Homework 3

Due 18:00, October 6, 2023

Problem 3.1

Prove by Well Ordering Principle:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \forall n \in \mathbb{N}, n \geqslant 1.$$

Problem 3.2

Prove by Principle of Mathematical Induction:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}, \quad \forall n \in \mathbb{N}, n > 1.$$

Problem 3.3

Give an inductive proof that the Fibonacci numbers F_n and F_{n+1} are relatively prime for all $n \geqslant 0$. The Fibonacci numbers are defined as follows:

$$F_0 = 1, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \forall n \geqslant 2.$$

Two integeres are called relatively prime if they don't have common divisors except 1. For example, 14 and 25 are relatively prime, or 80 and 81 are also relatively prime.

Problem 3.4

Prove by induction that for all integers $n \geqslant 2$

$$4^n + 7 \leqslant 5^n.$$

Problem 3.5

Find the flaw in the following bogus proof that $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.

Proof. The bogus proof is by induction on n, with hypothesis

$$P(n) : \forall k \leq n \quad a^k = 1$$

where k is a nonnegative integer valued variable.

Base Case: $P(0)$ is equivalent to $a^0 = 1$, which is true by definition of a^0, (by convention, this holds even if $a = 0$).

Inductive Step: By induction hypothesis, $a^k = 1$ for all $k \in \mathbb{N}$ such that $k \leqslant n$. But then

$$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1$$

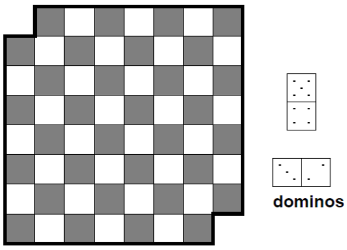
which implies that $P(n+1)$ holds. It follows by induction that $P(n)$ holds for all $n \in \mathbb{N}$, and in particular, $a^n = 1$ holds for all $n \in \mathbb{N}$.

Problem 3.6

A robot Romeo moves on two-dimensional integer grid. It starts out at $(0,0)$ and is allowed to move in any of these four ways:
1. $(+2,-1)$: right 2, down 1;
2. $(-2,+1)$: left 2, up 1;
3. $(+1,+3)$;
4. $(-1,-3)$.
Julietta waits him at $(1,1)$. Prove that Romeo never will be able to reach his love.

Problem 3.7

Can you tile an 8×8 chessboard with 31 dominos if opposite corners are removed? Argue your answer rigourously.



Problem 3.8

Let BM be the set of all sequences (or strings) of square brackets [and] that are matched (left with right). For example, the following three strings are in BM:

$$[\,]\,[\,] \text{ and } [\,[\,]\,] \text{ and } [\,[\,[\,]\,[\,]\,]\,],$$

but these strings are not in BM:

$$[\,[\,]\,]\,] \text{ and }]\,[\,[\,]\,].$$

Recursively define the set BM of matched brackets as follows:

Base case. $\lambda \in BM$ (empty string is in BM).
Constructor case. If $s, t \in BM$, then $[\,s\,]\,t \in BM$.

Prove by structural induction that (recursively-defined) matched strings from BM always have an equal number of left and right brackets.

Problem 3.9

BONUS PROBLEM

Elementary functions EF are the set of functions of one real variable defined recursively as follows:
Base Cases.

- Identity function, $id(x) = x$ is in EF.
- Any constant function is in EF.
- The sine function $\sin(x)$ is in EF

Constructor Cases. If $f, g \in EF$, then so are

- $f+g$, fg, 2^g;
- The inverse function f^{-1};
- The composition $f \circ g$.

Prove the following:

- Function $1/x$ is in $ELFUN$.
- Function $\tan(\frac{x-5}{x+5})$ is in EF.
- If $f \in EF$, then its derivative is also in EF.