Linear Algebra and Analytic Geometry

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Dot Product

scalar product

Definition of Dot Product

Given $\mathbf{v} = \langle a, b \rangle$ and $\mathbf{w} = \langle c, d \rangle$, the **dot product** of $\mathbf{v}$ and $\mathbf{w}$ is given by

$$\mathbf{v} \cdot \mathbf{w} = ac + bd.$$

- Example: Find the dot product of $\mathbf{v} = \langle 6, -2 \rangle$ and $\mathbf{w} = \langle -3, 4 \rangle$;

$$\mathbf{v} \cdot \mathbf{w} = 6 \cdot (-3) + (-2) \cdot 4 = -18 - 8 = -26;$$

Properties of the Dot Product

In the following $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors and a is a scalar:

1. $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$;
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$;
3. $a(\mathbf{u} \cdot \mathbf{v}) = (a\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (a\mathbf{v})$;
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$;
5. $\mathbf{0} \cdot \mathbf{v} = 0$;
6. $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$;
7. $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$;

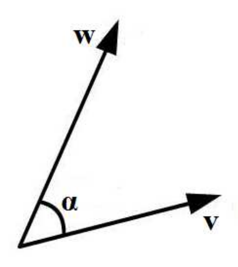
Magnitude, Angle and the Dot Product

Magnitude of a Vector in Terms of the Dot Product

If $\mathbf{v} = \langle a, b \rangle$, then $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.

Alternative Formula for the Dot Product

If $\mathbf{v}$ and $\mathbf{w}$ are two nonzero vectors and α is the smallest nonnegative angle between $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \alpha$.



Angle Between Two Vectors

Angle Between Two Vectors

If $\mathbf{v}$ and $\mathbf{w}$ are two nonzero vectors and α is the smallest nonnegative angle between $\mathbf{v}$ and $\mathbf{w}$, then $\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$ and $\alpha = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)$.

- Example: Find the measure of the smallest nonnegative angle between the vectors $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{w} = -\mathbf{i} + 5\mathbf{j}$;

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{2\cdot(-1)+(-3)\cdot 5}{\sqrt{2^2+(-3)^2}\sqrt{(-1)^2+5^2}} = \frac{-17}{\sqrt{13}\sqrt{26}} = \frac{-17}{13\sqrt{2}} = -\frac{17\sqrt{2}}{26};$$

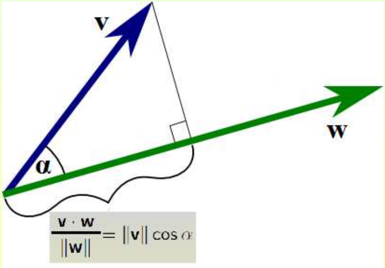
$$\alpha = \cos^{-1}\left(-\tfrac{17\sqrt{2}}{26}\right) \approx 157.6^\circ;$$

Scalar Projection

Scalar Projection of **v** Onto **w**

If **v** and **w** are two nonzero vectors and α is the angle between **v** and **w**, then the scalar projection of **v** onto **w**, $\text{proj}_{\mathbf{w}}\mathbf{v}$, is given by

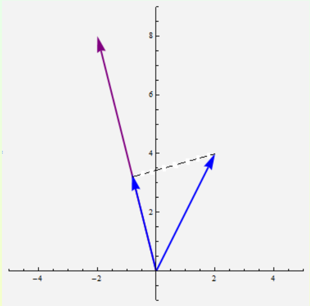
$$\text{proj}_{\mathbf{w}}\mathbf{v} = \|\mathbf{v}\| \cos\alpha;$$



- Since $\cos\alpha = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$, we also get $\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}$;

Example

Given $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{w} = -2\mathbf{i} + 8\mathbf{j}$, find $\text{proj}_{\mathbf{w}}\mathbf{v}$;



$$\text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|} = \frac{2\cdot(-2)+4\cdot 8}{\sqrt{(-2)^2+8^2}} = \frac{28}{\sqrt{68}} = \frac{28}{2\sqrt{17}} = \frac{14\sqrt{17}}{17};$$

Parallel and Perpendicular Vectors

- Two vectors are **parallel** when the angle α between them is 0° or 180°;
- Two vectors are **perpendicular** or **orthogonal** when the angle between them is 90°;
- Two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel if and only if there exists a real number c, such that $\mathbf{w} = c\mathbf{v}$;
- Two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if $\mathbf{v} \cdot \mathbf{w} = 0$.

Application: Work

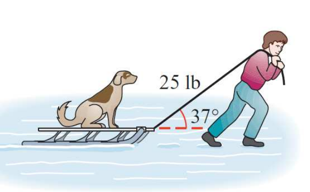
Definition of Work

The work W done by a force $\mathbf{F}$ applied along a displacement $\mathbf{s}$ is

$$W = \mathbf{F} \cdot \mathbf{s} = \|\mathbf{F}\| \|\mathbf{s}\| \cos\alpha, \ \alpha \text{ angle between } \mathbf{F} \text{ and } \mathbf{s};$$

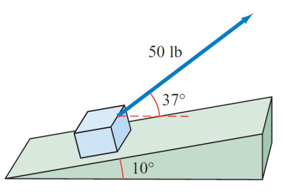
- If the child pulls the sled a horizontal distance of 7 feet, what is the work done?

  $$W = \mathbf{F} \cdot \mathbf{s} = \|\mathbf{F}\| \|\mathbf{s}\| \cos\alpha = 25 \cdot 7 \cdot \cos 37^\circ \approx 140 \text{ ft-lb};$$



- What is the work done in moving the box 15 feet along the ramp?

  $$W = \mathbf{F} \cdot \mathbf{s} = \|\mathbf{F}\| \|\mathbf{s}\| \cos\alpha = 50 \cdot 15 \cdot \cos 27^\circ \approx 670 \text{ ft-lb};$$

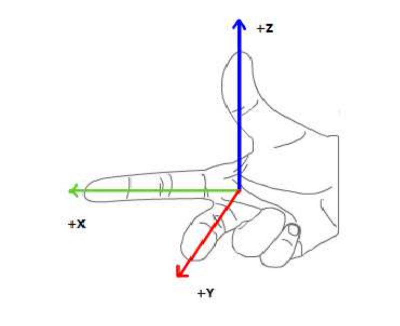


Subsection 2

Vectors in Three Dimensions

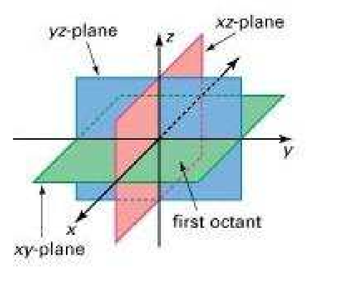
Three-Dimensional Coordinate System

- Fix a point O called the **origin**.
- Through O, fix three mutually perpendicular straight lines, called the **coordinate axes**. These are termed the x-, the y- and the z-**axis**.
- The positive direction on the z-axis is taken so that the **right-hand side rule is satisfied** with respect to the positive directions on the x- and the y-axis:



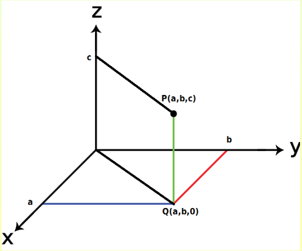
Coordinate Planes and Octants

- The planes formed by each pair of the axes are called the **coordinate planes**. These are the *xy*-, the *xz*- and the *yz*-**planes**.
- Each of the eight regions in which space is separated by the coordinate planes is called an **octant**.
- The **first octant** is the one enclosed by the positive *xy*-, the positive *yz*- and the positive *xz*-planes.



Coordinates of Points

- Let P be a point in space.
  Its **coordinates** are $P = (a, b, c)$ if
  - a is the signed distance from P to the yz-plane;
  - b is the signed distance from P to the xz-plane;
  - c is the signed distance from P to the xy-plane.
- $Q = (a, b, 0)$ is the **projection** of $P(a, b, c)$ on the xy-plane.
- We call the set of all triples $\mathbb{R}^3 = \{(x, y, z) : x, y, z \text{ in } \mathbb{R}\}$ the **three-dimensional space**.



The Plane $z = 3$

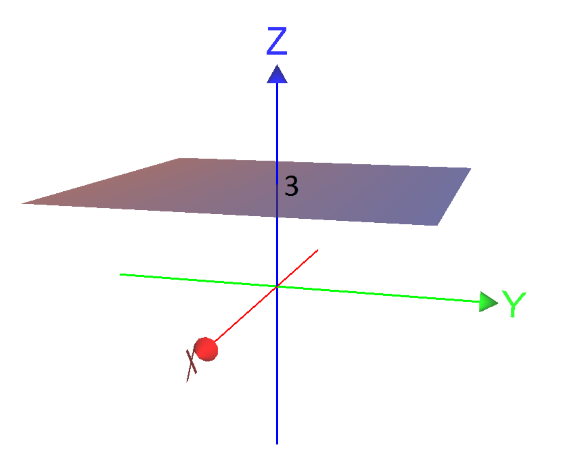


Figure: The Plane $z = 3$.

The Plane $y = 5$

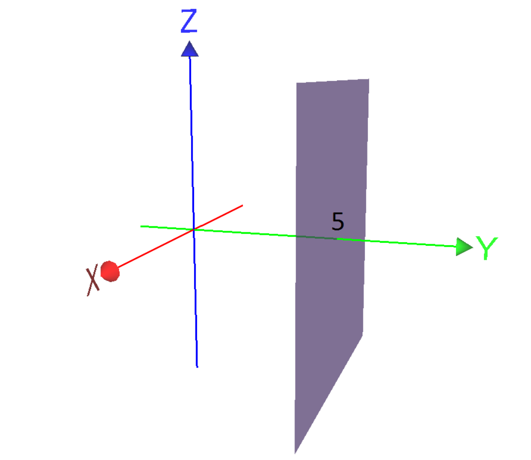


Figure: The Plane $y = 5$.

The Plane $y = x$

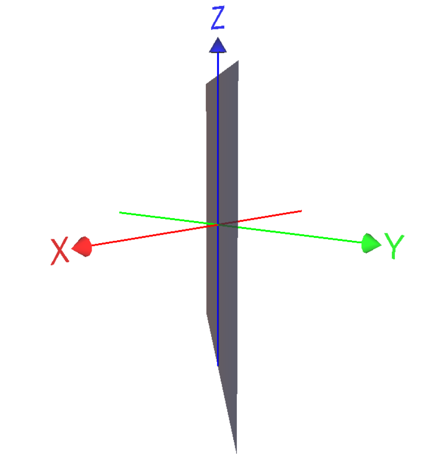
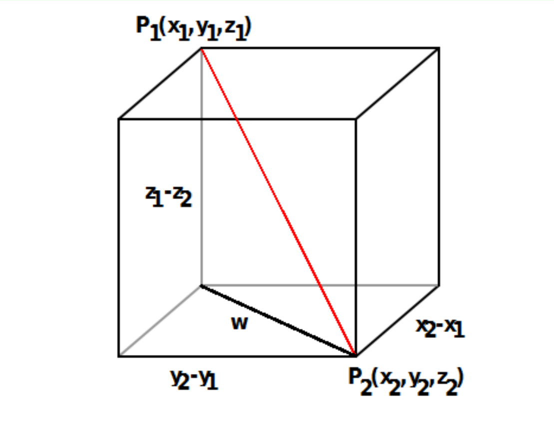


Figure: The Plane $y = x$.

Distance Formula in Three Dimensions: A Figure

- Suppose $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are points in space.



Distance Formula in Three Dimensions

- Suppose $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are points in space.
- The distance $|P_1P_2|$ between P_1 and P_2 is calculated by

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example: If $P = (2, -1, 2)$ and $Q = (1, -3, 5)$,

$$\begin{aligned} |PQ| &= \sqrt{(1-2)^2 + (-3+1)^2 + (5-2)^2} \\ &= \sqrt{1+4+9} \\ &= \sqrt{14}. \end{aligned}$$

Equation of a Sphere

- An equation of a sphere with center $C = (a, b, c)$ and radius R is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2.$$

- In particular, if $C = (0, 0, 0)$, then

$$x^2 + y^2 + z^2 = R^2.$$

Example: Find the center and the radius of the sphere with equation

$$x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0.$$

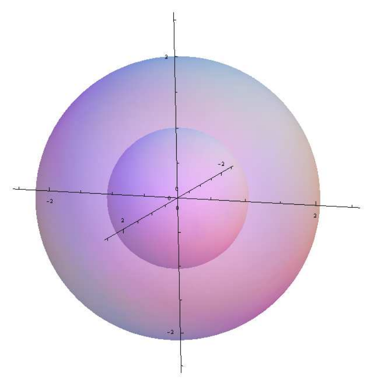
$$\begin{aligned}
&x^2 + 4x + y^2 - 6y + z^2 + 2z = -6 \\
&(x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1) = -6 + 14 \\
&(x+2)^2 + (y-3)^2 + (z+1)^2 = (\sqrt{8})^2.
\end{aligned}$$

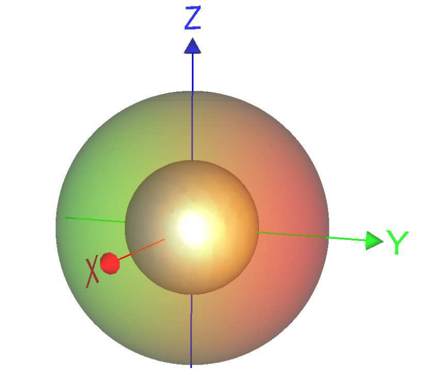
Thus the center is $C = (-2, 3, -1)$ and the radius $R = \sqrt{8}$.

Another Example

- What is the shape of the solid represented by

$$1 \leq x^2 + y^2 + z^2 \leq 4?$$

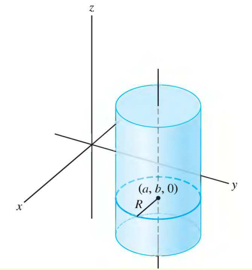




Equation of a Cylinder

- An equation of a right circular cylinder of radius R, whose central axis is the vertical line through $(a, b, 0)$ is

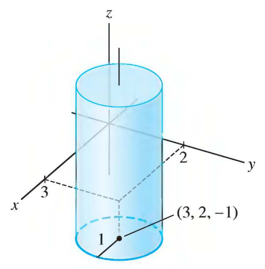
$$(x-a)^2+(y-b)^2=R^2.$$



Example: Find the set of points defined by

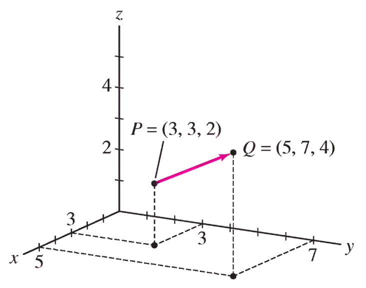
$$(x-3)^2+(y-2)^2=1, \quad z \geq -1.$$

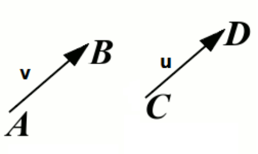
The given equation defines a right circular cylinder of radius 1 with central axis the vertical line through $(3, 2, 0)$. The condition $z \geq -1$ allows only the part of the cylinder on or above the plane $z=-1$.



Vectors in Three-Dimensional Space

- A **vector** has **magnitude** and **direction**.
- The vector $\boldsymbol{v} = \overrightarrow{PQ}$ has **initial point** P and **terminal point** Q.





- If $\boldsymbol{v} = \overrightarrow{AB}$ and $\boldsymbol{u} = \overrightarrow{CD}$, have the same length and direction, they are called **equivalent** (or **equal**), written $\boldsymbol{v} = \boldsymbol{u}$.
- The **zero vector 0** has 0 length and unspecified direction.

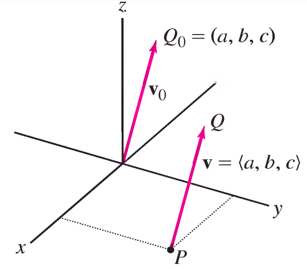
Components

- A vector $\boldsymbol{v} = \overrightarrow{OP}$ whose initial point is the origin and final point is $P = (a, b, c)$ is called the **position vector** of P.

  Then $\boldsymbol{v} = \overrightarrow{OP}$ will be denoted by $\langle a, b, c \rangle$ and the real numbers a, b, c are called the **components** of $\boldsymbol{v}$.

- Every vector $\overrightarrow{PQ}$, for $P = (a_1, b_1, c_1)$ and $Q = (a_2, b_2, c_2)$ has an equivalent **position vector** $\boldsymbol{v} = \overrightarrow{OQ_0}$. In that case

$$\boldsymbol{v} = \langle a_2 - a_1, b_2 - b_1, c_2 - c_1 \rangle.$$



Example: Find the position vector equivalent to $\overrightarrow{PQ}$, if $P = (3, -4, -4)$ and $Q = (2, 5, -1)$.

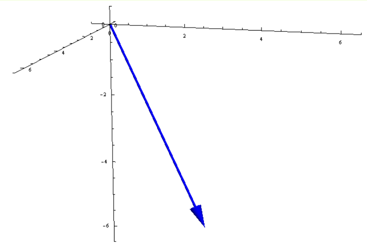
We get $\overrightarrow{OQ_0} = \langle 2 - 3, 5 - (-4), -1 - (-4) \rangle = \langle -1, 9, 3 \rangle$.

Length of a Vector

- Given a vector $\boldsymbol{v} = \overrightarrow{PQ}$, with $P = (a_1, b_1, c_1)$ and $Q = (a_2, b_2, c_2)$, its **length** $\|\boldsymbol{v}\| = \|\overrightarrow{PQ}\|$ is given by the formula:

$$\|\overrightarrow{PQ}\| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2}.$$

Example: The position vector $\boldsymbol{v} = \langle 2, 3, -5 \rangle$



has length $\|\boldsymbol{v}\| = \sqrt{2^2 + 3^2 + (-5)^2} = \sqrt{38}$.

Remark on Lengths

- Consider $P = (a_1, b_1, c_1)$ and $Q = (a_2, b_2, c_2)$.

  Suppose that the vector $\boldsymbol{v} = \overrightarrow{PQ}$ has equivalent position vector $\overrightarrow{OQ_0}$.

  Then $Q_0 = (a_2 - a_1, b_2 - b_1, c_2 - c_1)$.

  We have

  $$\|\overrightarrow{PQ}\| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2}.$$

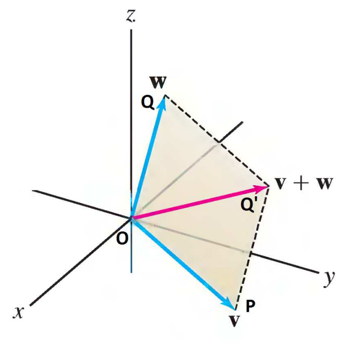
  Furthermore,

  $$\|\overrightarrow{OQ_0}\| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2}.$$

- This was to be expected. After all $\overrightarrow{OQ_0} = \overrightarrow{PQ}$!

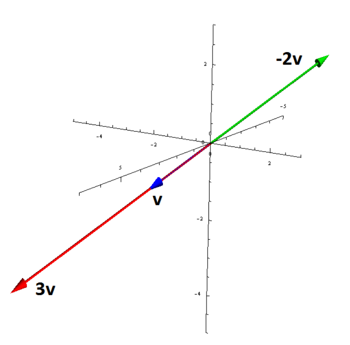
Sum of Vectors

- There are two ways to compute the **sum** $\boldsymbol{v} + \boldsymbol{w}$ of two vectors $\boldsymbol{v} = \overrightarrow{OP}$ and $\boldsymbol{w} = \overrightarrow{OQ}$:
  - **The Triangle Law**: Take $\boldsymbol{w} = \overrightarrow{PQ'}$; Then $\boldsymbol{v} + \boldsymbol{w} = \overrightarrow{OQ'}$.
  - **The Parallelogram Law**: Draw the parallelogram $OPQ'Q$ with sides $\overrightarrow{OP}$ and $\overrightarrow{OQ}$; Then $\boldsymbol{v} + \boldsymbol{w} = \overrightarrow{OQ'}$.



Scalar Multiplication

- The product of a real number λ times the vector $\boldsymbol{v} = \overrightarrow{PQ}$, written $\lambda\boldsymbol{v} = \lambda\overrightarrow{PQ}$, is the vector that has
  - length $|\lambda|\|\boldsymbol{v}\|$;
  - the same direction as $\boldsymbol{v}$, if $\lambda > 0$;
  - the opposite direction of $\boldsymbol{v}$, if $\lambda < 0$.



Addition, Subtraction and Scalar Multiplication

- We add two vectors $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$, $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$ component-wise

$$\boldsymbol{a} + \boldsymbol{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle.$$

- We multiply a vector $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ by a real constant λ

$$\lambda \boldsymbol{a} = \langle \lambda a_1, \lambda a_2, \lambda a_3 \rangle.$$

- In accordance with the definition of subtraction $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$, we then have

$$\boldsymbol{a} - \boldsymbol{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle.$$

Example: $\boldsymbol{a} = \langle 4, 0, 3 \rangle$ and $\boldsymbol{b} = \langle -2, 1, 5 \rangle$.
$\boldsymbol{a} + \boldsymbol{b} = \langle 4, 0, 3 \rangle + \langle -2, 1, 5 \rangle = \langle 2, 1, 8 \rangle$.
$\boldsymbol{a} - \boldsymbol{b} = \langle 4, 0, 3 \rangle - \langle -2, 1, 5 \rangle = \langle 6, -1, -2 \rangle$.
$3\boldsymbol{b} = 3\langle -2, 1, 5 \rangle = \langle -6, 3, 15 \rangle$.
$2\boldsymbol{a} + 5\boldsymbol{b} = 2\langle 4, 0, 3 \rangle + 5\langle -2, 1, 5 \rangle = \langle 8, 0, 6 \rangle + \langle -10, 5, 25 \rangle = \langle -2, 5, 31 \rangle$.

Basis Vectors

- The **basis vectors** are the three vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ of length 1 and pointing towards the positive direction on the x-, the y- and the z-axes, respectively.
- In component form these are:

$$\boldsymbol{i} = \langle 1, 0, 0 \rangle, \quad \boldsymbol{j} = \langle 0, 1, 0 \rangle, \quad \boldsymbol{k} = \langle 0, 0, 1 \rangle.$$

- Every vector $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ can be written in a unique way as a **linear combination** of the basis vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$. We have

$$\begin{aligned} \boldsymbol{a} &= \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle = a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k}. \end{aligned}$$

Example: If $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{b} = 4\boldsymbol{i} + 7\boldsymbol{k}$, then:
$2\boldsymbol{a} + 3\boldsymbol{b} = 2(\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}) + 3(4\boldsymbol{i} + 7\boldsymbol{k}) = (2\boldsymbol{i} + 4\boldsymbol{j} - 6\boldsymbol{k}) + (12\boldsymbol{i} + 21\boldsymbol{k}) = 14\boldsymbol{i} + 4\boldsymbol{j} + 15\boldsymbol{k}$.

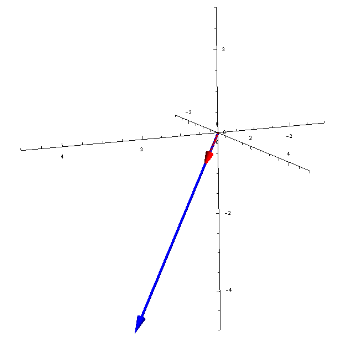
Unit Vectors

- Find the unit vector $\boldsymbol{e_v}$ in the direction of $\boldsymbol{v} = 3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$.
  First compute the magnitude:

$$\|\boldsymbol{v}\| = \sqrt{3^2 + 2^2 + (-4)^2} = \sqrt{29}.$$

Then set up

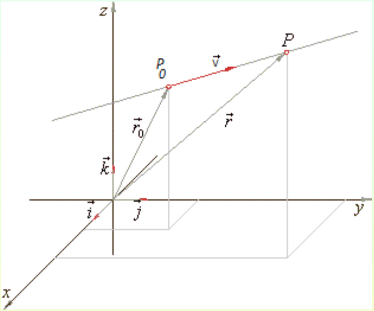
$$\begin{aligned} \boldsymbol{e_v} &= \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v} \\ &= \frac{1}{\sqrt{29}}\langle 3, 2, -4\rangle \\ &= \left\langle \frac{3}{\sqrt{29}}, \frac{2}{\sqrt{29}}, \frac{-4}{\sqrt{29}} \right\rangle. \end{aligned}$$



Equations of Lines

- Given a point $P_0 = (x_0, y_0, z_0)$ in space, with position vector $\boldsymbol{r}_0 = \langle x_0, y_0, z_0 \rangle$, and a direction vector $\boldsymbol{v}$, the line passing through P_0 with direction $\mathbf{v}$ is given by the parametric vector equation

$$\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{v}.$$



- If $\boldsymbol{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\boldsymbol{v} = \langle a, b, c \rangle$, then the vector equation $\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{v}$ gives $\langle x, y, z \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$. So we get the three parametric equations:

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

An Example of a Straight Line

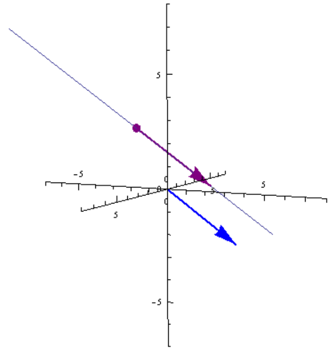
- Find an equation for the line through $P_0 = (5, 1, 3)$ and parallel to the vector $\boldsymbol{v} = \boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$.

Its vector equation is

$$\boldsymbol{r} = \langle 5, 1, 3 \rangle + t\langle 1, 4, -2 \rangle.$$

The corresponding system of parametric equations is

$$x = 5 + t, \quad y = 1 + 4t, \quad z = 3 - 2t.$$



Different Parameterizations of the Same Line

- Show that $\boldsymbol{r}_1(t) = \langle 1,1,0\rangle + t\langle -2,1,3\rangle$ and $\boldsymbol{r}_2(t) = \langle -3,3,6\rangle + t\langle 4,-2,-6\rangle$ parametrize the same line.

  The line $\boldsymbol{r}_1$ has direction vector $\boldsymbol{v} = \langle -2,1,3\rangle$. The line $\boldsymbol{r}_2$ has direction vector $\boldsymbol{w} = \langle 4,-2,-6\rangle$. These vectors are parallel because $\boldsymbol{w} = -2\boldsymbol{v}$. Therefore, the lines described by $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are parallel.

  We must check that they have a point in common. Choose any point on $\boldsymbol{r}_1$, say $P = (1,1,0)$ (corresponding to $t = 0$). This point lies on $\boldsymbol{r}_2$ if there is a t such that $\langle 1,1,0\rangle = \langle -3,3,6\rangle + t\langle 4,-2,-6\rangle$. This yields three equations $\left\{\begin{array}{l} 1 = -3+4t \\ 1 = 3-2t \\ 0 = 6-6t \end{array}\right\}$. All three are satisfied when $t = 1$. Therefore P also lies on $\boldsymbol{r}_2$.

  We conclude that $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ parametrize the same line.

Intersection of Two Lines

- Determine whether the following two lines $\boldsymbol{r}_1(t) = \langle 1, 0, 1\rangle + t\langle 3, 3, 5\rangle$, $\boldsymbol{r}_2(t) = \langle 3, 6, 1\rangle + t\langle 4, -2, 7\rangle$ intersect.

  The two lines intersect if there exist parameter values t_1 and t_2 such that $\boldsymbol{r}_1(t_1) = \boldsymbol{r}_2(t_2)$. This gives

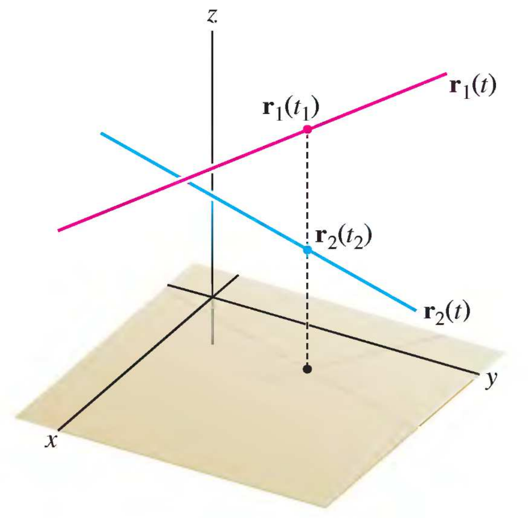
$$\langle 1, 0, 1\rangle + t_1\langle 3, 3, 5\rangle = \langle 3, 6, 1\rangle + t_2\langle 4, -2, 7\rangle.$$

  This is equivalent to three equations for the components:

$$x = 1 + 3t_1 = 3 + 4t_2,\ y = 3t_1 = 6 - 2t_2,\ z = 1 + 5t_1 = 1 + 7t_2.$$

  Solve the first two equations for t_1 and t_2. We get $t_1 = \frac{14}{9},\ t_2 = \frac{2}{3}$. These values satisfy the first two equations. However, t_1 and t_2 do not satisfy the third equation $1 + 5 \cdot \frac{14}{9} \neq 1 + 7 \cdot \frac{2}{3}$. Therefore, the lines do not intersect.

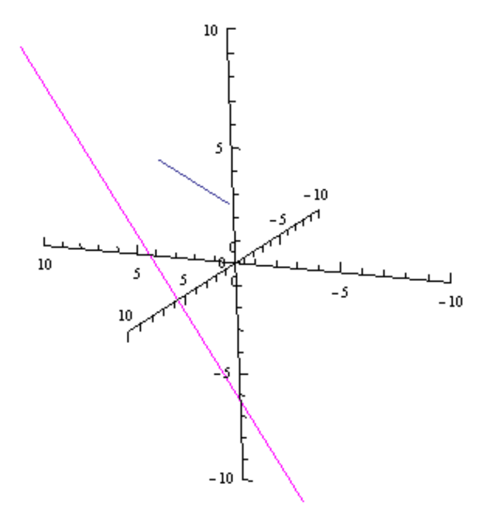
The Two Non-Intersecting Lines



Skew Lines

- Two straight lines in space are called **skew lines** if they do not intersect and are not parallel.
  Example: Show that the lines with parametric equations
  $$x = 1 + t, \quad y = -2 + 3t, \quad z = 4 - t;$$
  $$x = 2s, \quad y = 3 + s, \quad z = -3 + 4s$$
  are skew lines.
  To see that the lines do not intersect, we try to find a point of intersection by setting
  $$1 + t = 2s, \quad -2 + 3t = 3 + s, \quad 4 - t = -3 + 4s.$$
  The first two taken together give $(t, s) = (\frac{11}{5}, \frac{8}{5})$. These values do not satisfy the third equation! So there is no point of intersection.
  To see that they are not parallel, look at the direction vectors. The first has direction vector $\langle 1, 3, -1 \rangle$ and the second $\langle 2, 1, 4 \rangle$. These are not parallel vectors (Why?).

Plots of the Skew Lines



Line Segment Between Two Points

- Let $P_1 = (a_1, b_1, c_1)$ with position vector $\boldsymbol{r}_1 = \langle a_1, b_1, c_1 \rangle$ and $P_2 = (a_2, b_2, c_2)$ with position vector $\boldsymbol{r}_2 = \langle a_2, b_2, c_2 \rangle$.
  The vector equation of the line segment joining P_1 and P_2 is

$$\boldsymbol{r} = (1-t)\boldsymbol{r}_1 + t\boldsymbol{r}_2, \quad 0 \le t \le 1.$$

Example: Find the vector equation and the parametric equations for the line segment joining $P_1 = (10, 3, 1)$ with $P_2 = (5, 6, -3)$.
For the vector equation, we have

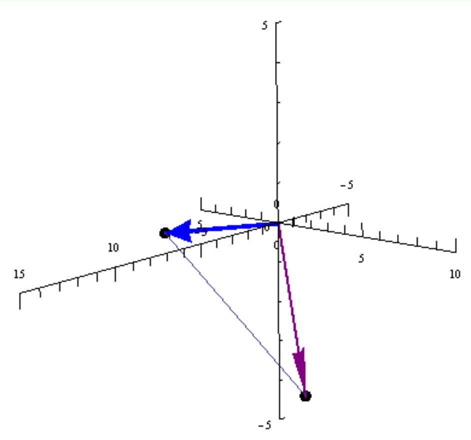
$$\boldsymbol{r} = (1-t)\langle 10, 3, 1 \rangle + t\langle 5, 6, -3 \rangle, \quad 0 \le t \le 1.$$

For the parametric equations, we have

$$x = 10 - 5t, \quad y = 3 + 3t, \quad z = 1 - 4t, \quad 0 \le t \le 1.$$

The Line Segment

- The line segment joining $(10, 3, 1)$ with $(5, 6, -3)$.



Example

- Parametrize the segment $\overrightarrow{PQ}$ where $P = (1, 0, 4)$ and $Q = (3, 2, 1)$. Find the midpoint of the segment.

  The line through $P = (1, 0, 4)$ and $Q = (3, 2, 1)$ has the parametrization

  $$\boldsymbol{r}(t) = (1 - t)\langle 1, 0, 4\rangle + t\langle 3, 2, 1\rangle = \langle 1 + 2t, 2t, 4 - 3t\rangle.$$

  The segment $\overline{PQ}$ is traced for $0 \leq t \leq 1$.

  The midpoint of $\overline{PQ}$ is the terminal point of the vector

  $$\boldsymbol{r}\left(\frac{1}{2}\right) = \frac{1}{2}\langle 1, 0, 4\rangle + \frac{1}{2}\langle 3, 2, 1\rangle = \langle 2, 1, \frac{5}{2}\rangle.$$

  In other words, the midpoint is $(2, 1, \frac{5}{2})$.

Symmetric Equations

- Given a point $P_0 = (x_0, y_0, z_0)$ in space, with position vector $\boldsymbol{r}_0 = \langle x_0, y_0, z_0 \rangle$, and a direction vector $\boldsymbol{v} = \langle a, b, c \rangle$, the **symmetric equations** for the straight line in space through P_0 with direction $\boldsymbol{v}$ are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Example: Find the symmetric equations for the line passing through $A = (2, 4, -3)$ and $B = (3, -1, 1)$ and the point at which this line intersects the xy-plane.

The line has the direction of the vector $\overrightarrow{AB} = \langle 1, -5, 4 \rangle$. Therefore, the symmetric equations are:

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{z + 3}{4}.$$

It intersects the xy-plane when $z = 0$. Therefore, we get $x - 2 = \frac{3}{4}$ and $-\frac{1}{5}(y - 4) = \frac{3}{4}$, which yield $(x, y, z) = (\frac{11}{4}, \frac{1}{4}, 0)$.

Picture of the Line

- The line passing through $A = (2, 4, -3)$ and $B = (3, -1, 1)$ and the point $(\frac{11}{4}, \frac{1}{4}, 0)$ at which this line intersects the *xy*-plane.