Probability theory

Prof.dr.hab. Viorel Bostan

Technical University of Moldova

viorel.bostan@adm.utm.md

Lecture 6

How many operations are needed to place an elephant in a refrigerator?

Answer: 3 operations:

- one, open the refrigerator;
- two, place the elephant inside;
- three, close the refrigerator.

How many operations are needed to place a giraffe in a refrigerator?

Answer: 4 operations ...

- ...open it, take the elephant out, place giraffe, close it.

Another Joke of the Day

Who will run faster a 1 km distance, elephant or giraffe?

Answer: elephant!

Why?

Because, giraffe is still in the refrigerator!

Previous Lectures Outline

- Conditional Probability:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0;$$

- Product rule (why tree diagrams work).

$$P(A \cap B) = P(A)P(B \mid A) = P(B)P(A \mid B);$$

- Law of total probability;

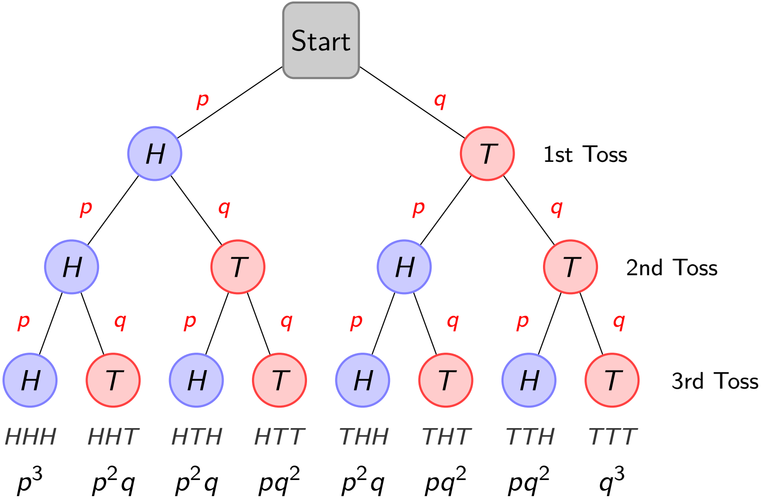
$$P(A) = P(E)P(A \mid E) + P(E^c)P(A \mid E^c);$$

- A posteriori probabilities;
- Bayes formula

$$P(A_i \mid E) = \frac{P(A_i)P(E \mid A_i)}{P(E)} = \frac{P(A_i)P(E \mid A_i)}{\sum_{k=1}^{m} P(A_k)P(E \mid A_k)}.$$

Tossing a coin 3 times

Consider a biased coin with $P(H) = p$ and $P(T) = q$ with $p + q = 1$.Toss it three times.



$P(THT) = ?$

$P(\text{3rd H} \mid \text{first two } H) = ?$

$P(\text{one } H) = ?$

$P(\text{1st } H \mid \text{one } H) = ?$

A poker problem

Problem

You shuffle a deck of cards and deal your friend a 5-card hand. Suppose your friend says, "I have the ace of spades". What is the probability that your friend has another ace?

Solution. The sample space for this experiment is the set of all 5 card hands. All outcomes are equally likely, so the probability of each outcome is $1/\binom{52}{5}$.

Let S be the event that your friend has the ace of spades, and let A be the event that your friend has another ace.

Our objective is to compute:

$$P(A \mid S) = \frac{P(A \cap S)}{P(S)}.$$

The number of hands containing the ace of spades is equal to the number of ways to select 4 of the remaining 51 cards and therefore,

$$P(S) = \frac{\binom{51}{4}}{\binom{52}{5}}.$$

A poker problem

Number of hands containing ace of spades and at least 1 more ace is:

$$|A \cap S| = \binom{3}{1}\binom{50}{3} + \binom{3}{2}\binom{49}{2} + \binom{3}{3}\binom{48}{1}$$

and thus,

$$P(A \cap S) = \frac{\binom{3}{1}\binom{50}{3} + \binom{3}{2}\binom{49}{2} + \binom{3}{3}\binom{48}{1}}{\binom{52}{5}}.$$

Substituting these results into our original equation gives the solution:

$$P(A \mid S) = \frac{P(A \cap S)}{P(S)} = \frac{\frac{\binom{3}{1}\binom{50}{3}+\binom{3}{2}\binom{49}{2}+\binom{3}{3}\binom{48}{1}}{\binom{52}{5}}}{\frac{\binom{51}{4}}{\binom{52}{5}}}$$

$$= \frac{\binom{3}{1}\binom{50}{3} + \binom{3}{2}\binom{49}{2} + \binom{3}{3}\binom{48}{1}}{\binom{51}{4}}$$

$$= \frac{3 \cdot \frac{50 \cdot 49 \cdot 48}{6} + 3 \cdot \frac{49 \cdot 48}{2} + 1 \cdot 48}{\frac{51 \cdot 50 \cdot 49 \cdot 48}{4}} \approx 0.0425.$$

Minesweeper game problem

In the past, all PC came with a game called Minesweeper.

In this version of the game, there is an 8×8 grid of squares.

10 randomly selected squares contain mines, and all configurations of mines are equally likely.

For example, after 3 moves we can get the following configuration:

			1	1	3		

In the past all PC came with a game called Minesweeper.

In this version of the game, there is an 8×8 grid of squares.

10 randomly selected squares contain mines, and all configurations of mines are equally likely.

Question 1. What is the sample space for this game and what is the probability of each outcome?

The sample space Ω consists of all arrangements of 10 mines on the 8×8 board. All such configurations have the same probability, which therefore must be

$$\frac{1}{\binom{64}{10}}$$

			1	1	3		

3 numbered squares do not contain mines. Each number indicates how many squares adjacent to that number do contain mines.

Let $E \subset \Omega$ be event that configuration of mines is consistent with this numbering.

Question 2. Describe all outcomes in the event E, in other words how many such configurations are there?

Minesweeper game problem

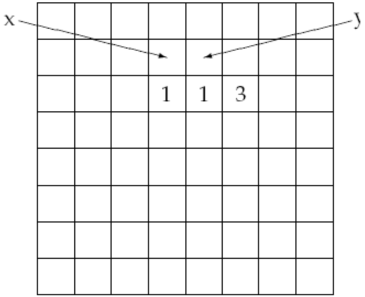
		a	b	c	d	e	
		a	1	1	3	e	
		a	b	c	d	e	

nr. of mines						
a	b	c	d	e	other	nr. of configurations
1	0	0	1	2	6	$\binom{3}{1}\binom{2}{0}\binom{2}{0}\binom{2}{1}\binom{3}{2}\binom{49}{6}$
0	1	0	0	3	6	$\binom{3}{0}\binom{2}{1}\binom{2}{0}\binom{2}{0}\binom{3}{3}\binom{49}{6}$
0	0	1	$\cancel{1}$ 0	2	7	$\binom{3}{0}\binom{2}{0}\binom{2}{1}\binom{2}{0}\binom{3}{2}\binom{49}{7}$

There are $\binom{3}{1}\binom{2}{0}\binom{2}{0}\binom{2}{1}\binom{3}{2}\binom{49}{6} + \binom{3}{0}\binom{2}{1}\binom{2}{0}\binom{2}{0}\binom{3}{3}\binom{49}{6} + \binom{3}{0}\binom{2}{0}\binom{2}{1}\binom{2}{0}\binom{3}{2}\binom{49}{7}$ configurations consistent with event E.

Minesweeper game problem

On the next move, must click an unnumbered square.
If that square contains a mine, game is lost!
The squares marked x and y below look like reasonable choices.



Question 3. What is the probability that there is a mine at square x?

If there is a mine at square x, then the state of all the other squares next to numbers is complete determined and 6 of the other 49 squares are mined.
Therefore, the probability that x is mined is:

$$P(x \text{ mined} \mid E) = \frac{P(x \text{ mined} \cap E)}{P(E)} = \frac{\binom{49}{6}}{P(E)} \approx 0.0175$$

Question 4. What is the probability that there is a mine at square y?
If there is a mine at square y, then two of the three squares marked e are mined as well as 7 of the remaining 49 squares.
Thus, the probability that there is a mine at y is:

$$P(y \text{ mined} \mid E) = \frac{P(y \text{ mined} \cap E)}{P(E)} = \frac{\binom{3}{2}\binom{49}{7}}{P(E)} \approx 0.324.$$

Thus, a mine at position y is 18 times more likely than a mine at position x.

Independent Events

Suppose two fair coins are flipped simultaneously on opposite sides of a room. Intuitively, the way one coin lands does not affect the way the other coin lands.

The mathematical concept that captures this intuition is called **independence**.

Two events A and B are independent, if knowing that B happens does not alter the probability that A happens, and vice versa.

Definition

Events A and B are called **independent** if

$$P(A \mid B) = P(A),$$
$$P(B \mid A) = P(B).$$

Independent Events

Consider two independent events A and B. Then, by definition

$$P(A \mid B) = P(A).$$

On the other hand, by definition of conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$
$$P(A \cap B) = P(A \mid B) \cdot P(B) = P(A) \cdot P(B).$$

Definition

Events A and B are **independent** if

$$P(A \cap B) = P(A) \cdot P(B).$$

Wrong idea!: disjoint events are independent.

Independent Events

Actually, the opposite statement is true:
if $A \cap B = \varnothing$, then knowing that A happens means you know that B does not happen.

$$P(A \cap B) = P(A)P(B).$$

So, disjoint events are never independent—unless one of them has probability zero.

Generally, independence is something that you assume in modeling a phenomenon.

For example, consider the experiment of flipping 2 fair coins. Let A be the event that the first coin comes up heads, and let B be the event that the second coin is heads.

If A and B are independent, then the probability that both coins come up heads is:

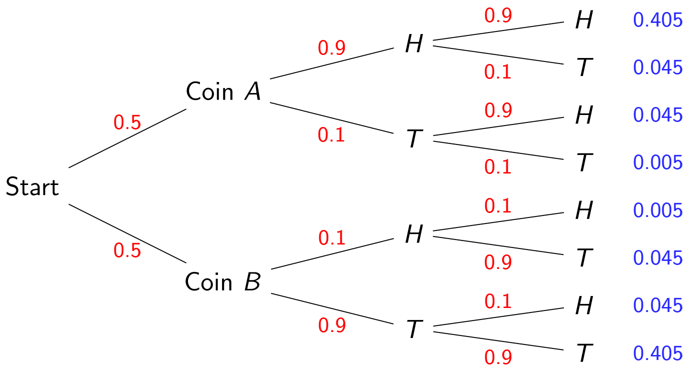
$$P(A \cap B) = P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Here the assumption of independence is reasonable. The result of one coin toss should have negligible impact on the outcome of the other coin toss. And if we were to repeat the experiment many times, we would be likely to have $A \cap B$ about $1/4$ of the time.

Unfair coins

Consider two unfair coins A and B:

$$P(H \mid \text{Coin } A) = 0.9, \qquad P(H \mid \text{Coin } B) = 0.1.$$



A probability professor plans to travel to a conference by plane.

When he passes the security check, they discover a bomb in his carry-on-baggage.

Of course, he is hauled off immediately for interrogation.

"I don't understand it!" the interrogating officer exclaims. "You're an accomplished professional, a caring family man, a pillar of your community – and now you want to destroy that all by blowing up an airplane!"

"Sorry", the professor interrupts him. "I had never intended to blow up the plane."

"So, for what reason did you try to bring a bomb on board?!"

"Let me explain. My calculations show that the probability of a bomb being on an airplane is

$$\frac{1}{10\,000}.$$

That's so high that I wouldn't have any peace of mind on a flight."

Joke of the Day

"And what does this have to do with you bringing a bomb on board of a plane?"

"You see, since the probability of one bomb being on my plane is

$$\frac{1}{10\,000},$$

then the chance that there are two bombs is

$$\frac{1}{10\,000} \cdot \frac{1}{10\,000} = \frac{1}{100\,000\,000}.$$

"If I already bring one, the chance of another bomb being around is actually

$$\frac{1}{100\,000\,000}$$

and I am much safer..."

Professor with a bomb problem

Define events: $X =$"bomb 1 is on the plane" and $Y =$"bomb 2 is on the plane"

Events X and Y are independent events, with

$$P(X) = P(Y) = \frac{1}{10000}.$$

Probability of having both bombs on plane is

$$P(X \cap Y) = P(X) \cdot P(Y) = \frac{1}{10000} \cdot \frac{1}{10000} = \frac{1}{100000000}.$$

But, probability of "having bomb 2 on the plane" given event that "bomb 1 is on plane" is is $P(Y \mid X)$:

$$P(Y \mid X) = \frac{P(Y \cap X)}{P(X)} = \frac{\frac{1}{100000000}}{\frac{1}{10000}} = \frac{1}{10000}.$$

Thus, professor was definitely wrong!