Section 2.2

Exercise 2.2.1

Prove Proposition 2.2.5. (Hint: fix two of the variables and induct on the third.)

Proof.

We use induction on a. First we do the base case $a = 0$ to show $(0 + b) + c = 0 + (b + c)$. By the definition of addition, we have $(0 + b) + c = b + c$ and $0 + (b + c) = b + c$. Thus the base case is done. Now suppose inductively that $(a + b) + c = a + (b + c)$, now we have to prove $((a{+}{+}) + b) + c = (a{+}{+}) + (b + c)$. By the definition of addition, $((a{+}{+}) + b) + c = ((a + b){+}{+}) + c = ((a + b) + c){+}{+}$ and $(a{+}{+}) + (b + c) = (a + (b + c)){+}{+}$. By axiom 2.4 and the hypothesis, $((a{+}{+}) + b) + c = (a{+}{+}) + (b + c)$ thus we have closed the induction. □

Exercise 2.2.2

Prove Lemma 2.2.10. (Hint: use induction.)

Proof. We use induction on a. Let the property $P(a)$ to be "if a is a positive number, then there exists exactly one natural number b such that $b{+}{+} = a$ ". First we do the base case $a = 0$. $P(0)$ is true since the hypothesis is vacuous, thus the base case is done. Now suppose inductively $P(a)$ is true, now we need to show $P(a{+}{+})$ is true. By Axiom 2.3 and Definition 2.2.7, $a{+}{+}$ is a positive number and hence we need to prove the conclusion is ture. Obviously there exists at least one natural number $b{+}{+} = a{+}{+}$ by Axiom 2.2. For the sake of contradiction that there exists more than one natural number of which the successor is $a{+}{+}$, by Axiom 2.4, a contradiction. Thus there exists exactly one natural number b such that $b{+}{+} = a{+}{+}$, thus we have closed the induction. □

Exercise 2.2.3

Prove Proposition 2.2.12. (Hint: you will need many of the preceding propositions, corollaries, and lemmas.)

Proof.

For (a) (Order is reflexive), by Lemma 2.2.2 and $a = a + 0$, and Definition 2.2.11, $a \geq a$ is proved.

For (b) (Order is transitive), if $a \geq b$ and $b \geq c$, by Definition 2.2.11 there exists a natural number k_0 such that $a = b + k_0$ and k_1 such that $b = c + k_1$. By substitution, we have $a = (c + k_1) + k_0$. By Proposition 2.2.5, $a = (c + k_1) + k_0 = c + (k_1 + k_0)$. Since $k_0 + k_1$ is a natural number (can be proved by induction), we have $a \geq c$.

For (c) (Order is anti-symmetric), For the sake of contradiction that $a \neq b$. By the trichotomy of order of natural numbers we have $a > b$ or $a < b$. If $a > b$, a contradiction to $a \leq b$. If $a < b$, a contradiction to $a \geq b$. Thus we have proved $a = b$.

For (d) (Addition preserves order), to prove $a \geq b$ iff $a + c \geq b + c$, we need to show both $a \geq b$ implies $a + c \geq b + c$ and $a + c \geq b + c$ implies $a \geq b$. For the first implication, let $x = a + c$ and $y = b + c$, we need to show $x \geq y$. By definition of ordering, $a \geq b$ iff there exists a natural number k such that $a = b + k$. Then we have $x = (b + k) + c = (b + c) + k = y + k$. Thus we have proved the first implication. For the second implication, suppose $a < b$, we have $a + c < b + c$ (can be proved in the way similar to the first implication), a contradiction. Thus we have proved the second implication.

For (e), to prove $a < b$ iff $a{+}{+} \leq b$, we need to prove both $a < b$ implies $a{+}{+} \leq b$ and $a{+}{+} \leq b$ implies $a < b$. For the first implication, there exists a positive number k_1 (by (f)) such that $b = a + k_1$. By Lemma 2.2.10 there exists a natural number k_2 such that $b = a + k_1 = a + k_2{+}{+} = (a + 1) + k_2 = a{+}{+} + k_2$, thus we have proved $a{+}{+} \leq b$. For the second implication, there exists a natural number k_3 such that $b = a{+}{+} + k_3 = (a+1) + k_3 = a + (1 + k_3)$. By Proposition 2.2.8, $1 + k_3$ is a positive number, thus $a < b$. Thus we have proved the second implication.

For (f), to prove $a < b$ iff $b = a + d$ for some positive number d, we need to prove $a < b$ implies $b = a + d$ and $b = a + d$ implies $a < b$, for some positive number d. For the first implication, we have $a = b + d$ for some natural number d and $a \neq b$. For the sake of contradiction that $d = 0$, we have $a = b$, a contradiction. Thus d is positive by Definition 2.2.7. For the second implication, by Definition 2.2.11 we have $a \leq b$, and we need to show $a \neq b$. For the sake of contradiction that $a = b$, by Proposition 2.2.6 $a + 0 = b + d$ implies $d = 0$, a contradiction. Thus we have proved $a < b$.

□

Exercise 2.2.4

Justify the three statements marked (why?) in the proof of Proposition 2.2.13.

Proof.

To prove "$0 \leq b$ for all natural number b" we use induction. First we do the base case $0 \leq 0$, by Proposition 2.2.12 (a) it is true. Now suppose inductively $0 \leq b$, we need to show $0 \leq b{+}{+}$. Since $b{+}{+} = b + 1$, by Proposition 2.2.12 (f) we have $b < b{+}{+}$, and then by Proposition 2.2.12 (b) we have $0 \leq b{+}{+}$. Thus we have closed the induction.

To prove "if $a > b$, then $a{+}{+} > b$", similar to the proof of the first statement, we have $a{+}{+} > a$ thus $a{+}{+} \geq b$. For the sake of contradiction that $a{+}{+} = b$, we have $b = a + 1$ thus $b > a$, a contradiction. Thus we have proved the statement.

To prove "if $a = b$, then $a{+}{+} > b$ ", we have $a{+}{+} = a + 1 = b + 1$, thus by definition we have proved $a{+}{+} > b$.

□

Exercise 2.2.5

Prove Proposition 2.2.14. (Hint: define $Q(n)$ to be the property that $P(m)$ is true for all $m_0 \leq m < n$; note that $Q(n)$ is vacuously true when $n < m_0$.)

Proof. Use the definition of $Q(n)$ in the hint. We induct on the natural n. First we do the base case $n = 0$. $Q(n)$ is vacuously true for all natural number

m_0 since $m_0 \leq m < 0$ is vacuous, thus we have proved the base case. Now suppose $Q(n)$ is true, we now need to show $Q(n{+}{+})$ is ture. If $Q(n)$ is true, then $P(m)$ is ture for all natural number $m_0 \leq m \leq n$ by the implication. By Proposition 2.2.12 (b) and $n < n{+}{+}$, we have $m_0 \leq m \leq n{+}{+}$. For the sake of contradiction that $m = n{+}{+}$, $m = n + 1$ hence $m > n$, by the trichotomy of ordering, a contradiction. Thus we have proved $Q(n{+}{+})$ is ture and closed the induction.

□

Exercise 2.2.6

Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m{+}{+})$ is true, then $P(m)$ is true. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for all natural numbers $m \leq n$; this is known as the principle of backwards induction. (Hint: apply induction to the variable n.)

Proof. We use induction on n. For the base case we need to show $P(m)$ is true for all natural numbers $m \leq 0$, which is true by definition. Now suppose $P(m)$ is true for all natural numbers $m \leq n$, inductively we need to show that if $P(n{+}{+})$ is ture, then $P(m)$ is true for all natural numbers $m \leq n{+}{+}$. If $P(n{+}{+})$ is true, then $P(n)$ is true. By the hypothesis we have $P(m)$ is true for $m \leq n$. Thus $P(m)$ is true for $m \leq n{+}{+}$, closing the induction.

□