore,

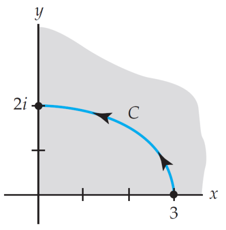
Theorem (Existence of Antiderivative)

Suppose that a function f is analytic in a simply connected domain D. Then f has an antiderivative in D, i.e., there exists a function F such that $F'(z) = f(z)$, for all z in D.

- We have seen that, for $|z| > 0$, $-\pi < \arg(z) < \pi$, $\frac{1}{z}$ is the derivative of $\text{Ln}z$. Thus, under some circumstances $\text{Ln}z$ is an antiderivative of $\frac{1}{z}$, but one must be careful!
If D is the entire complex plane without the origin, $\frac{1}{z}$ is analytic in this multiply connected domain. If C is any simple closed contour containing the origin, it does not follow that $\oint_C \frac{1}{z}dz = 0$. In this case, $\text{Ln}z$ is not an antiderivative of $\frac{1}{z}$ in D since $\text{Ln}z$ is not analytic in D ($\text{Ln}z$ fails to be analytic on the non-positive real axis).

Using the Logarithmic Function

- Evaluate $\int_C \frac{1}{z}dz$, where C is the contour shown:



Suppose that D is the simply connected domain defined by $x > 0$, $y > 0$, i.e., the first quadrant. In this case, $\text{Ln}z$ is an antiderivative of $\frac{1}{z}$ since both these functions are analytic in D.

Therefore,

$$\int_C \frac{1}{z}dz = \int_3^{2i} \frac{1}{z}dz = \text{Ln}z\Big|_3^{2i} = \text{Ln}(2i) - \text{Ln}3.$$

Recall $\text{Ln}(2i) = \log_e 2 + \frac{\pi}{2}i$ and $\text{Ln}3 = \log_e 3$. Hence,
$\int_C \frac{1}{z}dz = \log_e 2 + \frac{\pi}{2}i - \log_e 3 = \log_e \frac{2}{3} + \frac{\pi}{2}i.$

Using an Antiderivative of $z^{-1/2}$

- Evaluate $\int_C \frac{1}{z^{1/2}} dz$, where C is the line segment between $z_0 = i$ and $z_1 = 9$.

  We take $f_1(z) = z^{1/2}$ to be the principal branch of the square root function. In the domain $|z| > 0$, $-\pi < \arg(z) < \pi$, the function $\frac{1}{f_1(z)} = \frac{1}{z^{1/2}} = z^{-1/2}$ is analytic and possesses the antiderivative $F(z) = 2z^{1/2}$. Hence,

$$\begin{aligned} \int_C \frac{1}{z^{1/2}} dz &= \int_i^9 \frac{1}{z^{1/2}} dz \\ &= 2z^{1/2}\Big|_i^9 \\ &= 2[3 - (\tfrac{\sqrt{2}}{2} + i\tfrac{\sqrt{2}}{2})] \\ &= (6 - \sqrt{2}) - i\sqrt{2}. \end{aligned}$$

Integration-By-Parts

- In calculus indefinite integrals of certain kinds can be evaluated by integration by parts:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

More compactly, $\int udv = uv - \int vdu$.

- Suppose f and g are analytic in a simply connected domain D. Then

$$\int f(z)g'(z)dz = f(z)g(z) - \int g(z)f'(z)dz.$$

- In addition, if z_0 and z_1 are the initial and terminal points of a contour C lying entirely in D, then

$$\int_{z_0}^{z_1} f(z)g'(z)dz = f(z)g(z)\Big|_{z_0}^{z_1} - \int_{z_0}^{z_1} g(z)f'(z)dz.$$

The Mean Value Theorem for Definite Integrals

- The **Mean Value Theorem for Definite Integrals**: If f is a real function continuous on the closed interval $[a, b]$, then there exists a number c in the open interval (a, b), such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

- Let f be a complex function analytic in a simply connected domain D. Then, f is continuous at every point on a contour C in D with initial point z_0 and terminal point z_1.

  Unfortunately, no analog of the Mean Value Theorem exists for the contour integral $\int_{z_0}^{z_1} f(z)dz$.

Subsection 5

Cauchy's Integral Formulas

Cauchy's First Formula

- If f is analytic in a simply connected domain D and z_0 is a point in D, the quotient $\frac{f(z)}{z-z_0}$ is not defined at z_0 and, hence, is not analytic in D.
- Therefore, we cannot conclude that the integral of $\frac{f(z)}{z-z_0}$ around a simple closed contour C that contains z_0 is zero.
- Indeed,the integral of $\frac{f(z)}{z-z_0}$ around C has the value $2\pi i f(z_0)$.

Theorem (Cauchy's Integral Formula)

Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then, for any point z_0 within C,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz.$$

- Let D be a simply connected domain, C a simple closed contour in D, and z_0 an interior point of C. In addition, let C_1 be a circle centered at z_0 with radius small enough so that C_1 lies within the interior of C. By the principle of deformation of contours, $\oint_C \frac{f(z)}{z-z_0} dz = \oint_{C_1} \frac{f(z)}{z-z_0} dz$.

Proof of Cauchy's Integral Formula

- From $\oint_C \frac{f(z)}{z-z_0} dz = \oint_{C_1} \frac{f(z)}{z-z_0} dz$, we get by adding and subtracting $f(z_0)$ in the numerator: $\oint_C \frac{f(z)}{z-z_0} dz = \oint_{C_1} \frac{f(z_0)-f(z_0)+f(z)}{z-z_0} dz =$ $f(z_0) \oint_{C_1} \frac{1}{z-z_0} dz + \oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0} dz$. We know that $\oint_{C_1} \frac{1}{z-z_0} dz = 2\pi i$, whence $\oint_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0) + \oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0} dz$.
  Since f is continuous at z_0, for any $\varepsilon > 0$, there exists a $\delta > 0$, such that $|f(z) - f(z_0)| < \varepsilon$, whenever $|z - z_0| < \delta$. In particular, if we choose C_1 to be $|z - z_0| = \frac{1}{2}\delta < \delta$, then by the ML-inequality, $\left|\oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0} dz\right| \leq \frac{\varepsilon}{\delta/2} 2\pi \frac{\delta}{2} = 2\pi\varepsilon$. Thus, the absolute value of the integral can be made arbitrarily small by taking the radius of the circle C_1 to be sufficiently small. This implies that the integral is 0. We conclude that $\oint_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$.

Using Cauchy's Integral Formula

- Cauchy's integral formula shows that the values of an analytic function f at points z_0 inside a simple closed contour C are determined by the values of f on the contour C.
- Since we often work problems without a simply connected domain explicitly defined, a more practical restatement is:

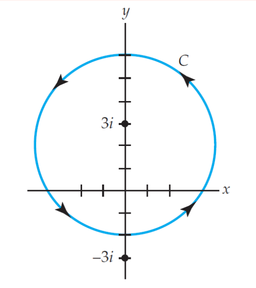
> If f is analytic at all points within and on a simple closed contour C, and z_0 is any point interior to C, then $f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz$.

- Example: Evaluate $\oint_C \frac{z^2-4z+4}{z+i} dz$, where C is the circle $|z| = 2$.

  We identify $f(z) = z^2 - 4z + 4$ and $z_0 = -i$ as a point within the circle C. Next, we observe that f is analytic at all points within and on the contour C. Thus, by the Cauchy integral formula,
  $\oint_C \frac{z^2-4z+4}{z+i} dz = 2\pi i f(-i) = 2\pi i(3+4i) = \pi(-8+6i)$.

Another Application of Cauchys Integral Formula

- Evaluate $\oint_C \frac{z}{z^2+9}dz$, where C is the circle $|z-2i|=4$.



By factoring the denominator as $z^2+9=(z-3i)(z+3i)$, we see that $3i$ is the only point within the closed contour C at which the integrand fails to be analytic. By rewriting the integrand as $\dfrac{z}{z^2+9}=\dfrac{\frac{z}{z+3i}}{z-3i}$, we identify $f(z)=\frac{z}{z+3i}$

The function f is analytic at all points within and on the contour C. Hence, by Cauchy's integral formula

$$\oint_C \frac{z}{z^2+9}dz=\oint_C \frac{\frac{z}{z+3i}}{z-3i}dz=2\pi i f(3i)=2\pi i\frac{3i}{6i}=\pi i.$$

Cauchy's Second Formula

- We prove that the values of the derivatives $f^{(n)}(z_0)$, $n = 1, 2, 3, \dots$ of an analytic function are also given by an integral formula.

Theorem (Cauchy's Integral Formula for Derivatives)

Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then, for any point z_0 within C,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

- Partial Proof (for $n = 1$): By the definition of the derivative and Cauchy's Integral Formula, $f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} =$ $\lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z} \left[\oint_C \frac{f(z)}{z - (z_0 + \Delta z)} dz - \oint_C \frac{f(z)}{z - z_0} dz \right] =$ $\lim_{\Delta z \to 0} \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)} dz.$

Prof of Cauchy's Second Formula for $n = 1$

- We work out some preliminaries:
  - Continuity of f on the contour C guarantees that f is bounded, i.e., there exists real number M, such that $|f(z)| \leq M$, for all points z on C.
  - In addition, let L be the length of C and let δ denote the shortest distance between points on C and the point z_0. Thus, for all points z on C, we have $|z - z_0| \geq \delta$, or $\frac{1}{|z-z_0|^2} \leq \frac{1}{\delta^2}$.
  - Furthermore, if we choose $|\Delta z| \leq \frac{1}{2}\delta$, then $|z - z_0 - \Delta z| \geq ||z - z_0| - |\Delta z|| \geq \delta - |\Delta z| \geq \frac{1}{2}\delta$, whence $\frac{1}{|z-z_0-\Delta z|} \leq \frac{2}{\delta}$.

Now, $\left|\oint_C \frac{f(z)}{(z-z_0)^2}dz - \oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}dz\right| = \left|\oint_C \frac{-\Delta z\ f(z)}{(z-z_0-\Delta z)(z-z_0)^2}dz\right| \leq \frac{2ML|\Delta z|}{\delta^3}$. The last expression approaches zero as $\Delta z \to 0$, whence

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz.$$

Using Cauchy's Integral Formula for Derivatives

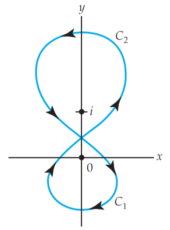
- Evaluate $\oint_C \frac{z+1}{z^4+2iz^3}dz$, where C is the circle $|z|=1$.

  Inspection of the integrand shows that it is not analytic at $z=0$ and $z=-2i$, but only $z=0$ lies within the closed contour. By writing the integrand as $\dfrac{z+1}{z^4+2iz^3}=\dfrac{\frac{z+1}{z+2i}}{z^3}$ we can identify, $z_0=0,\ n=2$, and $f(z)=\frac{z+1}{z+2i}$. The quotient rule gives $f'(z)=\frac{-1+2i}{(z+2i)^2}$ and $f''(z)=\frac{2-4i}{(z+2i)^3}$, whence $f''(0)=\frac{2i-1}{4i}$. Therefore, we get

$$\begin{aligned}\oint_C \frac{z+1}{z^4+4z^3}dz &= \frac{2\pi i}{2!}f''(0)\\ &= \frac{2\pi i}{2!}\frac{2i-1}{4i}\\ &= -\frac{\pi}{4}+\frac{\pi}{2}i.\end{aligned}$$

Another Application of the Integral Formula for Derivatives

- Evaluate $\oint_C \frac{z^3+3}{z(z-i)^2} dz$, where C is the figure-eight contour shown below:



Although C is not a simple closed contour, we can think of it as the union of two simple closed contours C_1 and C_2. We write $\oint_C \frac{z^3+3}{z(z-i)^2} dz = \oint_{C_1} \frac{z^3+3}{z(z-i)^2} dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2} dz = -\oint_{-C_1} \frac{\frac{z^3+3}{(z-i)^2}}{z} dz + \oint_{C_2} \frac{\frac{z^3+3}{z}}{(z-i)^2} dz = -I_1 + I_2$.

  - $I_1 = \oint_{-C_1} \frac{\frac{z^3+3}{(z-i)^2}}{z} dz = 2\pi i f(0) = 2\pi i(-3) = -6\pi i$.
  - For I_2, $f(z) = \frac{z^3+3}{z}$, whence $f'(z) = \frac{2z^3-3}{z^2}$, and $f'(i) = 3 + 2i$. Thus,
    $$I_2 = \oint_{C_2} \frac{\frac{z^3+3}{z}}{(z-i)^2} dz = \frac{2\pi i}{1!} f'(i) = 2\pi i(3+2i) = -4\pi + 6\pi i.$$

Finally, $\oint_C \frac{z^3+3}{z(z-i)^2} dz = -I_1 + I_2 = 6\pi i + (-4\pi + 6\pi i) = -4\pi + 12\pi i$.

Subsection 6

Consequences of the Integral Formulas

The Derivatives of an Analytic Function are Analytic

Theorem (Derivative of an Analytic Function Is Analytic)

Suppose that f is analytic in a simply connected domain D. Then f possesses derivatives of all orders at every point z in D. The derivatives $f', f'', f''', \ldots$ are analytic functions in D.

- If $f(z) = u(x,y) + iv(x,y)$ is analytic in a simply connected domain D, its derivatives of all orders exist at any point z in D. Thus, f', f'', f''', ... are continuous. From

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}, \\ f''(z) &= \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} - i\frac{\partial^2 u}{\partial y \partial x} \\ &\vdots \end{aligned}$$

we can also conclude that the real functions u and v have continuous partial derivatives of all orders at a point of analyticity.

Cauchy's Inequality

Theorem (Cauchy's Inequality)

Suppose that f is analytic in a simply connected domain D and C is a circle defined by $|z - z_0| = r$ that lies entirely in D. If $|f(z)| \leq M$, for all points z on C, then

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}.$$

- From the hypothesis, $\left|\frac{f(z)}{(z-z_0)^{n+1}}\right| = \frac{|f(z)|}{r^{n+1}} \leq \frac{M}{r^{n+1}}$. Thus, by Cauchy's Formula for Derivatives and the ML-inequality,

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \leq \frac{n!}{2\pi}\frac{M}{r^{n+1}}2\pi r = \frac{n!M}{r^n}.$$

- The number M depends on the circle $|z - z_0| = r$. But, if $n = 0$, then $M \geq |f(z_0)|$, for any circle C centered at z_0, as long as C lies within D. Thus, an upper bound M of $|f(z)|$ on C cannot be smaller than $|f(z_0)|$.

Liouville's Theorem

- Although the next result is known as "Liouville's Theorem", it was probably first proved by Cauchy.
- The gist of the theorem is that an entire function f, one that is analytic for all z, cannot be bounded unless f itself is a constant:

Theorem (Liouville's Theorem)

The only bounded entire functions are constants.

- Suppose f is an entire bounded function, i.e., $|f(z)| \leq M$, for all z. Then, for any point z_0, by Cauchy's Inequality, $|f'(z_0)| \leq \frac{M}{r}$. By making r arbitrarily large we can make $|f'(z_0)|$ as small as we wish. This means $f'(z_0) = 0$, for all points z_0 in the complex plane. Hence, by a preceding theorem, f must be a constant.

Fundamental Theorem of Algebra

- Liouville's Theorem enables us to establish the celebrated

Fundamental Theorem of Algebra

If $p(z)$ is a nonconstant polynomial, then the equation $p(z) = 0$ has at least one root.

- Suppose that the polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, $n > 0$, is not 0 for any complex number z. This implies that the reciprocal of p, $f(z) = \frac{1}{p(z)}$, is an entire function. Now

$$\begin{aligned} |f(z)| &= \frac{1}{|a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0|} \\ &= \frac{1}{|z|^n \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right|}. \end{aligned}$$

Thus, $|f(z)| \to 0$ as $|z| \to \infty$. So the function f must be bounded for finite z. By Liouville's Theorem, f is a constant. Hence, p is a constant. But this contradicts p not being a constant polynomial. Therefore, there must exist at least one z for which $p(z) = 0$.

Morera's Theorem

- Morera's theorem, which gives a sufficient condition for analyticity, is often taken to be the converse of the Cauchy-Goursat Theorem:

Theorem (Morera's Theorem)

If f is continuous in a simply connected domain D and if $\oint_C f(z)dz = 0$, for every closed contour C in D, then f is analytic in D.

- By the hypotheses of continuity of f and $\oint_C f(z)dz = 0$, for every closed contour C in D, we conclude that $\int_C f(z)dz$ is independent of the path. Then, the function F, defined by $F(z) = \int_{z_0}^{z} f(s)ds$ (where s denotes a complex variable, z_0 is a fixed point in D, and z any point in D) is an antiderivative of f, i.e., $F'(z) = f(z)$. Hence, F is analytic in D. In addition, $F'(z)$ is analytic in view of the analyticity of the derivative of any analytic function. Since $f(z) = F'(z)$, we see that f is analytic in D.

The Maximum Modulus Theorem

- We saw that, if a function f is continuous on a closed and bounded region R, then f is bounded, i.e., there exists some constant M, such that $|f(z)| \leq M$, for z in R.
- If the boundary of R is a simple closed curve C, then the modulus $|f(z)|$ assumes its maximum value at some z on the boundary C:

Theorem (Maximum Modulus Theorem)

Suppose that f is analytic and nonconstant on a closed region R bounded by a simple closed curve C. Then the modulus $|f(z)|$ attains its maximum on C.

- If the stipulation that $f(z) \neq 0$, for all z in R, is added to the hypotheses, then the modulus $|f(z)|$ also attains its minimum on C.

Finding The Maximum Modulus

- Find the maximum modulus of $f(z) = 2z + 5i$ on the closed circular region defined by $|z| \leq 2$.

  We know that $|z|^2 = z \cdot \overline{z}$. By replacing z by $2z + 5i$, we have $|2z + 5i|^2 = (2z + 5i)(\overline{2z + 5i}) = (2z + 5i)(2\overline{z} - 5i) =$ $4z\overline{z} - 10i(z - \overline{z}) + 25$. But, $z - \overline{z} = 2i\text{Im}(z)$, whence $|2z + 5i|^2 = 4|z|^2 + 20\text{Im}(z) + 25$. Because f is a polynomial, it is analytic on the region defined by $|z| \leq 2$. Thus, $\max\limits_{|z|\leq 2}|2z + 5i|$ occurs on the boundary $|z| = 2$. There, $|2z + 5i| = \sqrt{41 + 20\text{Im}(z)}$. This attains its maximum when $\text{Im}(z)$ attains its maximum on $|z| = 2$, namely, at the point $z = 2i$. Thus, $\max_{|z|\leq 2}|2z + 5i| = \sqrt{81} = 9$.
- Note that $f(z) = 0$ only at $z = -\frac{5}{2}i$ and that this point is outside the region defined by $|z| \leq 2$. Hence we can conclude that we have a minimum when $\text{Im}(z)$ attains its minimum on $|z| = 2$ at $z = -2i$. As a result, $\min_{|z|\leq 2}|2z + 5i| = \sqrt{1} = 1$.