Calculus I

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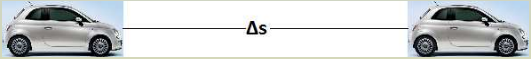
Subsection 1

Limits, Rates of Change and Tangent Lines

Average Velocity

- An object moving on a straight line is at position $s(t)$ at time t;
- Then in the time interval $[t_0, t_1]$ it has moved from position $s(t_0)$ to position $s(t_1)$ having a **displacement** (or **net change in position**) $\Delta s = s(t_1) - s(t_0)$;



- Its average velocity in $[t_0, t_1]$ is given by

$$v_{\text{avg}}[t_0, t_1] = \frac{\Delta s}{\Delta t} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}.$$

Example: If an object is at position $s(t) = 5t^2$ miles from the origin at time t in hours, what is $v_{\text{avg}}[1, 5]$?

$$v_{\text{avg}}[1, 5] = \frac{s(5) - s(1)}{5 - 1} = \frac{5 \cdot 5^2 - 5 \cdot 1^2}{4} = 30\text{mph}.$$

Instantaneous Velocity

- An object moving on a straight line is at position $s(t)$ at time t;
- To estimate the instantaneous velocity of the object at t_0, we consider a very short time interval $[t_0, t_1]$ and compute $v_{\mathrm{avg}}[t_0, t_1]$;
- If $[t_0, t_1]$ is very short, then the change in velocity might be negligible and so a good approximation of the instantaneous velocity at t_0;
- Thus $v(t_0) \underbrace{\cong}_{\Delta t \text{ small}} \frac{\Delta s}{\Delta t}$;

  Example: Estimate the instantaneous velocity $v(1)$ of the object whose position function is $s(t) = 5t^2$ miles from the origin at time t in hours.

$$v(1) \cong \frac{s(1.01) - s(1)}{1.01 - 1} = \frac{5 \cdot (1.01)^2 - 5 \cdot 1^2}{0.01} = 10.05\text{mph}.$$

Another Example of a Rate of Change

- Suppose that the length of the side of a melting cube as function of time is given by $s(t) = \frac{1}{t+2}$ inches at t minutes since the start of the melting process. What is the average change in the volume of the ice cube from $t = 0$ to $t = 3$ minutes?

The volume $V(t)$ in cubic inches as a function of time t in minutes is given by $V(t) = s(t)^3 = (\frac{1}{t+2})^3$.
Therefore

$$\left(\frac{\Delta V}{\Delta t}\right)_{\text{avg}}[0,3] = \frac{V(3)-V(0)}{3-0} = \frac{(\frac{1}{5})^3-(\frac{1}{2})^3}{3}$$
$$= \frac{\frac{1}{125}-\frac{1}{8}}{3} = \frac{\frac{8}{1000}-\frac{125}{1000}}{3} = -\frac{117}{3000}\text{in}^3/\text{min}.$$

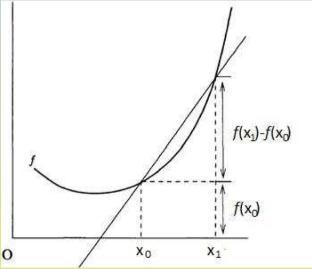
Instantaneous Rate of Change of Volume

- In the previous example, to estimate the instantaneous rate of change of the volume of the ice cube at $t = 1$, we may consider the average rate of change between $t = 1$ minute and $t = 1.01$ minute:

$$\begin{aligned}
\left(\frac{\Delta V}{\Delta t}\right)\Big|_{t=1} &\cong \left(\frac{\Delta V}{\Delta t}\right)_{\text{avg}} [1, 1.01] \\
&= \frac{V(1.01) - V(1)}{1.01 - 1} \\
&= \frac{(\frac{1}{3.01})^3 - (\frac{1}{3})^3}{0.01} \\
&\cong -0.037\text{in}^3/\text{min}.
\end{aligned}$$

Slope of a Secant Line

- Consider the graph of $y = f(x)$ and two points on the graph $(x_0, f(x_0))$ and $(x_1, f(x_1))$;

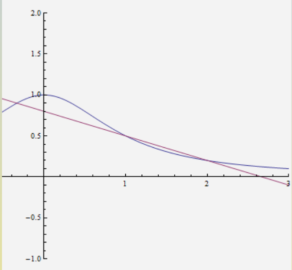


- The line passing through these two points is called the **secant line** to $y = f(x)$ through x_0 and x_1;
- Its slope is equal to

$$m_f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

An Example

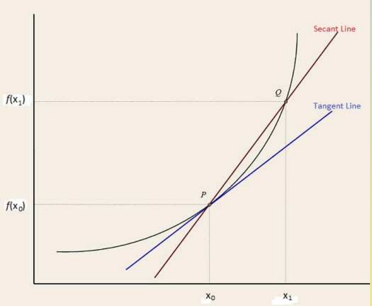
- Example: Find an equation for the secant line to $f(x) = \frac{1}{1+x^2}$ through $x_0 = 1$ and $x_1 = 2$;



We have
$$m_f[1,2] = \frac{f(2) - f(1)}{2-1} = \frac{\frac{1}{5} - \frac{1}{2}}{2-1} = -\frac{3}{10}.$$
Therefore $y - \frac{1}{2} = -\frac{3}{10}(x-1)$ is the point-slope form of the equation of the secant line.

Slope of a Tangent Line

- To approximate the slope $m_f(x_0)$ of the tangent line to the graph of $y = f(x)$ at x_0 we use a process similar to that approximating the instantaneous rate of change by using the average rate of change for points x_0, x_1 very close to each other;

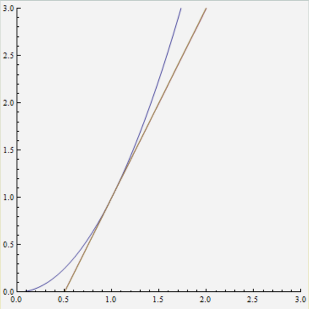


Therefore, we have

$$m_f(x_0) \underbrace{\cong}_{\Delta x \text{ small}} m_f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Approximating the Slope of a Tangent Line

- Let us approximate the slope to $y = x^2$ at $x = 1$ using the process outlined in the previous slide;



We have

$$m_f(1) \cong m_f[1, 1.01] = \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{(1.01)^2 - 1^2}{0.01} = 2.01.$$

Subsection 2

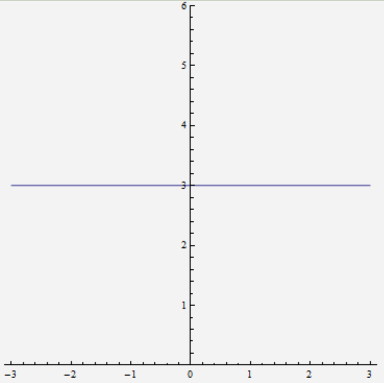
A Graphical Approach to Limits

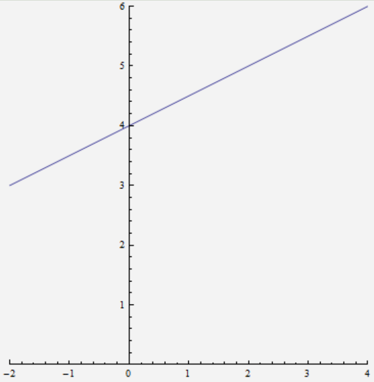
Definition of Limit

- Suppose that $f(x)$ is defined in an open interval containing a number c, but not necessarily c itself;
- The **limit of** $f(x)$ **as** x **approaches** c **is equal to** L if $f(x)$ has value arbitrarily close to L when x assumes values sufficiently close (but not equal) to c.
- In this case, we write
$$\lim_{x \to c} f(x) = L.$$
- An alternative terminology is that $f(x)$ **approaches** or **converges to** L **as** x **approaches** c.

Two Easy Examples

- Draw the graph of $f(x) = 3$ and find graphically the limit $\lim_{x \to c} f(x)$.
- Draw the graph of $g(x) = \frac{1}{2}x + 4$ and find graphically $\lim_{x \to 2} f(x)$.

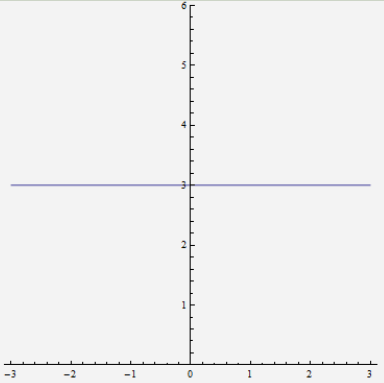


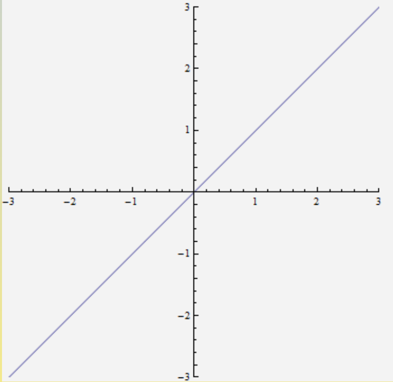


We have $\lim_{x \to c} 3 = 3$ and $\lim_{x \to 2} \left(\frac{1}{2}x + 4\right) = 5$.

Two Easy Rules

- Draw the graph of $f(x) = k$ (a constant) and find graphically the limit $\lim_{x \to c} k$.
- Draw the graph of $g(x) = x$ and find graphically $\lim_{x \to c} x$.

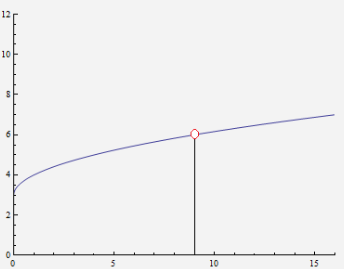


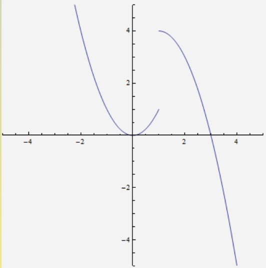


We have $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$.

Two More Complicated Examples

- Draw the graph of $f(x) = \frac{x-9}{\sqrt{x}-3}$ and find graphically the limit $\lim_{x\to 9} f(x)$.
- Draw the graph of $g(x) = \begin{cases} x^2, \text{if } x \leq 1 \\ -x^2+2x+3, & \text{if } x > 1 \end{cases}$ and find graphically $\lim_{x\to 1} f(x)$.

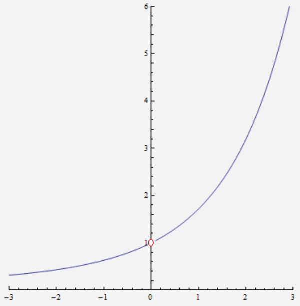


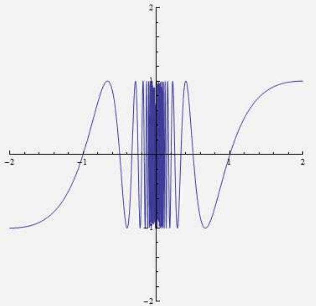


We have $\lim_{x\to c} \frac{x-9}{\sqrt{x}-3} = 6$ and $\lim_{x\to 1} g(x)$ does not exist since $g(x)$ does not approach a single number when x approaches 1.

Two Additional Examples

- Draw the graph of $f(x) = \frac{e^x - 1}{x}$ and find graphically the limit $\lim_{x \to 0} f(x)$.
- Draw the graph of $g(x) = \sin \frac{\pi}{x}$ and find graphically $\lim_{x \to 0} g(x)$.





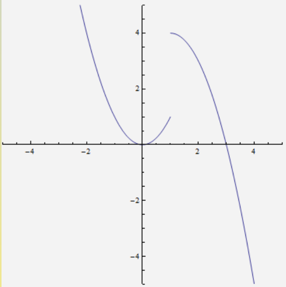
We have $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$ and $\lim_{x \to 0} \sin \frac{\pi}{x}$ does not exist since the values of $g(x) = \sin \frac{\pi}{x}$ oscillate between -1 and 1 as x approaches 0.

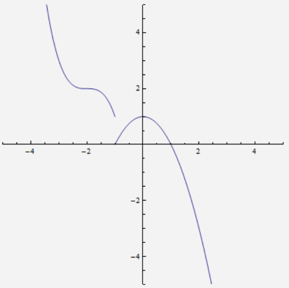
Definition of Side-Limits

- Suppose that $f(x)$ is defined in an open interval containing a number c, but not necessarily c itself;
- The **right-hand limit of** $f(x)$ **as** x **approaches** c (**from the right**) **is equal to** L if $f(x)$ has value arbitrarily close to L when x approaches sufficiently close (but is not equal) to c from the right hand side. In this case, we write $\lim_{x\to c^+} f(x) = L$.
- The **left-hand limit of** $f(x)$ **as** x **approaches** c (**from the left**) **is equal to** L if $f(x)$ has value arbitrarily close to L when x approaches sufficiently close (but is not equal) to c from the left hand side. In this case, we write $\lim_{x\to c^-} f(x) = L$.
- The limits we saw before are "two sided limits"; It is the case that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c^+} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$, i.e., a function has limit L as x approaches c if and only if the left and right hand side limits as x approaches c exist and are equal.

Two Examples

- Draw the graph of $f(x) = \begin{cases} x^2, & \text{if } x \leq 1 \\ -x^2 + 2x + 3, & \text{if } x > 1 \end{cases}$ and find graphically $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.
- Draw the graph of $g(x) = \begin{cases} -(x+2)^3 + 2, & \text{if } x < -1 \\ -x^2 + 1, & \text{if } x > -1 \end{cases}$ and find graphically $\lim_{x\to -1^-} g(x)$ and $\lim_{x\to -1^+} g(x)$.

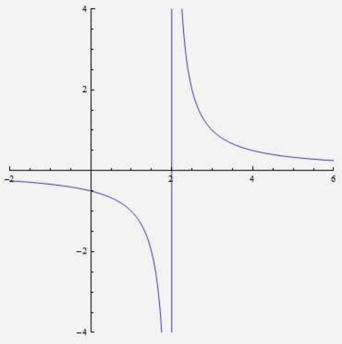


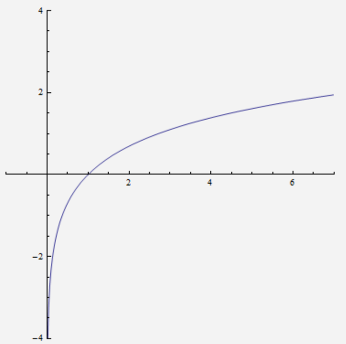


$\lim_{x\to 1^-} f(x) = 1$, $\lim_{x\to 1^+} f(x) = 4$, so $\lim_{x\to 1} f(x)$ DNE, and $\lim_{x\to -1^-} g(x) = 1$, $\lim_{x\to -1^+} g(x) = 0$, so $\lim_{x\to -1} g(x)$ DNE.

Examples of Limits Involving Infinity

- Draw the graph of $f(x) = \frac{1}{x-2}$ and find graphically $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$.
- Draw the graph of $g(x) = \ln x$ and find graphically $\lim_{x\to 0^+} g(x)$ and $\lim_{x\to +\infty} g(x)$.





$\lim_{x\to 2^-} f(x) = -\infty$, $\lim_{x\to 2^+} f(x) = +\infty$, and
$\lim_{x\to 0^+} g(x) = -\infty$, $\lim_{x\to +\infty} g(x) = +\infty$.

Subsection 3

Basic Limit Laws

Theorem (Basic Limit Laws)

Suppose that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then

- **Sum Law:** $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$;
- **Constant Factor Law:** $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x)$;
- **Product Law:** $\lim_{x\to c} f(x)g(x) = (\lim_{x\to c} f(x))(\lim_{x\to c} g(x))$;
- **Quotient Law:** If $\lim_{x\to c} g(x) \neq 0$, then $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$;
- **Power and Root Law:** For p, q integers, with $q \neq 0$, $\lim_{x\to c}[f(x)]^{p/q} = (\lim_{x\to c} f(x))^{p/q}$, under the assumption that $\lim_{x\to c} f(x) \geq 0$ if q is even and $\lim_{x\to c} f(x) \neq 0$ if $\frac{p}{q} < 0$.
  In particular, for n a positive integer,
  - $\lim_{x\to c}[f(x)]^n = (\lim_{x\to c} f(x))^n$;
  - $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$;

Examples of Calculating Limits I

- Compute $\lim_{x\to 2} x^3$;
  We apply the power rule:
  $$\lim_{x\to 2}(x^3) = (\lim_{x\to 2} x)^3 = 2^3 = 8.$$

- Compute $\lim_{x\to -1}(-2x^3 + 7x - 5)$;
  We apply the sum rule, the constant factor and the power rules:
  $$\begin{aligned}\lim_{x\to -1}(-2x^3 + 7x - 5) &= \lim_{x\to -1}(-2x^3) + \lim_{x\to -1}(7x) - \lim_{x\to -1} 5 \\ &= -2\lim_{x\to -1}(x^3) + 7\lim_{x\to -1} x - \lim_{x\to -1} 5 \\ &= -2\cdot(-1)^3 + 7(-1) - 5 \\ &= -10.\end{aligned}$$

Examples of Calculating Limits II

- Compute $\lim_{x\to 2}\frac{x+30}{2x^4}$;
  We apply the quotient rule:

$$\lim_{x\to 2}\frac{x+30}{2x^4}=\frac{\lim_{x\to 2}(x+30)}{\lim_{x\to 2}(2x^4)}=\frac{2+30}{2\cdot 2^4}=1.$$

- Compute $\lim_{x\to 3}(x^{-1/4}(x+5)^{1/3})$;
  We apply the product and the power rules:

$$\begin{aligned}\lim_{x\to 3}(x^{-1/4}(x+5)^{1/3}) &= (\lim_{x\to 3}x^{-1/4})(\lim_{x\to 3}\sqrt[3]{x+5})\\ &= ((\lim_{x\to 3}x)^{-1/4}(\sqrt[3]{\lim_{x\to 3}x+5})\\ &= 3^{-1/4}\sqrt[3]{8}\\ &= \tfrac{2}{\sqrt[4]{3}}.\end{aligned}$$

Treacherous Applications of the Laws

- We must take the hypotheses of the Basic Limit Laws into account when applying the rules;
- For instance, if $f(x) = x$ and $g(x) = x^{-1}$, then

$$\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} xx^{-1} = \lim_{x\to 0} 1 = 1,$$

but, if we tried to apply the product rule, we would be stuck:

$$\lim_{x\to 0} f(x)g(x) = (\lim_{x\to 0} x)(\lim_{x\to 0} x^{-1}),$$

The last limit on the right does not exist since $\lim_{x\to 0^+} x^{-1} = +\infty$ and $\lim_{x\to 0^-} x^{-1} = -\infty$.

Subsection 4

Limits and Continuity

Continuity at a Point

- A function $f(x)$ defined on an open interval containing $x = c$ is **continuous at** $x = c$ if

$$\lim_{x \to c} f(x) = f(c).$$

- If either the limit does not exist, or exists but is not equal to $f(c)$, then f has a **discontinuity** or is **discontinuous** at $x = c$.
- Not that the limit above exists if and only if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$;
- Therefore, the condition for continuity is equivalent to

$$\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x).$$

Example: Let $f(x) = k$ a constant. Recall that $\lim_{x \to c} k = k$. Also $f(c) = k$. Therefore, $f(x) = k$ is continuous at all $x = c$.

Some Additional Examples

- Consider $f(x) = x^n$, where n is a natural number. Then $\lim_{x\to c} x^n = (\lim_{x\to c} x)^n = c^n$. Also $f(c) = c^n$. Therefore, $f(x) = x^n$ is continuous at all $x = c$.
- Consider $f(x) = x^5 + 7x - 12$. Applying some of the Limit Laws, we get

$$\begin{aligned}\lim_{x\to c} (x^5 + 7x - 12) &= (\lim_{x\to c} x)^5 + 7(\lim_{x\to c} x) - \lim_{x\to c} 12 \\ &= c^5 + 7c - 12 \\ &= f(c).\end{aligned}$$

  Therefore $f(x)$ is continuous at $x = c$.
- Consider also $f(x) = \frac{x^2+5}{x+3}$. Applying some of the Limit Laws, we get

$$\begin{aligned}\lim_{x\to 2} \frac{x^2+5}{x+3} &= \frac{\lim_{x\to 2} (x^2+5)}{\lim_{x\to 2} (x+3)} = \frac{(\lim_{x\to 2} x)^2 + \lim_{x\to 2} 5}{\lim_{x\to 2} x + \lim_{x\to 2} 3} \\ &= \frac{2^2+5}{2+3} = f(2).\end{aligned}$$

  Thus $f(x)$ is continuous at $x = 2$.

Types of Discontinuities

- Recall $f(x)$ is continuous at $x = c$ if

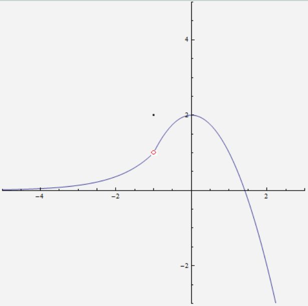
$$\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x).$$

- If $\lim_{x \to c} f(x)$ exists but is not equal to $f(c)$, then $f(x)$ has a **removable discontinuity** at $x = c$;
- If $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$ (in this case, of course, $\lim_{x \to c} f(x)$ does not exist), then f has a **jump discontinuity** at $x = c$;
- If either $\lim_{x \to c^-} f(x)$ or $\lim_{x \to c^+} f(x)$ is infinite, then f has an **infinite discontinuity** at $x = c$.

Removable Discontinuity

- Consider the piece-wise defined function

$$f(x) = \begin{cases} e^{x+1}, & \text{if } x < -1 \\ 2, & \text{if } x = -1 \\ -x^2 + 2, & \text{if } x > -1 \end{cases}$$

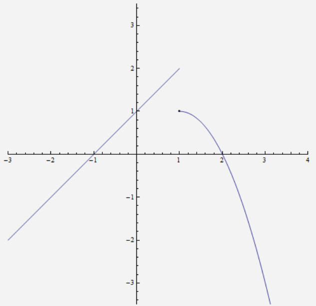


We have $\lim_{x\to -1^-} f(x) = 1$ and $\lim_{x\to -1^+} f(x) = 1,$ whence $\lim_{x\to -1} f(x) = 1$. But $f(-1) = 2$. So $\lim_{x\to -1} f(x)$ exists, but it does not equal $f(-1)$. This shows that $f(x)$ has a removable discontinuity at $x = -1$.

Jump Discontinuity

- Consider the piece-wise defined function

$$f(x) = \begin{cases} x+1, & \text{if } x < 1 \\ -x^2+2x, & \text{if } x \geq 1 \end{cases}$$

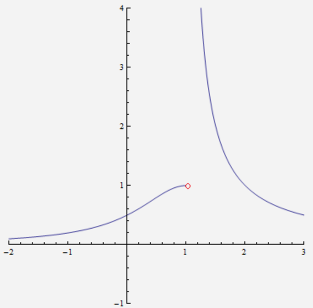


We have $\lim_{x\to 1^-} f(x) = 2$ and $\lim_{x\to 1^+} f(x) = 1,$ whence $\lim_{x\to -1} f(x) = \text{DNE}$. So the side limits of $f(x)$ as x approaches 1 exist, but they are not equal. This shows that $f(x)$ has a jump discontinuity at $x = 1$.

Infinite Discontinuity

- Consider the piece-wise defined function

$$f(x) = \begin{cases} \frac{1}{x^2-2x+2}, & \text{if } x < 1 \\ \frac{1}{x-1}, & \text{if } x > 1 \end{cases}$$



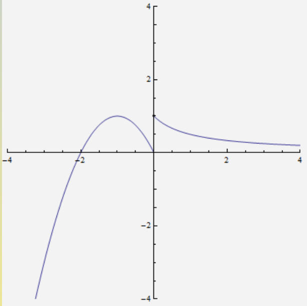
We have $\lim_{x\to 1^-} f(x) = 1$ and $\lim_{x\to 1^+} f(x) = +\infty$, Thus, at least one of the side limits as x approaches 1 is $\pm\infty$. This shows that $f(x)$ has an infinite discontinuity at $x = 1$.

One-Sided Continuity

- A function $f(x)$ is called
  - **left-continuous** at $x = c$ if $\lim_{x \to c^-} f(x) = f(c)$;
  - **right-continuous** at $x = c$ if $\lim_{x \to c^+} f(x) = f(c)$;

Example: Consider the function

$$f(x) = \begin{cases} -x^2 - 2x, & \text{if } x < 0 \\ \frac{1}{x+1}, & \text{if } x \geq 0 \end{cases}$$

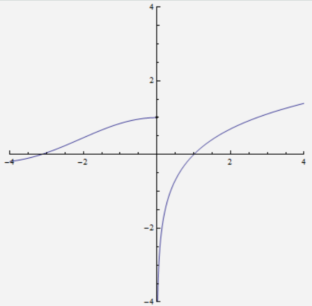


We have $\lim_{x \to 0^-} f(x) = 0$ and $\lim_{x \to 0^+} f(x) = 1$. Moreover, $f(0) = 1$. Therefore $f(x)$ is right-continuous at $x = 0$, but not left continuous at $x = 0$.

One More Example

- Consider the piece-wise defined function

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ 1, & \text{if } x = 0 \\ \ln x, & \text{if } x > 0 \end{cases}$$



We have $\lim_{x\to 0^-} f(x) = 1$ and $\lim_{x\to 0^+} f(x) = -\infty$, Moreover, $f(0) = 1$. Therefore, $f(x)$ is left-continuous at $x = 0$, but not right-continuous at $x = 0$.

Basic Continuity Laws

Theorem (Basic Laws of Continuity)

If $f(x)$ and $g(x)$ are continuous at $x = c$, then the following functions are also continuous at $x = c$:

$$\begin{array}{llll} \text{(i)} & f(x) \pm g(x) & \text{(iii)} & f(x)g(x) \\ \text{(ii)} & kf(x) & \text{(iv)} & \frac{f(x)}{g(x)}, \text{ if } g(c) \neq 0. \end{array}$$

- For instance, knowing that $f(x) = x$ and $g(x) = k$ are continuous functions at all real numbers, the previous rules allow us to conclude that
  - any polynomial function $P(x)$ is continuous at all real numbers;
  - any rational function $\frac{P(x)}{Q(x)}$ is continuous at all values in its domain.

Example: $f(x) = 3x^4 - 2x^3 + 8x$ is continuous at all real numbers. $g(x) = \frac{x+3}{x^2-1}$ is continuous at all numbers $x \neq \pm 1$.

Continuity of Roots, Trig, Exp and Log Functions

Theorem (Continuity of Various Functions)

- $f(x) = \sqrt[n]{x}$ is continuous on its domain;
- $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at all real numbers;
- $f(x) = b^x$ is continuous at all real numbers $(0 < b \neq 1)$;
- $f(x) = \log_b x$ is continuous at all $x > 0$ $(0 < b \neq 1)$;

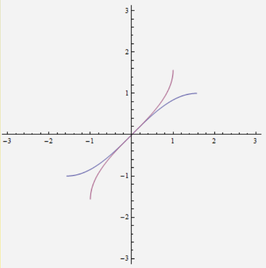
- Based on this theorem and the theorem on quotients, we may conclude, for example, that $\tan x = \dfrac{\sin x}{\cos x}$ is continuous at all points in its domain, i.e., at all $x \neq (2k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$.
- We can also conclude that $\csc x = \dfrac{1}{\sin x}$ is continuous at all points in its domain, i.e., at all $x \neq k\pi$, $k \in \mathbb{Z}$.

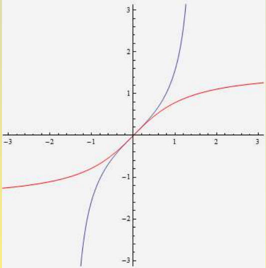
Continuity of Inverse Functions

Theorem (Continuity of Inverse Functions)

If $f(x)$ is continuous on an interval I with range R, then if $f^{-1}(x)$ exists, then $f^{-1}(x)$ is continuous with domain R.

- For instance $f(x) = \sin x$ is continuous on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with range $[-1, 1]$ and has an inverse; So, $f^{-1}(x) = \sin^{-1} x$ is continuous on $[-1, 1]$.
- Similarly $g(x) = \tan x$ is continuous on $(-\frac{\pi}{2}, \frac{\pi}{2})$ with range $\mathbb{R}$ and has an inverse; Therefore $g^{-1}(x) = \tan^{-1} x$ is continuous on $\mathbb{R}$.



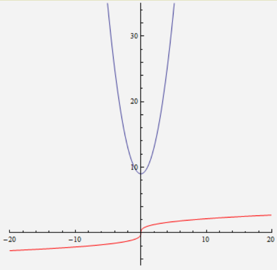


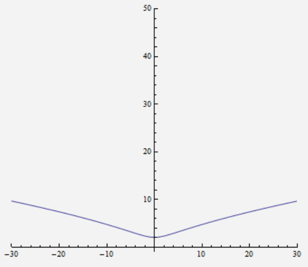
Continuity of Composite Functions

Theorem (Continuity of Composite Functions)

If $g(x)$ is continuous at the point $x = c$ and $f(x)$ is continuous at the point $x = g(c)$, then the function $F(x) = f(g(x))$ is continuous at $x = c$.

- For instance, the function $g(x) = x^2 + 9$ is continuous at all real numbers, since it is a polynomial function; Moreover, the function $f(x) = \sqrt[3]{x}$ is continuous at all real numbers as a root function; Therefore, the function $F(x) = f(g(x)) = \sqrt[3]{x^2 + 9}$ is also a continuous function, as the composite of two continuous functions.





Substitution Method: Using Continuity to Evaluate Limits

- Recall that $f(x)$ is continuous at $x = c$ if

$$\lim_{x \to c} f(x) = f(c).$$

- Suppose that you know that $f(x)$ is continuous at $x = c$ and want to compute $\lim_{x \to c} f(x)$.

  Then, because of the definition of continuity, to find $\lim_{x \to c} f(x)$, you may compute, instead, $f(c)$.
- This is called the **substitution property** (or **method**) for evaluating limits of continuous functions.

Examples of Using the Substitution Method

Example: Let us evaluate the limit $\lim_{x \to \frac{\pi}{3}} \sin x$.
Since $f(x)$ is continuous (by the basic theorem on trig functions) at all $x \in \mathbb{R}$, we may use the substitution property:

$$\lim_{x \to \frac{\pi}{3}} \sin x = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Example: Let us evaluate the limit $\lim_{x \to -1} \frac{3^x}{\sqrt{x+5}}$.
Since $f(x)$ is continuous (as a ratio of an exponential over a root function, both of which are continuous in their domain), we may use the substitution property:

$$\lim_{x \to -1} \frac{3^x}{\sqrt{x+5}} = \frac{3^{-1}}{\sqrt{-1+5}} = \frac{1}{6}.$$

Subsection 5

Algebraic Evaluation of Limits

Indeterminate Forms

- The following are **Indeterminate Forms**:
  - $\frac{0}{0}$
    - Example: $\lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 + x - 12}$
  - $\frac{\infty}{\infty}$
    - Example: $\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec x}$
  - $\infty \cdot 0$
    - Example: $\lim_{x \to 2} (\frac{1}{2x - 4} \cdot (x-2)^2)$
  - $\infty - \infty$
    - Example: $\lim_{x \to 1} (\frac{1}{x - 1} - \frac{2}{x^2 - 1})$

The Indeterminate Form $\frac{0}{0}$: Factor and Cancel

- To lift the indeterminacy, we transform algebraically, cancel and, finally, use the substitution property;

  Example: Compute $\lim_{x\to 3}\frac{x^2-4x+3}{x^2+x-12}$;

  We have

$$\begin{aligned}\lim_{x\to 3}\frac{x^2-4x+3}{x^2+x-12} &= \lim_{x\to 3}\frac{(x-1)(x-3)}{(x+4)(x-3)}\\ &= \lim_{x\to 3}\frac{x-1}{x+4}\\ &= \frac{3-1}{3+4}\\ &= \frac{2}{7}.\end{aligned}$$

The Indeterminate Form $\frac{0}{0}$: Another Example

- To lift the indeterminacy, we transform algebraically, cancel and, finally, use the substitution property;

  Example: Compute $\lim_{x\to 7}\frac{x-7}{x^2-49}$;

  We have

$$\begin{aligned}
\lim_{x\to 7}\frac{x-7}{x^2-49} &= \lim_{x\to 7}\frac{x-7}{(x+7)(x-7)}\\
&= \lim_{x\to 7}\frac{1}{x+7}\\
&= \frac{1}{7+7}\\
&= \frac{1}{14}.
\end{aligned}$$

The Indeterminate Form $\dfrac{\infty}{\infty}$

- To lift the indeterminacy, we transform algebraically, cancel and, finally, use the substitution property;

Example: Compute $\displaystyle\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\sec x}$;

We have

$$\begin{aligned}
\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\sec x} &= \lim_{x\to\frac{\pi}{2}}\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}}\\
&= \lim_{x\to\frac{\pi}{2}}\sin x\\
&= \sin\frac{\pi}{2}\\
&= 1.
\end{aligned}$$

The Indeterminate Form $\frac{0}{0}$: Multiply by Conjugate

- To lift the indeterminacy, we transform algebraically, cancel and, finally, use the substitution property;

  Example: Compute $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$;

  We have

$$
\begin{aligned}
\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4} &= \lim_{x \to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} \\
&= \lim_{x \to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)} \\
&= \lim_{x \to 4} \frac{1}{\sqrt{x}+2} \\
&= \frac{1}{\sqrt{4}+2} = \frac{1}{4}.
\end{aligned}
$$

The Indeterminate Form $\frac{0}{0}$: Multiply by Conjugate

- To lift the indeterminacy, we transform algebraically, cancel and, finally, use the substitution property;

  Example: Compute $\lim_{x\to 7}\frac{x-7}{\sqrt{x+9}-4}$;

$$\begin{aligned}
\lim_{x\to 7}\frac{x-7}{\sqrt{x+9}-4} &= \lim_{x\to 7}\frac{(x-7)(\sqrt{x+9}+4)}{(\sqrt{x+9}-4)(\sqrt{x+9}+4)} \\
&= \lim_{x\to 7}\frac{(x-7)(\sqrt{x+9}+4)}{x+9-16} \\
&= \lim_{x\to 7}\frac{(x-7)(\sqrt{x+9}+4)}{x-7} = \lim_{x\to 7}(\sqrt{x+9}+4) \\
&= \sqrt{7+9}+4 = 8.
\end{aligned}$$

The Indeterminate Form $\infty - \infty$

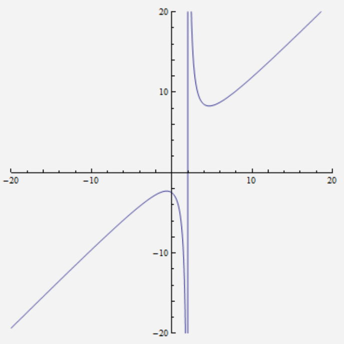
- To lift the indeterminacy, we transform algebraically, cancel and, finally, use the substitution property;

  Example: Compute $\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right)$;

$$
\begin{aligned}
\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right) &= \lim_{x\to 2}\left(\frac{x+2}{(x-2)(x+2)}-\frac{4}{(x-2)(x+2)}\right)\\
&= \lim_{x\to 2}\frac{x+2-4}{(x+2)(x-2)}\\
&= \lim_{x\to 2}\frac{x-2}{(x+2)(x-2)}\\
&= \lim_{x\to 2}\frac{1}{x+2}=\frac{1}{2+2}=\frac{1}{4}.
\end{aligned}
$$

Forms $\frac{c}{0}$, with $c \neq 0$ are Infinite but not Indeterminate

- $\lim_{x \to 2} \dfrac{x^2 - x + 5}{x - 2}$ is of the form $\dfrac{7}{0}$;
- These forms are not indeterminate, but rather they suggest that the side-limits as $x \to 2$ are infinite;
- If $x \to 2^-$, then $x < 2$, whence $x - 2 < 0$. Thus, $\lim_{x \to 2^-} \dfrac{x^2 - x + 5}{x - 2} (= (\dfrac{7}{0^-})) = -\infty$;
- If $x \to 2^+$, then $x > 2$, whence $x - 2 > 0$. Thus, $\lim_{x \to 2^+} \dfrac{x^2 - x + 5}{x - 2} (= (\dfrac{7}{0^+})) = \infty$;



Subsection 6

Trigonometric Limits

The Squeeze Theorem

The Squeeze Theorem

Assume that for $x \neq c$ in some open interval containing c,

$$\ell(x) \leq f(x) \leq u(x) \quad \text{and} \quad \lim_{x \to c} \ell(x) = \lim_{x \to c} u(x) = L.$$

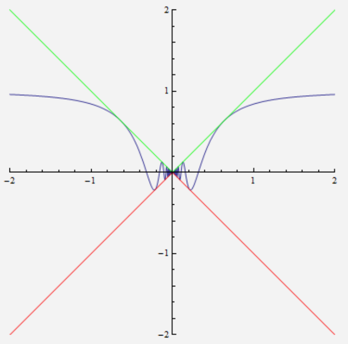
Then $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} f(x) = L$.

Example: We show $\lim_{x \to 0} \left(x \sin \frac{1}{x}\right)$

Note that $-|x| \leq x \sin \frac{1}{x} \leq |x|$;

Note, also that $\lim_{x \to 0} (-|x|) = \lim_{x \to 0} |x| = 0$;

Therefore, by Squeeze, $\lim_{x \to 0} \left(x \sin \frac{1}{x}\right) = 0$.

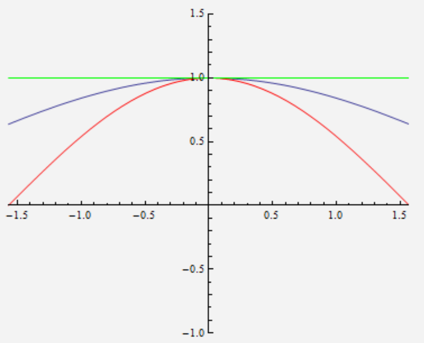


An Important Squeeze Identity

Theorem

For all $\theta \neq 0$, with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we have

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1.$$



Important Trigonometric Limits

Important Trigonometric Limits

We have

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \quad \text{and} \quad \lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0.$$

- Note that the first limit above follows by the Squeeze Theorem using the Squeeze Identity of the previous slide;
- For the second one, we have

$$\begin{aligned}
\lim_{\theta \to 0} \tfrac{1-\cos \theta}{\theta} &= \lim_{\theta \to 0} \tfrac{(1-\cos \theta)(1+\cos \theta)}{\theta(1+\cos \theta)} \\
&= \lim_{\theta \to 0} \tfrac{1-\cos^2 \theta}{\theta(1+\cos \theta)} \\
&= \lim_{\theta \to 0} \tfrac{\sin^2 \theta}{\theta(1+\cos \theta)} \\
&= \lim_{\theta \to 0} \left(\tfrac{\sin \theta}{\theta} \cdot \tfrac{\sin \theta}{1+\cos \theta}\right) \\
&= \lim_{\theta \to 0} \tfrac{\sin \theta}{\theta} \cdot \lim_{\theta \to 0} \tfrac{\sin \theta}{1+\cos \theta} \\
&= 1 \cdot \tfrac{0}{1+1} = 0.
\end{aligned}$$

Evaluation of Limits by a Change of Variable

- Compute the limit $\lim_{\theta\to 0}\frac{\sin 4\theta}{\theta}$;

  We have

$$\begin{aligned}\lim_{\theta\to 0}\frac{\sin 4\theta}{\theta} &= \lim_{\theta\to 0}\frac{4\sin 4\theta}{4\theta} \\ &= 4\lim_{\theta\to 0}\frac{\sin 4\theta}{4\theta} \\ &\overset{x=4\theta}{=} 4\lim_{x\to 0}\frac{\sin x}{x} \\ &= 4\cdot 1 = 4.\end{aligned}$$

- Compute the limit $\lim_{\theta\to 0}\frac{\sin 7\theta}{\sin 3\theta}$;

  We have

$$\begin{aligned}\lim_{\theta\to 0}\frac{\sin 7\theta}{\sin 3\theta} &= \lim_{\theta\to 0}\frac{7\theta\frac{\sin 7\theta}{7\theta}}{3\theta\frac{\sin 3\theta}{3\theta}} = \lim_{\theta\to 0}\frac{7}{3}\frac{\frac{\sin 7\theta}{7\theta}}{\frac{\sin 3\theta}{3\theta}} \\ &= \frac{7}{3}\frac{\lim_{\theta\to 0}\frac{\sin 7\theta}{7\theta}}{\lim_{\theta\to 0}\frac{\sin 3\theta}{3\theta}} \overset{x=7\theta}{\underset{y=3\theta}{=}} \frac{7}{3}\frac{\lim_{x\to 0}\frac{\sin x}{x}}{\lim_{y\to 0}\frac{\sin y}{y}} \\ &= \frac{7}{3}\frac{1}{1} = \frac{7}{3}.\end{aligned}$$

Subsection 7

Limits at Infinity

Limits at Infinity

Limit of $f(x)$ as $x \to \pm\infty$

- We write $\lim_{x \to \infty} f(x) = L$ if $f(x)$ gets closer and closer to L as $x \to \infty$, i.e., as x increases without bound;
- We write $\lim_{x \to -\infty} f(x) = L$ if $f(x)$ gets closer and closer to L as $x \to -\infty$, i.e., as x decreases without bound;

In either case, the line $y = L$ is called a **horizontal asymptote** of $y = f(x)$.

- Horizontal asymptotes describe the asymptotic behavior of $f(x)$, i.e., the behavior of the graph as we move way out to the left or to the right.

Example of Limits at Infinity

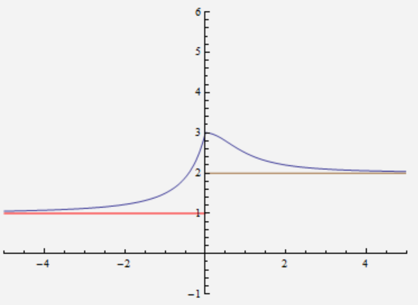
- Consider the function $f(x)$ whose graph is given on the right:

  We have

  $$\lim_{x\to-\infty} f(x) = 1$$

  and

  $$\lim_{x\to\infty} f(x) = 2.$$



Thus, both $y = 1$ and $y = 2$ are horizontal asymptotes of $y = f(x)$.

Powers of x

Theorem

Assume $n > 0$. Then we have

$$\lim_{x\to\infty} x^n = \infty \quad \text{and} \quad \lim_{x\to\infty} x^{-n} = \lim_{x\to\infty} \frac{1}{x^n} = 0.$$

For $n > 0$ an integer,

$$\lim_{x\to-\infty} x^n = \begin{cases} \infty, & \text{if } n \text{ is even} \\ -\infty, & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad \lim_{x\to-\infty} x^{-n} = \lim_{x\to-\infty} \frac{1}{x^n} = 0.$$

Example: $\lim_{x\to\infty} (3 - 4x^{-3} + 5x^{-5}) =$
$\lim_{x\to\infty} 3 - 4 \lim_{x\to\infty} x^{-3} + 5 \lim_{x\to\infty} x^{-5} = 3 - 4 \cdot 0 + 5 \cdot 0 = 3.$

Example

- Calculate $\lim_{x\to\pm\infty} \dfrac{20x^2 - 3x}{3x^5 - 4x^2 + 5}$.
  We follow the method of dividing numerator and denominator by the highest power x^5:

$$
\begin{aligned}
\lim_{x\to\pm\infty} \frac{20x^2-3x}{3x^5-4x^2+5} &= \lim_{x\to\pm\infty} \frac{\frac{20x^2-3x}{x^5}}{\frac{3x^5-4x^2+5}{x^5}} \\
&= \lim_{x\to\pm\infty} \frac{\frac{20x^2}{x^5} - \frac{3x}{x^5}}{\frac{3x^5}{x^5} - \frac{4x^2}{x^5} + \frac{5}{x^5}} \\
&= \lim_{x\to\pm\infty} \frac{\frac{20}{x^3} - \frac{3}{x^4}}{3 - \frac{4}{x^3} + \frac{5}{x^5}} \\
&= \frac{\lim_{x\to\pm\infty} \frac{20}{x^3} - \lim_{x\to\pm\infty} \frac{3}{x^4}}{\lim_{x\to\pm\infty} 3 - \lim_{x\to\pm\infty} \frac{4}{x^3} + \lim_{x\to\pm\infty} \frac{5}{x^5}} \\
&= \frac{0-0}{3-0+0} = 0.
\end{aligned}
$$

Limits at Infinity of Rational Functions

Theorem

If $a_n, b_m \neq 0$, then it is the case that

$$\lim_{x\to\pm\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m}.$$

Example:

- $\lim_{x\to\infty} \frac{3x^4-7x+9}{7x^4-4} = \frac{3}{7}\lim_{x\to\infty} x^0 = \frac{3}{7}$;
- $\lim_{x\to\infty} \frac{3x^3-7x+9}{7x^4-4} = \frac{3}{7}\lim_{x\to\infty} x^{-1} = \frac{3}{7}\lim_{x\to\infty} \frac{1}{x} = 0$;
- $\lim_{x\to-\infty} \frac{3x^8-7x+9}{7x^3-4} = \frac{3}{7}\lim_{x\to-\infty} x^5 = -\infty$;
- $\lim_{x\to-\infty} \frac{3x^7-7x+9}{7x^3-4} = \frac{3}{7}\lim_{x\to-\infty} x^4 = \infty$;

Two More Examples

- Compute the limit $\displaystyle\lim_{x\to\infty}\frac{3x^{7/2}+7x^{-1/2}}{x^2-x^{1/2}}$;

  We have

$$\begin{aligned}
\lim_{x\to\infty}\frac{3x^{7/2}+7x^{-1/2}}{x^2-x^{1/2}} &= \lim_{x\to\infty}\frac{(x^{-2})(3x^{7/2}+7x^{-1/2})}{(x^{-2})(x^2-x^{1/2})}\\
&= \lim_{x\to\infty}\frac{3x^{3/2}+7x^{-5/2}}{1-x^{-3/2}}\\
&= \frac{\lim_{x\to\infty}3x^{3/2}+\lim_{x\to\infty}7x^{-5/2}}{\lim_{x\to\infty}1-\lim_{x\to\infty}x^{-3/2}}\\
&= \frac{\infty}{1}=\infty.
\end{aligned}$$

- Compute the limit $\displaystyle\lim_{x\to\infty}\frac{x^2}{\sqrt{x^3+1}}$;

  We have

$$\begin{aligned}
\lim_{x\to\infty}\frac{x^2}{\sqrt{x^3+1}} &= \lim_{x\to\infty}\frac{x^{-3/2}x^2}{x^{-3/2}\sqrt{x^3+1}}=\lim_{x\to\infty}\frac{x^{1/2}}{\sqrt{x^{-3}(x^3+1)}}\\
&= \lim_{x\to\infty}\frac{x^{1/2}}{\sqrt{1+x^{-3}}}=\frac{\infty}{1}=\infty.
\end{aligned}$$

One More Example

- Calculate the limits at infinity of $f(x) = \dfrac{12x+25}{\sqrt{16x^2+100x+500}}$;

  We have

$$\begin{aligned}
\lim_{x\to-\infty} \frac{12x+25}{\sqrt{16x^2+100x+500}} &= \lim_{x\to-\infty} \frac{12x(1+\frac{25}{12x})}{\sqrt{16x^2(1+\frac{100}{16x}+\frac{500}{16x^2})}} \\
&= \lim_{x\to-\infty} \frac{12x(1+\frac{25}{12x})}{-4x\sqrt{1+\frac{100}{16x}+\frac{500}{16x^2}}} \\
&= -3\lim_{x\to-\infty} \frac{1+\frac{25}{12x}}{\sqrt{1+\frac{100}{16x}+\frac{500}{16x^2}}} \\
&= -3; \\
\lim_{x\to\infty} \frac{12x+25}{\sqrt{16x^2+100x+500}} &= \lim_{x\to\infty} \frac{12x(1+\frac{25}{12x})}{\sqrt{16x^2(1+\frac{100}{16x}+\frac{500}{16x^2})}} \\
&= \lim_{x\to\infty} \frac{12x(1+\frac{25}{12x})}{4x\sqrt{1+\frac{100}{16x}+\frac{500}{16x^2}}} \\
&= 3\lim_{x\to\infty} \frac{1+\frac{25}{12x}}{\sqrt{1+\frac{100}{16x}+\frac{500}{16x^2}}} = 3.
\end{aligned}$$

Subsection 8

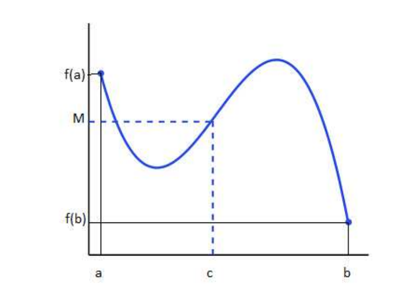
Intermediate Value Theorem

The Intermediate Value Theorem

Intermediate Value Theorem

If $f(x)$ is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, then for every value M between $f(a)$ and $f(b)$, there exists at least one value $c \in (a, b)$, such that $f(c) = M$.

Example: Show that $\sin x = \frac{1}{8}$ has at least one solution.
Consider $f(x) = \sin x$ in the closed interval $[0, \frac{\pi}{2}]$.
We have $f(0) = 0 < \frac{1}{8} < 1 = f(\frac{\pi}{2})$.
Thus, by the Intermediate Value Theorem,

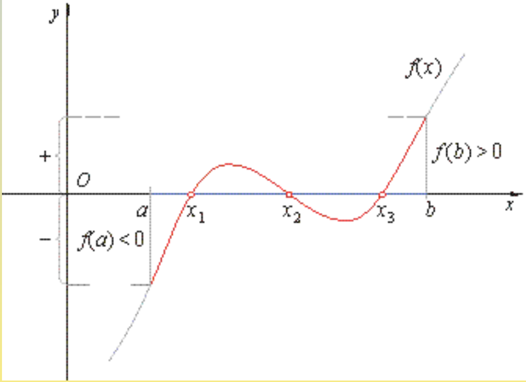


there exists $c \in (0, \frac{\pi}{2})$, such that $f(c) = \frac{1}{8}$, i.e., $\sin c = \frac{1}{8}$. This c is a solution of the equation $\sin x = \frac{1}{8}$;.

Existence of Zeros

Existence of Zeros

If $f(x)$ is continuous on $[a, b]$ and if $f(a)$ and $f(b)$ are nonzero and have opposite signs, then $f(x)$ has a zero in (a, b).



Applying the Existence of Zeros Theorem

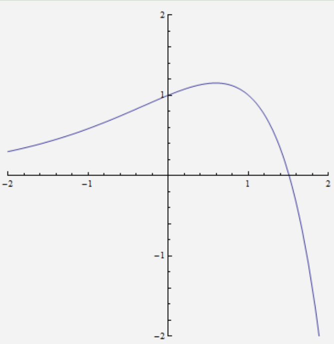
- Show that the equation $2^x + 3^x = 4^x$ has at least one zero.

  Consider $f(x) = 2^x + 3^x - 4^x$ in the closed interval $[1, 2]$.

  We have $f(1) = 1 > 0$, whereas $f(2) = -3 < 0$.

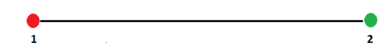
  Thus, by the Existence of Zeros Theorem, there exists $c \in (1, 2)$, such that $f(c) = 0$, i.e., $2^c + 3^c - 4^c = 0$.

  But, then, c satisfies $2^c + 3^c = 4^c$, i.e., it is a zero of $2^x + 3^x = 4^x$.

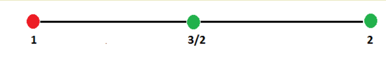


The Bisection Method

- Find an interval of length $\frac{1}{4}$ in $[1, 2]$ containing a root of the equation $x^7 + 3x - 10 = 0$;
  Consider the function $f(x) = x^7 + 3x - 10$ in $[1, 2]$.
  Since $f(1) = -6 < 0$ and $f(2) = 112 > 0$, by the Existence of Zeros Theorem, it has a root in $(1, 2)$.



Since $f(1) = -6 < 0$ and $f(\frac{3}{2}) = 11.586 > 0$, it has a root in the interval $(1, \frac{3}{2})$.



Finally, since $f(\frac{5}{4}) = -1.482 < 0$ and $f(\frac{3}{2}) = 11.586 > 0$, the root is in the interval $(\frac{5}{4}, \frac{3}{2})$, which has length $\frac{1}{4}$.