Homework 6

Due November 23, 2023

Problem 6.1

For the following pairs of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, either define an isomorphism between them, or prove that there is none.

$$V_1 = \{1, 2, 3, 4, 5, 6\}, E_1 = \{(1, 2), (2, 3), (3, 4), (1, 4), (4, 5), (5, 6), (2, 6)\}$$
$$V_2 = \{a, b, c, d, e, f\}, E_2 = \{ab, bc, cd, de, ae, ef, cf\}$$

Problem 6.2

Prove that the average degree of a tree is less than 2.

Problem 6.3

Show that if a connected planar graph with more than two vertices is bipartite, then

$$e \leqslant 2v - 4.$$

Problem 6.4

Prove that if an arbitrary graph G is disconnected, then its compliment $\overline{G}$ is connected.

Problem 6.5

Show that any tree T has at least $\Delta(T)$ leaves.

Problem 6.6

Let $G = \{V, E\}$, where $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and
$E = \{(1, 2), (2, 3), (3, 4), (4, 1), (2, 6), (3, 6), (3, 9), (4, 8), (4, 9), (8, 9), (6, 7), (6, 9)\}$
Answer the following questions:
a. Draw this graph.
b. Write the adjacency matrix.
c. What is $|G|$?
d. Are vertices 1 and 9 adjacent?
e. Compute $d(3), d(5), d(6), \delta(G), \Delta(G), \varepsilon(G), girth(G), diam(G)$.
f. Is this graph connected?
g. Is this graph regular?
h. Does this graph contain a regular subgraph?
i. Does this graph contain C^5, P^7?
j. Is this graph eulerian?
k. Is this graph hamiltonian?
l. Is this graph planar? If yes, verify the Euler identity.
m. Color the vertices of this graph. What can be said on its chromatic number?

Problem 6.7

Let G be the graph with vertices labelled by $\{1, 2, 3, \dots, 7\}$, two distinct vertices i and j are adjacent if $|i - j|$ is odd. Draw the graph G and give the adjacency matrix and adjacency list of G. Is this graph connected? If not, then how many connected components does G have?

Problem 6.8

For each of the following, try to give two different (not isomorphic) unlabeled graphs with the given properties, or explain why doing so is impossible.

1. Two different trees with the same number of vertices and the same number of edges. A tree is a connected graph with no cycles.
2. Two different graphs with 8 vertices all of degree 2.
3. Two different graphs with 5 vertices all of degree 4.
4. Two different graphs with 5 vertices all of degree 3.

Problem 6.9

Is it possible for a planar graph to have 6 vertices, 10 edges and 5 faces? Explain.

Problem 6.10

The graph G has 6 vertices with degrees $2, 2, 3, 4, 4, 5$. How many edges does G have? Could G be planar? If so, how many faces would it have. If not, explain.

Problem 6.11

Euler's formula ($v - e + f = 2$) holds for all connected planar graphs. What if a graph is not connected? Suppose a planar graph has two connected components. What is the value of $v - e + f$ now? What if it has k connected components?

Problem 6.12

Draw a graph with chromatic number 6. Could your graph be planar? Explain.

Problem 6.13

Suppose a graph has a Hamilton path. What is the maximum number of vertices of degree one the graph can have? Explain why your answer is correct. Also, find a graph which does not have a Hamilton path even though no vertex has degree one. Explain why your example works.