# Lecture 1, Jan 6.

## Binomial coefficients (review)

Define $\mathbb{N} := \{0, 1, \dots\}$ and $\mathbb{P} = \{1, 2, 3, \dots\}$.

The permutations of $\{1,2,3\}$ are

$$(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)$$

How many permutations of $\{1, 2, \dots, k\}$ are there?

Answer: $k! := k.(k-1). \dots .1$. (Consider choosing the elements of the permutation one at a time; when choosing element $i+1$ we have $k-i$ elements left to choose from.)

Theorem (Theorem 1.5 in the course notes)

For $k, n \in \mathbb{N}$, the number of $k$-element subsets of $\{1, \dots, n\}$ is

$$\frac{n(n-1) \dots (n-k+1)}{k!}.$$

Proof. Let $L$ denote the collection of all lists of $k$ distinct elements chosen from $\{1, 2, \dots, n\}$, and let $S$ denote the collection of all $k$-element subsets of $\{1, 2, \dots, n\}$. Since each set in $S$ has $k!$ distinct permutations in $L$, we have

$$|L| = k! \, |S|. \tag{1}$$

For a list $(a_1, a_2, \dots, a_k) \in L$ we have $n$ choices for $a_1$, $n-1$ remaining choices for $a_2$, $(n-2)$ remaining choices for $a_3$, etc. Therefore

$$|L| = n(n-1). \dots .(n-k+1). \tag{2}$$

By (1) and (2), $|S| = \frac{1}{k!} |L| = \frac{1}{k!} \cdot n.(n-1). \dots \cdot 1$, as required. $\square$

**Definition:** Let $\binom{n}{k}$ denote the number of $k$-element subsets of $\{1, 2, \dots, n\}$; thus $\binom{n}{k} = \frac{n.(n-1)\dots(n-k+1)}{k!}$

Note that

- $\binom{n}{0} = 1$ (the empty set)
- $\binom{n}{k} = \binom{n}{n-k}$ (the complement of a $k$-element set is an $(n-k)$-element set)

**Theorem.** For $n, k \in \mathbb{N}$, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

**Proof.** Let $S$ denote the collection of all $k$-element subsets of $\{1, \dots, n\}$, let $S_1$ denote the collection of all sets in $S$ that contain $n$ and let $S_2$ denote the collection of all sets in $S$ that do not contain $n$. Thus

$$|S| = |S_1| + |S_2|. \tag{1}$$

By definition,

$$|S| = \binom{n}{k}. \tag{2}$$

Note that $S_2$ contains all $k$-element subsets of $\{1, \dots, n-1\}$, so

$$|S_2| = \binom{n-1}{k}. \tag{3}$$

Moreover, for each set $A \in S_1$, the set $A \setminus \{n\}$ is a $(k-1)$-element subset of $\{1, \dots, n-1\}$, so

$$|S_1| = \binom{n-1}{k-1}. \tag{4}$$

From (1)-(4), $\binom{n}{k} = |S| = |S_1| + |S_2| = \binom{n-1}{k-1} + \binom{n-1}{k}$, as required. $\square$

# Pascal's Triangle

$n=0$ $\qquad\qquad\qquad \binom{0}{0}=1$

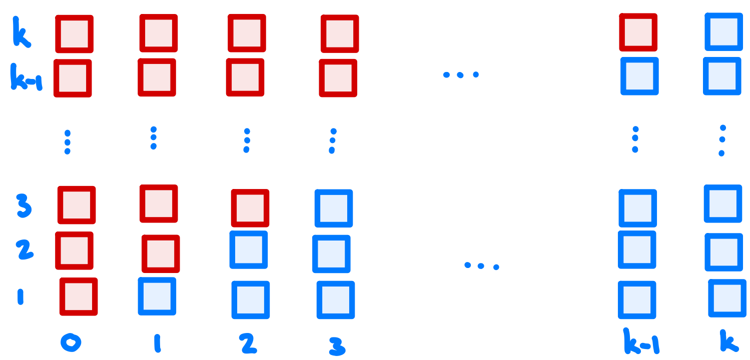
$n=1$ $\qquad\qquad \binom{1}{0}=1 \qquad + \qquad \binom{1}{1}=1$

$n=2$ $\qquad \binom{2}{0}=1 \qquad \binom{2}{1}=2 \qquad \binom{2}{2}=1$

$n=3$ $\quad \binom{3}{0}=1 \qquad \binom{3}{1}=3 \qquad \binom{3}{2}=3 \qquad \binom{3}{3}=1$

The above proof is an example of a combinatorial proof; we proved an algebraic identity by counting a set in two ways.

**Example (Combinatorial proof)** $1+2+\cdots+k=\frac{k(k+1)}{2}$.




*Proof.* The number of blue boxes is $1+2+\cdots+k$. On the other hand, the total number of boxes is $k(k+1)$, and half of them are blue. Therefore

$$1+2+\cdots+k=\frac{k(k+1)}{2}. \qquad \square$$

# Lecture 2, Jan. 8

Recall: a **combinatorial proof** is a proof of an algebraic identity by counting a set in two ways.

**Another example.** For $n, k \in \mathbb{N}$,

$$\binom{n+k}{n} = \binom{n-1}{n-1} + \binom{n}{n-1} + \cdots + \binom{n+k-1}{n-1}.$$

**Proof.** Let $S$ denote the collection of all $n$-element subsets of $\{1, 2, \dots, n+k\}$.

Thus $$|S| = \binom{n+k}{n}. \tag{1}$$

For a set $A \in S$, the largest element in $A$ is either $n, n+1, \dots, n+k$.

For each $i \in \{n, n+1, \dots, n+k\}$, let $S_i$ denote the collection of sets in $S$ whose largest element is $i$. Thus

$$|S| = |S_n| + |S_{n+1}| + \cdots + |S_{n+k}|. \tag{2}$$

Moreover, each set $A \in S_i$ contains the element $i$ and $n-1$ other elements from $\{1, \dots, i-1\}$. Therefore

$$|S_i| = \binom{i-1}{n-1}. \tag{3}$$

By (1), (2), and (3),

$$\begin{aligned} |S| &= |S_n| + |S_{n+1}| + \cdots + |S_{n+k}| \\ &= \binom{n-1}{n-1} + \binom{n}{n-1} + \cdots + \binom{n+k-1}{n-1}, \end{aligned}$$

as required. $\square$

## Application of binomial coefficients

**Question:** How many non-negative integer solutions are there to the equation

$$x_1 + x_2 + x_3 = 5\ ?$$

Consider a sequence $(1,1,1,1,1)$; we can generate a solution by inserting two zeros to divide the five 1s into three sets.

For example $(1,1,0,1,1,0,1)$ encodes the solution $x_1=2, x_2=2, x_3=1$ and $(1,1,0,0,1,1,1)$ encodes $x_1=2,\ x_2=0,\ x_3=3$.

Therefore each solution is encoded (uniquely) as a $(0,1)$-sequence of length 7 with exactly two zeros. There are $\binom{7}{2}$ such sequences, since we need to choose which two of the 7 entries are zero.

**Answer:** $\binom{7}{2} = \frac{7 \cdot 6}{2} = 21$.

**Remark:** This is a "bijective proof"; we gave a bijection from the set of solutions to certain $(0,1)$-sequences that we knew how to count.

**More generally:** For $t, n \in \mathbb{N}$, consider the equation

$$x_1 + x_2 + \cdots + x_t = n. \tag{1}$$

**Theorem** (Theorem 1.9) The number of non-negative integer solutions to (1) is $\binom{n+t-1}{t-1}$.

**Proof.** We can uniquely encode any solution as a sequence of $n$ ones and $t-1$ zeros. The sequence has length $n+t-1$ and we need to choose where to put the $t-1$ zeros, so the number of such sequences is $\binom{n+t-1}{t-1}$, as required. $\square$

# Counting using algebra

**Example.** The subsets of $\{1,2,3\}$ are $\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$

the empty set

Let $f(x_1,x_2,x_3) = (1+x_1)(1+x_2)(1+x_3)$

$$= 1 + x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3.$$

There is a bijection between the subsets of $\{1,2,3\}$ and the terms in the expansion of $f(x_1,x_2,x_3)$.

e.g. $\{1,3\} \rightarrow x_1x_3$.

Note that $f(x,x,x) = 1 + 3x^1 + 3x^2 + x^3$

$$= \binom{3}{0} + \binom{3}{1}x^1 + \binom{3}{2}x^2 + \binom{3}{3}x^3.$$

Thus the coefficient of $x^k$ in $(1+x)^3$ counts the number of $k$-element subsets of $\{1,2,3\}$.

Moreover $f(1,1,1) = (1+1)^3 = 8$ counts the number of subsets of $\{1,2,3\}$.

**Binomial Theorem** (Theorem 2.2) For $n \in \mathbb{N}$,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

**Proof.** For a set $A \subseteq \{1,\dots,n\}$ we write $x^A = \prod_{i\in A} x_i$.

For example $x^{\{2,5,6\}} = x_2x_5x_6$.

As we saw earlier $(1+x_1)(1+x_2)\cdots(1+x_n) = \sum_{A\subseteq\{1,2,\dots,n\}} x^A$.

Substituting $x_1 = x_2 = \cdots = x_n = x$ gives

$$(1+x)^n = \sum_{A\subseteq\{1,\dots,n\}} x^{|A|}.$$

The coefficient of $x^k$, after collecting like terms, is the number of $k$-element subsets of $\{1,\dots,n\}$, which is $\binom{n}{k}$. So

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n,$$

as required. $\square$

## Applications

(1) There are $2^n$ subsets of $\{1,\dots,n\}$.

(Proof. The number of subsets is $\binom{n}{0}+\binom{n}{1}+\dots+\binom{n}{n} = (1+1)^n = 2^n$.)

(2) There are equal numbers of even and odd subsets of $\{1,\dots,n\}$.

(Proof. Note that $(1+x)^n = \sum_{A\subseteq\{1,\dots,n\}} x^{|A|}$. Thus $0=(1-1)^n = \sum_{A\subseteq\{1,\dots,n\}} (-1)^{|A|}$. Each even set contributes 1, each odd set contributes $-1$, and the sum is zero.)

Exercise. Find a "bijective proof" of (2). That is, find an explicit bijection from the even subsets to the odd subsets.

# Lecture 3, Jan. 10.

**Recall.**

subsets of $\{1,2,3\}$: $\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$

$$(1+x_1)(1+x_2)(1+x_3) = 1 + x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3.$$

$$(1+x)^3 = 1 + 3x^1 + 3x^2 + x^3$$
$$= \binom{3}{0} + \binom{3}{1}x^1 + \binom{3}{2}x^2 + \binom{3}{3}x^3.$$

**Binomial Theorem (Theorem 2.2)** For $n \in \mathbb{N}$,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

**Proof.** For a set $A \subseteq \{1,\dots,n\}$ we write $x^A = \prod_{i\in A} x_i$.

For example $x^{\{2,5,6\}} = x_2x_5x_6$.

As we saw earlier $(1+x_1)(1+x_2)\cdots(1+x_n) = \sum_{A\subseteq\{1,2,\dots,n\}} x^A$.

Substituting $x_1 = x_2 = \cdots = x_n = x$ gives

$$(1+x)^n = \sum_{A\subseteq\{1,\dots,n\}} x^{|A|}.$$

The coefficient of $x^k$, after collecting like terms, is the number of $k$-element subsets of $\{1,\dots,n\}$, which is $\binom{n}{k}$. So

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n;$$

as required. $\square$

## Applications

(1) There are $2^n$ subsets of $\{1,\dots,n\}$.

(Proof. The number of subsets is $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=(1+1)^n=2^n$. )

(2) There are equal numbers of even and odd subsets of $\{1,\dots,n\}$.

( Proof. Note that $(1+x)^n=\sum_{A\subseteq\{1,\dots,n\}} x^{|A|}$. Thus $0=(1-1)^n=\sum_{A\subseteq\{1,\dots,n\}}(-1)^{|A|}$. Each even set contributes 1, each odd set contributes $-1$, and the sum is zero.)

## Generating series

Typical counting problem: We have a set $S$ of "congurations" and each configurations $a\in S$ has a given non-negative integer "weight" $w(a)$. For some given $k\in\mathbb{N}$, compute the number of configurations in $S$ that have weight $k$.

Subsets example: Take $S$ to be the collection of all subsets of $\{1,2,\dots,n\}$, and, for each $A\in S$, let $w(A):=|A|$.

In this case the number of configurations of weight $k$ is $\binom{n}{k}$.

Remark. We allow $S$ to be infinite. We call $w$ a **weight function** when there are only finitely many configurations of each weight, which we require for counting.

**Definition (Generating series)** Given a weight function $w$ for a set $S$, the generating series is

$$\Phi_S(x) = \sum_{a \in S} x^{w(a)}.$$

By collecting like powers we can write

$$\Phi_S(x) = a_0 + a_1 x^1 + a_2 x^2 + \cdots$$

where $a_k$ is the number of configurations of weight $k$ in $S$.

Thus, in the subsets example,

$$\Phi_S(x) = \binom{n}{0} + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

In this example $\Phi_S(x)$ is a polynomial, which will be the case whenever $S$ is finite.

When $S$ is infinite $\Phi_S(x)$ is a power series.

**Example.** Let $S = \{0, 1, 2, \ldots\}$ and for each $a \in S$ let $w(a) = a$.

Then $\Phi_S(x) = 1 + x^1 + x^2 + x^3 + \cdots$

**Remark:** We do not think of $\Phi_S(x)$ as a function, but rather a "formal power series"; we care about the coefficients, not the evaluations, and, therefore, we don't care about the "radius of convergence".

Note that $(1-x)(1 + x + x^2 + \cdots) = 1$, so

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

But $\frac{1}{1-x}$ is not a power series! What is going on here?

## Formal power series (see supplementary notes)

A **formal power series** is a series

$$A(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

where $a_0, a_1, \dots$ are complex numbers.

We say that $a_k$ is the **coefficient** of $x^k$ in $A(x)$ and write $a_k = [x^k]A(x)$.

We are not interested in evaluations of $A(x)$, we only care about the coefficients.

Consider, for example, $A(x) = 1 + 2^2 x^2 + 3^4 \cdot x^4 + 4^6 \cdot x^6 + \dots$

Note that, for $x \in \mathbb{R}$, $A(x) = \begin{cases} 1, & \text{if } x=0 \\ \infty, & \text{otherwise.} \end{cases}$

Therefore $A(x) = 2A(x) - 1$ for all real numbers $x$. Nevertheless, we consider $A(x)$ and $2A(x)-1$ to be different formal power series, since they have different coefficients.