Secure Distributed Training at Scale

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Abstract

Some of the hardest problems in deep learning can be solved with the combined effort of many independent parties, as is the case for volunteer computing and federated learning. These setups rely on high numbers of peers to provide computational resources or train on decentralized datasets. Unfortunately, participants in such systems are not always reliable. Any single participant can jeopardize the entire training run by sending incorrect updates, whether deliberately or by mistake. Training in presence of such peers requires specialized distributed training algorithms with Byzantine tolerance. These algorithms often sacrifice efficiency by introducing redundant communication or passing all updates through a trusted server. As a result, it can be infeasible to apply such algorithms to large-scale distributed deep learning, where models can have billions of parameters. In this work, we propose a novel protocol for secure (Byzantine-tolerant) decentralized training that emphasizes communication efficiency. We rigorously analyze this protocol: in particular, we provide theoretical bounds for its resistance against Byzantine and Sybil attacks and show that it has a marginal communication overhead. To demonstrate its practical effectiveness, we conduct large-scale experiments on image classification and language modeling in presence of Byzantine attackers.

1 Introduction

Many hard scientific problems were solved through collaboration between many nations, groups and individual researchers. This is especially evident in natural sciences, where researchers formed multinational collaborations to run large-scale experiments and share compute infrastructure [1, 2, 3]. Projects like Folding@home [4] and BOINC [5] push this trend even further by recruiting volunteers that donate their compute to collectively run computational experiments at an unprecedented scale [6].

Similar techniques were recently proposed for machine learning, aiming to solve two key challenges. The first challenge is the sheer computational complexity of many machine learning tasks, such as pretraining transformers for NLP [7, 8, 9] or learning on huge datasets in vision [10, 11, 12]. Recent works propose several systems [13, 14, 15] that can share the computation across many volunteers that donate the idle time of their computers. Another challenge arises in Federated Learning, where participants train a shared model over decentralized data that cannot be shared for privacy reasons [16, 17, 18].

Despite their strengths, both volunteer computing and federated learning systems have so far seen limited practical applications [19, 13]. A major roadblock towards the global adoption of these techniques is trust in reliability of each participant. For distributed training, all progress made by the collaboration can be undermined if a single peer sends incorrect outputs due to an error in computation [20] or malicious intent [21].

Prior art in decentralized optimization proposed several optimization algorithms that are resistant to such "Byzantine" faults. However, this additional security comes at a high price: most Byzantine-

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tolerant training protocols require either passing all updates through a trusted central server or exchanging additional messages that increase the network load by several times [22, 23]. This is a major problem for large-scale distributed deep learning, where hundreds of peers must exchange updates for millions of parameters at regular intervals [24, 25, 26]. Thus, in many practical scenarios, the computation and communication overhead of Byzantine-tolerant algorithms outweighs the potential benefits of collaborating with others.

In this work, we set out to solve this problem by proposing a novel distributed training protocol designed for large-scale deep learning workloads. Our approach combines the scalability and communication efficiency of modern distributed training techniques such as ALL-REDUCE SGD [25] with resilience against Byzantine and Sybil attackers. To achieve this, we leverage secure multi-party computing (MPC) to verify the integrity of training with minimal overhead that does not depend on the model size. Our protocol does not require trusted peers, operating under the assumption that anyone can be an attacker. The overall contributions of our work can be summarized as follows:

- We propose a novel strategy for decentralized Byzantine-tolerant training where the extra communication cost does not depend on the number of parameters.
- We rigorously analyze the proposed strategy and prove convergence bounds for convex and nonconvex losses with Byzantine attackers. Furthermore, we derive accelerated convergence rates for the same task under realistic assumptions about model gradients.
- Based on the above algorithm, we describe a system that allows multiple parties to train a shared model with zero trust assumptions. We prove that this system is resistant to both Byzantine and Sybil attacks from a computationally constrained attacker.
- We verify the effectiveness of our algorithm through both controlled experiments² and actual large-scale training runs. Specifically, we start with ResNet-18 for CIFAR-10 classification and follow up with pretraining ALBERT-large in a setup where almost half of all peers are malicious.

2 Related work

2.1 Distributed deep learning

Nowadays, training deep neural networks often requires so much raw computation that it is infeasible for any single compute accelerator. As a result, such models can only be trained by splitting the workload over multiple machines using one of the existing distributed training algorithms. Most of these methods fall into two broad categories: in *data-parallel* training, each worker trains the entire model over a different part of the training batch [25, 27]; in contrast, *model-parallel* training splits the model itself, making every worker responsible for certain layers [28, 29] or a single slice of a layer [26]. For the purpose of this study, we only consider *data-parallel* algorithms³.

In general, training neural networks with data parallelism consists of two interleaving phases: *computation* and *communication*. During the computation phase, each worker independently computes gradients over its fraction of the current minibatch. Once the gradients are computed, workers initiate the communication phase to aggregate these gradients and perform an SGD step. Naturally, there are exceptions to this rule: some algorithms run multiple local steps before each synchronization [32, 33, 34] while others replace global aggregation with peer-to-peer communication [35, 36]. However, the majority of real-world distributed training setups follow the above two-stage procedure.

As we increase the scale of data-parallel training, so does the importance of the communication phase. Deep neural networks can have anywhere between tens of millions and trillions [31, 37] of trainable parameters, and hence, gradients. As such, it is crucial that nodes aggregate their gradients in the most communication-efficient way. To address this concern, distributed training systems can use one of several protocols, depending on the available hardware and network infrastructure.

The most obvious one is known as Parameter Servers (PS) [38, 39, 40, 41]. As the name suggests, this strategy assigns one of the servers to storing and updating the model parameters (along with the optimizer statistics). All other workers simply compute gradients and send them to the PS for aggregation, then download the updated model parameters and repeat the same procedure for the next step. In terms of training efficiency, this strategy performs well enough when training with a small number of workers. However, as we add more workers, the parameter server will eventually

²Source code for the experiments is available at github.com/yandex-research/btard

³This choice does not limit the generality of our results as most real-world model-parallel systems still rely on data-parallel training between groups of servers that run pipeline or tensor parallel groups [30, 31].

become unable to download gradients and send updates quickly enough to keep all workers busy. This problem can partially alleviated through gradient compression [42, 43, 44, 45, 46, 47, 48] or local updates [32], but it remains a fundamental bottleneck of any parameter server.

For this reason, most real-world distributed training adopt a different strategy based on ALL-REDUCE (AR) [27, 49, 50]. This is a family of communication protocols that allow multiple servers to collectively compute the average (or similar functions) of their local vectors and distribute the result to each server. As such, AR-SGD runs ALL-REDUCE on local gradients of each peer and computes the global mean. Depending on the network topology, modern distributed training systems typically use Ring-, Butterfly- or Tree-based variant of ALL-REDUCE. All three variants have good scalability: when averaging a vector of d gradients with n peers, the former two protocols will require each peer to communicate at most $O(d \cdot (n-1)/n)$ over the network, while the third protocol needs $O(d \cdot \log((n-1)/n))$ communication for each peer, compared to $O(d \cdot n)$ with a parameter server.

2.2 Byzantine-tolerant optimization

Training a shared model across many distributed agents is already a difficult task, but adding Byzantine agents makes it even harder. And yet, the research community invented specialized algorithms that can train models even in this setup. These algorithms are different in nature, but all of them provide an extra layer of complexity on top of distributed training methods described in Section 2.1.

Parameter-server (PS) based approaches. The majority of existing algorithms designed to be Byzantine-resilient rely on the existence of a trusted parameter-server. Since in the classical parallel SGD even 1 Byzantine worker can break the convergence of the whole method by shifting the mean of the resulting vector in an arbitrary way, it is natural to substitute averaging of the vectors received from the workers by a more robust aggregation rule, e.g., Krum [51], coordinate-wise median, trimmed median [52], Multi-Krum [53], Bulyan [54], geometric median [55]. Despite their simplicity, all these methods were shown to be brittle and not robust to special types of Byzantine attacks [56, 57, 58]. Moreover, in [58], authors show that all permutation-invariant algorithms cannot converge to any predefined accuracy of the solution, meaning that simple application of some aggregation rules on top of SGD does not lead to Byzantine tolerance.

There are several approaches to circumvent this issue. In [59], the authors proposed BYZANTINESGD and prove the convergence results for convex problems. This approach was recently extended to handle non-convex problems as well [60]. In both papers, the key idea is based on applying the concentration properties of the sums depending on the stochastic gradients as well as iterative removing of Byzantine peers. However, theoretical guarantees from [59, 60] rely on the restrictive assumption that the noise in the stochastic gradients is uniformly bounded with probability 1. Next, in [61], the authors propose a Byzantine-tolerant version of parallel SAGA [62], i.e., variance-reduced version of SGD, with geometric median as an aggregation rule — BYRD-SAGA – and prove the convergence of the new method for convex objectives. However, variance-reduced methods are known to converge slowly in practice [63], which limits the practical utility of BYRD-SAGA. Finally, in [58], authors propose a new aggregation rule called CENTEREDCLIP, apply it to SGD with client momentum, and prove convergence results for the obtained method in the non-convex case under reasonable assumptions. Alternative lines of work achieve Byzantine-tolerant optimization through redundant computations [22, 23] or reputation-based approaches [64, 65, 66]. Unfortunately, these papers either do not contain theoretical convergence results for the proposed methods or rely on too restrictive assumptions in the analysis. One can find more references in the recent survey [67].

Decentralized approaches. Byzantine-tolerant optimization methods for decentralized communication architectures are studied only in a couple of papers. Among them we emphasize [68, 69, 70]. In [68, 69], the authors consider a specific scenario when workers compute full gradients, local loss functions on peers are heterogeneous, and the trimmed coordinate-wise median is used as an aggregation rule. In this setup, the authors prove convergence results in the strongly convex case for the proposed algorithms to some accuracy depending on the heterogeneity level of local loss functions, which is natural in the presence of Byzantine peers. However, these results are not applicable to a wide range of practically important problems where stochastic gradients have to be used. This issue was partially resolved in [70], where the authors propose a version of GOSSIP SGD applied to the equivalent reformulation of the original problem based on TV-regularization [71]. However, the established convergence results in the strongly convex case do not show any benefits of using communications with other workers in the homogeneous data regime that appears in large-batch training of deep learning models. The same idea is used in [72] but for a parameter-server architecture.

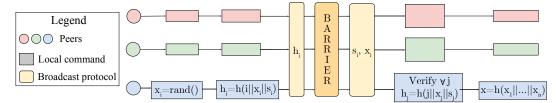


Figure 1: An intuitive scheme of JPRNG from [73]. The || operation denotes concatenation, h(x) denotes a cryptographic hash function. The hashed value includes the peer identifier i to protect from replay attacks (an attacker repeats someone else's message) and a large random string s_i to resist dictionary attacks (an attacker reverses the hash function using a dictionary of its possible values).

2.3 Security in distributed systems

In this work, we circumvent the restrictions of existing Byzantine-tolerant techniques with secure multi-party computing (MPC). This is a family of cryptographic protocols that allow multiple parties to perform secure collective computations in the presence of malicious participants. While most MPC protocols are designed to preserve data privacy (which is irrelevant to us), there are two ideas from MPC that are crucial to our approach.

Broadcast channels. Several key stages of our algorithm require peers to send a certain value to all their groupmates. Since we rely exclusively on peer-to-peer connections, a malicious peer could violate this process by deliberately sending different values to each participant. To protect against such an attack, distributed systems can employ secure *broadcast protocols* [74, 75]. These protocols ensure that peers receive the same broadcasted value even if some of them are malicious. We provide more details on broadcast protocols and their guarantees in Appendix B.1.

Multi-party pseudorandom number generation. To ensure that peers compute correct gradients, our approach verifies a random subset of all computed gradients. However, such verifications would be ineffective if malicious peers could predict (or influence) whether or not they are going to be verified. Hence, we need a way to select who is going to be checked in such a way that malicious peers can neither predict nor influence the random draw. Fortunately, the MPC community has already proposed solutions for joint pseudorandom number generation (JPRNG) [73]. For example, a distributed system may use JPRNG to choose a lottery winner. We provide an intuitive scheme of JPRNG from [73] in Figure 1 and a more detailed overview of these techniques in Appendix B.2.

3 Method

We focus on secure distributed training on public datasets, where each peer can access the entire training data. In this scenario, multiple parties cooperate by combining their computational resources for a single large-scale training run. More precisely, we consider data-parallel training setup with All-Reduce SGD, as described previously in Section 2.1. We describe our strategy in several stages, progressively moving from the theoretical setup to real-world distributed training:

- Section 3.1 outlines our approach for Byzantine-Tolerant All-Reduce (BTARD) and analyze its communication overhead for training models with a large number of parameters.
- In Section 3.2, we formulate the underlying optimization problem and derive convergence bounds.
- In Section 3.3, we propose a decentralized system design for distributed training with zero trust.

3.1 Byzantine-Tolerant All-Reduce

The core design principle behind our algorithm is that all types of Byzantine faults must have limited effect and a chance of being discovered. Together, these properties impose a limit on the total damage that an attacker can do over the entire training run. To control the magnitude of attacks over a single SGD step, we modify All-Reduce with a robust aggregation technique known as Centeredlip. Specifically, we use Butterfly All-Reduce [76] and apply Centeredlip in parallel to each partition of the gradient vector. We refer to this procedure as ButterflyClip (Algorithm 2).

However, Byzantine peers can circumvent robust aggregation by attacking over many iterations. To protect against this, BTARD periodically chooses random peers to serve as *validators*. The validators must then recalculate the gradients of other peers and report any discrepancies. However, such tests are only effective if the attackers cannot predict when they will be validated⁴. To ensure that, we use the multi-party random number generation protocol described in Section 2.3.

⁴Otherwise, Byzantine peers can simply defer the attack to subsequent steps when they are not validated.

After each training step, peers use JPRNG to choose p workers that will validate p other workers. As a result, malicious peers are unable to predict "safe" iterations before they commit to an attack. Thus, any persistent attacker will eventually be found by an honest validator.

However, since validators can also be malicious, BTARD uses a separate *accuse* procedure to root out false reports. Before each averaging round, peers broadcast the hash of their gradients for that round. These values serve the same purpose as commitments in JPRNG. If peer i accuses peer j of modifying gradients, all other peers must also recalculate j's gradients. If the majority finds that j is innocent, the accusing peer i is banned instead [77].

The resulting algorithm is resilient to attacks made through incorrect gradients. However, malicious peers may also harm training by violating the CENTEREDCLIP procedure for the portion of gradients they are aggregating. Fortunately, we can design a test through which peers can verify that the vector they received was indeed the output of CENTEREDCLIP. To formulate this test, we need to view CENTEREDCLIP as a fixed-point iteration for the equation: $\sum_{i=1}^{n} (\vec{g}_i - \vec{x}) \min \left\{ 1, \frac{\tau}{\|\vec{q}_i - \vec{x}\|} \right\} = 0.$

To test if the vector \vec{v} solves the above equation, workers must check if the sum of their actual $(\vec{g}_i - \vec{x}) \min \left\{ 1, \frac{\tau}{\|\vec{g}_i - \vec{x}\|} \right\}$ equals zero. However, doing so naively will result in $O(d \cdot n)$ extra communication, defeating the purpose our algorithm. Instead, workers use randomness from JPRNG to sample a random direction \vec{z} in the space of model gradients. Each peer computes the inner product $s_i = \left\langle \vec{z}, (\vec{g}_i - \vec{x}) \min \left\{ 1, \frac{\tau}{\|\vec{g}_i - \vec{x}\|} \right\} \right\rangle$ and sends it through the *broadcast channel* (as described in

Section 2.3). Finally, all peers verify that
$$\sum_{i=1}^{n} s_i = \left\langle \vec{z}, \sum_{i=1}^{n} (\vec{g}_i - \vec{x}) \min\left\{1, \frac{\tau}{\|\vec{g}_i - \vec{x}\|}\right\} \right\rangle = 0.$$

Similarly to our previous use of JPRNG, all aggregators must commit to their vectors \vec{v} before they learn \vec{z} . This is to make sure that a malicious aggregator cannot modify the results in such a way that the difference would be orthogonal to \vec{z} . We review this and more complex attacks vectors in Appendix D. We combine all these procedures in Algorithm 1 (more formal version in Algorithm 6).

Algorithm 1 BTARD-SGD for peer i (informal)

```
Input: i rank, n peers, x^0 model, K steps, \xi_i^0 seed
 1: for k \in {0, \ldots, K-1} do
              g_i^k = \texttt{compute\_gradients}(x^k, \xi_i^k) 

\mathbf{broadcast} \texttt{hash}(g_i^k) as c_i^k 

\hat{g}_i^k = \texttt{ButterflyClip}(i, n, g_i^k)
 3:
              r^k = \text{JPRNG}(\xi_i^k)
              z^k = \texttt{random\_vector}(r^k)
 6:
 7:
              for j \in 1, \dots, n do
                     \Delta_{i}^{j} \! = \! (g_{i}^{k}[j] \! - \! \hat{g}^{k}[j]) \min \left\{ 1, \frac{\tau}{\|g_{i}^{k}[j] \! - \! \hat{g}^{k}[j]\|_{2}} \right\}
 8:
                     //q^k[j] denotes a part from peer.
 9:
                     broadcast\langle z^k[j], \Delta_i^j \rangle as s_i^j
10:
                     for j \in 1, ..., n do
11:
                             // peer i knows \Delta_i^j from CenteredClip
12:
                            if s_i^j \neq \langle z^k[j], \Delta_i^j \rangle then
13:
                                 ACCUSE(i, j, s_i^j is wrong)
14:
                     if \sum_{i=1}^{n} s_{i}^{j} \neq 0 then
15:
                            // peer _i verified that s^j are correct
16:
                             ACCUSE(i, j, g^k[j] \text{ is wrong})
17:
              \begin{aligned} \boldsymbol{x}^{k+1} &= \mathtt{sgd\_step}(\boldsymbol{x}^{k}, \hat{\boldsymbol{g}}^{k}) \\ \boldsymbol{\xi}_{i}^{k+1} &= \mathtt{hash}(\boldsymbol{r}^{k} || i) \end{aligned}
18:
19:
              \hat{\mathbf{if}}\ i \in \mathtt{chosen\_validators}(r^k) then
20:
                     j = \texttt{chosen\_target}(r^k, i) \texttt{validate\_peer}(j, x^k, \xi^k_j, c^k_j, h^*_j, s^*_j)
21:
22:
23: return x^K
```

$\overline{\textbf{Algorithm 2}}$ BUTTERFLYCLIP for peer i

```
Input: i rank, n peers, g_i \in \mathbb{R}^d gradients

1: g_i[1],...,g_i[n] = \mathtt{split}(g_i,n)

2: \mathbf{broadcast} \ \forall j, \ \ \mathbf{hash}(g_i[j]) \ \ \mathbf{as} \ \ h_i^j

3: \mathbf{scatter} \ \forall j, \ \ g_i[j] \to \mathbf{peer}_j

4: \mathbf{gather} \ g_1[i],...,g_n[i] \ \ \mathbf{from} \ \ \mathbf{peers}

5: \mathbf{verify} \ \forall j, \ \ \mathbf{hash}(g_j[i]) = h_j^i

6: \hat{g}_i = \mathbf{CenteredClip}(g_1[i],...,g_n[i])

7: \mathbf{broadcast} \ \forall j, \ \ \hat{g}_i \to \mathbf{peer}_j

8: \mathbf{gather} \ \hat{g}_1,...,\hat{g}_n \ \ \mathbf{from} \ \ \mathbf{peers}

9: \mathbf{return} \ \mathbf{merge}(\hat{g}_1,...,\hat{g}_n)
```

Algorithm 3 ACCUSE (i, j, allegation)

```
Input: i accuser, j target, n peers, k step,
      signed by i's private key
  1: g_j^k = \text{compute\_gradients}(x^k, \xi_i^k)
 2: if \operatorname{hash}(g_j^k) \neq c_j^k or \operatorname{hash}(g_j^k[\cdot]) \neq h_j

3: or \langle \Delta_j^l, z^k \rangle \neq s_j^l or \sum_{l=1}^n s_l^j \neq 0
             vote ban peer, // and any ...
  4:
  5:
             // peer, that covered up for peer,
  6: else
             vote ban peer.
  7:
  8: for \hat{i} = 1 \dots n do
            if \operatorname{votes}(\operatorname{peer}_{\hat{j}}) \geq n/2 then
 9:
10:
                  BAN peer
```

3.2 Convergence analysis

For convenience, let us paraphrase our task from the perspective of optimization theory. Consider the following expectation minimization problem: $\min_{x \in Q \subset \mathbb{R}^d} \{f(x) := \mathbb{E}_{\xi \sim \mathcal{D}}[f(x,\xi)]\}, (1)$

where the objective function f is smooth and uniformly lower bounded, $Q \subseteq \mathbb{R}^d$ is a closed convex set of admissible parameters and ξ is the source of stochasticity, such as minibatch indices. We assume that the problem (1) can be solved in a distributed manner, i.e., one can use n workers performing (mini-batched) stochastic gradients calculations in parallel and communicating according to some protocol. For simplicity, we will define the set of workers as $[n] := \{1, 2, \ldots, n\}$.

Furthermore, we assume that some workers can be malicious, i.e., they can deviate arbitrarily from the predefined algorithm: send arbitrary vectors instead of stochastic gradients or violate the communication protocol. Such workers are usually called *Byzantine nodes* or just *Byzantines*. We define the set of all "good" workers as \mathcal{G} and the set of Byzantine workers as \mathcal{B} : $[n] = \mathcal{G} \sqcup \mathcal{B}$. We further assume that strictly less than half nodes are Byzantines: $b = |\mathcal{B}| \leq \delta n$, where $\delta \in [0, 1/2)$.

There are many ways for Byzantines to affect the training. To simplify further analysis, we group these strategies into 4 broad categories: (i) gradient attacks, where Byzantines modify their g_i^k , but otherwise behave normally; (ii) aggregation attacks, where Byzantine aggregator returns wrong \hat{g}_i and relies on others to cover it up by misreporting s_i ; (iii) reputation attacks such as frame-up or slander via false ACCUSE (i, j, \cdot) ; and finally, (iv) protocol errors are any other deviations from the steps of Algorithm 1, e.g. refusing to send any data. We elaborate on each attack type in Appendix D.

For the purpose of this analysis, the latter two attacks can be repelled with an extra policy that allows an active worker to eliminate any other worker at the cost of also being banned. Whenever a benign \mathtt{peer}_i encounters a protocol error from another \mathtt{peer}_j , it invokes that policy to remove both himself and \mathtt{peer}_j from training. The design of this policy ensures that every invocation, whether by normal or Byzantine peers, eliminates at least 1 Byzantine peer and at most 1 benign one. Thus, if a Byzantine minority uses this against benign peers, this will only decrease their relative numbers: $(\delta n-1)/(n-2) < \delta$. This leaves us with two attacks that both target the aggregated gradients.

We provide convergence guarantees for variants of BTARD-SGD with $Q = \mathbb{R}^d$ under different sets of assumptions about the function f and its stochastic gradients. Our first two setups assume, that: **Assumption 3.1.** There exist such constant $\sigma \geq 0$, $s_0 \in [d]$ that for any set of indices $S = (i_1, \ldots, i_d)$, $1 \leq i_1 < i_2 < \ldots < i_s \leq d$, $s \geq s_0$ stochastic gradient $\nabla f(x, \xi)$ satisfy

either
$$\mathbb{E}[\nabla f(x,\xi)] = \nabla f(x)$$
, $\mathbb{E}\left[\left\|\nabla_{[S]}f(x,\xi) - \nabla_{[S]}f(x)\right\|^4\right] \le \left(\frac{s\sigma^2}{d}\right)^2$, (Option I)

$$or \quad \mathbb{E}[\nabla f(x,\xi)] = \nabla f(x), \quad \mathbb{E}\left[\left\|\nabla_{[S]}f(x,\xi) - \nabla_{[S]}f(x)\right\|^2\right] \leq \frac{s\sigma^2}{d}, \quad (\text{Option II})$$

where $\nabla_{[S]}f(x,\xi) = (\nabla_{i_1}f(x,\xi), \dots, \nabla_{i_s}f(x,\xi))^{\top}$, $\nabla_{[S]}f(x) = (\nabla_{i_1}f(x), \dots, \nabla_{i_s}f(x))^{\top}$, and $\nabla f_j(x,\xi), \nabla_j f(x)$ are j-th components of $\nabla f(x,\xi)$ and f(x) respectively.

Here, (Option II) is an extension of the classical uniformly bounded variance assumption [78, 79, 80] ensuring that the noise in all subvectors of large enough dimension has the variance dependent on the ratio between the dimension of the subvector s and the dimension of the full vector d. (Option I) is more restrictive but under this assumption we obtain better convergence results. In order to further reduce overhead from **Verification 3** in the full Algorithm 6, we also assume that the stochastic gradients distributions have sub-quadratically decreasing tails (see details in Appendix F).

Assumption 3.2. There exist such constant $\sigma \geq 0$, $s_0 \in [d]$ that for any set of indices $S = (i_1, \ldots, i_d)$, $1 \leq i_1 < i_2 < \ldots < i_s \leq d$, $s \geq s_0$ stochastic gradient $\nabla f(x, \xi)$ satisfy

$$\mathbb{P}\left\{\left\|\frac{1}{k}\sum_{i=1}^{k}\nabla_{[S]}f(x,\xi_{i})-\nabla_{[S]}f(x)\right\|^{2}>\frac{ts\sigma^{2}}{kd}\right\}<\frac{1}{t^{2}},\quad\forall t>0,$$
(2)

where ξ_1, \ldots, ξ_k are i.i.d. samples from \mathcal{D} , and $\nabla_{[S]} f(x, \xi)$, $\nabla_{[S]} f(x)$ are defined in As. 3.1.

Under these assumptions, we derive the following convergence bounds for strongly convex, generally convex, and non-convex objectives (see Table 1). The respective proofs are deferred to Appendix F.3.

Table 1: Summary of complexity bounds for BTARD-SGD in different scenarios. By complexity we mean the number of iterations sufficient to find such point \widehat{x} that $\mathbb{E}[\|\nabla f(\widehat{x})\|^2] \leq \varepsilon^2$ for non-convex problems and $\mathbb{E}[f(\widehat{x}) - f(x^*)] \leq \varepsilon$ for convex and μ -strongly convex problems (see Def. F.2) with x^* being the solution. Notation: "known $|\mathcal{B}_k^a|$ " = the exact number of attacking Byzantine workers at iteration k is known to each participant, L = smoothness constant (see Def. F.1), $\Delta_0 = f(x^0) - f_*$, f_* = uniform lower bound for f, σ^2 = variance parameter from As. 3.1, n = the initial number of peers, b = the initial number of Byzantine workers, $\delta = b/n$, m = number of peers checked at each iteration, $R_0 = \|x^0 - x^*\|$.

Assumptions	Convexity of f		
	Non-convex	Convex	Strongly convex
As. 3.1 (Option I) + As. 3.2 + known $ \mathcal{B}_k^a $	$\frac{L\Delta_0}{\varepsilon^2} + \frac{L\Delta_0\sigma^2}{n\varepsilon^4} + \frac{n\delta\sigma^2}{m\varepsilon^2}$	$\frac{LR_0^2}{\varepsilon} + \frac{\sigma^2 R_0^2}{n\varepsilon^2} + \frac{n\sqrt{\delta}\sigma R_0}{m\varepsilon}$	$\frac{L}{\mu}\log\frac{\mu R_0^2}{\varepsilon} + \frac{\sigma^2}{n\mu\varepsilon} + \frac{n\sqrt{\delta}\sigma}{m\sqrt{\mu\varepsilon}}$
As. 3.1 (Option II) + As. 3.2	$\frac{L\Delta_0}{\varepsilon^2} + \frac{L\Delta_0\sigma^2}{n\varepsilon^4} + \frac{nb\sigma^2}{m\varepsilon^2}$	$\frac{LR_0^2}{\varepsilon} + \frac{\sigma^2 R_0^2}{n\varepsilon^2} + \frac{nb\sigma R_0}{m\varepsilon}$	$\frac{L}{\mu}\log\frac{\mu R_0^2}{\varepsilon} + \frac{\sigma^2}{n\mu\varepsilon} + \frac{nb\sigma}{m\sqrt{\mu\varepsilon}}$

Let us briefly discuss the main properties of the derived results. When $\delta=0$, i.e., there are no Byzantine peers, we recover the tightest known rates for parallel SGD for strongly convex, generally convex, and non-convex objectives with both sets of assumptions. Next, we notice that in all complexity bounds in (Option I), the term depending on the ratio of Byzantine workers δ (the third one in all bounds) has better dependence on the accuracy of the solution ε than the classical variance term (the second one in all bounds). Therefore, for sufficiently small ε , the derived complexity bounds are the same as in the case when there are no Byzantine workers and parallel SGD is used. However, these bound are obtained under the assumption that all participants know the exact number of attacking Byzantine workers at each iteration, which is unrealistic.

As for the weaker (Option II), the third term is much worse than the corresponding term in the previous setup. Nevertheless, the term that depends on the ratio of Byzantine workers δ has the same dependence on ε as in (Option I). This implies that for sufficiently small ε the derived complexity bounds are the same as in the case when there are no Byzantine workers and parallel SGD is used. For complete formulations, proofs and other details we refer the reader to Appendix F.2.

So far, all our convergence results rely on As. 3.2, i.e., that the stochastic gradients have not too heavy tails. This assumption holds for many real-world neural networks. However, there are important NLP tasks such as BERT training [81], where the noise in the stochastic gradient has such a heavy noise that As. 3.2 becomes unlrealistic. The third and final setup in our analysis aims to address such heavy-tailed problems with BTARD-CLIPPED-SGD (see full Algorithm 8 in appendix). We analyse the method under the assumption that α -th moments of the stochastic gradients are uniformly upper-bounded for some $\alpha \in (1,2]$. We notice that for $\alpha < 2$ this assumption allows the variance of the stochastic gradient to be unbounded. In this setting, we prove that BTARD-CLIPPED-SGD finds an ε -solution of the convex problem after $\mathcal{O}\left(\varepsilon^{-\alpha/(\alpha-1)}\left(1+\binom{n\sqrt{\delta}/m}{n}\right)^{\alpha/(\alpha-1)}\right)\right)$ iterations when the number of attacking Byzantine peers is known at each iteration and $\mathcal{O}\left(\varepsilon^{-\alpha/(\alpha-1)}\left(1+\binom{nb/m}{n}\right)^{\alpha/(\alpha-1)}\right)$ iterations otherwise. One can find the full statements and complete proofs of our results in Appendix F.

3.3 System design considerations

The algorithm described in Section 3.1 operates with a pre-defined list of peers that can only decrease in size. However, many real-world scenarios can benefit from training in a public environment where new peers can join midway through training. Unfortunately, these scenarios make the system exposed to a new type of malicious behavior known as Sybil attacks [82]. To exploit this attack vector, a single computationally constrained attacker adopts multiple pseudonymous identities in order to establish a dishonest majority and break the algorithm.

To combat this behavior, we augment BTARD with a protocol which dictates how new peers can join. The core idea of this protocol is that any new participant must prove that it has honestly trained the network over multiple continuous iterations before it will be given full privileges. This ensures that the influence of Sybil attackers is proportional to their actual computing power. We formulate these claims in more detail in Appendix G.

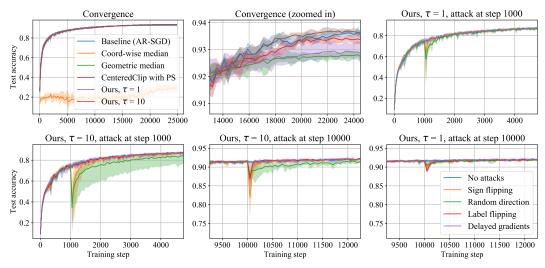


Figure 2: (**Upper-Left, Upper-Middle:**) ResNet-18 test accuracy with different robust aggregation techniques. (**Remaining plots:**) Effectiveness of Byzantine attacks against against BTARD-SGD.

To operate in this setting, we augment Algorithm 1 with a specialized distributed system designed to be resistant against Sybil attacks. Inspired by other public distributed projects, we build upon Distributed Hash Table [83, 84, 85] — a protocol that establishes a shared key-value storage over distributed unreliable devices. We describe the implementation details in Appendix H.

4 Experiments

4.1 CIFAR10 classification

First, we evaluate our approach with a realistic image-classification workload in controlled conditions. Our setup is a ResNet-18 [86] model trained to solve the CIFAR10 classification task [87]. We train the model on 16 peers (each peer processes 8 samples per batch) using the SGD with Nesterov [88] momentum and the cosine annealing learning rate [89]. We deliberately use tuned setup that achieves 93.5% test accuracy in order to measure how byzantine attacks affect this training outcome.

We evaluate our method with constant τ =10 (weaker clipping) and with τ =1 (stronger clipping). These values were chosen based on the maximal standard deviation of the gradient parts averaged by the workers during normal training, so that almost no vectors are clipped for the weaker clipping and approximately half of the vectors are clipped for the stronger clipping scenario. BTARD randomly selects 1 validator on each step. If the validator happens to be Byzantine, it does not accuse its peers.

We compare our method to the regular All-Reduce without clipping, the coordinate-wise median and geometric median approaches, and the original variant of CENTEREDCLIP that uses a trusted parameter server [58]. Some other popular robust aggregation techniques are omitted because they were shown to be inferior [58]. We run all iterative algorithms (e.g. CENTEREDCLIP) to convergence with $\epsilon=10^{-6}$, as we have found that limiting the number of iterations can significantly decrease the final model quality (see Fig. 6 in Appendix I).

In addition to training convergence, we evaluate our setup in presence of malicious peers. To test pessimistic conditions, we pick a setting where 7 of 16 peers are Byzantine (other setups can be found in Appendix I). We experiment with the following attack types:

- SIGN FLIPPING: each attacker sends the opposite of its true gradient.
- RANDOM DIRECTION: all attackers send large vectors pointed at a common random direction.
- LABEL FLIPPING: each attacker computes its gradient based on the cross-entropy loss with flipped labels. For CIFAR-10, we replace label $l \in \{0, ..., 9\}$ with 9 l.
- DELAYED GRADIENT: attackers send their real gradients delayed by 1000 steps.

We further amplify the Byzantine gradients from the first two attacks by a large coefficient $\lambda=1000$ so they would dominate the aggregated gradient if no clipping is used. While in practice such attacks can be identified right away due to the large gradient norms, we deliberately avoid doing that to test our clipping approach. We omit some common low-magnitude attacks [56, 57, 60] designed to bypass variance and magnitude checks, as BTARD does not use these checks anyway.

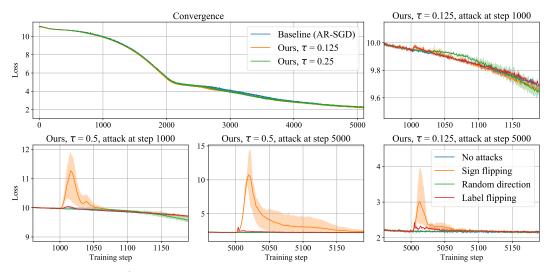


Figure 3: (**Upper-Left:**) ALBERT-large training objective using AR-SGD and BTARD-Clipped-SGD. (**Remaining plots:**) Effectiveness of Byzantine attacks against BTARD.

For each experiment configuration, Byzantines behave honestly prior to step s, then simultaneously attack on each subsequent step until they are banned. Other setups (e.g. attacking periodically) are considered in Appendix I. We consider attacks in two training regions: early stages (s=1000) and closer to convergence (s=10,000). We repeat each experiment 5 times and report the mean and range of the test accuracy during at least 2000 steps after all Byzantines are banned. In our experiments, this usually happened within 150 steps after s.

The results are shown in the Fig. 2. comparing to the All-Reduce baseline, we note that our method does not worsen the speed of convergence but introduces a minor negative effect on the final test accuracy. For τ =10, this effect is indistinguishable from random variation. In terms of test accuracy, the two most effective attacks are the random direction and sign flipping. The effect of label flipping is smaller, and the effect of delayed gradients is almost undetectable.

4.2 Pre-training transformers

For our second experiment, we choose a more compute-intensive and hyperparameter-sensitive model with adaptive optimizers to demonstrate that our approach may be applied to models that are commonly used in distributed training scenarios. Our setup is pre-training the ALBERT-large model [90] on the Wikitext 103 dataset [91] using LAMB [50]. Since the original ALBERT setup uses gradient clipping, we use BTARD-CLIPPED-SGD (see Alg. 8 in Appendix). We train the model on 16 machines that jointly accumulate 4096 samples for every batch. Similarly to the previous section, we evaluate two configurations with τ =0.5 (weaker clipping) and τ =0.125 respectively, in addition to All-Reduce baseline. For Byzantine tolerance evaluation we also use 7 out of 16 Byzantine servers, 1 validator and two attack regions: s=1000 and s=5,000. The only major difference is that here we do not report the delayed gradient attack as we have found it completely ineffective. We provide full configuration of this experiment in Appendix J.

The results shown in Figure 3 demonstrate a similar pattern to what we have seen in Section 4.1. During normal training, both τ values had no significant effect on the training progress. However, $\tau=0.125$ shows significantly faster recovery against all three attacks. One important observation from these experiments is that while some attacks can significantly increase the loss function, model can recover from those attacks much faster than the time it takes to training from scratch with the same loss function. As such, we hypothesize that Byzantine attacks do much less harm the learned feature representation than to the classification layers.

5 Conclusion

In this work we formulated BTARD-SGD — a byzantine-tolerant training strategy for large neural networks. We verified its robustness and effectiveness through rigorous theoretical analysis and large-scale distributed training experiments. While our research is mostly algorithmical, it can open new opportunities in many deep learning applications.

Perhaps the most important one is making it possible to train large neural networks in a cooperative manner. BTARD-SGD could allow small research groups to host open cooperative training projects where the training hardware is crowdsourced by volunteers around the world [92] or by a group of small companies that can collectively compete with larger corporations by combining their compute clusters in a single training "supercomputer". While these two applications also need huge engineering effort to become practical, our algorithm ensures that these projects can run securely and effectively without the need to carefully screen every potential volunteer.

Acknowledgments

Eduard Gorbunov was supported by the Ministry of Science and Higher Education of the Russian Federation (Goszadaniye) 075-00337-20-03, project No. 0714-2020-0005. We thank Sai Praneeth Karimireddy for useful discussions and suggestions, Lie He for providing the code with CENTERED-CLIP, William Cappelletti for pointing out several relevant papers, Gennady Pekhimenko for his technical expertize and infrastructure support for actual distributed training experiments, and Dmitrii Emelianenko for helpful discussions. The computational resources for the experiments were provided entirely by the Amazon Research Awards program.

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Supplementary Material

A Table of contents

We organize the supplementary materials as follows:

- Appendix B provides a more detailed explanation of distributed security protocols that we outlines in Section 2.3,
- Appendix C analyzes the computation and communication cost of BTARD-SGD and BTARD-CLIPPED-SGD in comparison with regular AR-SGD,
- Appendix D enumerates possible strategies that Byzantine peers can adopt to attack BTARD-SGD,
- Appendix E contains a detailed version of BTARD-SGD and auxiliary algorithms that were previously announced in 3.1,
- Appendix F formulates and proves the convergence bounds annouced in Table 1 (Section 3.2),
- Appendix G describes how to apply BTARD-SGD for public cooperative training where peers can join at any time,
- Appendix H explains Distributed Hash Tables and the additional security measures necessary for our setup,
- Appendix I reports experimental evaluation of BTARD-SGD for ResNet-18 in alternative settings,
- Appendix J contains a detailed description of ALBERT training hyerparameters used in Section 4.2.

B Secure distributed systems

B.1 Broadcast channels

Many distributed systems rely exclusively on direct peer-to-peer connections, avoiding any centralized servers to increase reliability and avoid the performance bottleneck. In presence of malicious participants, this introduces additional security challenges since an attacker can send corrupted data to one participant and behave honestly with others. If a peer accuses another peer in sending corrupted data, it is impossible for remaining peers to determine whether the accusation is fair since only these two peers had access to the contents of the communication channel between them.

To overcome this, [75] provide algorithms to build broadcast channels over the peer-to-peer connections and [74] improves their performance with the usage of digital signatures [94, 95]. Broadcast channels guarantee that if a peer p sends a message, (a) all honest peers receive the same message and (b) the received message coincides with the original one if p is honest. If t peers out of n are malicious, the algorithm takes O(t) consecutive rounds of O(n) messages each to broadcast the message.

B.2 Multi-party pseudorandom number generation

Distributed systems benefit from the algorithms [73] for the joint pseudorandom number generation (JPRNG) where a group of malicious peers would have little influence (bias) on the generator output. For example, the JPRNG allows to decide which participant wins a lottery or whose calculations are going to be validated by other peers to detect possible cheating.

While there are JPRNGs [96] with a negligible bias for the case when more than a half parties are honest (assuming the presence of the broadcast channel), [97] proves that it is impossible to reach the negligible bias for the case of dishonest majority, which may be reached in practice with the Sybil attacks.

[98] propose variants of JPRNGs robust to a dishonest majority with the bias close to the lower bound from [97]. The bias appears because sometimes an attacker may learn the result earlier than other peers and abort the procedure. However, if we use a JPRNG to select a peer to be checked for cheating, using these algorithms is not necessary since we may ban peers who have aborted the procedure and restart it from scratch without them.

C Compute and network overhead of BTARD

Despite having complex structure, BTARD-SGD has only limited communication overhead, when compared to regular ALL-REDUCE SGD. When operating without Byzantines, a single step of BTARD requires each peer to send each gradient tensor exactly once for aggregation, then download the results, exactly as in Butterfly All-Reduce. In contrast, all additional communication only requires sending scalars that are independent of the total size of the trained model. As a result, the communication complexity of a single BTARD-SGD step for each peer is $O(\frac{n-1}{n} \cdot d + n^2 \cdot b)$ where n is the number of peers, b is the number of "active" byzantines and d is the total vector size. The $n^2 \cdot b$ component comes from the broadcast channel worst case complexity under b malicious peers. In all setups considered in Section 4, these costs were dominated by the vector size d, as both models contain millions of trainable parameters.

In terms of computation, BTARD-SGD introduces two main overheads: from validators and CENTEREDCLIP respectively. As we have shown empirically, both BTARD-SGD and BTARD-CLIPPED-SGD can withstand attacks even with 1 random validator chosen from 16 peers. As such, the computation overhead for these validators is under 10% of the total compute. As for the CENTEREDCLIP, our algorithm executes the same computation as the original CENTEREDCLIP [58], except that now the extra load is distributed evenly across all peers.

D Overview of attack vectors

In Section 3.2 we have outlined 4 main types of byzantine attacks that can affect BTARD-SGD. Here, we analyze each of these types in detail and provide a list of attacks that fit these types.

Gradient attacks. This attack vector encompasses all attacks where byzantine peers replace their true gradients with something else, but otherwise act normally. With this attack, b byzantine peers can collectively shift the outputs of CENTEREDCLIP by up to $\tau \cdot b/n$ in any chosen direction. However, since byzantine peers will need to commit hash of their incorrect gradients, *every honest validator* can accuse one of these peers with probability b/n.

Aggregation attacks. A similar, but opposite attack type can be attempted when a byzantine peer performs gradient aggregation. Instead of honestly computing CENTEREDCLIP, an attacker may modify the returned vector to incorporate the same kinds of changes as in gradient attacks (see above). This time, the maximum difference that can be applied through such attacks is larger, but it only affects b/n of vector coordinates that are aggregated by Byzantines.

Done naively, such attacks can be detected and banned by the gradient checksum (see L15-17 in Alg. 1). In order to ensure that the above check passes, Byzantines can misreport their s_i^j in such a way that $\sum_i s_i^j = 0$. However, since actual s_i^j depend only on g_i^k and \hat{g}^k , these values can be verified by the chosen validators, and, in case of mismatch, reported via ACCUSE.

Furthermore, if an honest validator finds that a certain peer has broadcast incorrect s_i^j , the validator can simultaneously accuse the corresponding byzantine aggregator j that *should have* detected incorrect s_i^j (see L12-14 in Alg. 1), but didn't. We analyze this attack further in Appendix F.

Reputation abuse. Since BTARD-SGD provides means by which benign participants can ban byzantine attackers, it is important to ensure that the same means cannot be exploited by byzantine peers to eliminate benign ones or otherwise abuse the system. There are three potential attack vectors that fit this description:

- Falsely accusing a benign peer,
- Persistently calling the ACCUSE procedure to slow down training,
- Automatically approving gradients without actual validation,

In BTARD-SGD, we protect against slander (issues 1. and 2.) by the design of ACCUSE protocol, by which a peer that initiates false allegations will itself be banned. As such, Byzantines can only invoke ACCUSE protocol a limited number of times before they are all permanently banned.

In turn, the attack vector (3.) is more effective: if one Byzantine was chosen as validator for another Byzantine, they can automatically report successful validation without negative consequences for either of them. However, since all validators are chosen through JPRNG, an attacker has no way of predicting whether its validator will be benign or byzantine. Thus, any malicious activity will always have a chance of being caught by an honest validator.

Protocol violations. Finally, a byzantine attacker can deviate from the protocol prescribed by BTARD-SGD in simpler ways ways, for instance:

- 1. Not committing the hash of its gradient when required by 4,
- 2. Not sending data to a particular peer when required (or sending data twice),
- 3. Deliberately broadcasting a hash that mismatches the subsequently sent data,
- 4. Sending metadata (e.g. gradient norm) that is inconsistent with previously sent gradient part,
- 5. Sending s_i that is inconsistent with previously sent gradient,
- 6. Not validating when chosen as validator, validating when **not** chosen, or validating a different peer than was chosen by BTARD-SGD.

For protocol deviations that are visible to all benign participants, such as in (1.) or (6.), benign peers can ban the offender instantaneously. However, this is not the case for attacks such as (2.), where the deviation is only visible to one or few peers.

As described earlier in Section 3.2, we address this issue with a special procedure that allows any peer to ban any other peer at the cost of also being banned. Thus, if an attacker sends inconsistent gradients, norms or inner products to only one benign peer, that peer can still get the attacker banned even though it wouldn't be able to call ACCUSE.

Protecting from attacks 3, 4 and 5 from the above list also relies on this mutual elimination procedure. Specifically, if an attacker sends provably incorrect data to a benign peer, that peer will immediately trigger the mutual elimination procedure. The only exception to this rule is if one byzantine peer sends incorrect data to another byzantine peer: this behavior is neither punishable nor, in itself, harmful. In turn, the mutuality of this elimination procedure prevents potential misuse by Byzantines: if an attacker decides to ban someone through this procedure, that attacker will also be banned.

E Detailed algorithm description

In this section, we provide more formal versions of BTARD (Alg. 4) and BTARD-SGD (Alg. 6) algorithms, as well as auxiliary subroutines. For completeness, we describe our approach in a bottom-up manner. First, we describe a single aggregation step in Algorithm 4. Then, we formulate the ACCUSE subroutine for blocking malicious peers in Algorithm 5. Finally, we formulate the full BTARD-SGD in Algorithm 6 using the above subroutines as building blocks.

Algorithm 4 defines a single gradient aggregation step (outlined earlier in Alg. 2) with additional verifications needed to reduce the negative influence of Byzantine peers. For simplicity, we assume that workers run each line in a synchronous manner (e.g. wait for all peers to broadcast $hash(g_i)$ before communicating the actual gradients). In practice, this restriction can be lifted in favor of asynchronous steps with several explicit synchronization barriers, but that would further complicate the pseudo-code.

CenteredClip is a robust aggregation rule proposed in [58]. Unlike a number of other aggregation rules as coordinate-wise median, Krum, geometric median, CENTEREDCLIP is provably robust against Byzantine attacks (see Theorem III from [58] and Lemma F.1).

Let \mathcal{G} be the set of good peers, \mathcal{B} be the set of Byzantine workers, and, for simplicity, let $[n] = \mathcal{G} \sqcup \mathcal{B}$, $|\mathcal{B}| = \delta n \leq \delta_0 n < n/2$. Assume that we have n random vectors x_1, \ldots, x_n , such that $\forall i, j \in \mathcal{G}$

$$\mathbb{E}[x_i] = \mathbb{E}[x_j] = x, \quad \mathbb{E}[\|x_i - x_j\|^2] \le \sigma^2,$$

and for all $i \in \mathcal{B}$ vectors x_i can be arbitrary. CENTEREDCLIP works as follows: it is an iterative procedure generating a sequence $\{v_l\}_{l>0}$ satisfying

$$v^{l+1} = v^l + \frac{1}{n} \sum_{i=1}^n (x_i - v^l) \min\left\{1, \frac{\tau_l}{\|x_i - v^l\|}\right\},$$
 (CenteredClip)

where

$$\tau_l = \sqrt{\frac{4\sqrt{2}(B_l^2 + \sigma^2)}{\delta}}, \quad B_{l+1}^2 = (4\delta^2 + 4\sqrt{2}\delta)B_l^2 + 4\sqrt{2}\delta\sigma^2.$$
(3)

The goal of this procedure is natural: find good enough approximation \widehat{x} of $\overline{x} = \frac{1}{|\mathcal{G}|} \sum_{i \in \mathcal{G}} x_i$. In [58], it is shown⁵ that for $\delta \leq 0.15$ the sequence $\{v_l\}_{l \geq 0}$ generated by CENTEREDCLIP satisfies

$$\mathbb{E}[\|v^l - \overline{x}\|^2] \le 0.94^l \mathbb{E}[\|v_0 - \overline{x}\|^2] + 95\delta\sigma^2.$$

Moreover, the authors of [58] prove that for all possible aggregation rules producing \hat{x} and given δ_0 , σ there exists such set of vectors x_1, \ldots, x_n and such a partition $[n] = \mathcal{G} \sqcup \mathcal{B}$ that

$$\mathbb{E}[\|\widehat{x} - \overline{x}\|^2] = \Omega(\delta\sigma^2).$$

Therefore, CENTEREDCLIP can be seen as an optimal aggregation rule neglecting numerical constants. The usage of CENTEREDCLIP helps the good peer i to produce a good enough approximation of the ideal average of the i-th parts of stochastic gradients among good peers in BTARD.

Moreover, since $\delta \leq 0.15$ we have that $4\delta^2 + 4\sqrt{2}\delta \leq 0.94$ implying that $B_l^2 \to B^2 \sim \delta\sigma^2$ when $l \to \infty$, and $\tau_l \to \tau \sim \sqrt{\sigma^2 + \frac{\sigma^2}{\delta}}$. These limits can be easily computed from (3). Next, for $l \to \infty$ CenteredClip converges to the solution of the following equation:

$$\sum_{i=1}^{n} (x_i - v) \min\left\{1, \frac{\tau}{\|x_i - v\|}\right\} = 0.$$
 (4)

In other words, CenteredClip for large enough l approximates the fixed point iteration process of solving (4). This property plays a key role in **Verification 2** of BTARD.

Verifications 1 and 2. While good peers always run CENTEREDCLIP, Byzantine peers can arbitrary violate the protocol meaning that they can send an arbitrary vector instead of sending the result of CENTEREDCLIP. **Verification 1** and **2** are needed to prevent such violations and make it possible to identify them during the check of computations.

First of all, both verifications are split into 2 rounds in order to let the aggregators of the corresponding part accuse those peers who send inconsistent norms or inner products. Next, in theory, we assume that all good peers find exactly the solution of Centered Clip equaition (4). Therefore, it is possible to compute the weights from (4) for each worker i and each component j knowing only a norm of the difference of corresponding vectors, i.e., one can compute $\min\{1, \frac{\tau}{\|g_i(j) - \widehat{g}(i)\|}\}$ by $\|g_i(j) - \widehat{g}(i)\|$. That is, if Byzantine peer i sends $\text{norm}_{ij} \neq \|g_i(j) - \widehat{g}(j)\|$, it will be either revealed by j-th worker if $j \in \mathcal{G}$ or it will be revealed with some probability during the subsequent checks of computations.

However, **Verification 1** is insufficient to prevent malicious behavior: at iteration k Byzantine peer can send $g_i^k(j)$ such that $\|g_i^k(j) - \widehat{g}^k(j)\| = \|\nabla_{(j)} f(x^k, \xi_{i,k}) - \widehat{g}^k(j)\|$. If $j \in \mathcal{B}$, then it can be the case that i-th worker commits the hash of $\nabla_{(j)} f(x^k, \xi_{i,k})$ and the check of gradient computation will not identify the violation of the protocol. That is why, **Verification 2** is required.

The goal of **Verification 2**, is to check that CENTEREDCLIP equation (4) holds for the received vector. The idea is simple: if

$$\sum_{l=1}^{n} (g_l(i) - \widehat{g}(i)) \min\left\{1, \frac{\tau}{\|g_l(i) - \widehat{g}(i)\|}\right\} = 0,$$
 (5)

⁵On the moment of writing this paper, the proof of Theorem III from [58] contained a minor inaccuracy. The issue can be easily resolved under assumption that for all $i, j \in \mathcal{G}$ we have $\mathbb{E}[\|x_i - x_j\|^4] \leq \sigma^4$.

then for any z_i of an appropriate dimension

$$\sum_{l=1}^{n} \langle g_l(i) - \widehat{g}(i), z_i \rangle \min \left\{ 1, \frac{\tau}{\|g_l(i) - \widehat{g}(i)\|} \right\} = 0.$$
 (6)

Since z_i in BTARD is generated from the uniform distribution on the unit Euclidean sphere, we have $\mathbb{P}\{(5) \text{ does not hold } \& (6) \text{ holds}\} = 0.$ (7)

However, it is impossible to verify (6) explicitly for workers $j \neq i$. Therefore, in the algorithm, good workers check

$$\sum_{l=1}^{n} s_{l}^{i} = 0, \quad \text{where } s_{l}^{i} = \begin{cases} \langle g_{l}(i) - \widehat{g}(i), z_{i} \rangle \min\left\{1, \frac{\tau}{\|g_{l}(i) - \widehat{g}(i)\|}\right\}, & \text{if } l \in \mathcal{G}, \\ *, & \text{if } l \in \mathcal{B}. \end{cases}$$
(8)

Unfortunately, Byzantine peers can send arbitrary s_l^i . This can lead to the situations when (8) holds while (6) and, as a consequence, (5) do not. Below, we rigorously show that all possible violations of the protocol that are not detected by verifications of BTARD can be detected by the auxiliary check of computations with some probability.

random_vector as described earlier in Section 3.1, this function generates a pseudo-random unit vector in the space of model parameters. This vector is based on a random seed obtained from JPRNG.

ACCUSE and ELIMINATE are the two protocols by which peers ban bock Byzantine attackers from training. The ACCUSE protocol is only invoked if there the malicious activity of the target peer can be proven to others (we detail the exact mechanism in Algorithm 3). In contrast, ELIMINATE is a mechanism that allows any peer i to ban any other peer j from training without proof — but at the cost of peer i also being banned. We have described this protocol earlier as a countermeasure for protocol violations (see Appendix D).

Verification 3. This is an additional verification that serves to limit the potential scope of *aggregation attacks* (as described in Appendix D). If the result of CenteredClip landed far from too many benign participants, BTARD will verify it by re-running the same aggregation across all peers. While this procedure is costly, our analysis proves that it is has a very small probability of triggering unless some of the peers perform aggregation attacks. In the latter case, verifying the gradient accumulation will root out such attacks and ban the corresponding peers.

Check of computations. As we mentioned earlier, it is possible to violate the protocol without being detected by the verifications of BTARD. Therefore, extra checks of computations are required. In particular, after each aggregation in BTARD-SGD 2m workers are selected uniformly at random: m workers check the computations at the previous step of other m workers. That is, each Byzantine peer is checked at iteration k with probability $\sim m/n$ by some good worker (see the proof of Thm. F.1). Consider an arbitrary Byzantine peer j and all possible violations of the protocol at iteration k that are not detected by verifications of BTARD.

First of all, we notice that if $c_j(i) \neq \operatorname{hash}(\nabla_{(i)} f(x^k, \xi_{j,k}))$, then it will be detected during the check of computations with some probability⁶. Moreover, if $i \in \mathcal{G}$, then j-th worker has to send $c_j(i) = \operatorname{hash}(g_j(i))$ to avoid ban.

Therefore, the only non-trivial case is when $i \in \mathcal{B}$ as well. In this case, j-th worker can commit $c_j(i) = \mathsf{hash}(\nabla_{(i)} f(x^k, \xi_{j,k}))$ since it is meaningless for i-th worker to accuse j-th one. Since $\mathsf{norm}_{ij}, s^j_i$ and $\widehat{g}(i)$ are known for all i and j, j-th worker has to broadcast $\mathsf{norm}_{ji} = \|\nabla_{(i)} f(x^k, \xi_{j,k}) - \widehat{g}(i)\|$ and $s^i_j = \langle \nabla_{(i)} f(x^k, \xi_{j,k}) - \widehat{g}(i), z_i \rangle \min\left\{1, \frac{\tau}{\|\nabla_{(i)} f(x^k, \xi_{j,k}) - \widehat{g}(i)\|}\right\}$ to avoid the ban during the check of the computations. Therefore, regardless to the choice $g_j(i)$, to pass **Verification 2** i-th worker should send such $\widehat{g}(i)$ that

$$\sum_{l \in \mathcal{G} \cup \{j\}} \langle \nabla_{(i)} f(x^k, \xi_{l,k}) - \widehat{g}(i), z_i \rangle \min \left\{ 1, \frac{\tau}{\|\nabla_{(i)} f(x^k, \xi_{l,k}) - \widehat{g}(i)\|} \right\} + \sum_{l \in \mathcal{B} \setminus \{j\}} s_l^i = 0.$$

⁶Here and below, this means that the attack/violation will be detected iff a non-Byzantine peer is chosen to validate the perpetrator.

In this case, the behavior of the j-th worker along i-th component is equivalent to the behavior of the good one. It means, that to avoid ban during the check of computations, each Byzantine worker l should broadcast $\operatorname{norm}_{li} = \|\nabla_{(i)} f(x^k, \xi_{l,k}) - \widehat{g}(i)\|$ and $s^i_l = \langle \nabla_{(i)} f(x^k, \xi_{l,k}) - \widehat{g}(i) \rangle$ implying that i-th worker should send such $\widehat{g}(i)$ that

$$\sum_{l=1}^{n} \langle \nabla_{(i)} f(x^k, \xi_{l,k}) - \widehat{g}(i), z_i \rangle \min \left\{ 1, \frac{\tau}{\|\nabla_{(i)} f(x^k, \xi_{l,k}) - \widehat{g}(i)\|} \right\} = 0.$$

In view of (7), it implies that

$$\widehat{g}(i) = \text{CenteredClip}(\nabla_{(i)} f(x^k, \xi_{1,k}), \nabla_{(i)} f(x^k, \xi_{2,k}), \dots, \nabla_{(i)} f(x^k, \xi_{2,k})),$$

i.e., there are no violations of the protocol along the i-th component.

BTARD-SGD. Finally, the Algorithm 6 incorporates the two above procedures into a secure decentralized SGD training loop. This algorithm is intended as a more formal version of Alg. 1 from Section 3.1.

Algorithm 4 Byzantine-Tolerant All-Reduce (BTARD)

```
Input: n -number of workers, g_1, g_2, \ldots, g_n \in \mathbb{R}^d - vectors on the workers, d > n, \Delta_{\max} > 0 -
      parameter for verification 3
 1: for workers i = 1, \dots, n in parallel do
           Split g_i into n parts: g_i = (g_i(1)^\top, \dots, g_i(n)^\top)^\top, g_i(j) \in \mathbb{R}^{d_j} for all i, j \in [n]
 3:
 4:
           Broadcast hash(g_i) as c_i
 5:
           for j = 1, ..., n do
                Broadcast hash(g_i(j)) as c_i(j)
 6:
 7:
 8:
           Aggregate gradients (same as Alg. 2):
 9:
           Send g_i(j) to the j-th worker for all j \neq i and receive g_i(i) for all j \neq i from other workers
10:
           for j = 1, \ldots, n do
11:
                if hash(g_i(i)) \neq c_i(i) then
                     ELIMINATE(i, j) signed with peer<sub>i</sub> private key
12:
13:
           Compute \widehat{g}(i) = \text{CENTEREDCLIP}(g_1(i), g_2(i), \dots, g_n(i))
14:
           Broadcast hash(\widehat{g}(i)) as \widehat{c}(i)
           Send \widehat{g}(i) to each worker and receive \widehat{g}(j) for all j \neq i from other workers
15:
16:
           for j = 1, \ldots, n do
17:
                if hash(\widehat{g}(j)) \neq \widehat{c}(j) then
                     ELIMINATE(i, j) signed with peer, private key
18:
19:
           Compute \widehat{g} = \text{merge}(\widehat{g}(1), \dots, \widehat{g}(n))
20:
21:
           Send metadata for verification:
           Generate r via JPRNG
22:
23:
           for j \in 1, ..., n do
                \begin{split} & \Delta_i^j {=} (g_i(j) - \widehat{g}(j)) \cdot \min \left\{ 1, \frac{\tau}{\|g_i(j) - \widehat{g}(j)\|_2} \right\} \\ & z_j = \texttt{random\_vector}(r\|j) \end{split}
24:
25:
                Broadcast \langle z_j, \Delta_i^j \rangle as s_i^j
Broadcast \|g_i(j) - \widehat{g}(j)\|_2 as \operatorname{norm}_{ij}
for l = 1, \dots, n do
26:
27:
28:
                     Compute w_{lj} = \min\left\{1, \frac{\tau}{\mathtt{norm}_{lj}}\right\}
29:
30:
31:
           for j=1,\ldots,n do
32:
                Verification 1:
33:
                if norm_{ji} \neq ||g_j(i) - \widehat{g}(i)||_2 then
34:
                     ACCUSE(i, j, norm<sub>ji</sub> does not mach c_i(i))
                Verification 2:
35:
                // peer i knows \Delta_i^i from CenteredClip
36:
37:
                if s_i^i \neq \langle z^k[j], \Delta_i^i \rangle then
                     ACCUSE(i, j, s_i^j does not match c_i(i))
38:
                if \sum_{i=1}^{n} s_{i}^{j} \neq 0 then
39:
                     // peer _i already verified that all s^{j} are correct
40:
                     ACCUSE(i, j, \widehat{g}(j) is wrong)
41:
42:
                Verification 3:
43:
                Broadcast binary(\|g_i(j) - \widehat{g}(j)\|_2 > \Delta_{\max}) as check_{ij}
                if \sum_l \mathrm{check}_l j > \frac{n}{2} then
44:
                     CHECKAVERAGING(j)
45:
46:
           return \hat{q}
```

Algorithm 5 ACCUSE (i, j, allegation), detailed version

```
Input: i accuser, j target, n peers, all values exchanged in Algorithm 4.
 1: if Alleged wrong gradient then
           Recalculate g_i^k = \text{compute\_gradients}(x^k, \xi_i^k)
 3: else
 4:
           peer, must broadcast its g_i^k
 5:
           /* this branch is a time-saving optimization. If this optimization is not required,
           all peers should instead compute g_i^k from scratch as in the previous branch*/
 6:
 7: Split g_i into n parts: g_i = (g_i(1)^\top, \dots, g_i(n)^\top)^\top, g_i(j) \in \mathbb{R}^{d_j} for all i, j \in [n]
 8:
 9: for l = 1 \dots n do
           \begin{array}{c} \textbf{if} \; \text{hash}(g_j^k) \!\!\neq\!\! c_j^k \; \textbf{or} \; \text{hash}(g_j^k(l) \!\!\neq\!\! h_j^l \; \textbf{then} \\ \textbf{vote} \; \text{ban} \; \text{peer}_j \quad \  \  /^* \; \text{for gradient attack */} \end{array}
10:
11:
12:
           Compute \Delta_l^j = (g_l(j) - \widehat{g}(j)) \cdot \min \left\{ 1, \frac{\tau}{\|g_l(j) - \widehat{g}(j)\|_2} \right\}
13:
           if \|g_j(l) - \widehat{g}(l)\|_2 \neq \operatorname{norm}_{jl} or \langle \Delta_l^j, z_j \rangle \neq s_l^j or \sum_{l=1}^n s_l^j \neq 0 then vote ban \operatorname{peer}_j /* for aggregation attack */
14:
15:
16:
                 for o = 1, \ldots, n do
17:
                       if peer<sub>o</sub> approved norm<sub>jo</sub> or s_i^o then
                            vote ban peer o /* for covering up peer 's aggregation attack */
18:
19:
20: if did not vote against peer, then
21:
           /* caught reputation abuse by peer, */
            vote ban peer, /* for false allegation */
22:
23: for l = 1 \dots n do
24:
           if votes(peer<sub>1</sub>) \geq n/2 then
25:
                 BAN peer;
```

Algorithm 6 BTARD-SGD

```
Input: x^0 – starting point, \gamma – stepsize, K – number of iterations, \{s_{i,k}\}_{i,k=0,0}^{n,K-1} – seeds for batches
         computations
  1: C_0 = \text{Banned}_{-1} = \emptyset
  2: for k = 0, 1, \dots, K - 1 do
                 Worker i computes g_i^k = \begin{cases} \nabla f(x^k, \xi_{i,k}), & \text{if } i \in \mathcal{G}_k \setminus \mathcal{C}_k, \\ *, & \text{if } i \in \mathcal{B}_k \setminus \mathcal{C}_k, \end{cases}, where \xi_{i,k} is generated via seed
  3:
         s_{i,k} available to every worker
                 \widehat{g}^k, public_info_k = \operatorname{BTARD}(g^k_{i^k_1}, g^k_{i^k_1}, \dots, g^k_{i^k_{a_k}}), where \{i^k_1, \dots, i^k_{a_k}\} = (\mathcal{G}_k \cup \mathcal{B}_k) \setminus \mathcal{C}_k
  4:
        /* The above step is described in more detail in Algorithm 4 Choose 2m workers c_1^{k+1},\ldots,c_m^{k+1},u_1^{k+1},\ldots,u_m^{k+1} uniformly at random without replacement, \mathcal{C}_{k+1}=\{c_1^{k+1},\ldots,c_m^{k+1}\},\mathcal{U}_{k+1}=\{u_1^{k+1},\ldots,u_m^{k+1}\} Banned<sub>k</sub> = CHECKCOMPUTATIONS (\mathcal{C}_{k+1},\mathcal{U}_{k+1},\operatorname{public\_info}_k)
  5:
  7:
                 x^{k+1} = \operatorname{proj}_Q(x^k - \gamma \widehat{g}^k) := \operatorname{argmin}_{x \in Q} \|x - (x^k - \gamma \widehat{g}^k)\|
  8:
                 \mathcal{G}_{k+1} = \mathcal{G}_k \setminus \text{Banned}_{k-1}, \mathcal{B}_{k+1} = \mathcal{B}_k \setminus \text{Banned}_{k-1}
  9:
```

F Convergence analysis: missing proofs and extra details

F.1 Preliminaries

For convenience, we provide the classical definitions and facts on smooth and strongly convex functions below.

Definition F.1 (L-smoothness). We say that function $f:Q\to\mathbb{R}$, $Q\subseteq\mathbb{R}^d$ is L-smooth if it is differentiable and

$$\forall x, y \in Q \quad \|\nabla f(x) - \nabla f(y)\| \le L\|x - y\|. \tag{9}$$

One can show [99] that L-smoothness implies

$$\forall x, y \in Q \quad f(y) \le f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} ||y - x||^2, \tag{10}$$

$$\forall x \in Q \quad \|\nabla f(x)\|^2 \le 2L (f(x) - f_*),$$
 (11)

where f_* is a uniform lower bound for f.

Definition F.2 (μ -strong convexity). Differentiable function $f:Q\to\mathbb{R}$, $Q\subseteq\mathbb{R}^d$ is called μ -strongly convex if

$$\forall x, y \in Q \quad f(y) \ge f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} ||y - x||^2. \tag{12}$$

F.2 Convergence guarantees for BTARD-SGD

F.2.1 On Assumption 3.2

As we show in Lemmas F.2 and F.4, under As. 3.2 **Verification 3** at BTARD leads to extra checking of computations with probability $\sim 1/n$ at each iteration when all workers honestly follow the protocol and under a proper choice of $\Delta_{\rm max}$. Therefore, extra computations either appear due to malicious manipulations of byzantine peers, and lead eventually to the ban for the byzantine peers who deviate from the protocol, or, when all workers honestly follow the protocol, only once per n iterations on average. There are a number of important machine learning tasks, e.g., training ResNet-50 on Imagenet [81] and many others image classification problems, where the noise in the stochastic gradient has much "lighter" (sub-Gaussian) tails. That is, As. 3.2 is reasonable for a large class of practically important problems.

F.2.2 Quality of the aggregation

The quality of the aggregation at each iteration of BTARD-SGD significantly affects the rate of the method. That is, properties of \tilde{g}^k are highly important for the convergence of BTARD-SGD. This aggregator is obtained via BTARD that requires to know a tight estimate of the total number of Byzantine workers violating the protocol at iteration k – clipping parameter τ depends on this quantity. Therefore, it is natural to start with relatively simple setup when the number of Byzantine workers violating the protocol is known at each iteration.

Before we formulate the first result we introduce some useful notations. Let n_k be the total number of peers at iteration k, b_k be the total number of Byzantine peers at iteration k, \widehat{b}^k be the total number of Byzantine peers violating the protocol at iteration k, and $\delta_k = \frac{b_k}{n_k}$, $\widehat{\delta}_k = \frac{\widehat{b}_k}{n_k - m}$. In view of new notation, we start with the ideal situation when \widehat{b}_k is known for each worker at each iteration k. First of all, it is needed to to estimate the quality of the aggregation for good workers⁷.

Lemma F.1 (Theorem III from [58]). Let (Option I) from As. 3.1 hold, $\delta \leq 0.15(n-m)$, and $i \in \mathcal{G}_k \setminus \mathcal{C}_k$. Assume that \widehat{b}_k is known for each worker at iteration k and $\delta = \widehat{\delta}_k$ is used to compute clipping parameter τ_l for CenteredClip. If the total number of iterations T of CenteredClip satisfies $T \geq \log_{0.94} \frac{2\delta\sigma^2}{\mathbb{E}[\|v^0 - \overline{g}^k\|^2]}$, then

$$\mathbb{E}\left[\|\widehat{g}^k(i) - \overline{g}^k(i)\|^2 \mid x^k\right] \le 192\widehat{\delta}_k \frac{\sigma^2}{n_k - m},\tag{13}$$

⁷On the moment of writing this paper, the proof of Theorem III from [58] contained a minor inaccuracy. The simplest way of resolving this inaccuracy is based on the assumption of boundedness of the 4-th central moment of the stochastic gradient.

where
$$\overline{g}^k(i) = \frac{1}{|\mathcal{G}_k \setminus \mathcal{C}_k|} \sum_{j \in \mathcal{G}_k \setminus \mathcal{C}_k} g_j^k(i)$$
.

Proof. The proof follows exactly the same steps as the proof of Theorem III from [58] with few technical modifications. \Box

Unlike the good peers, Byzantine workers can cooperate and shift the result of CENTEREDCLIP in the components they aggregate without being revealed at **Verification 2** of BTARD. However, they cannot produce an arbitrary large shifts due to **Verification 3**. The next lemma estimates the maximal possible magnitude of a shift together with probability of triggering CHECKAVERAGING at iteration k for at least one worker.

Lemma F.2. Let As. 3.2, (Option I) from As. 3.1 hold, $b \le 0.15(n-m)$, and $i \in \mathcal{B}_k \setminus \mathcal{C}_k$. Assume that \widehat{b}_k is known for each worker at iteration k, $\Delta_{\max}^k = \frac{(1+\sqrt{3})\sqrt{2}\sigma}{\sqrt{n_k-m}}$ and $\delta = \widehat{\delta}_k$ is used to compute clipping parameter τ_l for CenteredClip. If the total number of iterations T of CenteredClip satisfies $T \ge \log_{0.94} \frac{2\delta\sigma^2}{\mathbb{E}[\|v^0 - \overline{q}^k\|^2]}$ and CHECKAVERAGING(i) is not triggered, then

$$\mathbb{E}\left[\|\widehat{g}^{k}(i) - \overline{g}^{k}(i)\|^{2} \mid x^{k}\right] \le \frac{4\left((1+\sqrt{3})^{2}+4\right)\sigma^{2}}{n_{k}-m},\tag{14}$$

where $\overline{g}^k(i) = \frac{1}{|\mathcal{G}_k \setminus \mathcal{C}_k|} \sum_{j \in \mathcal{G}_k \setminus \mathcal{C}_k} g_j^k(i)$. Moreover, if $\widehat{b}_k = 0$ and $n_k - m \ge 170$, then $\widehat{g}^k(i) = \overline{g}^k(i)$ and

$$\mathbb{P}\left\{\mathsf{CHECKAVERAGING} \ is \ triggered \ for \ \geq 1 \ peer \mid x^k\right\} \leq \frac{149}{49(n_k-m)}. \tag{15}$$

Proof. If CHECKAVERAGING(i) is not triggered at iteration k, then for $r_k \geq \frac{n_k - m}{2}$ good workers $i_1, i_2, \ldots, i_{r_k} \in \mathcal{G}_k \setminus \mathcal{C}_k$ we have $\|g_{i_j}^k(i) - \widehat{g}^k(i)\| \leq \Delta_{\max}^k$. Therefore, due to the independence of g_i^k , $i \in \mathcal{G}_k \setminus \mathcal{C}_k$ for fixed x^k we have

$$\begin{split} \mathbb{E}\left[\left\|\widehat{g}^{k}(i) - \overline{g}^{k}(i)\right\|^{2} \mid x^{k}\right] & \leq & 2\mathbb{E}\left[\left\|\widehat{g}^{k}(i) - \frac{1}{r_{k}}\sum_{j=1}^{r_{k}}g_{i_{j}}^{k}(i)\right\|^{2} \mid x^{k}\right] \\ & + 2\mathbb{E}\left[\left\|\frac{1}{r_{k}}\sum_{j=1}^{r_{k}}g_{i_{j}}^{k}(i) - \overline{g}^{k}(i)\right\|^{2} \mid x^{k}\right] \\ & \leq & 2\mathbb{E}\left[\frac{1}{r_{k}}\sum_{j=1}^{r_{k}}\|\widehat{g}^{k}(i) - g_{i_{j}}^{k}(i)\|^{2}\right] + 4\mathbb{E}\left[\left\|\nabla_{(i)}f(x^{k}) - \overline{g}^{k}(i)\right\|^{2} \mid x^{k}\right] \\ & 4\mathbb{E}\left[\left\|\frac{1}{r_{k}}\sum_{j=1}^{r_{k}}g_{i_{j}}^{k}(i) - \nabla_{(i)}f(x^{k})\right\|^{2} \mid x^{k}\right] \\ & \leq & 2(\Delta_{\max}^{k})^{2} + \frac{4\sigma^{2}}{|\mathcal{G}_{k}\setminus\mathcal{C}_{k}|} + \frac{8\sigma^{2}}{n_{k} - m} \\ & \leq & \frac{4\left((1 + \sqrt{3})^{2} + 4\right)\sigma^{2}}{n_{k} - m}, \end{split}$$

where we use $|\mathcal{G}_k \setminus \mathcal{C}_k| \ge r_k \ge \frac{n_k - m}{2}$ and $\nabla f_{(i)}(x^k) = \mathbb{E}[g_{i_j}^k \mid x^k]$. Finally, let us estimate the probability of triggering CHECKAVERAGING when all workers follow the protocol. In this case, $\widehat{g}^i(i) = \overline{g}^k(i)$. Next, due to As. 3.2 and $b \le 0.15(n-m)$ we have

$$\mathbb{P}\left\{ \|\overline{g}^{k}(i) - \nabla f_{(i)}(x^{k})\|_{2} > \sqrt{\frac{\sigma^{2}}{n_{k} - m}} \mid x^{k} \right\} \leq \frac{1}{|\mathcal{G}_{k} \setminus \mathcal{C}_{k}|^{2}} \leq \frac{100}{49(n_{k} - m)^{2}}$$

and for all $j \in \mathcal{G}_k \setminus \mathcal{C}_k$

$$\mathbb{P}\left\{\|g_j^k(i) - \nabla f_{(i)}(x^k)\|_2 > \sqrt{\frac{3\sigma^2}{n_k - m}} \mid x^k\right\} \le \frac{1}{9}.$$

Consider the independent random variables η_j , $j \in \mathcal{G}_k \setminus \mathcal{C}_k$, where

$$\eta_j = \begin{cases} 1, & \text{if } \|g_j^k(i) - \nabla f_{(i)}(x^k)\|_2 \le \sqrt{\frac{3\sigma^2}{n_k - m}}, \\ 0, & \text{otherwise,} \end{cases}$$

where x^k is fixed. Then, η_j is a Bernoulli random variable with parameter of "success" $q \ge 8/9$. Applying Hoeffding's inequality we get that

$$\mathbb{P}\left\{\sum_{j\in\mathcal{G}_k\setminus\mathcal{C}_k}\eta_j \leq \frac{n_k-m}{2} \mid x^k\right\} \leq \exp\left(-2(n_k-m)\left(q-\frac{n_k-m}{2|\mathcal{G}_k\setminus\mathcal{C}_k|}\right)^2\right) \\
\leq \exp\left(-2(n_k-m)\left(\frac{8}{9}-\frac{n-m}{1.4(n-m)}\right)^2\right) \\
= \exp\left(-\frac{242(n_k-m)}{3969}\right).$$

Since for all $j \in \mathcal{G}_k \setminus \mathcal{C}_k$ we have $\|\overline{g}^k(i) - g_j^k(i)\|_2 \le \|\overline{g}^k(i) - \nabla_{(i)} f(x^k)\|_2 + \|\nabla_{(i)} f(x^k) - g_j^k(i)\|_2$ the obtained bounds imply that CHECKAVERAGING is triggered for at least one worker at iteration k with probability not greater than

$$\frac{100}{49(n_k - m)} + (n_k - m) \exp\left(-\frac{242(n_k - m)}{3969}\right) \le \frac{149}{49(n_k - m)},$$
 where we use that $\exp\left(-\frac{242x}{3969}\right) \le \frac{1}{\pi^2}$ for all $x \ge 170$.

We notice that Byzantine peers can trigger CHECKAVERAGING by violating the protocol. However, each byzantine is checked at iteration k with probability $p \sim m/n$ (see Thm. F.1). Therefore, Byzantine workers can trigger only $\mathcal{O}\left(bn/m\right)$ extra rounds of communications and computations on average via triggering CHECKAVERAGING. In contrast, when there are no Byzantine workers or all workers follow the protocol CHECKAVERAGING is triggered only once per $\mathcal{O}(n-m)$ iterations that is a negligible communication an computation overhead when n is large.

Combining two previous lemmas we get the following result.

Lemma F.3. Let (Option I) from As. 3.1 hold and $b \leq 0.15(n-m)$. Assume that \widehat{b}_k is known for each worker at iteration k, $\Delta_{\max}^k = \frac{(1+\sqrt{3})\sqrt{2}\sigma}{\sqrt{n_k-m}}$ and $\delta = \widehat{\delta}_k$ is used to compute clipping parameter τ_l for CenteredClip. If the total number of iterations T of CenteredClip satisfies $T \geq \log_{0.94} \frac{2\delta\sigma^2}{\mathbb{E}[\|v^0-\overline{g}^k\|^2]}$ and CHECKAVERAGING is not triggered for any worker, then

$$\mathbb{E}\left[\|\widehat{g}^k - \overline{g}^k\|^2 \mid x^k\right] \le C\widehat{\delta}_k \sigma^2,\tag{16}$$

$$\mathbb{E}\left[\|\widehat{g}^k\|^2 \mid x^k\right] \le 2C\widehat{\delta}_k \sigma^2 + 2\|\nabla f(x^k)\|^2 + \frac{2\sigma^2}{n - 2b - m},$$
where $\overline{g}^k = \frac{1}{|\mathcal{G}_k \setminus \mathcal{C}_k|} \sum_{j \in \mathcal{G}_k \setminus \mathcal{C}_k} g_j^k$ and $C = 192 + 4\left((1 + \sqrt{3})^2 + 4\right).$ (17)

Proof. We have

$$\mathbb{E}\left[\|\widehat{g}^{k} - \overline{g}^{k}\|^{2} \mid x^{k}\right] = \sum_{i \in \mathcal{G}_{k} \setminus \mathcal{C}_{k}} \mathbb{E}\left[\|\widehat{g}^{k}(i) - \overline{g}^{k}(i)\|^{2} \mid x^{k}\right] + \sum_{i \in \mathcal{B}_{k} \setminus \mathcal{C}_{k}} \mathbb{E}\left[\|\widehat{g}^{k}(i) - \overline{g}^{k}(i)\|^{2} \mid x^{k}\right]$$

$$\stackrel{(13),(14)}{\leq} (1 - \widehat{\delta}_{k})(n_{k} - m) \cdot 192\widehat{\delta}_{k} \frac{\sigma^{2}}{n_{k} - m} + \widehat{\delta}_{k}(n_{k} - m) \cdot \frac{4\left((1 + \sqrt{3})^{2} + 4\right)\sigma^{2}}{n_{k} - m}$$

$$= C\widehat{\delta}_{k} \sigma^{2}$$

Next, using the independence of g_j^k for $j \in \mathcal{G}_k \setminus \mathcal{C}_k$ and fixed x^k we derive

$$\mathbb{E} \left[\| \widehat{g}^{k} \|^{2} \mid x^{k} \right] \leq 2\mathbb{E} \left[\| \widehat{g}^{k} - \overline{g}^{k} \|^{2} \mid x^{k} \right] + 2\mathbb{E} \left[\| \overline{g}^{k} \|^{2} \mid x^{k} \right]$$

$$\leq 2C \widehat{\delta}_{k} \sigma^{2} + 2\| \nabla f(x^{k}) \|^{2} + 2\mathbb{E} \left[\| \overline{g}^{k} - \nabla f(x^{k}) \|^{2} \mid x^{k} \right]$$

$$\leq 2C \widehat{\delta}_{k} \sigma^{2} + 2\| \nabla f(x^{k}) \|^{2} + \frac{2\sigma^{2}}{|\mathcal{G}_{k} \setminus \mathcal{C}_{k}|}$$

$$\leq 2C \widehat{\delta}_{k} \sigma^{2} + 2\| \nabla f(x^{k}) \|^{2} + \frac{2\sigma^{2}}{n - 2b - m}.$$

In view of the definition of (δ, c) -robust aggregator from [58], the result of BTARD at iteration k is $(\hat{b_k}, C)$ -robust. However, we derive this property under assumption that $\hat{b_k}$ is known to all workers at each iteration k, which is impractical.

When \hat{b}_k is unknown the situation changes dramatically: in general, good peers can only know some upper bound for the fraction of Byzantine peers at iteration k. Unfortunately, if used without bans this is not enough to converge to any accuracy of the solution since BTARD-SGD is permutation invariant algorithm (see [58]). Therefore, in this case, we always use CENTEREDCLIP with $\tau_l = \infty$ for all $l \ge 0$, i.e., good peers compute an exact average. In this settings, even 1 byzantine worker can significantly shift the average in all parts of the vector. The next lemma quantifies the negative effect of Byzantine workers in this case.

Lemma F.4. Let As. 3.2, (Option II) from As. 3.1 hold and $b \le 0.15(n-m)$, $m \le (n-2b)/2$. Assume that $\Delta_{\max}^k = \frac{(1+\sqrt{3})\sqrt{2}\sigma}{\sqrt{n_k-m}}$ and $\delta=0$ is used to compute clipping parameter τ_l for CenteredClip. If CHECKAVERAGING is not triggered for any worker, then

$$\mathbb{E}\left[\|\widehat{g}^k - \overline{g}^k\|^2 \mid x^k\right] \le C\sigma^2 \mathbb{1}_{k,v},\tag{18}$$

$$\mathbb{E}\left[\|\widehat{g}^k\|^2 \mid x^k\right] \le 2C\sigma^2 \mathbb{1}_{k,v} + 2\|\nabla f(x^k)\|^2 + \frac{2\sigma^2}{n - 2b - m},\tag{19}$$

 $\mathbb{E}\left[\|\widehat{g}^k\|^2 \mid x^k\right] \leq 2C\sigma^2 \mathbb{1}_{k,v} + 2\|\nabla f(x^k)\|^2 + \frac{2\sigma^2}{n - 2b - m},\tag{19}$ where $\overline{g}^k = \frac{1}{|\mathcal{G}_k \setminus \mathcal{C}_k|} \sum_{j \in \mathcal{G}_k \setminus \mathcal{C}_k} g_j^k$, $C = 4\left((1 + \sqrt{3})^2 + 4\right)$, and $\mathbb{1}_{k,v}$ is an indicator function of the

event that at least 1 Byzantine peer violates the protocol at iteration k. Moreover, if $\hat{b}_k = 0$ and $n_k - m \ge 170$, then $\hat{g}^k(i) = \overline{g}^k(i)$ and

$$\mathbb{P}\left\{\mathsf{CHECKAVERAGING} \ \textit{is triggered for } \geq 1 \ \textit{peer} \mid x^k\right\} \leq \frac{149}{49(n_k - m)}. \tag{20}$$

Proof. If CHECKAVERAGING is not triggered for any worker, then $\|\widehat{g}^k(i) - \overline{g}^k(i)\|^2 \le (\Delta_{\max}^k)^2 \mathbb{1}_{k,v}$ for all $i \in (\mathcal{G}_k \cup \mathcal{C}_k) \setminus \mathcal{C}_k$ implying

$$\begin{split} \mathbb{E}\left[\|\widehat{g}^k - \overline{g}^k\|^2 \mid x^k\right] &= \sum_{i \in (\mathcal{G}_k \cup \mathcal{C}_k) \backslash \mathcal{C}_k} \mathbb{E}\left[\|\widehat{g}^k(i) - \overline{g}^k(i)\|^2 \mid x^k\right] \\ &\leq \left(n_k - m\right) \cdot \frac{2\left((1 + \sqrt{3})^2 + 8\right)\sigma^2}{n_k - m} \mathbbm{1}_{k,v} \leq C\sigma^2 \mathbbm{1}_{k,v}. \end{split}$$

Next, using the independence of g_j^k for $j \in \mathcal{G}_k \setminus \mathcal{C}_k$ and fixed x^k we derive

$$\mathbb{E}\left[\|\widehat{g}^{k}\|^{2} \mid x^{k}\right] \leq 2\mathbb{E}\left[\|\widehat{g}^{k} - \overline{g}^{k}\|^{2} \mid x^{k}\right] + 2\mathbb{E}\left[\|\overline{g}^{k}\|^{2} \mid x^{k}\right] \\
\leq 2C\sigma^{2}\mathbb{1}_{k,v} + 2\|\nabla f(x^{k})\|^{2} + 2\mathbb{E}\left[\|\overline{g}^{k} - \nabla f(x^{k})\|^{2} \mid x^{k}\right] \\
\leq 2C\sigma^{2}\mathbb{1}_{k,v} + 2\|\nabla f(x^{k})\|^{2} + \frac{2\sigma^{2}}{|\mathcal{G}_{k} \setminus \mathcal{C}_{k}|} \\
\leq 2C\sigma^{2}\mathbb{1}_{k,v} + 2\|\nabla f(x^{k})\|^{2} + \frac{2\sigma^{2}}{n - 2b - m}.$$

The proof of the final part of the lemma is identical to the proof of the same result from Lemma F.2.

28

F.2.3 Non-convex case

In this section, we provide the complete statements and the full proofs of the convergence results for BTARD-SGD when the objective function f is smooth, but can be non-convex. We start with the case when the number of attacking Byzantine workers is known at each iteration.

Theorem F.1. Let As. 3.2, (Option I) from As. 3.1 hold, $Q = \mathbb{R}^d$, and f be L-smooth (see Def. F.1) and uniformly lower bounded by f_* . Moreover, assume that $b \le 0.15(n-m)$, $m \le \frac{(n-2b)}{2}$, and the exact number of attacking byzantine peers is known to all good peers at each iteration. Next, assume that

$$\gamma = \min \left\{ \frac{1}{4L}, \sqrt{\frac{\Delta_0 n}{L\sigma^2 K}} \right\}, \quad \Delta_{\max}^k = \frac{(1+\sqrt{3})\sqrt{2}\sigma}{\sqrt{n_k - m}}, \tag{21}$$

where $\Delta_0 = f(x^0) - f_*$ and Δ_{\max}^k is the parameter for verification 3 at iteration k of BTARD-SGD. Then, we have $\mathbb{E}[\|\nabla f(\overline{x}^K)\|^2] \leq \varepsilon$ after K iterations of BTARD-SGD, where

$$K = \mathcal{O}\left(\frac{L\Delta_0}{\varepsilon^2} + \frac{L\Delta_0\sigma^2}{n\varepsilon^4} + \frac{n\delta\sigma^2}{m\varepsilon^2}\right)$$
 (22)

and \overline{x}^K is picked uniformly at random from $\{x^0, x^1, \dots, x^{K-1}\}$.

Proof. From L-smoothness of f we have

$$f(x^{k+1}) \stackrel{(10)}{\leq} f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2} \|x^{k+1} - x^k\|^2$$

$$= f(x^k) - \gamma \langle \nabla f(x^k), \hat{g}^k \rangle + \frac{L\gamma^2}{2} \|\hat{g}^k\|^2.$$

Taking the conditional expectation $\mathbb{E}[\cdot \mid x^k]$ from the both sides of the previous inequality we obtain

$$\mathbb{E}\left[f(x^{k+1}) \mid x^{k}\right] \leq f(x^{k}) - \gamma \|\nabla f(x^{k})\|^{2} - \gamma \left\langle \nabla f(x^{k}), \mathbb{E}\left[\hat{g}^{k} - \overline{g}^{k} \mid x^{k}\right]\right\rangle \\ + \frac{L\gamma^{2}}{2} \mathbb{E}\left[\|\hat{g}^{k}\|^{2} \mid x^{k}\right] \\ \leq f(x^{k}) - \frac{\gamma}{2} \|\nabla f(x^{k})\|^{2} + \frac{\gamma}{2} \left\|\mathbb{E}\left[\hat{g}^{k} - \overline{g}^{k} \mid x^{k}\right]\right\|^{2} \\ + CL\gamma^{2} \hat{\delta}_{k} \sigma^{2} + L\gamma^{2} \|\nabla f(x^{k})\| + \frac{L\gamma^{2} \sigma^{2}}{n - 2b - m} \\ \leq f(x^{k}) - \frac{\gamma}{2} \left(1 - 2L\gamma\right) \|\nabla f(x^{k})\|^{2} + \frac{\gamma}{2} \mathbb{E}\left[\|\hat{g}^{k} - \overline{g}^{k}\|^{2} \mid x^{k}\right] \\ + CL\gamma^{2} \hat{\delta}_{k} \sigma^{2} + \frac{L\gamma^{2} \sigma^{2}}{n - 2b - m}.$$

Since $\gamma \leq \frac{1}{4L}$ we continue our derivations as

$$\mathbb{E}\left[f(x^{k+1}) \mid x^{k}\right] \leq f(x^{k}) - \frac{\gamma}{4} \|\nabla f(x^{k})\|^{2} + \gamma C \sigma^{2} (1 + L\gamma) \widehat{\delta}_{k} + \frac{L\gamma^{2} \sigma^{2}}{n - 2b - m}$$

$$\leq f(x^{k}) - \frac{\gamma}{4} \|\nabla f(x^{k})\|^{2} + 2\gamma C \sigma^{2} \widehat{\delta}_{k} + \frac{L\gamma^{2} \sigma^{2}}{n - 2b - m}.$$

Taking the full expectation from the both sides of the obtained inequality and summing up the results for $k = 0, 1, \dots, K - 1$ we get

$$\begin{split} \frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E} \left[\| \nabla f(x^k) \|^2 \right] & \leq & \frac{4}{\gamma K} \sum_{k=0}^{K-1} \mathbb{E} \left[f(x^k) - f(x^{k+1}) \right] + \frac{8C\sigma^2}{K} \mathbb{E} \left[\sum_{k=0}^{K-1} \widehat{\delta}_k \right] + \frac{4L\gamma\sigma^2}{n - 2b - m} \\ & = & \frac{4 \left(f(x^0) - \mathbb{E}[f(x^K)] \right)}{\gamma K} + \frac{8C\sigma^2}{K} \mathbb{E} \left[\sum_{k=0}^{K-1} \frac{\widehat{b}_k}{n_k - m} \right] + \frac{4L\gamma\sigma^2}{n - 2b - m} \\ & \leq & \frac{4(f(x^0) - f_*)}{\gamma K} + \frac{8C\sigma^2}{K(n - 2b - m)} \mathbb{E} \left[\sum_{k=0}^{K-1} \widehat{b}_k \right] + \frac{4L\gamma\sigma^2}{n - 2b - m}. \end{split}$$

If a Byzantine peer deviates from the protocol at iteration k, it will be detected with some probability p_k during the next iteration. One can lower bound this probability as

$$p_k \ge m \cdot \frac{|\mathcal{G}_k|}{n_k} \cdot \frac{1}{n_k} = \frac{m(1 - \delta_k)}{n_k} \ge \frac{m}{n}.$$

Therefore, each individual Byzantine worker can violate the protocol no more than 1/p times on average implying that

$$\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E} \left[\|\nabla f(x^k)\|^2 \right] \leq \frac{4(f(x^0) - f_*)}{\gamma K} + \frac{8Cnb\sigma^2}{Km(n - 2b - m)} + \frac{4L\gamma\sigma^2}{n - 2b - m} \\
\leq \frac{4(f(x^0) - f_*)}{\gamma K} + \frac{16Cnb\sigma^2}{Km(n - 2b)} + \frac{8L\gamma\sigma^2}{n - 2b} \\
\leq \frac{4(f(x^0) - f_*)}{\gamma K} + \frac{160Cn\delta\sigma^2}{7Km} + \frac{80L\gamma\sigma^2}{7n}.$$

Since \overline{x}^K is picked uniformly at random from $\{x^0, x^1, \dots, x^{K-1}\}$ we have

$$\mathbb{E}\left[\|\nabla f(\overline{x}^K)\|^2\right] \leq \frac{4(f(x^0) - f_*)}{\gamma K} + \frac{160Cn\delta\sigma^2}{7Km} + \frac{80L\gamma\sigma^2}{7n}.$$

Using the stepsize rule

$$\gamma = \min \left\{ \frac{1}{4L}, \sqrt{\frac{\Delta_0 n}{L\sigma^2 K}} \right\}$$

we derive

$$\mathbb{E}\left[\|\nabla f(\overline{x}^K)\|^2\right] = \mathcal{O}\left(\frac{L\Delta_0}{K} + \frac{\sqrt{L\Delta_0}\sigma}{\sqrt{nK}} + \frac{n\delta\sigma^2}{mK}\right)$$

meaning that after

$$K = \mathcal{O}\left(\frac{L\Delta_0}{\varepsilon^2} + \frac{L\Delta_0\sigma^2}{n\varepsilon^4} + \frac{n\delta\sigma^2}{m\varepsilon^2}\right)$$

iterations BTARD-SGD guarantees $\mathbb{E}\left[\|\nabla f(\overline{x}^K)\|^2\right] \leq \varepsilon^2$

In the main part of the paper, we notice that the rate of BTARD-SGD in the presence of bad workers is asymptotically the same as for SGD without Byzantine peers when ε is sufficiently small⁸. This phenomenon has a clear intuition. When the target accuracy ε is small, the stepsize γ is also needed to be small enough. However, as we show in Lemmas F.3 and F.4, Byzantine workers can produce only a bounded shift independent of the stepsize. Moreover, they can violate the protocol at only $\sim n/m$ iterations on average. Therefore, the overall impact of Byzantine workers on the convergence of BTARD-SGD decreases when the stepsize γ decreases.

Next, we derive the result without assuming that \hat{b}^k is known to all peers at each iteration.

Theorem F.2. Let As. 3.2, (Option II) from As. 3.1 hold, $Q = \mathbb{R}^d$, and f be L-smooth (see Def. F.1) and uniformly lower bounded by f_* . Moreover, assume that $b \leq 0.15(n-m)$, $m \leq \frac{(n-2b)}{2}$, and $\delta = 0$ is used to compute clipping parameter τ_l for CenteredClip. Next, assume that

$$\gamma = \min \left\{ \frac{1}{4L}, \sqrt{\frac{\Delta_0 n}{L\sigma^2 K}} \right\}, \quad \Delta_{\max}^k = \frac{(1 + \sqrt{3})\sqrt{2}\sigma}{\sqrt{n_k - m}}, \tag{23}$$

where $\Delta_0 = f(x^0) - f_*$ and Δ_{\max}^k is the parameter for verification 3 at iteration k of BTARD-SGD. Then, we have $\mathbb{E}[\|\nabla f(\overline{x}^K)\|^2] \leq \varepsilon$ after K iterations of BTARD-SGD, where

$$K = \mathcal{O}\left(\frac{L\Delta_0}{\varepsilon^2} + \frac{L\Delta_0\sigma^2}{n\varepsilon^4} + \frac{nb\sigma^2}{m\varepsilon^2}\right)$$
 (24)

and \overline{x}^K is picked uniformly at random from $\{x^0, x^1, \dots, x^{K-1}\}$.

⁸This is true for convex and strongly convex cases as well.

Proof. The proof is almost identical to the proof of Theorem F.1. Following the same steps and using (18) and (19) instead of (16) and (17) respectively we obtain the same sequence of inequalities up to the following change: instead of $\hat{\delta}_k$ we should use $\mathbb{1}_{k,v}$. Therefore, we have

$$\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E} \left[\| \nabla f(x^k) \|^2 \right] \leq \frac{4(f(x^0) - f_*)}{\gamma K} + \frac{8C\sigma^2}{K} \mathbb{E} \left[\sum_{k=0}^{K-1} \mathbb{1}_{k,v} \right] + \frac{4L\gamma \sigma^2}{n - 2b - m}.$$

If a Byzantine peer deviates from the protocol at iteration k, it will be detected with some probability p_k during the next iteration. One can lower bound this probability as

$$p_k \ge m \cdot \frac{|\mathcal{G}_k|}{n_k} \cdot \frac{1}{n_k} = \frac{m(1 - \delta_k)}{n_k} \ge \frac{m}{n}.$$

That is, each individual Byzantine worker can violate the protocol no more than $^1/p$ times on average. However, even one Byzantine peer can create a shift of the order Δ^k_{\max} at each part of the resulting vector. Therefore, all Byzantine peers can violate the protocol no more than $^b/p$ times on average implying that

$$\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E} \left[\|\nabla f(x^k)\|^2 \right] \leq \frac{4(f(x^0) - f_*)}{\gamma K} + \frac{8Cnb\sigma^2}{Km} + \frac{4L\gamma\sigma^2}{n - 2b - m} \\
\leq \frac{4(f(x^0) - f_*)}{\gamma K} + \frac{8Cnb\sigma^2}{Km} + \frac{8L\gamma\sigma^2}{n - 2b} \\
\leq \frac{4(f(x^0) - f_*)}{\gamma K} + \frac{8Cnb\sigma^2}{Km} + \frac{80L\gamma\sigma^2}{7n}.$$

Since \overline{x}^K is picked uniformly at random from $\{x^0, x^1, \dots, x^{K-1}\}$ we have

$$\mathbb{E}\left[\|\nabla f(\overline{x}^K)\|^2\right] \leq \frac{4(f(x^0) - f_*)}{\gamma K} + \frac{8Cnb\sigma^2}{Km} + \frac{80L\gamma\sigma^2}{7n}.$$

Using the stepsize rule

$$\gamma = \min \left\{ \frac{1}{4L}, \sqrt{\frac{\Delta_0 n}{L\sigma^2 K}} \right\}$$

we derive

$$\mathbb{E}\left[\|\nabla f(\overline{x}^K)\|^2\right] = \mathcal{O}\left(\frac{L\Delta_0}{K} + \frac{\sqrt{L\Delta_0}\sigma}{\sqrt{nK}} + \frac{nb\sigma^2}{mK}\right)$$

meaning that after

$$K = \mathcal{O}\left(\frac{L\Delta_0}{\varepsilon^2} + \frac{L\Delta_0\sigma^2}{n\varepsilon^4} + \frac{nb\sigma^2}{m\varepsilon^2}\right)$$

iterations BTARD-SGD guarantees $\mathbb{E}\left[\|\nabla f(\overline{x}^K)\|^2\right] \leq \varepsilon^2$.

As we notice in the main part of the paper, the third term of the obtained complexity result is significantly worse than in (22): it is proportional to b instead of $\delta = b/n$. However, (24) is derived using (Option II) that is less restrictive than (Option I), and to derive is derived using (Option II) we do not need to assume that \hat{b}_k is known for all workers at each iteration. Moreover, as in (22), the third term in (24) has better dependence on ε than the second term implying that for small enough ε the rate of BTARD-SGD in the presence of bad workers without assuming that \hat{b}_k is known at each iteration is asymptotically the same as for SGD without Byzantine peers⁹.

F.2.4 Convex case

In this section, we provide the complete statements and the full proofs of the convergence results for BTARD-SGD when the objective function f is smooth and convex. We start with the case when the number of attacking Byzantine workers is known at each iteration.

⁹This is true for convex and strongly convex cases as well.

Theorem F.3. Let As. 3.2, (Option I) from As. 3.1 hold, $Q = \mathbb{R}^d$, f be L-smooth (see Def. F.1), convex, and x^* be some optimum of f. Moreover, assume that $b \le 0.15(n-m)$, $m \le \frac{(n-2b)}{2}$, and the exact number of attacking Byzantine peers is known to all good peers at each iteration. Next, assume that

$$\gamma = \min \left\{ \frac{1}{4L}, \sqrt{\frac{7nR_0^2}{120\sigma^2 K}}, \sqrt{\frac{m^2 R_0^2}{1440C\sigma^2 n^2 \delta}} \right\}, \quad \Delta_{\max}^k = \frac{(1+\sqrt{3})\sqrt{2}\sigma}{\sqrt{n_k - m}}, \tag{25}$$

where $R_0 \ge \|x^0 - x^*\|$ and Δ_{\max}^k is the parameter for verification 3 at iteration k of BTARD-SGD. Then, we have $\mathbb{E}[f(\overline{x}^K) - f(x^*)] \le \varepsilon$ after K iterations of BTARD-SGD, where

$$K = \mathcal{O}\left(\frac{LR_0^2}{\varepsilon} + \frac{\sigma^2 R_0^2}{n\varepsilon^2} + \frac{n\sqrt{\delta}\sigma R_0}{m\varepsilon}\right)$$
 (26)

and $\overline{x}^K = \frac{1}{K} \sum_{k=0}^{K-1}$.

Proof. Lemma F.3 implies

$$\begin{split} \mathbb{E} \left[\| x^{k+1} - x^* \|^2 \mid x^k \right] &= \mathbb{E} \left[\| x^k - x^* - \gamma \widehat{g}^k \|^2 \mid x^k \right] \\ &= \| x^k - x^* \|^2 - 2\gamma \mathbb{E} \left[\langle x^k - x^*, \widehat{g}^k \rangle \mid x^k \right] + \gamma^2 \mathbb{E} \left[\| \widehat{g}^k \|^2 \mid x^k \right] \\ &\stackrel{(17)}{\leq} \| x^k - x^* \|^2 - 2\gamma \langle x^k - x^*, \nabla f(x^k) \rangle + 2\gamma^2 \| \nabla f(x^k) \|^2 \\ &- 2\gamma \mathbb{E} \left[\langle x^k - x^*, \widehat{g}^k - \overline{g}^k \rangle \mid x^k \right] + 2\gamma^2 C \widehat{\delta}_k \sigma^2 + \frac{2\gamma^2 \sigma^2}{n - 2b - m}. \end{split}$$

Next, we use convexity (see (12) with $\mu = 0$) and L-smoothness of f:

$$\mathbb{E}\left[\|x^{k+1} - x^*\|^2 \mid x^k\right] \stackrel{\text{(11),(12)}}{\leq} \|x^k - x^*\|^2 - 2\gamma \left(1 - 2L\gamma\right) \left(f(x^k) - f(x^*)\right) \\ -2\gamma \mathbb{E}\left[\langle x^k - x^*, \widehat{g}^k - \overline{g}^k \rangle \mid x^k\right] + 2\gamma^2 C\sigma^2 \frac{\widehat{b}_k}{n_k - m} + \frac{2\gamma^2 \sigma^2}{n - 2b - m}$$

To estimate the inner product in the right-hand side we apply Cauchy-Schwarz inequality:

$$-2\gamma \mathbb{E}\left[\langle x^k - x^*, \widehat{g}^k - \overline{g}^k \rangle \mid x^k\right] \leq 2\gamma \|x^k - x^*\| \mathbb{E}\left[\|\widehat{g}^k - \overline{g}^k\| \mid x^k\right]$$

$$\leq 2\gamma \|x^k - x^*\| \sqrt{\mathbb{E}\left[\|\widehat{g}^k - \overline{g}^k\|^2 \mid x^k\right]}$$

$$\leq 2\gamma \sqrt{C}\sigma \|x^k - x^*\| \sqrt{\widehat{\delta}_k} \leq \frac{2\gamma \sqrt{C}\sigma}{\sqrt{n_k - m}} \|x^k - x^*\| \sqrt{\widehat{b}_k}$$

$$\leq \frac{2\gamma \sqrt{C}\sigma}{\sqrt{n - 2b - m}} \|x^k - x^*\| \sqrt{\widehat{b}_k}.$$

Putting all together and using $b \le 0.15(n-m)$, $m \le (n-2b)/2$, $\gamma \le 1/4L$, $n_k - m \ge n - 2b - m$, we obtain

$$\mathbb{E}\left[\|x^{k+1} - x^*\|^2 \mid x^k\right] \leq \|x^k - x^*\|^2 - \gamma \left(f(x^k) - f(x^*)\right) + \frac{4\gamma\sqrt{5C}\sigma}{\sqrt{n}} \|x^k - x^*\|\sqrt{\widehat{b}_k} + \frac{40\gamma^2C\sigma^2}{7n}\widehat{b}_k + \frac{40\gamma^2\sigma^2}{7n}.$$

Taking the full expectation from the both sides of the above inequality and summing up the results for k = 0, 1, ..., K - 1 we derive

$$\begin{split} \frac{\gamma}{K} \sum_{k=0}^{K-1} \mathbb{E}[f(x^k) - f(x^*)] & \leq & \frac{1}{K} \sum_{k=0}^{K-1} \left(\mathbb{E}\left[\|x^k - x^*\|^2 \right] - \mathbb{E}\left[\|x^{k+1} - x^*\|^2 \right] \right) + \frac{40\gamma^2 \sigma^2}{7n} \\ & + \frac{4\gamma\sqrt{5C}\sigma}{\sqrt{n}K} \sum_{k=0}^{K-1} \mathbb{E}\left[\|x^k - x^*\|\sqrt{\hat{b}_k} \right] + \frac{40\gamma^2 C\sigma^2}{7nK} \sum_{k=0}^{K-1} \mathbb{E}[\hat{b}_k] \\ & \leq & \frac{\|x^0 - x^*\|^2 - \mathbb{E}[\|x^K - x^*\|^2]}{K} + \frac{40\gamma^2 \sigma^2}{7n} \\ & + \frac{4\gamma\sqrt{5C}\sigma}{\sqrt{n}K} \sum_{k=0}^{K-1} \sqrt{\mathbb{E}\left[\|x^k - x^*\|^2 \right] \mathbb{E}[\hat{b}_k]} + \frac{40\gamma^2 C\sigma^2}{7nK} \sum_{k=0}^{K-1} \mathbb{E}[\hat{b}_k]. \end{split}$$

From Jensen's inequality we have $f(\overline{x}^K) \leq \frac{1}{K} \sum_{k=0}^{K-1} f(x^k)$, where $\overline{x}^K = \frac{1}{K} \sum_{k=0}^{K-1} x^k$. Using this and new notation $R_k = \|x^k - x^*\|$, k > 0, $R_0 \geq \|x^0 - x^*\|$ we get

$$0 \leq \gamma \mathbb{E}\left[f(\overline{x}^K) - f(x^*)\right] \leq \frac{R_0^2 - \mathbb{E}[R_K^2]}{K} + \frac{40\gamma^2 \sigma^2}{7n} + \frac{4\gamma\sqrt{5C}\sigma}{\sqrt{n}K} \sum_{k=0}^{K-1} \sqrt{\mathbb{E}\left[R_k^2\right]\mathbb{E}[\widehat{b}_k]} + \frac{40\gamma^2 C\sigma^2}{7nK} \sum_{k=0}^{K-1} \mathbb{E}[\widehat{b}_k] (27)$$

implying (after changing the indices) that

$$\mathbb{E}[R_k^2] \le R_0^2 + \frac{40\gamma^2 \sigma^2 k}{7n} + \frac{4\gamma\sqrt{5C}\sigma}{\sqrt{n}} \sum_{l=0}^{k-1} \sqrt{\mathbb{E}[R_l^2] \,\mathbb{E}[\hat{b}_l]} + \frac{40\gamma^2 C\sigma^2}{7n} \sum_{l=0}^{k-1} \mathbb{E}[\hat{b}_l]$$
 (28)

holds for all $k \ge 0$. In the remaining part of the proof we derive by induction that

$$R_0^2 + \frac{40\gamma^2 \sigma^2 k}{7n} + \frac{4\gamma\sqrt{5C}\sigma}{\sqrt{n}} \sum_{l=0}^{k-1} \sqrt{\mathbb{E}[R_l^2] \mathbb{E}[\hat{b}_l]} + \frac{40\gamma^2 C\sigma^2}{7n} \sum_{l=0}^{k-1} \mathbb{E}[\hat{b}_l] \le 2R_0^2$$
 (29)

for all $k=0,\ldots,K$. For k=0 this inequality trivially holds. Next, assume that it holds for all $k=0,1,\ldots,T-1,T\leq K-1$. Let us show that it holds for k=T as well. From (28) and (29) we have that $\mathbb{E}[R_k^2]\leq 2R_0^2$ for all $k=0,1,\ldots,T-1$. Therefore,

$$\mathbb{E}[R_T^2] \leq R_0^2 + \frac{40\gamma^2 \sigma^2 T}{7n} + \frac{4\gamma\sqrt{5C}\sigma}{\sqrt{n}} \sum_{l=0}^{T-1} \sqrt{\mathbb{E}[R_l^2] \mathbb{E}[\widehat{b}_l]} + \frac{40\gamma^2 C\sigma^2}{7n} \sum_{l=0}^{T-1} \mathbb{E}[\widehat{b}_l] \\
\leq R_0^2 + \frac{40\gamma^2 \sigma^2 T}{7n} + \frac{4\gamma\sqrt{10C}\sigma R_0}{\sqrt{n}} \sum_{l=0}^{T-1} \sqrt{\mathbb{E}[\widehat{b}_l]} + \frac{40\gamma^2 C\sigma^2}{7n} \sum_{l=0}^{T-1} \mathbb{E}[\widehat{b}_l].$$

If a Byzantine peer deviates from the protocol at iteration k, it will be detected with some probability p_k during the next iteration. One can lower bound this probability as

$$p_k \ge m \cdot \frac{|\mathcal{G}_k|}{n_k} \cdot \frac{1}{n_k} = \frac{m(1 - \delta_k)}{n_k} \ge \frac{m}{n}.$$

Therefore, each individual Byzantine worker can violate the protocol no more than 1/p times on average implying that

$$\mathbb{E}[R_T^2] \leq R_0^2 + \frac{40\gamma^2 \sigma^2 T}{7n} + \frac{4n\gamma\sqrt{10Cb}\sigma R_0}{m\sqrt{n}} + \frac{40\gamma^2 C\sigma^2 nb}{7nm}$$

$$= R_0^2 + \frac{40\gamma^2 \sigma^2 T}{7n} + \frac{4n\gamma\sqrt{10C\delta}\sigma R_0}{m} + \frac{40\gamma^2 C\sigma^2 n\delta}{7m}$$

Taking

$$\gamma = \min \left\{ \frac{1}{4L}, \sqrt{\frac{7nR_0^2}{120\sigma^2 K}}, \sqrt{\frac{m^2 R_0^2}{1440C\sigma^2 n^2 \delta}} \right\}$$

we ensure that

$$\frac{40\gamma^2\sigma^2T}{7n} + \frac{4n\gamma\sqrt{10C\delta}\sigma R_0}{m} + \frac{40\gamma^2C\sigma^2n\delta}{7m} \le \frac{R_0^2}{3} + \frac{R_0^2}{3} + \frac{R_0^2}{3} = R_0^2,$$

and, as a result, we get $\mathbb{E}[R_T^2] \leq 2R_0^2$. Therefore, (29) holds for all $k = 0, 1, \dots, K$. Together with (27) it implies

$$\mathbb{E}\left[f(\overline{x}^K) - f(x^*)\right] \le \frac{2R_0^2}{\gamma K}.$$

Next, from our stepsize rule (25) it follows that

$$\mathbb{E}\left[f(\overline{x}^K) - f(x^*)\right] = \mathcal{O}\left(\frac{LR_0^2}{K} + \frac{\sigma R_0}{\sqrt{nK}} + \frac{n\sqrt{\delta}\sigma R_0}{mK}\right)$$

meaning that after

$$K = \mathcal{O}\left(\frac{LR_0^2}{\varepsilon} + \frac{\sigma^2 R_0^2}{n\varepsilon^2} + \frac{n\sqrt{\delta}\sigma R_0}{m\varepsilon}\right)$$

iterations BTARD-SGD guarantees $\mathbb{E}[f(\overline{x}^K) - f(x^*)] \leq \varepsilon$.

In the convex case, similar observations hold as in the non-convex case. Next, we derive the result without assuming that \hat{b}^k is known to all peers at each iteration.

Theorem F.4. Let As. 3.2, (Option II) from As. 3.1 hold, $Q = \mathbb{R}^d$, f be L-smooth (see Def. F.1), convex, and x^* be some optimum of f. Moreover, assume that $b \leq 0.15(n-m)$, $m \leq (n-2b)/2$, and $\delta = 0$ is used to compute clipping parameter τ_l for CenteredClip. Next, assume that

$$\gamma = \min \left\{ \frac{1}{4L}, \sqrt{\frac{7nR_0^2}{120\sigma^2 K}}, \sqrt{\frac{m^2 R_0^2}{72C\sigma^2 n^2 b^2}} \right\}, \quad \Delta_{\max}^k = \frac{(1+\sqrt{3})\sqrt{2}\sigma}{\sqrt{n_k - m}}, \tag{30}$$

where $R_0 \ge \|x^0 - x^*\|$ and Δ_{\max}^k is the parameter for verification 3 at iteration k of BTARD-SGD. Then, we have $\mathbb{E}[f(\overline{x}^K) - f(x^*)] \le \varepsilon$ after K iterations of BTARD-SGD, where

$$K = \mathcal{O}\left(\frac{LR_0^2}{\varepsilon} + \frac{\sigma^2 R_0^2}{n\varepsilon^2} + \frac{nb\sigma R_0}{m\varepsilon}\right)$$
(31)

and $\overline{x}^K = \frac{1}{K} \sum_{k=0}^{K-1}$.

Proof. The proof is almost identical to the proof of Theorem F.3. Following the same steps and using (18) and (19) instead of (16) and (17) respectively we obtain the same sequence of inequalities up to the following change: instead of $\hat{\delta}_k$ we should use $\mathbb{1}_{k,v}$. Therefore, we have

$$\begin{split} \mathbb{E}\left[\|x^{k+1} - x^*\|^2 \mid x^k\right] & \leq \|x^k - x^*\|^2 - 2\gamma \left(1 - 2L\gamma\right) \left(f(x^k) - f(x^*)\right) \\ & - 2\gamma \mathbb{E}\left[\langle x^k - x^*, \widehat{g}^k - \overline{g}^k \rangle \mid x^k\right] + 2\gamma^2 C\sigma^2 \mathbb{1}_{k,v} + \frac{2\gamma^2 \sigma^2}{n - 2b - m} \\ & - 2\gamma \mathbb{E}\left[\langle x^k - x^*, \widehat{g}^k - \overline{g}^k \rangle \mid x^k\right] & \leq 2\gamma \sqrt{C}\sigma \|x^k - x^*\| \mathbb{1}_{k,v}, \end{split}$$

that result in

$$\begin{split} \mathbb{E}\left[\|x^{k+1} - x^*\|^2 \mid x^k\right] & \leq \|x^k - x^*\|^2 - \gamma \left(f(x^k) - f(x^*)\right) \\ & + 2\gamma \sqrt{C}\sigma \|x^k - x^*\| \mathbbm{1}_{k,v} + 2\gamma^2 C\sigma^2 \mathbbm{1}_{k,v} + \frac{40\gamma^2\sigma^2}{7n}. \end{split}$$

Taking the full expectation from the both sides of the above inequality and summing up the results for k = 0, 1, ..., K - 1 we derive

$$\begin{split} \frac{\gamma}{K} \sum_{k=0}^{K-1} \mathbb{E}[f(x^k) - f(x^*)] & \leq & \frac{1}{K} \sum_{k=0}^{K-1} \left(\mathbb{E} \left[\|x^k - x^*\|^2 \right] - \mathbb{E} \left[\|x^{k+1} - x^*\|^2 \right] \right) + \frac{40 \gamma^2 \sigma^2}{7n} \\ & + \frac{2 \gamma \sqrt{C} \sigma}{K} \sum_{k=0}^{K-1} \mathbb{E} \left[\|x^k - x^*\| \mathbb{1}_{k,v} \right] + \frac{2 \gamma^2 C \sigma^2}{K} \sum_{k=0}^{K-1} \mathbb{E}[\mathbb{1}_{k,v}] \\ & \leq & \frac{\|x^0 - x^*\|^2 - \mathbb{E}[\|x^K - x^*\|^2]}{K} + \frac{40 \gamma^2 \sigma^2}{7n} \\ & + \frac{2 \gamma \sqrt{C} \sigma}{K} \sum_{k=0}^{K-1} \sqrt{\mathbb{E} \left[\|x^k - x^*\|^2 \right] \mathbb{E}[\mathbb{1}_{k,v}]} + \frac{2 \gamma^2 C \sigma^2}{K} \sum_{k=0}^{K-1} \mathbb{E}[\mathbb{1}_{k,v}]. \end{split}$$

From Jensen's inequality we have $f(\overline{x}^K) \leq \frac{1}{K} \sum_{k=0}^{K-1} f(x^k)$, where $\overline{x}^K = \frac{1}{K} \sum_{k=0}^{K-1} x^k$. Using this and new notation $R_k = \|x^k - x^*\|$, $k \geq 0$ we get

$$\begin{split} 0 \leq \gamma \mathbb{E}\left[f(\overline{x}^K) - f(x^*)\right] & \leq & \frac{R_0^2 - \mathbb{E}[R_K^2]}{K} + \frac{40\gamma^2\sigma^2}{7n} \\ & + \frac{2\gamma\sqrt{C}\sigma}{K} \sum_{k=0}^{K-1} \sqrt{\mathbb{E}\left[R_k^2\right]\mathbb{E}[\mathbb{1}_{k,v}]} + \frac{2\gamma^2C\sigma^2}{K} \sum_{k=0}^{K-1} \mathbb{E}[\mathbb{1}_{k,v}] \\ 32) \end{split}$$

implying (after changing the indices) that

$$\mathbb{E}[R_k^2] \le R_0^2 + \frac{40\gamma^2 \sigma^2 k}{7n} + 2\gamma \sqrt{C} \sigma \sum_{l=0}^{k-1} \sqrt{\mathbb{E}[R_l^2] \mathbb{E}[\mathbb{1}_{l,v}]} + 2\gamma^2 C \sigma^2 \sum_{l=0}^{k-1} \mathbb{E}[\mathbb{1}_{l,v}]$$
(33)

holds for all $k \ge 0$. In the remaining part of the proof we derive by induction that

$$R_0^2 + \frac{40\gamma^2 \sigma^2 k}{7n} + 2\gamma \sqrt{C} \sigma \sum_{l=0}^{k-1} \sqrt{\mathbb{E}[R_l^2] \mathbb{E}[\mathbb{1}_{l,v}]} + 2\gamma^2 C \sigma^2 \sum_{l=0}^{k-1} \mathbb{E}[\mathbb{1}_{l,v}] \le 2R_0^2$$
 (34)

for all $k=0,\ldots,K$. For k=0 this inequality trivially holds. Next, assume that it holds for all $k=0,1,\ldots,T-1,T\leq K-1$. Let us show that it holds for k=T as well. From (33) and (34) we have that $\mathbb{E}[R_k^2]\leq 2R_0^2$ for all $k=0,1,\ldots,T-1$. Therefore,

$$\mathbb{E}[R_T^2] \leq R_0^2 + \frac{40\gamma^2 \sigma^2 k}{7n} + 2\gamma \sqrt{C} \sigma \sum_{l=0}^{T-1} \sqrt{\mathbb{E}[R_l^2]} \, \mathbb{E}[\mathbb{1}_{l,v}] + 2\gamma^2 C \sigma^2 \sum_{l=0}^{T-1} \mathbb{E}[\mathbb{1}_{l,v}] \\
\leq R_0^2 + \frac{40\gamma^2 \sigma^2 k}{7n} + 2\gamma \sqrt{2C} \sigma R_0 \sum_{l=0}^{T-1} \sqrt{\mathbb{E}[\mathbb{1}_{l,v}]} + 2\gamma^2 C \sigma^2 \sum_{l=0}^{T-1} \mathbb{E}[\mathbb{1}_{l,v}].$$

If a Byzantine peer deviates from the protocol at iteration k, it will be detected with some probability p_k during the next iteration. One can lower bound this probability as

$$p_k \ge m \cdot \frac{|\mathcal{G}_k|}{n_k} \cdot \frac{1}{n_k} = \frac{m(1 - \delta_k)}{n_k} \ge \frac{m}{n}.$$

That is, each individual Byzantine worker can violate the protocol no more than $^1/p$ times on average. However, even one Byzantine peer can create a shift of the order Δ^k_{\max} at each part of the resulting vector. Therefore, all Byzantine peers can violate the protocol no more than $^b/p$ times on average implying that

$$\mathbb{E}[R_T^2] \leq R_0^2 + \frac{40\gamma^2 \sigma^2 T}{7n} + \frac{2\gamma nb\sqrt{2C}\sigma R_0}{m} + \frac{2\gamma^2 nbC\sigma^2}{m}.$$

Taking

$$\gamma = \min \left\{ \frac{1}{4L}, \sqrt{\frac{7nR_0^2}{120\sigma^2 K}}, \sqrt{\frac{m^2 R_0^2}{72C\sigma^2 n^2 b^2}} \right\}$$

we ensure that

$$\frac{40\gamma^2\sigma^2T}{7n} + \frac{2\gamma nb\sqrt{2C}\sigma R_0}{m} + \frac{2\gamma^2 nbC\sigma^2}{m} \le \frac{R_0^2}{3} + \frac{R_0^2}{3} + \frac{R_0^2}{3} = R_0^2,$$

and, as a result, we get $\mathbb{E}[R_T^2] \leq 2R_0^2$. Therefore, (34) holds for all $k = 0, 1, \dots, K$. Together with (32) it implies

$$\mathbb{E}\left[f(\overline{x}^K) - f(x^*)\right] \le \frac{2R_0^2}{\gamma K}.$$

Next, from our stepsize rule (30) it follows that

$$\mathbb{E}\left[f(\overline{x}^K) - f(x^*)\right] = \mathcal{O}\left(\frac{LR_0^2}{K} + \frac{\sigma R_0}{\sqrt{nK}} + \frac{nb\sigma R_0}{mK}\right)$$

meaning that after

$$K = \mathcal{O}\left(\frac{LR_0^2}{\varepsilon} + \frac{\sigma^2 R_0^2}{n\varepsilon^2} + \frac{nb\sigma R_0}{m\varepsilon}\right)$$

iterations BTARD-SGD guarantees $\mathbb{E}[f(\overline{x}^K) - f(x^*)] \leq \varepsilon$.

F.2.5 Strongly convex case: Restarted-BTARD-SGD

In this section, we provide the complete statements and the full proofs of the convergence results for the restarted version of BTARD-SGD (RESTARTED-BTARD-SGD, Alg. 7) when the objective function f is smooth and strongly convex.

Algorithm 7 RESTARTED-BTARD-SGD

Input: x^0 – starting point, r – number of restarts, $\{\gamma_t\}_{t=1}^r$ – stepsizes for BTARD-SGD, $\{K_t\}_{t=1}^r$ – number of iterations for BTARD-SGD, $\{s_{i,k,t}\}_{i,k,t=0,0,0}^{n,K-1,r}$ – seeds for batches computations

- 1: $\hat{x}^0 = x^0$
- 2: **for** t = 1, 2, ..., r **do**
- 3: Run BTARD-SGD (Alg. 6) for K_t iterations with stepsize γ_t , starting point \widehat{x}^{t-1} , and seeds for batches computations $\{s_{i,k,t}\}_{i,k=0,0}^{n,K-1}$. Define \widehat{x}^t as $\widehat{x}^t = \frac{1}{K_t} \sum_{k=0}^{K_t} x^{k,t}$, where $x^{0,t}, x^{1,t}, \dots, x^{K_t,t}$ are the iterates produced by BTARD-SGD.

Output: \hat{x}^r

We start with the case when the number of attacking Byzantine workers is known at each iteration.

Theorem F.5. Let As. 3.2, (Option I) from As. 3.1 hold, $Q = \mathbb{R}^d$, f be L-smooth (see Def. F.1), μ -strongly convex (see Def. F.2), and x^* be some optimum of f. Moreover, assume that $b \leq 0.15(n-m)$, $m \leq (n-2b)/2$, and the exact number of attacking Byzantine peers is known to all good peers at each iteration. Next, assume that

$$\gamma_t = \min \left\{ \frac{1}{4L}, \sqrt{\frac{7nR_0^2}{120 \cdot 2^t \sigma^2 K_t}}, \sqrt{\frac{m^2 R_0^2}{1440 \cdot 2^t C \sigma^2 n^2 \delta}} \right\}, \quad \Delta_{\max}^{k,t} = \frac{(1 + \sqrt{3})\sqrt{2}\sigma}{\sqrt{n_k^t - m}}, \quad (35)$$

$$K_{t} = \left[\max \left\{ \frac{16L}{\mu}, \frac{32\sigma^{2}2^{t}}{\mu^{2}R_{0}^{2}}, \frac{48\sqrt{10C}n\sqrt{\delta}\sigma^{2\frac{t}{2}}}{m\mu R_{0}} \right\} \right], \quad r = \left[\log_{2} \frac{\mu R_{0}^{2}}{\varepsilon} \right] - 1$$
 (36)

where $R_0 \geq \|x^0 - x^*\|$, $\Delta_{\max}^{k,t}$ is the parameter for verification 3 at iteration k of BTARD-SGD during the t-th restart, n_k^t is the total number of workers at iteration k of t-th restart. Then, we have $\mathbb{E}[f(\widehat{x}^r) - f(x^*)] \leq \varepsilon$ after r restarts of BTARD-SGD and the total number of executed iterations of BTARD-SGD is

$$\sum_{t=1}^{r} K_{t} = \mathcal{O}\left(\frac{L}{\mu}\log\frac{\mu R_{0}^{2}}{\varepsilon} + \frac{\sigma^{2}}{n\mu\varepsilon} + \frac{n\sqrt{\delta}\sigma}{m\sqrt{\mu\varepsilon}}\right). \tag{37}$$

Proof. Theorem F.3 implies that BTARD-SGD with

$$\gamma = \min \left\{ \frac{1}{4L}, \sqrt{\frac{7nR_0^2}{120\sigma^2 K}}, \sqrt{\frac{m^2 R_0^2}{1440C\sigma^2 n^2 \delta}} \right\}$$

guarantees

$$\mathbb{E}\left[f(\overline{x}^K) - f(x^*)\right] \le \frac{2R_0^2}{\gamma K}$$

after K iterations. Therefore, after the first restart we have

$$\mathbb{E}[f(\widehat{x}^1) - f(x^*)] \le \frac{2R_0^2}{\gamma_1 K_1} \le \frac{\mu R_0^2}{4}.$$

From μ -strong convexity of f and $\nabla f(x^*) = 0$ we have

$$\frac{\mu}{2} \|\widehat{x}^1 - x^*\|^2 \le f(\widehat{x}^1) - f(x^*) \Longrightarrow \mathbb{E}[\|\widehat{x}^1 - x^*\|^2] \le \frac{R_0^2}{2}.$$

Next, assume that we have $\mathbb{E}[f(\widehat{x}^t) - f(x^*)] \leq \frac{\mu R_0^2}{2^{t+1}}$, $\mathbb{E}[\|\widehat{x}^t - x^*\|^2] \leq \frac{R_0^2}{2^t}$ for some $t \leq r-1$. Then, Theorem F.3 implies that

$$\mathbb{E}[f(\widehat{x}^{t+1}) - f(x^*) \mid x^t] \le \frac{2\|\widehat{x}^t - x^*\|^2}{\gamma_t K_t}.$$

Taking the full expectation from the both sides of previous inequality we get

$$\mathbb{E}[f(\widehat{x}^{t+1}) - f(x^*)] \leq \frac{2\mathbb{E}[||\widehat{x}^t - x^*||^2]}{\gamma_t K_t} \leq \frac{2R_0^2}{2^t \gamma_t K_t} \leq \frac{\mu R_0^2}{2^{t+2}}$$

From μ -strong convexity of f and $\nabla f(x^*) = 0$ we have

$$\frac{\mu}{2} \|\widehat{x}^{t+1} - x^*\|^2 \le f(\widehat{x}^{t+1}) - f(x^*) \Longrightarrow \mathbb{E}[\|\widehat{x}^{t+1} - x^*\|^2] \le \frac{R_0^2}{2^{t+1}}$$

Therefore, by mathematical induction we have that for all t = 1, ..., r

$$\mathbb{E}[f(\widehat{x}^t) - f(x^*)] \le \frac{\mu R_0^2}{2^{t+1}}, \quad \mathbb{E}\left[\|\widehat{x}^t - x^*\|^2\right] \le \frac{R_0^2}{2^t}.$$

Then, after $r = \left\lceil \log_2 \frac{\mu R_0^2}{\varepsilon} \right\rceil - 1$ restarts of BTARD-SGD we have $\mathbb{E}[f(\widehat{x}^r) - f(x^*)] \le \varepsilon$. The total number of iterations executed by BTARD-SGD is

$$\begin{split} \sum_{t=1}^{r} K_{t} &= \mathcal{O}\left(\sum_{t=1}^{r} \max\left\{\frac{L}{\mu}, \frac{\sigma^{2} 2^{t}}{\mu^{2} R_{0}^{2}}, \frac{n\sqrt{\delta}\sigma 2^{\frac{t}{2}}}{m\mu R_{0}}\right\}\right) \\ &= \mathcal{O}\left(\frac{L}{\mu} r + \frac{\sigma^{2} 2^{r}}{\mu^{2} R_{0}^{2}} + \frac{n\sqrt{\delta}\sigma 2^{\frac{r}{2}}}{m\mu R_{0}}\right) \\ &= \mathcal{O}\left(\frac{L}{\mu} \log \frac{\mu R_{0}^{2}}{\varepsilon} + \frac{\sigma^{2}}{\mu^{2} R_{0}^{2}} \cdot \frac{\mu R_{0}^{2}}{\varepsilon} + \frac{n\sqrt{\delta}\sigma}{m\mu R_{0}} \cdot \sqrt{\frac{\mu R_{0}^{2}}{\varepsilon}}\right) \\ &= \mathcal{O}\left(\frac{L}{\mu} \log \frac{\mu R_{0}^{2}}{\varepsilon} + \frac{\sigma^{2}}{n\mu\varepsilon} + \frac{n\sqrt{\delta}\sigma}{m\sqrt{\mu\varepsilon}}\right). \end{split}$$

In the strongly convex case, similar observations hold as in the non-convex case. Next, we derive the result without assuming that \hat{b}^k is known to all peers at each iteration.

Theorem F.6. Let As. 3.2, (Option II) from As. 3.1 hold, $Q = \mathbb{R}^d$, f be L-smooth (see Def. F.1), μ -strongly convex (see Def. F.2), and x^* be some optimum of f. Moreover, assume that $b \leq 0.15(n-m)$, $m \leq (n-2b)/2$, and $\delta = 0$ is used to compute clipping parameter τ_l for CenteredClip. Next, assume that

$$\gamma_t = \min \left\{ \frac{1}{4L}, \sqrt{\frac{7nR_0^2}{120 \cdot 2^t \sigma^2 K_t}}, \sqrt{\frac{m^2 R_0^2}{72 \cdot 2^t C \sigma^2 n^2 b^2}} \right\}, \quad \Delta_{\max}^{k,t} = \frac{(1 + \sqrt{3})\sqrt{2}\sigma}{\sqrt{n_k^t - m}}, \quad (38)$$

$$K_{t} = \left[\max \left\{ \frac{16L}{\mu}, \frac{32\sigma^{2}2^{t}}{\mu^{2}R_{0}^{2}}, \frac{24\sqrt{2C}nb\sigma^{2}2^{\frac{t}{2}}}{m\mu R_{0}} \right\} \right], \quad r = \left[\log_{2} \frac{\mu R_{0}^{2}}{\varepsilon} \right] - 1$$
 (39)

where $R_0 \geq \|x^0 - x^*\|$, $\Delta_{\max}^{k,t}$ is the parameter for verification 3 at iteration k of BTARD-SGD during the t-th restart, n_k^t is the total number of workers at iteration k of t-th restart. Then, we have $\mathbb{E}[f(\widehat{x}^r) - f(x^*)] \leq \varepsilon$ after r restarts of BTARD-SGD and the total number of executed iterations of BTARD-SGD is

$$\sum_{t=1}^{r} K_{t} = \mathcal{O}\left(\frac{L}{\mu} \log \frac{\mu R_{0}^{2}}{\varepsilon} + \frac{\sigma^{2}}{n\mu\varepsilon} + \frac{nb\sigma}{m\sqrt{\mu\varepsilon}}\right). \tag{40}$$

Proof. Theorem F.4 implies that BTARD-SGD with

$$\gamma = \min \left\{ \frac{1}{4L}, \sqrt{\frac{7nR_0^2}{120\sigma^2 K}}, \sqrt{\frac{m^2R_0^2}{72C\sigma^2n^2b^2}} \right\}$$

guarantees

$$\mathbb{E}\left[f(\overline{x}^K) - f(x^*)\right] \le \frac{2R_0^2}{\gamma K}$$

after K iterations. Therefore, after the first restart we have

$$\mathbb{E}[f(\widehat{x}^1) - f(x^*)] \le \frac{2R_0^2}{\gamma_1 K_1} \le \frac{\mu R_0^2}{4}.$$

From μ -strong convexity of f and $\nabla f(x^*) = 0$ we have

$$\frac{\mu}{2} \|\widehat{x}^1 - x^*\|^2 \le f(\widehat{x}^1) - f(x^*) \Longrightarrow \mathbb{E}[\|\widehat{x}^1 - x^*\|^2] \le \frac{R_0^2}{2}.$$

Next, assume that we have $\mathbb{E}[f(\widehat{x}^t) - f(x^*)] \leq \frac{\mu R_0^2}{2^{t+1}}$, $\mathbb{E}[\|\widehat{x}^t - x^*\|^2] \leq \frac{R_0^2}{2^t}$ for some $t \leq r-1$. Then, Theorem F.4 implies that

$$\mathbb{E}[f(\widehat{x}^{t+1}) - f(x^*) \mid x^t] \le \frac{2\|\widehat{x}^t - x^*\|^2}{\gamma_t K_t}.$$

Taking the full expectation from the both sides of previous inequality we get

$$\mathbb{E}[f(\widehat{x}^{t+1}) - f(x^*)] \le \frac{2\mathbb{E}[\|\widehat{x}^t - x^*\|^2]}{\gamma_t K_t} \le \frac{2R_0^2}{2^t \gamma_t K_t} \le \frac{\mu R_0^2}{2^{t+2}}.$$

From μ -strong convexity of f and $\nabla f(x^*) = 0$ we have

$$\frac{\mu}{2} \|\widehat{x}^{t+1} - x^*\|^2 \le f(\widehat{x}^{t+1}) - f(x^*) \Longrightarrow \mathbb{E}[\|\widehat{x}^{t+1} - x^*\|^2] \le \frac{R_0^2}{2^{t+1}}$$

Therefore, by mathematical induction we have that for all $t = 1, \ldots, r$

$$\mathbb{E}[f(\widehat{x}^t) - f(x^*)] \le \frac{\mu R_0^2}{2^{t+1}}, \quad \mathbb{E}\left[\|\widehat{x}^t - x^*\|^2\right] \le \frac{R_0^2}{2^t}.$$

Then, after $r = \left\lceil \log_2 \frac{\mu R_0^2}{\varepsilon} \right\rceil - 1$ restarts of BTARD-SGD we have $\mathbb{E}[f(\widehat{x}^r) - f(x^*)] \le \varepsilon$. The total number of iterations executed by BTARD-SGD is

$$\begin{split} \sum_{t=1}^{r} K_{t} &= \mathcal{O}\left(\sum_{t=1}^{r} \max\left\{\frac{L}{\mu}, \frac{\sigma^{2} 2^{t}}{\mu^{2} R_{0}^{2}}, \frac{nb\sigma 2^{\frac{t}{2}}}{m\mu R_{0}}\right\}\right) \\ &= \mathcal{O}\left(\frac{L}{\mu} r + \frac{\sigma^{2} 2^{r}}{\mu^{2} R_{0}^{2}} + \frac{nb\sigma 2^{\frac{r}{2}}}{m\mu R_{0}}\right) \\ &= \mathcal{O}\left(\frac{L}{\mu} \log \frac{\mu R_{0}^{2}}{\varepsilon} + \frac{\sigma^{2}}{\mu^{2} R_{0}^{2}} \cdot \frac{\mu R_{0}^{2}}{\varepsilon} + \frac{nb\sigma}{m\mu R_{0}} \cdot \sqrt{\frac{\mu R_{0}^{2}}{\varepsilon}}\right) \\ &= \mathcal{O}\left(\frac{L}{\mu} \log \frac{\mu R_{0}^{2}}{\varepsilon} + \frac{\sigma^{2}}{n\mu\varepsilon} + \frac{nb\sigma}{m\sqrt{\mu\varepsilon}}\right). \end{split}$$

F.3 Convergence guarantees for BTARD-Clipped-SGD

The results for BTARD-SGD and RESTARTED-BTARD-SGD rely on As. 3.2 that the stochastic gradients have not too heavy tails, i.e., sub-quadratically decreasing tails. The main reason why it is needed in the analysis is to prevent too often extra computations because of **Verification 3** from BTARD when all workers honestly follow the protocol. However, in many important NLP tasks

such as BERT training [81], the noise in the stochastic gradient has such a heavy noise that As. 3.2 becomes unnatural.

Algorithm 8 BTARD-CLIPPED-SGD

Input: x^0 – starting point, γ – stepsize, K – number of iterations, $\{s_{i,k}\}_{i,k=0,0}^{n,K-1}$ – seeds for batches computations, $\{\lambda_k\}_{k=0}^{K-1}$ – gradient clipping parameter 1: $C_0 = \mathrm{Banned}_{-1} = \varnothing$ 2: for $k=0,1,\ldots,K-1$ do

- Worker i computes $\widetilde{g}_{i}^{k} = \begin{cases} \min\left\{1, \frac{\lambda_{k}}{\|\nabla f(x^{k}, \xi_{i,k})\|}\right\} \nabla f(x^{k}, \xi_{i,k}), & \text{if } i \in \mathcal{G}_{k} \setminus \mathcal{C}_{k}, \\ *, & \text{if } i \in \mathcal{B}_{k} \setminus \mathcal{C}_{k}, \end{cases}$, where

- $\xi_{i,k}$ is generated via seed $s_{i,k}$ available to every worker \widehat{g}^k , public_info $_k = \operatorname{BTARD}(\widetilde{g}^k_{i^k_1}, g^k_{i^k_1}, \ldots, \widetilde{g}^k_{i^k_{a_k}})$, where $\{i^k_1, \ldots, i^k_{a_k}\} = (\mathcal{G}_k \cup \mathcal{B}_k) \setminus \mathcal{C}_k$
- Choose 2m workers $c_1^{k+1}, \ldots, c_m^{k+1}, u_1^{k+1}, \ldots, u_m^{k+1}$ uniformly at random without replacement, $\mathcal{C}_{k+1} = \{c_1^{k+1}, \ldots, c_m^{k+1}\}, \mathcal{U}_{k+1} = \{u_1^{k+1}, \ldots, u_m^{k+1}\}$ Banned $_k = \mathsf{CHECKCOMPUTATIONS}(\mathcal{C}_{k+1}, \mathcal{U}_{k+1}, \mathsf{public_info}_k)$ $x^{k+1} = \mathsf{proj}_Q(x^k \gamma \widehat{g}^k) := \mathrm{argmin}_{x \in Q} \|x (x^k \gamma \widehat{g}^k)\|$ $\mathcal{G}_{k+1} = \mathcal{G}_k \setminus \mathsf{Banned}_{k-1}, \mathcal{B}_{k+1} = \mathcal{B}_k \setminus \mathsf{Banned}_{k-1}$
- 7:
- 8:

To handle the problems with heavy-tailed noise distributions we consider BTARD-CLIPPED-SGD (see Alg. 8 in Appendix) applied to solve (1) such that Q is bounded. Essentially, this algorithm coincides with BTARD-SGD up to the following change: all good peers $i \in G_k \setminus C_k$ use clipped stochastic gradients $\widetilde{g}_i^k = (\widetilde{g}_i^k(1)^\top, \dots, \widetilde{g}_i^k(n_k - m)^\top)^\top$, where $\widetilde{g}_i^k(l) = \min\left\{1, \frac{\lambda_k}{\|g_i^k(l)\|}\right\} g_i^k(l)$, $l = 1, \dots, n_k - m$, and g_i^k is the stochastic gradient. Next, we introduce the following assumption. **Assumption F.1.** There exist such constant G > 0, $s_0 \in [d]$, and $\alpha \in (1,2]$ that for any set of indices $S = (i_1, \ldots, i_d), \ 1 \leq i_1 < i_2 < \ldots < i_s \leq d, \ s \geq s_0$ and arbitrary $x \in Q$ stochastic gradient $\nabla f(x,\xi)$ satisfy

$$\mathbb{E}[\nabla f(x,\xi)] = \nabla f(x), \quad \mathbb{E}\left[\left\|\nabla_{[S]}f(x,\xi)\right\|^{\alpha}\right] \le \left(\frac{\sqrt{s}G}{\sqrt{d}}\right)^{\alpha},\tag{41}$$

where $\nabla_{[S]} f(x,\xi)$ is defined in As. 3.1.

This is a modified version of the assumption used in [81]. When $\alpha < 2$ the variance of the stochastic gradient can be unbounded. One can show that in such a regime vanilla SGD can diverge [81].

Under As. F.1 we derive the convergence results for convex and strongly convex problems.

Quality of the aggregation F.3.1

Since now we have As. F.1 instead of As. 3.1 and 3.2 it is needed to derive new guarantees for the quality of the aggregation. We start with the following useful lemma about the properties of clipped stochastic gradeints.

Lemma F.5 (See also Lemma 9 from [81]). Let As. F.1 holds and $i, j \in \mathcal{G}_k \setminus \mathcal{C}_k$. Then, for all $l=1,2,\ldots,n_k-m$ we have

$$\sqrt{\mathbb{E}\left[\|\widetilde{g}_{i}^{k}(l) - \widetilde{g}_{j}^{k}(l)\|^{4} \mid x^{k}\right]} \leq 4\lambda_{k}^{\frac{4-\alpha}{2}} \left(\frac{G}{\sqrt{n_{k} - m}}\right)^{\frac{\alpha}{2}},\tag{42}$$

$$\mathbb{E}\left[\|\overline{g}^k(l)\|^2 \mid x^k\right] \leq \frac{G^\alpha \lambda_k^{2-\alpha}}{(n_k - m)^{\frac{\alpha}{2}}},\tag{43}$$

$$\mathbb{E}\left[\|\overline{g}^{k}(l)\|^{2} \mid x^{k}\right] \leq \frac{G^{\alpha}\lambda_{k}^{2-\alpha}}{(n_{k}-m)^{\frac{\alpha}{2}}}, \tag{43}$$

$$\|\mathbb{E}[\overline{g}^{k}(l) \mid x^{k}] - \nabla_{(l)}f(x^{k})\|^{2} \leq \frac{G^{2\alpha}}{(n_{k}-m)^{\alpha}\lambda_{k}^{2(\alpha-1)}}, \tag{44}$$

where $\overline{g}^k(l) = \frac{1}{|\mathcal{G}_k \setminus \mathcal{C}_k|} \sum_{i \in \mathcal{G}_k \setminus \mathcal{C}_k} \widetilde{g}_i^k(l)$ for all $l = 1, \dots, n_k - m$.

Proof. First of all, we derive

$$\begin{split} \mathbb{E}\left[\|\widetilde{g}_{i}^{k}(l)-\widetilde{g}_{j}^{k}(l)\|^{4}\mid x^{k}\right] &= \mathbb{E}\left[\|\widetilde{g}_{i}^{k}(l)-\widetilde{g}_{j}^{k}(l)\|^{\alpha}\|\widetilde{g}_{i}^{k}(l)-\widetilde{g}_{j}^{k}(l)\|^{4-\alpha}\mid x^{k}\right] \\ &\leq 8\lambda_{k}^{4-\alpha}\mathbb{E}\left[\|\nabla_{(l)}f(x^{k},\xi_{i,k})\|^{\alpha}+\|\nabla_{(l)}f(x^{k},\xi_{j,k})\|^{\alpha}\mid x^{k}\right] \\ &\stackrel{(41)}{\leq} 16\lambda_{k}^{4-\alpha}\left(\frac{G}{\sqrt{n_{k}-m}}\right)^{\alpha} \end{split}$$

implying (42). Next, for all $i \in \mathcal{G}_k \setminus \mathcal{C}_k$ we have

$$\mathbb{E}\left[\|\widetilde{g}_{i}^{k}(l)\|^{2} \mid x^{k}\right] = \mathbb{E}\left[\|\widetilde{g}_{i}^{k}(l)\|^{\alpha}\|\widetilde{g}_{i}^{k}(l)\|^{2-\alpha} \mid x^{k}\right] \leq \lambda_{k}^{2-\alpha}\mathbb{E}\left[\|\nabla_{(l)}f(x^{k},\xi_{i,k})\|^{\alpha} \mid x^{k}\right] \leq \frac{G^{\alpha}\lambda_{k}^{2-\alpha}}{(n_{k}-m)^{\frac{\alpha}{2}}}$$

implying

$$\mathbb{E}\left[\|\overline{g}^k(l)\|^2 \mid x^k\right] \le \frac{1}{|\mathcal{G}_k \setminus \mathcal{C}_k|} \sum_{i \in \mathcal{G}_k \setminus \mathcal{C}_k} \mathbb{E}\left[\|\widetilde{g}_i^k(l)\|^2 \mid x^k\right] \le \frac{G^{\alpha} \lambda_k^{2-\alpha}}{(n_k - m)^{\frac{\alpha}{2}}}.$$

Finally, for all $i \in \mathcal{G}_k \setminus \mathcal{C}_k$ we derive

$$\begin{split} \left\| \mathbb{E}[\widetilde{g}_{i}^{k}(l) \mid x^{k}] - \nabla_{(l)} f(x^{k}) \right\| &= \left\| \mathbb{E}[\widetilde{g}_{i}^{k}(l) - \nabla_{(l)} f(x^{k}, \xi_{i,k}) \mid x^{k}] \right\| \\ &\leq \mathbb{E}\left[\left\| \widetilde{g}_{i}^{k}(l) - \nabla_{(l)} f(x^{k}, \xi_{i,k}) \right\| \mid x^{k} \right] \\ &= \mathbb{E}\left[\left\| \widetilde{g}_{i}^{k}(l) - \nabla_{(l)} f(x^{k}, \xi_{i,k}) \right\| \mathbb{1}_{\left\{ \| \nabla_{(l)} f(x^{k}, \xi_{i,k}) \| \geq \lambda_{k} \right\}} \mid x^{k} \right] \\ &\leq \mathbb{E}\left[\left\| \nabla_{(l)} f(x^{k}, \xi_{i,k}) \right\| \mathbb{1}_{\left\{ \| \nabla_{(l)} f(x^{k}, \xi_{i,k}) \| \geq \lambda_{k} \right\}} \mid x^{k} \right] \\ &\leq \frac{\mathbb{E}\left[\left\| \nabla_{(l)} f(x^{k}, \xi_{i,k}) \right\|^{\alpha} \mathbb{1}_{\left\{ \| \nabla_{(l)} f(x^{k}, \xi_{i,k}) \| \geq \lambda_{k} \right\}} \mid x^{k} \right]}{\lambda_{k}^{\alpha - 1}} \\ &\leq \frac{G^{\alpha}}{(n_{k} - m)^{\frac{\alpha}{2}} \lambda_{l}^{\alpha - 1}} \end{split}$$

implying

$$\begin{aligned} \left\| \mathbb{E}[\overline{g}^{k}(l) \mid x^{k}] - \nabla_{(l)} f(x^{k}) \right\|^{2} & \leq \frac{1}{|\mathcal{G}_{k} \setminus \mathcal{C}_{k}|} \sum_{i \in \mathcal{G}_{k} \setminus \mathcal{C}_{k}} \mathbb{E}\left[\|\widetilde{g}_{i}^{k}(l) - \nabla_{(l)} f(x^{k}) \|^{2} \mid x^{k} \right] \\ & \leq \frac{G^{2\alpha}}{(n_{k} - m)^{\alpha} \lambda_{k}^{2(\alpha - 1)}}. \end{aligned}$$

Next, we derive the guarantees for the quality of the aggregation in the case when the number of Byzantine peers violating the protocol \hat{b}_k is known at each iteration.

Lemma F.6. Let As. F.1 hold and $b \le 0.15(n-m)$. Assume that \widehat{b}_k is known for each worker at iteration k, $\Delta_{\max}^k = 2\lambda_k = \frac{2\lambda}{\sqrt{n_k-m}}$ and $\delta = \widehat{\delta}_k$ is used to compute clipping parameter τ_l for CenteredClip. If the total number of iterations T of CenteredClip satisfies $T \ge \log_{0.94} \frac{2\delta\sigma^2}{\mathbb{E}[\|v^0-\overline{g}^k\|^2]}$ and CHECKAVERAGING is not triggered for any worker, then

$$\mathbb{E}\left[\|\widehat{g}^k - \overline{g}^k\|^2 \mid x^k\right] \le \widehat{\delta}_k(C_1\lambda^{\frac{4-\alpha}{2}}G^{\frac{\alpha}{2}} + C_2\lambda^2),\tag{45}$$

$$\mathbb{E}\left[\|\widehat{g}^k\|^2 \mid x^k\right] \leq 2\widehat{\delta}_k\left(C_1\lambda^{\frac{4-\alpha}{2}}G^{\frac{\alpha}{2}} + C_2\lambda^2\right) + 2G^{\alpha}\lambda^{2-\alpha},\tag{46}$$
where $\overline{g}^k = \frac{1}{|\mathcal{G}_k \setminus \mathcal{C}_k|} \sum_{j \in \mathcal{G}_k \setminus \mathcal{C}_k} g_j^k$, $C_1 = 384$, and $C_2 = 4$.

Proof. Consider the *i*-th part of \widehat{g}^k , i.e., consider $\widehat{g}^k(i)$. If $i \in \mathcal{G}_k \setminus \mathcal{C}_k$, then, in view of (42), we can directly apply Lemma F.1 and get

$$\mathbb{E}\left[\|\widehat{g}^{k}(i) - \overline{g}^{k}(i)\|^{2} \mid x^{k}\right] \leq 384\widehat{\delta}_{k}\lambda_{k}^{\frac{4-\alpha}{2}} \frac{G^{\frac{\alpha}{2}}}{(n_{k} - m)^{\frac{\alpha}{4}}} = \frac{384\widehat{\delta}_{k}\lambda^{\frac{4-\alpha}{2}}G^{\frac{\alpha}{2}}}{n_{k} - m}.$$

Next, if $i \in \mathcal{B}_k \setminus \mathcal{C}_k$, then

$$\mathbb{E}\left[\|\widehat{g}^{k}(i) - \overline{g}^{k}(i)\|^{2} \mid x^{k}\right] \le (\Delta_{\max}^{k})^{2} = 4\lambda_{k}^{2} = \frac{4\lambda^{2}}{n_{k} - m}.$$

Putting all together, we derive

$$\begin{split} \mathbb{E}\left[\|\widehat{g}^k - \overline{g}^k\|^2 \mid x^k\right] &= \sum_{i \in \mathcal{G}_k \setminus \mathcal{C}_k} \mathbb{E}\left[\|\widehat{g}^k(i) - \overline{g}^k(i)\|^2 \mid x^k\right] + \sum_{i \in \mathcal{B}_k \setminus \mathcal{C}_k} \mathbb{E}\left[\|\widehat{g}^k(i) - \overline{g}^k(i)\|^2 \mid x^k\right] \\ &\leq (1 - \widehat{\delta}_k)(n_k - m) \cdot \frac{384\widehat{\delta}_k \lambda^{\frac{4-\alpha}{2}} G^{\frac{\alpha}{2}}}{n_k - m} + \widehat{\delta}_k(n_k - m) \cdot \frac{4\lambda^2}{n_k - m} \\ &< \widehat{\delta}_k(C_1 \lambda^{\frac{4-\alpha}{2}} G^{\frac{\alpha}{2}} + C_2 \lambda^2). \end{split}$$

Using (43) we obtain

$$\mathbb{E} \left[\| \widehat{g}^{k} \|^{2} \mid x^{k} \right] \leq 2\mathbb{E} \left[\| \widehat{g}^{k} - \overline{g}^{k} \|^{2} \mid x^{k} \right] + 2\mathbb{E} \left[\| \overline{g}^{k} \|^{2} \mid x^{k} \right] \\
\leq 2\widehat{\delta}_{k} (C_{1} \lambda^{\frac{4-\alpha}{2}} G^{\frac{\alpha}{2}} + C_{2} \lambda^{2}) + 2 \sum_{i \in (\mathcal{G}_{k} \cup \mathcal{B}_{k}) \setminus \mathcal{C}_{k}} \frac{G^{\alpha} \lambda_{k}^{2-\alpha}}{(n_{k} - m)^{\frac{\alpha}{2}}} \\
= 2\widehat{\delta}_{k} (C_{1} \lambda^{\frac{4-\alpha}{2}} G^{\frac{\alpha}{2}} + C_{2} \lambda^{2}) + 2G^{\alpha} \lambda^{2-\alpha}.$$

We notice that **Verification 3** can be simplified in the following way: if at least on good peer i notices that $\|\widetilde{g}_i^k(j) - \widehat{g}^k(j)\| > \Delta_{\max}^k = 2\lambda_k$, then peer i should accuse j-th peer and both are removed from the training process. In this scenario, there is no sense for Byzantine workers in triggering to deviate significantly from the clipped stochastic gradients of the good peers.

As for BTARD-SGD, when \widehat{b}_k is unknown we always use CENTEREDCLIP with $\tau_l=\infty$ for all $l\geq 0$, i.e., good peers compute an exact average. In this settings, even 1 byzantine worker can significantly shift the average in all parts of the vector. The next lemma quantifies the negative effect of Byzantine workers in this case.

Lemma F.7. Let As. F.1 hold and $b \leq 0.15(n-m)$. Assume that \widehat{b}_k is known for each worker at iteration k, $\Delta_{\max}^k = 2\lambda_k = \frac{2\lambda}{\sqrt{n_k-m}}$ and $\delta = \widehat{\delta}_k$ is used to compute clipping parameter τ_l for CenteredClip. If the total number of iterations T of CenteredClip satisfies $T \geq \log_{0.94} \frac{2\delta\sigma^2}{\mathbb{E}[\|v^0 - \overline{g}^k\|^2]}$ and CHECKAVERAGING is not triggered for any worker, then

$$\mathbb{E}\left[\|\widehat{g}^k - \overline{g}^k\|^2 \mid x^k\right] \le C_2 \lambda^2 \mathbb{1}_{k,v},\tag{47}$$

$$\mathbb{E}\left[\|\widehat{g}^{k}\|^{2} \mid x^{k}\right] \le 2C_{2}\lambda^{2}\mathbb{1}_{k,v} + 2G^{\alpha}\lambda^{2-\alpha},\tag{48}$$

where $\overline{g}^k = \frac{1}{|\mathcal{G}_k \setminus \mathcal{C}_k|} \sum_{j \in \mathcal{G}_k \setminus \mathcal{C}_k} g_j^k$, $C_2 = 4$, and $\mathbb{1}_{k,v}$ is an indicator function of the event that at least 1 Byzantine peer violates the protocol at iteration k.

Proof. For all $i \in (\mathcal{G}_k \cup \mathcal{B}_k) \setminus \mathcal{C}_k$ we have

$$\mathbb{E}\left[\|\widehat{g}^{k}(i) - \overline{g}^{k}(i)\|^{2} \mid x^{k}\right] \leq (\Delta_{\max}^{k})^{2} \mathbb{1}_{k,v} = 4\lambda_{k}^{2} \mathbb{1}_{k,v} = \frac{4\lambda^{2}}{n_{k} - m} \mathbb{1}_{k,v}$$

implying

$$\mathbb{E}\left[\|\widehat{g}^k - \overline{g}^k\|^2 \mid x^k\right] = \sum_{i \in (\mathcal{G}_k \cup \mathcal{B}_k) \setminus \mathcal{C}_k} \mathbb{E}\left[\|\widehat{g}^k(i) - \overline{g}^k(i)\|^2 \mid x^k\right]$$

$$\leq (n_k - m) \cdot \frac{4\lambda^2}{n_k - m} \mathbb{1}_{k,v} = C_2 \lambda^2 \mathbb{1}_{k,v}.$$

Using (43) we obtain

$$\mathbb{E}\left[\|\widehat{g}^{k}\|^{2} \mid x^{k}\right] \leq 2\mathbb{E}\left[\|\widehat{g}^{k} - \overline{g}^{k}\|^{2} \mid x^{k}\right] + 2\mathbb{E}\left[\|\overline{g}^{k}\|^{2} \mid x^{k}\right] \\
\leq 2C_{2}\lambda^{2}\mathbb{1}_{k,v} + 2\sum_{i \in (\mathcal{G}_{k} \cup \mathcal{B}_{k}) \setminus \mathcal{C}_{k}} \frac{G^{\alpha}\lambda_{k}^{2-\alpha}}{(n_{k} - m)^{\frac{\alpha}{2}}} = 2C_{2}\lambda^{2}\mathbb{1}_{k,v} + 2G^{\alpha}\lambda^{2-\alpha}.$$

F.3.2 Convex case

In this section, we provide the complete statements and the full proofs of the convergence results for BTARD-CLIPPED-SGD when the objective function f is smooth and convex. We start with the case when the number of Byzantine peers violating the protocol \hat{b}_k is known at each iteration.

Theorem F.7. Let As. F.1 hold, Q is bounded, f be convex, x^* be some optimum of f, and $\nabla f(x^*) = 0$. Moreover, assume that $b \leq 0.15(n-m)$, $m \leq (n-2b)/2$, and the exact number of attacking Byzantine peers is known to all good peers at each iteration. Next, assume that

$$\gamma = \min \left\{ \frac{R_0}{\sqrt{6}GK^{\frac{1}{\alpha}}}, \frac{mR_0}{12Gn\sqrt{10\delta(C_1K^{\frac{4-\alpha}{2\alpha}} + C_2K^{\frac{2}{\alpha}})}} \right\}, \quad \Delta_{\max}^k = 2\lambda_k = \frac{2\lambda}{\sqrt{n_k - m}}, \quad (49)$$

$$\lambda = GK^{\frac{1}{\alpha}},\tag{50}$$

where $R_0 \geq \|x^0 - x^*\|$ and Δ_{\max}^k is the parameter for verification 3 at iteration k of BTARD-CLIPPED-SGD. Then, we have $\mathbb{E}[f(\overline{x}^K) - f(x^*)] \leq \varepsilon$ after K iterations of BTARD-CLIPPED-SGD, where

$$K = \mathcal{O}\left(\left(\frac{GR_0}{\varepsilon}\right)^{\frac{\alpha}{\alpha-1}} + \left(\frac{n\sqrt{\delta}GR_0}{m\varepsilon}\right)^{\frac{\alpha}{\alpha-1}}\right)$$
 (51)

and $\overline{x}^K = \frac{1}{K} \sum_{k=0}^{K-1}$.

Proof. Non-expansiveness of the projection operator and convexity of f imply

$$\begin{split} \|x^{k+1} - x^*\|^2 &= \left\| \mathrm{proj}_Q(x^k - \gamma \widehat{g}^k) - \mathrm{proj}_Q(x^*) \right\|^2 \\ &\leq \|x^k - x^* - \gamma \widehat{g}^k\|^2 \\ &= \|x^k - x^*\|^2 - 2\gamma \langle x^k - x^*, \widehat{g}^k \rangle + \gamma^2 \|\widehat{g}^k\|^2 \\ &= \|x^k - x^*\|^2 - 2\gamma \langle x^k - x^*, \nabla f(x^k) \rangle - 2\gamma \langle x^k - x^*, \widehat{g}^k - \nabla f(x^k) \rangle + \gamma^2 \|\widehat{g}^k\|^2 \\ &\leq \|x^k - x^*\|^2 - 2\gamma \left(f(x^k) - f(x^*)\right) - 2\gamma \langle x^k - x^*, \widehat{g}^k - \nabla f(x^k) \rangle + \gamma^2 \|\widehat{g}^k\|^2. \end{split}$$

Taking conditional expectation $\mathbb{E}[\cdot \mid x^k]$ from the both sides of previous inequality we derive

$$\mathbb{E}\left[\|x^{k+1} - x^*\|^2 \mid x^k\right] \leq \|x^k - x^*\|^2 - 2\gamma \left(f(x^k) - f(x^*)\right) \\ -2\gamma \mathbb{E}\left[\left\langle x^k - x^*, \widehat{g}^k - \nabla f(x^k)\right\rangle \mid x^k\right] + \gamma^2 \mathbb{E}\left[\|\widehat{g}^k\|^2 \mid x^k\right] \\ \leq \|x^k - x^*\|^2 - 2\gamma \left(f(x^k) - f(x^*)\right) + 2\gamma^2 G^{\alpha} \lambda^{2-\alpha} \\ -2\gamma \left\langle x^k - x^*, \mathbb{E}\left[\widehat{g}^k - \overline{g}^k \mid x^k\right]\right\rangle + 2\gamma^2 \widehat{\delta}_k (C_1 \lambda^{\frac{4-\alpha}{2}} G^{\frac{\alpha}{2}} + C_2 \lambda^2) \\ = \|x^k - x^*\|^2 - 2\gamma \left(f(x^k) - f(x^*)\right) + 2\gamma^2 G^2 K^{\frac{2-\alpha}{\alpha}} \\ -2\gamma \left\langle x^k - x^*, \mathbb{E}\left[\widehat{g}^k - \overline{g}^k \mid x^k\right]\right\rangle + \frac{2\gamma^2 G^2 (C_1 K^{\frac{4-\alpha}{2\alpha}} + C_2 K^{\frac{2}{\alpha}})}{n_k - m} \widehat{b}_k.$$

To estimate the inner product in the right-hand side we apply Cauchy-Schwarz inequality:

$$-2\gamma \left\langle x^{k} - x^{*}, \mathbb{E}\left[\widehat{g}^{k} - \overline{g}^{k} \mid x^{k}\right]\right\rangle \leq 2\gamma \|x^{k} - x^{*}\| \cdot \left\|\mathbb{E}\left[\widehat{g}^{k} - \overline{g}^{k} \mid x^{k}\right]\right\|$$

$$\leq 2\gamma \|x^{k} - x^{*}\|\mathbb{E}\left[\|\widehat{g}^{k} - \overline{g}^{k}\| \mid x^{k}\right]$$

$$\leq 2\gamma \|x^{k} - x^{*}\| \sqrt{\mathbb{E}\left[\|\widehat{g}^{k} - \overline{g}^{k}\|^{2} \mid x^{k}\right]}$$

$$\leq 2\gamma \|x^{k} - x^{*}\| \sqrt{\widehat{\delta}_{k}(C_{1}\lambda^{\frac{4-\alpha}{2}}G^{\frac{\alpha}{2}} + C_{2}\lambda^{2})}$$

$$= \frac{2\gamma G \|x^{k} - x^{*}\| \sqrt{C_{1}K^{\frac{4-\alpha}{2\alpha}} + C_{2}K^{\frac{2}{\alpha}}}}{\sqrt{n_{k} - m}} \sqrt{\widehat{b}_{k}}$$

$$\leq \frac{2\gamma G \|x^{k} - x^{*}\| \sqrt{20(C_{1}K^{\frac{4-\alpha}{2\alpha}} + C_{2}K^{\frac{2}{\alpha}})}}{\sqrt{7n_{k}}} \sqrt{\widehat{b}_{k}},$$

where in the last inequality we use $b \le 0.15(n-m)$, $m \le (n-2b)/2$, $\gamma \le 1/4L$, $n_k-m \ge n-2b-m \ge \frac{7}{20}n$. Putting all together we obtain

$$\mathbb{E}\left[\|x^{k+1} - x^*\|^2 \mid x^k\right] \leq \|x^k - x^*\|^2 - 2\gamma \left(f(x^k) - f(x^*)\right) + 2\gamma^2 G^2 K^{\frac{2-\alpha}{\alpha}} + \frac{2\gamma G \|x^k - x^*\| \sqrt{20(C_1 K^{\frac{4-\alpha}{2\alpha}} + C_2 K^{\frac{2}{\alpha}})}}{\sqrt{7n}} \sqrt{\widehat{b}_k} + \frac{40\gamma^2 G^2 (C_1 K^{\frac{4-\alpha}{2\alpha}} + C_2 K^{\frac{2}{\alpha}})}{7n} \widehat{b}_k.$$

Taking the full expectation from the both sides of the above inequality and summing up the results for $k = 0, 1, \dots, T-1$ we derive

$$\begin{split} \frac{2\gamma}{T} \sum_{k=0}^{T-1} \mathbb{E}[f(x^k) - f(x^*)] & \leq & \frac{1}{T} \sum_{k=0}^{T-1} \left(\mathbb{E}\left[\|x^k - x^*\|^2 \right] - \mathbb{E}\left[\|x^{k+1} - x^*\|^2 \right] \right) + 2\gamma^2 G^2 K^{\frac{2-\alpha}{\alpha}} \\ & + \frac{4\gamma G \sqrt{5(C_1 K^{\frac{4-\alpha}{2\alpha}} + C_2 K^{\frac{2}{\alpha}})}}{\sqrt{n} T} \sum_{k=0}^{T-1} \mathbb{E}\left[\|x^k - x^*\| \sqrt{\hat{b}_k} \right] \\ & + \frac{40\gamma^2 G^2 (C_1 K^{\frac{4-\alpha}{2\alpha}} + C_2 K^{\frac{2}{\alpha}})}{7nT} \sum_{k=0}^{T-1} \mathbb{E}[\hat{b}_k] \\ & \leq & \frac{\|x^0 - x^*\|^2 - \mathbb{E}[\|x^K - x^*\|^2]}{K} + 2\gamma^2 G^2 K^{\frac{2-\alpha}{\alpha}} \\ & + \frac{4\gamma G \sqrt{5(C_1 K^{\frac{4-\alpha}{2\alpha}} + C_2 K^{\frac{2}{\alpha}})}}{\sqrt{n} T} \sum_{k=0}^{T-1} \sqrt{\mathbb{E}\left[\|x^k - x^*\|^2 \right] \mathbb{E}\left[\hat{b}_k\right]} \\ & + \frac{40\gamma^2 G^2 (C_1 K^{\frac{4-\alpha}{2\alpha}} + C_2 K^{\frac{2}{\alpha}})}{7nT} \sum_{k=0}^{T-1} \mathbb{E}[\hat{b}_k]. \end{split}$$

From Jensen's inequality we have $f(\overline{x}^T) \leq \frac{1}{T} \sum_{k=0}^{T-1} f(x^k)$, where $\overline{x}^T = \frac{1}{T} \sum_{k=0}^{T-1} x^k$. Using this and new notation $R_k = \|x^k - x^*\|$, k > 0, $R_0 \geq \|x^0 - x^*\|$ we get

$$0 \leq 2\gamma \mathbb{E}\left[f(\overline{x}^{T}) - f(x^{*})\right] \leq \frac{R_{0}^{2} - \mathbb{E}[R_{T}^{2}]}{T} + 2\gamma^{2}G^{2}K^{\frac{2-\alpha}{\alpha}} + \frac{4\gamma G\sqrt{5(C_{1}K^{\frac{4-\alpha}{2\alpha}} + C_{2}K^{\frac{2}{\alpha}})}}{\sqrt{n}T} \sum_{k=0}^{T-1} \sqrt{\mathbb{E}\left[R_{k}^{2}\right]\mathbb{E}\left[\widehat{b}_{k}\right]} + \frac{40\gamma^{2}G^{2}(C_{1}K^{\frac{4-\alpha}{2\alpha}} + C_{2}K^{\frac{2}{\alpha}})}{7nT} \sum_{k=0}^{T-1} \mathbb{E}[\widehat{b}_{k}]$$
 (52)

implying (after changing the indices) that

$$\mathbb{E}[R_{k}^{2}] \leq R_{0}^{2} + 2\gamma^{2}G^{2}kK^{\frac{2-\alpha}{\alpha}} + \frac{4\gamma G\sqrt{5(C_{1}K^{\frac{4-\alpha}{2\alpha}} + C_{2}K^{\frac{2}{\alpha}})}}{\sqrt{n}}\sum_{l=0}^{k-1}\sqrt{\mathbb{E}[R_{l}^{2}]\mathbb{E}\left[\hat{b}_{l}\right]} + \frac{40\gamma^{2}G^{2}(C_{1}K^{\frac{4-\alpha}{2\alpha}} + C_{2}K^{\frac{2}{\alpha}})}{7n}\sum_{l=0}^{k-1}\mathbb{E}[\hat{b}_{l}]$$
(53)

holds for all $k \ge 0$. In the remaining part of the proof we derive by induction that

$$R_{0}^{2} + 2\gamma^{2}G^{2}kK^{\frac{2-\alpha}{\alpha}} + \frac{4\gamma G\sqrt{5(C_{1}K^{\frac{4-\alpha}{2\alpha}} + C_{2}K^{\frac{2}{\alpha}})}}{\sqrt{n}} \sum_{l=0}^{k-1} \sqrt{\mathbb{E}\left[R_{l}^{2}\right]\mathbb{E}\left[\widehat{b}_{l}\right]} + \frac{40\gamma^{2}G^{2}(C_{1}K^{\frac{4-\alpha}{2\alpha}} + C_{2}K^{\frac{2}{\alpha}})}{7n} \sum_{l=0}^{k-1} \mathbb{E}\left[\widehat{b}_{l}\right] \leq 2R_{0}^{2} \quad (54)$$

for all $k=0,\ldots,K$. For k=0 this inequality trivially holds. Next, assume that it holds for all $k=0,1,\ldots,T-1,T\leq K-1$. Let us show that it holds for k=T as well. From (28) and (29) we have that $\mathbb{E}[R_k^2]\leq 2R_0^2$ for all $k=0,1,\ldots,T-1$. Therefore,

$$\mathbb{E}[R_T^2] \leq R_0^2 + 2\gamma^2 G^2 T K^{\frac{2-\alpha}{\alpha}} + \frac{4\gamma G \sqrt{5(C_1 K^{\frac{4-\alpha}{2\alpha}} + C_2 K^{\frac{2}{\alpha}})}}{\sqrt{n}} \sum_{l=0}^{T-1} \sqrt{\mathbb{E}[R_l^2] \mathbb{E}\left[\hat{b}_l\right]} + \frac{40\gamma^2 G^2 (C_1 K^{\frac{4-\alpha}{2\alpha}} + C_2 K^{\frac{2}{\alpha}})}{7n} \sum_{l=0}^{T-1} \mathbb{E}[\hat{b}_l]$$

$$\leq R_0^2 + 2\gamma^2 G^2 T K^{\frac{2-\alpha}{\alpha}} + \frac{4\gamma G R_0 \sqrt{10(C_1 K^{\frac{4-\alpha}{2\alpha}} + C_2 K^{\frac{2}{\alpha}})}}{\sqrt{n}} \sum_{l=0}^{T-1} \sqrt{\mathbb{E}\left[\hat{b}_l\right]} + \frac{40\gamma^2 G^2 (C_1 K^{\frac{4-\alpha}{2\alpha}} + C_2 K^{\frac{2}{\alpha}})}{7n} \sum_{l=0}^{T-1} \mathbb{E}[\hat{b}_l]$$

If a Byzantine peer deviates from the protocol at iteration k, it will be detected with some probability p_k during the next iteration. One can lower bound this probability as

$$p_k \ge m \cdot \frac{|\mathcal{G}_k|}{n_k} \cdot \frac{1}{n_k} = \frac{m(1 - \delta_k)}{n_k} \ge \frac{m}{n}$$

Therefore, each individual Byzantine worker can violate the protocol no more than 1/p times on average implying that

$$\mathbb{E}[R_T^2] \leq R_0^2 + 2\gamma^2 G^2 T K^{\frac{2-\alpha}{\alpha}} + \frac{4\gamma G R_0 n \sqrt{10(C_1 K^{\frac{4-\alpha}{2\alpha}} + C_2 K^{\frac{2}{\alpha}})b}}{m\sqrt{n}} + \frac{40\gamma^2 G^2 (C_1 K^{\frac{4-\alpha}{2\alpha}} + C_2 K^{\frac{2}{\alpha}})nb}{7nm}$$

$$\stackrel{T \leq K}{\leq} R_0^2 + 2\gamma^2 G^2 K^{\frac{2}{\alpha}} + \frac{4\gamma G R_0 n \sqrt{10(C_1 K^{\frac{4-\alpha}{2\alpha}} + C_2 K^{\frac{2}{\alpha}})\delta}}{m} + \frac{40\gamma^2 G^2 (C_1 K^{\frac{4-\alpha}{2\alpha}} + C_2 K^{\frac{2}{\alpha}})n\delta}{7m}.$$

Taking

$$\gamma = \min \left\{ \frac{R_0}{\sqrt{6}GK^{\frac{1}{\alpha}}}, \frac{mR_0}{12Gn\sqrt{10\delta(C_1K^{\frac{4-\alpha}{2\alpha}} + C_2K^{\frac{2}{\alpha}})}} \right\}$$

we ensure that

$$\begin{split} 2\gamma^2 G^2 K^{\frac{2}{\alpha}} + \frac{4\gamma G R_0 n \sqrt{10 (C_1 K^{\frac{4-\alpha}{2\alpha}} + C_2 K^{\frac{2}{\alpha}}) \delta}}{m} \\ + \frac{40\gamma^2 G^2 (C_1 K^{\frac{4-\alpha}{2\alpha}} + C_2 K^{\frac{2}{\alpha}}) n \delta}{7m} & \leq & \frac{R_0^2}{3} + \frac{R_0^2}{3} + \frac{R_0^2}{3} = R_0^2 \end{split}$$

and, as a result, we get $\mathbb{E}[R_T^2] \leq 2R_0^2$. Therefore, (54) holds for all $k = 0, 1, \dots, K$. Together with (52) it implies

$$\mathbb{E}\left[f(\overline{x}^K) - f(x^*)\right] \le \frac{R_0^2}{\gamma K}.$$

Next, from our stepsize rule (49) it follows that

$$\mathbb{E}\left[f(\overline{x}^K) - f(x^*)\right] = \mathcal{O}\left(\frac{GR_0}{K^{\frac{1-\alpha}{\alpha}}} + \frac{n\sqrt{\delta}GR_0}{mK^{\frac{1-\alpha}{\alpha}}}\right)$$

meaning that after

$$K = \mathcal{O}\left(\left(\frac{GR_0}{\varepsilon}\right)^{\frac{\alpha}{\alpha-1}} + \left(\frac{n\sqrt{\delta}GR_0}{m\varepsilon}\right)^{\frac{\alpha}{\alpha-1}}\right)$$

iterations BTARD-CLIPPED-SGD guarantees $\mathbb{E}[f(\overline{x}^K) - f(x^*)] \leq \varepsilon$.

First of all, when there are no Byzantine peers ($\delta=0$) the theorem establishes new result for the convergence of CLIPPED-SGD for convex objectives, and in the strongly convex case the theorem recovers the rates from [81] that are optimal in this setting [81]. Next, when the number of attacking Byzantines is known at each iteration and $n\sqrt{\delta}/m=\mathcal{O}(1)$ the complexity bound is the same as in the case when $\delta=0$ meaning that the negative impact of Byzantine workers is negligible. Finally, the derived theoretical guarantees do not benefit from the increase of the total number of peers n. However, the result holds even for non-smooth problems and this is known that parallelization does not help to improve the complexity bounds in such generality. Nevertheless, our results show that BTARD-CLIPPED-SGD provably converges to any predefined accuracy $\varepsilon>0$ – the property that the majority of previous methods does not have [58].

Next, we derive the result without assuming that \hat{b}^k is known to all peers at each iteration.

Theorem F.8. Let As. F.1 hold, Q is bounded, f be convex, x^* be some optimum of f, and $\nabla f(x^*) = 0$. Moreover, assume that $b \leq 0.15(n-m)$, $m \leq (n-2b)/2$, and $\delta = 0$ is used to compute clipping parameter τ_l for CenteredClip. Next, assume that

$$\gamma = \min \left\{ \frac{R_0}{\sqrt{6}GK^{\frac{1}{\alpha}}}, \frac{mR_0}{12\sqrt{2C_2}GnbK^{\frac{1}{\alpha}}} \right\}, \quad \Delta_{\max}^k = 2\lambda_k = \frac{2\lambda}{\sqrt{n_k - m}}, \tag{55}$$

$$\lambda = GK^{\frac{1}{\alpha}},\tag{56}$$

where $R_0 \geq \|x^0 - x^*\|$ and Δ_{\max}^k is the parameter for verification 3 at iteration k of BTARD-CLIPPED-SGD. Then, we have $\mathbb{E}[f(\overline{x}^K) - f(x^*)] \leq \varepsilon$ after K iterations of BTARD-CLIPPED-SGD, where

$$K = \mathcal{O}\left(\left(\frac{GR_0}{\varepsilon}\right)^{\frac{\alpha}{\alpha-1}} + \left(\frac{nbGR_0}{m\varepsilon}\right)^{\frac{\alpha}{\alpha-1}}\right)$$
 (57)

and $\overline{x}^K = \frac{1}{K} \sum_{k=0}^{K-1}$.

Proof. The proof is almost identical to the proof of Theorem F.7. Following the same steps and using (47) and (48) instead of (45) and (46) respectively we obtain the same sequence of inequalities up to the following change: instead of $\hat{\delta}_k$ we should use $\mathbb{1}_{k,v}$. Therefore, we have

$$\mathbb{E}\left[\|x^{k+1} - x^*\|^2 \mid x^k\right] \leq \|x^k - x^*\|^2 - 2\gamma \left(f(x^k) - f(x^*)\right) + 2\gamma^2 G^2 K^{\frac{2-\alpha}{\alpha}} - 2\gamma \left\langle x^k - x^*, \mathbb{E}\left[\widehat{g}^k - \overline{g}^k \mid x^k\right]\right\rangle + 2\gamma^2 C_2 G^2 K^{\frac{2}{\alpha}} \mathbb{1}_{k,v}.$$

$$-2\gamma \left\langle x^k - x^*, \mathbb{E}\left[\widehat{g}^k - \overline{g}^k \mid x^k\right]\right\rangle \leq 2\gamma G \|x^k - x^*\| \sqrt{C_2} K^{\frac{1}{\alpha}} \mathbb{1}_{k,v},$$

and

$$\mathbb{E}\left[\|x^{k+1} - x^*\|^2 \mid x^k\right] \leq \|x^k - x^*\|^2 - 2\gamma \left(f(x^k) - f(x^*)\right) + 2\gamma^2 G^2 K^{\frac{2-\alpha}{\alpha}} \\ + 2\gamma G\sqrt{C_2} K^{\frac{1}{\alpha}} \|x^k - x^*\| \mathbbm{1}_{k,v} + 2\gamma^2 C_2 G^2 K^{\frac{2}{\alpha}} \mathbbm{1}_{k,v}.$$

Taking the full expectation from the both sides of the above inequality and summing up the results for k = 0, 1, ..., T - 1 we derive

$$\begin{split} \frac{2\gamma}{T} \sum_{k=0}^{T-1} \mathbb{E}[f(x^k) - f(x^*)] & \leq & \frac{1}{T} \sum_{k=0}^{T-1} \left(\mathbb{E}\left[\|x^k - x^*\|^2 \right] - \mathbb{E}\left[\|x^{k+1} - x^*\|^2 \right] \right) + 2\gamma^2 G^2 K^{\frac{2-\alpha}{\alpha}} \\ & + \frac{2\gamma G \sqrt{C_2} K^{\frac{1}{\alpha}}}{T} \sum_{k=0}^{T-1} \mathbb{E}\left[\|x^k - x^*\| \mathbbm{1}_{k,v} \right] + \frac{2\gamma^2 C_2 G^2 K^{\frac{2}{\alpha}}}{T} \sum_{k=0}^{T-1} \mathbb{E}[\mathbbm{1}_{k,v}] \\ & \leq & \frac{\|x^0 - x^*\|^2 - \mathbb{E}[\|x^K - x^*\|^2]}{K} + 2\gamma^2 G^2 K^{\frac{2-\alpha}{\alpha}} \\ & + \frac{2\gamma G \sqrt{C_2} K^{\frac{1}{\alpha}}}{T} \sum_{k=0}^{T-1} \sqrt{\mathbb{E}\left[\|x^k - x^*\|^2 \right] \mathbb{E}\left[\mathbbm{1}_{k,v} \right]} \\ & + \frac{2\gamma^2 C_2 G^2 K^{\frac{2}{\alpha}}}{T} \sum_{k=0}^{T-1} \mathbb{E}[\mathbbm{1}_{k,v}]. \end{split}$$

From Jensen's inequality we have $f(\overline{x}^T) \leq \frac{1}{T} \sum_{k=0}^{T-1} f(x^k)$, where $\overline{x}^T = \frac{1}{T} \sum_{k=0}^{T-1} x^k$. Using this and new notation $R_k = \|x^k - x^*\|$, k > 0, $R_0 \geq \|x^0 - x^*\|$ we get

$$0 \leq 2\gamma \mathbb{E}\left[f(\overline{x}^{T}) - f(x^{*})\right] \leq \frac{R_{0}^{2} - \mathbb{E}[R_{T}^{2}]}{T} + 2\gamma^{2}G^{2}K^{\frac{2-\alpha}{\alpha}} + \frac{2\gamma G\sqrt{C_{2}}K^{\frac{1}{\alpha}}}{T} \sum_{k=0}^{T-1} \sqrt{\mathbb{E}\left[R_{k}^{2}\right]\mathbb{E}\left[\mathbb{1}_{k,v}\right]} + \frac{2\gamma^{2}C_{2}G^{2}K^{\frac{2}{\alpha}}}{T} \sum_{k=0}^{T-1} \mathbb{E}[\mathbb{1}_{k,v}]$$

$$(58)$$

implying (after changing the indices) that

$$\mathbb{E}[R_k^2] \leq R_0^2 + 2\gamma^2 G^2 k K^{\frac{2-\alpha}{\alpha}} + 2\gamma G \sqrt{C_2} K^{\frac{1}{\alpha}} \sum_{l=0}^{k-1} \sqrt{\mathbb{E}[R_l^2] \mathbb{E}[\mathbb{1}_{l,v}]}$$

$$+ 2\gamma^2 C_2 G^2 K^{\frac{2}{\alpha}} \sum_{l=0}^{k-1} \mathbb{E}[\mathbb{1}_{l,v}]$$
(59)

holds for all $k \ge 0$. In the remaining part of the proof we derive by induction that

$$R_0^2 + 2\gamma^2 G^2 k K^{\frac{2-\alpha}{\alpha}} + 2\gamma G \sqrt{C_2} K^{\frac{1}{\alpha}} \sum_{l=0}^{k-1} \sqrt{\mathbb{E}[R_l^2] \mathbb{E}[\mathbb{1}_{l,v}]}$$

$$+ 2\gamma^2 C_2 G^2 K^{\frac{2}{\alpha}} \sum_{l=0}^{k-1} \mathbb{E}[\mathbb{1}_{l,v}] \leq 2R_0^2$$
(60)

for all $k=0,\ldots,K$. For k=0 this inequality trivially holds. Next, assume that it holds for all $k=0,1,\ldots,T-1,T\leq K-1$. Let us show that it holds for k=T as well. From (33) and (34)

we have that $\mathbb{E}[R_k^2] \leq 2R_0^2$ for all $k = 0, 1, \dots, T - 1$. Therefore,

$$\mathbb{E}[R_T^2] \leq R_0^2 + 2\gamma^2 G^2 T K^{\frac{2-\alpha}{\alpha}} + 2\gamma G \sqrt{C_2} K^{\frac{1}{\alpha}} \sum_{l=0}^{T-1} \sqrt{\mathbb{E}[R_l^2]} \mathbb{E}[\mathbb{1}_{l,v}]$$

$$+ 2\gamma^2 C_2 G^2 K^{\frac{2}{\alpha}} \sum_{l=0}^{T-1} \mathbb{E}[\mathbb{1}_{l,v}]$$

$$\leq R_0^2 + 2\gamma^2 G^2 T K^{\frac{2-\alpha}{\alpha}} + 2\gamma G R_0 \sqrt{2C_2} K^{\frac{1}{\alpha}} \sum_{l=0}^{T-1} \sqrt{\mathbb{E}[\mathbb{1}_{l,v}]}$$

$$+ 2\gamma^2 C_2 G^2 K^{\frac{2}{\alpha}} \sum_{l=0}^{T-1} \mathbb{E}[\mathbb{1}_{l,v}]$$

If a Byzantine peer deviates from the protocol at iteration k, it will be detected with some probability p_k during the next iteration. One can lower bound this probability as

$$p_k \ge m \cdot \frac{|\mathcal{G}_k|}{n_k} \cdot \frac{1}{n_k} = \frac{m(1 - \delta_k)}{n_k} \ge \frac{m}{n}.$$

That is, each individual Byzantine worker can violate the protocol no more than $^1/_p$ times on average. However, even one Byzantine peer can create a shift of the order Δ^k_{\max} at each part of the resulting vector. Therefore, all Byzantine peers can violate the protocol no more than $^b/_p$ times on average implying that

$$\mathbb{E}[R_T^2] \leq R_0^2 + 2\gamma^2 G^2 T K^{\frac{2-\alpha}{\alpha}} + \frac{2\gamma G R_0 \sqrt{2C_2} K^{\frac{1}{\alpha}} nb}{m} + \frac{2\gamma^2 C_2 G^2 K^{\frac{2}{\alpha}} nb}{m}.$$

Taking

$$\gamma = \min \left\{ \frac{R_0}{\sqrt{6}GK^{\frac{1}{\alpha}}}, \frac{mR_0}{12\sqrt{2C_2}GnbK^{\frac{1}{\alpha}}} \right\}$$

we ensure that

$$2\gamma^2 G^2 T K^{\frac{2-\alpha}{\alpha}} + \frac{2\gamma G R_0 \sqrt{2C_2} K^{\frac{1}{\alpha}} nb}{m} + \frac{2\gamma^2 C_2 G^2 K^{\frac{2}{\alpha}} nb}{m} \leq \frac{R_0^2}{3} + \frac{R_0^2}{3} + \frac{R_0^2}{3} = R_0^2$$

and, as a result, we get $\mathbb{E}[R_T^2] \leq 2R_0^2$. Therefore, (60) holds for all $k = 0, 1, \dots, K$. Together with (58) it implies

$$\mathbb{E}\left[f(\overline{x}^K) - f(x^*)\right] \le \frac{R_0^2}{\gamma K}$$

Next, from our stepsize rule (55) it follows that

$$\mathbb{E}\left[f(\overline{x}^K) - f(x^*)\right] = \mathcal{O}\left(\frac{GR_0}{K^{\frac{1-\alpha}{\alpha}}} + \frac{nbGR_0}{mK^{\frac{1-\alpha}{\alpha}}}\right)$$

meaning that after

$$K = \mathcal{O}\left(\left(\frac{GR_0}{\varepsilon}\right)^{\frac{\alpha}{\alpha-1}} + \left(\frac{nbGR_0}{m\varepsilon}\right)^{\frac{\alpha}{\alpha-1}}\right)$$

iterations BTARD-CLIPPED-SGD guarantees $\mathbb{E}[f(\overline{x}^K) - f(x^*)] \leq \varepsilon$.

That is, when the number of attacking Byzantines is unknown the complexity bound becomes $\binom{nb/m}{\alpha}^{\alpha/(\alpha-1)}$ times worse in comparison to (51).

F.3.3 Strongly convex case: Restarted-BTARD-Clipped-SGD

In this section, we provide the complete statements and the full proofs of the convergence results for the restarted version of BTARD-CLIPPED-SGD (RESTARTED-BTARD-CLIPPED-SGD, Alg. 7) when the objective function f is smooth and strongly convex.

Algorithm 9 RESTARTED-BTARD-CLIPPED-SGD

Input: x^0 – starting point, r – number of restarts, $\{\gamma_t\}_{t=1}^r$ – stepsizes for BTARD-CLIPPED-SGD, $\{K_t\}_{t=1}^r$ – number of iterations for BTARD-CLIPPED-SGD, $\{s_{i,k,t}\}_{i,k,t=0,0,1}^{n,K-1,r}$ – seeds for batches computations, $\{\lambda_{k,t}\}_{k,t=0,1}^{K_t,r}$ – gradient clipping parameters

1: $\hat{x}^0 = x^0$

2: **for** t = 1, 2, ..., r **do**

3: Run BTARD-CLIPPED-SGD (Alg. 8) for K_t iterations with stepsize γ_t , starting point \widehat{x}^{t-1} , gradient clipping parameters $\{\lambda_{k,t}\}_{k=0}^{K-1}$, and seeds for batches computations $\{s_{i,k,t}\}_{i,k=0,0}^{n,K-1}$.

Define \widehat{x}^t as $\widehat{x}^t = \frac{1}{N} \sum_{k=0}^{K_t} x^{k,t}$ where $x^{0,t}$ $x^{1,t}$ are the iterates produced by BTARD.

Define \widehat{x}^t as $\widehat{x}^t = \frac{1}{K_t} \sum_{k=0}^{K_t} x^{k,t}$, where $x^{0,t}, x^{1,t}, \dots, x^{K_t,t}$ are the iterates produced by BTARD-CLIPPED-SGD.

Output: \hat{x}^r

We start with the case when the number of attacking Byzantine workers is known at each iteration.

Theorem F.9. Let As. F.1 hold, Q is bounded, f be μ -strongly convex (see Def. F.2), x^* be some optimum of f, and $\nabla f(x^*) = 0$. Moreover, assume that $b \leq 0.15(n-m)$, $m \leq \frac{(n-2b)}{2}$, and the exact number of attacking Byzantine peers is known to all good peers at each iteration. Next, assume that

$$\gamma = \min \left\{ \frac{R_0}{\sqrt{6} \cdot 2^{\frac{t}{2}} G K_t^{\frac{1}{\alpha}}}, \frac{m R_0}{12 \cdot 2^{\frac{t}{2}} G n \sqrt{10 \delta (C_1 K_t^{\frac{4-\alpha}{2\alpha}} + C_2 K_t^{\frac{2}{\alpha}})}} \right\}, \quad \Delta_{\max}^{k,t} = 2\lambda_{k,t} = \frac{2\lambda_t}{\sqrt{n_k^t - m}},$$
(61)

$$K_{t} = \max \left\{ \left(\frac{2\sqrt{6}G \cdot 2^{\frac{t}{2}}}{\mu R_{0}} \right)^{\frac{\alpha}{\alpha - 1}}, \left(\frac{24Gn\sqrt{10\delta(C_{1} + C_{2})}2^{\frac{t}{2}}}{m\mu R_{0}} \right)^{\frac{\alpha}{\alpha - 1}} \right\}, \quad \lambda_{t} = GK_{t}^{\frac{1}{\alpha}}, \quad (62)$$

$$r = \left\lceil \log_2 \frac{\mu R_0^2}{\varepsilon} \right\rceil - 1,\tag{63}$$

where $R_0 \geq \|x^0 - x^*\|$ and $\Delta_{\max}^{k,t}$ is the parameter for verification 3 at iteration k of BTARD-CLIPPED-SGD, n_k^t is the total number of workers at iteration k of t-th restart. Then, we have $\mathbb{E}[f(\hat{x}^r) - f(x^*)] \leq \varepsilon$ after r restarts of BTARD-CLIPPED-SGD and the total number of executed iterations of BTARD-CLIPPED-SGD is

$$\sum_{t=1}^{r} K_{t} = \mathcal{O}\left(\left(\frac{G^{2}}{\mu\varepsilon}\right)^{\frac{\alpha}{2(\alpha-1)}} + \left(\frac{n\sqrt{\delta}}{m}\right)^{\frac{\alpha}{\alpha-1}} \left(\frac{G^{2}}{\mu\varepsilon}\right)^{\frac{\alpha}{2(\alpha-1)}}\right)$$
(64)

Proof. Theorem F.7 implies that BTARD-CLIPPED-SGD with

$$\gamma = \min \left\{ \frac{R_0}{\sqrt{6}GK^{\frac{1}{\alpha}}}, \frac{mR_0}{12Gn\sqrt{10\delta(C_1K^{\frac{4-\alpha}{2\alpha}} + C_2K^{\frac{2}{\alpha}})}} \right\}$$

guarantees

$$\mathbb{E}\left[f(\overline{x}^K) - f(x^*)\right] \le \frac{R_0^2}{\gamma K}$$

after K iterations. Therefore, after the first restart we have

$$\mathbb{E}[f(\widehat{x}^1) - f(x^*)] \le \frac{R_0^2}{\gamma_1 K_1} \le \frac{\mu R_0^2}{4}.$$

From μ -strong convexity of f and $\nabla f(x^*) = 0$ we have

$$\frac{\mu}{2} \|\widehat{x}^1 - x^*\|^2 \le f(\widehat{x}^1) - f(x^*) \Longrightarrow \mathbb{E}[\|\widehat{x}^1 - x^*\|^2] \le \frac{R_0^2}{2}.$$

Next, assume that we have $\mathbb{E}[f(\widehat{x}^t) - f(x^*)] \leq \frac{\mu R_0^2}{2^{t+1}}$, $\mathbb{E}[\|\widehat{x}^t - x^*\|^2] \leq \frac{R_0^2}{2^t}$ for some $t \leq r-1$. Then, Theorem F.7 implies that

$$\mathbb{E}[f(\widehat{x}^{t+1}) - f(x^*) \mid x^t] \le \frac{\|\widehat{x}^t - x^*\|^2}{\gamma_t K_t}.$$

Taking the full expectation from the both sides of previous inequality we get

$$\mathbb{E}[f(\widehat{x}^{t+1}) - f(x^*)] \le \frac{\mathbb{E}[\|\widehat{x}^t - x^*\|^2]}{\gamma_t K_t} \le \frac{R_0^2}{2^t \gamma_t K_t} \le \frac{\mu R_0^2}{2^{t+2}}$$

From μ -strong convexity of f and $\nabla f(x^*) = 0$ we have

$$\frac{\mu}{2} \|\widehat{x}^{t+1} - x^*\|^2 \le f(\widehat{x}^{t+1}) - f(x^*) \Longrightarrow \mathbb{E}[\|\widehat{x}^{t+1} - x^*\|^2] \le \frac{R_0^2}{2^{t+1}}.$$

Therefore, by mathematical induction we have that for all t = 1, ..., r

$$\mathbb{E}[f(\widehat{x}^t) - f(x^*)] \le \frac{\mu R_0^2}{2^{t+1}}, \quad \mathbb{E}\left[\|\widehat{x}^t - x^*\|^2\right] \le \frac{R_0^2}{2^t}.$$

Then, after $r = \left\lceil \log_2 \frac{\mu R_0^2}{\varepsilon} \right\rceil - 1$ restarts of BTARD-CLIPPED-SGD we have $\mathbb{E}[f(\widehat{x}^r) - f(x^*)] \leq \varepsilon$. The total number of iterations executed by BTARD-CLIPPED-SGD is

$$\begin{split} \sum_{t=1}^{r} K_{t} &= \mathcal{O}\left(\sum_{t=1}^{r} \max\left\{\left(\frac{G \cdot 2^{\frac{t}{2}}}{\mu R_{0}}\right)^{\frac{\alpha}{\alpha-1}}, \left(\frac{Gn\sqrt{\delta}2^{\frac{t}{2}}}{m\mu R_{0}}\right)^{\frac{\alpha}{\alpha-1}}\right\}\right) \\ &= \mathcal{O}\left(\max\left\{\left(\frac{G}{\mu R_{0}}\right)^{\frac{\alpha}{\alpha-1}} \cdot 2^{\frac{r\alpha}{2(\alpha-1)}}, \left(\frac{Gn\sqrt{\delta}}{m\mu R_{0}}\right)^{\frac{\alpha}{\alpha-1}} \cdot 2^{\frac{r\alpha}{2(\alpha-1)}}\right\}\right) \\ &= \mathcal{O}\left(\max\left\{\left(\frac{G}{\mu R_{0}}\right)^{\frac{\alpha}{\alpha-1}} \cdot \left(\frac{\mu R_{0}^{2}}{\varepsilon}\right)^{\frac{\alpha}{2(\alpha-1)}}, \left(\frac{Gn\sqrt{\delta}}{m\mu R_{0}}\right)^{\frac{\alpha}{\alpha-1}} \cdot \left(\frac{\mu R_{0}^{2}}{\varepsilon}\right)^{\frac{\alpha}{2(\alpha-1)}}\right\}\right) \\ &= \mathcal{O}\left(\left(\frac{G^{2}}{\mu \varepsilon}\right)^{\frac{\alpha}{2(\alpha-1)}} + \left(\frac{n\sqrt{\delta}}{m}\right)^{\frac{\alpha}{\alpha-1}} \left(\frac{G^{2}}{\mu \varepsilon}\right)^{\frac{\alpha}{2(\alpha-1)}}\right). \end{split}$$

In the strongly convex case, similar observations hold as in the convex case. Next, we derive the result without assuming that \hat{b}^k is known to all peers at each iteration.

Theorem F.10. Let As. F.1 hold, Q is bounded, f be μ -strongly convex (see Def. F.2), x^* be some optimum of f, and $\nabla f(x^*) = 0$. Moreover, assume that $b \leq 0.15(n-m)$, $m \leq (n-2b)/2$, and $\delta = 0$ is used to compute clipping parameter τ_l for CenteredClip. Next, assume that

$$\gamma = \min \left\{ \frac{R_0}{\sqrt{6} \cdot 2^{\frac{t}{2}} G K_t^{\frac{1}{\alpha}}}, \frac{m R_0}{12 \cdot 2^{\frac{t}{2}} G n b \sqrt{2C_2} K_t^{\frac{1}{\alpha}}} \right\}, \quad \Delta_{\max}^{k,t} = 2\lambda_{k,t} = \frac{2\lambda_t}{\sqrt{n_k^t - m}}, \quad (65)$$

$$K_t = \max \left\{ \left(\frac{2\sqrt{6}G \cdot 2^{\frac{t}{2}}}{\mu R_0} \right)^{\frac{\alpha}{\alpha - 1}}, \left(\frac{24Gnb\sqrt{2C_2}2^{\frac{t}{2}}}{m\mu R_0} \right)^{\frac{\alpha}{\alpha - 1}} \right\}, \quad \lambda_t = GK_t^{\frac{1}{\alpha}}, \tag{66}$$

$$r = \left\lceil \log_2 \frac{\mu R_0^2}{\varepsilon} \right\rceil - 1,\tag{67}$$

where $R_0 \geq \|x^0 - x^*\|$ and $\Delta_{\max}^{k,t}$ is the parameter for verification 3 at iteration k of BTARD-CLIPPED-SGD, n_k^t is the total number of workers at iteration k of t-th restart. Then, we have $\mathbb{E}[f(\hat{x}^r) - f(x^*)] \leq \varepsilon$ after r restarts of BTARD-CLIPPED-SGD and the total number of executed iterations of BTARD-CLIPPED-SGD is

$$\sum_{t=1}^{r} K_{t} = \mathcal{O}\left(\left(\frac{G^{2}}{\mu\varepsilon}\right)^{\frac{\alpha}{2(\alpha-1)}} + \left(\frac{nb}{m}\right)^{\frac{\alpha}{\alpha-1}} \left(\frac{G^{2}}{\mu\varepsilon}\right)^{\frac{\alpha}{2(\alpha-1)}}\right)$$
(68)

Proof. Theorem F.8 implies that BTARD-CLIPPED-SGD with

$$\gamma = \min \left\{ \frac{R_0}{\sqrt{6}GK^{\frac{1}{\alpha}}}, \frac{mR_0}{12\sqrt{2C_2}GnbK^{\frac{1}{\alpha}}} \right\}$$

guarantees

$$\mathbb{E}\left[f(\overline{x}^K) - f(x^*)\right] \le \frac{R_0^2}{\gamma K}$$

after K iterations. Therefore, after the first restart we have

$$\mathbb{E}[f(\widehat{x}^1) - f(x^*)] \le \frac{R_0^2}{\gamma_1 K_1} \le \frac{\mu R_0^2}{4}.$$

From μ -strong convexity of f and $\nabla f(x^*) = 0$ we have

$$\frac{\mu}{2} \|\widehat{x}^1 - x^*\|^2 \le f(\widehat{x}^1) - f(x^*) \Longrightarrow \mathbb{E}[\|\widehat{x}^1 - x^*\|^2] \le \frac{R_0^2}{2}.$$

Next, assume that we have $\mathbb{E}[f(\widehat{x}^t)-f(x^*)] \leq \frac{\mu R_0^2}{2^{t+1}}, \, \mathbb{E}[\|\widehat{x}^t-x^*\|^2] \leq \frac{R_0^2}{2^t}$ for some $t \leq r-1$. Then, Theorem F.8 implies that

$$\mathbb{E}[f(\widehat{x}^{t+1}) - f(x^*) \mid x^t] \le \frac{\|\widehat{x}^t - x^*\|^2}{\gamma_t K_t}.$$

Taking the full expectation from the both sides of previous inequality we get

$$\mathbb{E}[f(\widehat{x}^{t+1}) - f(x^*)] \le \frac{\mathbb{E}[\|\widehat{x}^t - x^*\|^2]}{\gamma_t K_t} \le \frac{R_0^2}{2^t \gamma_t K_t} \le \frac{\mu R_0^2}{2^{t+2}}.$$

From μ -strong convexity of f and $\nabla f(x^*) = 0$ we have

$$\frac{\mu}{2} \|\widehat{x}^{t+1} - x^*\|^2 \le f(\widehat{x}^{t+1}) - f(x^*) \Longrightarrow \mathbb{E}[\|\widehat{x}^{t+1} - x^*\|^2] \le \frac{R_0^2}{2^{t+1}}$$

Therefore, by mathematical induction we have that for all t = 1, ..., r

$$\mathbb{E}[f(\widehat{x}^t) - f(x^*)] \le \frac{\mu R_0^2}{2^{t+1}}, \quad \mathbb{E}\left[\|\widehat{x}^t - x^*\|^2\right] \le \frac{R_0^2}{2^t}.$$

Then, after $r = \left\lceil \log_2 \frac{\mu R_0^2}{\varepsilon} \right\rceil - 1$ restarts of BTARD-CLIPPED-SGD we have $\mathbb{E}[f(\widehat{x}^r) - f(x^*)] \leq \varepsilon$. The total number of iterations executed by BTARD-CLIPPED-SGD is

$$\begin{split} \sum_{t=1}^{r} K_{t} &= \mathcal{O}\left(\sum_{t=1}^{r} \max\left\{\left(\frac{G \cdot 2^{\frac{t}{2}}}{\mu R_{0}}\right)^{\frac{\alpha}{\alpha-1}}, \left(\frac{Gn\sqrt{\delta}2^{\frac{t}{2}}}{m\mu R_{0}}\right)^{\frac{\alpha}{\alpha-1}}\right\}\right) \\ &= \mathcal{O}\left(\max\left\{\left(\frac{G}{\mu R_{0}}\right)^{\frac{\alpha}{\alpha-1}} \cdot 2^{\frac{r\alpha}{2(\alpha-1)}}, \left(\frac{Gnb}{m\mu R_{0}}\right)^{\frac{\alpha}{\alpha-1}} \cdot 2^{\frac{r\alpha}{2(\alpha-1)}}\right\}\right) \\ &= \mathcal{O}\left(\max\left\{\left(\frac{G}{\mu R_{0}}\right)^{\frac{\alpha}{\alpha-1}} \cdot \left(\frac{\mu R_{0}^{2}}{\varepsilon}\right)^{\frac{\alpha}{2(\alpha-1)}}, \left(\frac{Gnb}{m\mu R_{0}}\right)^{\frac{\alpha}{\alpha-1}} \cdot \left(\frac{\mu R_{0}^{2}}{\varepsilon}\right)^{\frac{\alpha}{2(\alpha-1)}}\right\}\right) \\ &= \mathcal{O}\left(\left(\frac{G^{2}}{\mu\varepsilon}\right)^{\frac{\alpha}{2(\alpha-1)}} + \left(\frac{nb}{m}\right)^{\frac{\alpha}{\alpha-1}} \left(\frac{G^{2}}{\mu\varepsilon}\right)^{\frac{\alpha}{2(\alpha-1)}}\right). \end{split}$$

G Reputation system for public collaborations

In this section, we address byzantine-tolerant training in a setup where new participants can join or leave collaboration midway through training. This requirement arises naturally if a given training run relies on volunteers or an open pool of paid participants [13, 14, 15]. In addition to all existing concerns from Section 3, this new setup allows Byzantine attackers to assume new identity each time they are blocked. Further yet, Byzantine participants can simultaneously use multiple identities in order to obtain majority in the voting procedure, which is known as Sybil attacks [82, 100, 101].

For the purpose of this analysis, we consider a training run where byzantine peers collectively possess $\delta < B_m ax$ of all compute resources (we explore the role of $\Delta_{max} < \frac{1}{2}$ later in this section). Intuitively, one can think of this setting as distributed training with n identical computers, $\lfloor \delta \cdot n \rfloor$ of which are controlled by Byzantines. The "Byzantine GPUs" can be allocated between an arbitrary number of identities. For instance, one accelerator can run full BTARD-SGD protocol for one peer or drop some of the computation and use the freed "compute cycles" to run computation for another participant. Theoretically, a device can run computation for an arbitrarily large number of peers, as long as it actually computes as many gradients as one benign participant does in the same time-frame.

To protect against this new attack type, we augment BTARD-SGD with a reputation system designed to limit the impact of pseudonymous identities with the actual underlying compute. We base this system on the following three assumptions:

- Unique and optimal computations: the gradients computed by peer_i at step k cannot be circumvented or reused from other peers and/or previous steps.
- Public key infrastructure: peers have unique public/private key pairs and know each other's public keys.
- 3. Cryptographic hash: peers have access to a hash function hash s.t. finding a vector x such that satisfies hash(x) = y is infeasible for $\lfloor \delta \cdot n \rfloor$ compute over the entire training duration.

We associate each participant with a public record that is used to verify that peer's legitimacy. When a new peer joins the network, it begins with an empty record and is therefore "untrusted". Untrusted peers compute gradients normally, but cannot aggregate vectors from others and cannot serve as validators. More importantly, other peers exclude untrusted gradients from aggregation, using them only for the purpose of validating those peers.

Each time a peer computes gradients g_i^k over publicly known batch ξ_i^k , it must write hash (g_i^k) to its own public record and sign it with its private key. As in the original BTARD-SGD, some of those entries will be validated by other peers chosen by JPRNG. In turn, the chosen validators will either approve their entry or invoke ACCUSE to ban the peer.

In order to become trusted, a given peer must report consecutive gradients until it accumulates T entries approved by (provably) random peers. Once a peer becomes trusted, it must continue reporting gradient hashes to maintain trust. Even a single missing or invalidated hash breaks the chain and results in the corresponding peer being banned. To maintain this invariant, peers chosen as a validators add the recalculated hashes into their own record instead of the skipped iteration.

To protect against dilution attacks, a cooperative training run can simultaneously consider at most as many "untrusted" peers as there are trusted ones: all subsequent peers wait in a queue until one of the untrusted peers becomes either trusted or banned.

Analysis. Under this formalism, a Sybil attacker will attempt to maximize the number of trusted identities it can control with a limited amount of compute. In the simplest case, an attacker has exactly one GPU that can be used to either run all computations for identity or partial computation for multiple identities.

In the latter case, an attacker can honestly compute gradients for identity A with probability $p \in [0, 1]$ and for identity B with probability 1 - p. To breaking the chain, the identity that does **not** compute gradients at a given step can report arbitrary (e.g. random) entries instead of hash (q_i^k) .

Consider the expected number of "trusted" identities after enough steps for T validations by *honest* validators (on average, $T \cdot \frac{n}{k \cdot (1-\delta)}$ steps). Identity A becomes trusted with probability p^T , otherwise

it is banned. Similarly, identitiy B survives with probability $(1-p)^T$. Thus, the expected number of trusted identities after T steps is $p^T + (1-p)^T$.

For T>1, this expectation is maximal iff $p\in\{0,1\}$. Thus, if a peer needs more than one validation to become trusted, the "optimal strategy" for a Sybil attacker is to fully support one identity instead of spreading the resources between multiple ones. This observation can be generalized for distributing $\lfloor \delta \cdot n \rfloor$ over an $m \geq \lfloor \delta \cdot n \rfloor$ pseudonymous identities, where maximizing the expected number of trusted identities requires fully supporting any $\lfloor \delta \cdot n \rfloor$ identities and disregarding the rest (for T>1, as before).

Overhead computation. When training without byzantine participants, this modified version of BTARD-SGD requires, on average, $T \cdot \frac{n}{k}$ additional gradient computations per participant at the very beginning. However, once all peers become trusted, the algorithm computes exactly the same number of gradients as regular BTARD-SGD, effectively training at $\frac{n-k}{n}$ efficiency of AR-SGD, plus the same communication overhead.

Remark 1: Temporary majority. Despite the fact that spreading 1 "compute unit" across multiple identities reduces the *expected* number of trusted identities, it may still be useful to establish a temporary majority, albeit with a small probability. For instance, splitting one compute unit evenly among m identities (each with p=1/m) may result in both m identities temporarily gaining trust with probability:

$$P(\mathtt{peer}_1 \wedge \dots \wedge \mathtt{peer}_m) = \prod_{i=1}^m \frac{1}{m^T} = m^{-Tm}$$
 (69)

A Sybil attacker can simply repeat this procedure on every step until it can establish a temporary majority and use this majority to harm training (e.g. ban non-malicious peers). A natural way to remedy this is to increase T to such an extent that (69) becomes negligibly small.

Remark 2: extra compute for byzantine nodes. Unlike benign peers, byzantine attackers do not need to honestly validate each other. When a byzantine peer is chosen as validator, it can approve its target without actually computing the gradients. In turn, the freed compute resources can be used to support additional byzantine identities.

Thus, if a given training run has n trusted peers and chooses k validators on each step, Sybil attackers can control slightly more than $\lfloor \delta \cdot n \rfloor$ of all identities by using the free compute cycles from validation to support additional peers. Thus, the proposed reputation system requires that the total computational power B_{max} available to Byzantines is less than $\frac{1}{2}$ by a (typically small) margin that depends on n, k, and T.

Remark 3: perpetual attacks. When training in open collaborations, one cannot ban the byzantine peers entirely: a byzantine attacker will always be able to assume a new identity at the cost of running honestly for $T \cdot \frac{n}{k \cdot (1-\delta)}$ gradient steps. Thus, unlike in Appendix F, we cannot make BTARD-SGD unbiased by increasing τ . However, as we demonstrated in Section 4, the biased variant of BTARD-SGD with constant τ can still train real-world deep learning models with the same or virtually the same learning curves as regular SGD.

H Secure distributed hash tables

Distributed Hash Tables (DHT) are protocols that establish a decentralized key-value storage over decentralized unreliable participants [83, 102, 84, 103]. To determine which DHT peers are responsible for a given key-value pair, each participant samples a unique binary identifier (ID) sampled uniformly from the space of hash function outputs. When "storing a (key, value)" on the DHT, one finds k peers whose IDs are nearest to hash(key) and sends the data to each one of those peers. In turn, a peer that wants to read the value or a given key will also search for neighbors whose IDs are close to hash(key) and request the data from those peers. Thus, the data can be accessed as long as at least one o k chosen peers remains active, with some DHT variants introducing additional replication protocols.

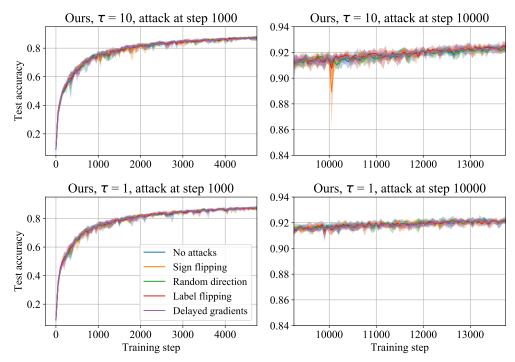


Figure 4: Effectiveness of attacks against BTARD-SGD for the case when 3 of 16 participants are Byzantine.

Our specific implementation is based on Kademlia [83]: a popular DHT variant that determines nearest neighbors based on XOR distance function or their IDs: $d(x,y) = \operatorname{int}(x \oplus y)$. More importantly, Kademlia protocol organizes nodes in such a way that each individual peer only "knows" a small subset of $O(\log_2 n)$ direct neighbors, however, it is possible to navigate the neighborhood graph to find the globally nearest neighbors in $O(\log_2 N)$ network requests.

DHT protocols were originally designed for large-scale distributed systems such as BitTorrent, IPFS and several cryptocurrencies. To maintain integrity in these applications, modern DHT protocols also employ security measures that make them resistant to Byzantine and Sybil attacks [104].

In our specific scenario, the most sensitive DHT entries are personal records that determine whether or not a given peer is trusted. We protect thee records by enforcing that every value stored in the DHT must be signed by their author's digital signature [94]. Thus, if a malicious peer attempts to modify a record it was not supposed to, all other peers will be able to detect that and eliminate such peers from the collective.

However, digital signature are known to be vulnerable to replay attacks: every time a non-byzantine peer stores an given key-value pair signed with its private key, a byzantine eavesdropper can record the signed entry and replay it in future. For ordinary DHTs, this would allow an attacker to revert any key-value pair to its previous state by replaying such pre-recorded messages.

Our algorithm protects against replay attacks by associating each key-value pair with a third value denoted as **expiration time**. Given two entries for the same key, DHT nodes will now prioritize the ones with the latest expiration time and consider it valid up to that time. Furthermore, in order to store a new entry to the DHT, a peer must now sign the entire key-value-expiration tuple. Thus, if a byzantine peer replays a pre-recorded message, it will not be able to overwrite newer DHT entries that were signed for a more recent expiration time.

I Extra evaluations on the CIFAR10 classification task

In this section, we perform several additional experiments with BTARD-SGD used to train the ResNet-18 model to solve the CIFAR10 classification task.

To better explore the space of possible attack vectors, we also evaluate two alternative settings. First, we consider a situation where Byzantine peers are less numerous. For this experiment, we

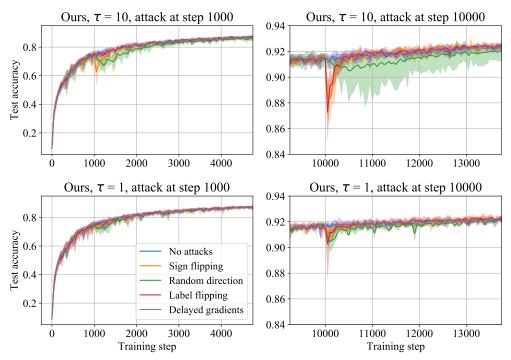


Figure 5: Effectiveness of attacks against BTARD-SGD for the case when Byzantines send incorrect gradients once per T=10 steps.

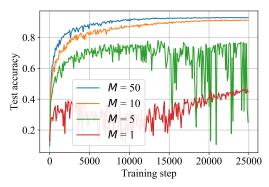


Figure 6: Convergence of BTARD-SGD with $\tau=1$ depending on the maximal number of iterations M in the CenteredClip procedure.

use the same configuration as in Section 4.1, but with only 3 Byzantine peers out of 16 (just under 20%). Figure 4 demonstrates similar behavior to our original setup, but with significantly weaker in magnitude across all attacks.

Next, we explore a situation where Byzantine peers send incorrect gradients periodically, e.g. once per T iterations. This reduces the attack intensity, but allows them to stay undetected for longer. In this setting, we consider 7 Byzantine peers and reuse all parameters from the original setup, except for the new attack period. We consider T=10 for both scenarios (early and late attacks). The attacks are performed at steps $s+k\cdot T, k\in\mathbb{N}$ until the attacker is eventually banned. As expected, this setup increases the duration of each attack by a factor of T, but decreases the peak attack influence (see Figure 5).

Finally, we evaluate the convergence and the final test accuracy of the less computationally intensive variants of BTARD-SGD that limit the maximal number of iterations in the CenteredClip procedure to M, where M varies from 1 to 50. In the setup with $\tau=1$, we observe that M=50 iterations are always enough for CenteredClip to converge with $\epsilon=10^{-6}$ in absence of the attacks. Figure 6 demonstrates that stopping the procedure earlier has negative effect on the final test accuracy. The effect becomes more significant for the smaller values of M.

J ALBERT experiment setup

In the experiment described in Section 4.2, we pretrain ALBERT [90] — a self-supervised Transformer model for learning representations of language data. We specifically choose ALBERT instead of other models like BERT [7] due to its high communication efficiency, which is caused by layerwise weight sharing and embedding layer factorization. In particular, we focus on a communication-efficient model, because the connection speed between the workers can become a noticeable constraint when averaging gradients of models with hundreds of millions of parameters. We train ALBERT-large on sequences of 512 tokens from the WikiText-103 [91] dataset. The training procedure starts from a random initialization, but the subword vocabulary [105] is the same as created by the authors of the original ALBERT models.

This model is trained with two objectives: masked language modeling (given a sentence with several masked tokens, predict the tokens that were masked) and sentence order prediction (given two segments from the same document, determine if they were swapped). We use LAMB optimizer [50] with batches that contain 4,096 examples, training with a peak learning rate equal to 0,00176 and a warmup of 5,000 gradient descent steps. In addition, we use gradient clipping with a maximum norm of 1 and weight decay regularization with the weight of 0,01. We run distributed training on 16 cloud instances, each equipped with a single Tesla T4 GPU. Each training run takes 2–3 days, depending on the instance availability.