Mathematical analysis I

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1. Infinite Series
   - Sequences
   - Summing an Infinite Series
   - Convergence of Series with Positive Terms
   - Absolute and Conditional Convergence
   - The Ratio and Root Tests
   - Power Series
   - Taylor Series

Subsection 1

Sequences

Sequences

- A **sequence** is an ordered collection M of numbers defined by a function $f(n)$ on a set of integers; $f : \mathbb{N} \to M$, $a_n = f(n)$
- The values $a_n = f(n)$ are the **terms** of the sequence and n the **index**;
- We think of $\{a_n\}$ as a list $a_1,\ a_2,\ a_3,\ a_4,\ \ldots$
- The sequence may not start at $n = 1$; It may start at $n = 0, n = 2$ or any other integer;
- When a_n is given by a formula, then it is referred to as the **general term** of the sequence;
- Examples:

General Term	Domain	Sequence
$a_n = 1 - \frac{1}{n}$	$n \geq 1$	$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
$a_n = (-1)^n n$	$n \geq 0$	$0, -1, 2, -3, 4, \ldots$
$a_n = \frac{n^2}{n^2-4}$	$n \geq 3$	$\frac{9}{5}, \frac{16}{12}, \frac{25}{21}, \frac{36}{32}, \frac{49}{45}, \ldots$

Recursively Defined Sequences

- A sequence is defined **recursively** if one or more of its first few terms are given and the n-th term a_n is computed in terms of one or more of the preceding terms $a_{n-1}, a_{n-2}, \ldots$;
- Example: Compute a_2, a_3, a_4 for the sequence defined recursively by

$$a_1 = 1, \quad a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right);$$

$$\begin{aligned}
a_2 &= \frac{1}{2}\left(a_1 + \frac{2}{a_1}\right) = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2};\\
a_3 &= \frac{1}{2}\left(a_2 + \frac{2}{a_2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{1}{2} \cdot \frac{17}{6} = \frac{17}{12};\\
a_4 &= \frac{1}{2}\left(a_3 + \frac{2}{a_3}\right) = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{17/12}\right) = \frac{1}{2} \cdot \frac{577}{204} = \frac{577}{408};
\end{aligned}$$

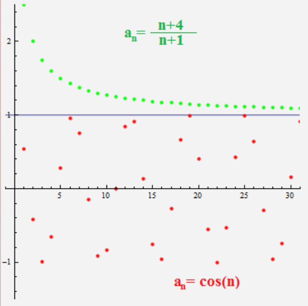
Fibonacci sequences

Stewart, p.691

1,1,2,3,5,8,13,21,...

Limit of a Sequence

- We say that the sequence $\{a_n\}$ **converges** to a limit L, written $\lim_{n\to\infty} a_n = L$ or $a_n \to L$, if the values of a_n get arbitrarily close to the value L when n is taken sufficiently large;
- If a sequence does not converge, we say it **diverges**;
- If the terms increase without bound, $\{a_n\}$ **diverges to infinity**;



Sequence Defined by a Function

Theorem (Limit of a Sequence Defined by a Function)

If $\lim_{x\to\infty} f(x)$ exists, then the sequence $a_n = f(n)$ converges to the same limit, i.e., $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$;

- Example: Show that $\lim_{n\to\infty} a_n = 1$, where $a_n = \dfrac{n+4}{n+1}$;

  We consider the function $f(x) = \dfrac{x+4}{x+1}$; Clearly, $a_n = f(n)$;

  Therefore, by the Theorem, it suffices to show that $\lim_{x\to\infty} f(x) = 1$;

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+4}{x+1} = \lim_{x\to\infty} \frac{1+\frac{4}{x}}{1+\frac{1}{x}} = \frac{1+0}{1+0} = 1;$$

Example I

- Find the limit of the sequence $\frac{2^2-2}{2^2}, \frac{3^2-2}{3^2}, \frac{4^2-2}{4^2}, \frac{5^2-2}{5^2}, \ldots;$

  The general term of the given sequence is $a_n = \frac{n^2-2}{n^2}$; We consider the function $f(x) = \frac{x^2-2}{x^2} = 1 - \frac{2}{x^2}$; Clearly, $a_n = f(n)$; Therefore, it suffices to find the limit $\lim_{x\to\infty} f(x)$;

  $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (1 - \tfrac{2}{x^2}) = 1 - 0 = 1;$$

  Thus, $\lim_{n\to\infty} a_n = 1$;

Example II

- Find the limit $\lim\limits_{n\to\infty}\dfrac{n+\ln n}{n^2}$;

  We consider the function $f(x)=\dfrac{x+\ln x}{x^2}$; Clearly, $a_n=f(n)$;
  Therefore, it suffices to find the limit $\lim\limits_{x\to\infty} f(x)$;

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x+\ln x}{x^2}=$$
$$\left(\frac{\infty}{\infty}\right)\overset{\text{L'Hôpital}}{=}\lim_{x\to\infty}\frac{(x+\ln x)'}{(x^2)'}=\lim_{x\to\infty}\frac{1+(1/x)}{2x}=0;$$

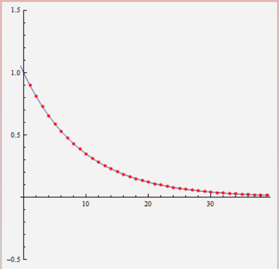
Thus, $\lim\limits_{n\to\infty}\dfrac{n+\ln n}{n^2}=0$;

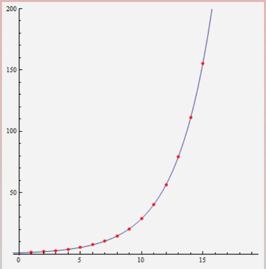
Geometric Sequences

- For $r \geq 0$ and $c > 0$,

$$\lim_{n\to\infty} cr^n = \begin{cases} 0, & \text{if } 0 \leq r < 1 \\ c, & \text{if } r = 1 \\ \infty, & \text{if } r > 1 \end{cases}$$

To see this, one considers the corresponding function $f(x) = cr^x$; If $r < 1$, then, $\lim_{x\to\infty} cr^x = 0$, and, if $r > 1$, then, $\lim_{x\to\infty} cr^x = \infty$;





Limits Laws for Sequences

Limit Laws for Sequences

Assume $\{a_n\}$ and $\{b_n\}$ are convergent sequences with

$$\lim_{n\to\infty} a_n = L, \qquad \lim_{n\to\infty} b_n = M;$$

Then, we have:

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = L \pm M;$
2. $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = LM;$
3. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}$, if $M \neq 0;$
4. $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n = cL$, ($c$ a constant;)

Squeeze Theorem for Sequences

Squeeze Theorem for Sequences

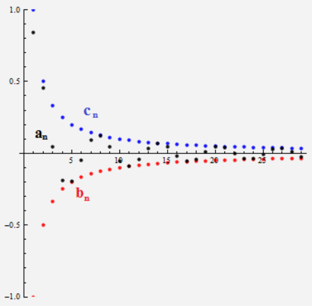
Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences, such that, for some number M,

$$b_n \leq a_n \leq c_n, \text{ for all } n > M$$

and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L;$$

Then $\lim_{n\to\infty} a_n = L;$



- Example: Show that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.
  Note that $-|a_n| \leq a_n \leq |a_n|$; By hypothesis $\lim_{n\to\infty} |a_n| = 0$; This also implies $\lim_{n\to\infty}(-|a_n|) = -\lim_{n\to\infty} |a_n| = 0$; Now, by the Squeeze Theorem for Sequences, $\lim_{n\to\infty} a_n = 0$;

Geometric Sequences with $r < 0$

- For $c \neq 0$,

$$\lim_{n\to\infty} cr^n = \begin{cases} 0, & \text{if } -1 < r < 0 \\ \text{diverges}, & \text{if } r \leq -1 \end{cases}$$

  - If $-1 < r < 0$, then $0 < |r| < 1$ and, therefore $\lim_{n\to\infty} |cr^n| = \lim_{n\to\infty} |c| \cdot |r|^n = 0;$ Thus, since $-|cr^n| \leq cr^n \leq |cr^n|$, by the Squeeze Theorem, we get $\lim_{n\to\infty} cr^n = 0;$
  - If $r = -1$, then $\lim_{n\to\infty} (-1)^n c$ diverges, since $|(-1)^n c| = |c|$ and its sign keeps alternating;
  - If $r < -1$, then $|r| > 1$, whence $|cr^n| = |c| \cdot |r|^n \to \infty$, whence $\lim_{n\to\infty} cr^n$ diverges in this case also;

Exploiting Continuity

Theorem

If $f(x)$ is a continuous function and $\lim_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L);$$

This says, informally speaking, that if f is continuous, we can "push the limit in";

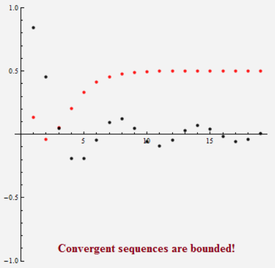
- Example: Since $f(x) = e^x$ and $g(x) = x^2$ are both continuous, we may use this theorem to compute:
  - $\lim_{n\to\infty} e^{\frac{3n}{n+1}} = \lim_{n\to\infty} f(\frac{3n}{n+1}) = f(\lim_{n\to\infty} \frac{3n}{n+1}) = f(3) = e^3;$
  - $\lim_{n\to\infty} (\frac{3n}{n+1})^2 = \lim_{n\to\infty} g(\frac{3n}{n+1}) = g(\lim_{n\to\infty} \frac{3n}{n+1}) = g(3) = 9;$

Bounded Sequences

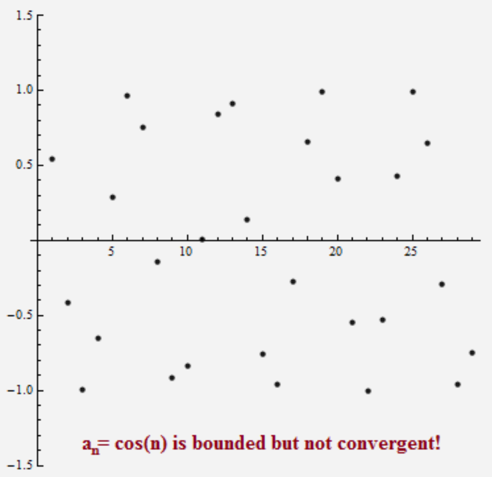
- A sequence $\{a_n\}$ is
  - **bounded from above** if there is a number M, such that $a_n \leq M$, for all n; In this case M is called an **upper bound**;
  - **bounded from below** if there is a number m, such that $a_n \geq m$, for all n; In this case m is called a **lower bound**;
- $\{a_n\}$ is **bounded** if it is bounded from above and from below; A sequence is **unbounded** if it is not bounded;

Theorem

If $\{a_n\}$ converges, then $\{a_n\}$ is bounded;



Is Every Bounded Sequence Convergent?



Bounded Monotonic Sequences

- A sequence $\{a_n\}$ is
  - **increasing** if $a_n < a_{n+1}$, for all n;
  - **decreasing** if $a_n > a_{n+1}$, for all n;
  - **monotonic** if it is either increasing or decreasing;

Theorem (Bounded Monotonic Sequences Converge)

- If $\{a_n\}$ is increasing and $a_n \leq M$, then a_n converges and $\lim_{n\to\infty} a_n \leq M$;
- If $\{a_n\}$ is decreasing and $a_n \geq m$, then a_n converges and $\lim_{n\to\infty} a_n \geq m$;

Example I

- Show that $a_n = \sqrt{n+1} - \sqrt{n}$ is decreasing and bounded from below; Does $\lim\limits_{n\to\infty} a_n$ exist? $[\infty - \infty]$

  We show that a_n is decreasing by two different methods; The first uses the sequence itself, the second uses the corresponding function;

  - Method 1: Rewrite $a_n = \sqrt{n+1} - \sqrt{n} =$
    $\dfrac{(\sqrt{n+1}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \dfrac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$;
    Now we see
    $$a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} > \frac{1}{\sqrt{(n+1)+1}+\sqrt{n+1}} = a_{n+1};$$
    So $\{a_n\}$ is decreasing;
  - Method 2: Consider $f(x) = \sqrt{x+1} - \sqrt{x}$ and compute
    $f'(x) = \dfrac{1}{2\sqrt{x+1}} - \dfrac{1}{2\sqrt{x}} < 0$, for $x > 0$; Thus, since $f' < 0$, we get that $f \searrow [0, \infty)$, showing that $\{a_n\}$ is a decreasing sequence;

  Clearly $a_n = \sqrt{n+1} - \sqrt{n} > 0$, which shows that $\{a_n\}$ is bounded from below;

Example II

- Show that the following sequence is bounded and increasing; Then find its limit:

St.p.701
Ex.79,80

$$a_1 = \sqrt{2}, \quad a_2 = \sqrt{2\sqrt{2}}, \quad a_3 = \sqrt{2\sqrt{2\sqrt{2}}}, \quad \dots$$

The key here is to realize that $a_{n+1} = \sqrt{2a_n}$, for all n;
We show $\{a_n\}$ is bounded: Clearly, $a_1 = \sqrt{2} < 2$; If $a_n < 2$, then $a_{n+1} = \sqrt{2a_n} < \sqrt{2 \cdot 2} = 2$; Therefore, $a_n < 2$, for every $n \geq 1$;
Next, we show that $\{a_n\}$ is increasing:

$$a_n = \sqrt{a_n \cdot a_n} < \sqrt{2 \cdot a_n} = a_{n+1};$$

Since $\{a_n\}$ is increasing and bounded from above, the theorem asserts that it converges; Let $\lim\limits_{n\to\infty} a_n = L$; Then
$a_{n+1} = \sqrt{2a_n} \Rightarrow \lim\limits_{n\to\infty} a_{n+1} = \sqrt{2 \lim\limits_{n\to\infty} a_n} \Rightarrow L = \sqrt{2L} \Rightarrow L^2 = 2L \Rightarrow$
$L^2 - 2L = 0 \Rightarrow L(L-2) = 0 \Rightarrow L = 0$ or $L = 2$; So $\lim\limits_{n\to\infty} a_n = 2$;

Subsection 2

Summing an Infinite Series

Stewart, 11.2, p.703

$\pi = 3,14\ldots$

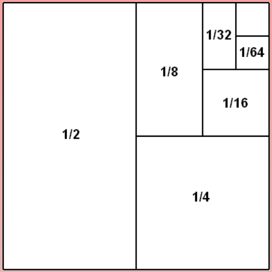
$\pi = 3 + \frac{1}{10} + \frac{4}{100} + \ldots$

Introducing Infinite Series and Partial Sums

- If we look carefully at the figure on the right we realize that

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots;$$

Infinite sums of this type are called **infinite series**;



- The **partial sum** S_N of an infinite series is the sum of the terms up to and including the N-th term:

$$\begin{aligned} S_1 &= \tfrac{1}{2}; \\ S_2 &= \tfrac{1}{2} + \tfrac{1}{4}; \\ S_3 &= \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}; \\ S_4 &= \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16}; \\ &\vdots \end{aligned}$$

Definition of Infinite Series and Partial Sums

- An **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots ,$$

where $\{a_n\}$ is any *sequence*;

- Example:

Sequence	General Term	Infinite Series
$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$	$a_n = \frac{1}{3^n}$	$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$
$\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$	$a_n = \frac{1}{n^2}$	$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots$

- The N**-th partial sum** S_N is defined as the finite sum of the terms up to and including a_N:

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N;$$

Convergence of an Infinite Series

Convergence of an Infinite Series

An infinite series $\sum_{n=k}^{\infty} a_n$ **converges** to the sum S if its partial sums converge to S:

$$\lim_{N\to\infty} S_N = S;$$

In this case, we write $S = \sum_{n=k}^{\infty} a_n$;

- If the limit $\lim_{N\to\infty} S_N$ does not exist, then we say the infinite series **diverges**;
- If $\lim_{N\to\infty} S_N = \infty$, then we say that the infinite series **diverges to infinity**;

Telescoping Series

- Compute the sum S of the infinite series Stewart, p.707, example 7

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \frac{1}{4(5)} + \cdots ;$$

Note that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$; Therefore, we have

$$\frac{1}{1\cdot 2} = 1 - \frac{1}{2}, \quad \frac{1}{2\cdot 3} = \frac{1}{2} - \frac{1}{3}, \quad \frac{1}{3\cdot 4} = \frac{1}{3} - \frac{1}{4}, \quad \ldots$$

Now, we compute the N-th partial sum:

$$S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)} = (1 - \cancel{\tfrac{1}{2}}) + (\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}) + (\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}) + \cdots + (\cancel{\tfrac{1}{N}} - \frac{1}{N+1}) = 1 - \frac{1}{N+1};$$

Therefore, $S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} (1 - \frac{1}{N+1}) = 1 - 0 = 1;$ conv., S=1

Sequence $\{a_n\}$ versus Series $\sum a_n$

- The previous example provides an opportunity to discuss the difference between the sequence $\{a_n\}$ and the infinite series
$$S = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots;$$
- The sequence $a_n = \dfrac{1}{n(n+1)}$ is the list of numbers
$\dfrac{1}{1 \cdot 2}, \quad \dfrac{1}{2 \cdot 3}, \quad \dfrac{1}{3 \cdot 4}, \quad \ldots$ Clearly $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{1}{n(n+1)} = 0;$
- On the other hand, for the *sum of the infinite series* $S = \sum_{n=1}^{\infty} a_n$, we look **not** at $\lim_{n\to\infty} a_n$, but rather at $\lim_{N\to\infty} S_N$, where
$$S_N = \sum_{n=1}^{N} a_n = \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \cdots + \frac{1}{N(N+1)};$$
We saw that this limit is 1, not 0!

Linearity of Infinite Series

Theorem 8, p.709 Stewart

Linearity of Infinite Series

If the infinite series $\sum a_n$ and $\sum b_n$ converge, then the series $\sum (a_n \pm b_n)$ and $\sum ca_n$ also converge and we have

- $\sum a_n + \sum b_n = \sum (a_n + b_n)$;
- $\sum a_n - \sum b_n = \sum (a_n - b_n)$;
- $\sum ca_n = c \sum a_n$;

- In the sequel, we will be interested in establishing techniques for determining whether an infinite series converges or diverges;

Geometric Series

- A **geometric series** with **ratio** $r \neq 0$ is a series defined by the geometric sequence cr^n, where $c \neq 0$;
- The series looks like

$$S = \sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + cr^4 + \cdots ;$$

- The following work determines the N-th partial sum S_N of the geometric series:

$$\begin{aligned} S_N &= c + cr + cr^2 + cr^3 + \cdots + cr^N \\ rS_N &= \quad\; cr + cr^2 + cr^3 + \cdots + cr^N + cr^{N+1} \\ S_N - rS_N &= c - cr^{N+1} \\ S_N(1-r) &= c(1 - r^{N+1}) \\ S_N &= \frac{c(1-r^{N+1})}{1-r}; \end{aligned}$$

the sum of the first N terms of the geometric progression

  - If $|r| < 1$, the the Geometric Series converges and $S = \dfrac{c}{1-r}$;
  - If $|r| \geq 1$, it diverges;

Stewart, example 2, p.705-706

Examples I

- Evaluate $\sum_{n=0}^{\infty} 5^{-n}$; Stewart, p.707, examples 3-6

$$\sum_{n=0}^{\infty} 5^{-n} = \sum_{n=0}^{\infty} (\frac{1}{5})^n \overset{c=1,\, r=\frac{1}{5}<1}{=} \frac{1}{1-\frac{1}{5}} = \frac{5}{4};$$

- Evaluate $\sum_{n=3}^{\infty} 7\left(-\frac{3}{4}\right)^n$;

$$\begin{aligned}
\sum_{n=3}^{\infty} 7(-\tfrac{3}{4})^n &= 7(-\tfrac{3}{4})^3 + 7(-\tfrac{3}{4})^4 + 7(-\tfrac{3}{4})^5 + \cdots \\
&= 7(-\tfrac{3}{4})^3[1 + (-\tfrac{3}{4}) + (-\tfrac{3}{4})^2 + \cdots] \\
&\overset{c=1, r=-\frac{3}{4}}{=} 7(-\tfrac{3}{4})^3 \frac{1}{1-(-\frac{3}{4})} \\
&= -\frac{189}{64} \cdot \frac{4}{7} = -\frac{27}{16};
\end{aligned}$$

Examples II

- Evaluate $S = \sum_{n=0}^{\infty} \frac{2+3^n}{5^n}$;

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{2+3^n}{5^n} \\ &= \sum_{n=0}^{\infty} \frac{2}{5^n} + \sum_{n=0}^{\infty} \frac{3^n}{5^n} \\ &= 2\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n \\ &= 2 \cdot \frac{1}{1-\frac{1}{5}} + \frac{1}{1-\frac{3}{5}} \\ &= 2 \cdot \frac{5}{4} + \frac{5}{2} \\ &= 5; \end{aligned}$$

Theorem.
If the series $\sum_{n=1}^{\infty} a_n$ is convergent $\xrightarrow{\text{to}} S$, then $a_n \to 0$, $n \to \infty$

Proof

$$S_n = \underbrace{a_1 + a_2 + \cdots + a_{n-1}} + a_n$$

$$a_n = S_n - S_{n-1}$$

$n \to \infty$

$$\begin{array}{ccccc} a_n & = & S_n & - & S_{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & S & - & S \end{array}$$

$$P \to Q \iff \neg Q \to \neg P$$

Divergence Test

Stewart, Theorem 7,p.709

Divergence Test

If the n-th term a_n does not converge to 0, i.e., if $\lim_{n\to\infty} a_n \neq 0$, or does not exist then the series $\sum_{n=1}^{\infty} a_n$ diverges;

- Example: Prove the divergence of $S = \sum_{n=1}^{\infty} \frac{n}{4n+1}$;

  Clearly, $\lim_{n\to\infty} \frac{n}{4n+1} = \frac{1}{4} \neq 0$; Thus, by the Divergence Test, S diverges;

Another Example

- Example: Determine the convergence or divergence of
$$S = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1} = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \cdots;$$
The n-th term $a_n = (-1)^{n-1} \dfrac{n}{n+1}$ does not approach a limit; To see this, note that:
    - for even indices,
    $$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} (-1)^{2n-1} \frac{2n}{2n+1} = \lim_{n\to\infty} \frac{-2n}{2n+1} = -1;$$
    - for odd indices,
    $$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} (-1)^{2n+1-1} \frac{2n+1}{2n+1+1} = \lim_{n\to\infty} \frac{2n+1}{2n+2} = 1;$$

  Since $\lim_{n\to\infty} a_n \neq 0$, by the Divergence Test, S diverges;

If $\lim_{n\to\infty} a_n = 0$, Cannot Apply Divergence Test

- Prove the divergence of $S = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$;

  Note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$; Therefore, the Divergence Test cannot be applied; We must find another way to prove that the series diverges; We will use comparison instead!

$$\begin{aligned} S_N &= \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{N}} \\ &\geq \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} + \cdots + \frac{1}{\sqrt{N}} \\ &= N\frac{1}{\sqrt{N}} = \sqrt{N}; \end{aligned}$$

  Now note that $\lim_{N\to\infty} \sqrt{N} = \infty$; Therefore, since $S_N \geq \sqrt{N}$, we also have $\lim_{N\to\infty} S_N = \infty$, showing that S diverges to infinity;