Linear Algebra and Analytic Geometry

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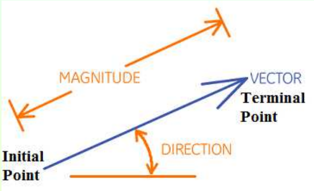
Subsection 0

Vectors

Vectors

- **Vector quantities** have a **magnitude** and a **direction**;



- Examples are force, velocity, displacement etc.;

Definition of a Vector

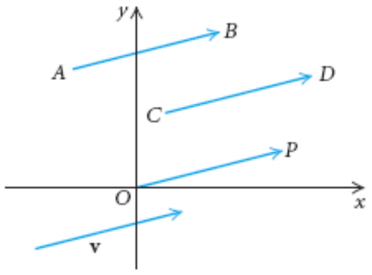
A **vector** is a directed line segment; The length of the line segment is the **magnitude** of the vector and the **direction** is measured by an angle.

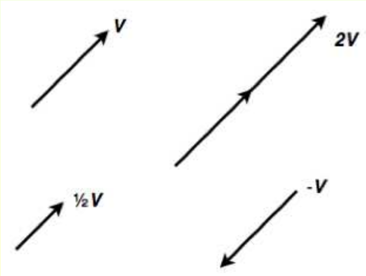
- The starting point A is called the **initial point** or **tail** and the ending point B is called the **terminal point** or the **head** of the vector;
- A vector with tail A and head B is denoted $\overrightarrow{AB}$ or $\mathbf{AB}$;
- The magnitude of this vector is denoted $\|\overrightarrow{AB}\| = \|\mathbf{AB}\|$;

$|\overrightarrow{AB}|$

Equivalence and Scalar Multiplication

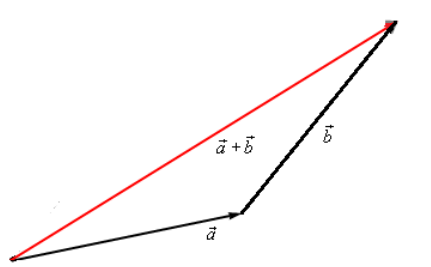
- Two vectors are **equivalent** if they have the same magnitude and the same direction;
- **Scalar multiplication** is the multiplication of a vector by a real number; If the real number is positive, then the magnitude changes but the direction does not; If the number is negative then the magnitude changes and the direction is reversed;

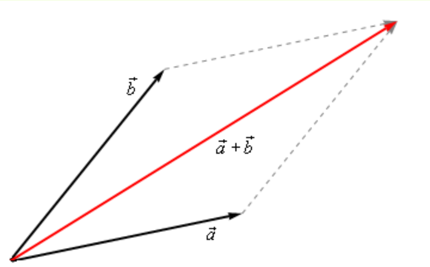




Sum or Resultant Vector

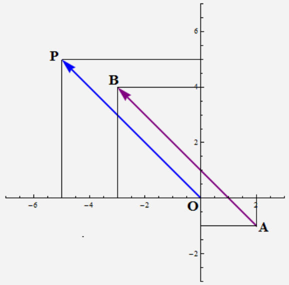
- The **resultant** or **sum** of two vectors is the vector that has the same effect as the combined application of the two vectors;
- The resultant can be computed using
  - the triangle method; or
  - the parallelogram method;





Standard Position

- A vector can be moved in the plane as long as its magnitude and direction are not changed;
- For instance, the vector **AB** with $A(2, -1)$ and $B(-3, 4)$ may be moved so that its initial point is at the origin O; Then its terminal point becomes $P(-5, 5)$;



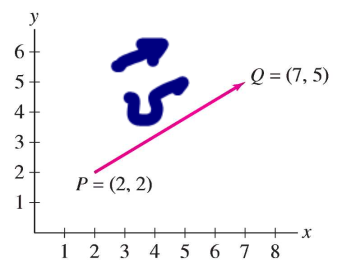
- Because **OP** and **AB** have same magnitude and direction, they are equivalent: $\mathbf{OP} = \mathbf{AB}$;

Subsection 1

Vectors in the Plane

Vectors

- A two-dimensional **vector** $\boldsymbol{v}$ is determined by two points in the plane:
  - an **initial point** P (also called the **tail** or **basepoint**);
  - a **terminal point** Q (also called the **head**).

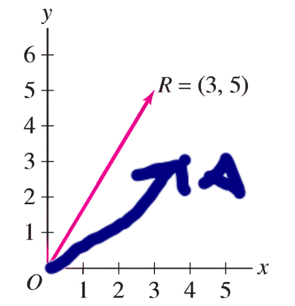


We write $\boldsymbol{v} = \overrightarrow{PQ}$ and we draw $\boldsymbol{v}$ as an arrow pointing from P to Q. This vector is said to be **based at** P.

- The **length** or **magnitude** of $\boldsymbol{v}$, denoted $\|\boldsymbol{v}\|$, is the distance from P to Q.

Position Vectors

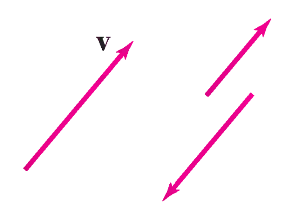
- The vector $\boldsymbol{v} = \overrightarrow{OR}$ pointing from the origin to a point R is called the **position vector** of R.

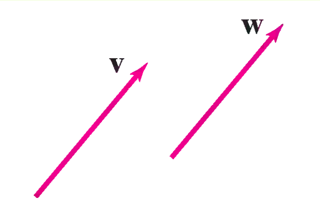


Example: The figure shows the position vector of $R = (3, 5)$.

Parallel and Translate Vectors

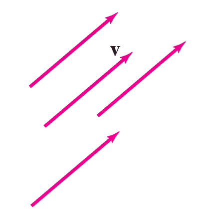
- Two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ of nonzero length are called **parallel** if the lines through $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel.
  Parallel vectors point either in the same or in opposite directions.
- A vector $\boldsymbol{v}$ is said to undergo a **translation** when it is moved parallel to itself without changing its length or direction.
  The resulting vector $\boldsymbol{w}$ is called a **translate** of $\boldsymbol{v}$.
  Translates have the same length and direction but different basepoints.

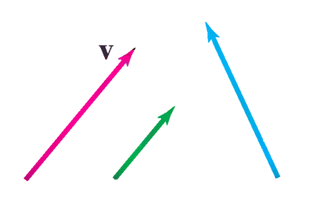




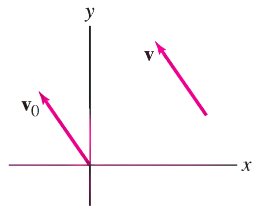
Equivalent Vectors

- Two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are **equivalent** if $\boldsymbol{w}$ is a translate of $\boldsymbol{v}$.





- Every vector can be translated so that its tail is at the origin. Therefore, every vector $\boldsymbol{v}$ is equivalent to a unique vector $\boldsymbol{v}_0$ based at the origin.

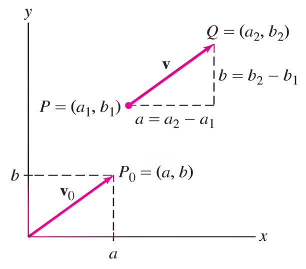


Components, Length and Zero Vector

- The **components** of $\boldsymbol{v} = \overrightarrow{PQ}$, where $P = (a_1, b_1)$ and $Q = (a_2, b_2)$, are the quantities

$$a = a_2 - a_1 \quad (x\text{-}\textbf{component}),$$
$$b = b_2 - b_1 \quad (y\text{-}\textbf{component}).$$

  The pair of components is denoted $\langle a, b \rangle$.



- The length of a vector in terms of its components (by the distance formula) is

$$\|\overrightarrow{PQ}\| = \sqrt{a^2 + b^2}.$$

- The **zero vector** (whose head and tail coincide) is the vector $\mathbf{0} = \langle 0, 0 \rangle$ of length zero.

$\vec{0} = (0, 0)$

Representation Convention

- The components $\langle a, b\rangle$ determine the length and direction of $\boldsymbol{v}$, but not its basepoint.

  Two vectors have the same components if and only if they are equivalent.

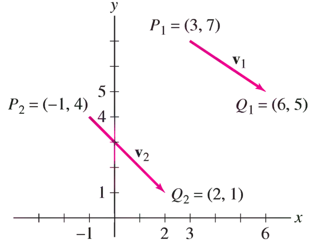
- Nevertheless, the standard practice is to describe a vector by its components, and thus we write $\boldsymbol{v} = \langle a, b\rangle$. $\vec{v} = (a, b)$

  Although this notation is ambiguous (because it does not specify the basepoint), it rarely causes confusion in practice.

  In the sequel we assume all vectors are based at the origin unless otherwise stated.

Example I

- Determine whether $\boldsymbol{v}_1 = \overrightarrow{P_1Q_1}$ and $\boldsymbol{v}_2 = \overrightarrow{P_2Q_2}$ are equivalent, where $P_1 = (3,7)$, $Q_1 = (6,5)$ and $P_2 = (-1,4)$, $Q_2 = (2,1)$. What is the magnitude of $\boldsymbol{v}_1$?

  We compare components:

  $$\boldsymbol{v}_1 = \langle 6-3, 5-7\rangle = \langle 3, -2\rangle$$

  and

  $$\boldsymbol{v}_2 = \langle 2-(-1), 1-4\rangle = \langle 3, -3\rangle.$$

  Since $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ have different components, they are not equivalent vectors.

  The magnitude of $\boldsymbol{v}_1$ is

  $$\|\boldsymbol{v}_1\| = \sqrt{3^2 + (-2)^2} = \sqrt{13}.$$

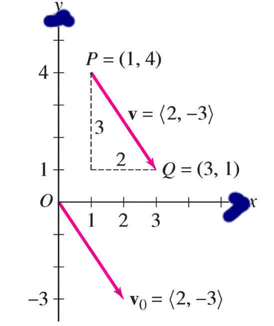
Example II

- Sketch the vector $\mathbf{v} = \langle 2, -3 \rangle$ based at $P = (1, 4)$ and the vector $\mathbf{v}_0$ equivalent to $\mathbf{v}$ based at the origin.

The vector $\mathbf{v} = \langle 2, 3 \rangle$ based at $P = (1, 4)$ has terminal point

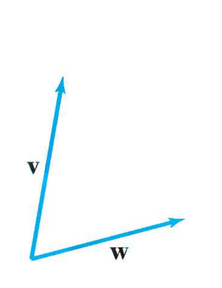
$$Q = (1 + 2, 4 - 3) = (3, 1).$$

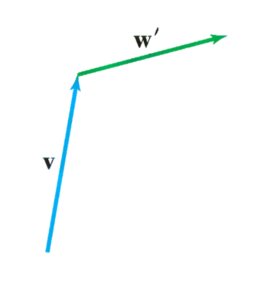
The vector $\mathbf{v}_0$ equivalent to $\mathbf{v}$ based at O has terminal point $(2, -3)$.

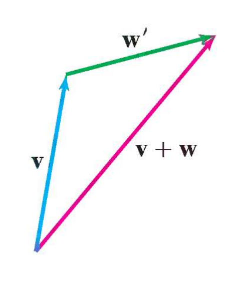


Addition of Two Vectors: Tail-to-Head Method

- The **vector sum** $\boldsymbol{v} + \boldsymbol{w}$ is defined when $\boldsymbol{v}$ and $\boldsymbol{w}$ have the same basepoint:
  - Translate $\boldsymbol{w}$ to the equivalent vector $\boldsymbol{w}'$ whose tail coincides with the head of $\boldsymbol{v}$.
  - The sum $\boldsymbol{v} + \boldsymbol{w}$ is the vector pointing from the tail of $\boldsymbol{v}$ to the head of $\boldsymbol{w}'$.



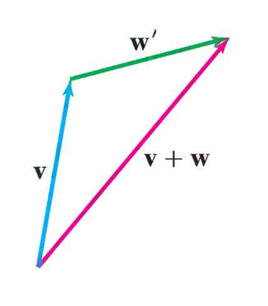


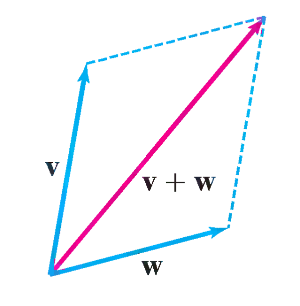


Addition of Two Vectors: Parallelogram Law

- Another way to add vectors is to use the **Parallelogram Law**:

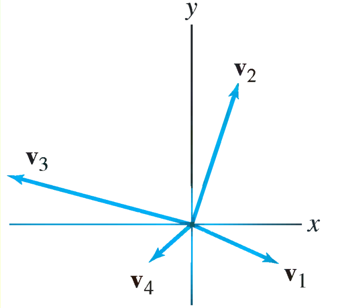
  $\boldsymbol{v} + \boldsymbol{w}$ is the vector pointing from the basepoint to the opposite vertex of the parallelogram formed by $\boldsymbol{v}$ and $\boldsymbol{w}$.

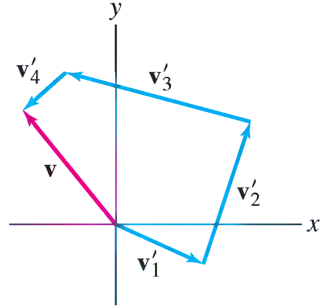




Addition of Multiple Vectors

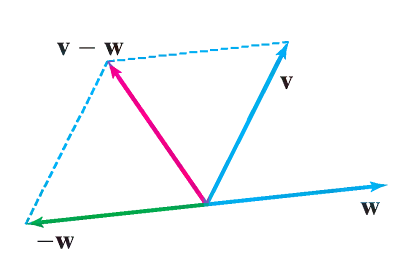
- To add several vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$:
  - translate the vectors to $\boldsymbol{v}_1 = \boldsymbol{v}_1', \boldsymbol{v}_2', \ldots, \boldsymbol{v}_n'$ so that they lie head to tail;
  - The vector sum $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2 + \cdots + \boldsymbol{v}_n$ is the vector whose terminal point is the terminal point of $\boldsymbol{v}_n'$.

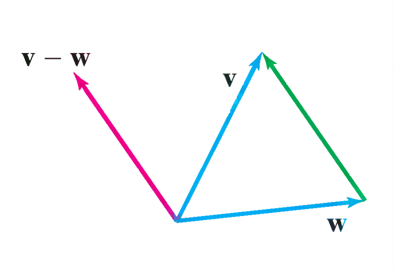




Subtraction of Vectors

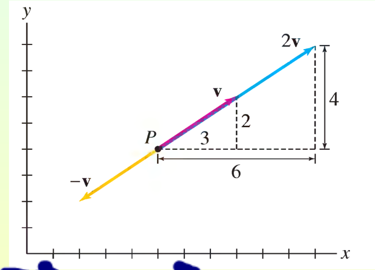
- **Vector subtraction** $\boldsymbol{v} - \boldsymbol{w}$ is carried out by adding $-\boldsymbol{w}$ to $\boldsymbol{v}$.
- More simply:
  - draw the vector pointing from $\boldsymbol{w}$ to $\boldsymbol{v}$;
  - translate it back to the basepoint to obtain $\boldsymbol{v} - \boldsymbol{w}$.





Scalar Multiplication

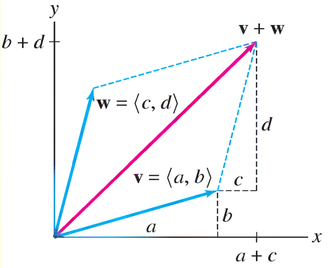
- The term **scalar** is another word for "real number".
- If λ is a scalar and $\boldsymbol{v}$ is a nonzero vector, **the scalar multiple** $\lambda\boldsymbol{v}$ is defined as follows:
  - $\lambda\boldsymbol{v}$ has length $|\lambda|\|\boldsymbol{v}\|$.
  - It points in the same direction as $\boldsymbol{v}$ if $\lambda > 0$.
  - It points in the opposite direction if $\lambda < 0$.

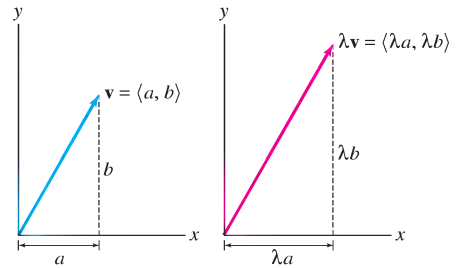


- Note that $0\boldsymbol{v} = \mathbf{0}$, for all $\boldsymbol{v}$; $0\vec{v} = \vec{0}$
- Also $\|\lambda\boldsymbol{v}\| = |\lambda|\|\boldsymbol{v}\|$.
- $-\boldsymbol{v}$ has the same length as $\boldsymbol{v}$ but points in the opposite direction.
- A vector $\boldsymbol{w}$ is parallel to $\boldsymbol{v}$ if and only if $\boldsymbol{w} = \lambda\boldsymbol{v}$, for some nonzero scalar λ.

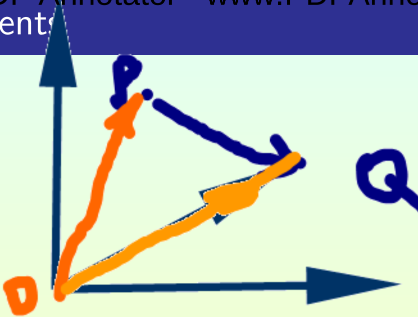
Vector Operations Using Components

- To add or subtract two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, we add or subtract their components.
- To multiply $\boldsymbol{v}$ by a scalar λ, we multiply the components of $\boldsymbol{v}$ by λ.





Vector Operations Using Components



- If $\boldsymbol{v} = \langle a, b \rangle$ and $\boldsymbol{w} = \langle c, d \rangle$, then:
  (i) $\boldsymbol{v} + \boldsymbol{w} = \langle a + c, b + d \rangle$;
  (ii) $\boldsymbol{v} - \boldsymbol{w} = \langle a - c, b - d \rangle$;
  (iii) $\lambda \boldsymbol{v} = \langle \lambda a, \lambda b \rangle$;
  (iv) $\boldsymbol{v} + \mathbf{0} = \mathbf{0} + \boldsymbol{v} = \boldsymbol{v}$.
- Also note that if $P = (a_1, b_1)$ and $Q = (a_2, b_2)$, then the components of the vector $\boldsymbol{v} = \overrightarrow{PQ}$ are conveniently computed as the difference

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle a_2, b_2 \rangle - \langle a_1, b_1 \rangle = \langle a_2 - a_1, b_2 - b_1 \rangle.$$

Example

- For $\boldsymbol{v} = \langle 1, 4 \rangle$, $\boldsymbol{w} = \langle 3, 2 \rangle$, calculate

$$(a)\ \boldsymbol{v} + \boldsymbol{w} \qquad (b)\ 5\boldsymbol{v}$$

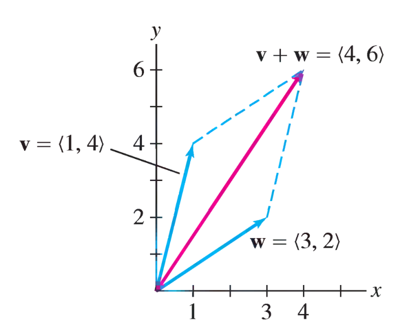
and sketch $\boldsymbol{v}$, $\boldsymbol{w}$ and their sum.

(i)

$$\begin{aligned} \boldsymbol{v} + \boldsymbol{w} &= \langle 1, 4 \rangle + \langle 3, 2 \rangle \\ &= \langle 1 + 3, 4 + 2 \rangle \\ &= \langle 4, 6 \rangle. \end{aligned}$$

(ii)

$$5\boldsymbol{v} = 5\langle 1, 4 \rangle = \langle 5, 20 \rangle.$$



Basic Properties of Vector Algebra

$\lambda \in \mathbb{R}$

- For all vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ and for all scalars λ,
  - $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$; (**Commutative Law**)
  - $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$; (**Associative Law**)
  - $\lambda(\boldsymbol{v} + \boldsymbol{w}) = \lambda\boldsymbol{v} + \lambda\boldsymbol{w}$. (**Distributive Law for Scalars**)
- These properties are easily checked using components:
  - $\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle = \langle c + a, d + b\rangle = \langle c, d\rangle + \langle a, b\rangle$;
  - $\langle a, b\rangle + (\langle c, d\rangle + \langle e, f\rangle) = \langle a, b\rangle + \langle c + e, d + f\rangle =$
    $\langle a + (c + e), b + (d + f)\rangle = \langle (a + c) + e, (b + d) + f\rangle =$
    $\langle a + c, b + d\rangle + \langle e, f\rangle = (\langle a, b\rangle + \langle c, d\rangle) + \langle e, f\rangle$.
  - $\lambda(\langle a, b\rangle + \langle c, d\rangle) = \lambda\langle a + c, b + d\rangle = \langle \lambda(a + c), \lambda(b + d)\rangle =$
    $\langle \lambda a + \lambda c, \lambda b + \lambda d\rangle = \langle \lambda a, \lambda b\rangle + \langle \lambda c, \lambda d\rangle = \lambda\langle a, b\rangle + \lambda\langle c, d\rangle$.

$\vec{a}$ $\vec{b}$, $\lambda \in \mathbb{R}$

$\mu \in \mathbb{R}$

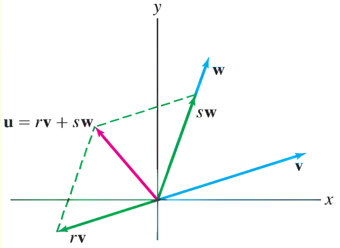
$\vec{c} = \lambda \vec{a} + \mu \vec{b}$

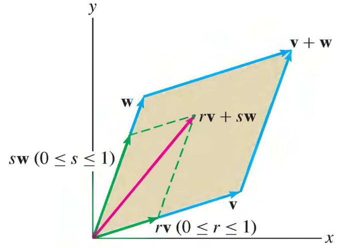
linear

comb

Linear Combinations

- A **linear combination** of vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is a vector $r\boldsymbol{v} + s\boldsymbol{w}$, where r and s are scalars.
- If $\boldsymbol{v}$ and $\boldsymbol{w}$ are not parallel, then every vector $\boldsymbol{u}$ in the plane can be expressed as a linear combination $\boldsymbol{u} = r\boldsymbol{v} + s\boldsymbol{w}$.
- The parallelogram $\mathcal{P}$ whose vertices are the origin and the terminal points of $\boldsymbol{v}$, $\boldsymbol{w}$ and $\boldsymbol{v} + \boldsymbol{w}$ is called the **parallelogram spanned** by $\boldsymbol{v}$ and $\boldsymbol{w}$. It consists of the linear combinations $r\boldsymbol{v} + s\boldsymbol{w}$, with $0 \leq r \leq 1$ and $0 \leq s \leq 1$.





Example

- Express the vector $\boldsymbol{u} = \langle 4, 4 \rangle$ as a linear combination of $\boldsymbol{v} = \langle 6, 2 \rangle$ and $\boldsymbol{w} = \langle 2, 4 \rangle$.

  We must find r and s, such that $r\boldsymbol{v} + s\boldsymbol{w} = \langle 4, 4 \rangle$. This gives $r\langle 6, 2 \rangle + s\langle 2, 4 \rangle = \langle 6r + 2s, 2r + 4s \rangle = \langle 4, 4 \rangle$. The components must be equal, so we have a system of two linear equations:

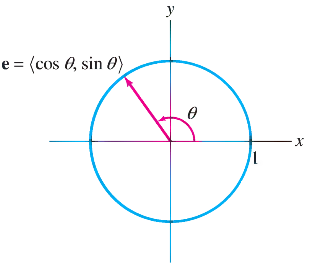
$$\left\{ \begin{array}{rcl} 6r + 2s & = & 4 \\ 2r + 4s & = & 4 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} 6r + 2s & = & 4 \\ -r - 2s & = & -2 \end{array} \right\}$$
$$\Rightarrow \left\{ \begin{array}{rcl} s & = & 2 - 3r \\ 5r & = & 2 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} s & = & 2 - 3 \cdot \frac{2}{5} = \frac{4}{5} \\ r & = & \frac{2}{5} \end{array} \right\}.$$

Therefore, $\boldsymbol{u} = \langle 4, 4 \rangle = \frac{2}{5}\langle 6, 2 \rangle + \frac{4}{5}\langle 2, 4 \rangle$.

Unit Vectors

- A vector of length 1 is called a **unit vector**.

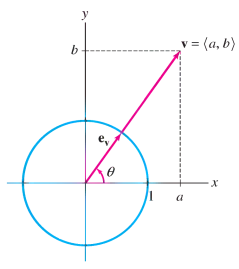
  The head of a unit vector $\boldsymbol{e}$ based at the origin lies on the unit circle and has components $\boldsymbol{e} = \langle \cos\theta, \sin\theta \rangle$, where θ is the angle between $\boldsymbol{e}$ and the positive x-axis.



- We can always scale a nonzero vector $\boldsymbol{v} = \langle a, b \rangle$ to obtain a unit vector pointing in the same direction: $\boldsymbol{e_v} = \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}$.

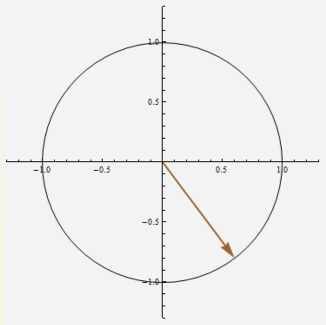
  If $\boldsymbol{v} = \langle a, b \rangle$ makes an angle θ with the positive x-axis, then

$$\boldsymbol{v} = \langle a, b \rangle = \|\boldsymbol{v}\|\boldsymbol{e_v} = \|\boldsymbol{v}\|\langle \cos\theta, \sin\theta \rangle.$$



Unit Vectors

- A **unit vector** is one whose magnitude is 1;
- Example: Verify that $\mathbf{v} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$ is a unit vector;



We have

$$\|\mathbf{v}\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{1} = 1;$$

Example

- Find the unit vector in the direction of $\boldsymbol{v} = \langle 3, 5 \rangle$.

  Compute the magnitude

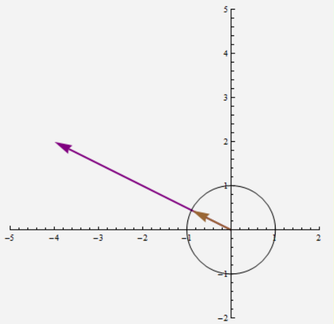
$$\|\boldsymbol{v}\| = \sqrt{3^2 + 5^2} = \sqrt{34}.$$

  Then, we get

$$\boldsymbol{e}_{\boldsymbol{v}} = \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v} = \frac{1}{\sqrt{34}}\langle 3, 5 \rangle = \langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \rangle.$$

Unit Vector in a Given Direction

- Example: Find a unit vector **u** in the direction of the vector $\mathbf{v} = \langle -4, 2\rangle$;



$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{(-4)^2 + 2^2}}\langle -4, 2\rangle = \langle -\frac{4}{\sqrt{20}}, \frac{2}{\sqrt{20}}\rangle = \langle -\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\rangle;$$

Standard Basis Vectors

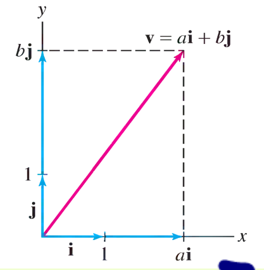
- We introduce a special notation for the unit vectors in the direction of the positive x- and y-axes:

$\vec{i}$

$$\boldsymbol{i} = \langle 1, 0 \rangle, \quad \boldsymbol{j} = \langle 0, 1 \rangle.$$

$\vec{j}$

The vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are called the **standard basis vectors**.



- Every vector in the plane is a linear combination of $\boldsymbol{i}$ and $\boldsymbol{j}$: ∃!

$$\boldsymbol{v} = \langle a, b \rangle = \langle a, 0 \rangle + \langle 0, b \rangle = a\langle 1, 0 \rangle + b\langle 0, 1 \rangle = a\boldsymbol{i} + b\boldsymbol{j}.$$

Example: $\langle 4, -2 \rangle = 4\boldsymbol{i} - 2\boldsymbol{j}$.

- Moreover vector addition is performed by adding the $\boldsymbol{i}$ and $\boldsymbol{j}$ coefficients:

$$(4\boldsymbol{i} - 2\boldsymbol{j}) + (5\boldsymbol{i} + 7\boldsymbol{j}) = (4+5)\boldsymbol{i} + (-2+7)\boldsymbol{j} = 9\boldsymbol{i} + 5\boldsymbol{j}.$$

Unit Vectors i and j

Definitions of Vectors i and j

$$\mathbf{i} = \langle 1, 0 \rangle, \qquad \mathbf{j} = \langle 0, 1 \rangle$$

- Example: Write the vector $\langle 3, 7 \rangle$ in terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$;

$$\langle 3, 7 \rangle = \langle 3, 0 \rangle + \langle 0, 7 \rangle = 3\langle 1, 0 \rangle + 7\langle 0, 1 \rangle = 3\mathbf{i} + 7\mathbf{j};$$

Representation of a Vector in Terms of i and j

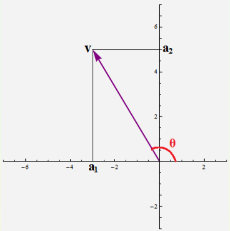
If $\mathbf{v}$ is a vector and $\mathbf{v} = \langle a_1, a_2 \rangle$, then $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j}$.

- Example: Given $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{w} = 5\mathbf{i} + 3\mathbf{j}$, find $3\mathbf{v} - 2\mathbf{w}$;

$$3\mathbf{v} - 2\mathbf{w} = 3(3\mathbf{i} - 4\mathbf{j}) - 2(5\mathbf{i} + 3\mathbf{j}) = (9\mathbf{i} - 12\mathbf{j}) - (10\mathbf{i} + 6\mathbf{j}) =$$
$$(9 - 10)\mathbf{i} + (-12 - 6)\mathbf{j} = -\mathbf{i} - 18\mathbf{j};$$

Horizontal and Vertical Components

- Consider the vector $\mathbf{v} = \langle a_1, a_2 \rangle$;



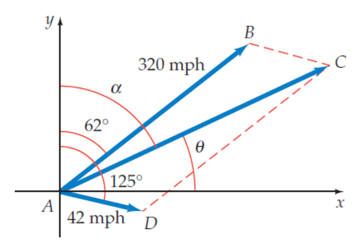
  - Its magnitude is $\|\mathbf{v}\| = \sqrt{a_1^2 + a_2^2}$;
  - Recall the definitions of sine and cosine of the angle θ with initial side the positive x-axis and terminal side the vector $\mathbf{v}$:
$$\cos\theta = \frac{a_1}{\|\mathbf{v}\|} \quad \text{and} \quad \sin\theta = \frac{a_2}{\|\mathbf{v}\|};$$
- Thus, we obtain $a_1 = \|\mathbf{v}\| \cos\theta$ and $a_2 = \|\mathbf{v}\| \sin\theta$;

Horizontal and Vertical Components of a Vector

If $\mathbf{v} = \langle a_1, a_2 \rangle$, with $\mathbf{v} \neq \mathbf{0}$, then
$$a_1 = \|\mathbf{v}\| \cos\theta \quad \text{and} \quad a_2 = \|\mathbf{v}\| \sin\theta;$$
The **horizontal component** of $\mathbf{v}$ is $\|\mathbf{v}\| \cos\theta$ and the **vertical component** is $\|\mathbf{v}\| \sin\theta$.

Application: Air Speed

An airplane is traveling with an airspeed of 320 mph and a heading of 62°; A wind of 42 mph is blowing at a heading of 125°; Find the ground speed and the course of the airplane;



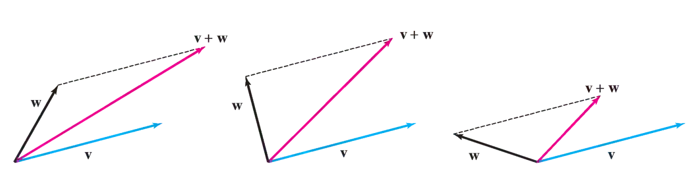
$$\mathbf{AB} = 320\cos 28^\circ \mathbf{i} + 320\sin 28^\circ \mathbf{j};$$
$$\mathbf{AD} = 42\cos(-35^\circ)\mathbf{i} + 42\sin(-35^\circ)\mathbf{j};$$
$$\mathbf{AC} = [320\cos 28^\circ + 42\cos(-35^\circ)]\mathbf{i} + [320\sin 28^\circ + 42\sin(-35^\circ)]\mathbf{j} \approx (282.5 + 34.4)\mathbf{i} + (150.2 - 24.1)\mathbf{j} = 316.9\mathbf{i} + 126.1\mathbf{j};$$

Therefore,

$$\|\mathbf{AC}\| = \sqrt{(316.9)^2 + (126.1)^2} \approx 340;$$
$$\alpha = 90^\circ - \theta \approx 90^\circ - \tan^{-1}\frac{126.1}{316.9} \approx 68^\circ;$$

Triangle Inequality

- The vector sum $\boldsymbol{v} + \boldsymbol{w}$ for three different vectors $\boldsymbol{w}$ of the same length is shown below.
  Clearly, the length $\|\boldsymbol{v} + \boldsymbol{w}\|$ varies, depending on the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.
  So in general, $\|\boldsymbol{v} + \boldsymbol{w}\|$ is not equal to the sum $\|\boldsymbol{v}\| + \|\boldsymbol{w}\|$.



- **Triangle Inequality:** For any two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$,

$$\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|.$$

Equality holds only if $\boldsymbol{v} = \mathbf{0}$ or $\boldsymbol{w} = \mathbf{0}$, or if $\boldsymbol{w} = \lambda \boldsymbol{v}$, where $\lambda > 0$.