Mathematical analysis I

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Subsection 5

The Gradient and Directional Derivatives

The Gradient Vector

- The **gradient of a function** $f(x, y)$ **at a point** $P = (a, b)$ is the vector

$$\text{grad } f\,(a,b) = \nabla f_P = \langle f_x(a, b), f_y(a, b) \rangle.$$

∇ nabla or del is a vector differential operator

In three variables, if $P = (a, b, c)$,

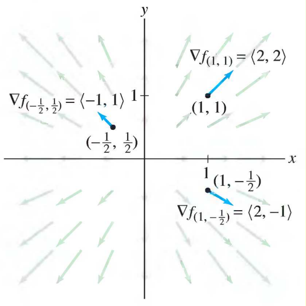
$$\nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y}$$

$$\nabla f_P = \langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c) \rangle.$$

- We also write $\nabla f_{(a,b)}$ or $\nabla f(a, b)$ for the gradient. Sometimes, we omit reference to the point P and write

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle.$$

The gradient ∇f assigns a vector ∇f_P to each point in the domain of f.



Examples

Ex.

- Let $f(x,y) = x^2 + y^2$. Calculate the gradient ∇f and compute ∇f_P at $P = (1,1)$.

  The partial derivatives are $f_x(x,y) = 2x$ and $f_y(x,y) = 2y$. So $\nabla f = \langle 2x, 2y \rangle$. At $(1,1)$, $\nabla f_P = \nabla f(1,1) = \langle 2, 2 \rangle$.

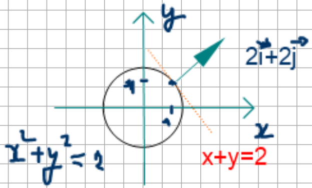
- If $f(x,y) = \sin x + e^{xy}$, compute ∇f.

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \langle \cos x + ye^{xy}, xe^{xy} \rangle.$$

- Calculate $\nabla f_{(3,-2,4)}$, where $f(x,y,z) = ze^{2x+3y}$.

  The partial derivatives and the gradient are $\frac{\partial f}{\partial x} = 2ze^{2x+3y}$, $\frac{\partial f}{\partial y} = 3ze^{2x+3y}$, $\frac{\partial f}{\partial z} = e^{2x+3y}$. So $\nabla f = \langle 2ze^{2x+3y}, 3ze^{2x+3y}, e^{2x+3y} \rangle$. Finally, $\nabla f_{(3,-2,4)} = \langle 8, 12, 1 \rangle$.

Ex.



Properties of the Gradient Vector

- If $f(x, y, z)$ and $g(x, y, z)$ are differentiable and c is a constant, then:
  (i) $\nabla(f+g) = \nabla f + \nabla g$ (**Sum Rule**)
  (ii) $\nabla(cf) = c\nabla f$ (**Constant Multiple Rule**)
  (iii) $\nabla(fg) = f\nabla g + g\nabla f$ (**Product Rule**)
  (iv) If $F(t)$ is a differentiable function of one variable, then

$$\nabla(F(f(x, y, z))) = F'(f(x, y, z))\nabla f \quad (\textbf{Chain Rule}).$$

Using the Chain Rule

- Find the gradient of

$$g(x, y, z) = (x^2 + y^2 + z^2)^8.$$

The function g is a composite $g(x, y, z) = F(f(x, y, z))$, with:
  - $F(t) = t^8$;
  - $f(x, y, z) = x^2 + y^2 + z^2$.

Now we have

$$\begin{aligned} \nabla g &= \nabla((x^2 + y^2 + z^2)^8) \\ &= 8(x^2 + y^2 + z^2)^7 \nabla(x^2 + y^2 + z^2) \\ &= 8(x^2 + y^2 + z^2)^7 \langle 2x, 2y, 2z \rangle \\ &= 16(x^2 + y^2 + z^2)^7 \langle x, y, z \rangle. \end{aligned}$$

Chain Rule for Paths

- If $z = f(x, y)$ is a differentiable function of x and y, where $x = x(t)$ and $y = y(t)$ are differentiable functions of t, then $z = f(x(t), y(t))$ is a differentiable function of t and
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla f \cdot \langle x'(t), y'(t)\rangle.$$
- Alternatve formulation: If $f(x, y)$ is a differentiable function of x and y and $\boldsymbol{c}(t) = \langle x(t), y(t)\rangle$ a differentiable function of t, then
$$\frac{d}{dt} f(\boldsymbol{c}(t)) = \nabla f_{\boldsymbol{c}(t)} \cdot \boldsymbol{c}'(t)$$
also written
$$\frac{d}{dt} f(\boldsymbol{c}(t)) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle x'(t), y'(t)\rangle.$$

Applying The Chain Rule for Paths

- Suppose that $f(x,y) = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$. Compute $\frac{dz}{dt}$ at $t = 0$.
  We have

$$\frac{\partial f}{\partial x} = 2xy + 3y^4, \ \frac{\partial f}{\partial y} = x^2 + 12xy^3, \ \frac{dx}{dt} = 2\cos 2t, \ \frac{dy}{dt} = -\sin t.$$

At $t = 0$, $x = \sin 0 = 0$, $y = \cos 0 = 1$, whence

$$\left.\frac{\partial f}{\partial x}\right|_{(0,1)} = 3, \quad \left.\frac{\partial f}{\partial y}\right|_{(0,1)} = 0, \quad \frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 0.$$

Since $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$, we get, $\left.\frac{dz}{dt}\right|_{t=0} = 3 \cdot 2 + 0 \cdot 0 = 6$.

Application

- The pressure P in kilopascals, the volume V in liters and the temperature T in kelvins of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/sec and the volume is 100 L and increasing at a rate of 0.2 L/sec.

  Note, first, that $P = \frac{8.31T}{V}$.

  Thus, we have

  $$\frac{\partial P}{\partial T} = \frac{8.31}{V}, \ \frac{\partial P}{\partial V} = -\frac{8.31T}{V^2}, \ \frac{dT}{dt} = 0.1, \ \frac{dV}{dt} = 0.2.$$

  Moreover, since $T = 300$ and $V = 100$,

  $$\frac{\partial P}{\partial T} = \frac{8.31}{100}, \quad \frac{\partial P}{\partial V} = -\frac{8.31 \cdot 300}{100^2}.$$

  Therefore, $\frac{dP}{dt} = \frac{8.31}{100} \cdot 0.1 + (-\frac{8.31 \cdot 300}{100^2}) \cdot 0.2$ kPa/sec.

The Chain Rule for Paths in Three Variables

- In general, if $f(x_1,\dots,x_n)$ is a differentiable function of n variables and $\boldsymbol{c}(t)=\langle x_1(t),\dots,x_n(t)\rangle$ is a differentiable path, then

$$\frac{d}{dt}f(\boldsymbol{c}(t))=\nabla f\cdot\boldsymbol{c}'(t)=\frac{\partial f}{\partial x_1}\frac{dx_1}{dt}+\frac{\partial f}{\partial x_2}\frac{dx_2}{dt}+\cdots+\frac{\partial f}{\partial x_n}\frac{dx_n}{dt}.$$

Example: Calculate $\frac{d}{dt}f(\boldsymbol{c}(t))\Big|_{t=\pi/2}$, where $f(x,y,z)=xy+z^2$ and $\boldsymbol{c}(t)=\langle\cos t,\sin t,t\rangle$.
We have $\boldsymbol{c}(\frac{\pi}{2})=\langle\cos\frac{\pi}{2},\sin\frac{\pi}{2},\frac{\pi}{2}\rangle=\langle 0,1,\frac{\pi}{2}\rangle$.
Compute the gradient: $\nabla f=\langle y,x,2z\rangle$ and $\nabla f_{\boldsymbol{c}(0,1,\frac{\pi}{2})}=\langle 1,0,\pi\rangle$.
Then compute the tangent vector:

$$\boldsymbol{c}'(t)=\langle-\sin t,\cos t,1\rangle,\quad \boldsymbol{c}'(\frac{\pi}{2})=\langle-1,0,1\rangle.$$

By the Chain Rule,

$$\frac{d}{dt}(f(\boldsymbol{c}(t))\Big|_{t=\pi/2}=\nabla f_{\boldsymbol{c}(\frac{\pi}{2})}\cdot\boldsymbol{c}'(\frac{\pi}{2})=\langle 1,0,\pi\rangle\cdot\langle-1,0,1\rangle=\pi-1.$$

Application

- The temperature at (x, y) is $T(x,y) = 20 + 10e^{-0.3(x^2+y^2)}$ °C. A bug carries a tiny thermometer along the path $\boldsymbol{c}(t) = \langle \cos(t-2), \sin 2t \rangle$ (t in seconds). How fast is the temperature changing at time t?

$$\begin{aligned} \tfrac{dT}{dt} &= \nabla T_{\boldsymbol{c}(t)} \cdot \boldsymbol{c}'(t); \\ \nabla T_{\boldsymbol{c}(t)} &= \langle -6xe^{-0.3(x^2+y^2)}, -6ye^{-0.3(x^2+y^2)} \rangle_{\boldsymbol{c}(t)} \\ &= \langle -6\cos(t-2)e^{-0.3(\cos^2(t-2)+\sin^2(2t))}, \\ &\qquad -6\sin(2t)e^{-0.3(\cos^2(t-2)+\sin^2(2t))} \rangle; \\ \boldsymbol{c}'(t) &= \langle -\sin(t-2), 2\cos(2t) \rangle. \end{aligned}$$

So, we get

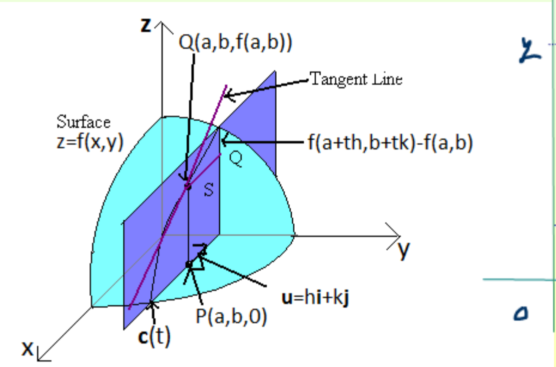
$$\begin{aligned} \tfrac{dT}{dt} &= 6\sin(t-2)\cos(t-2)e^{-0.3(\cos^2(t-2)+\sin^2(2t))} \\ &\qquad - 12\sin(2t)\cos(2t)e^{-0.3(\cos^2(t-2)+\sin^2(2t))}. \end{aligned}$$

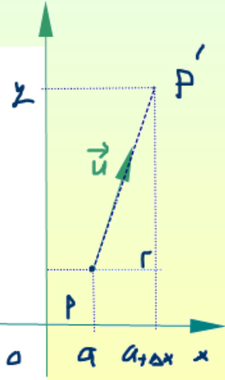
Directional Derivatives

Stewart, 14.6, p.933

- The **directional derivative of** f at $P = (a, b)$ **in the direction** of a unit vector $\mathbf{u} = \langle h, k \rangle$ is

$$D_{\mathbf{u}} f(a, b) = \lim_{t \to 0} \frac{f(a + th, b + tk) - f(a, b)}{t}.$$





Computing Directional Derivatives Using Partials

Theorem

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle h, k\rangle$ and

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)h + f_y(x,y)k = \nabla f \cdot \boldsymbol{u}.$$

Example: What is the directional derivative $D_{\mathbf{u}} f(x,y)$ of $f(x,y) = x^3 - 3xy + 4y^2$ in the direction of the unit vector with angle $\theta = \frac{\pi}{6}$? What is $D_{\mathbf{u}} f(1,2)$?
The unit vector $\mathbf{u}$ with direction $\theta = \frac{\pi}{6}$ is
$\mathbf{u} = \langle h, k\rangle = \langle 1\cos\frac{\pi}{6}, 1\sin\frac{\pi}{6}\rangle = \langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle$. Moreover, we have $\frac{\partial f}{\partial x} = 3x^2 - 3y$ and $\frac{\partial f}{\partial y} = -3x + 8y$. Therefore,

$$D_{\mathbf{u}} f(x,y) = \frac{\partial f}{\partial x}h + \frac{\partial f}{\partial y}k = \frac{\sqrt{3}}{2}(3x^2 - 3y) + \frac{1}{2}(-3x + 8y).$$

In particular, for $(x,y) = (1,2)$, $D_{\mathbf{u}}(1,2) = -\frac{3\sqrt{3}}{2} + \frac{13}{2}$.

Graphical Illustration

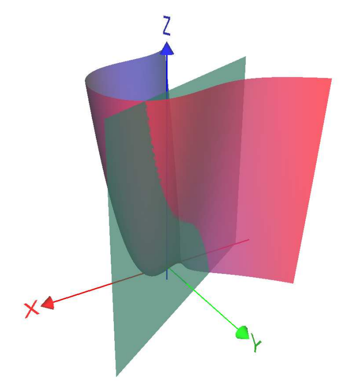
- The graph of the function $f(x, y) = x^3 - 3xy + 4y^2$.

  The plane passing through $(1, 2, 11)$, with direction $\mathbf{u} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$.

  The directional derivative

  $$D_{\mathbf{u}}(1, 2) = -\frac{3\sqrt{3}}{2} + \frac{13}{2}$$

  is the slope of the tangent to the curve of intersection of the surface $z = f(x, y)$ with the plane at $(1, 2, 11)$.



Directional Derivatives Generalized

- To evaluate directional derivatives, it is convenient to define $D_{\boldsymbol{v}} f(a,b)$ even when $\boldsymbol{v} = \langle h, k \rangle$ is not a unit vector:

$$D_{\boldsymbol{v}} f(a,b) = \lim_{t \to 0} \frac{f(a+th, b+tk) - f(a,b)}{t}.$$

  We call $D_{\boldsymbol{v}} f$ the **derivative with respect to $\boldsymbol{v}$**.

- We have

$$D_{\boldsymbol{v}} f(a,b) = \nabla f(a,b) \cdot \boldsymbol{v}.$$

- It $\boldsymbol{v} \neq \mathbf{0}$, then $\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$ is the unit vector in the direction of $\boldsymbol{v}$, and the directional derivative is given by

$$D_{\boldsymbol{u}} f(P) = \frac{1}{\|\boldsymbol{v}\|} \nabla f_P \cdot \boldsymbol{v}.$$

Example

- Let $f(x,y) = xe^y$, $P = (2,-1)$ and $\boldsymbol{v} = \langle 2, 3\rangle$.
  (a) Calculate $D_{\boldsymbol{v}} f(P)$.
  (b) Then calculate the directional derivative in the direction of $\boldsymbol{v}$.

(a) First compute the gradient at $P = (2,-1)$:

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle = \langle e^y, xe^y\rangle \quad \Rightarrow \quad \nabla f_P = \nabla f_{(2,-1)} = \langle \frac{1}{e}, \frac{2}{e}\rangle.$$

Now we get

$$D_{\boldsymbol{v}} f_P = \nabla f_P \cdot \boldsymbol{v} = \langle \frac{1}{e}, \frac{2}{e}\rangle \cdot \langle 2, 3\rangle = \frac{8}{e}.$$

(b) The directional derivative is $D_{\boldsymbol{u}} f(P)$, where $\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$.
We get

$$D_{\boldsymbol{u}} f(P) = \frac{1}{\|\boldsymbol{v}\|} D_{\boldsymbol{v}} f(P) = \frac{8/e}{\sqrt{2^2+3^2}} = \frac{8}{\sqrt{13}e}.$$

Applying $D_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u}$ Directly

- Find the directional derivative of $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\boldsymbol{v} = 2\boldsymbol{i} + 5\boldsymbol{j}$.

  For the gradient vector, we have $\nabla f(x, y) = \langle 2xy^3, 3x^2y^2 - 4\rangle$ and, hence, $\nabla f(2, -1) = \langle -4, 8\rangle$.

  The unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{v} = \langle 2, 5\rangle$ is $\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\rangle$.

  Therefore, the directional derivative $D_{\boldsymbol{u}} f(2, -1)$ of f in the direction of $\boldsymbol{u}$ is

$$D_{\boldsymbol{u}} f(2, -1) = \nabla f(2, -1) \cdot \boldsymbol{u} = \langle -4, 8\rangle \cdot \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\rangle = \frac{32}{\sqrt{29}}.$$

Applying $D_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u}$ in Three Variables

- If $f(x, y, z) = x \sin yz$, find ∇f and the directional derivative of f at $(1, 3, 0)$ in the direction of $\boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$.

  For the gradient vector, we have
  $\nabla f(x, y, z) = \langle \sin yz, xz \cos yz, xy \cos yz \rangle$ and, hence,
  $\nabla f(1, 3, 0) = \langle 0, 0, 3 \rangle$.

  The unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{v} = \langle 1, 2, -1 \rangle$ is
  $\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \rangle$.

  Therefore, the directional derivative $D_{\boldsymbol{u}} f(1, 3, 0)$ of f in the direction of $\boldsymbol{u}$ is

  $$D_{\boldsymbol{u}} f(1, 3, 0) = \nabla f(1, 3, 0) \cdot \boldsymbol{u} = \langle 0, 0, 3 \rangle \cdot \langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \rangle = -\frac{3}{\sqrt{6}}.$$

Maximum Directional Derivative

Theorem

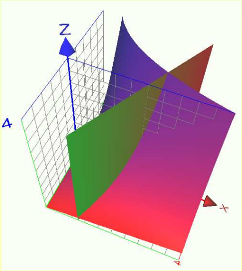
If f is a differentiable function of two or three variables, the maximum value of $D_{\boldsymbol{u}}f(\mathbf{x})$ is $\|\nabla f(x,y)\|$ and it occurs when $\boldsymbol{u}$ has the same direction as the gradient vector $\nabla f(x,y)$.

Example: Suppose that $f(x,y) = xe^y$. Find the rate of change of f at $P = (2,0)$ in the direction from P to $Q = (\frac{1}{2}, 2)$.

We have $\nabla f(x,y) = \langle e^y, xe^y\rangle$, whence $\nabla f(2,0) = \langle 1,2\rangle$. Moreover, $\overrightarrow{PQ} = \langle -\frac{3}{2}, 2\rangle$, whence the unit vector in the direction of $\overrightarrow{PQ}$ is

$\boldsymbol{u} = \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \langle -\frac{3}{5}, \frac{4}{5}\rangle$. Therefore, we get $D_{\boldsymbol{u}}f(2,0) = \langle 1,2\rangle \cdot \langle -\frac{3}{5}, \frac{4}{5}\rangle = 1$.

According to the Theorem, the max change occurs in the direction of $\nabla f(2,0) = \langle 1,2\rangle$ and equals $\|\nabla f(2,0)\| = \sqrt{5}$.



Example

- Let $f(x,y) = \frac{x^4}{y^2}$ and $P = (2,1)$. Find the unit vector that points in the direction of maximum rate of increase at P.

  The gradient at P points in the direction of maximum rate of increase:

$$\nabla f = \langle \frac{4x^3}{y^2}, -\frac{2x^4}{y^3} \rangle \quad \Rightarrow \quad \nabla f_{(2,1)} = \langle 32, -32 \rangle.$$

  The unit vector in this direction is

$$\boldsymbol{u} = \frac{\langle 32, -32 \rangle}{\|\langle 32, -32 \rangle\|} = \frac{\langle 32, -32 \rangle}{32\sqrt{2}} = \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle.$$

Application

- If the temperature at a point (x, y, z) is given by $T(x, y, z) = \frac{80}{1+x^2+2y^2+3z^2}$ in degrees Celsius, where x, y, z are in meters, in which direction does the temperature increase the fastest at $(1, 1, -2)$ and what is the maximum rate of increase?

  We have that $\nabla T(x, y, z) = \langle -\frac{160x}{(1+x^2+2y^2+3z^2)^2}, -\frac{320y}{(1+x^2+2y^2+3z^2)^2}, -\frac{480z}{(1+x^2+2y^2+3z^2)^2} \rangle$.

  Thus, $\nabla T(1, 1, -2) = \langle -\frac{5}{8}, -\frac{5}{4}, \frac{15}{4} \rangle$.

  Therefore, the temperature increases the fastest in the direction of the vector $\nabla T(1, 1, -2) = \langle -\frac{5}{8}, -\frac{5}{4}, \frac{15}{4} \rangle$ and the fastest rate of increase is

$$\|\nabla T(1, 1, -2)\| = \sqrt{\frac{25}{64} + \frac{25}{16} + \frac{225}{16}} = \frac{\sqrt{25 + 100 + 900}}{4} = \frac{5\sqrt{41}}{8}.$$

Gradient Vectors and Level Surfaces

- Consider a surface $\mathcal{S}$, with equation $F(x, y, z) = k$.

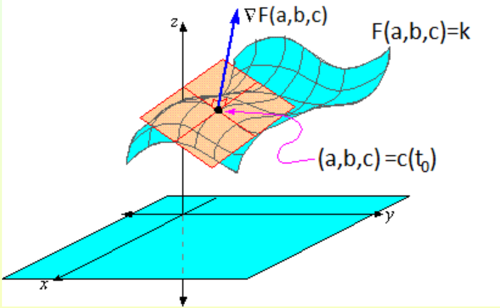
  Let $\mathcal{C}$ be a curve $\boldsymbol{c}(t) = \langle x(t), y(t), z(t) \rangle$ on the surface $\mathcal{S}$, passing through a point $\boldsymbol{c}(t_0) = \langle a, b, c \rangle$ on $\mathcal{C}$.

  Recall that

  $$\left.\frac{dF}{dt}\right|_{t=t_0} = \nabla F_{\boldsymbol{c}(t_0)} \cdot \boldsymbol{c}'(t_0).$$

  Hence, we get

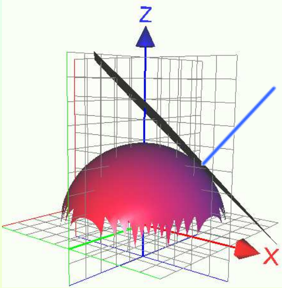
  $$\nabla F_{\boldsymbol{c}(t_0)} \cdot \boldsymbol{c}'(t_0) = 0.$$



Therefore, $\nabla F_{\boldsymbol{c}(t_0)}$ is perpendicular to the tangent vector $\boldsymbol{c}'(t_0)$ to any curve $\mathcal{C}$ on $\mathcal{S}$ passing through $\boldsymbol{c}(t_0)$.

Tangent Plane to a Level Surface

- We define the **tangent plane to the level surface** $F(x, y, z) = k$ **at** $P = (a, b, c)$ as the plane passing through P, with normal vector $\nabla F(a, b, c)$.



  This plane has equation

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0.$$

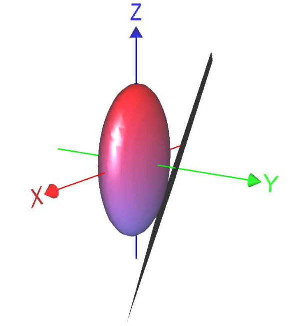
- Moreover, the **normal line** to $\mathcal{S}$ at P that passes through P and is perpendicular to the tangent plane has parametric equations

$$x = a + tF_x(a, b, c), \ \ y = b + tF_y(a, b, c), \ \ z = c + tF_z(a, b, c).$$

Finding a Tangent Plane and a Normal Line

- Let us find the equations of the tangent plane and of the normal line at $P = (-2, 1, -3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$;
  We consider $F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$.
  We have $F_x(x, y, z) = \frac{1}{2}x$, $F_y(x, y, z) = 2y$, $F_z(x, y, z) = \frac{2}{9}z$.
  So, $F_x(-2, 1, -3) = -1$, $F_y(-2, 1, -3) = 2$ and
  $F_z(-2, 1, -3) = -\frac{2}{3}$.
  Therefore, the equation of the tangent plane is $-(x+2)+2(y-1)-\frac{2}{3}(z+3) = 0$, i.e., $3x - 6y + 2z + 18 = 0$,
  and the parametric equations of the normal line are

$$\left\{ \begin{array}{lcl} x & = & -2 - t \\ y & = & 1 + 2t \\ z & = & -3 - \frac{2}{3}t \end{array} \right\}.$$



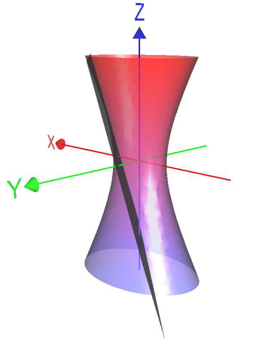
Finding a Normal Vector and a Tangent Plane

- Find an equation of the tangent plane to the surface $4x^2 + 9y^2 - z^2 = 16$ at $P = (2, 1, 3)$.

  Let $F(x, y, z) = 4x^2 + 9y^2 - z^2$. Then $\nabla F = \langle 8x, 18y, -2z \rangle$ and

  $$\nabla F_P = \nabla F(2, 1, 3) = \langle 16, 18, -6 \rangle.$$

  The vector $\langle 16, 18, -6 \rangle$ is normal to the surface $F(x, y, z) = 16$.



  So the tangent plane at P has equation

$$16(x - 2) + 18(y - 1) - 6(z - 3) = 0 \quad \text{or} \quad 16x + 18y - 6z = 32.$$

Subsection 6

The Chain Rule

The Chain Rule

- If $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t, then

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}, \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}.$$

Example: If $f(x, y) = e^x \sin y$, $x = st^2$, $y = s^2 t$, what are $\frac{\partial f}{\partial s}$, $\frac{\partial f}{\partial t}$?
We have

$$\frac{\partial f}{\partial x} = e^x \sin y, \quad \frac{\partial f}{\partial y} = e^x \cos y.$$

We also have

$$\frac{\partial x}{\partial s} = t^2, \quad \frac{\partial x}{\partial t} = 2st, \quad \frac{\partial y}{\partial s} = 2st, \quad \frac{\partial y}{\partial t} = s^2.$$

Therefore,

$$\frac{\partial f}{\partial s} = e^x \sin y \cdot t^2 + e^x \cos y \cdot 2st, \ \frac{\partial f}{\partial t} = e^x \sin y \cdot 2st + e^x \cos y \cdot s^2.$$

The Chain Rule: General Version

- If f is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$, then f is a differentiable function of $t_1, \ldots, t_m$ and, for all $i = 1, \ldots, m$,

$$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}.$$

  This may be expressed using the dot product:

$$\frac{\partial f}{\partial t_i} = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle \cdot \left\langle \frac{\partial x_1}{\partial t_i}, \frac{\partial x_2}{\partial t_i}, \ldots, \frac{\partial x_n}{\partial t_i} \right\rangle.$$

Using the Chain Rule

- Let $f(x, y, z) = xy + z$. Calculate $\frac{\partial f}{\partial s}$, where $x = s^2$, $y = st$, $z = t^2$. Compute the primary derivatives.

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial f}{\partial z} = 1.$$

Next, we get

$$\frac{\partial x}{\partial s} = 2s, \quad \frac{\partial y}{\partial s} = t, \quad \frac{\partial z}{\partial s} = 0.$$

Now apply the Chain Rule:

$$\begin{aligned} \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} \\ &= y \cdot 2s + x \cdot t + 1 \cdot 0 \\ &= (st) \cdot 2s + s^2 \cdot t = 3s^2 t. \end{aligned}$$

Evaluating the Derivative

- If $f = x^4y + y^2z^3$, $x = rse^t$, $y = rs^2e^{-t}$ and $z = r^2s\sin t$, find $\frac{\partial f}{\partial s}$ when $r = 2, s = 1$ and $t = 0$.
  Note, first, that for $(r, s, t) = (2, 1, 0)$, we have $(x, y, z) = (2, 2, 0)$.
  Moreover,

$$\frac{\partial f}{\partial x} = 4x^3y, \quad \frac{\partial f}{\partial y} = x^4 + 2yz^3, \quad \frac{\partial f}{\partial z} = 3y^2z^2.$$

Thus, for $(r, s, t) = (2, 1, 0)$, we get $\frac{\partial f}{\partial x} = 64, \quad \frac{\partial f}{\partial y} = 16, \quad \frac{\partial f}{\partial z} = 0$.
Furthermore,

$$\frac{\partial x}{\partial s} = re^t, \quad \frac{\partial y}{\partial s} = 2rse^{-t}, \quad \frac{\partial z}{\partial s} = r^2\sin t.$$

Thus, for $(r, s, t) = (2, 1, 0)$, we get $\frac{\partial x}{\partial s} = 2, \quad \frac{\partial y}{\partial s} = 4, \quad \frac{\partial z}{\partial s} = 0$.
Therefore, $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} = 64 \cdot 2 + 16 \cdot 4 + 0 \cdot 0 = 192$.

Polar Coordinates

- Let $f(x, y)$ be a function of two variables, and let (r, θ) be polar coordinates.
  (a) Express $\frac{\partial f}{\partial \theta}$ in terms of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
  (b) Evaluate $\frac{\partial f}{\partial \theta}$ at $(x, y) = (1, 1)$ for $f(x, y) = x^2 y$.

(a) Since $x = r\cos\theta$ and $y = r\sin\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$.
By the Chain Rule,

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}.$$

Since $x = r\cos\theta$ and $y = r\sin\theta$, we can write $\frac{\partial f}{\partial \theta}$ in terms of x and y alone: $\frac{\partial f}{\partial \theta} = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}$.

(b) Apply the preceding equation to $f(x, y) = x^2 y$:

$$\begin{aligned} \frac{\partial f}{\partial \theta} &= -y\frac{\partial}{\partial x}(x^2 y) + x\frac{\partial}{\partial y}(x^2 y) = -2xy^2 + x^3; \\ \left.\frac{\partial f}{\partial \theta}\right|_{(x,y)=(1,1)} &= -2 \cdot 1 \cdot 1^2 + 1^3 = -1. \end{aligned}$$

An Abstract Example on the Chain Rule

- If $g(s,t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the PDE $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$.

  Notice that $g(s,t) = f(x,y)$, where $x = s^2 - t^2$ and $y = t^2 - s^2$.
  Thus, by the chain rule, we get

$$\begin{aligned} \frac{\partial g}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ &= 2s\frac{\partial f}{\partial x} - 2s\frac{\partial f}{\partial y}; \end{aligned}$$

$$\begin{aligned} \frac{\partial g}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\ &= -2t\frac{\partial f}{\partial x} + 2t\frac{\partial f}{\partial y}. \end{aligned}$$

  Therefore,

$$\begin{aligned} t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} &= t(2s\frac{\partial f}{\partial x} - 2s\frac{\partial f}{\partial y}) + s(-2t\frac{\partial f}{\partial x} + 2t\frac{\partial f}{\partial y}) \\ &= 2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y} - 2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y} \\ &= 0. \end{aligned}$$

Implicit Differentiation: $y = y(x)$

- Suppose that the equation $F(x, y) = 0$ defines y implicitly as a function of x.

  By the chain rule $\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$, whence

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}.$$

  Example: Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$.

  We have $F(x, y) = x^3 + y^3 - 6xy = 0$, whence

$$\frac{\partial F}{\partial x} = 3x^2 - 6y, \quad \frac{\partial F}{\partial y} = 3y^2 - 6x.$$

  Therefore, $\displaystyle \frac{dy}{dx} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$.

Implicit Differentiation $z = z(x, y)$

- Suppose that the equation $F(x, y, z) = 0$ defines z implicitly as a function of x and y.
  By the chain rule $\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$.
  But, we also have $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$, whence $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$, giving

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}. \quad \text{Similarly} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.
We have $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1 = 0$, whence

$$\frac{\partial F}{\partial x} = 3x^2 + 6yz, \quad \frac{\partial F}{\partial y} = 3y^2 + 6xz, \quad \frac{\partial F}{\partial z} = 3z^2 + 6xy.$$

Therefore, $\frac{\partial z}{\partial x} = -\frac{3x^2+6yz}{3z^2+6xy} = -\frac{x^2+2yz}{z^2+2xy}$;
$\frac{\partial z}{\partial y} = -\frac{3y^2+6xz}{3z^2+6xy} = -\frac{y^2+2xz}{z^2+2xy}$.