Fourier Transform

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Spring 2023

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Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a given real-valued function. **Fourier Transform** of function $f(t)$ is defined by

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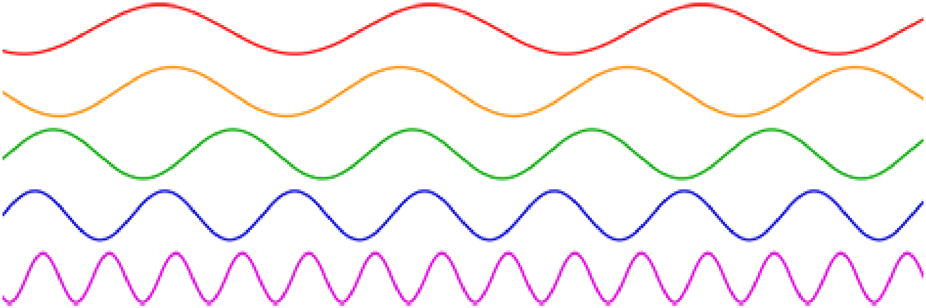
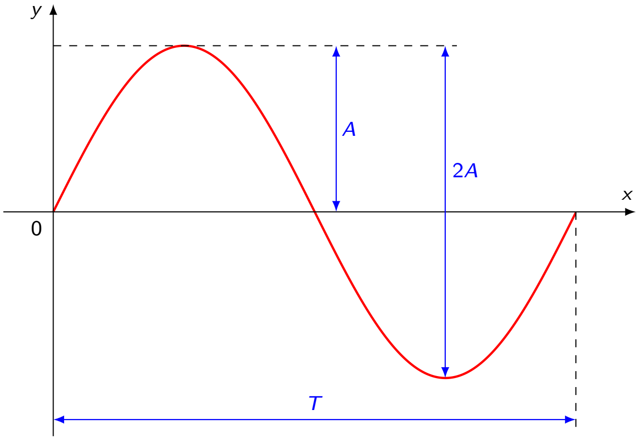
We are saying that Fourier Transform moves us from time domain to frequency domain.

Oscillations



Sinusoid periodic signals



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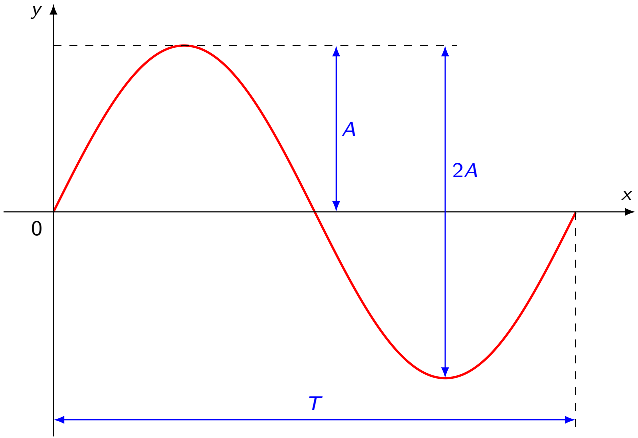
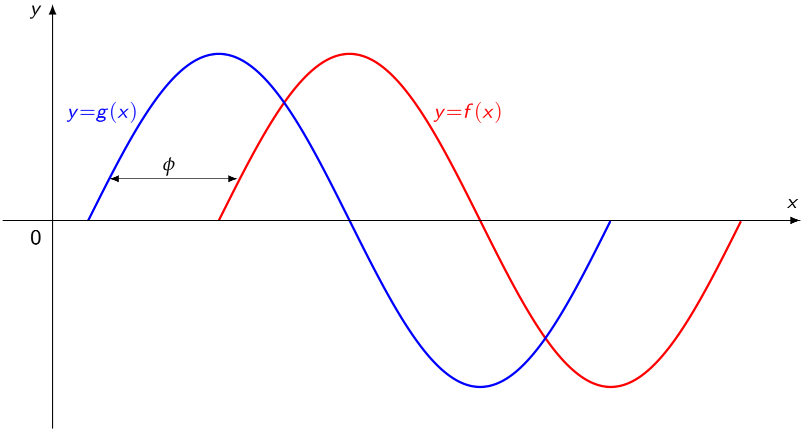
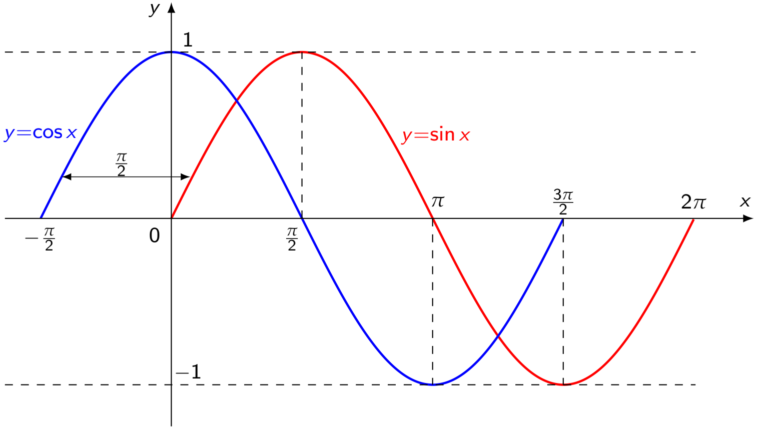


$$y = A\sin(kx), \qquad \text{Amplitude } A, \qquad \text{Period } T = \frac{2\pi}{k}$$

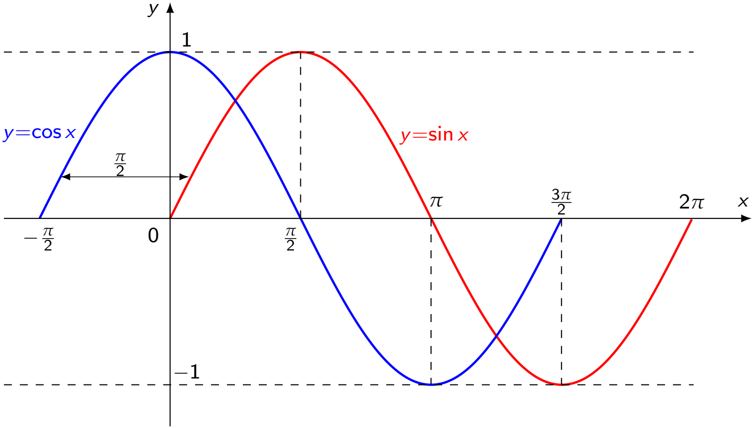
Sinusoid periodic signals. Phase shift



$$f(x+\phi) = g(x), \qquad g(x-\phi) = f(x)$$



Period $\boxed{T = 2\pi}$, Amplitude $\boxed{A = 1}$, Frequency $\boxed{\nu = \frac{1}{T} = \frac{1}{2\pi}}$



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$$\sin\left(x + \frac{\pi}{2}\right) = \cos(x), \qquad \cos\left(x - \frac{\pi}{2}\right) = \sin(x)$$

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We should try to filter out the noise and find the underlying signal properties, like peridoicity, frequency and so on.

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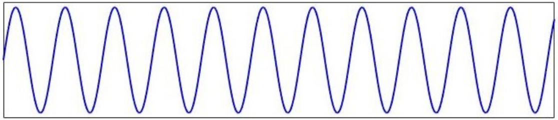

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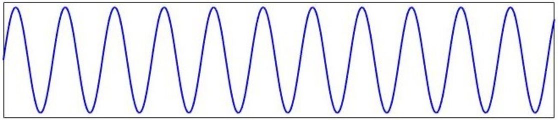

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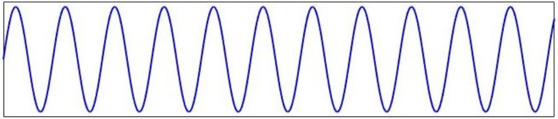

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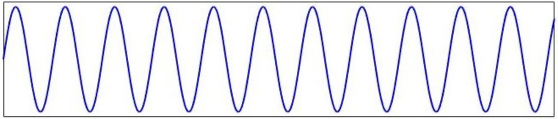

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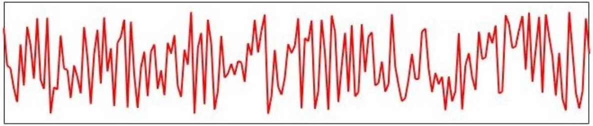

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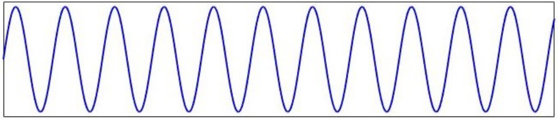

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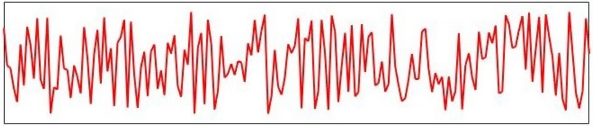
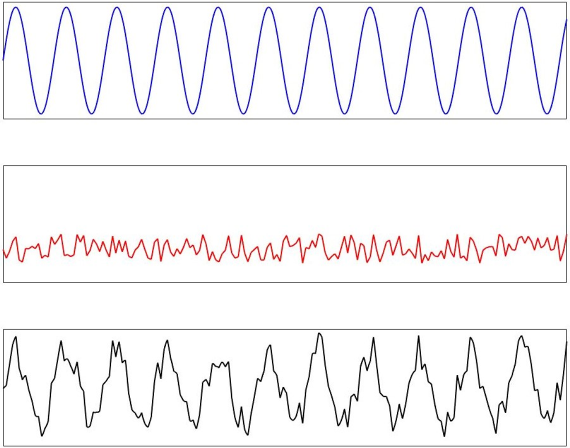


If we increase the strength of the noise, when do we lose the information contained in the original signal?

Signal and noise with amplitude ratio 4:1



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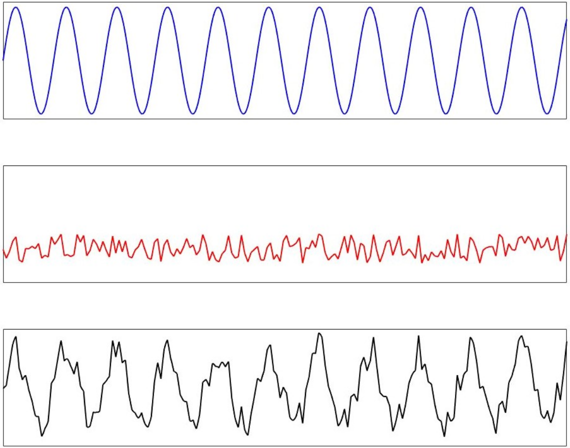

The combined signal + noise is no longer a pure sine wave, of course.

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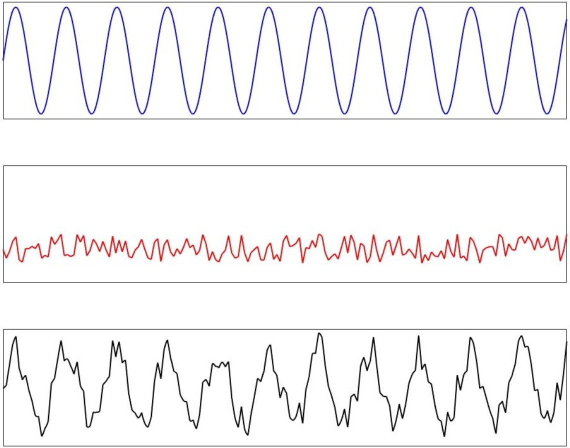
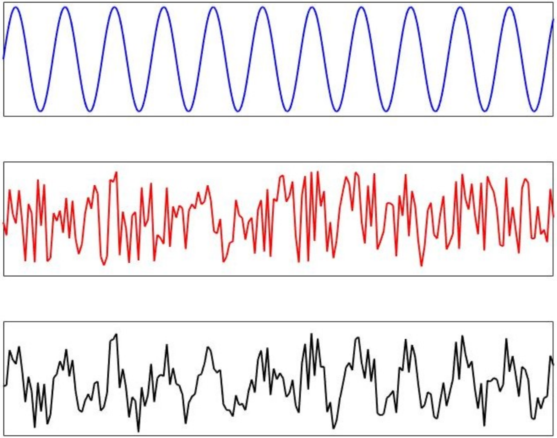


The combined signal + noise is no longer a pure sine wave, of course.

However, it is still regognizable as a noisy wave with the same frequency as the original signal. The information is not yet been lost!

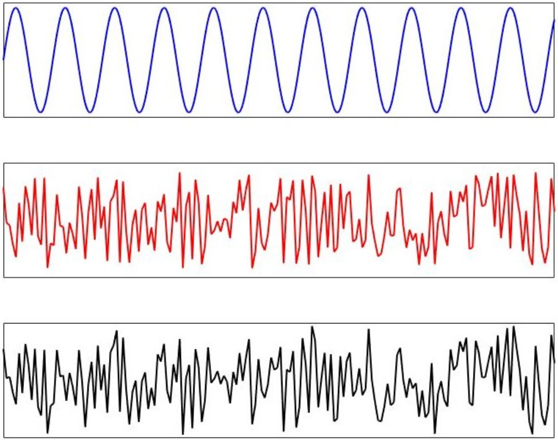
Signal and noise with amplitude ratio 1:1



The combined signal + noise is now very noisy.

It appears that we are losing information from our original signal.

Signal and noise with amplitude ratio 1:2.5



The combined signal + noise is now as random as the noise.

It appears that we definitely lost any information from our original signal.

Detecting the frequency of the signal

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Suppose, signal $S(t)$ is a pure sine function, where t is time measured in seconds. Our test probe is the function:

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What is happening to

$$S(t) \cdot P(t)?$$

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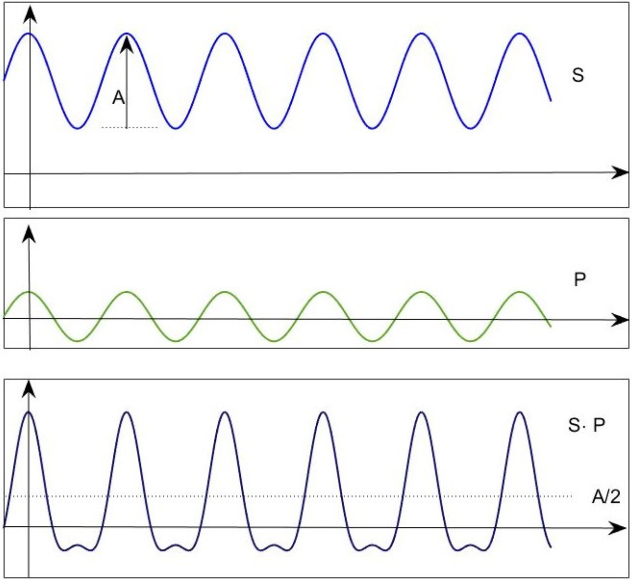

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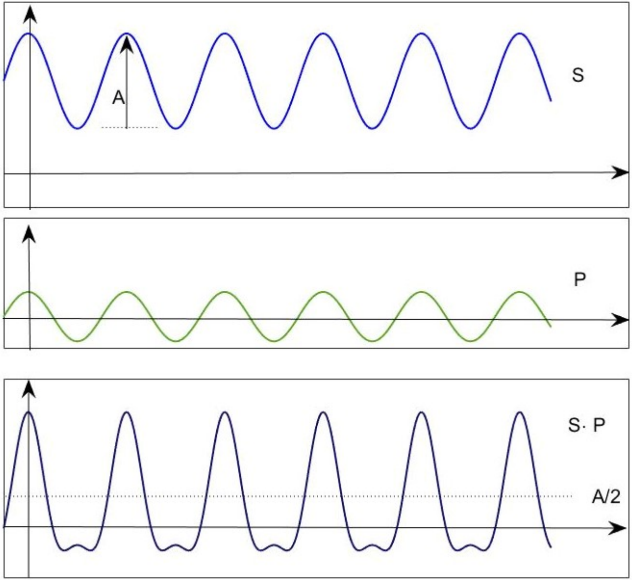

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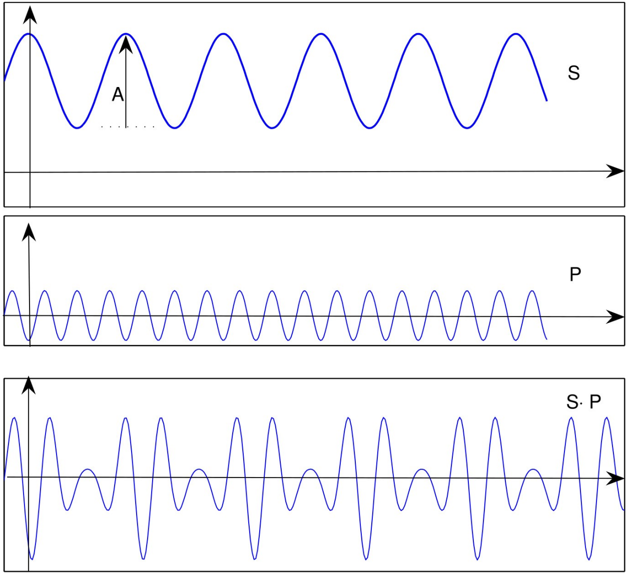

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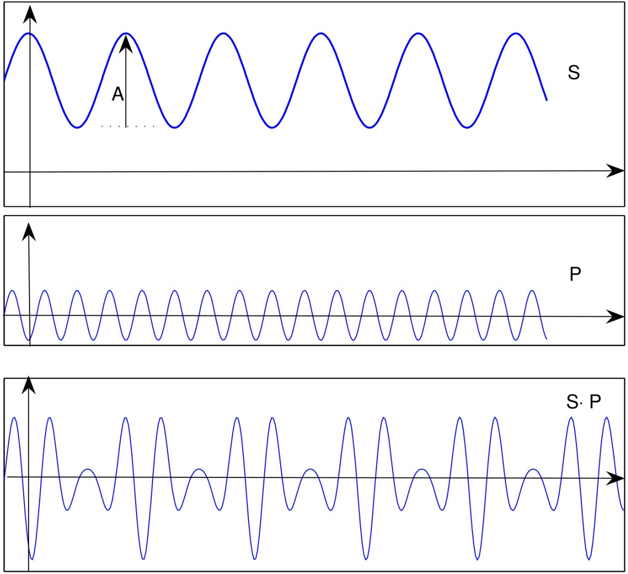

The average value of the product $S(t) \cdot P(t)$ is 0

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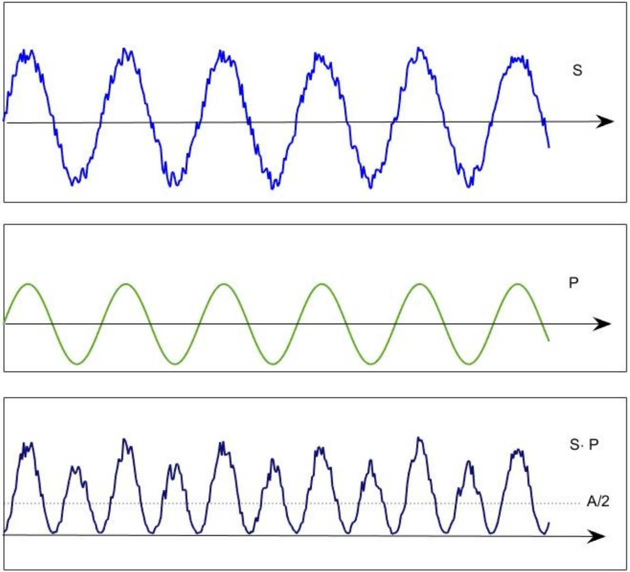

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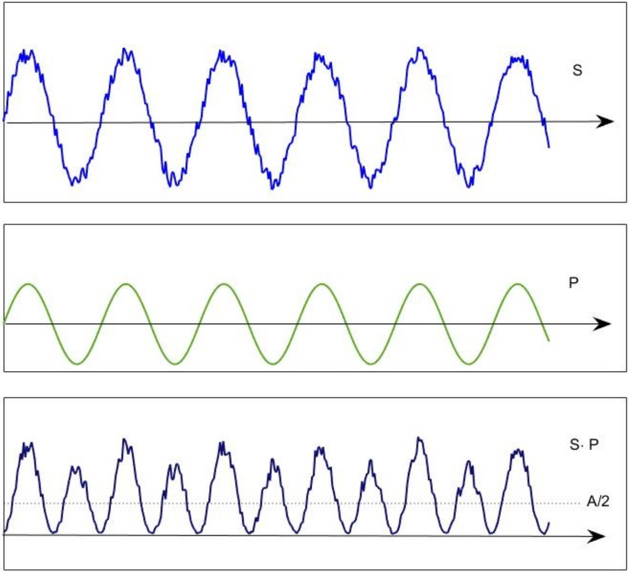

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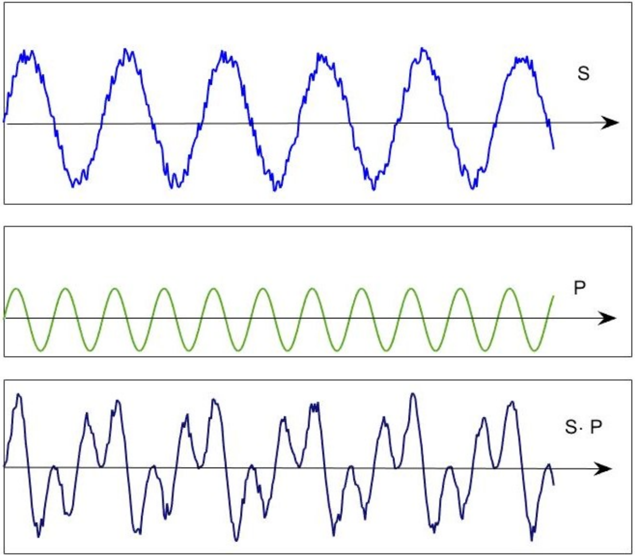

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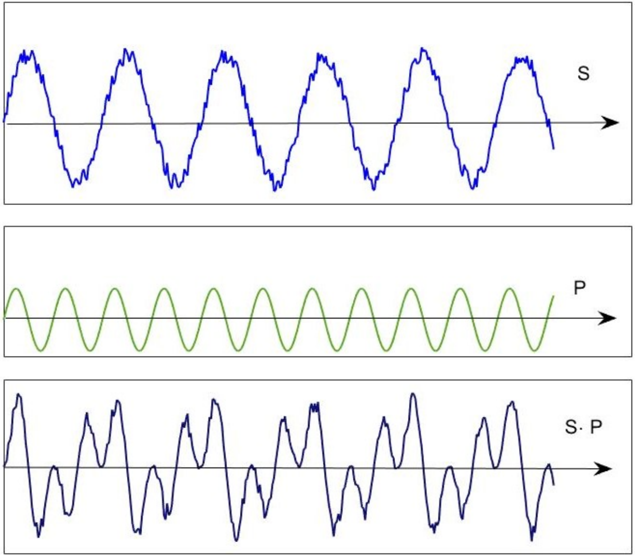

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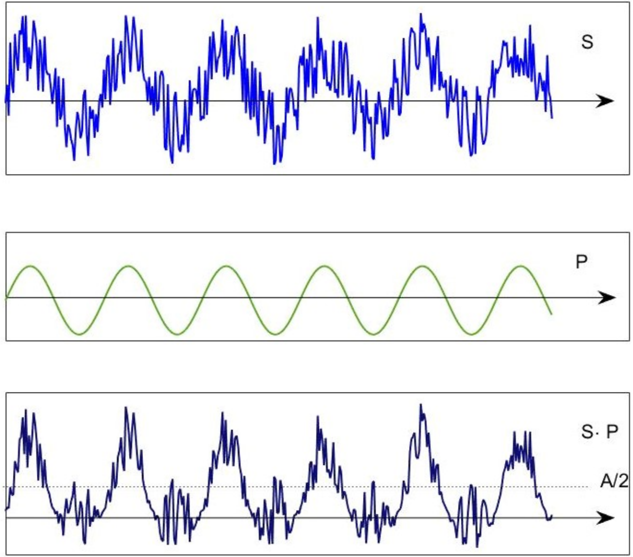

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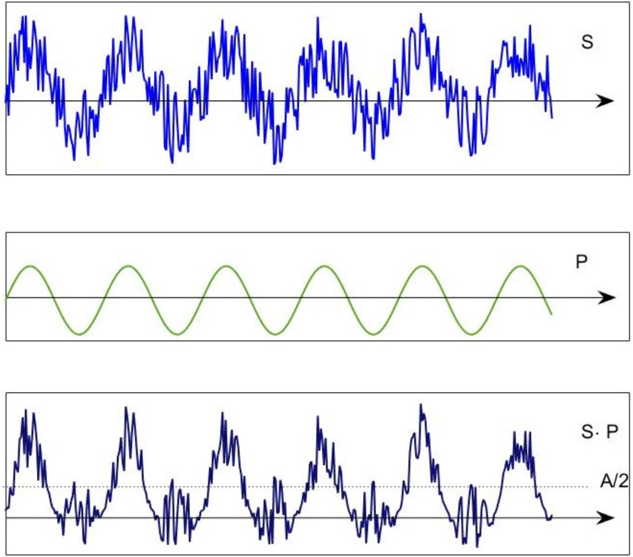

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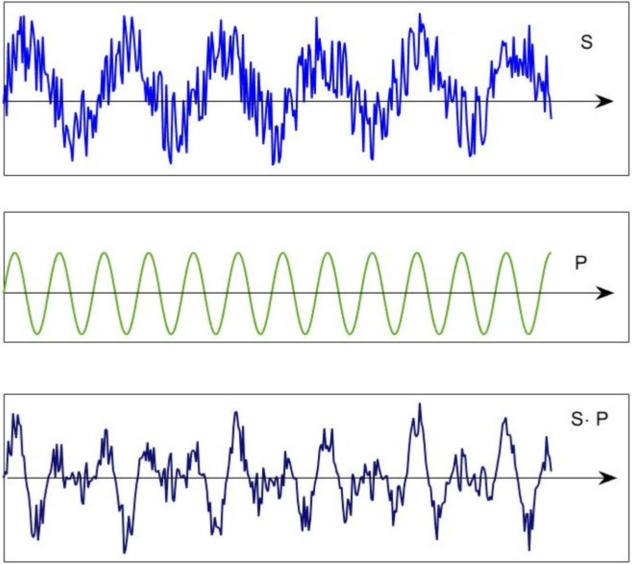

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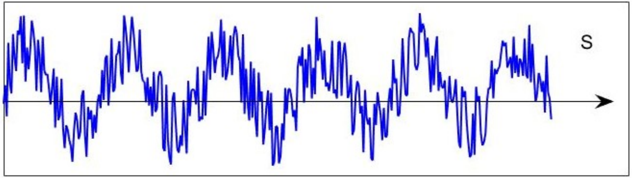

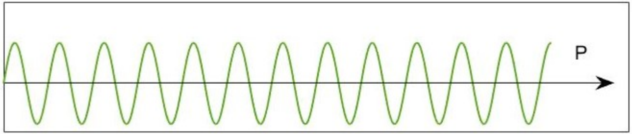

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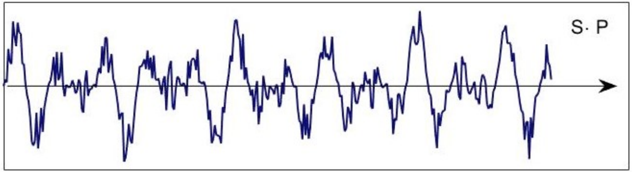

Frequency detector

Recall the following definition:

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Given a function $f : [a, b] \to \mathbb{R}$, the average value of function f on interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(t)dt.$$

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Our detector is the average value of the product of the signal $S(t)$ and the test probe $P(t) = \sin(2\pi\omega t)$:

$$\textbf{Frequency detector} \quad D(\omega) = \tfrac{1}{b-a}\int_a^b S(t)\sin(2\pi\omega t)dt$$

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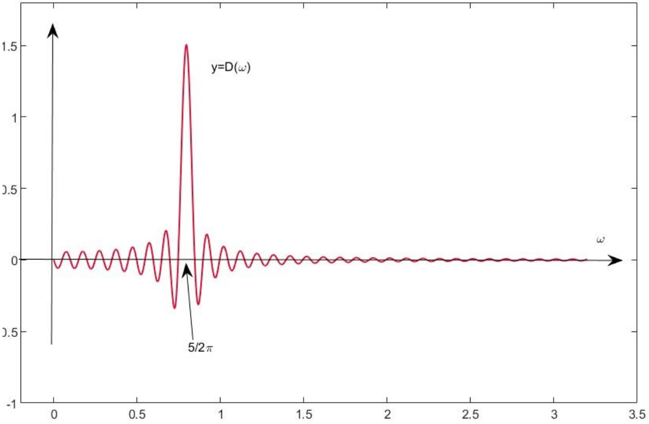
Integrating with respect to t, we get

$$D(\omega) = \frac{3}{10(4\pi^2\omega^2 - 25)}\Big(5\cos(50)\sin(20\pi\omega) - 2\pi\omega\sin(50)\cos(2\pi\omega)\Big).$$

Plot the graph of function $D(\omega)$ (on next page)

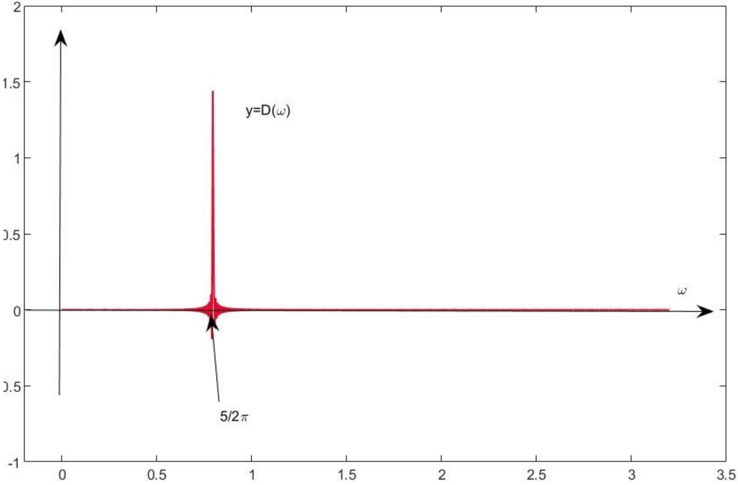
Example 1



For most frequencies the value of detector $D(\omega)$ is close to 0.
There is a single strong peak, which you can find at $w \approx 0.795$ *cycles/s*.
As it happens the frequency of signal $S(t) = 3\sin(5t)$ is $\frac{5}{2\pi} \approx 0.795$.
Moreover the height of the peak is $1.5 = \frac{3}{2}$, half of the amplitude.

Example 1

We can repeat this test for $t \in [0, 60]$ and note that $D(\omega)$ has a sharper peak.



For most signals $D(\omega)$ is computed numerically for a sequence of values of parameter ω.

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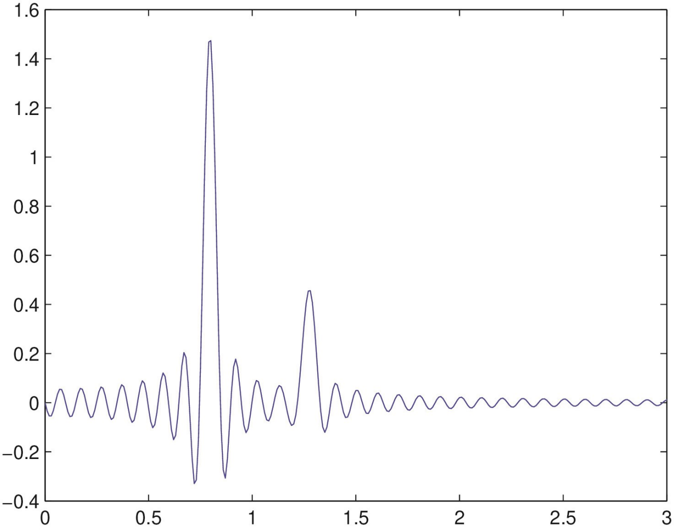

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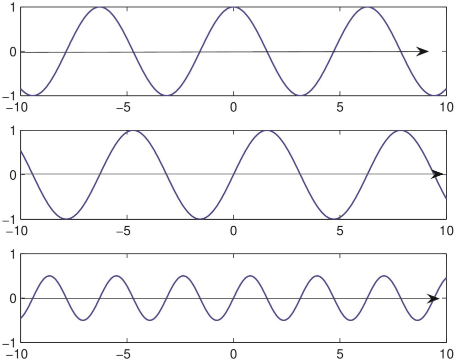
The product $S(t) \cdot P(t)$ (with the probe having the same frequency as the signal) will oscillate around 0 and will have the average value 0.



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In this case, the height of the detector at a peak will equal the amplitude of the signal at that frequency.

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Fourier transforms and their inverses are used in photo restoration, in the enhacement of the digitized pictures sent back from cameras in space and in filtering the audio signal in modern stereo systems.

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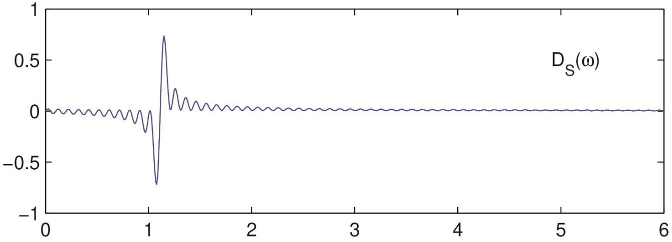

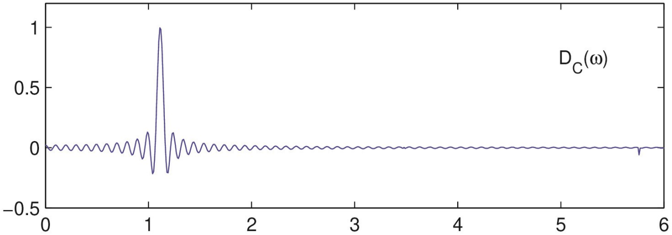

Homemade Frequency Detector. Resonance

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Resonance occurs all around us.

Power Spectrum

Sine and cosine detectors can be combined to determine the only the strength of the different frequencies that occur in the signal:

$$D_P(\omega) = \sqrt{D_S^2(\omega) + D_C^2(w)}.$$

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and

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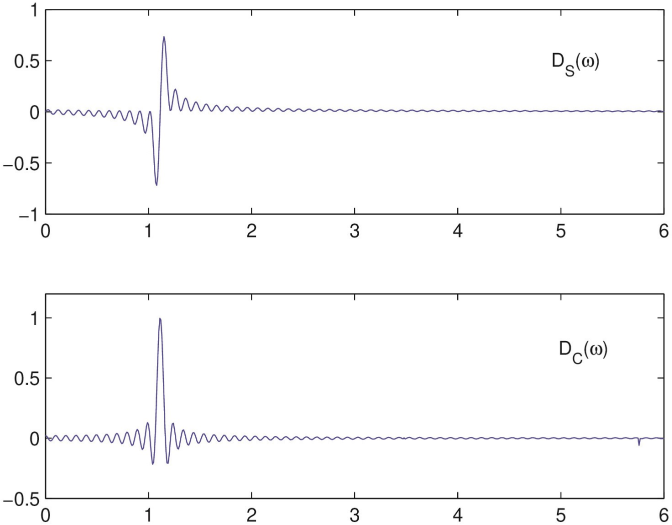
Signal $S(t) = \cos(7t - \varphi)$ and its power spectrum.



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```
function [DDS,DDC,DDP]=detector2(omega,a,b)
  DS = @(c)(quad(@(x) (myfun(x).*sin(2*pi*c.*x)), a, b));
  DC = @(c)(quad(@(x) (myfun(x).*cos(2*pi*c.*x)), a, b));
     for i = 1 : length(omega)
         DDS(i)=2/(b-a)*DS(omega(i));
         DDC(i)=2/(b-a)*DC(omega(i));
     end
  DDP=sqrt(DDS.^2+DDC.^2);
  function y=myfun(x)
      y=cos(4*pi*x);
  end
end
```

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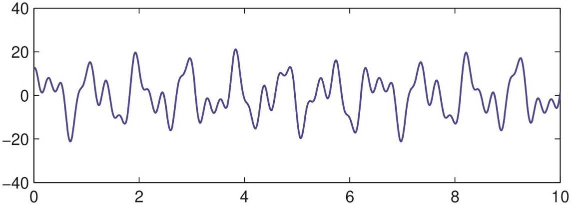

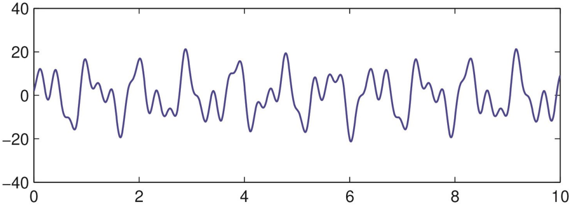

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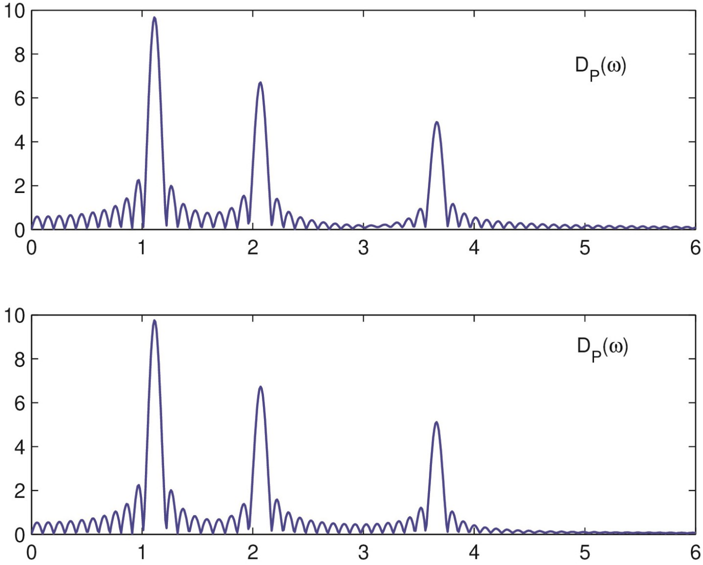

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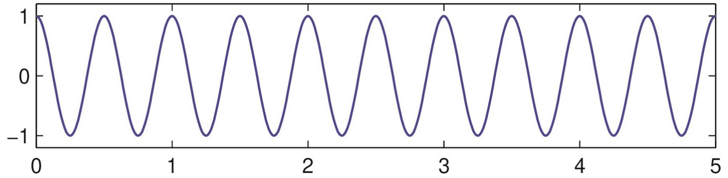
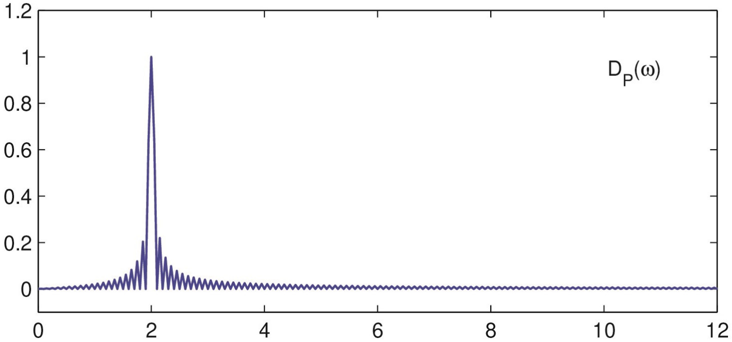
For simplicity we will take the frequency of the pure signal to be 2 cycles per second.

Power Spectrum. Pure signal





Power Spectrum. Signal to noise ration is 4:1

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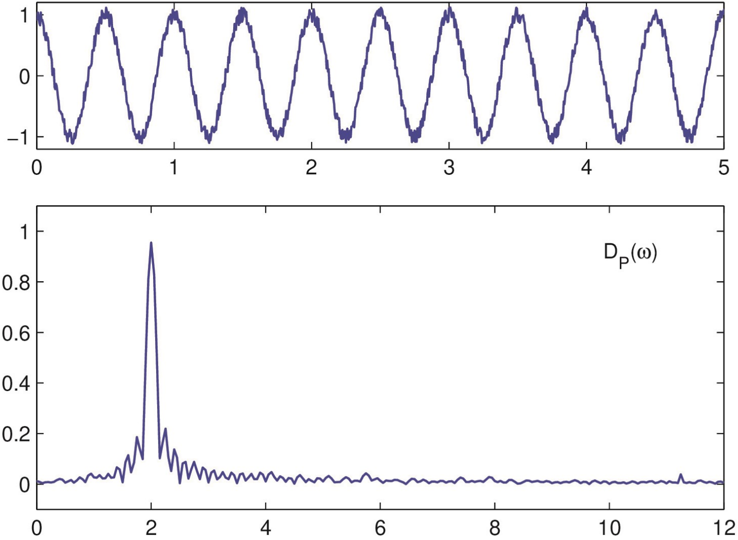

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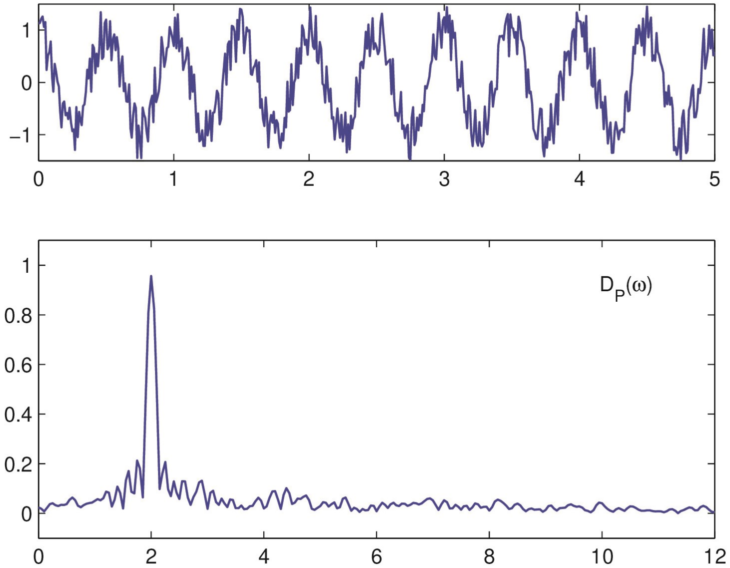

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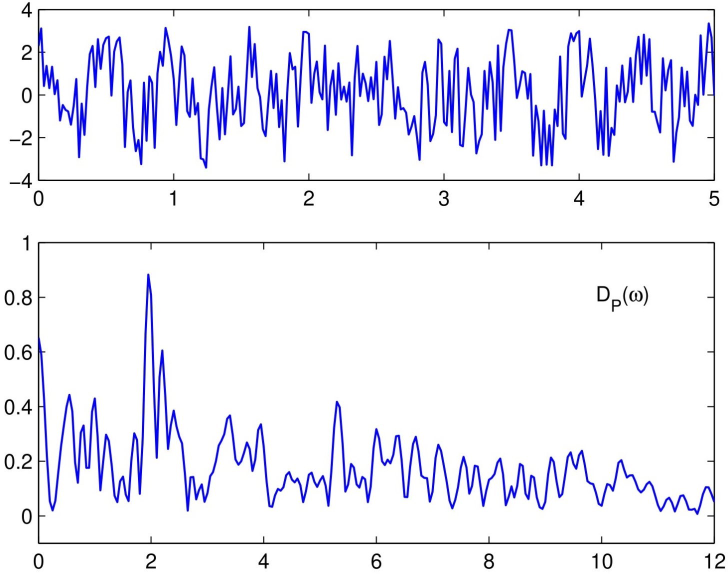

Definition

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We are saying that Fourier Transform moves us from time domain to frequency domain.

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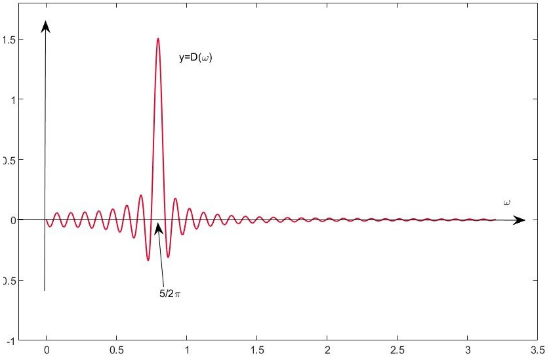
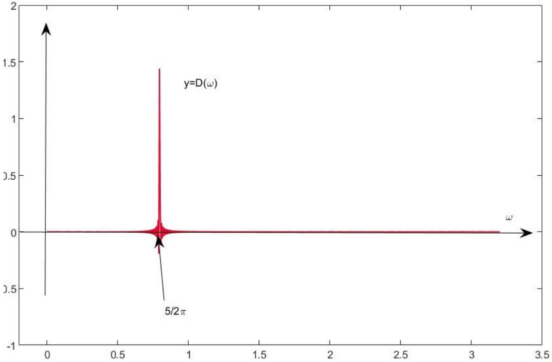
Thus, Cosine detector (Almost Cosine Fourier Transform) is the Real part of the Fourier Transform, whether Sine detector (Almost sine Fourier Transform) is the Imaginary part, respectively.

$$f(t) = 3\sin(5t)$$


y=D(ω)
ω
5/2π
1.5
1
).5
0
).5
-1
0
0.5
1
1.5
2
2.5
3
3.5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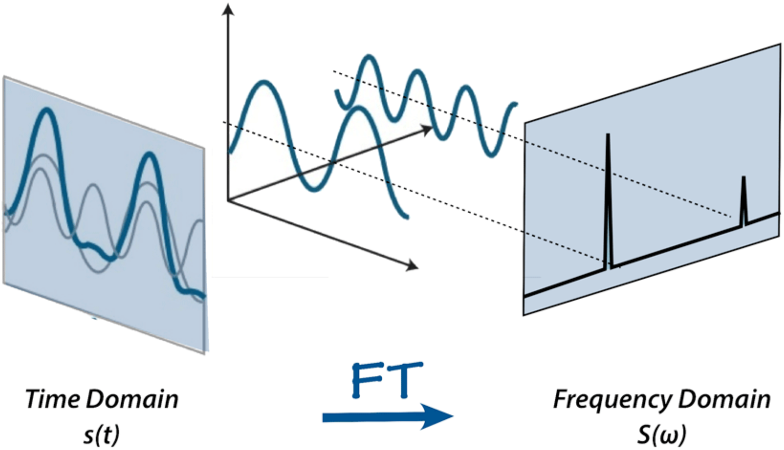

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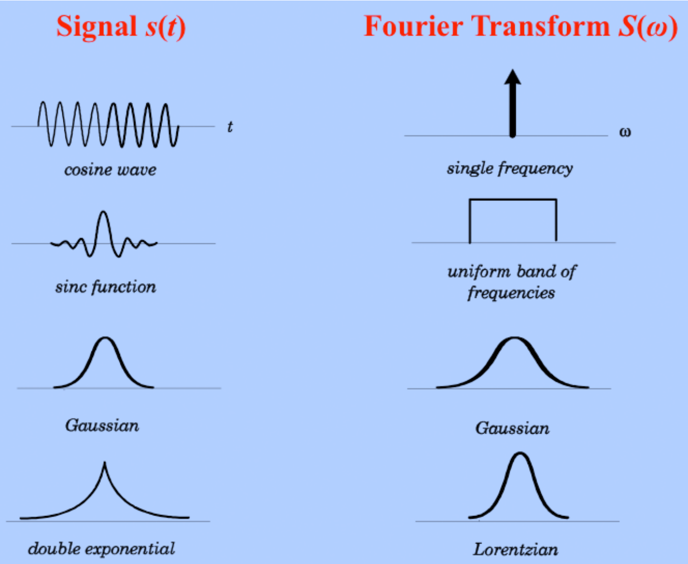

Fourier Transform