Mathematical Analysis II

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2022

HIGHER-ORDER

Differential Equations

$$F(x, y, y', y'', ..., y^{(n)})=0 \qquad (1)$$

$$y^{(n)} = f(x, y, y', y'', ..., y^{(n-1)}) \qquad (1')$$

Th. If in the equation $y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)})$

the function $f(x, y, y', y'', \ldots, y^{(n-1)})$ and its partial derivatives with respect to the arguments $y, y', y'', \ldots, y^{(n-1)}$ are continuous in some region containing the values $x = x_0$, $y = y_0$, $y' = y'_0$, ..., $y^{(n-1)} = y_0^{(n-1)}$, then there is one and only one solution, $y = y(x)$, of the equation that satisfies the conditions

$$\begin{cases} y_{x=x_0} = y_0 \\ y'_{x=x_0} = y'_0 \\ \cdots \\ y^{(n-1)}_{x=x_0} = y_0^{(n-1)} \end{cases} \qquad (2)$$

initial conditions

$y'' = f(x,y,y')$ the geometrical meaning

$y = y_0$,
$y' = y'_0$ where x_0, y_0, y'_0 are given numbers

Def. The general solution of a dif. eq. of nth order

$$y=\varphi(x, C_1, C_2, \dots, C_n) \qquad (3)$$

which is dependent of n arbitrary constants C_1, C_2, ..., C_n and

•) it satisfies the equation for any values of the constants C_1, C_2, ..., C_n ;

••) for specified initial conditions

$$y_{x=x_0}=y_0$$
$$y'_{x=x_0}=y'_0$$
$$\vdots$$
$$y^{(n-1)}_{x=x_0}=y_0^{(n-1)}$$

the constants C_1, C_2, ..., C_n may be chosen so that the function (3) will satisfy these conditions...

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the particular solution $y=\varphi(x, C_1^0, C_2^0, \dots, C_n^0)$ integral curve

ODE of order n which admit

A LOWERING OF THE ORDER

AN EQUATION OF THE FORM $y^{(n)} = f(x)$

$$y^{(n-1)} = \int f(x)\,dx + C_1 = F_1(x) + C_1$$

$$y^{(n-2)} = \int (F_1(x) + C_1)\,dx = F_2(x) + C_1 x + C_2$$

$$\cdots$$

$$y = F_n(x) + \overset{C_1}{\frac{C_1}{(n-1)!}} x^{n-1} + \overset{C_2}{\frac{C_2}{(n-2)!}} x^{n-2} + \ldots + C_{n-1} x + C_n$$

where

$$F_n(x) = \underbrace{\int\int\ldots\int}_{n\ \text{times}} f(x)\,\underbrace{dx\ldots dx}_{\text{n times}}$$

Example. $y'' = x + \sin x$

$$y' = \frac{x^2}{2} - \cos x + c_1$$

$$y = \frac{x^3}{6} - \sin x + c_1 x + c_2$$ the general solution

Exercises:

$y^{IV} = \cos^2 x$, $y(0)=1/32, y'(0)=0, y''(0)=1/8, y'''(0)=0$

$y''' = x \sin x$, $y(0)=0, y'(0)=0, y''(0)=2$

$y'' = 2 \sin x \cos^2 x - \sin^3 x$

$$F(x, y^{(k)}, \dots, y^{(n)}) = 0$$

does not contain y

$$y^{(k)}(x) = z(x)$$

$$F(x, z, z', \dots, z^{(n-k)}) = 0$$

of order n-k

Example $y'' - 2y' \operatorname{ctg} x = \sin^3 x$

$$z(x) = y' \Rightarrow z' = y''$$

$z' - 2z \operatorname{ctg} x = \sin^3 x$

...

$$y' = z = c_1 \sin^2 x - \sin^2 x \cos x$$

$$y = \frac{c_1}{4}(2x - \sin 2x) - \frac{1}{3}\sin^3 x + C_2$$

Exercises: $y'' - 3\frac{y'}{x} = x$

$$x y'' = y' \ln \frac{y'}{x}$$

$y'''(x-1) - y'' = 0$, $y(2)=2$, $y'(2)=1$, $y''(2)=1$

$$F(y', y'', \ldots, y^{(n)}) = 0$$

first substitution

$$y' = z(x)$$

and then take z as independent variable

$$y'''(1+y'^2) - 3y'y''^2 = 0$$

$$y' = z(x)$$

$$z''(1+z^2) - 3z(z')^2 = 0 \quad \mapsto$$

see next page

$$F(y, y', y'', \dots, y^{(n)}) = 0$$

$$y' = p$$, consider p as a function and take y as an independent variable

$$y'' = p\frac{dp}{dy}$$

$$y''' = p\left(\frac{dp}{dy}\right)^2 + p^2\frac{d^2p}{dy^2}$$

Ex. $y'' = 2yy'$

$yy'' - 2y'^2 = 0$

$$y'' = y'e^y, \quad y(0) = 0, \; y'(0) = 1$$

DE (1) homogeneous with respect $y, y', \ldots, y^{(n)}$

$$z(x) = \frac{y'}{y} \quad \text{or} \quad y = e^{\int z(x)\,dx}$$

reduce the DE (1) to the DE of order n-1

Ex

$$y'^2 + 2yy'' = 0$$

$$xyy'' - xy'^2 - yy' = 0$$

$$3y'^2 = 4yy'' + y^2.$$