## Gamma function

## Funcţia gamma

$\Gamma$ ($\approx$ 1811)

Adrien Marie Legendre (18 sept. 1752 - 10 jan. 1833)

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx \tag{1}$$

(Euler integral of the second kind.)

(1) conv. for $p > 0$
div. for $p \leq 0$.

Γ1 $\Gamma(p+1) = p\Gamma(p) \qquad \text{for } p > 0.$

Γ2 $\Gamma(p+n) = (p+n-1)\cdot(p+n-2)\ldots(p+1)\cdot p\cdot\Gamma(p).$

Γ3 $\Gamma(1) = \int\limits_0^\infty e^{-x}dx = 1 \quad \Rightarrow$

$\Rightarrow \Gamma(n+1) = n! \quad \text{for } n = 0, 1, 2, \ldots$

$[n = 0 \Rightarrow 0! = \Gamma(1) = 1]$

Γ4 Given that $\int\limits_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$ (the Euler-Poisson integral),

show that

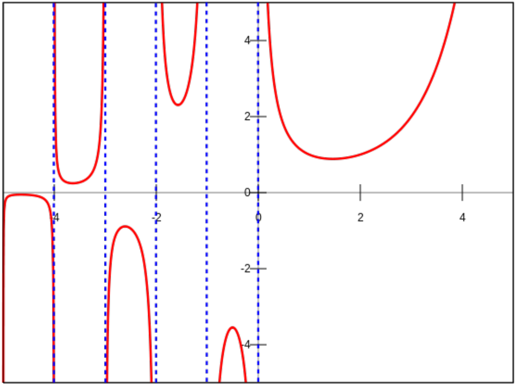
$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

and

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

In view of Γ3, $\Gamma(x+1)$ is often written $x!$ and regarded as a real-valued extension of the factorial function.

Some scientific calculators (in particular, HP calculators) with the factorial function $n!$ built in actually calculate the gamma functions rather than just the integral factorial.




The gamma function along part of the real axis.

The behavior of $\Gamma(x)$ for an increasing positive variable is simple: it grows quickly – faster than an exponential function.

Asymptotically as $x \to \infty$, the magnitude of the gamma function is given by Stirling's formula

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}.$$

The gamma function is a component in various probability-distribution functions, and as such it is applicable in the fields of probability and statistics, as well as combinatorics.

**Beta function**

The integral

$$\int_0^1 x^{p-1}(1-x)^{q-1}dx \tag{2}$$

is called Euler's first integral.

If $p > 0$ and $q > 0$ then (2) is convergent,

if $p \leq 0$ or $q \leq 0$ then (2) is divergent.

# Def. Beta function

Define

Beta function

$$B: \quad (0,\infty)\times(0,\infty)\to\mathbb{R}$$

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx \tag{B}$$

The beta function was studied by Euler and Legendre and was given its name by Jacques Binet; its symbol $B$ is a Greek capital beta rather than the similar Latin capital B or the Greek lowercase $\beta$.

(Jacques Philippe Marie Binet (2 February 1786 – 12 May 1856) was a French mathematician, physicist and astronomer born in Rennes.)

(B1) $B(p,1) = \dfrac{1}{p}$ for each $p > 0$

$B(1,1) = 1$

(B2) $B(\frac{1}{2}, \frac{1}{2}) = \pi$

(B3) $B(p,q) = B(q,p)$, for each $p > 0$ and $q > 0$

(B4) $B(p,q) = \dfrac{p-1}{p+q-1} B(p-1,q)$, for each $p > 1$ and $q > 0$

(B5) $B(p,q) = \dfrac{q-1}{p+q-1} B(p,q-1)$, for each $p > 0$ and $q > 1$

(B6) $B(m,n) = \dfrac{(m-1)!(n-1)!}{(m+n-1)!}$, for each $m, n \in \mathbb{N}^*$

B7 $$B(p,1-p)=\frac{\pi}{\sin p\pi}, \text{ for each } p>0 \text{ and } q>0$$

B $$\Rightarrow \quad x=\sin^2 t \quad \Rightarrow dx=2\sin t\cos t\ dt$$

$$0\le x\le 1 \qquad 0\le t\le \frac{\pi}{2}$$

$$B(p,q)=2\int_0^{\pi/2}\sin^{2p-1} t\cos^{2q-1} t\ dt \quad \Rightarrow$$

B8 $$\int_0^{\pi/2}\sin^m x\cos^n x\,dx=\frac{1}{2}B\left(\frac{m+1}{2},\frac{n+1}{2}\right) \ (m>0,n>0)$$

$$\text{B}\Gamma 9 \quad B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \text{ for each } p > 0 \text{ and } q > 0.$$