Adaptive Dynamic Programming based Integral Sliding Mode Control Law for Continuous Time Systems: A Design for Inverted Pendulum Systems

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***Abstract*— This work presents an adaptive optimal control algorithm based integral sliding mode control law for a class of continuous-time systems with input disturbance or uncertain and unknown parameters. The main objective is to find an general form of intergral sliding mode control law can assure that the system states are forced to reach a sliding surface in a finite time. Then, an adaptive optimal control based on the adaptive dynamic programming method is responsible for the robust stability of the closed-loop system. Finally, the theoretical analysis and simulation results demonstrate the performance of the proposed algorithm for an inverted pendulum system.**

***Keywords- adaptive dynamic programming (ADP), Integral sliding mode control, Robust control, Unknown system dynamics, Inverted pendulum*.**

I. INTRODUCTION

liding mode control was first proposed in the early 1950s (Utkin, 1977, 1992; Pisano and Usai, 2011), and many studies about the sliding mode control method have been published in recent years (Man and Yu, 1997; Drakunov, 1992; Ting et al., 2012). The most positive feature of sliding mode control consists in the complete compensation of the so-called matched disturbances (i.e., disturbances acting on the control input channel) when the system is in the sliding phase and a sliding mode is enforced. This latter takes place when the state is on a suitable subspace of the state space, called sliding manifold.

The integral sliding mode (ISM) technique was first proposed in [1], [2] as a solution to the reaching phase problem for systems with matched disturbances only. The integral sliding mode has find many application in industrial process like robots and electromechanical systems, etc, [6],[7]. In order to avoid the phase and to let a robustness from the initial time, the concept of integral sliding mode has been introduced [3],[4],[5].

In this paper, we propose the combined idea of adaptive dynamic programming and integral sliding mode control for the purpose of introducting controllers for unknown systems. At that time, besides our concern for system stability, we also pay attention to the bound of the cost function that other articles have been not mentioned.

II. PROBLEM STATEMENTS

We study a class of continuous-time systems described by:

$$\dot{x} = Ax + B\left(u + f\left(x,u,t\right)\right) \qquad (1)$$

Where $x \in \mathbb{R}^n$ is the measured component of the state available for feedback control, $u \in \mathbb{R}^m \left(m \le n\right)$ is the input. Suppose that $A \in \mathbb{R}^{n\times n}$ is unknown constant matrix, $f\left(x,u,t\right) \in \mathbb{R}^m$ is the disturbance or/and uncertain of system.

The control objective is to find an adaptive optimal control based integral sliding mode control law ensures that that the closed-loop system (1) is robustly stable and the cost function $J = \int_0^\infty \left(x^T Qx + u^T Ru\right) d\tau$ is bounded with Q, R are the symmetric definite matrices, $Q \ge 0, R > 0$ ..

Assumption 1: The matrix B has linearly independent columns,i.e. $rank\left(B\right) = m$.

Assumption 2: There exist a constant value $\rho > 0$, a continuous function $\mu(.)$ and a continuous function $\lambda\left(t\right)$ such that $0 < \lambda\left(t\right) < 1; \forall t$; and the disturbance and uncertain of system satisfied:

$$\left\| f\left(x;u;t\right)\right\| < \rho + \mu\left(\left\|x\left(t\right)\right\|\right) + \lambda\left(t\right)\left\|u\right\|$$

We define B^+ is the Moore-Penrose pseudoinverse of matrix B. By **assumption 1**, B^+ can be computed as: $B^+ = \left(B^T B\right)^{-1} B^T$.

Assumption 3: There exists a number $\sigma > 0$ such that: $\|MA\| < \sigma$, where $M = B^+ = \left(B^T B\right)^{-1} B^T$.

III. CONTROL DESIGN

Define the sliding mode $S(t)$ as follows:

$$S(t) = \left\{ x \in \mathbb{R}^n : s(t) = 0 \right\} \tag{2}$$

Where $s(t)$ is defined as:

$$s(t) = Mx - \int_0^t v(\tau)\, d\tau \tag{3}$$

with v is later designed.

Theorem 1: The control signal $u = v - k(t)\dfrac{s}{\|s\|}$, with

$$k(t) = \frac{1}{1-\lambda(t)}\left(\rho + \mu\left(\|x\|\right) + \sigma\|x\| + \lambda(t)\|v(t)\| + c\right) \quad (4)$$ and

c is a positive constant, can be guaranteed that the system states are forced to reach the sliding surface at time $t_s < \infty$.

Proof:

The time derivative of (3) given by:

$$\begin{aligned} \dot{s} &= B^+ B^T \left(Ax + B(u+f)\right) - v(t) \\ &= MAx + (u+f) - v \end{aligned} \tag{5}$$

It follows from **assumption 2**, we have:

$$\begin{aligned} &(1-\lambda)k(t) = \left(\rho + \mu\left(\|x(t)\|\right) + \sigma\|x(t)\| + \lambda\|v(t)\| + c\right) \\ &\Rightarrow k(t) = \rho + \mu\left(\|x(t)\|\right) + \lambda\left(\|v(t)\| + k(t)\right) + \sigma\|x(t)\| + c \\ &\Rightarrow k(t) > \rho + \mu\left(\|x(t)\|\right) + \lambda\left(\|u(t)\|\right) + \|MA\|\|x(t)\| + c \\ &\Rightarrow k(t) > \|f(x,u,t)\| + \|MA\|\|x(t)\| + c \end{aligned} \tag{6}$$

We consider a Lyapunov funtion candidate:

$$V = \frac{1}{2}s^T s \tag{7}$$

The derivative of V is computed as:

$$\begin{aligned} \dot{V} &= s^T\dot{s} = s^T\left[MAx + (u+f) - v\right] \\ &= s^T.\left[MAx + \left(v - k(t).\frac{s}{\|s\|} + f\right) - v\right] \\ &= s^T(MAx + f) - k(t)\|s\| \\ &\le \|s\|\left(\|MA\|\|x\| + \|f\|\right) - k(t)\|s\| \\ \dot{V} &< -c\|s\| = -cV^{1/2} \end{aligned} \tag{8}$$

Integrating (8) over the time interval $0 \le \tau \le t$ we obtain: $V^{1/2}(t) - V^{1/2}(0) \le -\dfrac{1}{2}ct$

Consequently, $V(t)$ can reach zero in a finite time t_s, that is bounded by: $t_s \le \dfrac{2V^{1/2}(0)}{c}$

When $s = \dot{s} = 0$, from (5) we have:

$$u_{eq} + f\left(x, u_{eq}, t\right) = -MAx + v$$

The system can be rewritten:

$$\dot{x} = Ax + B(-MAx + v) = (A - BMA)x + Bv$$

$$\dot{x} = \bar{A}x + \bar{B}v \tag{9}$$

where $\bar{A} = A - BMA; \bar{B} = B$.

Remark 1: it is necessary to ensure that the time of convergence of sliding surface is finite. The fact is described based on the following example:

We consider the system as follows:

$$\begin{cases} \dfrac{dx}{dt} = Ax + Bs \\ \dfrac{ds}{dt} = Cx + Ds \end{cases}$$ where:

$A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times r}, C \in \mathbb{R}^{r\times n}, D \in \mathbb{R}^{r\times r}$, s is the sliding surface. Selecting A is Hurwitz matrix and $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is not Hurwitz matrix. We obtain that although s converges to 0 in infinite time, x does not converge to 0.

Theorem 2: Let K_0 be any matrix such that $\bar{A} - \bar{B}K_0$ is Hurwitz, and repeat the following steps for $k = 0,1,...$

Step 1: Solve for the real symmetric positive definite solution P_k of the Lyapunov equation:

$$\bar{A}_k^T P_k + P_k \bar{A}_k + Q + K_k^T R K_k = 0 \tag{10}$$

where $\bar{A}_k = \bar{A} - \bar{B}K_k$.

Step 2: Update the matrix by:

$$K_{k+1} = R^{-1}\bar{B}^T P_k \tag{11}$$

Then, the following properties hold:

(1) $\bar{A} - \bar{B}K_k$ is Hurwitz

(2) $P^* \le P_{k+1} \le P_k$

(3) $\lim_{k\to\infty} K_k = K^*; \lim_{k\to\infty} P_k = P^*$

with $K^* = R^{-1}\bar{B}^T P^*$ and P^* is a unique symmetric, positive definite matrix such that:

$$\bar{A}^T P^* + P^*\bar{A} + Q - P^*\bar{B}R^{-1}\bar{B}^T P^* = 0 \tag{12}$$

Now, we find an approximate optimal control policy by using the online measurements of the closed-loop system (9).

We consider that: $V(x)=x^T P_k x$ (13)
So, we have:

$$x(t+T)^T P_k x(t+T)-x(t)^T P_k x(t)=\int_t^{t+T}\dot{V}(x(\tau))d\tau$$

$$=\int_t^{t+T} x^T\left(\bar{A}_k^T P_k+P_k\bar{A}_k\right)x+2\left(\bar{B}\left(v_0+K_k x\right)\right)^T P_k x d\tau$$

$$=\int_t^{t+T}\left[-x^T\left(Q+K_k^T R K_k\right)x+2\left(v_0+K_k x\right)^T\bar{B}^T P_k x\right]d\tau \quad (14)$$

Applying Kronecker product representation gives:

$$\left(x\left(t_{j+1,k}\right)^T\otimes x\left(t_{j+1,k}\right)^T-x\left(t_{j,k}\right)^T\otimes x\left(t_{j,k}\right)^T\right)vec\left(P_k\right)=$$

$$\int_{t_{j,k}}^{t_{j+1,k}}\begin{bmatrix}-\left(x^T\otimes x^T\right)vec\left(Q+K_k^T R K_k\right)\\ +2\left(x^T\otimes v_0^T\right)\left(\bar{B}\otimes I\right)vec\left(P_k\right)\\ +2\left(x^T\otimes x^T\right)\left(\bar{B}K_k\otimes I\right)vec\left(P_k\right)\end{bmatrix}d\tau$$

Define:

$$I_1\left(t_{j,k}\right)=x\left(t_{j+1,k}\right)^T\otimes x\left(t_{j+1,k}\right)^T-x\left(t_{j,k}\right)^T\otimes x\left(t_{j,k}\right)^T$$

$$I_2\left(t_{j,k}\right)=2\int_{x_{j,k}}^{x_{j+1,k}}\left(x^T\otimes v_0^T\right)\left(\bar{B}\otimes I\right)d\tau$$

$$I_3\left(t_{j,k}\right)=2\int_{t_{j,k}}^{t_{j+1,k}}\left(x^T\otimes x^T\right)\left(\bar{B}K_k\otimes I\right)d\tau$$

$$I\left(t_{j,k}\right)=I_1\left(t_{j,k}\right)-I_2\left(t_{j,k}\right)-I_3\left(t_{j,k}\right)$$

$$J\left(t_{j,k}\right)=-\int_{t_{j,k}}^{t_{j+1,k}}\left(x^T\otimes x^T\right)vec\left(Q+K_k^T R K_k\right)d\tau$$

Consequently, we have:

$$\begin{bmatrix}I\left(t_{1,k}\right)\\ I\left(t_{2,k}\right)\\ \vdots\\ I\left(t_{s,k}\right)\end{bmatrix}vec\left(P_k\right)=\begin{bmatrix}J\left(t_{1,k}\right)\\ J\left(t_{2,k}\right)\\ \vdots\\ J\left(t_{s,k}\right)\end{bmatrix}$$

So $\Phi_k vec\left(P_k\right)=\Psi_k$ (15) with:

$$\Phi_k=\begin{bmatrix}I\left(t_{1,k}\right)\\ I\left(t_{2,k}\right)\\ \vdots\\ I\left(t_{s,k}\right)\end{bmatrix};\Psi_k=\begin{bmatrix}J\left(t_{1,k}\right)\\ J\left(t_{2,k}\right)\\ \vdots\\ J\left(t_{s,k}\right)\end{bmatrix}$$

Assumption 4: There exists a number $s>0$ such that Φ_k has full column rank for all $k\in Z^+$.

By **assumption 4**, P_k can be uniquely determined by: $vec\left(P_k\right)=\left(\Phi_k^T\Phi_k\right)^{-1}\Phi_k^T\Psi_k$ (16).

Algorithm 1:

(1) Select K_0 such that $\bar{A}-\bar{B}K_0$ is Hurwitz and a threshold $v>0$. Let $k\to 0$.

repeat

(2) Solve P_k from (16)

(3) Update K_{k+1} by using (11).

$k\leftarrow k+1$

until $\left\|P_k-P_{k+1}\right\|<v$

$k^*\leftarrow k$

We obtain the approximated optimal control policy: $v=-K_{k^*}x$

Lemma 1: Under **assumption 4**, by using **algorithm 1**, we have $\lim_{k\to\infty}K_k=K^*;\lim_{k\to\infty}P_k=P^*$.

Proof:

From $(13),(14)$ one see that the $\left(P_k;K_{k+1}\right)$ obtained from $(10),(11)$ must satisfy the condition $(14),(15)$. In addition, by **assumption 4**, it is unique determined by (16). Therefore, from **theorem 2**, we obtain that $\lim_{k\to\infty}K_k=K^*;\lim_{k\to\infty}P_k=P^*$.

Lemma 2: There exists a sufficiently small constant $\varepsilon>0$ such that for all symmetric matrix $P>0$ satisfying $\left\|P-P^*\right\|<\varepsilon$ the system (9) can be stable by $u=-R^{-1}\bar{B}^T P x$.

Proof:

Because $Q+P^*\bar{B}R^{-1}\bar{B}^T P^*>0$, there exists $\alpha>0$ such that $Q+P^*\bar{B}R^{-1}\bar{B}^T P^*>\alpha I$. For any symmetric matrix $P>0$ we have:

$$\bar{A}^T P+P\bar{A}+\bar{Q}-P\bar{B}R^{-1}\bar{B}^T P=0$$

where:

$$\left(\bar{Q}+P\bar{B}R^{-1}\bar{B}^T P\right)=\left(Q+P^*\bar{B}R^{-1}\bar{B}^T P^*\right)+\left(P^*-P\right)\bar{A}+\bar{A}^T\left(P^*-P\right)+2\left(P\bar{B}R^{-1}\bar{B}^T P-P^*\bar{B}R^{-1}\bar{B}^T P^*\right)$$

Because of continuity, there exists a sufficiently small constant $\varepsilon>0$ such that for all symmetric matrix $P>0$ satisfying $\left\|P-P^*\right\|<\varepsilon$ we have:

$$\bar{Q}+P\bar{B}R^{-1}\bar{B}^T P>Q+P^*\bar{B}R^{-1}\bar{B}^T P^*-\alpha I>0$$

We consider the Lyapunov function $V=\frac{1}{2}x^T P x$, we have:

$$\dot{V}=x^T\left(\left(\bar{A}-\bar{B}R^{-1}\bar{B}^T P\right)^T P+P\left(\bar{A}-\bar{B}R^{=1}\bar{B}^T P\right)\right)x$$

$$\dot{V}=-x^T\left(\bar{Q}+P\bar{B}R^{-1}\bar{B}^T P\right)x$$

From $\bar{Q}+P\bar{B}R^{-1}\bar{B}^T P>0$ so the system (9) is globally asymptotically stabilizes.

Theorem 3: The feedback control is designed from **theorem 1** and **algorithm 1** can make sure that the closed-loop system (1) is robustly stable and the cost function $J=\int_0^\infty \left(x^T Qx+u^T Ru\right)d\tau$ is bounded.

Proof:

From **theorem 1**, the system states are forced to reach a sliding surface in a finite time. When on the sliding surface, the control is designed by **algorithm 1** can make the system is globally stable. Therefore, the closed-loop system (1) is robustly stable.

Moreover, by continuity we infer that:

$$J=\int_0^\infty \left(x^T Qx+u^T Ru\right)d\tau$$

$$J=\int_0^{t_s} \left(x^T Qx+u^T Ru\right)d\tau+\int_{t_s}^\infty \left(x^T Qx+v^T Rv\right)d\tau$$

$$J<\Omega+\int_0^\infty \left(x^T Qx+v^T Rv\right)d\tau$$

where:

$$\Omega=t_s\left(\lambda_{\max}(Q)\max_{0\le t\le t_s}\|x\|^2+\lambda_{\max}(R)\max_{0\le t\le t_s}\|u\|^2\right)$$

Due to the relation $v=-K_{k^*}x\approx K^* x$, there exists a positive number β such that $J<\beta$.

IV. SIMULATION RESULTS

TABLE I. PARAMETERS AND VARIABLES OF AN INVERTED PENDULUM

M	Weight of car	0.5	kg
m	Weight of link	0.2	kg
B	Friction coefficient	0.1	Ns/m
L	½ length of link	0.3	m
I	Inertial moment of link	0.006	$kg.m^2$
G	Gravity	9.8	m/s^2

In this section, we apply the proposed an adaptive optimal control based integral sliding mode control law to an inverted pendulum on a cart described as (17) and table 1. The Fig.1 and Fig.2 show the control and states of system using theorem 1 and algorithm 1. On the other hand, the Fig.3 and Fig.4 show the control and states of system when algorithm 1 has not been used. Fig.5 and Fig.6 show the convergence of matrix P and K of proposed algorithm 1, and the tracking errors converge to zero.

$$\begin{bmatrix}\dot{x}\\ \ddot{x}\\ \dot{\phi}\\ \ddot{\phi}\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & \dfrac{-\left(I+ml^2\right)b}{I\left(M+m\right)+Mml^2} & \dfrac{m^2gl^2}{I\left(M+m\right)+Mml^2} & 0\\ 0 & 0 & 0 & 1\\ 0 & \dfrac{-mlb}{I\left(M+m\right)+Mml^2} & \dfrac{mgl\left(M+m\right)}{I\left(M+m\right)+Mml^2} & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \phi\\ \dot{\phi}\end{bmatrix}$$

$$+\begin{bmatrix}0\\ \dfrac{I+ml^2}{I\left(M+m\right)+Mml^2}\\ 0\\ \dfrac{ml}{I\left(M+m\right)+Mml^2}\end{bmatrix}u \qquad (17)$$

$$y=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}x+\begin{bmatrix}0\\ 0\end{bmatrix}u$$

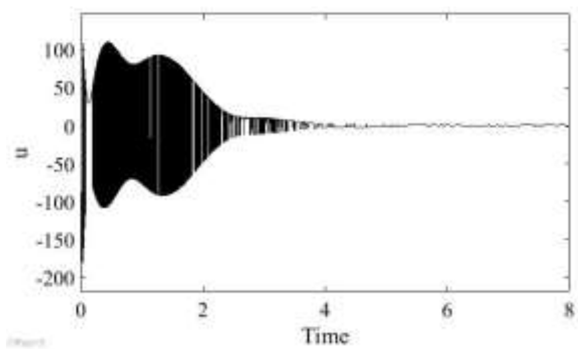

Figure 1. The control input

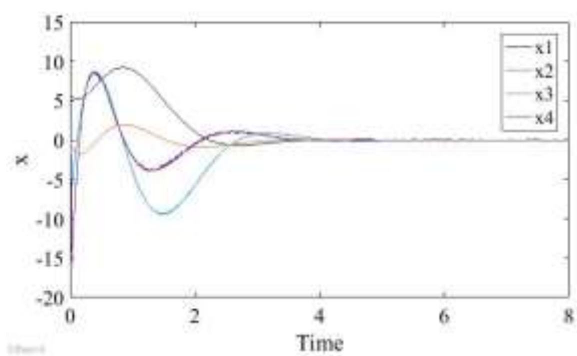

Figure 2. The behaviour of state variables

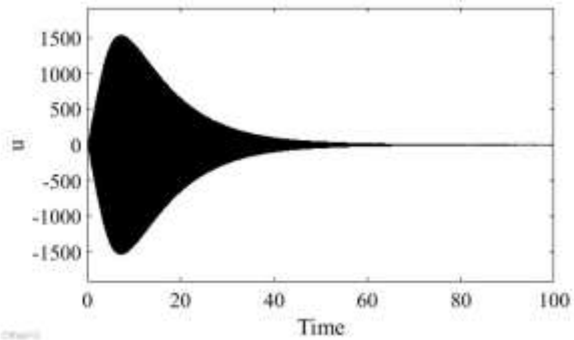

Figure 3. The Control input

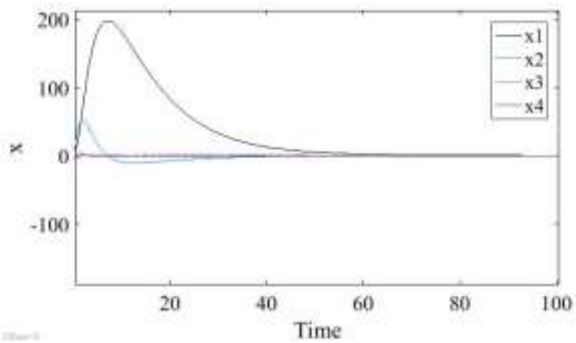

Figure 4. The behaviour of state variables

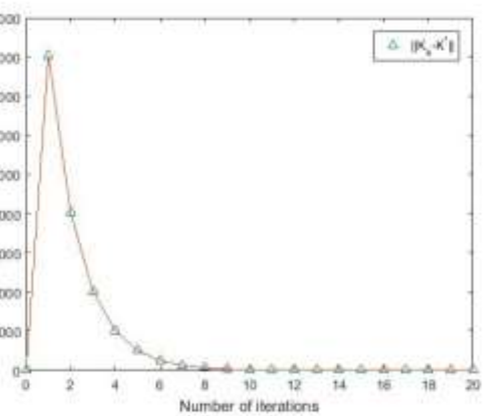

Figure 5. The convergence of proposed algorithm

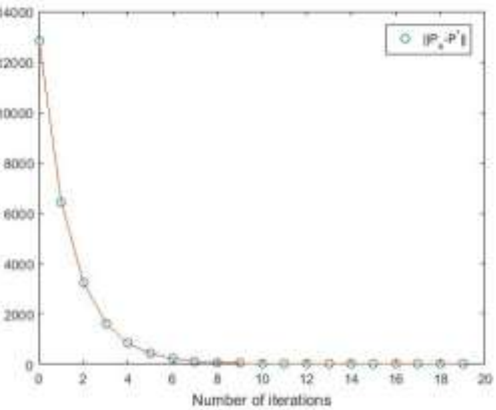

Figure 6. The convergence of proposed algorithm

CONCLUSIONS

This paper presents an adaptive optimal control algorithm based integral sliding mode control law for continuous-time systems with unknown system dynamics and external disturbance. The proposed algorithm pointed out the robustly stability of system and the bound of cost function. The theory analysis and simulation results illustrate the effectiveness of proposed algorithm.

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