Mathematical Analysis II

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LINEAR ODE

Stewart,
9.5 Exercises
1-4,
5-14,
15-20, 21(CAS)
23 (24,25),
26 if you wish
27,
37

Advanced Engineering Mathematics,

Problem Set 1.5, p.34
3-13,
14*, **21**
22-28,
29*, 30*,
31-40

Ex. 8 (Stewart), p.620

$$4x^3y + x^4y' = \sin^3 x$$

$$(uv)' = u'v + uv'$$

$$(x^4 y)' = \sin^3 x$$

$$x^4 y = \int \sin^3 x \, dx$$

$$\int \sin^3 x \, dx = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x \, dx \end{array} \right| = \int (t^2 - 1) \, dt =$$

$$\sin^2 x = 1 - \cos^2 x$$

$$= \frac{t^3}{3} - t + c = \frac{1}{3}\cos^3 x - \cos x + C$$

the general solution:

$$y = c x^{-4} - x^{-4}\cos x + \frac{1}{3} x^{-4} \cos^3 x$$

Ex. 9 (Stewart), p.620

$$xy' + y = \sqrt{x}$$

Integrating Factor Method

$$(xy)' = \sqrt{x}$$

$$xy = \frac{2}{3}\sqrt{x^3} + C$$

the general solution:

$$y = \frac{2}{3}\sqrt{x} + \frac{C}{x}$$

Ex. 9 (Stewart), p.620

$$xy' + y = \sqrt{x}$$

Integrating Factor Method

$$(xy)' = \sqrt{x}$$

$$xy = \frac{2}{3}\sqrt{x^3} + C$$

the general solution:

$$y = \frac{2}{3}\sqrt{x} + \frac{C}{x}$$

$$\int \sqrt{x}\, dx =$$

$$= \int x^{\frac{1}{2}} dx =$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C =$$

$$= \frac{x^{3/2}}{\frac{3}{2}} + C$$

$$= \frac{2}{3} x\sqrt{x} + C$$

Ex.10, p.620

$$y' + y = \sin e^{x}$$

Method I (Integrating Factor)

$$y' + P(x)y = Q(x)$$

$$I(x) = e^{\int P(x)\,dx} \ \cdots$$

$$I(x) = e^{x}$$

$$e^{x}y' + e^{x}y = e^{x}\sin e^{x}$$

$$(e^{x}y)' = e^{x}\sin e^{x}$$

$$e^{x}y = \int e^{x}\sin e^{x}\,dx = -\cos e^{x} + c$$

$$y = -e^{-x}\cos e^{x} + ce^{-x}$$

the general solution

Method II (Variation Of Parameter)

Linear Nonhomogeneous ODE, I $\longrightarrow$ $\underbrace{y' + P(x)y} = Q(x)$

Linear Homogeneous ODE, I $\longrightarrow$ $y' + P(x)y = 0$

$$\frac{dy}{dx} = -P(x)y$$

$$\frac{dy}{y} = -P(x)\,dx$$

$$\ln|y| = -\int P(x)\,dx + C$$

$$y = Ce^{-\int P(x)\,dx}, \quad C \in \mathbb{R}$$

the general solution
of Lin Hom. ODE, I

Method II (Variation Of Parameter)

Linear Nonhomogeneous ODE, I $\longrightarrow$ $y' + P(x)y = Q(x)$

Linear Homogeneous ODE, I $\longrightarrow$ $y' + P(x)y = 0$

the general solution of Lin Hom. ODE, I $y_0 = c\, e^{-\int P(x)dx}$, $c \in \mathbb{R}$

$c(x) - ?$

$y = c(x) \cdot e^{-\int P(x)dx}$ is a solution of Lin Nonhom. ODE, I

Method II (Variation Of Parameter)

Linear Nonhomogeneous ODE, I $\rightarrow$ $\underbrace{y' + P(x)y} = Q(x)$

Linear Homogeneous ODE, I $\rightarrow$ $y' + P(x)y = 0$

the general solution of Lin Hom. ODE, I $y_0 = C e^{-\int P(x)dx}, \ C \in \mathbb{R}$

$C(x) - ?$

$y = C(x) \cdot e^{-\int P(x)dx}$ is a solution of Lin Nonhom. ODE, I

$C'(x) e^{-\int P(x)dx} + C(x) e^{-\int P(x)dx} \cdot \left(-\int P(x)dx\right)' + \ldots$

Method II (Variation Of Parameter)

Linear Nonhomogeneous ODE, I $\longrightarrow$ $y' + P(x)y = Q(x)$

Linear Homogeneous ODE, I $\longrightarrow$ $y' + p(x)y = 0$

the general solution of Lin Hom. ODE, I $y_0 = c\, e^{-\int P(x)dx}, \; c \in \mathbb{R}$

$c(x) - ?$

$y = c(x) \cdot e^{-\int P(x)dx}$ is a solution of Lin Nonhom. ODE, I

$$\underbrace{c'(x) e^{-\int P(x)dx} + c(x) e^{-\int P(x)dx} \cdot (-P(x))} + \ldots$$

Method II (Variation Of Parameter)

Linear Nonhomogeneous ODE, I $\longrightarrow$ $y' + P(x)y = Q(x)$

Linear Homogeneous ODE, I $\longrightarrow$ $y' + P(x)y = 0$

the general solution of Lin Hom. ODE, I $y_0 = c\, e^{-\int P(x)dx}, \; c \in \mathbb{R}$

$c(x) - ?$

$y = c(x) \cdot e^{-\int P(x)dx}$ is a solution of Lin Nonhom. ODE, I

$$c'(x) e^{-\int P(x)dx} + c(x) e^{-\int P(x)dx} \cdot (-P(x)) +$$

$$+ P(x) \cdot c(x) \cdot e^{-\int P(x)dx} = Q(x)$$

Method II (Variation Of Parameter)

Linear Nonhomogeneous ODE, I $\rightarrow y' + P(x)y = Q(x)$

Linear Homogeneous ODE, I $\rightarrow y' + P(x)y = 0$

the general solution of Lin Hom. ODE, I $y_0 = C e^{-\int P(x)dx}, \; C \in \mathbb{R}$

$C(x) - ?$

$y = C(x) \cdot e^{-\int P(x)dx}$ is a solution of Lin Nonhom. ODE, I

$$C'(x) e^{-\int P(x)dx} + C(x) e^{-\int P(x)dx} \cdot (-P(x)) +$$

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Linear Nonhomogeneous ODE, I $\longrightarrow$ $y' + P(x)y = Q(x)$

Linear Homogeneous ODE, I $\longrightarrow$ $y' + P(x)y = 0$

the general solution of Lin Hom. ODE, I $y_0 = c\, e^{-\int P(x)dx}, \; c \in \mathbb{R}$

$c(x) - ?$

$y = C(x) \cdot e^{-\int P(x)dx}$ is a solution of Lin Nonhom. ODE, I

$$C'(x)e^{-\int P(x)dx} + \cancel{C(x)e^{-\int P(x)dx} \cdot (-P(x))} + \cancel{P(x) \cdot C(x) \cdot e^{-\int P(x)dx}} = Q(x)$$

Method II (Variation Of Parameter)

Linear Nonhomogeneous ODE, I $\longrightarrow$ $y' + P(x)y = Q(x)$

Linear Homogeneous ODE, I $\longrightarrow$ $y' + P(x)y = 0$

the general solution of Lin Hom. ODE, I $y_0 = c\,e^{-\int P(x)dx}$, $c \in \mathbb{R}$

$c(x) - ?$

$y = c(x) \cdot e^{-\int P(x)dx}$ is a solution of Lin Nonhom. ODE, I

$c'(x)e^{-\int P(x)dx} = Q(x)$

Method II (Variation Of Parameter)

Linear Nonhomogeneous ODE, I $\rightarrow$ $y' + P(x)y = Q(x)$

Linear Homogeneous ODE, I $\rightarrow$ $y' + P(x)y = 0$

the general solution of Lin Hom. ODE, I $y_0 = C e^{-\int P(x)dx}, \ C \in \mathbb{R}$

$C(x) - ?$

$y = C(x) \cdot e^{-\int P(x)dx}$ is a solution of Lin Nonhom. ODE, I

$C'(x) = Q(x) e^{\int P(x)dx}$

Method II (Variation Of Parameter)

Linear Nonhomogeneous ODE, I $\longrightarrow$ $y' + P(x)y = Q(x)$

Linear Homogeneous ODE, I $\longrightarrow$ $y' + P(x)y = 0$

the general solution of Lin Hom. ODE, I $y_0 = C e^{-\int P(x)dx}, \; C \in \mathbb{R}$

$c(x) - ?$

$y = C(x) \cdot e^{-\int P(x)dx}$ is a solution of Lin Nonhom. ODE, I

$$c'(x) = Q(x) e^{\int P(x)dx}$$

$$c(x) = \int Q(x) e^{\int P(x)dx} dx + C$$

Method II (Variation Of Parameter)

Linear Nonhomogeneous ODE, I $\rightarrow$ $y' + P(x)y = Q(x)$

Linear Homogeneous ODE, I $\rightarrow$ $y' + P(x)y = 0$

the general solution of Lin Hom. ODE, I $y_0 = c\, e^{-\int P(x)dx}$, $c \in \mathbb{R}$

$c(x) - ?$

$y = c(x) \cdot e^{-\int P(x)dx}$ is a solution of Lin Nonhom. ODE, I

$$c'(x) = Q(x)\, e^{\int P(x)dx}$$

$$c(x) = \int Q(x)\, e^{\int P(x)dx} dx + C$$

the general solution of Lin Nonhom. ODE, I :

$$y = e^{-\int P(x)dx}\left[\int Q(x)\, e^{\int P(x)dx} + C\right]$$

Ex.10, Stewart, p.620

$$y' + y = \sin e^{x}$$

Method II (Variation Of Parameter)

$$y' + y = 0$$

$$\frac{dy}{dx} = -y$$

$$\frac{dy}{y} = -dx$$

Ex.10, p.620 (cont.)

$$y' + y = \sin e^{x}$$

Method II (Variation Of Parameter)

$$y' + y = 0$$

$$\frac{dy}{dx} = -y$$

$$\frac{dy}{y} = -dx$$

$$\ln|y| = -x + C$$

$$y_0 = C \cdot e^{-x}, \quad C \in \mathbb{R}$$

Ex.10, p.620 (cont.)

$$y' + y = \sin e^{x}$$

Method II (Variation Of Parameter)

$$y' + y = 0$$

$$\frac{dy}{dx} = -y$$

$$\frac{dy}{y} = -dx$$

$$\ln |y| = -x + C$$

$$y_0 = C \cdot e^{-x}, \quad C \in \mathbb{R}$$

$$y = C(x)\, e^{-x}$$

$$C(x) - ?$$

Ex.10, p.620 (cont.)

$$y' + y = \sin e^{x}$$

Method II (Variation Of Parameter)

$$y' + y = 0, \quad y_0 = C \cdot e^{-x}$$

$$c(x) - ?, \quad y = c(x) e^{-x}$$

$$c'(x) e^{-x} - c(x) e^{-x} + c(x) e^{-x} = \sin e^{x}$$

$$c'(x) e^{-x} = \sin e^{x}$$

$$c'(x) = e^{x} \sin e^{x}$$

$$c(x) = -\cos e^{x} + C$$

Ex.10, p.620 (cont.)

$$y' + y = \sin e^{x}$$

Method II (Variation Of Parameter)

$$y' + y = 0, \quad y_0 = c \cdot e^{-x}$$

$$c(x) - ?, \quad y = c(x)e^{-x}$$

$$c'(x)e^{-x} - c(x)e^{-x} + c(x)e^{-x} = \sin e^{x}$$

$$c'(x)e^{-x} = \sin e^{x}$$

$$c'(x) = e^{x}\sin e^{x}$$

$$c(x) = -\cos e^{x} + c$$

the general solution:

$$y = -e^{-x}\cos e^{x} + c e^{-x}$$

Method I (Integrating Factor)

Ex.19, Stewart, p.621

$$xy' = y + x^2 \sin x, \qquad y(\pi) = 0$$

$$y' + P(x)y = Q(x)$$

$$y' - \frac{1}{x}y = x \sin x$$

$$I = e^{\int P(x)\,dx} = e^{-\int \frac{dx}{x}} = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$$

$$I = \frac{1}{x}$$

$$\frac{1}{x}y' - \frac{1}{x^2}y = \frac{1}{x}x \sin x$$

Method I (Integrating Factor)

Ex.19, Stewart, p.621 (cont.)

$$xy' = y + x^2 \sin x, \qquad y(\pi) = 0$$

$$y' - \frac{1}{x}y = x \sin x, \qquad I = \frac{1}{x}$$

$$I' = \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$\frac{1}{x}y' - \frac{1}{x^2}y = \sin x$$

$$(Iy)' = y'I + I'y$$

$$\left(\frac{1}{x}y\right)' = \sin x$$

$$\frac{1}{x}y = \int \sin x\, dx = -\cos x + C$$

$$y = Cx - x\cos x$$

Method I (Integrating Factor)

Ex.19, Stewart, p.621 (cont.)

$xy' = y + x^2 \sin x$, $\quad y(\pi) = 0$

$y' - \frac{1}{x} y = x \sin x$

the general solution:

$y = cx - x\cos x$

$y(\pi) = 0$

$0 = c \cdot \pi - \pi \cdot \cos \pi$

$\pi c = -\pi$

$c = -1$

the particular solution:

$y = -x - x\cos x$

Method II (Variation Of Parameter)

Ex.19, Stewart, p.621

$$xy' = y + x^2 \sin x, \quad y(\pi) = 0$$

$$y' - \frac{1}{x} y = x \sin x$$

$$y' - \frac{1}{x} y = 0$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\ln|y| = \ln|x| + C$$

$$y_0 = c x, \quad c \in \mathbb{R}$$

$$y = c(x) \cdot x$$

$$c'(x) x + \cancel{c(x) \cdot 1} - \frac{1}{x} \cancel{c(x) x} = x \sin x$$

$$c'(x) x = x \sin x$$

$$c'(x) = \sin x$$

$$c(x) = -\cos x + C$$

the general solution:

$$y = C \cdot x - x \cos x$$

Ex.19, Stewart, p.621 (cont.)

$$xy' = y + x^2 \sin x, \qquad y(\pi) = 0$$

$$y' - \frac{1}{x} y = x \sin x$$

the general solution:

$$y = cx - x\cos x$$

$$y(\pi) = 0$$

$$0 = c\cdot\pi - \pi\cdot\cos\pi$$

$$\pi c = -\pi$$

$$c = -1$$

the particular solution:

$$y = -x - x\cos x$$

Ex.21, Stewart, p.621

$$xy' + 2y = e^x$$

$$y' + \frac{2}{x}y = \frac{e^x}{x}$$

$$y = c(x)x^{-2}$$

$$c'(x)x^{-2} - 2x^{-3}c(x) + \frac{2}{x}c(x)x^{-2} = \frac{e^x}{x}$$

$$c'(x)x^{-2} = \frac{e^x}{x}$$

$$y' + \frac{2}{x}y = 0$$

$$\frac{dy}{y} = -\frac{2}{x}dx$$

$$\ln|y| = -2\ln|x| + C$$

$$y_0 = c\,x^{-2}$$

$$c'(x) = x\,e^x$$

$$c(x) = \int x e^x\,dx = \int x\,de^x =$$

$$= xe^x - \int e^x\,dx$$

$$= xe^x - e^x + c$$

Ex.21, Stewart, p.621 (cont)

$$xy' + 2y = e^x$$

$$y' + \frac{2}{x}y = \frac{e^x}{x}$$

$$y = c(x)x^{-2}$$

$$c(x) = xe^x - e^x + C$$

the general solution :

$$y = \frac{C}{x^2} + \frac{e^x}{x} - \frac{e^x}{x^2}$$

DE reducible to linear ones

Ex. 22, Adv. Eng. Math., p.35

$$y' + y = y^2, \quad y(0) = -\frac{1}{3}$$

$(x^{\alpha})' = \alpha x^{\alpha-1}$

Bernoulli Eq.

$$u(x) = [y(x)]^{1-2},$$

$$u = y^{-1} \Rightarrow y = \frac{1}{u}$$

the chain rule

$$y' = -\frac{1}{u^2} u'$$

$$y'(x) = \left(u^{-1}(x)\right)' = -u^{-1-1}(x) \cdot u'(x)$$

Ex. 22, Adv. Eng. Math., p.35 (CONT.)

$$y' + y = y^2, \quad y(0) = -\frac{1}{3}$$

Bernoulli Eq.

$$u(x) = [y(x)]^{1-2},$$

$$u = y^{-1} \Rightarrow y = \frac{1}{u}$$

$$y' = -\frac{1}{u^2} u'$$

$$-\frac{1}{u^2} u' + \frac{1}{u} = \frac{1}{u^2}$$

Ex. 22, Adv. Eng. Math., p.35 (CONT.)

$$y' + y = y^2 , \quad y(0) = -\frac{1}{3}$$

Bernoulli Eq.

$$u = y^{-1}$$

$$u' - u = -1$$

Linear DE

but also Separable DE

$$\frac{du}{dx} = u - 1$$

$$\frac{du}{u-1} = dx$$

$$\ln|u-1| = x + C$$

$$u - 1 = c\, e^x , \quad c \in \mathbb{R}$$

$$u = 1 + c\, e^x$$

$$y^{-1} = 1 + c\, e^x$$

Ex. 22, Adv. Eng. Math., p.35 (CONT.)

$$y' + y = y^2 \,, \quad y(0) = -\frac{1}{3}$$

the general solution is

$$y = \frac{1}{1 + ce^{x}}$$

$$-\frac{1}{3} = \frac{1}{1 + c \cdot e^{0}} \Rightarrow 1 + c = -3$$

$$c = -4$$

the particular solution:

$$y = \frac{1}{1 - 4e^{x}}$$

Ex. 27, Adv. Eng. Math., p.35

$$y' = \frac{1}{6e^{y} - 2x}$$

Nonlinear DE

Ex. 27, Adv. Eng. Math., p.35

$$y' = \frac{1}{6e^{y} - 2x}$$

Nonlinear DE

y (x) is unknown f.
and
x is ind. var.

$$y' = \frac{dy}{dx} \quad ?$$

$y(x)$

?

$x(y)$

Ex. 27, Adv. Eng. Math., p.35 (cont.)

$$\frac{dy}{dx} = \frac{1}{6e^y - 2x}$$

nonlinear dif. eq.

$$\frac{dx}{dy} = 6e^y - 2x$$

unknown f. is x(y)
ind. var is y

$$x' + 2x = 6e^y$$

Lin Dif Eq I

CONT.

$$x' + 2x = 6e^{y}$$

Linear DE, with unknown function x(y), where y is an independent variable

$$x' + 2x = 0$$

$$\frac{dx}{dy} = -2x$$

$$\frac{dx}{x} = -2dy$$

$$\ln|x| = -2y + C \Rightarrow x_0 = c\,e^{-2y}$$

$$x' + 2x = 6e^{y}$$

CONT.

$$x' + 2x = 0$$

Linear DE, with unknown function x(y), where y is an independent variable

the general solution of Linear Homogeneous DE:

$$x_0 = ce^{-2y}$$

$$x = C(y)e^{-2y}$$

$c(y) - ?$

$$c'(y)e^{-2y} + \cancel{c(y)e^{-2y}\cdot(-2)} + \cancel{2\cdot C(y)e^{-2y}} = 6e^{y}$$

$$c'(y)e^{-2y} = 6e^{y}$$

$$c'(y) = 6e^{3y} \Rightarrow$$

$$c(y) = 2e^{3y} + C$$

the general solution:

$$x = ce^{-2y} + 2e^{y}.$$