Introduction to Fuzzy Graph Theory

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Extended Abstract

Graph theory is a very important tool to represent many real world problems. Nowadays, graphs do not represent all the systems properly due to the uncertainty or haziness of the parameters of systems. For example, a social network may be represented as a graph where vertices represent accounts (persons, institutions, etc.) and edges represent the relation between the accounts. If the relations among accounts are to be measured as good or bad according to the frequency of contacts among the accounts, fuzzyness should be added to representation. This and many other problems motivated to define fuzzy graphs. Rosenfeld [16] first introduced the concept of fuzzy graphs. After that fuzzy graph theory becomes a vast research area. Applications of fuzzy graph include data mining, image segmentation, clustering, image capturing, networking, communication, planning, scheduling, etc.

Crisp graph and fuzzy graph both are structurally similar. But when there is an uncertainty on vertices and/or edges then fuzzy graph has a separate importance. Since the world is full of uncertainty so the fuzzy graph occurs in many real life situations.

Fuzzy graph theory is advanced with large number of branches. Bhutani and Battou have worked on M-strong fuzzy graphs [1]. Bhutani and Rosenfeld have worked on strong arcs in fuzzy graphs [2]. Mathew and Sunitha have classifies the types of arcs in fuzzy graphs and investigated their properties [5]. Koczy have used the fuzzy graphs in evaluation and optimization of networks [4]. Fuzzy tolerance graphs have been introduced by Samanta

and Pal in [7]. Also, they have introduced so many variations of fuzzy graphs such as, fuzzy threshold graphs [8], bipolar fuzzy hypergraphs [9], irregular bipolar fuzzy graphs [10]. Samanta et al. have presented some more results on bipolar fuzzy sets and bipolar fuzzy intersection graphs [11]. They also presents a new approach to social networks based on fuzzy graphs in [12]. Fuzzy k-competition graphs and p-competition graph are being studied by Samanta and Pal in [13]. Samanta and Pal have also worked on fuzzy planar graphs [14], telecommunication system based on fuzzy graphs [15].

In this presentation, an introduction of fuzzy graph is given. Let $G = (V, E)$ be a graph which consists of non-empty finite set V of elements called vertices and a finite set E of ordered pairs of distinct vertices called edges.

A fuzzy set A on a set X is characterized by a mapping $m : X \to [0, 1]$, which is called the *membership function*. A fuzzy set is denoted by $A = (X, m)$. *A fuzzy graph* [16] $\xi = (V, \sigma, \mu)$ is an algebraic structure of non-empty set V together with a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ such that for all $x, y \in V$, $\mu(x, y) \le \sigma(x) \wedge \sigma(y)$ and μ is a symmetric fuzzy relation on σ. Here $\sigma(x)$ and $\mu(x, y)$ represent the membership values of the vertex x and of the edge (x, y) in ξ respectively.

The fuzzy graph $\xi_1 = (V, \sigma_1, \mu_1)$ is called a *fuzzy subgraph* of $\xi = (V, \sigma, \mu)$ if $\sigma_1(x) \le \sigma(x)$ for all x and $\mu_1(x, y) \le \mu(x, y)$ for all edges $(x, y), x, y \in V$.

Fuzziness occurs in a fuzzy graph in many different ways, some of them are listed below:

Type I: crisp vertex set and fuzzy edge set

Type II: crisp vertices and edges with fuzzy connectivity

Type III: crisp graph with fuzzy weights

Type IV: fuzzy set of crisp graphs

We denote $x \wedge y = min\{x, y\}$ and $x \vee y = max\{x, y\}$.

For the fuzzy graph $\xi = (V, \sigma, \mu)$, an edge $(x, y), x, y \in V$ is called strong [3] if $\frac{1}{2}\{\sigma(x) \wedge \sigma(y)\} \le \mu(x, y)$ and it is called weak otherwise. The strength of an edge (u, v) is denoted by $I_{(u,v)} = \dfrac{\mu(u, v)}{\sigma(u) \wedge \sigma(v)}$.

The underlying crisp graph $G = (V', E)$ of a fuzzy graph $\xi = (V, \sigma, \mu)$ is such that $V' = \{v \in V \mid \sigma(v) > 0\}$ and $E = \{(u, v) \mid \mu(u, v) > 0\}$.

A *path* [6] in ξ is a sequence of vertices $x_0, x_1, \ldots, x_n$, such that $\mu(x_{i-1}, x_i) > 0$ and $i = 1, 2, \ldots, n$. The path is said to have length n. Two nodes that are joined by a path are said to be *connected*. A *component* of a fuzzy graph is the fuzzy subgraph such that any two vertices are connected by path. So, a

fuzzy graph is said to be *connected fuzzy graph* if it has one component and disconnected otherwise.

For $0 \leq \alpha \leq 1$, α-cut [6] graph of a fuzzy graph $\xi = (V, \sigma, \mu)$ is a crisp graph $\xi_\alpha = (V_\alpha, E_\alpha)$ such that $V_\alpha = \{u \in V \mid \sigma(u) \geq \alpha\}$ and $E_\alpha = \{(u, v) \mid \mu(u, v) \geq \alpha\}$.

An example of fuzzy graph from reality is shown in Figure 1.

Figure 1: An example of fuzzy graph

The strength of a path is defined as $\min\{\mu(x_{i1}, x_i) : i = 1, 2, \ldots, n\}$. In other words, strength of a path is the weight (membership value) of the weakest arc of the path. The strength of paths are illustrated in Figure 2.

The strength of connectedness between two nodes x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by $CONN_G(x, y)$. An xy path P is called a strongest xy path if its strength equals $CONN_G(x, y)$. For the above graph, $CONN_G(x, y) = \max\{0.3, 0.6, 0.2, 0.3\} = 0.6$. Thus, the strongest xy path is $x - c - y$.

Let $H - (x, y)$ be the fuzzy graph obtained from H by replacing $\mu(x, y)$ by 0. We call the arc (x, y) strong in H if $\mu(x, y) > 0$ and $\mu(x, y) \geq CONN_{H-(x,y)}(x, y)$.
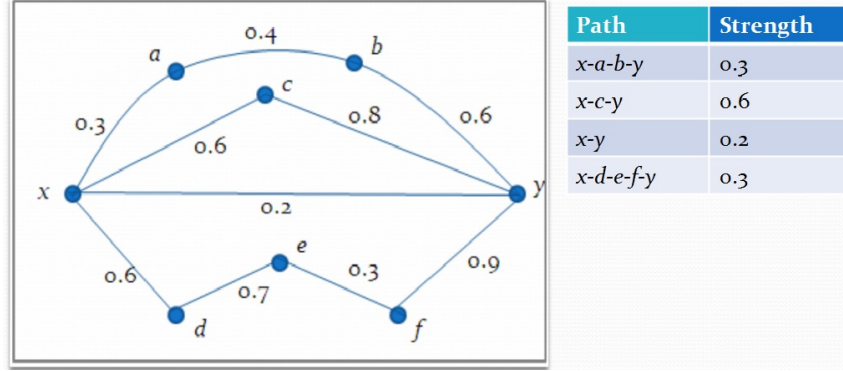
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Path	Strength
x-a-b-y	0.3
x-c-y	0.6
x-y	0.2
x-d-e-f-y	0.3

Figure 2: Illustration of path and their strength

A path $P : x = x_0, x_1, x_2, \ldots, x_n = y$ is called strong if (x_{i-1}, x_i) is strong for all i.

It is mentioned that $CONN_{G(x,y)}(x, y)$ is the strength of connectedness between x and y in the fuzzy graph obtained from G by deleting the arc (x, y). An arc (x, y) in G is called α-strong if $\mu(x, y) > CONN_{G(x,y)}(x, y)$. An arc (x, y) in G is called β-strong if $\mu(x, y) = CONN_{G(x,y)}(x, y)$. An arc (x, y) in G is called a δ-strong if $\mu(x, y) < CONN_{G(x,y)}(x, y)$.

A path in a fuzzy graph is called an α-strong path if all of its arcs are α-strong and is called a β-strong path if all of its arcs are β-strong.

A fuzzy graph G is said to be a strong fuzzy graph if $\mu(x, y) = \min\{\sigma(x), \sigma(y)\}$ for all $(x, y) \in E$. A complete fuzzy graph is a fuzzy graph such that $\mu(x, y) = \min\{\sigma(x), \sigma(y)\}$ for all $x, y \in V$.

Strength of an edge (u, v) in a fuzzy graph $\xi = (V, \sigma, \mu)$ is given by $I_{(u,v)} = \dfrac{\mu(u, v)}{\sigma(u) \wedge \sigma(v)}$. Again strength of a vertex x is considered as the maximum value among its membership value $\sigma(x)$ and the strengths $I_{(x,y)}$ of edges (x, y) incident to x. Now, let $\theta_x = max\{I_{(x,y)} \mid (x, y)$ is an edge in $\xi, y \in V\}$. The strength of a vertex $x \in V$ is denoted by I_x and defined by $I_x = \max\{\theta_x, \sigma(x)\}$.

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