Mathematical analysis I

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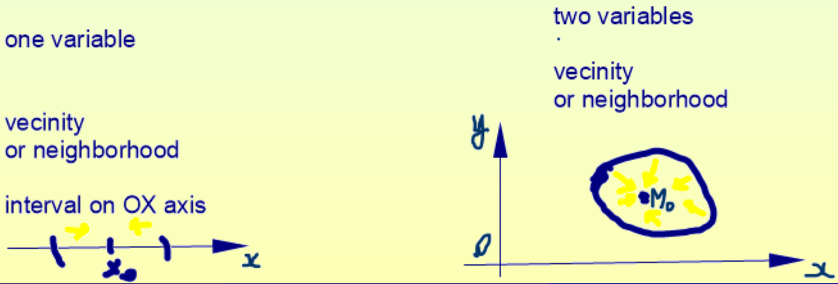
Subsection 2

Limits and Continuity in Several Variables



Limits

- Suppose f is a function of two variables whose domain $\mathcal{D}$ includes points arbitrarily close to the point (a, b).

  We say that the **limit of** $f(x, y)$ **as** (x, y) **approaches** (a, b) is L, written

  $$\lim_{(x,y)\to(a,b)} f(x, y) = L,$$

  (a,b)

  if the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within $\mathcal{D}$.

- The definition implies that, if
  - $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path $\mathcal{C}_1$ in $\mathcal{D}$,
  - $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path $\mathcal{C}_2$ in $\mathcal{D}$,
  - $L_1 \neq L_2$,

  then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Example of Non-Existence

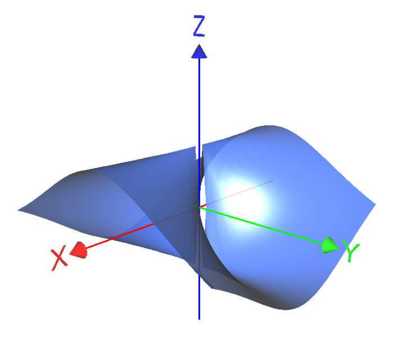
- Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

  If $(x, y) \to (0, 0)$ along the x-axis, then $y = 0$, whence

  $$\frac{x^2 - y^2}{x^2 + y^2} = \frac{x^2}{x^2} \to 1.$$

  If $(x, y) \to (0, 0)$ along the y-axis, then $x = 0$, whence

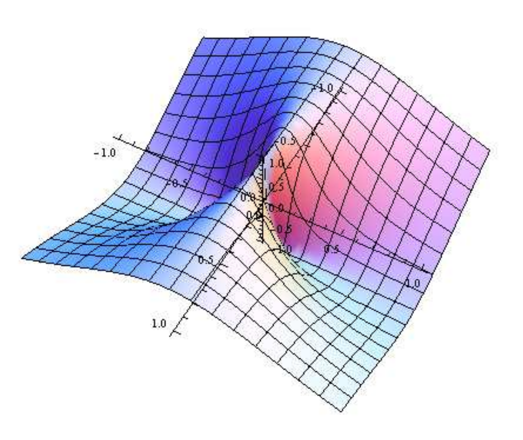
  $$\frac{x^2 - y^2}{x^2 + y^2} = \frac{-y^2}{y^2} \to -1.$$



  Since f approaches two different values along two different paths, the limit $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

Example of Non-Existence (Another Point of View)

$$f(x) = \frac{x^2 - y^2}{x^2 + y^2}$$



Another Example of Non-Existence

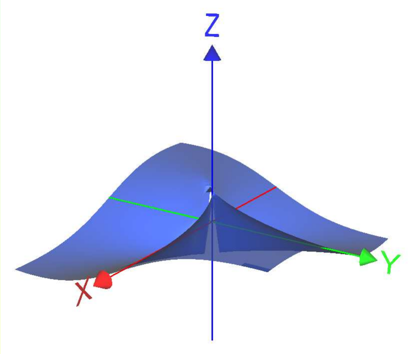
- Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.

If $(x, y) \to (0, 0)$ along the x-axis, then $y = 0$, whence

$$\frac{xy}{x^2+y^2} = \frac{x \cdot 0}{x^2+0} \to 0.$$

If $(x, y) \to (0, 0)$ along the line $y = x$, then

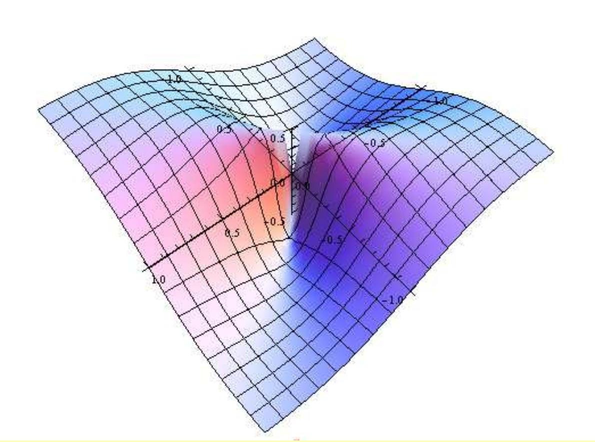
$$\frac{xy}{x^2+y^2} = \frac{x^2}{x^2+x^2} \to \frac{1}{2}.$$



Since f approaches two different values along two different paths, the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist;

Another Example of Non-Existence (Second Point of View)

$$f(x) = \frac{xy}{x^2 + y^2}.$$



A More Difficult Example of Non-Existence

- Show that $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ does not exist.

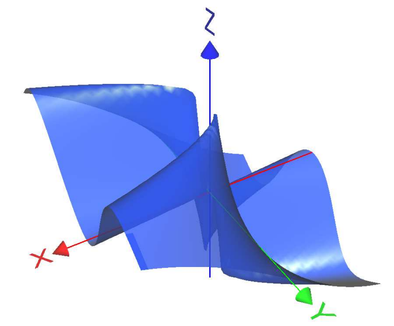
  If $(x,y)\to(0,0)$ along any line $y = mx$ through the origin,

$$\frac{xy^2}{x^2+y^4} = \frac{xm^2x^2}{x^2+m^4x^4} = \frac{m^2x}{1+m^4x^2} \to 0.$$

  If $(x,y)\to(0,0)$ along the parabola $x = y^2$, then

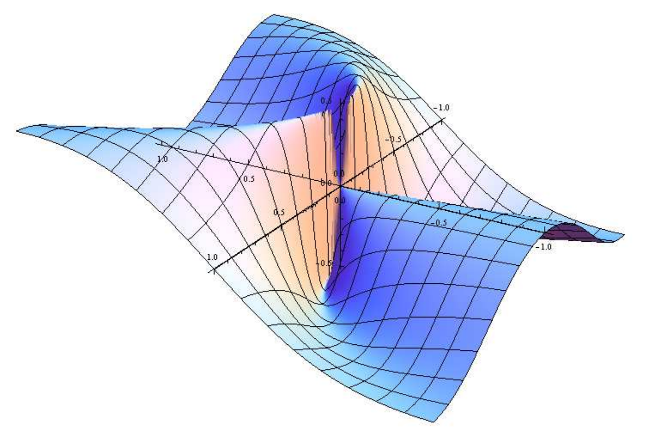
$$\frac{xy^2}{x^2+y^4} = \frac{y^2y^2}{y^4+y^4} = \frac{y^4}{2y^4} \to \frac{1}{2}.$$

  Since f approaches two different values along two different paths, $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ does not exist.



More Difficult Example (Second Point of View)

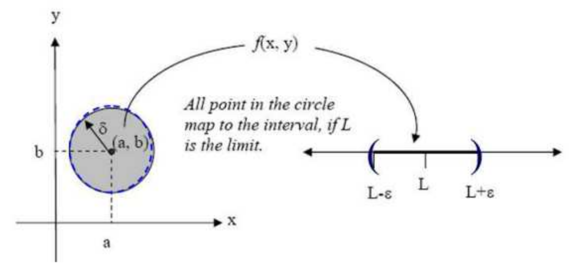
$$f(x) = \frac{xy^2}{x^2 + y^4}$$



Formal Definition of Limit

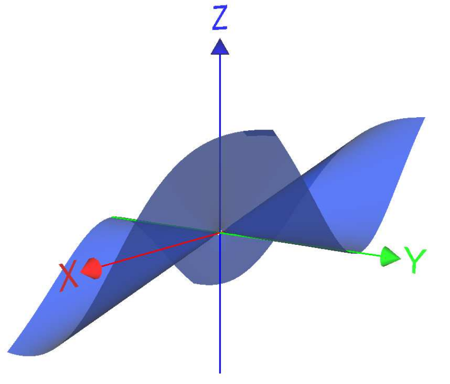
- Let f be a function of two variables whose domain $\mathcal{D}$ includes points arbitrarily close to (a, b).
  The **limit of** $f(x, y)$ **as** (x, y) **approaches** (a, b) is L, written $\lim_{(x,y)\to(a,b)} f(x, y) = L$, if for every number $\epsilon > 0$, there exists a number $\delta > 0$, such that

  if $(x, y) \in \mathcal{D}$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$.



Showing Existence of Limits

- Because there are many paths a point may follow to approach a fixed point, showing that a limit exists is rather difficult.
- We show formally that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$;



The Limit of the Function $f(x,y) = \frac{3x^2y}{x^2+y^2}$

- Assume that the distance from $(x,y) \neq (0,0)$ to $(0,0)$ is less than δ, i.e., $0 < \sqrt{x^2+y^2} < \delta$. Since $\dfrac{x^2}{x^2+y^2} \leq \dfrac{x^2}{x^2} = 1$, we obtain
$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| = \frac{3x^2|y|}{x^2+y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2+y^2}.$$
Thus, we have that the distance of $f(x,y)$ from 0 is
$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| \leq 3\sqrt{x^2+y^2} < 3\delta.$$
This shows that we can make $|f(x,y) - 0| < \epsilon$ (i.e., arbitrarily small) by taking $0 < \sqrt{x^2+y^2} < \delta = \frac{\epsilon}{3}$ (i.e., (x,y) sufficiently close to $(0,0)$) and verifies that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$.

Limit Laws

- Assume that $\lim_{(x,y)\to(a,b)} f(x,y)$ and $\lim_{(x,y)\to(a,b)} g(x,y)$ exist. Then:

  (i) **Sum Law:**

  $$\lim_{(x,y)\to(a,b)} (f(x,y)+g(x,y)) = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y).$$

  (ii) **Constant Multiple Law:** For any number k,

  $$\lim_{(x,y)\to(a,b)} kf(x,y) = k \lim_{(x,y)\to(a,b)} f(x,y).$$

  (iii) **Product Law:**

  $$\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = \left(\lim_{(x,y)\to(a,b)} f(x,y)\right)\left(\lim_{(x,y)\to(a,b)} g(x,y)\right).$$

  (iv) **Quotient Law:** If $\lim_{(x,y)\to(a,b)} g(x,y) \neq 0$, then

  $$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}.$$

Continuity

- A function f of two variables is called **continuous at** (a, b) if
$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$
- A function f is **continuous on** $\mathcal{D}$ if it is continuous at all (a, b) in $\mathcal{D}$.
Examples:
  - $f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$ is continuous on $\mathbb{R}^2$ because it is a polynomial.
  - $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ is continuous at all $(a, b) \neq (0, 0)$ as a rational function defined, for all $(a, b) \neq (0, 0)$. It is discontinuous at $(0, 0)$, since it is not defined at $(0, 0)$.
  - $f(x, y) = \begin{cases} \dfrac{3x^2y}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$ is continuous at all $(a, b) \neq (0, 0)$ as a rational function defined there. It is also continuous at $(a, b) = (0, 0)$, since $\lim_{(x,y)\to(0,0)} f(x, y) = 0 = f(0, 0)$.

Evaluating Limits by Substitution

- Show that $f(x,y) = \frac{3x+y}{x^2+y^2+1}$ is continuous.

  Then evaluate $\lim\limits_{(x,y)\to(1,2)} f(x,y)$.

  The function $f(x,y)$ is continuous at all points (a,b) because it is a rational function whose denominator $Q(x,y) = x^2+y^2+1$ is never zero.

  Therefore, we can evaluate the limit by substitution:

$$\lim_{(x,y)\to(1,2)} \frac{3x+y}{x^2+y^2+1} = f(1,2) = \frac{3\cdot 1+2}{1^2+2^2+1} = \frac{5}{6}.$$

Product Functions

- Evaluate $\lim_{(x,y)\to(3,0)} x^3 \frac{\sin y}{y}$.

  The limit is equal to a product of limits:

$$\begin{aligned}\lim_{(x,y)\to(3,0)} x^3 \frac{\sin y}{y} &= \left(\lim_{(x,y)\to(3,0)} x^3\right)\left(\lim_{(x,y)\to(3,0)} \frac{\sin y}{y}\right) \\ &= 3^3 \cdot 1 = 27.\end{aligned}$$

A Composite of Continuous Functions Is Continuous

- If
  - $f(x,y)$ is continuous at (a,b),
  - $G(u)$ is continuous at $c = f(a,b)$,

  then the composite function $G(f(x,y))$ is continuous at (a,b).

  Example: Write $H(x,y) = e^{-x^2+2y}$ as a composite function and evaluate $\lim\limits_{(x,y)\to(1,2)} H(x,y)$.

  We have $H(x,y) = G \circ f$, where
  - $G(u) = e^u$;
  - $f(x,y) = -x^2 + 2y$.

  Both f and G are continuous. So H is also continuous. This allows computing the limit as follows:

$$\lim_{(x,y)\to(1,2)} H(x,y) = \lim_{(x,y)\to(1,2)} e^{-x^2+2y} = e^{-(1)^2+2\cdot 2} = e^3.$$

Subsection 3

Partial Derivatives

Partial Derivative With Respect to x

- If f is a function of x and y, by keeping y constant, say $y = b$, we can consider a function of a single variable x:

$$g(x) = f(x, b).$$

- If g has a derivative at $x = a$, we call it the **partial derivative of** f **with respect to** x at (a, b) and denote it by $f_x(a, b)$.
- Thus, $f_x(a, b) = g'(a)$, where $g(x) = f(x, b)$.
- More formally, the **partial derivative** f_x of $f(x, y)$ is the function

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

- Sometimes we write $f_x(x, y) = \dfrac{\partial f}{\partial x} = D_1 f = D_x f$.

Partial Derivative With Respect to y

- If f is a function of x and y, by keeping x constant, say $x = a$, we can consider a function of a single variable y:

$$h(y) = f(a, y).$$

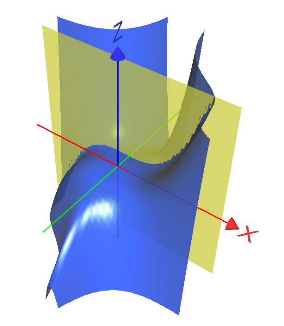
- If h has a derivative at $y = b$, we call it the **partial derivative of f with respect to y** at (a, b) and denote it by $f_y(a, b)$.
- Thus, $f_y(a, b) = h'(b)$, where $h(y) = f(a, y)$.
- More formally, the **partial derivative** f_y of $f(x, y)$ is the function

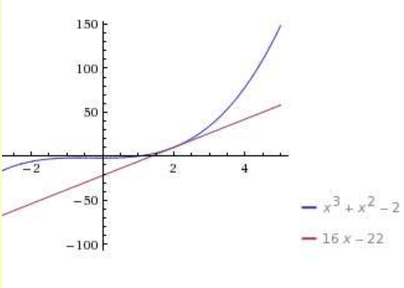
$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

- Sometimes we write $f_y(x, y) = \dfrac{\partial f}{\partial y} = D_2 f = D_y f$.

Computing the Partials

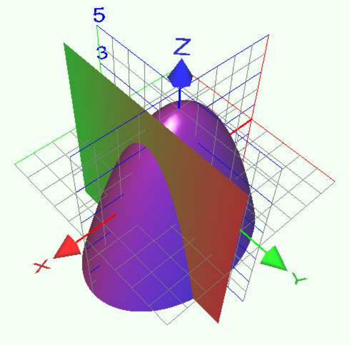
- To find f_x regard y as a constant and differentiate with respect to x.
  Example: If $f(x,y) = x^3 + x^2y^3 - 2y^2$, then $f_x(x,y) = 3x^2 + 2xy^3$ and $f_x(2,1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$.
- To find f_y regard x as a constant and differentiate with respect to y.
  Example: If $f(x,y) = x^3 + x^2y^3 - 2y^2$, then $f_y(x,y) = 3x^2y^2 - 4y$ and $f_y(2,1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$.

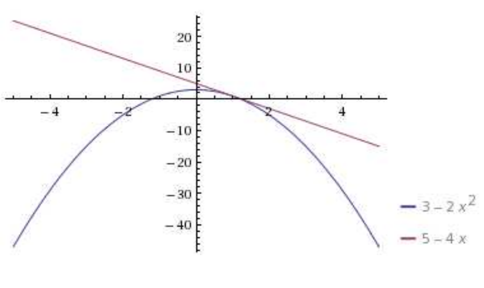




Another Example of Partials

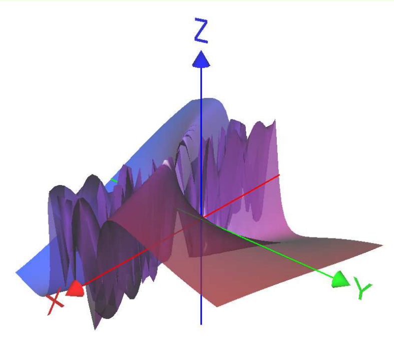
- Let $f(x,y) = 4 - x^2 - 2y^2$.
  Then $f_x(x,y) = -2x$ and $f_x(1,1) = -2$.
  Moreover, $f_y(x,y) = -4y$ and $f_y(1,1) = -4$.





A Third Example of Partials

- Let $f(x,y) = \sin\left(\frac{x}{1+y}\right)$.
  Then $\dfrac{\partial f}{\partial x} = \cos\left(\dfrac{x}{1+y}\right) \cdot \dfrac{\partial}{\partial x}\left(\dfrac{x}{1+y}\right) = \cos\left(\dfrac{x}{1+y}\right) \cdot \dfrac{1}{1+y}$ and
  $\dfrac{\partial f}{\partial y} = \cos\left(\dfrac{x}{1+y}\right) \cdot \dfrac{\partial}{\partial y}\left(\dfrac{x}{1+y}\right) = -\cos\left(\dfrac{x}{1+y}\right) \cdot \dfrac{x}{(1+y)^2}$.



$$z = x^y$$

$$z'_x = \left(x^y\right)'_x = y\,x^{y-1}$$

$$z'_y = \left(x^y\right)'_y = x^y \ln x$$

Implicit Partial Differentiation

Stewart, p.905, example 4

- Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x, y by

$$x^3 + y^3 + z^3 + 6xyz = 1.$$

Take partials with respect to x: $\dfrac{\partial}{\partial x}(x^3 + y^3 + z^3 + 6xyz) = \dfrac{\partial(1)}{\partial x}$.
Thus, we get $3x^2 + 3z^2\dfrac{\partial z}{\partial x} + 6y(z + x\dfrac{\partial z}{\partial x}) = 0$. To solve for $\frac{\partial z}{\partial x}$, we separate $(3z^2 + 6xy)\dfrac{\partial z}{\partial x} = -3x^2 - 6yz$ and, therefore,

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}.$$

- Do similar work for $\frac{\partial z}{\partial y}$.

Answer: $\dfrac{\partial z}{\partial y} = -\dfrac{y^2 + 2xz}{z^2 + 2xy}$.

Second Order Partial Derivatives

- For a function f of two variables x, y it is possible to consider four **second-order partial derivatives**:
  - $(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$
  - $(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$
  - $(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$
  - $(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$

  mixt partial derivatives

  Example: Calculate all four second order derivatives of $f(x, y) = x^3 + x^2y^3 - 2y^2$.
  - $f_x = \frac{\partial f}{\partial x} = 3x^2 + 2xy^3$ and $f_y = \frac{\partial f}{\partial y} = 3x^2y^2 - 4y$.
  - $f_{xx} = \frac{\partial^2 f}{\partial x^2} = 6x + 2y^3$ and $f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$.
  - $f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = 6xy^2$ and $f_{yy} = \frac{\partial^2 f}{\partial y^2} = 6x^2y - 4$.

  Note that $f_{xy} = f_{yx}$.

Clairaut's Theorem

Stewart, p.907

Clairaut's Theorem

If f is defined on a disk $\mathcal{D}$ containing the point (a, b) and the partial derivatives f_{xy} and f_{yx} are both continuous on $\mathcal{D}$, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Example: Show that, if $f(x, y) = x\sin(x + 2y)$, then $f_{xy} = f_{yx}$.
For the first-order partials, we have

$$f_x = \sin(x + 2y) + x\cos(x + 2y), \quad f_y = 2x\cos(x + 2y).$$

Therefore, we obtain

$$f_{xy} = 2\cos(x + 2y) - 2x\sin(x + 2y),$$

and

$$f_{yx} = 2\cos(x + 2y) - 2x\sin(x + 2y).$$

Verifying Clairaut's Theorem

- If $W(T,U) = e^{U/T}$, verify that $\frac{\partial^2 W}{\partial U \partial T} = \frac{\partial^2 W}{\partial T \partial U}$.

$$\begin{aligned}
\frac{\partial W}{\partial T} &= e^{U/T} \frac{\partial}{\partial T}\left(\frac{U}{T}\right) = -\frac{U}{T^2} e^{U/T}; \\
\frac{\partial W}{\partial U} &= e^{U/T} \frac{\partial}{\partial U}\left(\frac{U}{T}\right) = \frac{1}{T} e^{U/T}; \\
\frac{\partial^2 W}{\partial U \partial T} &= \frac{\partial}{\partial U}\left(-\frac{U}{T^2}\right) e^{U/T} + \left(-\frac{U}{T^2}\right) \frac{\partial}{\partial U}\left(e^{U/T}\right) \\
&= -\frac{1}{T^2} e^{U/T} - \frac{U}{T^3} e^{U/T}; \\
\frac{\partial^2 W}{\partial T \partial U} &= \frac{\partial}{\partial T}\left(\frac{1}{T}\right) e^{U/T} + \frac{1}{T} \frac{\partial}{\partial T}\left(e^{U/T}\right) \\
&= -\frac{1}{T^2} e^{U/T} - \frac{U}{T^3} e^{U/T}.
\end{aligned}$$

Using Clairaut's Theorem

- Although Clairaut's Theorem is stated for f_{xy} and f_{yx}, it implies more generally that partial differentiation may be carried out in any order, provided that the derivatives in question are continuous.

  Example: Calculate the partial derivative f_{zzwx}, where $f(x,y,z,w) = x^3w^2z^2 + \sin\left(\frac{xy}{z^2}\right)$.

  We differentiate with respect to w first:

$$\frac{\partial}{\partial w}\left(x^3w^2z^2 + \sin\left(\frac{xy}{z^2}\right)\right) = 2x^3wz^2.$$

  Next, differentiate twice with respect to z and once with respect to x:

$$\begin{aligned} f_{wz} &= \tfrac{\partial}{\partial z}(2x^3wz^2) = 4x^3wz; \\ f_{wzz} &= \tfrac{\partial}{\partial z}(4x^3wz) = 4x^3w; \\ f_{wzzx} &= \tfrac{\partial}{\partial x}(4x^3w) = 12x^2w. \end{aligned}$$

  We conclude that $f_{zzwx} = f_{wzzx} = 12x^2w$.

Partial Differential Equations (PDEs)

Stewart, p.908

- Verify that $f(x, y) = e^x \sin y$ is a solution of **Laplace's partial differential equation** $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

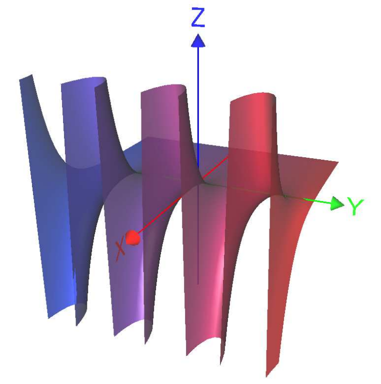
We have

$$f_x = e^x \sin y, \quad f_y = e^x \cos y,$$

$$f_{xx} = e^x \sin y, \quad f_{yy} = -e^x \sin y.$$

Thus,

$$f_{xx} + f_{yy} = 0.$$



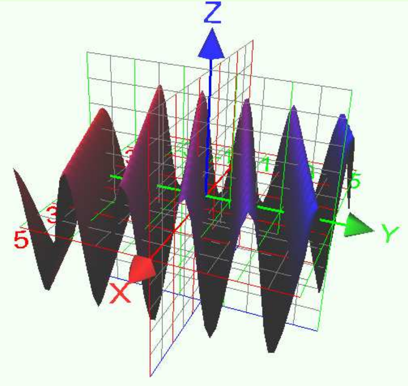
Partial Differential Equations (PDEs)

- Verify that $f(x,t) = \sin(x - at)$ is a solution of the **wave partial differential equation** $\frac{\partial^2 f}{\partial t^2} = a^2 \frac{\partial^2 f}{\partial x^2}$.

$$\frac{\partial f}{\partial t} = -a\cos(x - at),$$
$$\frac{\partial f}{\partial x} = \cos(x - at),$$

$$\frac{\partial^2 f}{\partial t^2} = -a^2 \sin(x - at),$$
$$\frac{\partial^2 f}{\partial x^2} = -\sin(x - at).$$

Thus, $\displaystyle\frac{\partial^2 f}{\partial t^2} = a^2 \frac{\partial^2 f}{\partial x^2}$.



Subsection 4

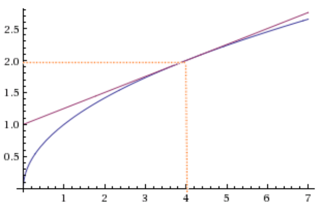
Differentiability and Tangent Planes

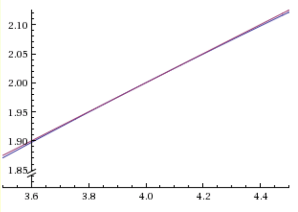
Stewart, p.915
14.4

Tangent Lines and Linear Approximations

remind

- Consider the function $f(x) = \sqrt{x}$.
  Calculate $f'(x) = \frac{1}{2\sqrt{x}}$ and $f'(4) = \frac{1}{4}$. Thus, the equation of the tangent line to f at $x = 4$ is
  $$y - 2 = \frac{1}{4}(x-4) \quad \text{or} \quad y = \frac{1}{4}x + 1.$$
- Very close to $x = 4$, $y = \sqrt{x}$ can be very accurately approximated by $y = \frac{1}{4}x + 1$.
  Therefore, e.g., $1.994993734 = \sqrt{3.98} \approx \frac{1}{4} \cdot 3.98 + 1 = 1.995$.





$z=f(x,y)$, S
P on S, $P(a,b,c)$, $c=f(a,b)$

interpretation of p.d.

$A(x-a)+B(y-b)+C(z-c)=0$ $\quad /-C$

$z-c=\underline{A_1}(x-a)+\underline{B_1}(y-b)$

$x=a$

$z-c= B1\ (y-b)$

$B_1=f_y(a,b)$

$z-c=f_x(a,b)(x-a)+f_y(a,b)(y-b)$

Tangent Planes and Linear Approximations

- Consider $f(x, y)$ with continuous partial derivatives.
- An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P = (a, b, c)$, where $c = f(a, b)$, is

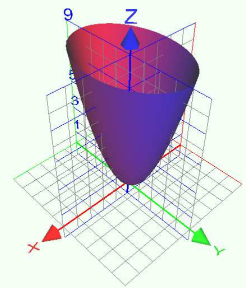
$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

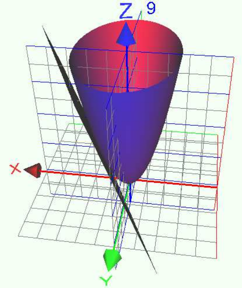
Example: Consider the elliptic paraboloid $f(x, y) = 2x^2 + y^2$. Since $f_x(x, y) = 4x$ and $f_y(x, y) = 2y$,

we have $f_x(1, 1) = 4$ and $f_y(1, 1) = 2$. Therefore, the plane

$$\begin{aligned} & z - 3 \\ & \quad = 4(x - 1) + 2(y - 1) \end{aligned}$$

is the tangent plane to the paraboloid at $(1, 1, 3)$.





Linearization of f at (a, b)

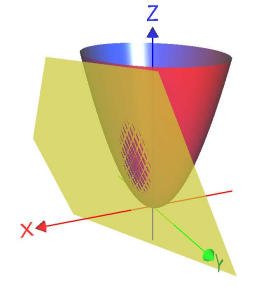
- Given a function $f(x, y)$ with continuous partial derivatives f_x, f_y, an equation of the tangent plane to $f(x, y)$ at $(a, b, f(a, b))$ is given by

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

- The linear function whose graph is this tangent plane

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

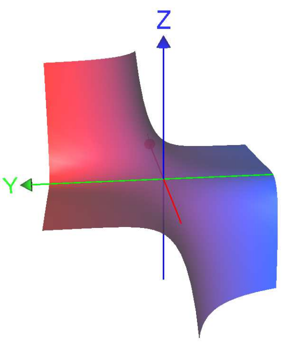
is called the **linearization** of f at (a, b). The approximation $f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$ is called the **linear approximation** of f at (a, b). Example: We saw for $f(x, y) = 2x^2 + y^2$, that $f(x, y) \approx 3 + 4(x - 1) + 2(y - 1)$ near $(1, 1, 3)$.

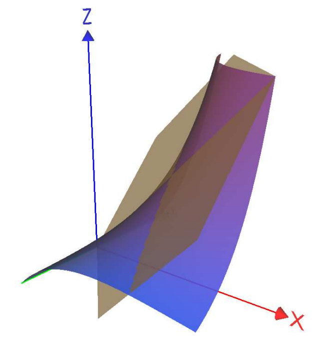


Another Example of a Linearization

- Consider the function $f(x, y) = xe^{xy}$.
  We have $f_x(x, y) = e^{xy} + xye^{xy}$ and $f_y(x, y) = x^2 e^{xy}$.
  Thus, $f_x(1, 0) = 1$ and $f_y(1, 0) = 1$.
  So the linearization of $f(x, y)$ at $(1, 0, 1)$ is

$$f(x, y) \approx 1 + (x - 1) + (y - 0) = x + y.$$





Differentiability

Stewart, p. 918

- Assume that $f(x, y)$ is defined in a disk $\mathcal{D}$ containing (a, b) and that $f_x(a, b)$ and $f_y(a, b)$ exist.

  $f(x, y)$ is **differentiable at** (a, b) if it is **locally linear**, i.e.,

  $$f(x, y) = L(x, y) + e(x, y),$$

  where $e(x, y)$ satisfies $\lim_{(x,y)\to(a,b)} \frac{e(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0.$

  In this case, the **tangent plane** to the graph at $(a, b, f(a, b))$ is the plane with equation

  $$z = L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

- If $f(x, y)$ is differentiable at all points in a domain $\mathcal{D}$, we say that $f(x, y)$ is **differentiable on** $\mathcal{D}$.

Criterion for Differentiability

- The following theorem provides a criterion for differentiability and shows that all familiar functions are differentiable on their domains.

Criterion for Differentiability

If $f_x(x, y)$ and $f_y(x, y)$ exist and are continuous on an open disk $\mathcal{D}$, then $f(x, y)$ is differentiable on $\mathcal{D}$.

Example: Show that $f(x, y) = 5x + 4y^2$ is differentiable and find the equation of the tangent plane at $(a, b) = (2, 1)$.

The partial derivatives exist and are continuous functions: $f_x(x, y) = 5,\ f_y(x, y) = 8y$. Therefore, $f(x, y)$ is differentiable for all (x, y), by the criterion.

To find the tangent plane, we evaluate the partial derivatives at $(2, 1)$: $f(2, 1) = 14,\ f_x(2, 1) = 5,$ and $f_y(2, 1) = 8$. The linearization at $(2, 1)$ is $L(x, y) = 14 + 5(x - 2) + 8(y - 1) = -4 + 5x + 8y$. Thus, the tangent plane through $P = (2, 1, 14)$ has equation $z = -4 + 5x + 8y$.

Tangent Plane

- Find a tangent plane of the graph of $f(x, y) = xy^3 + x^2$ at $(2, -2)$.

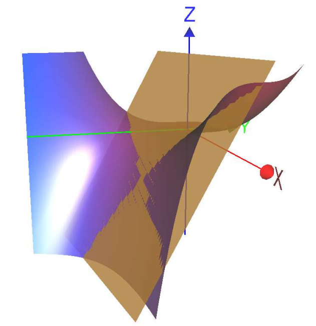
  The partial derivatives are continuous, so $f(x, y)$ is differentiable:

  $$f_x(x, y) = y^3 + 2x, \quad f_x(2, -2) = -4,$$
  $$f_y(x, y) = 3xy^2, \quad f_y(2, -2) = 24.$$

  Since $f(2, -2) = -12$, the tangent plane through $(2, -2, -12)$ has equation

  $$z = -12 - 4(x - 2) + 24(y + 2).$$

  This can be rewritten as $z = 44 - 4x + 24y$.



Differentials

Stewart,p.919

- For $z = f(x, y)$ a differentiable function of two variables, the **differentials** dx, dy are independent variables, i.e., can be assigned any values.
- The **differential** dz, also called the **total differential**, is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

- If we set $dx = x - a$ and $dy = y - b$ in the formula for the linear approximation of f, we have

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) = f(a, b) + dz.$$

Example: Consider $f(x, y) = x^2 + 3xy - y^2$. Then $dz = f_x(x, y)dx + f_y(x, y)dy = (2x + 3y)dx + (3x - 2y)dy$. If x changes from 2 to 2.05 and y changes from 3 to 2.96, then $dx = 0.05, dy = -0.04$ and $(a, b) = (2, 3)$, whence $dz = f_x(2, 3) \cdot 0.05 + f_y(2, 3) \cdot (-0.04) = 0.65$ and $f(2.05, 2.96) \approx f(2, 3) + dz = 13 + 0.65 = 13.65$.

Using Differentials for Error Estimation

- If the base radius and the height of a right circular cone are measured as 10 cm and 25 cm, respectively, with possible maximum error 0.1 cm in each, estimate the max possible error in calculating the volume of the cone, given that the volume formula is $V(r,h) = \frac{1}{3}\pi r^2 h$.

  We have $dV = V_r dr + V_h dh = \frac{2}{3}\pi r h dr + \frac{1}{3}\pi r^2 dh$.

  Therefore

$$\begin{aligned} dV &= \tfrac{2}{3}\pi \cdot 10 \cdot 25 \cdot (\pm 0.1) + \tfrac{1}{3}\pi \cdot 10^2 \cdot (\pm 0.1) \\ &= (\tfrac{500}{3}\pi + \tfrac{100}{3}\pi) \cdot (\pm 0.1) \\ &= \pm\, 20\pi \text{ cm}^3. \end{aligned}$$

Application: Change in Body Mass Index (BMI)

- A person's BMI is $I = \frac{W}{H^2}$, where W is the body weight (in kilograms) and H is the body height (in meters). Estimate the change in a child's BMI if (W, H) changes from $(40, 1.45)$ to $(41.5, 1.47)$.

  We have
  $$\frac{\partial I}{\partial W} = \frac{1}{H^2}, \quad \frac{\partial I}{\partial H} = -\frac{2W}{H^3}.$$
  At $(W, H) = (40, 1.45)$, we get
  $$\left.\frac{\partial I}{\partial W}\right|_{(40,1.45)} = \frac{1}{1.45^2}, \quad \left.\frac{\partial I}{\partial H}\right|_{(40,1.45)} = -\frac{2 \cdot 40}{1.45^3}.$$
  The differential $dI \approx \frac{1}{1.45^2}dW - \frac{80}{1.45^3}dH$.
  If (W, H) changes from $(40, 1.45)$ to $(41.5, 1.47)$, then $dW = 1.5$ and $dH = 0.02$. Therefore,
  $$\Delta I \approx dI = \frac{1}{1.45^2}dW - \frac{2 \cdot 40}{1.45^3}dH = \frac{1}{1.45^2} \cdot 1.5 - \frac{80}{1.45^3} \cdot 0.02.$$