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# Math for Machine Learning

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## **Linear algebra - Week 4**

# W4 Lesson 1



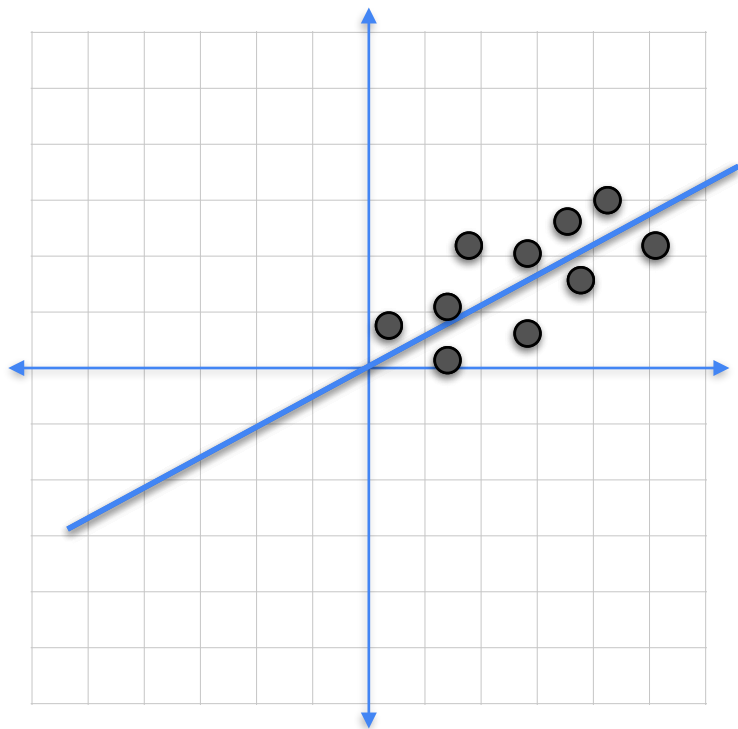
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# Determinants and Eigenvectors

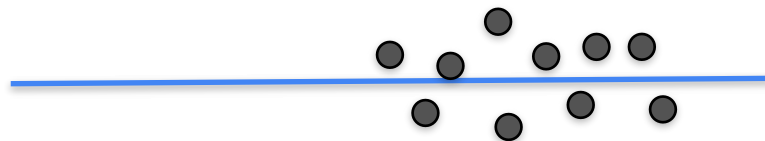
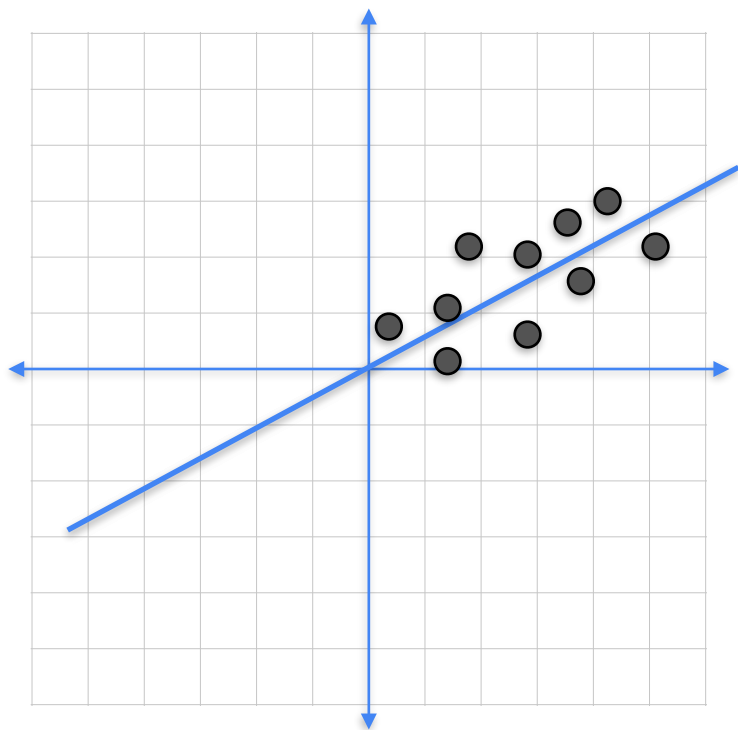
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## **Machine learning motivation**

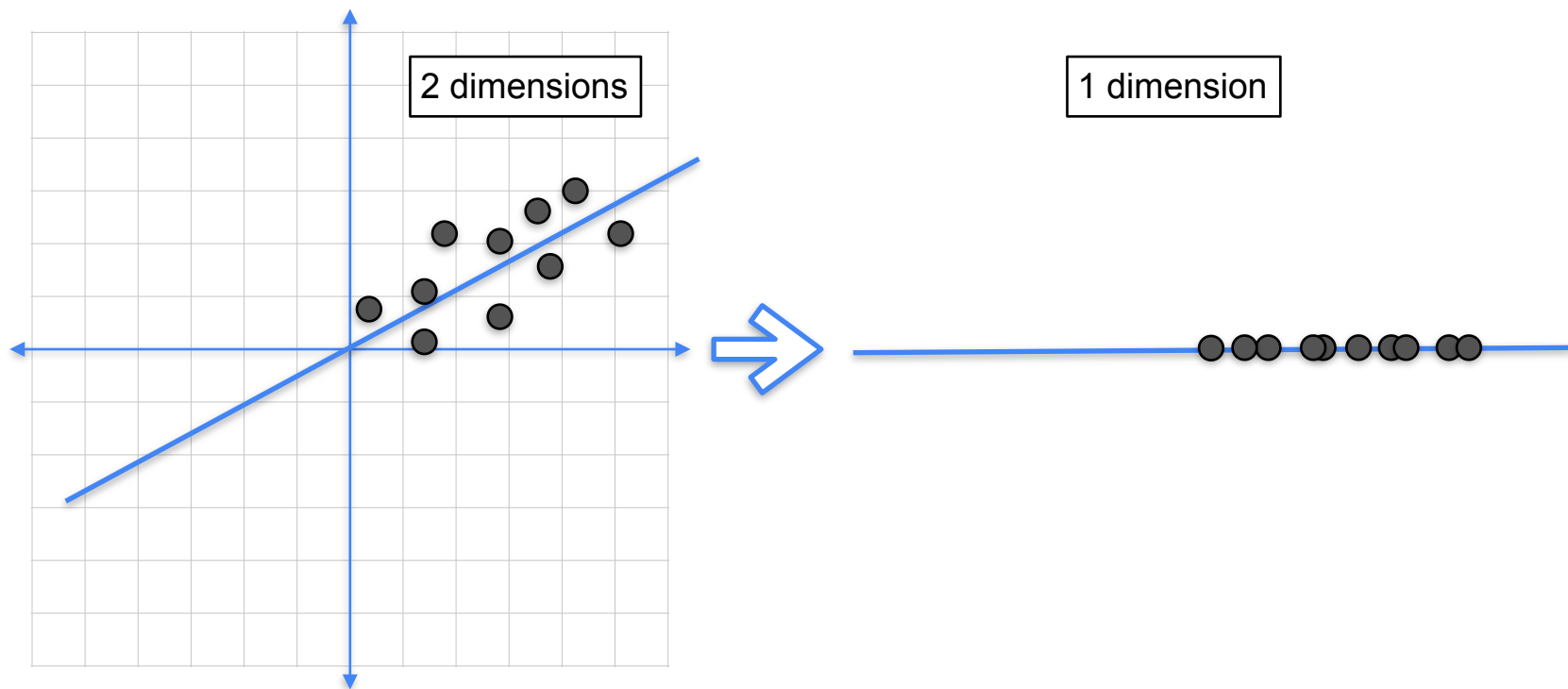
# Principal Component Analysis



# Principal Component Analysis

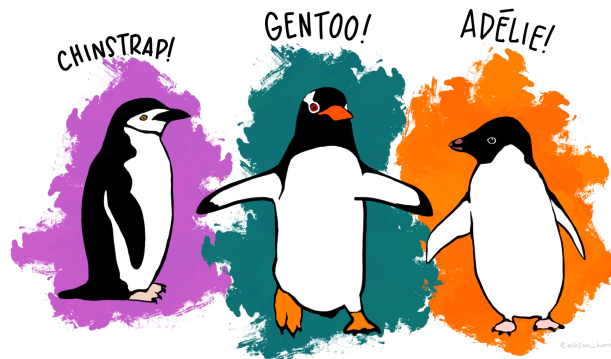


# Principal Component Analysis



# Principal Component Analysis

- Reduce dimensions (columns) of dataset
- Preserve as much information as possible



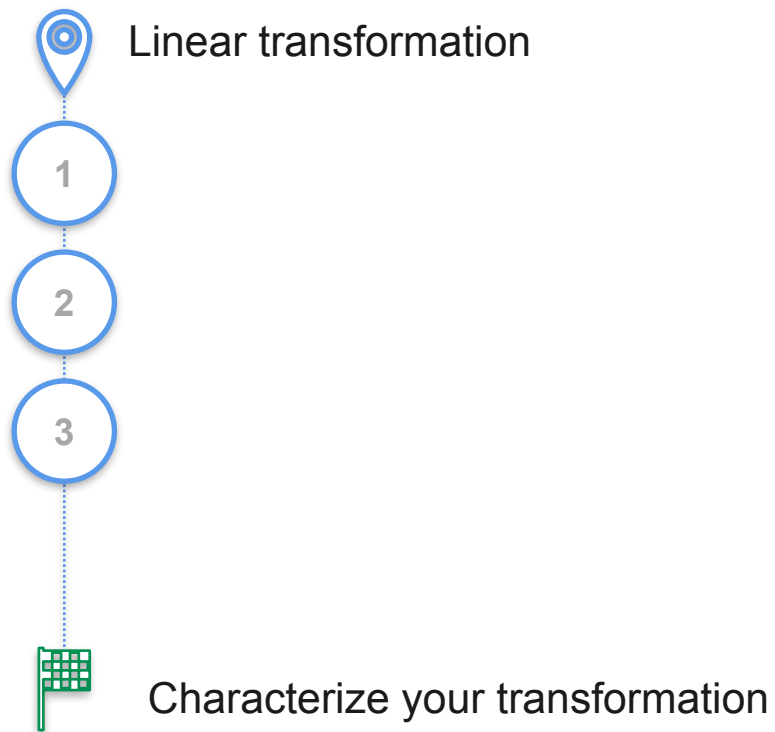
species	culmen_length_mm	culmen_depth_mm	flipper_length_mm	body_mass_g
Adelie	40.6	17.2	187.0	3475.0
Adelie	38.9	17.8	181.0	3625.0
Adelie	35.7	16.9	185.0	3150.0
Gentoo	50.0	15.3	220.0	5550.0
Adelie	34.5	18.1	187.0	2900.0



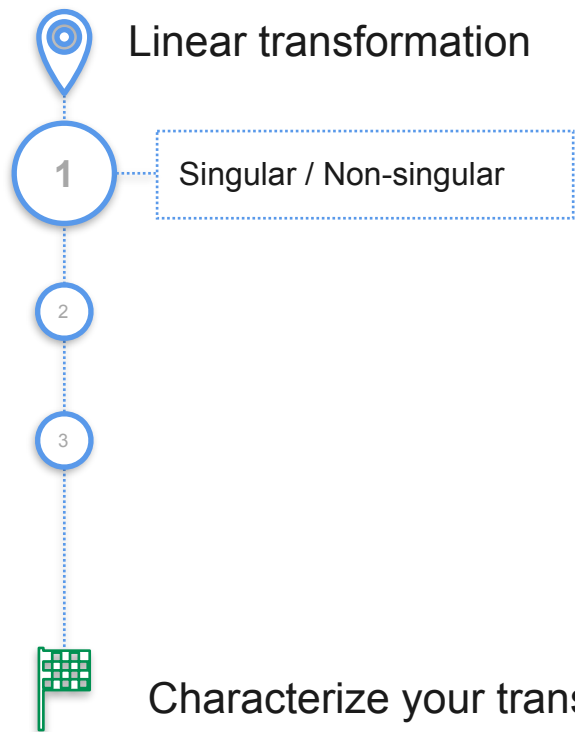
PC1	PC2	species
1.353843	-0.422253	Adelie
1.760446	-0.350965	Adelie
2.005766	-1.113797	Adelie
-2.585758	0.061768	Gentoo
2.438111	-0.786227	Adelie





# What to expect?





# What to expect?

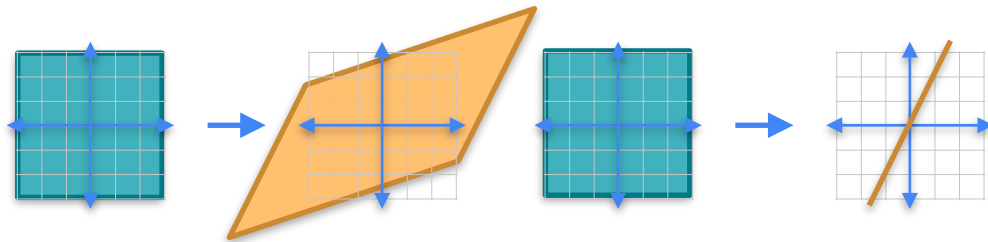


Non-singular

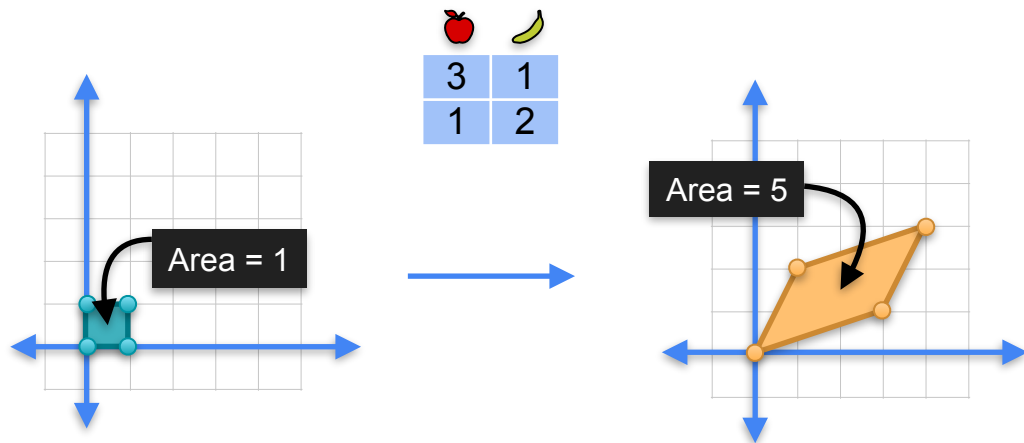
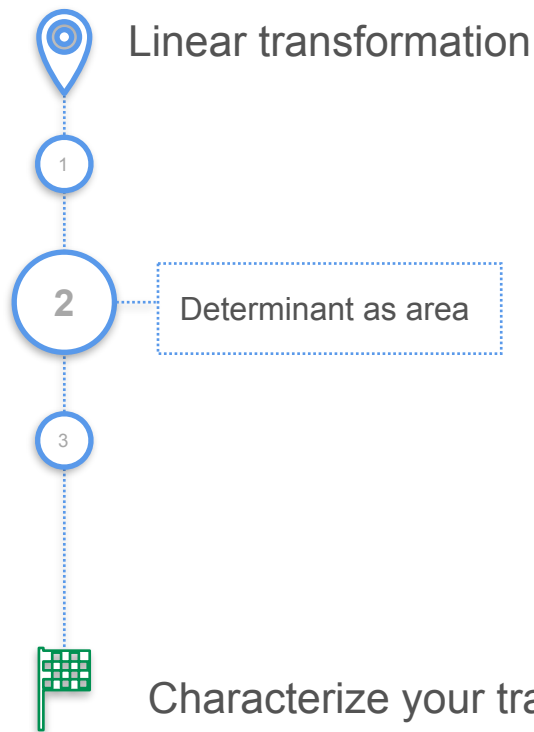
	
3	1
1	2

Singular

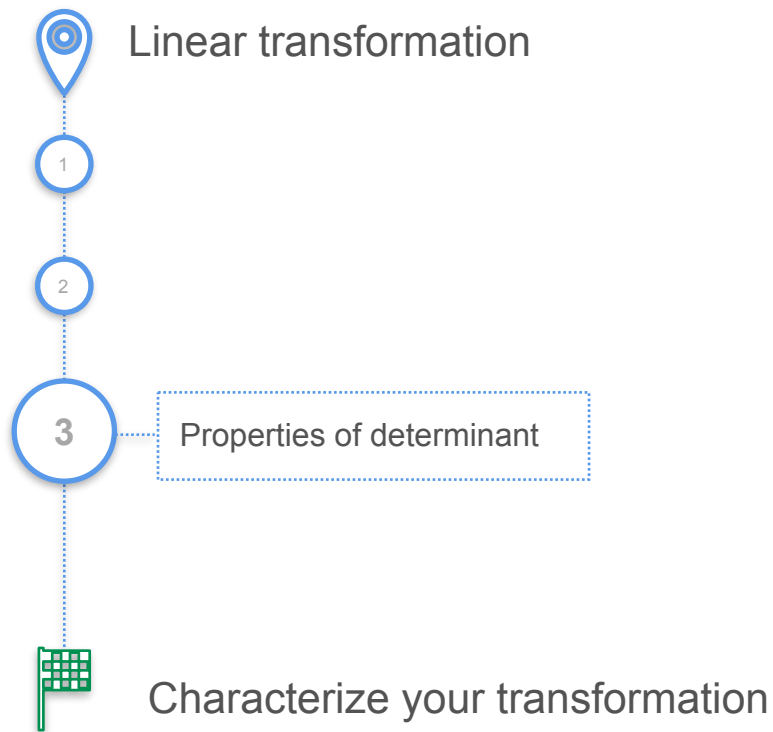
	
1	1
2	2



# What to expect?



# What to expect?



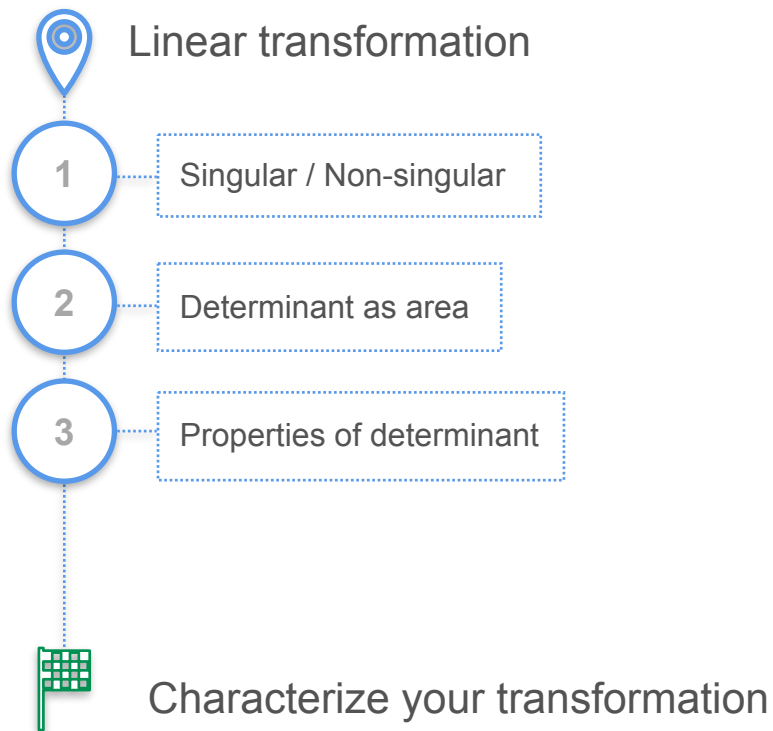
$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -3 & 3 \end{bmatrix}$$

Det = 5      Det = 3      Det = 15  
= 5 · 3

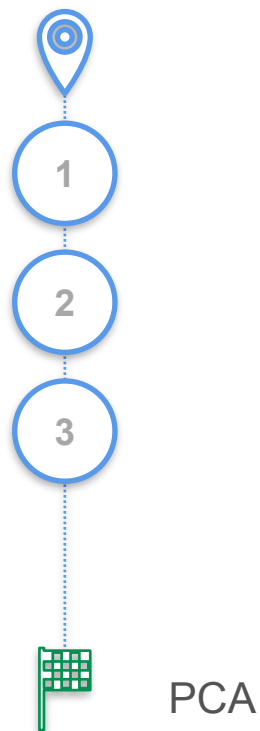
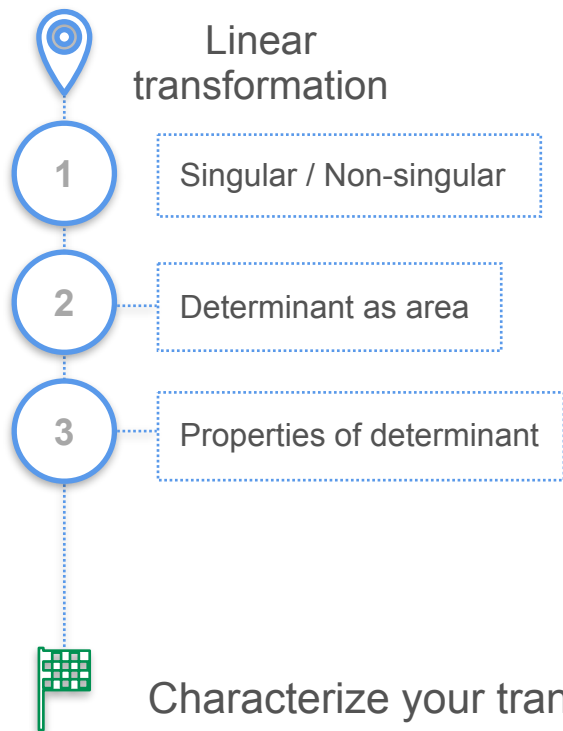
$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

Det = 5      Det = 0.2 =  $\frac{1}{5}$

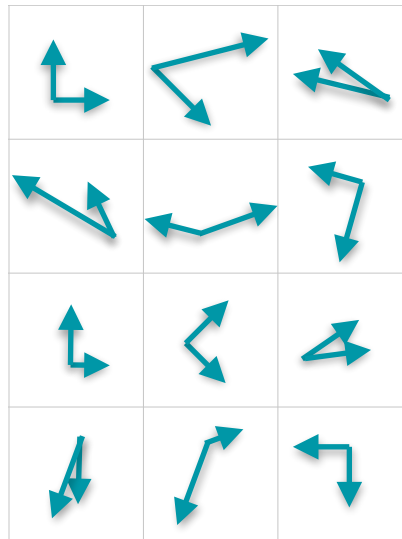
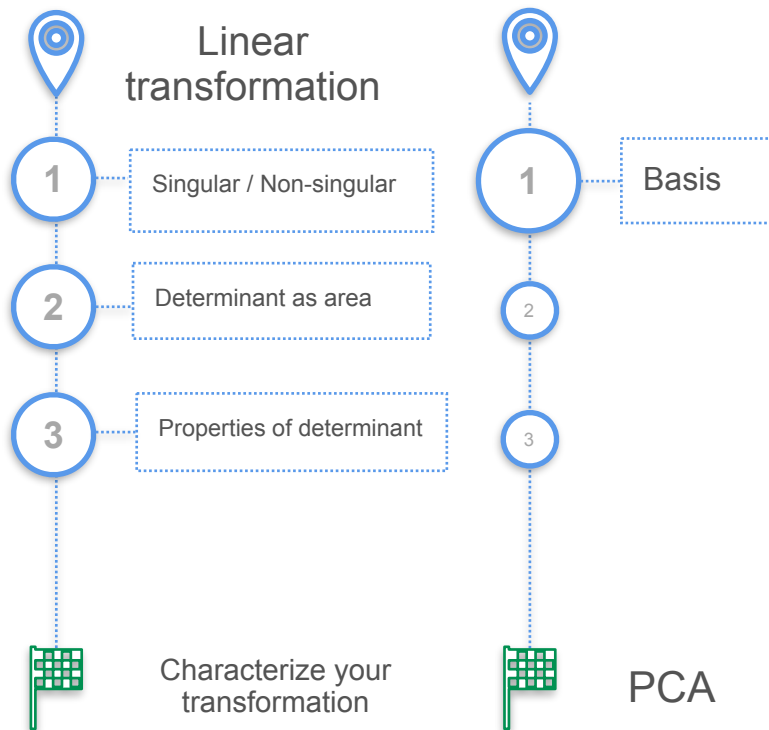
# What to expect?



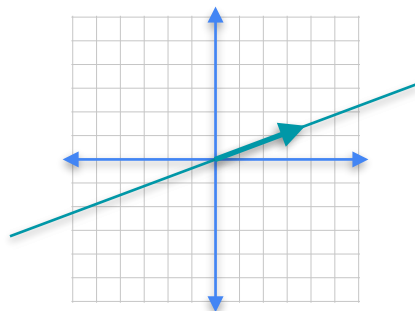
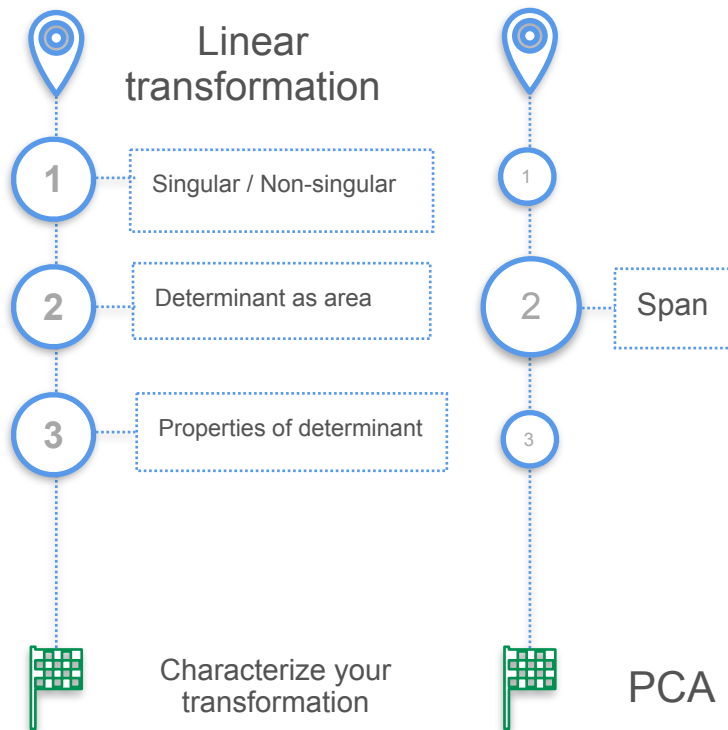
# What to expect?



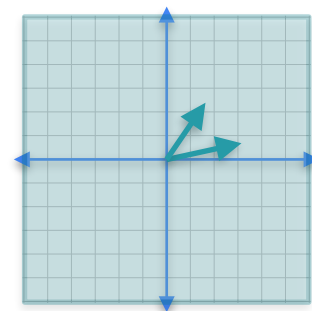
# What to expect?



# What to expect?



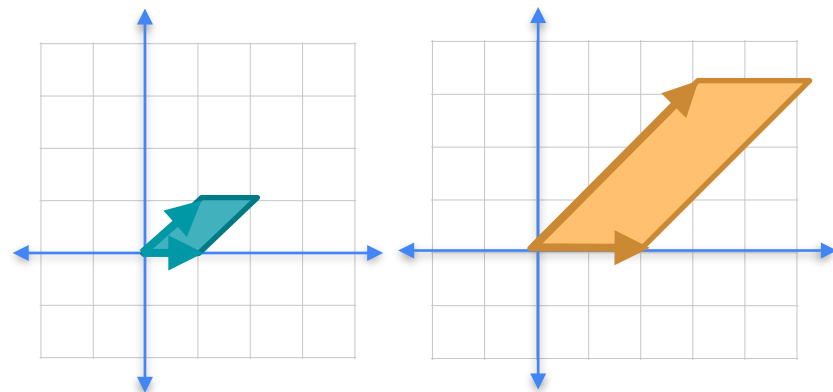
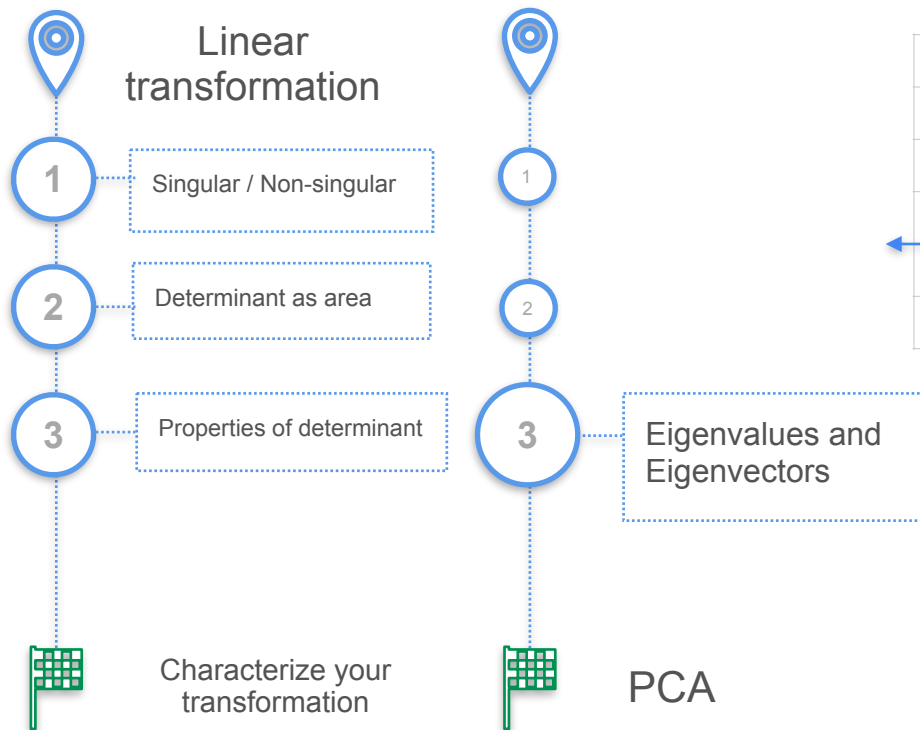
1 element  
Dimensions: 1



2 elements  
Dimensions: 2



# What to expect?

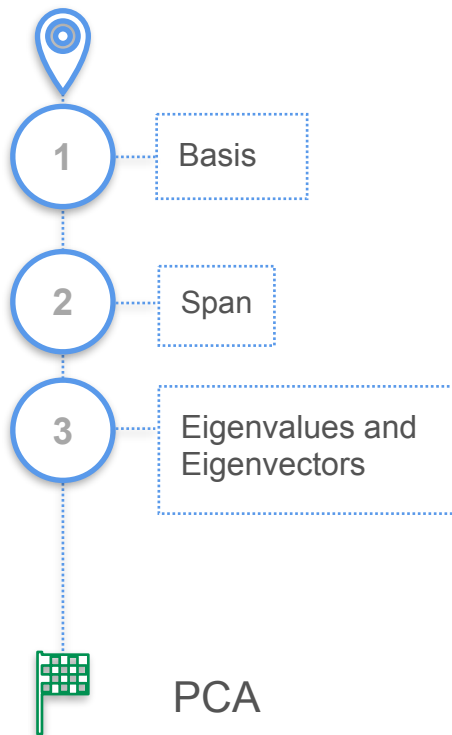
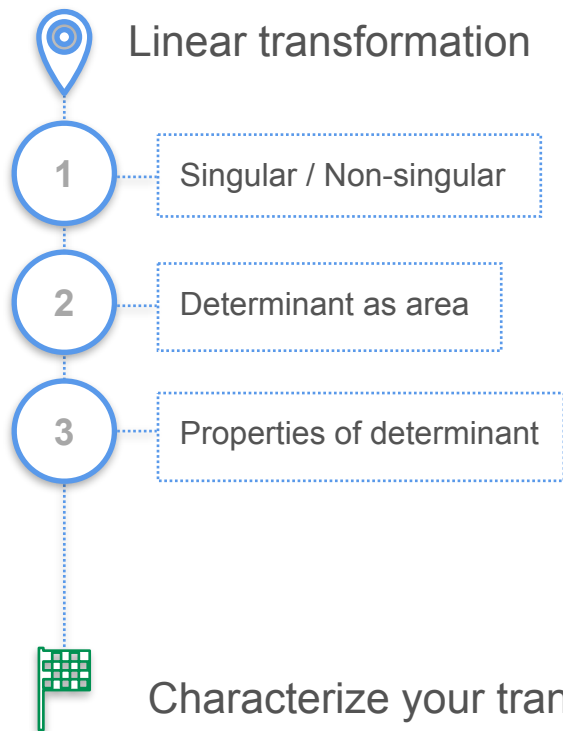


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$(1,0) \rightarrow (2,0)$$

$$A v_1 = \lambda_1 v_1$$

# What to expect?





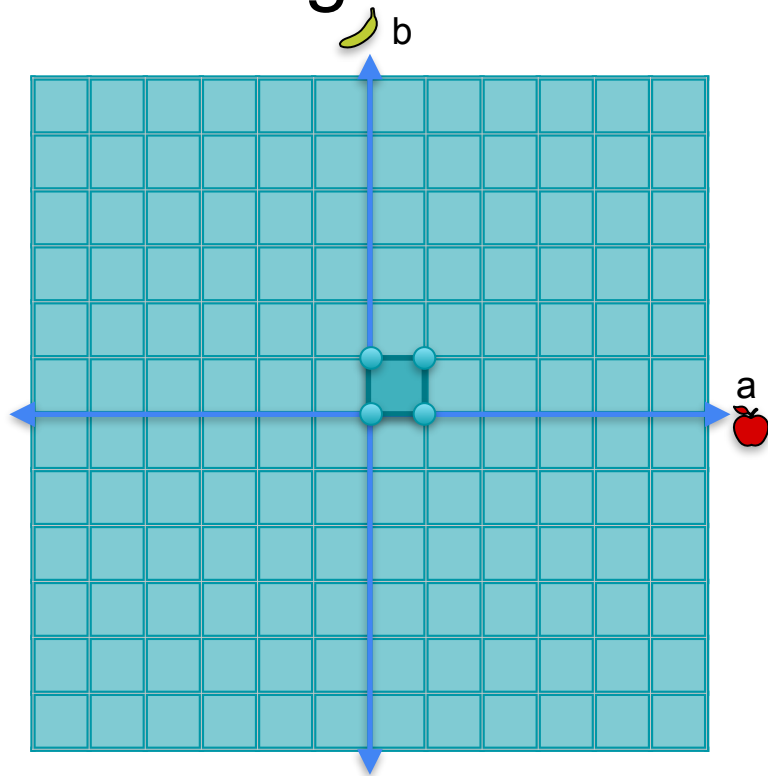
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# Determinants and Eigenvectors

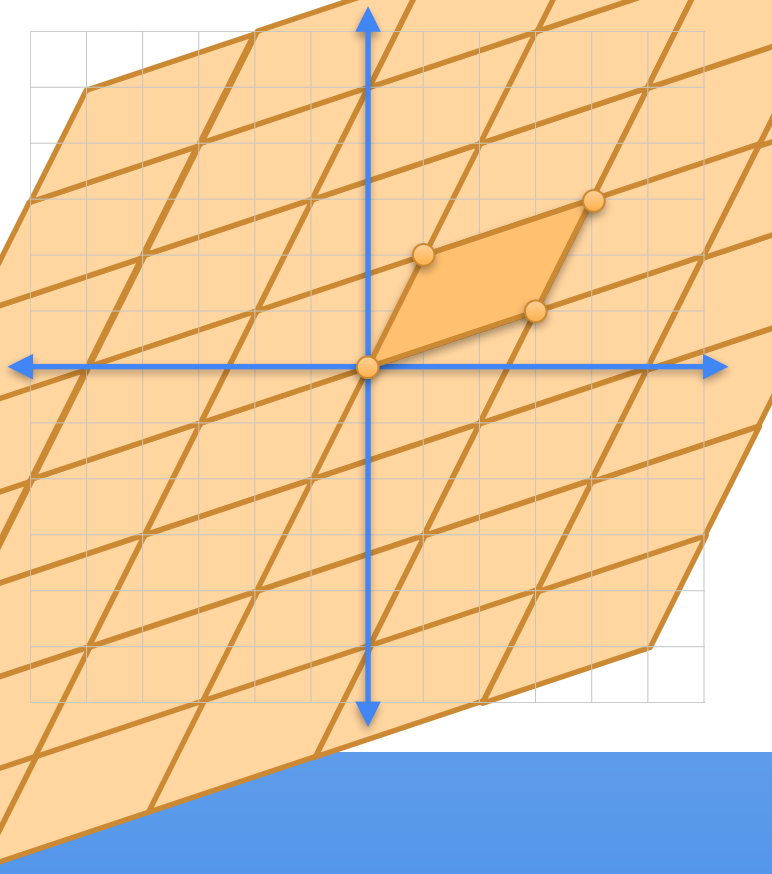
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**Singularity and rank of linear transformations**

# Non-singular transformation



	
3	1
1	2



# Singular transformation

b

🍎 🍌

1	1	1	=	2
2	2	1		4

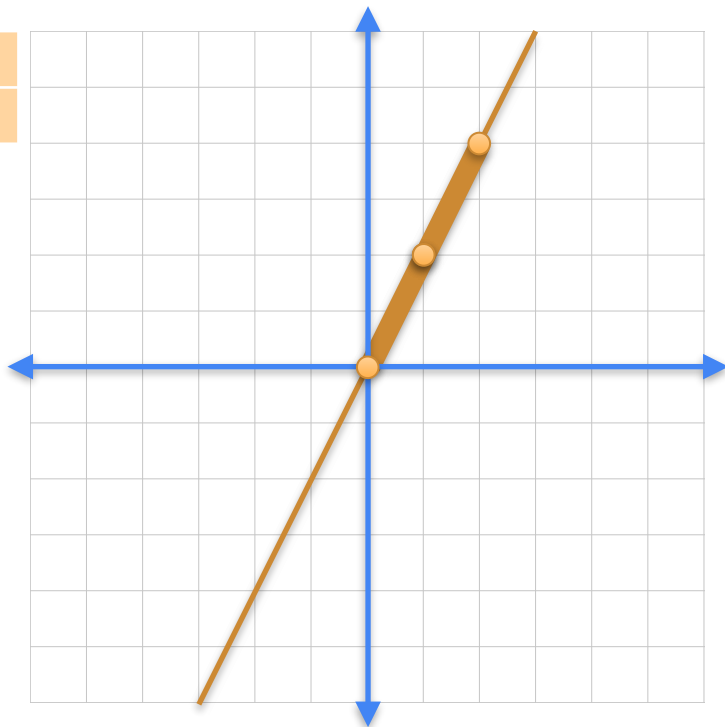
(0,0) → (0,0)

(1,0) → (1,2)

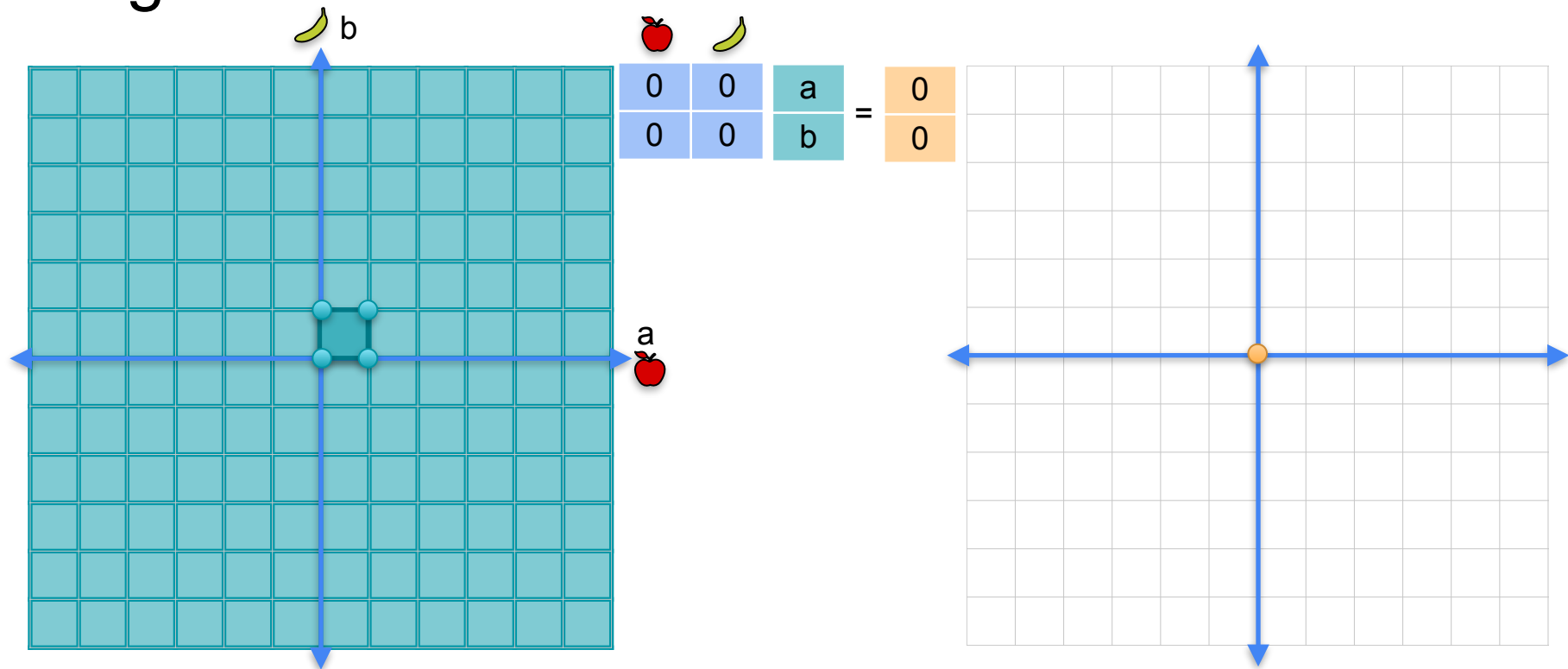
(0,1) → (1,2)

(1,1) → (2,4)

a





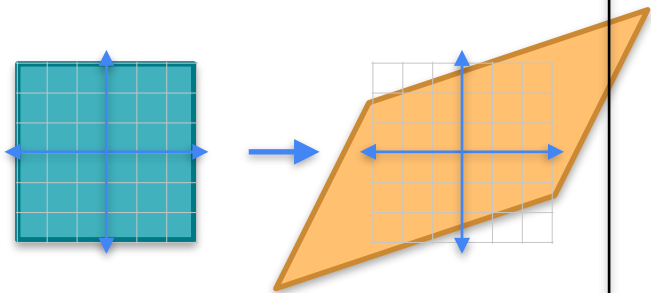
# Singular transformation





# Singular and non-singular transformations

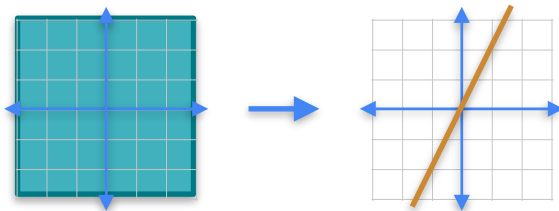
Non-singular

	
3	1
1	2





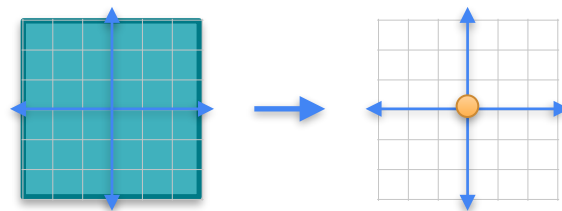
Singular

	
1	1
2	2





Singular

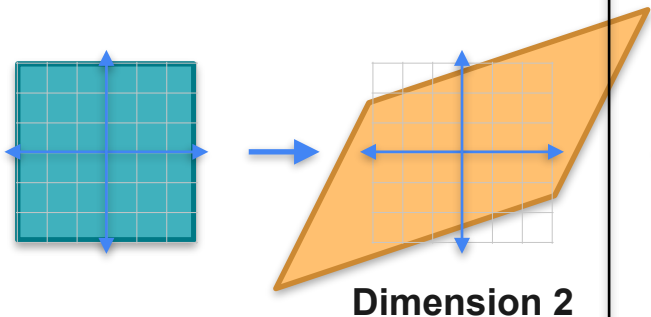
	
0	0
0	0





# Rank of linear transformations

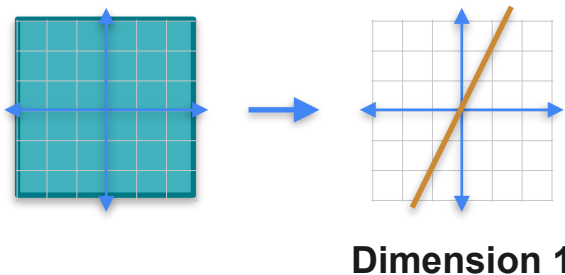
Rank 2

	
3	1
1	2





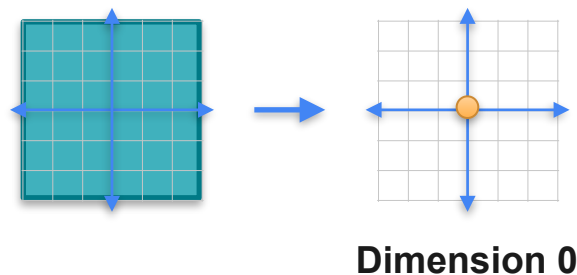
Rank 1

	
1	1
2	2



Rank 0

	
0	0
0	0







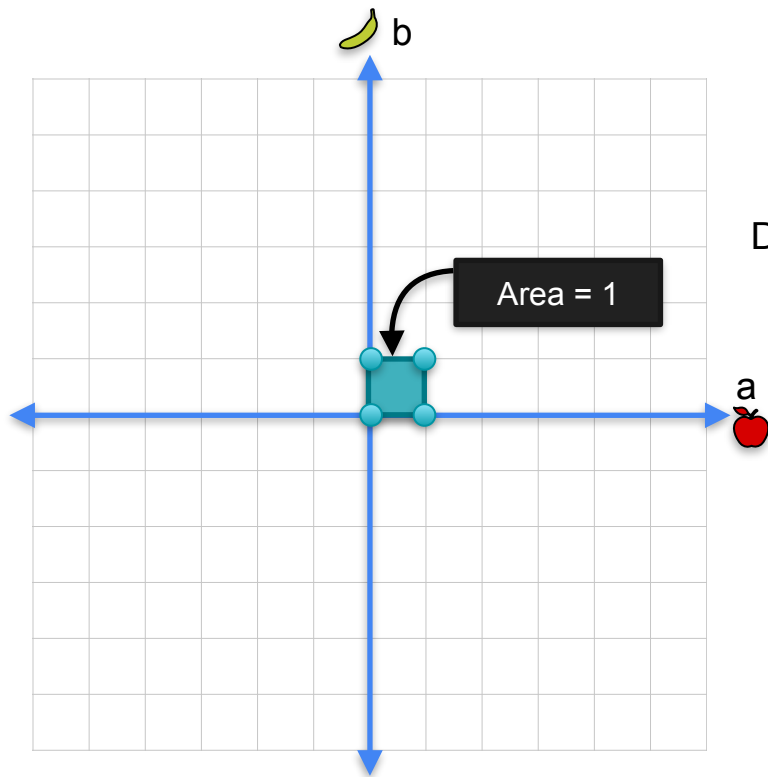
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# Determinants and Eigenvectors

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## **Determinant as an area**

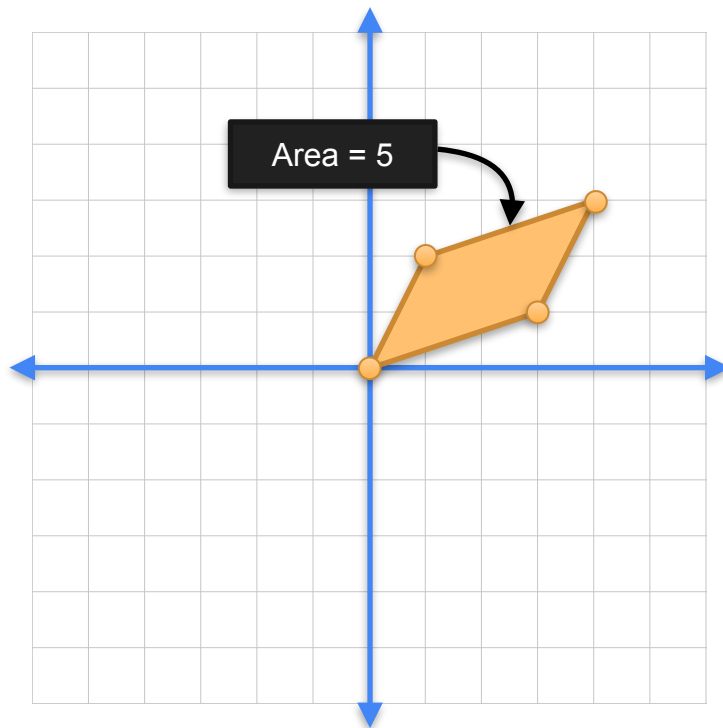
# Determinant as an area



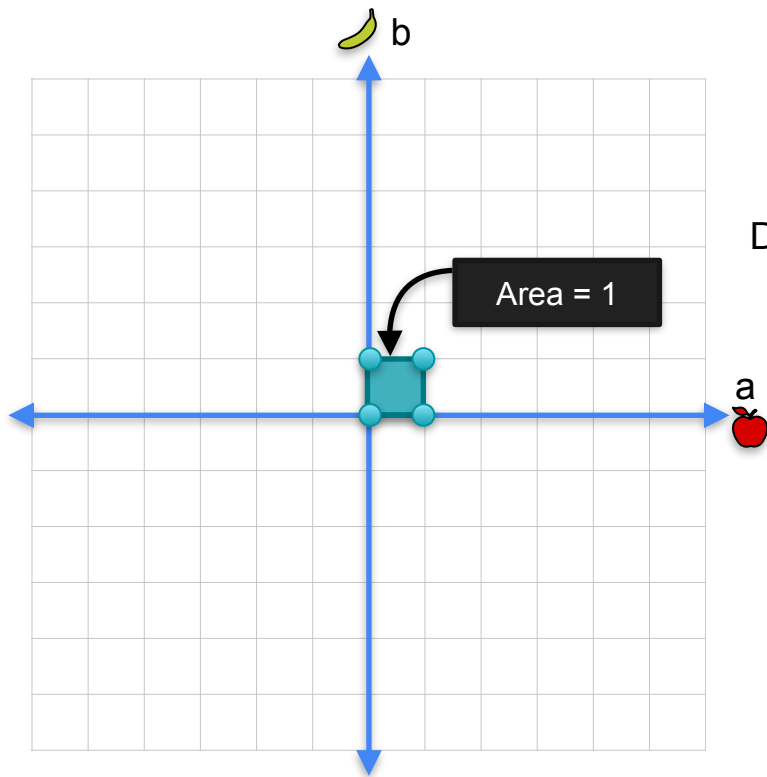
 3	 1
1	2



$$\text{Det} = 3 \cdot 2 - 1 \cdot 1$$

$$\text{Det} = 5$$



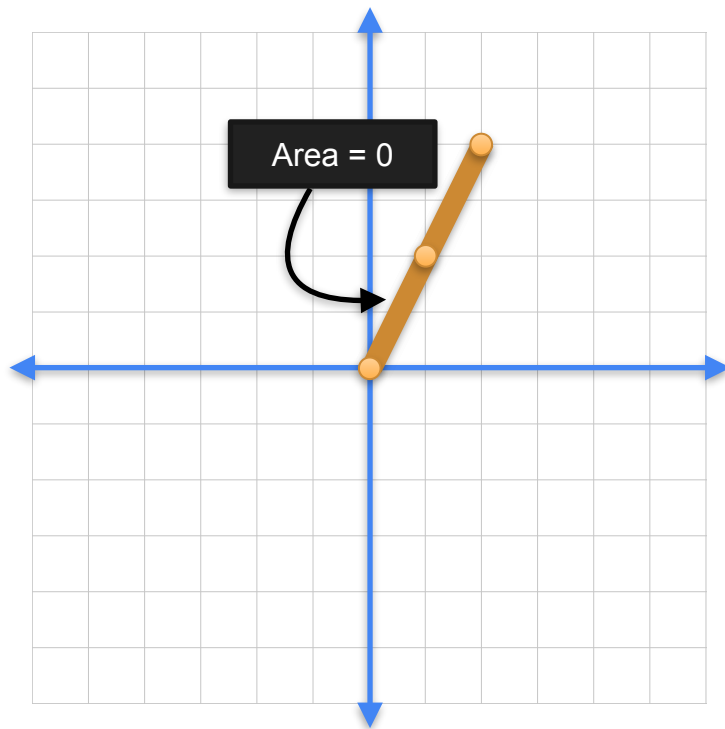
# Determinant as an area



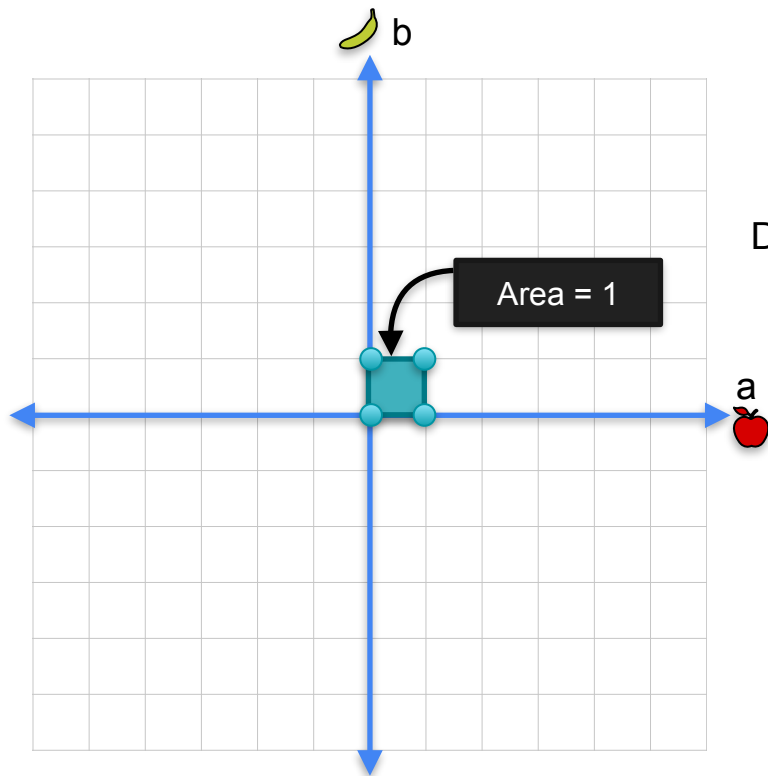
 1	 1
2	2



$$\text{Det} = 1 \cdot 2 - 1 \cdot 2$$

$$\text{Det} = 0$$



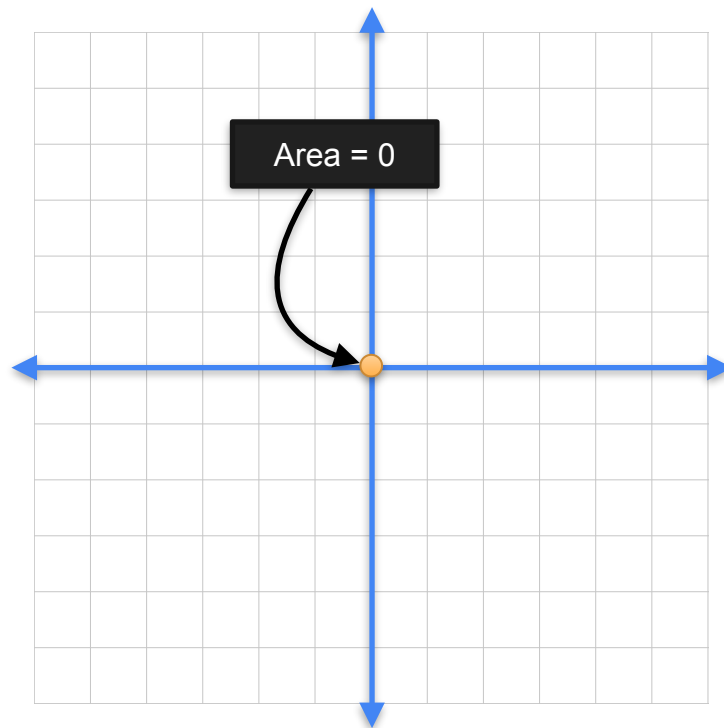
# Determinant as an area



	
0	0
0	0



$$\text{Det} = 0 \cdot 0 - 0 \cdot 0$$

$$\text{Det} = 0$$

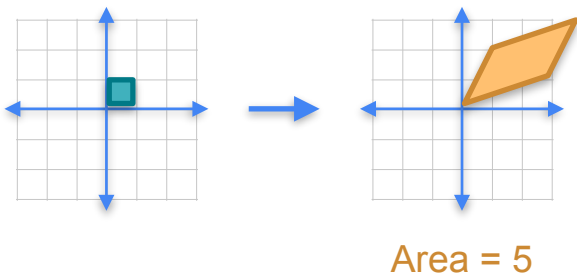


# Determinant as an area



Non-singular

	
3	1
1	2

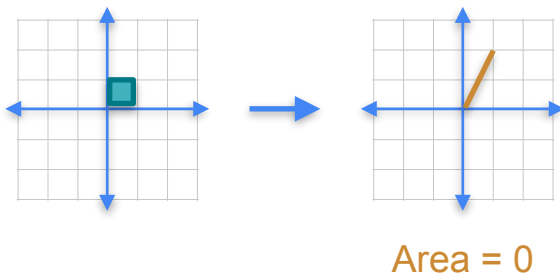
Determinant = 5





Singular

	
1	1
2	2

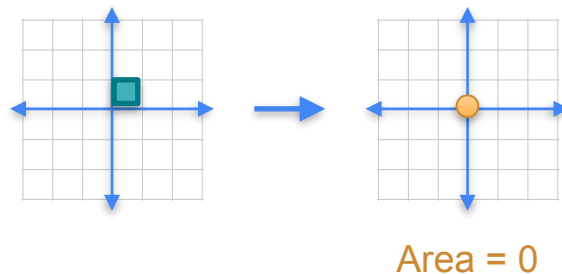
Determinant = 0





Singular

	
0	0
0	0

Determinant = 0





# Negative determinants?



3	1
1	2

$$\text{Det} = 3 \cdot 2 - 1 \cdot 1$$

$$\text{Det} = 5$$

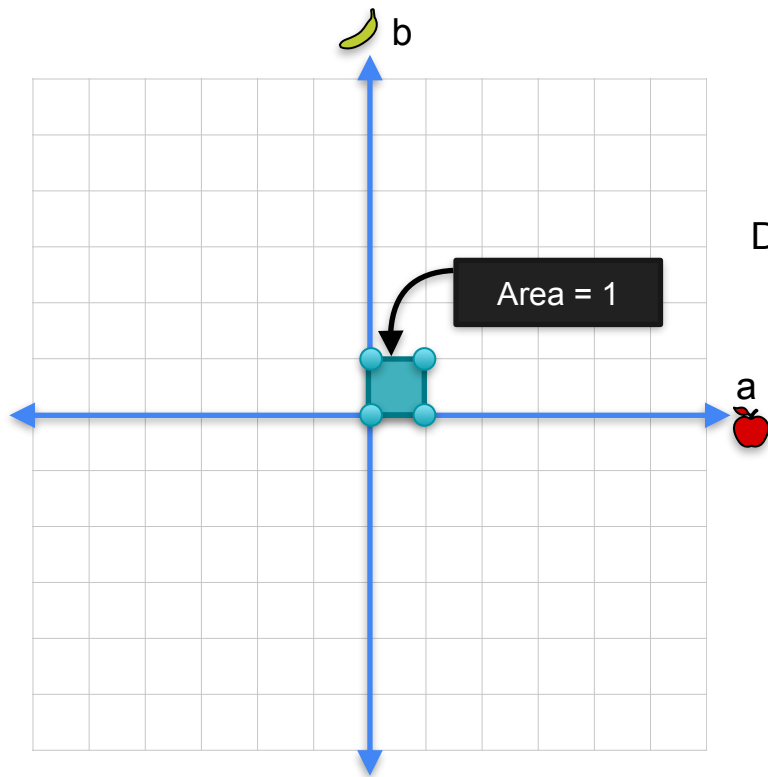




1	3
2	1

$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$

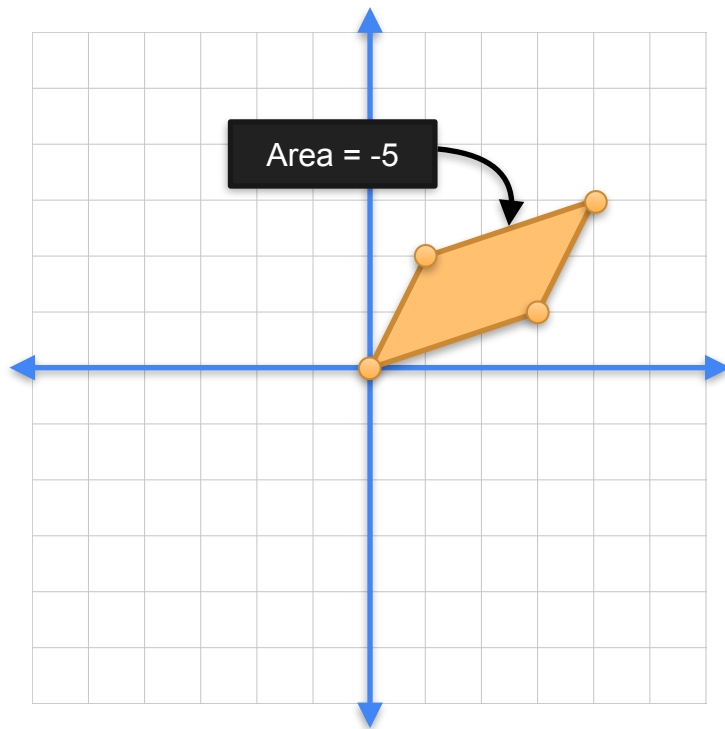
# Determinant as an area



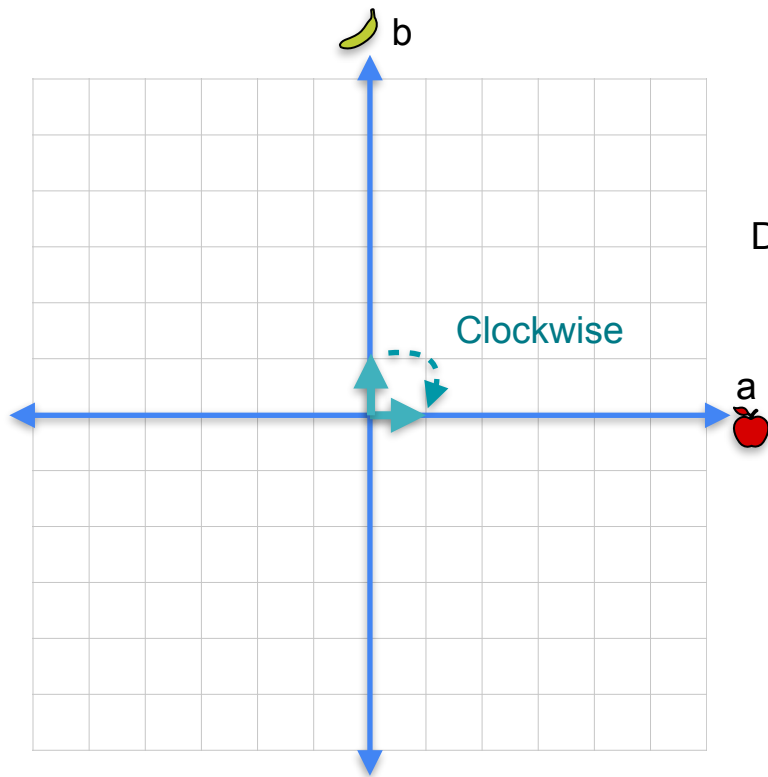
 1	 3
2	1



$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$



# Determinant as an area

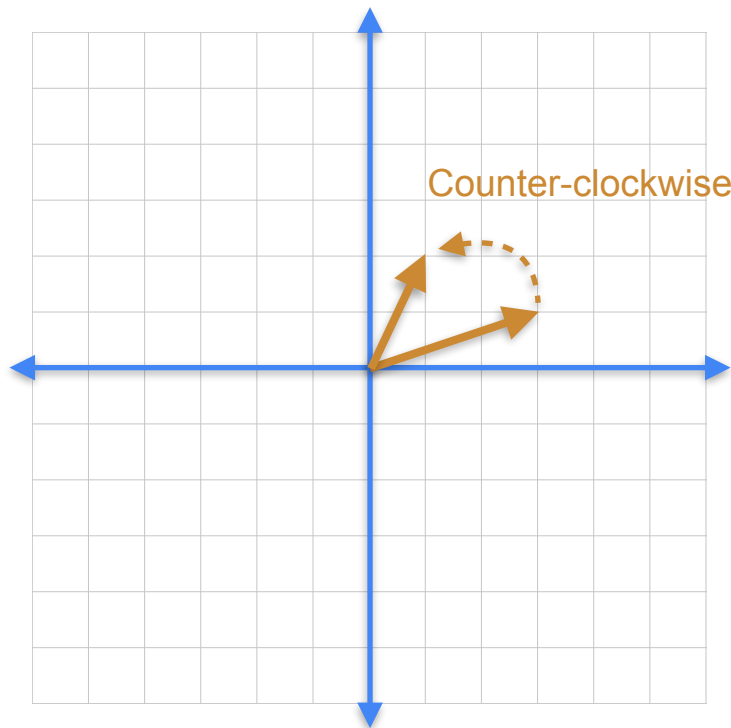


 1	 3
2	1

$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$

Negative







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# Determinants and Eigenvectors

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## **Determinant of a product**

# Determinant of a product

<table><tr><td>3</td><td>1</td></tr><tr><td>1</td><td>2</td></tr></table>	3	1	1	2	<table><tr><td>5</td><td>2</td></tr><tr><td>1</td><td>2</td></tr></table>	5	2	1	2	=	<table><tr><td>16</td><td>8</td></tr><tr><td>7</td><td>6</td></tr></table>	16	8	7	6
3	1														
1	2														
5	2														
1	2														
16	8														
7	6														
$\det = 5$ $3 \cdot 2 - 1 \cdot 1$	$\det = 8$ $5 \cdot 2 - 2 \cdot 1$		$\det = 40$ $16 \cdot 6 - 8 \cdot 7$												

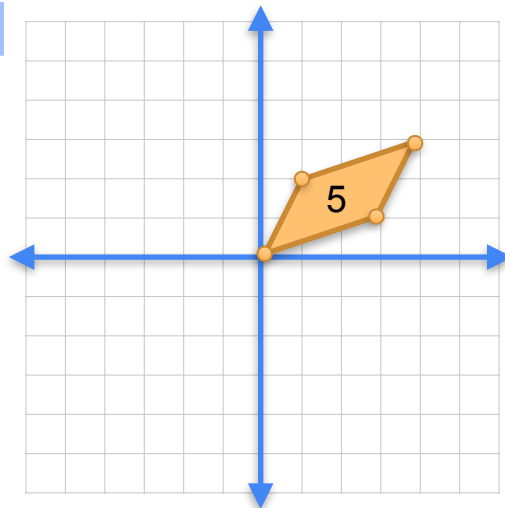
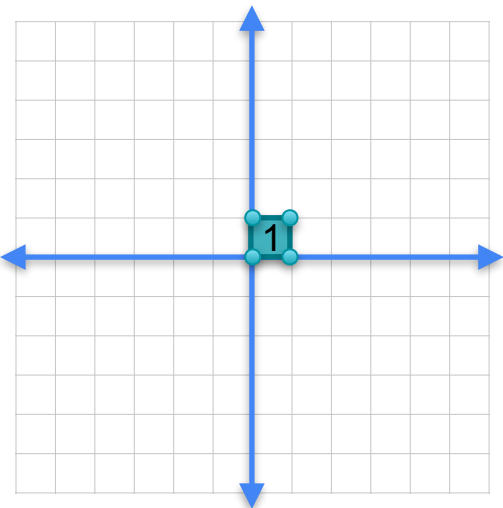
# Determinant of a product

$$\det(AB) = \det(A) \det(B)$$

# Determinant of a product

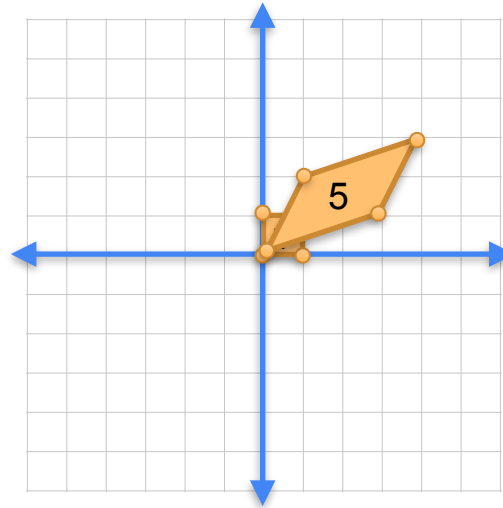
3	1
1	2

Det = 5



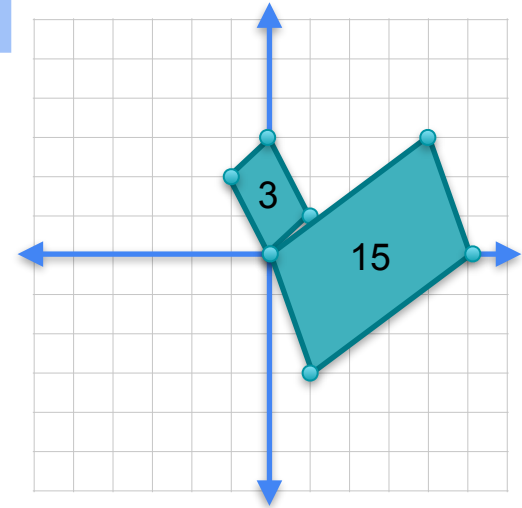
Area blows up by 5

# Determinant of a product



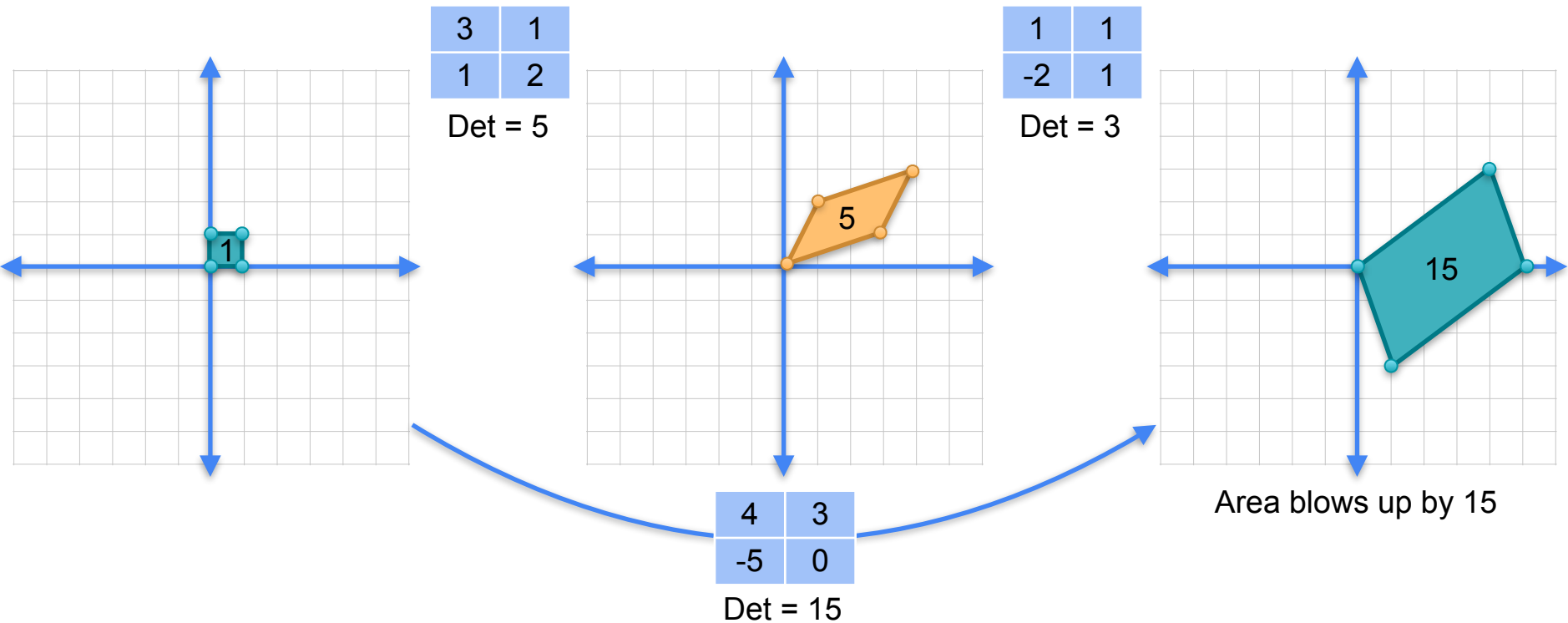
1	1
-2	1

Det = 3



Area blows up by 3

# Determinant of a product



# Quiz

- The product of a singular and a non-singular matrix (in any order) is:
  - Singular
  - Non-singular
  - Could be either one

# Solution

- If A is non-singular and B is singular, then  $\det(AB) = \det(A) \times \det(B) = 0$ , since  $\det(B) = 0$ . Therefore  $\det(AB) = 0$ , so AB is **singular**.



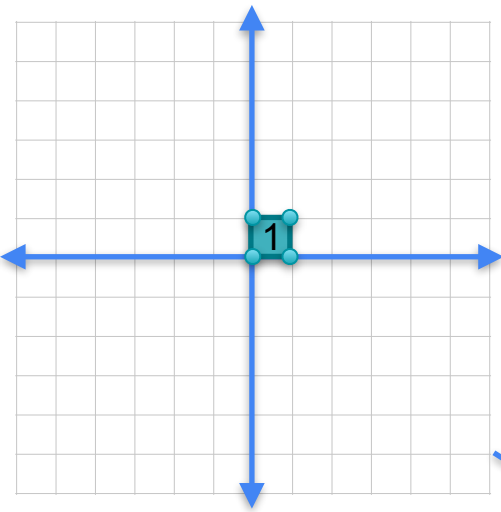
When one factor is zero

$$5 \cdot 0 = 0$$

# When one factor is singular...

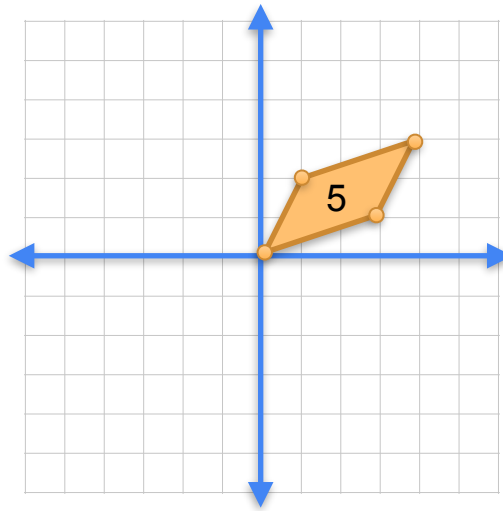
Non-singular		Singular		Singular												
<table><tr><td>3</td><td>1</td></tr><tr><td>1</td><td>2</td></tr></table>	3	1	1	2		<table><tr><td>1</td><td>2</td></tr><tr><td>1</td><td>2</td></tr></table>	1	2	1	2	=	<table><tr><td>4</td><td>8</td></tr><tr><td>3</td><td>6</td></tr></table>	4	8	3	6
3	1															
1	2															
1	2															
1	2															
4	8															
3	6															
Det = 5		Det = 0		Det = 0												

# If one factor is singular...



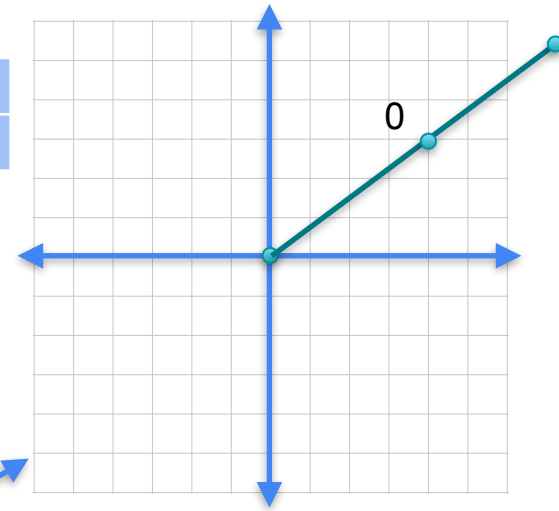
3	1
1	2

Det = 5



1	2
1	2

Det = 0



Area blows up by 0

4	8
3	6

Det = 0



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# Determinants and Eigenvectors

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## **Determinant of inverse**

# Quiz

- Find the determinants of the following matrices

0.4	-0.2
-0.2	0.6

0.25	-0.25
-0.125	0.625

# Solution

$$\text{Det} \begin{array}{|c|c|} \hline 0.4 & -0.2 \\ \hline -0.2 & 0.6 \\ \hline \end{array} = (0.4)(0.6) - (-0.2)(-0.2) = 0.2$$

$$\text{Det} \begin{array}{|c|c|} \hline 0.25 & -0.25 \\ \hline -0.125 & 0.625 \\ \hline \end{array} = (0.25)(0.625) - (-0.125)(-0.25) = 0.125$$

# Determinant of an inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

det = 5

det = 0.2

$$5^{-1} = 0.2$$

$$\begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{bmatrix}$$

det = 8

det = 0.125

$$8^{-1} = 0.125$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

det = 0

det = ???

$$0^{-1} = ???$$

# Determinant of an inverse

$$\det(A^{-1}) = \frac{1}{\det(A)}$$



# Why?

Why is this?

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$\det(AB) = \det(A) \det(B)$$

$$\det(AA^{-1}) = \det(A) \det(A^{-1})$$

$$\begin{array}{ccc} & \uparrow & \\ \det(I) & = & \det(A) \det(A^{-1}) \\ \uparrow & & \uparrow \\ 1 & & \frac{1}{\det(A)} \end{array}$$

# Determinant of the identity matrix

$$\det \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} = 1 \cdot 1 - 0 \cdot 0 = 1$$

$$\det(I) = 1$$

# W4 Lesson 2



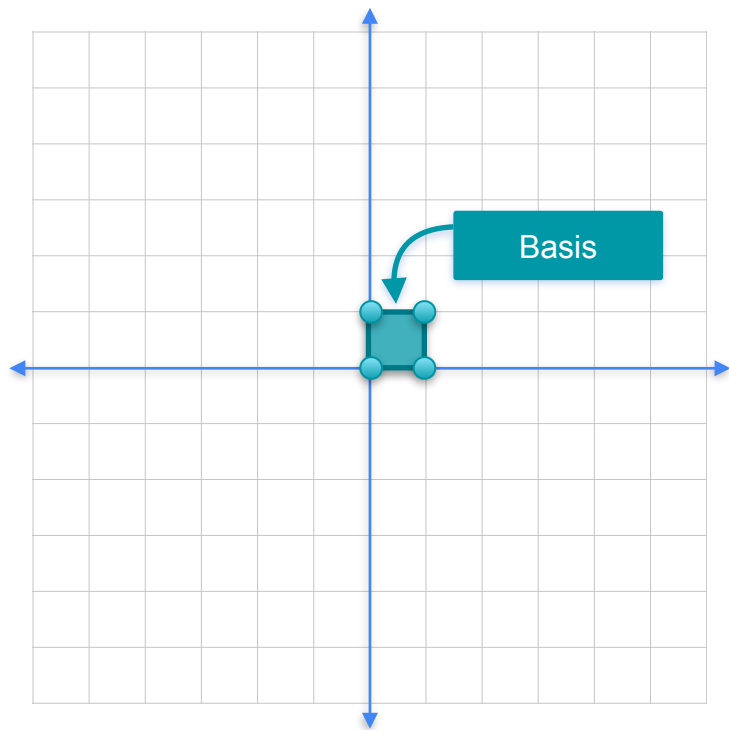
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# Determinants and Eigenvectors

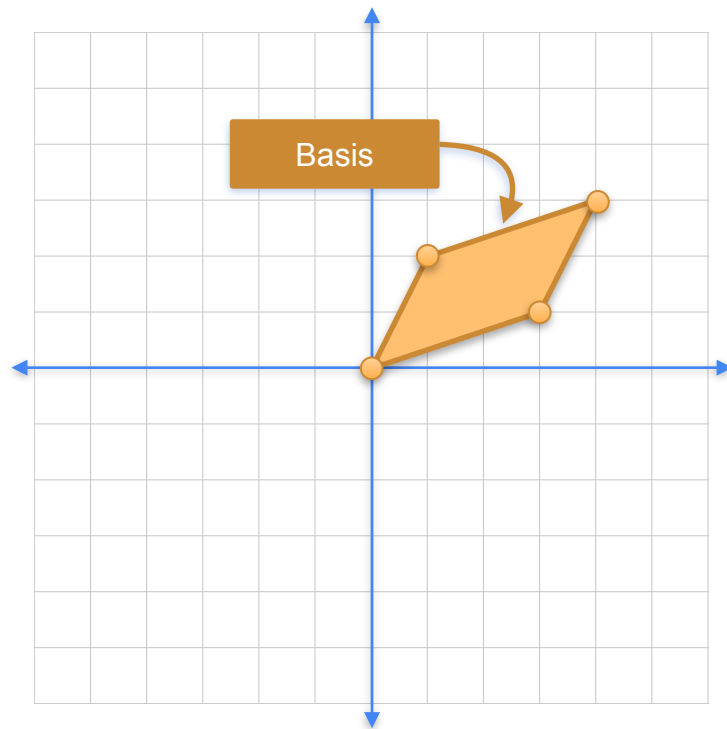
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## **Bases**

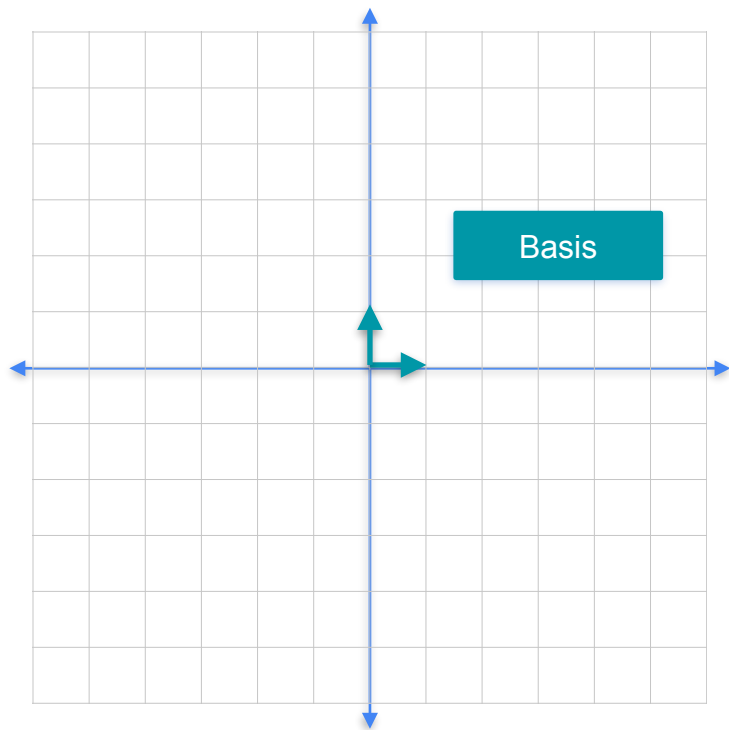
# Bases



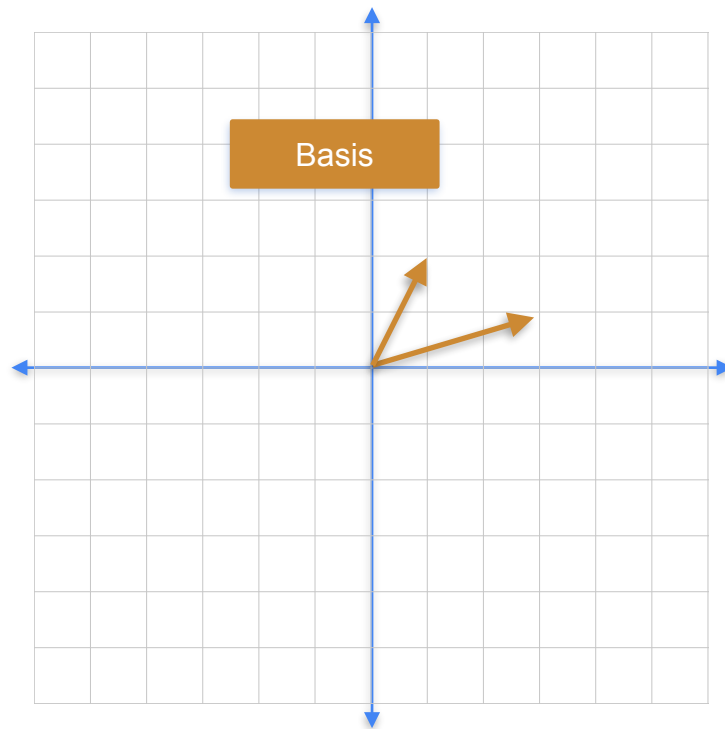
3	1
1	2



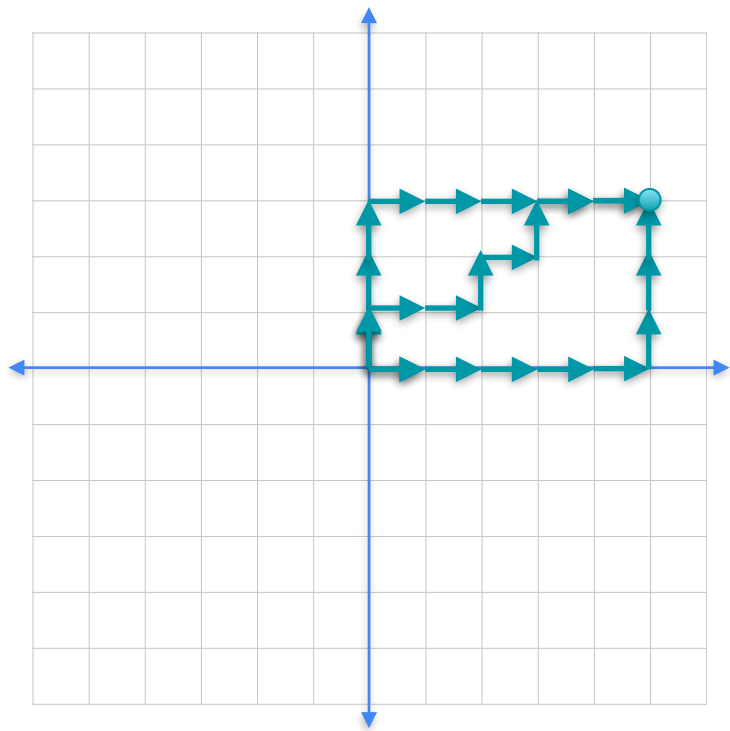
# Bases



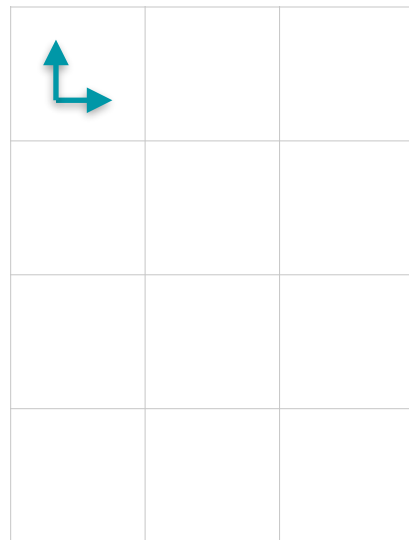
3	1
1	2



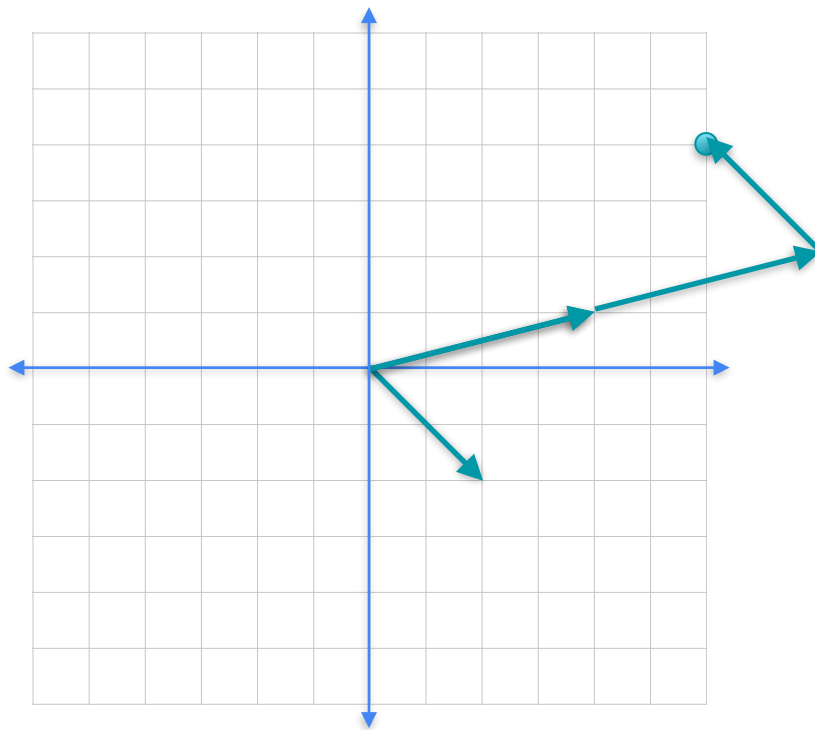
# Bases



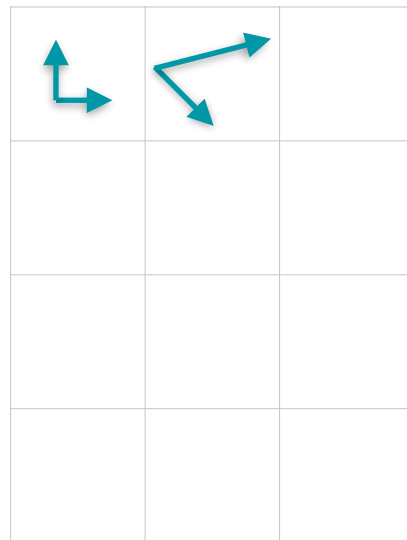
## Bases



# Bases

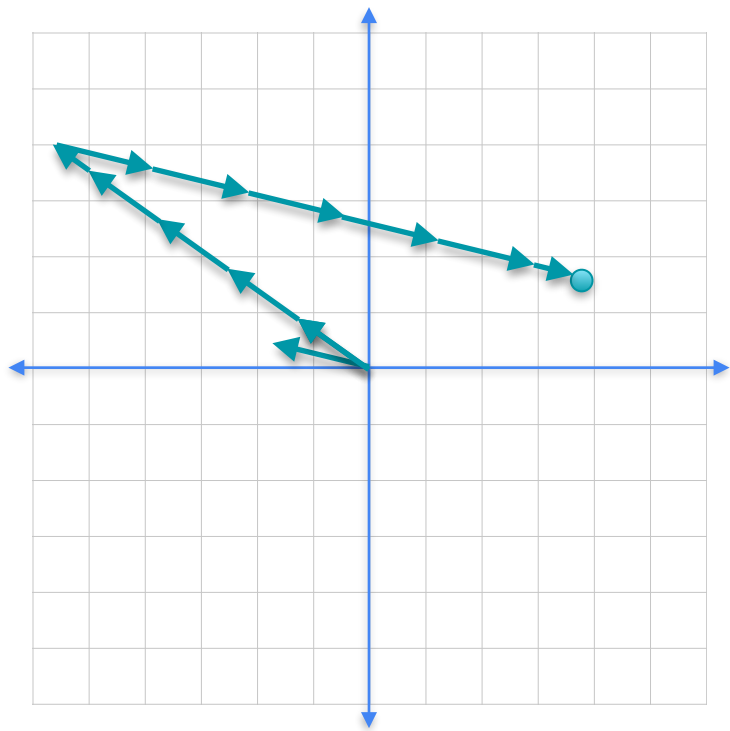


**Bases**

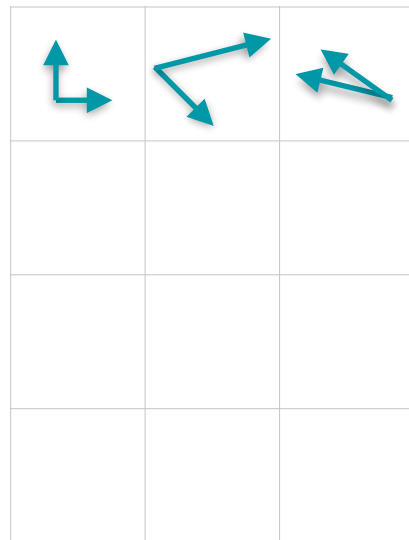




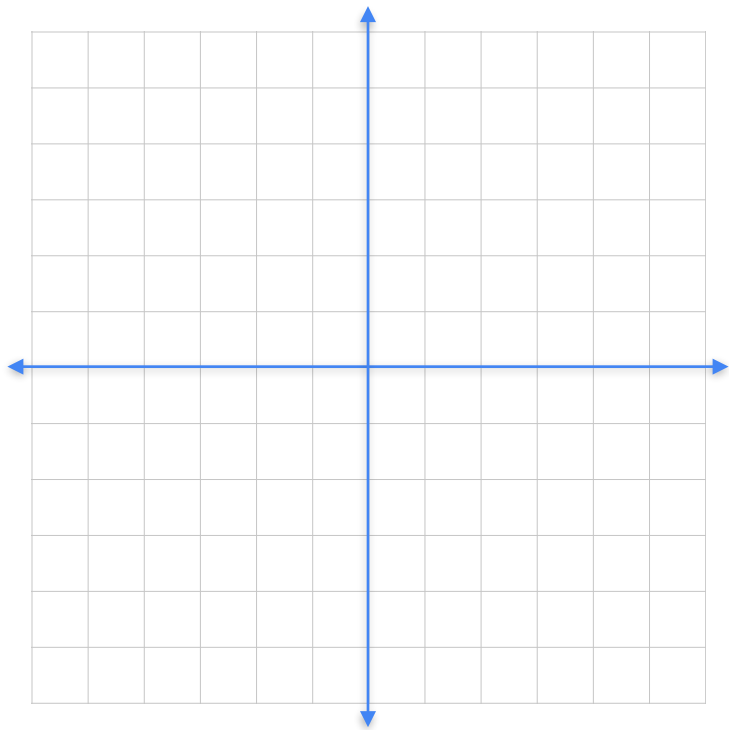
# Bases



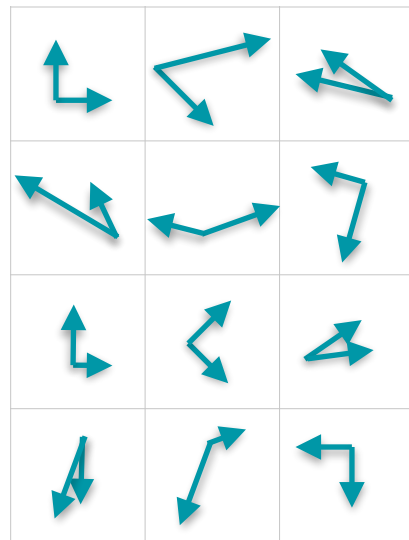
**Bases**



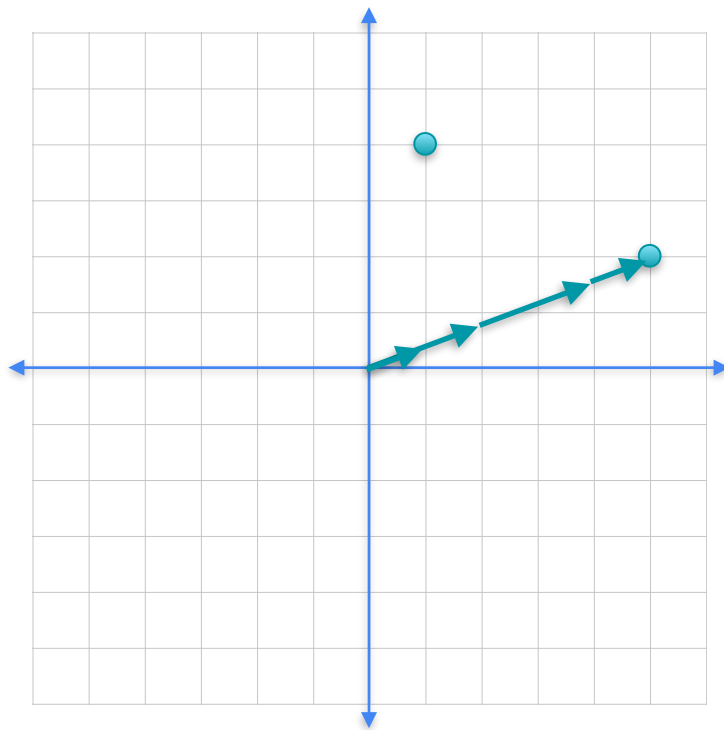
# Bases



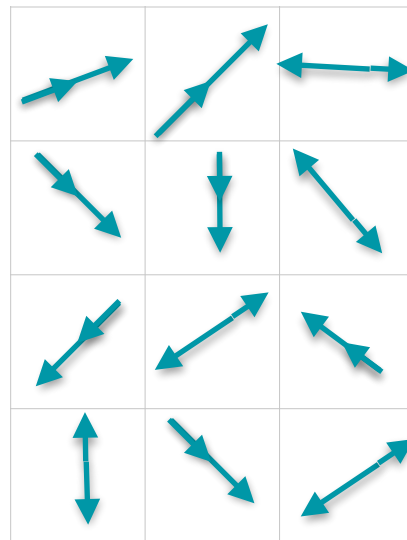
**Bases**



# What is not a basis?



## Not bases





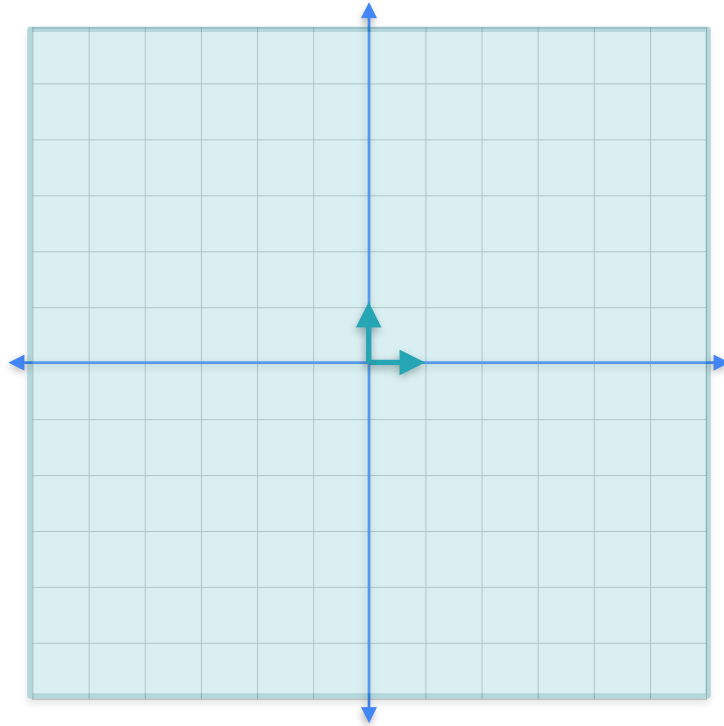
DeepLearning.AI

# Determinants and Eigenvectors

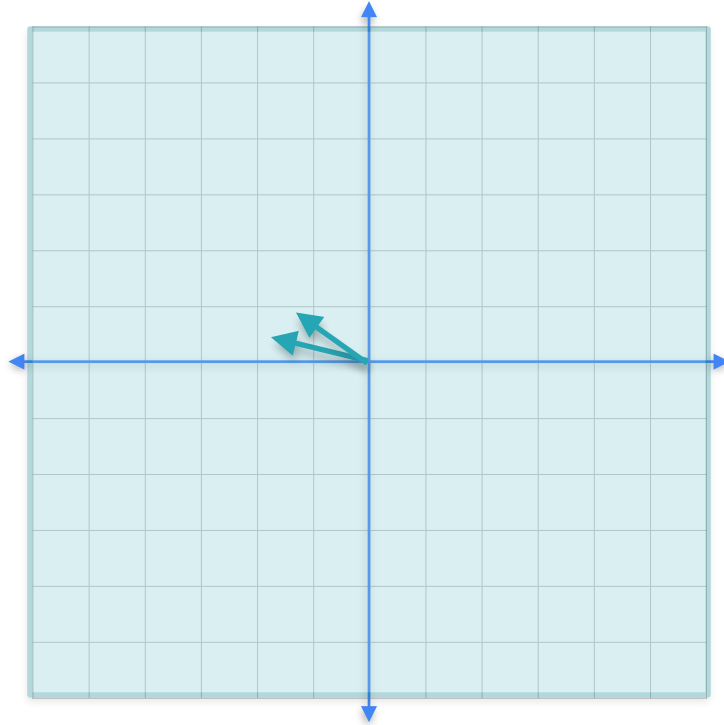
---

## Span

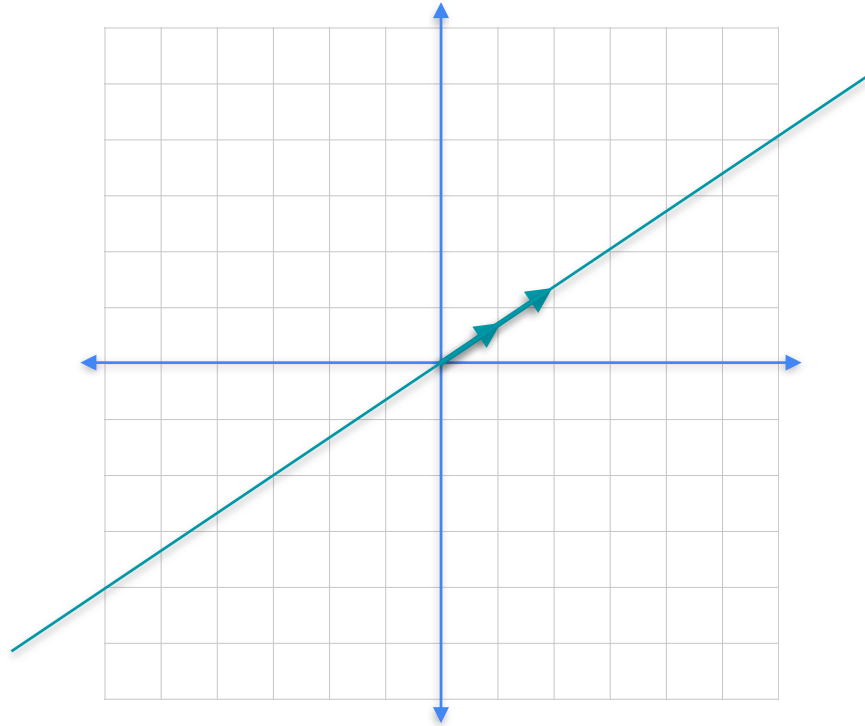
# Span



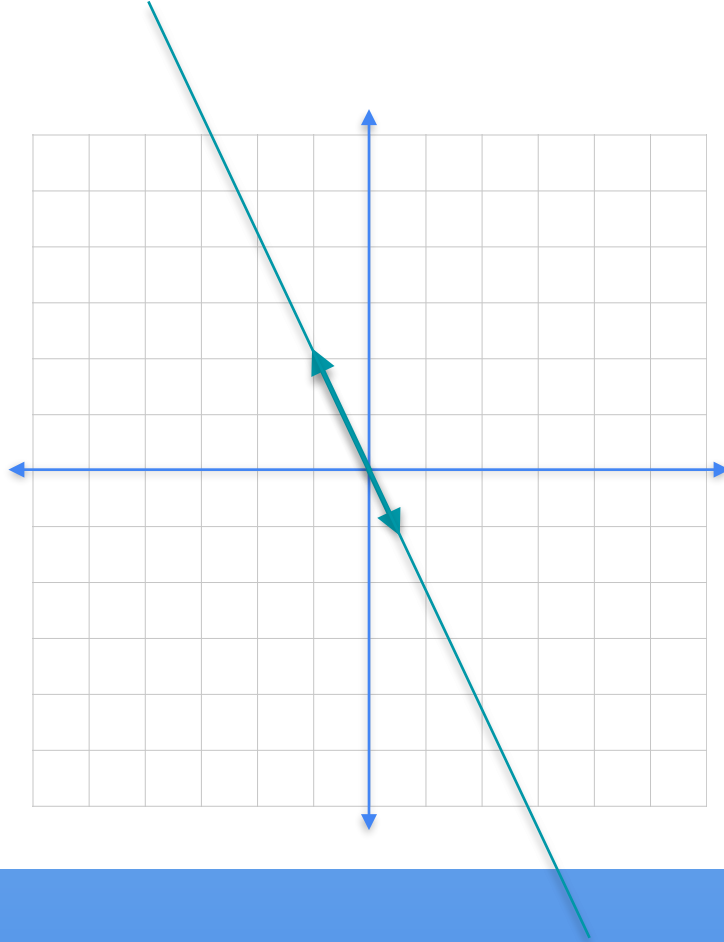
# Span



# Span

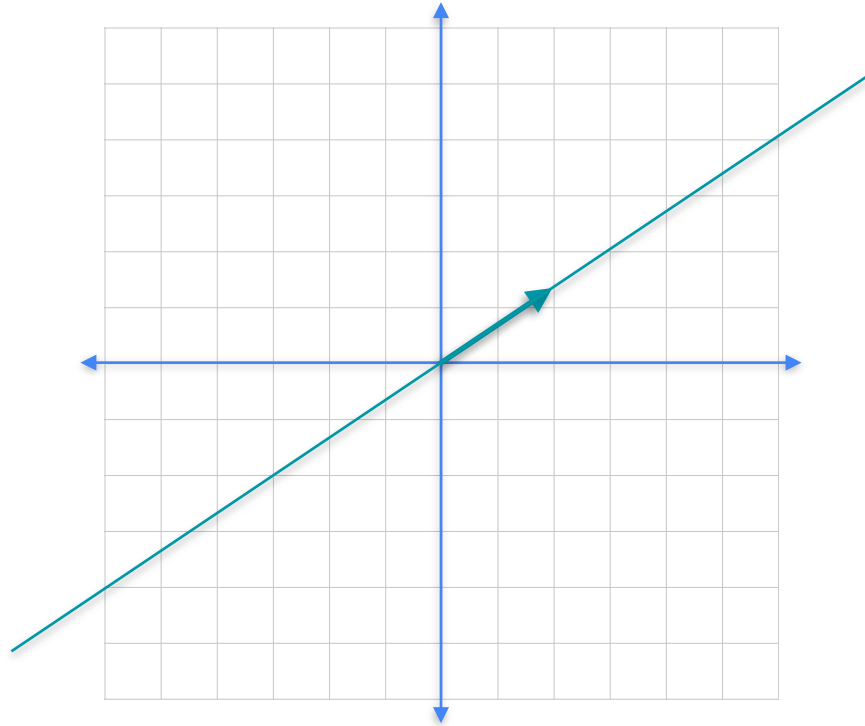


# Span

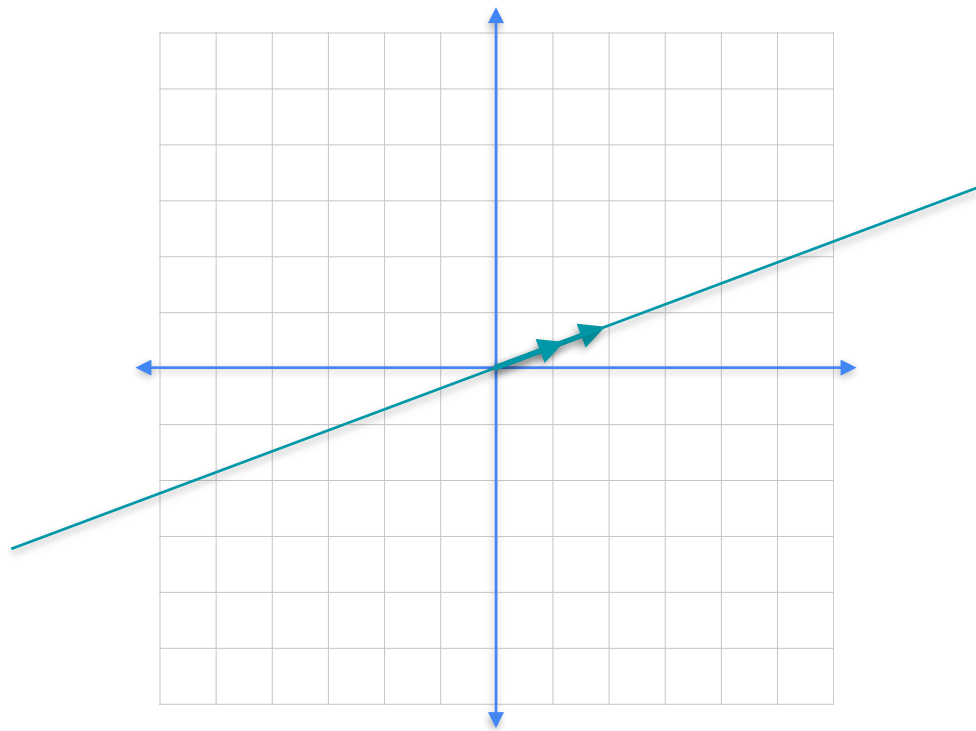




# Span

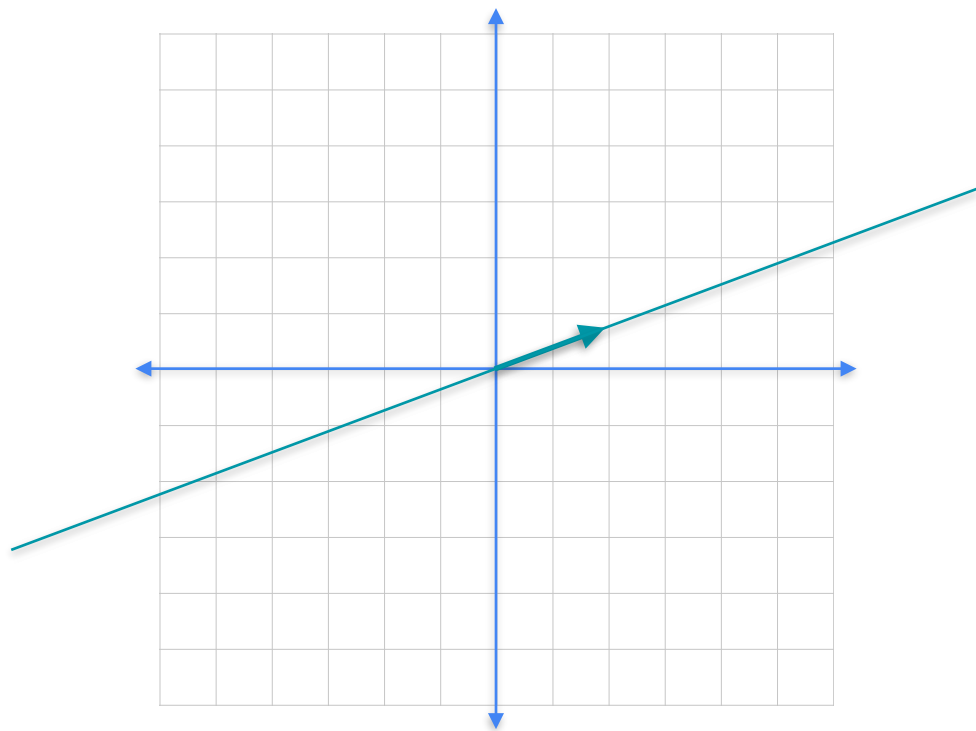


# Is this a basis?

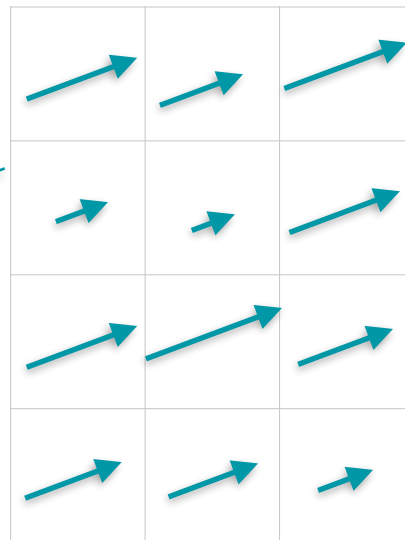


**No**

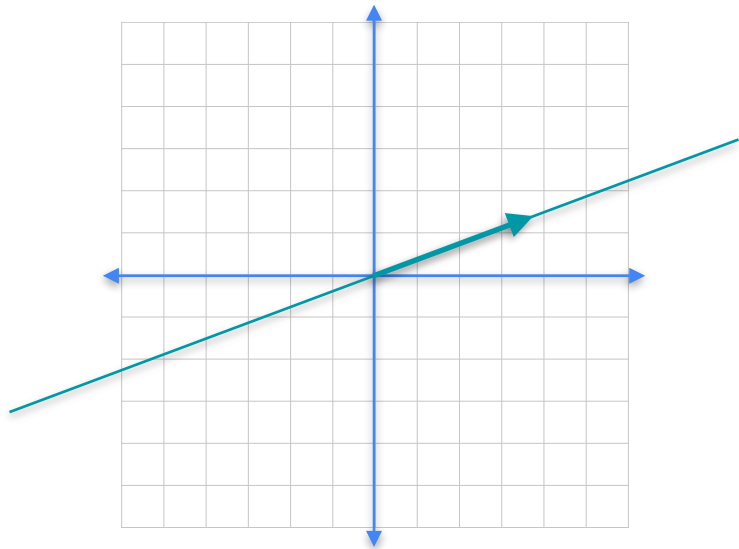
# Is this a basis for something?



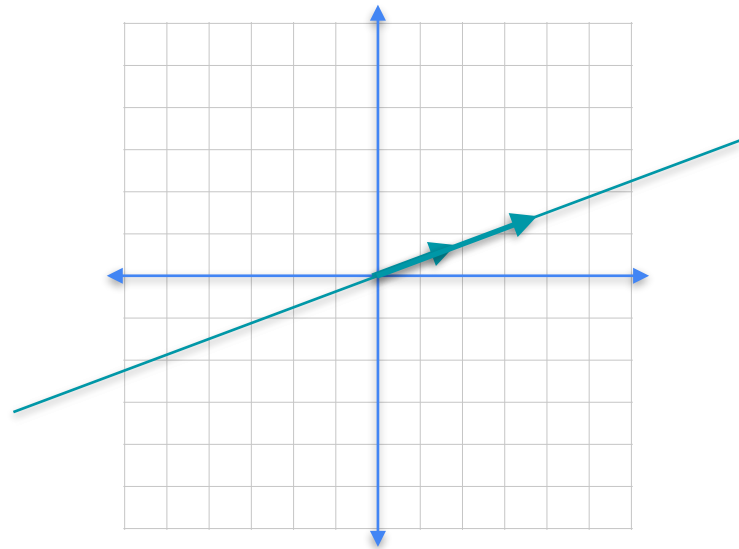
**Bases**



# A basis is a minimal spanning set

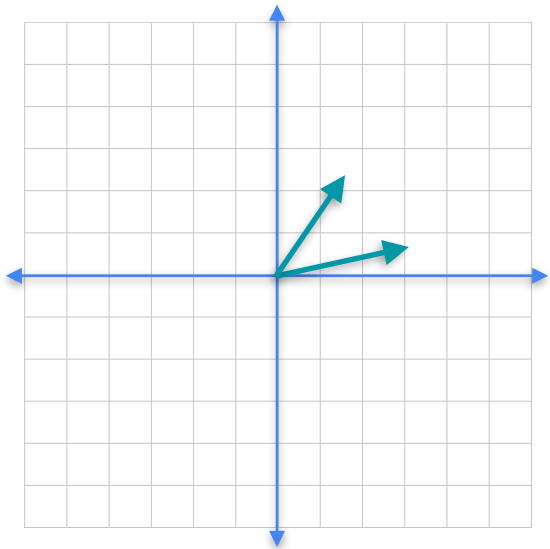


Basis

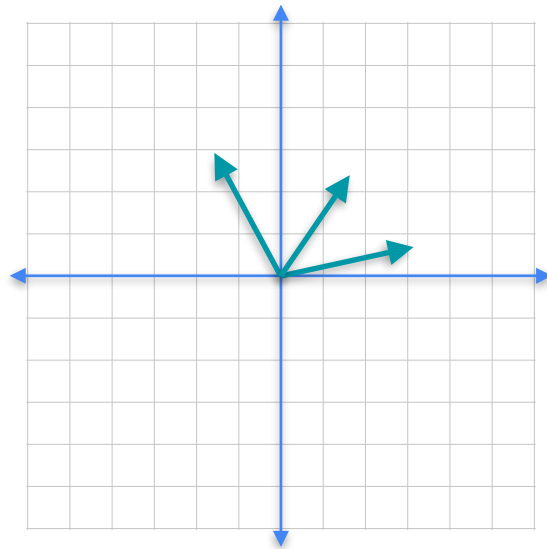


Not a basis

# A basis is a minimal spanning set

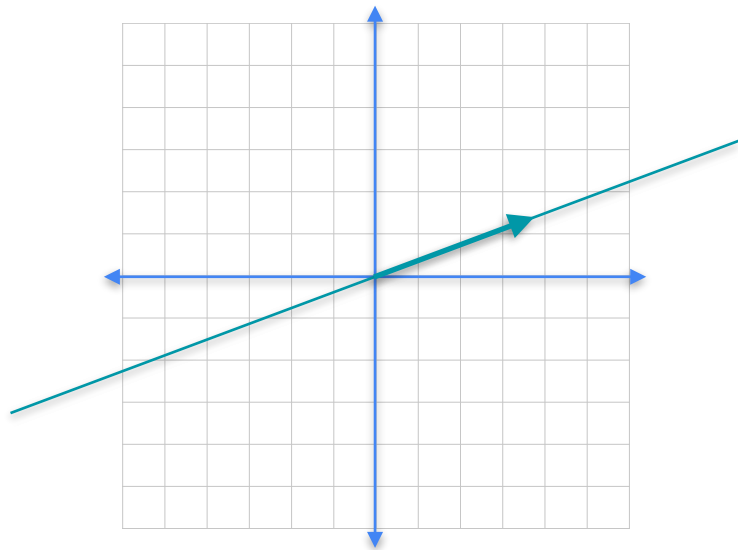


Basis

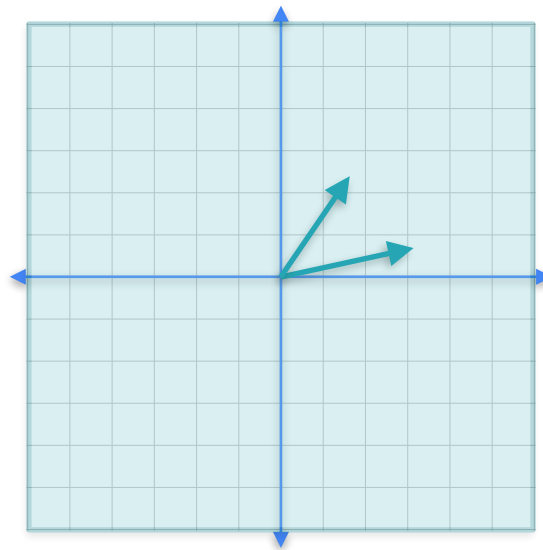


Not a basis

# Number of elements in the basis is the dimension

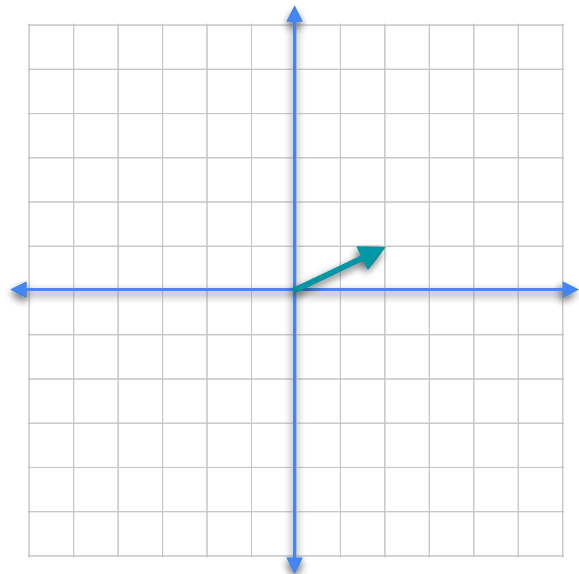


Dimensions: 1  
1 element in the basis



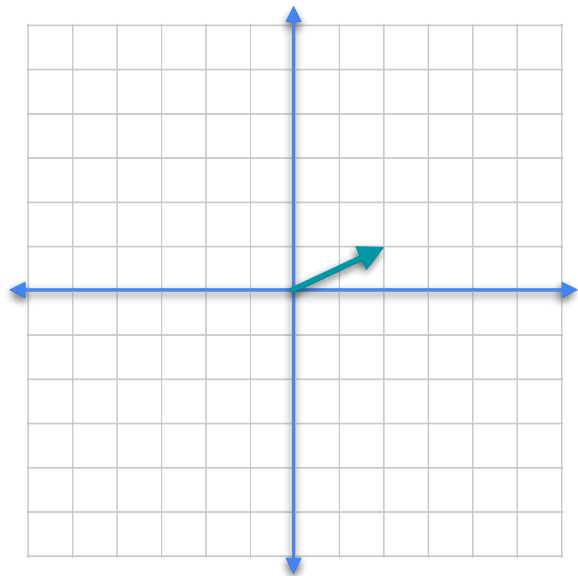
Dimensions: 2  
2 elements in the basis

# Linearly independent and linearly dependent vectors

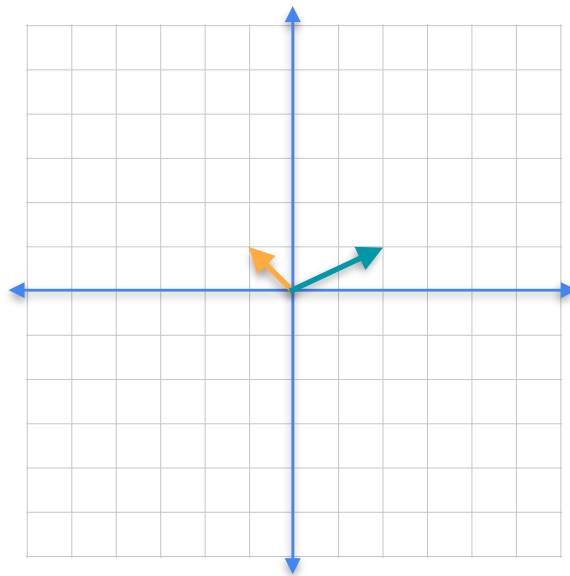


Linearly independent

# Linearly independent and linearly dependent vectors



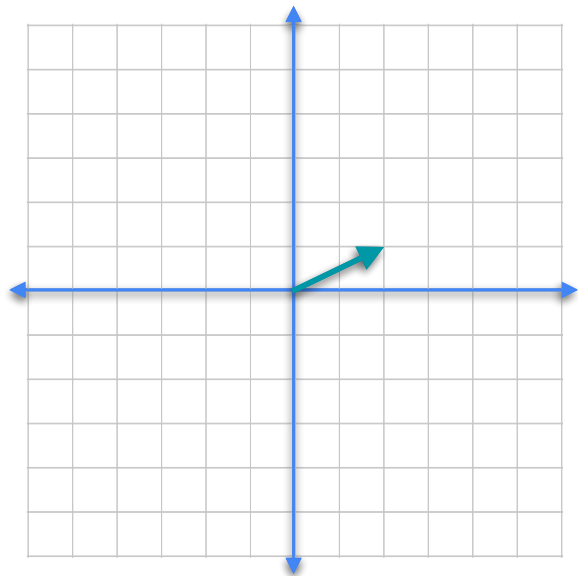
Linearly independent



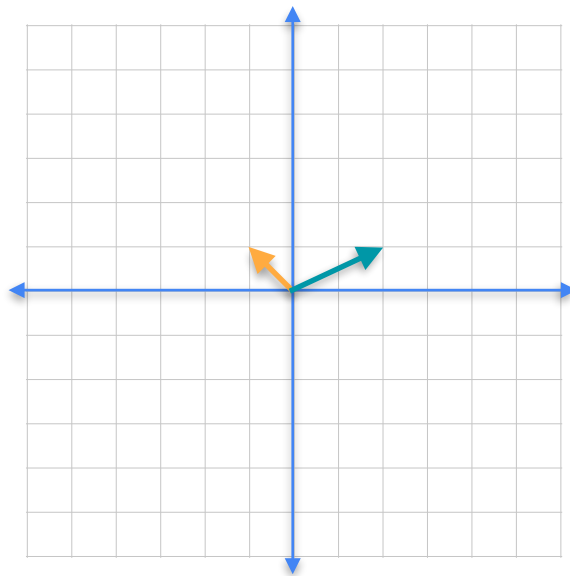
Linearly independent



# Linearly independent and linearly dependent vectors

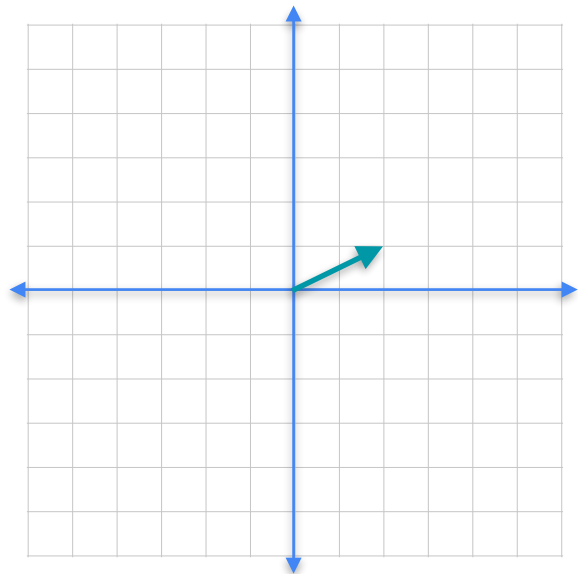


Linearly independent

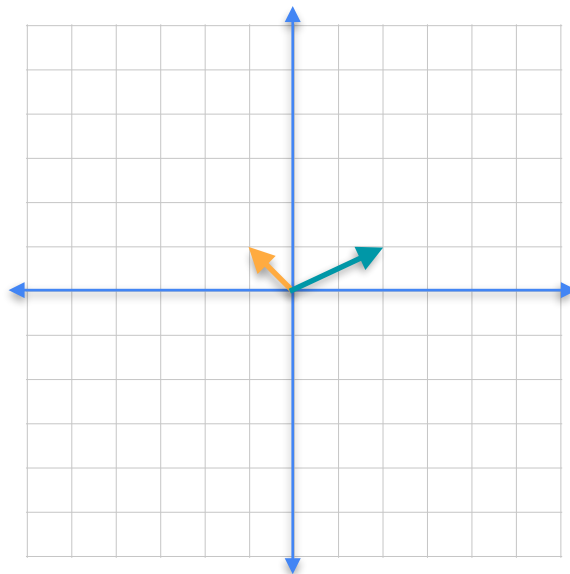


Linearly independent

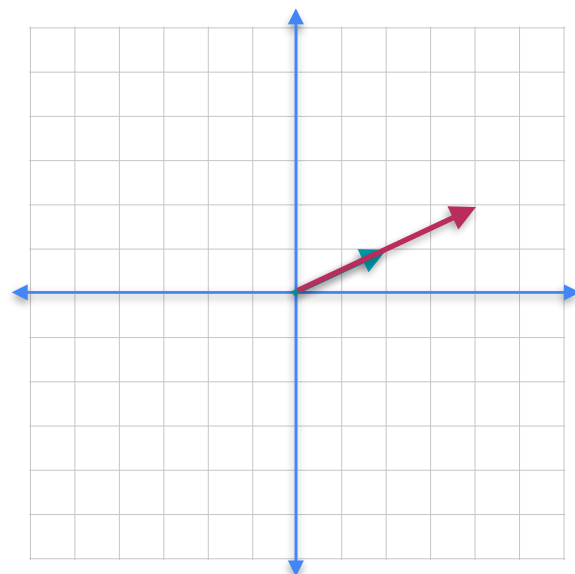
# Linearly independent and linearly dependent vectors



Linearly independent

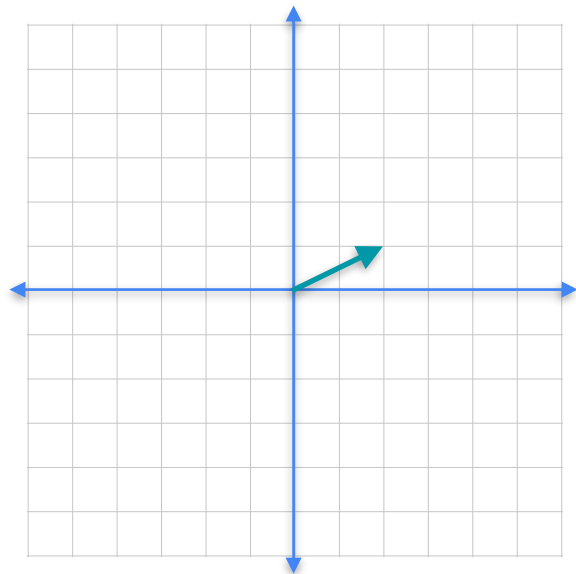


Linearly independent

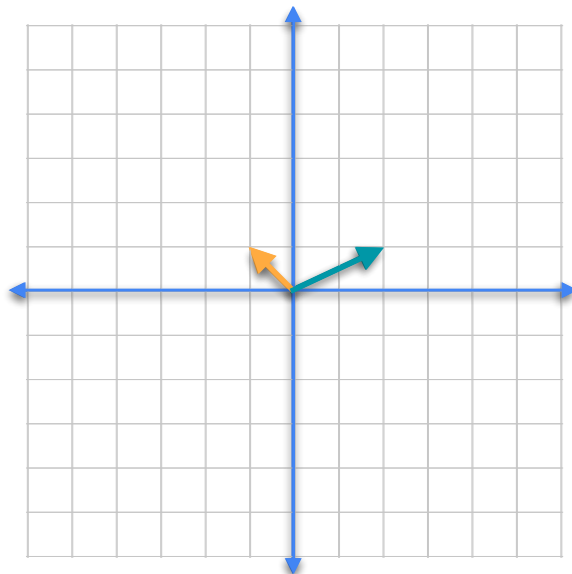


Linearly dependent

# Linearly independent and linearly dependent vectors

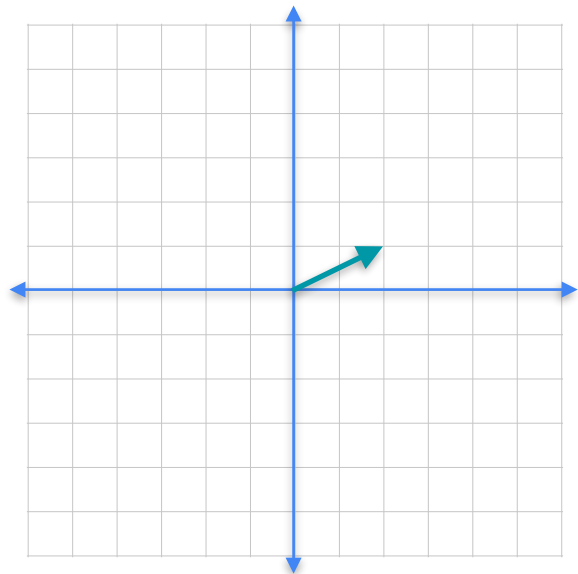


Linearly independent

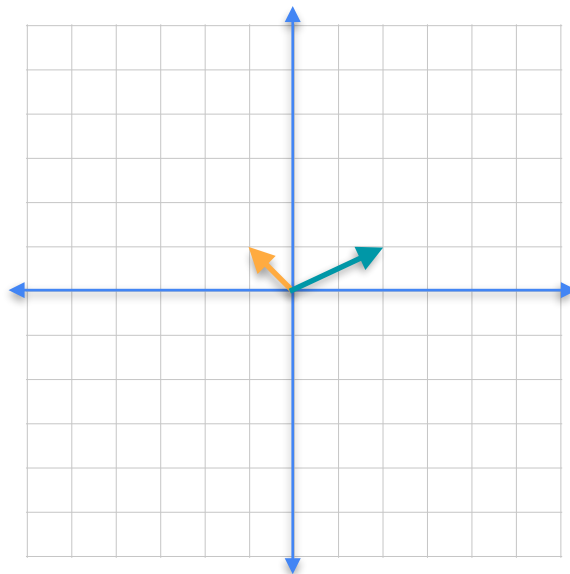


Linearly independent

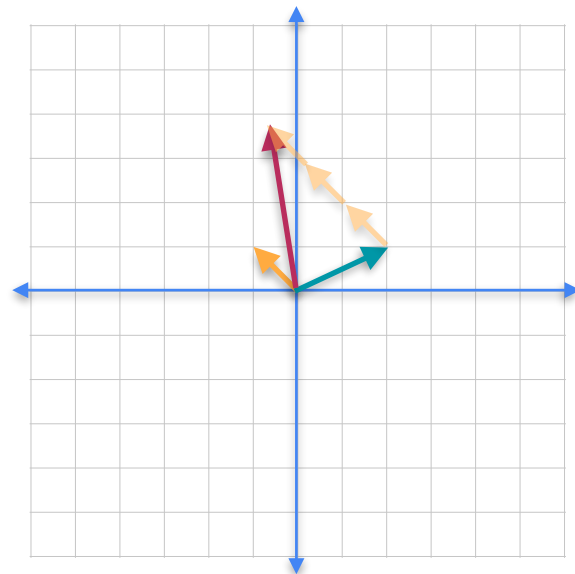
# Linearly independent and linearly dependent vectors



Linearly independent

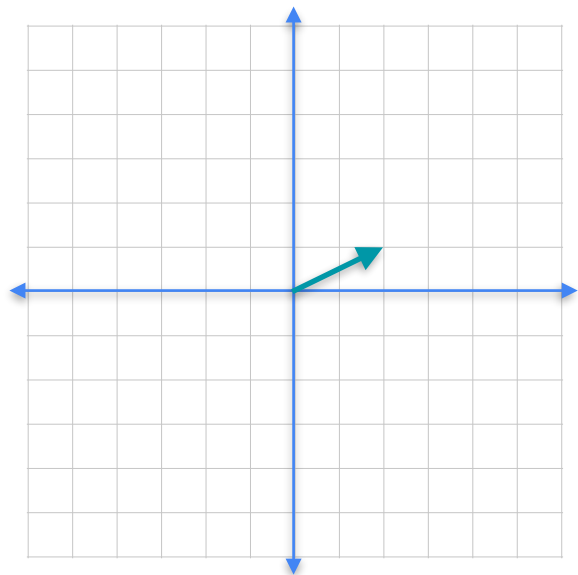


Linearly independent

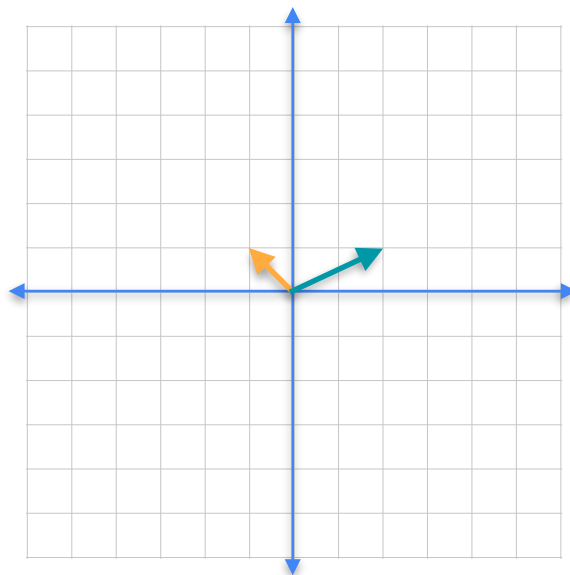


Linearly dependent

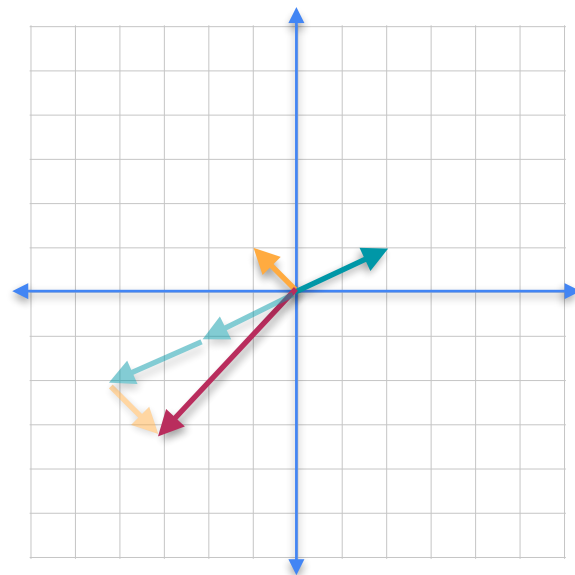
# Linearly independent and linearly dependent vectors



Linearly independent

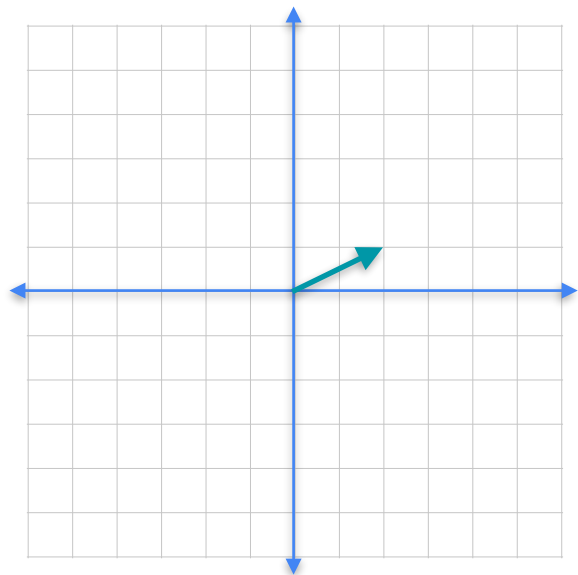


Linearly independent

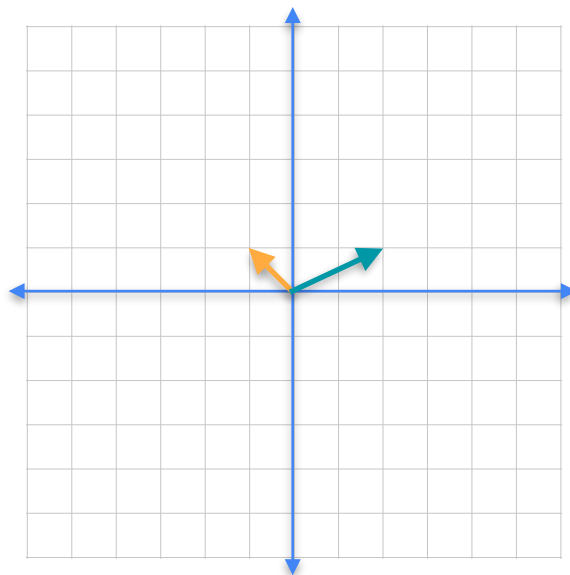


Linearly dependent

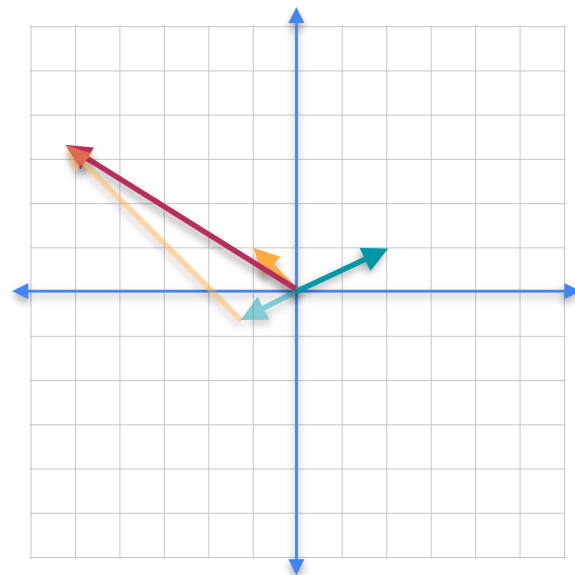
# Linearly independent and linearly dependent vectors



Linearly independent

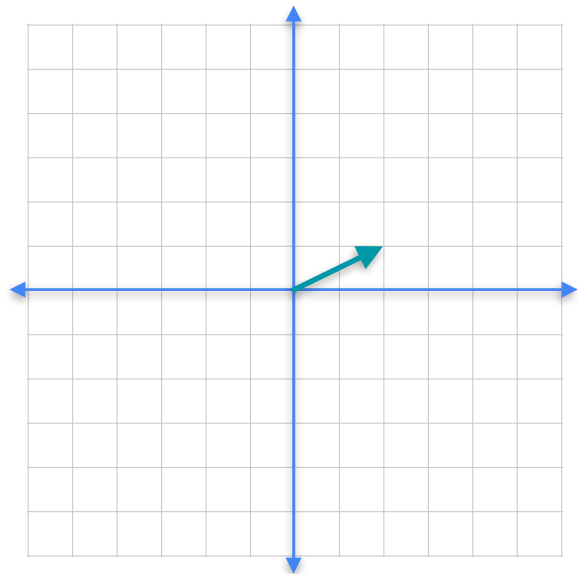


Linearly independent

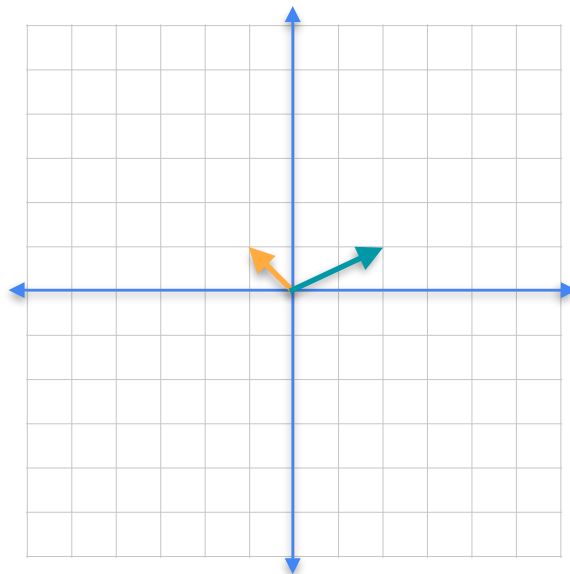


Linearly dependent

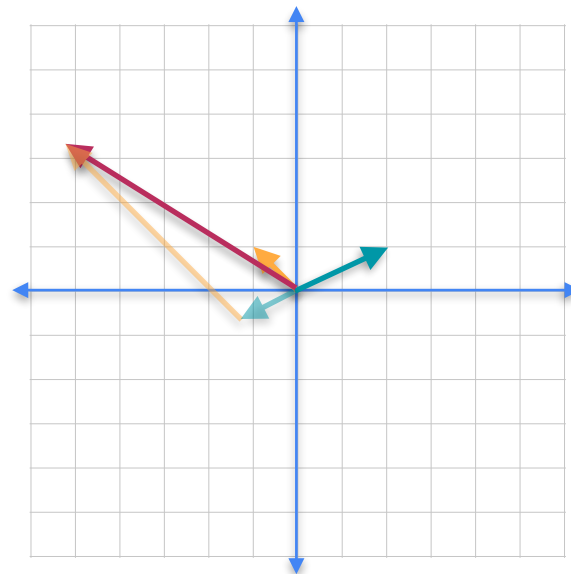
# Linearly independent and linearly dependent vectors



Linearly independent

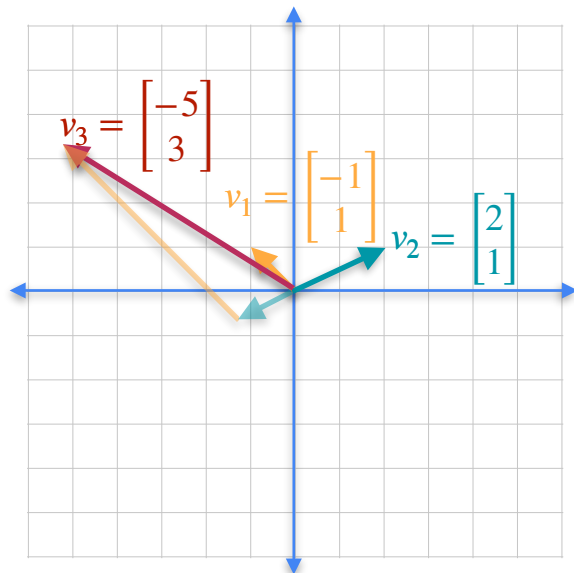


Linearly independent



Linearly dependent

# Let's see how to check for linear dependence

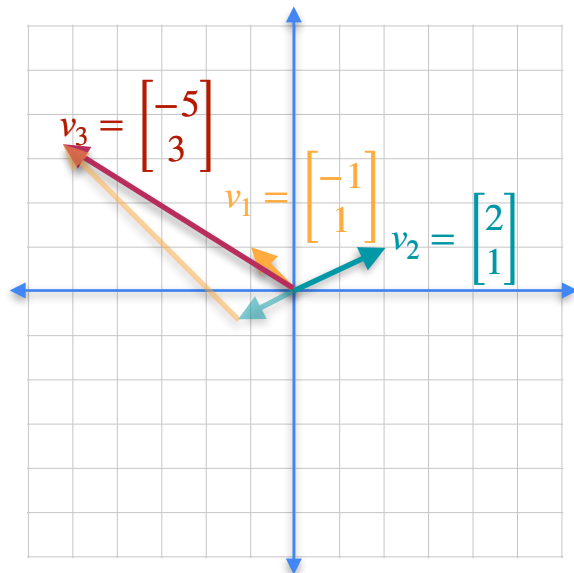


$$\alpha + \beta =$$

Linearly dependent



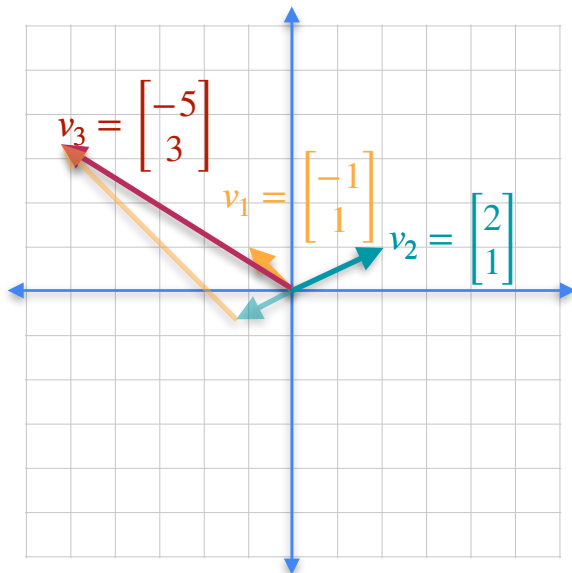
# Let's see how to check for linear dependence



$$\alpha v_1 + \beta v_2 = v_3$$

Linearly dependent

# Let's see how to check for linear dependence



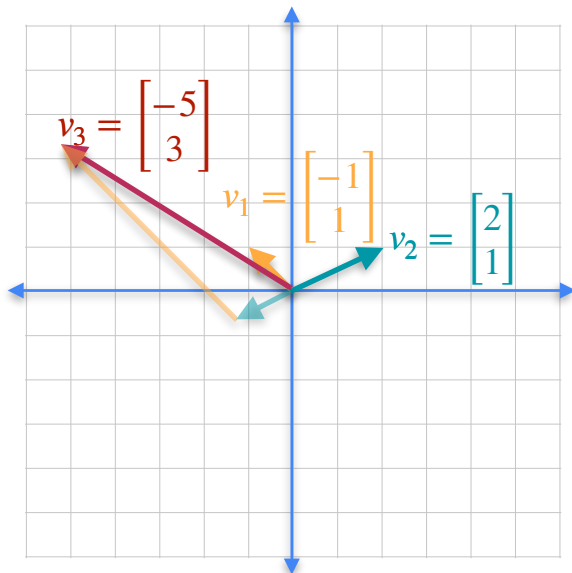
Linearly dependent

$$\alpha v_1 + \beta v_2 = v_3$$

$$\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

1	$-\alpha + 2\beta = -5$
2	$\alpha + \beta = 3$

# Let's see how to check for linear dependence



Linearly dependent

$$\alpha v_1 + \beta v_2 = v_3$$

$$\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

$v_3$  is a linear combination  
of  $v_1$  and  $v_2$

$$\begin{matrix} 1 \\ 2 \end{matrix} \quad \begin{matrix} -\alpha + 2\beta = -5 \\ \alpha + \beta = 3 \end{matrix}$$

$$\begin{matrix} 1 \\ 2 \end{matrix} \quad \alpha + \beta = 3$$

$$\begin{matrix} 1 \\ 2 \end{matrix} + \begin{matrix} 2 \\ 1 \end{matrix}$$

$$3\beta = -2 \rightarrow \beta = -\frac{2}{3}$$

$$\begin{matrix} 2 \\ 1 \end{matrix}$$

$$\alpha - \frac{2}{3} = 3 \rightarrow \alpha = \frac{11}{3}$$

# Quiz

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

# Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$1 \quad -1 \quad =$$

Linearly dependent

# Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Linearly dependent

# Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

**Not a basis!**

Linearly independent

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

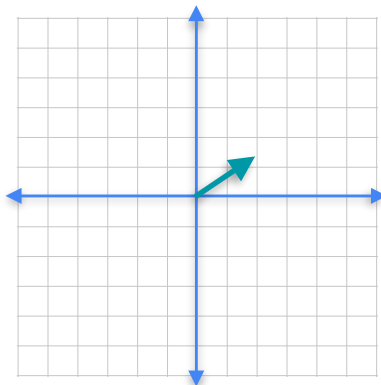
# Basis: a formal definition

A basis is a set of vectors that:

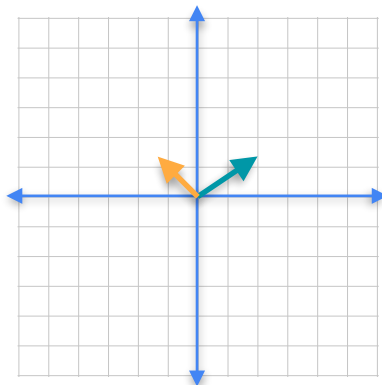
- Spans a vector space
- Is linearly independent



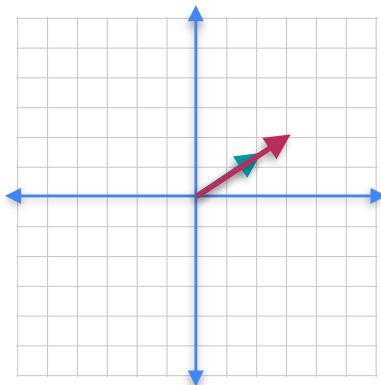
Not all sets of  $N$  vectors are a basis  
for  $N$ -dimensional space



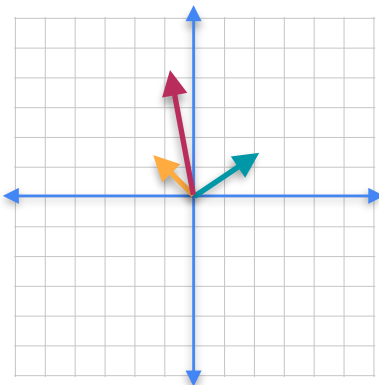
Spans a line  
Linearly independent  
Is a basis



Spans the plane  
Linearly independent  
Is a basis



Spans a line  
Linearly dependent  
Not a basis



Spans the plane  
Linearly dependent  
Not a basis





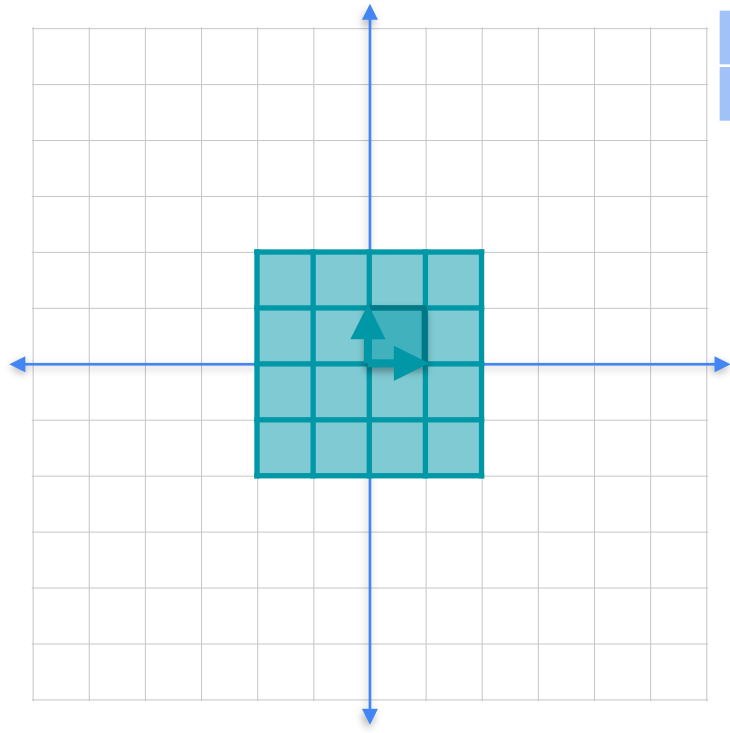
DeepLearning.AI

# Determinants and Eigenvectors

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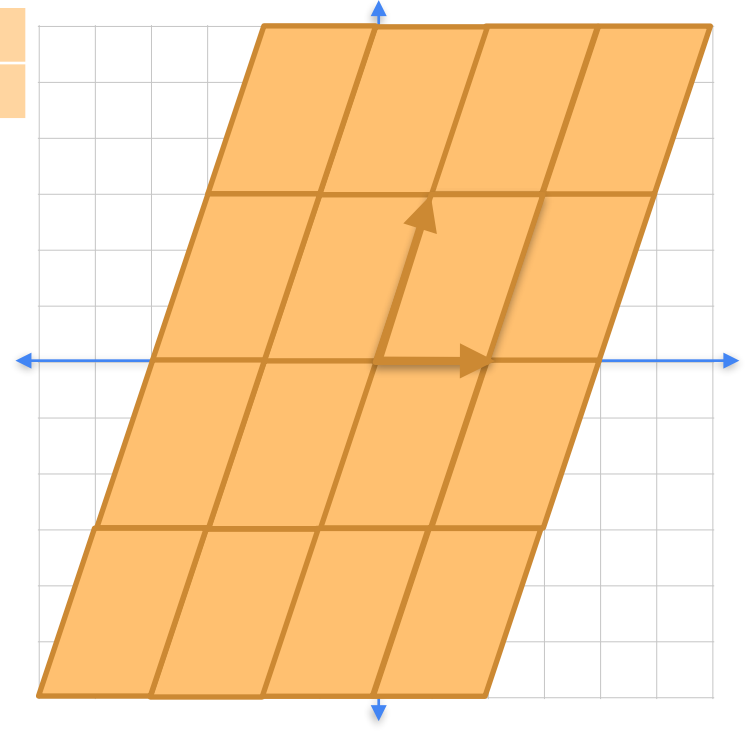
## **Eigenbasis**

# Basis

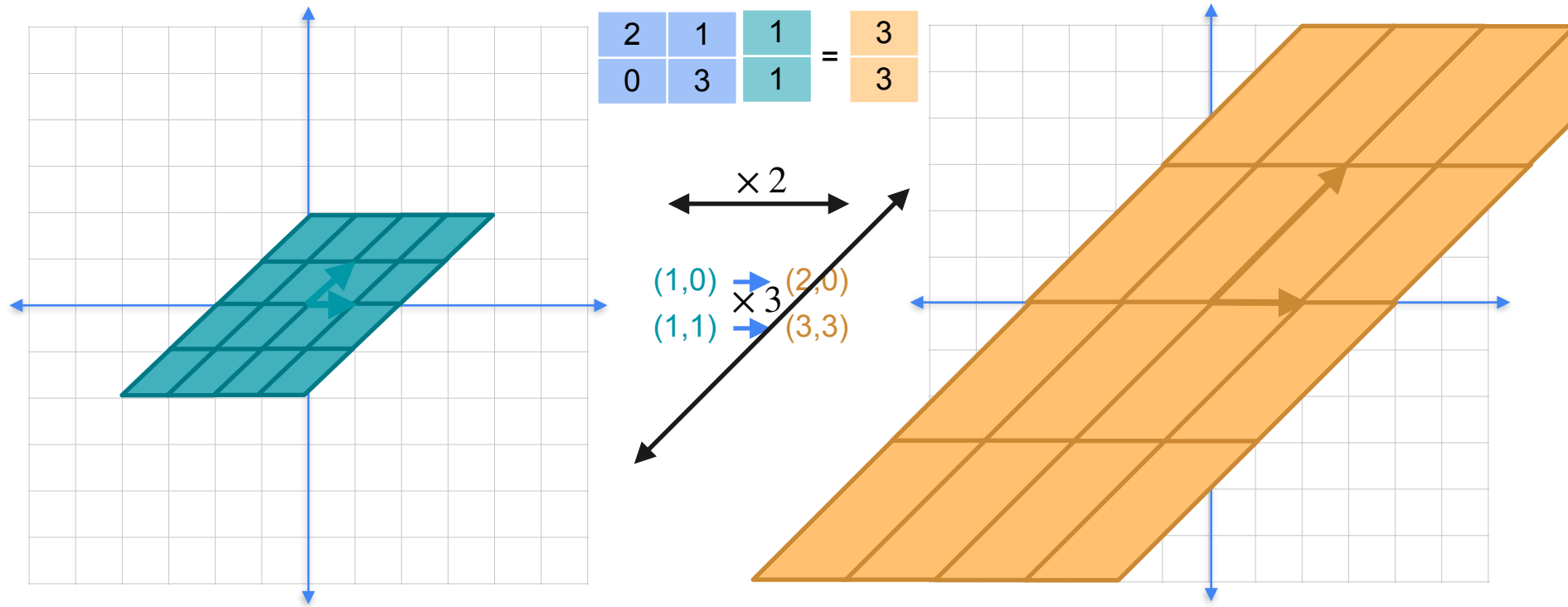


2	1	0	=	1
0	3	1		3

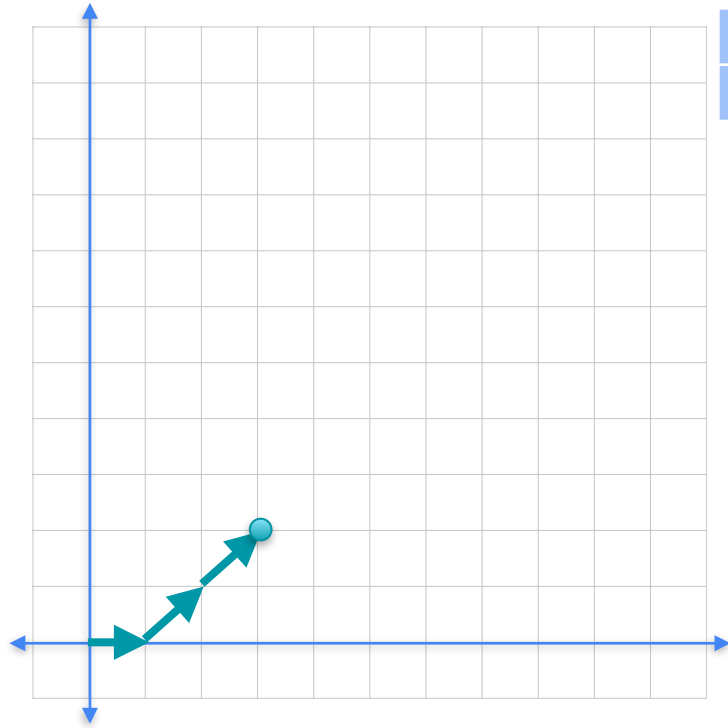
$(1,0) \rightarrow (2,0)$   
 $(0,1) \rightarrow (1,3)$



# Eigenbasis

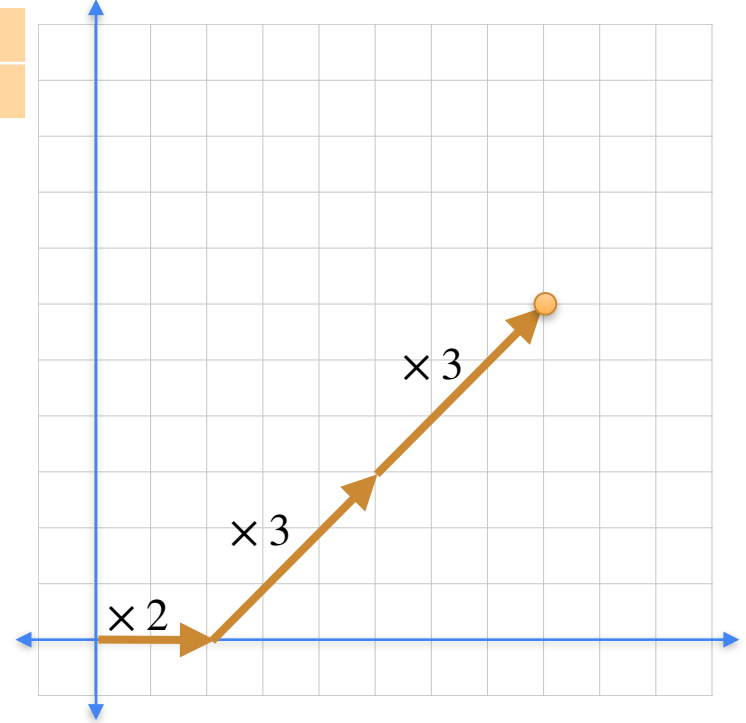


# Eigenbasis



2	1	3	=	8
0	3	2		6

$$(3,2) \rightarrow (8,6)$$





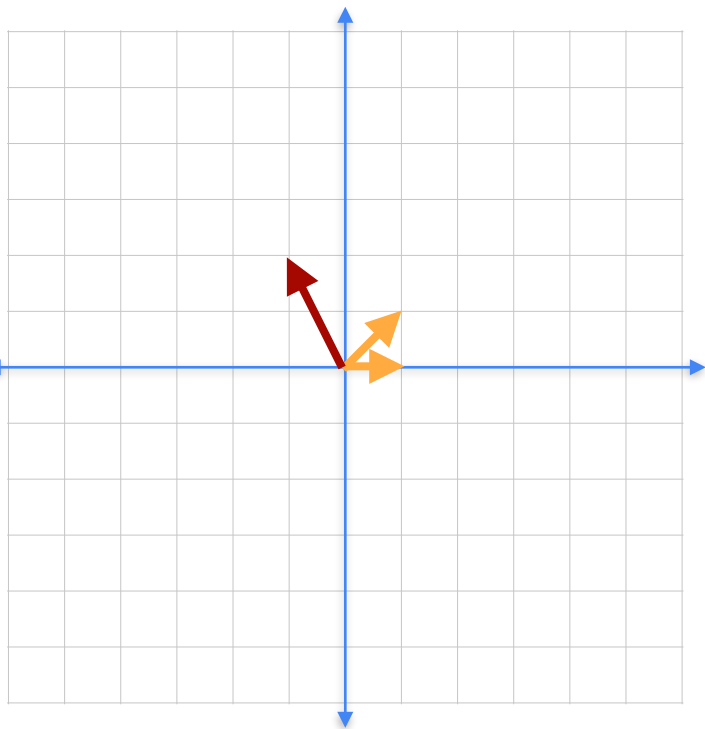
DeepLearning.AI

# Determinants and Eigenvectors

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## **Eigenvalues and Eigenvectors**

# Eigenvalues and eigenvectors

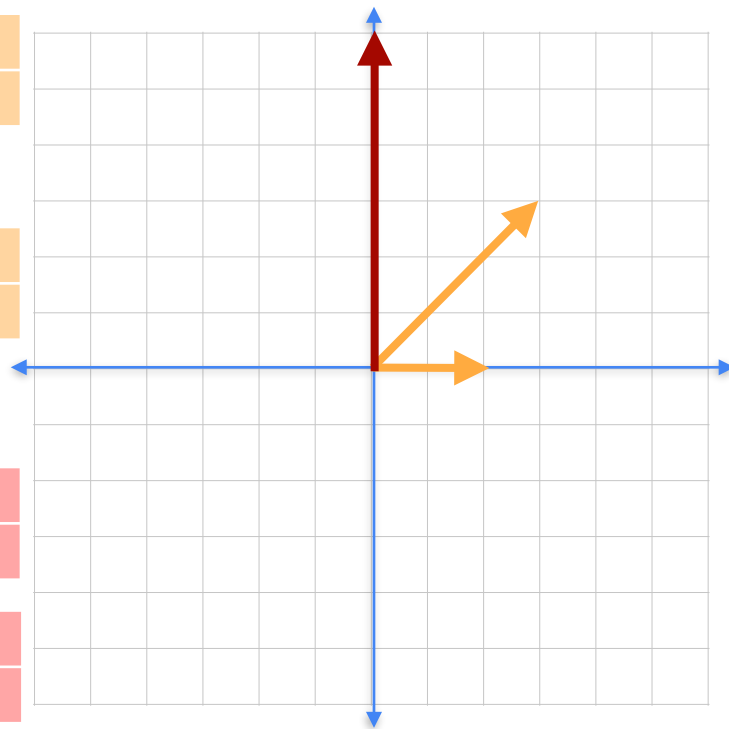


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

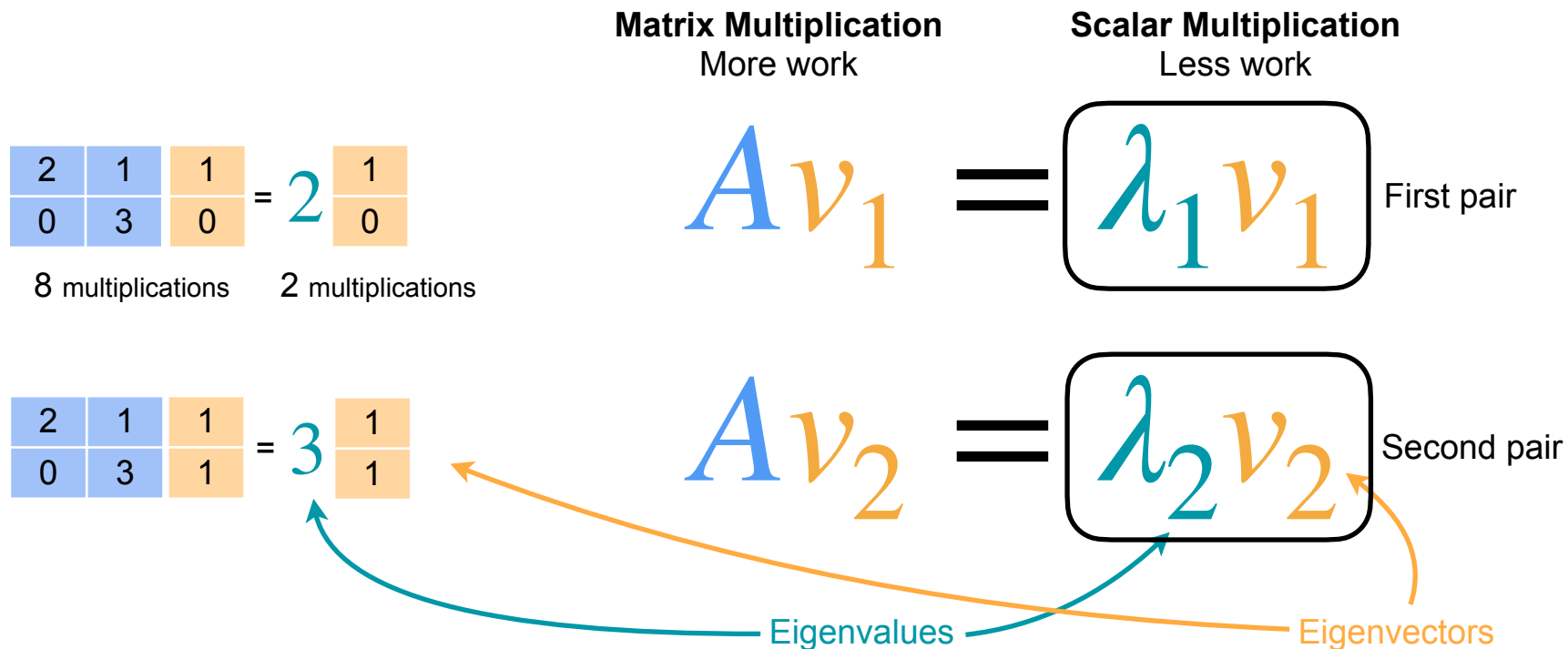
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

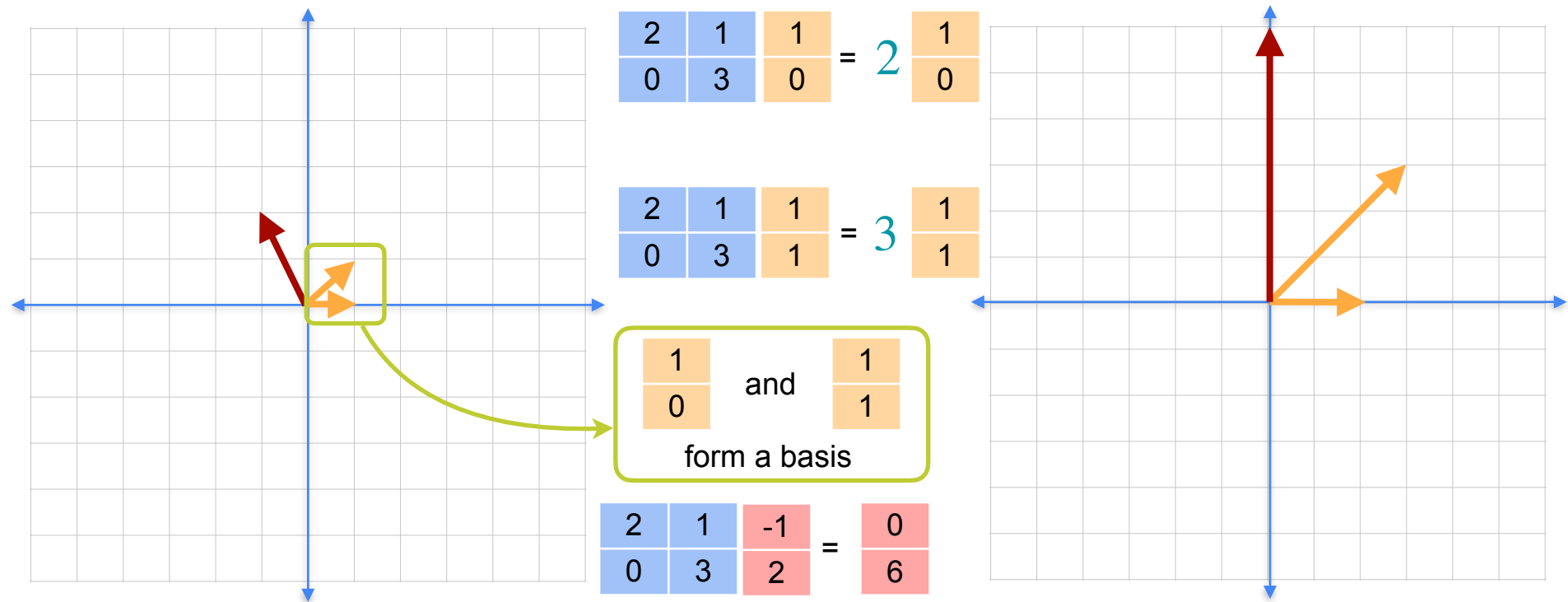
$$\neq \lambda \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



# Eigenvalues and eigenvectors

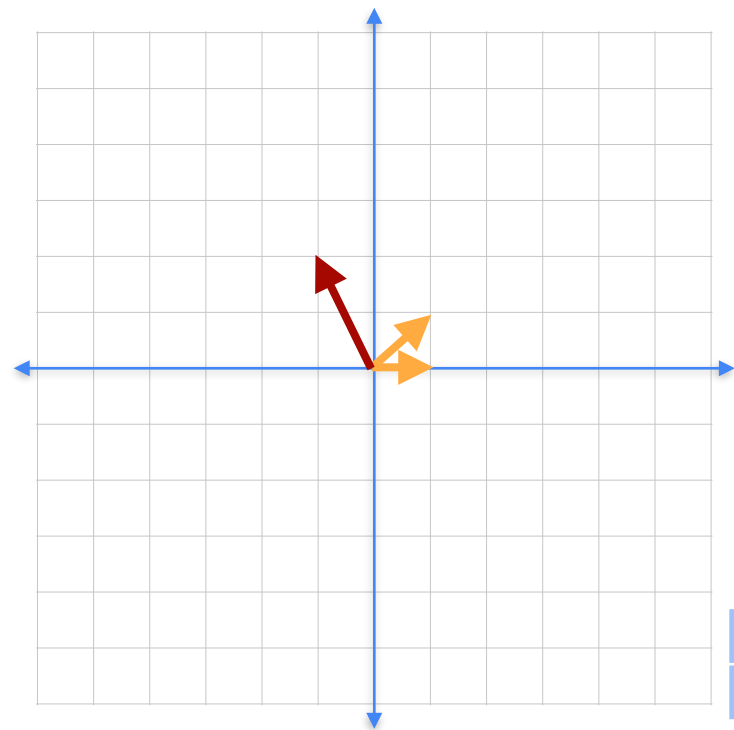


# Eigenvalues and eigenvectors





# Eigenvalues and eigenvectors

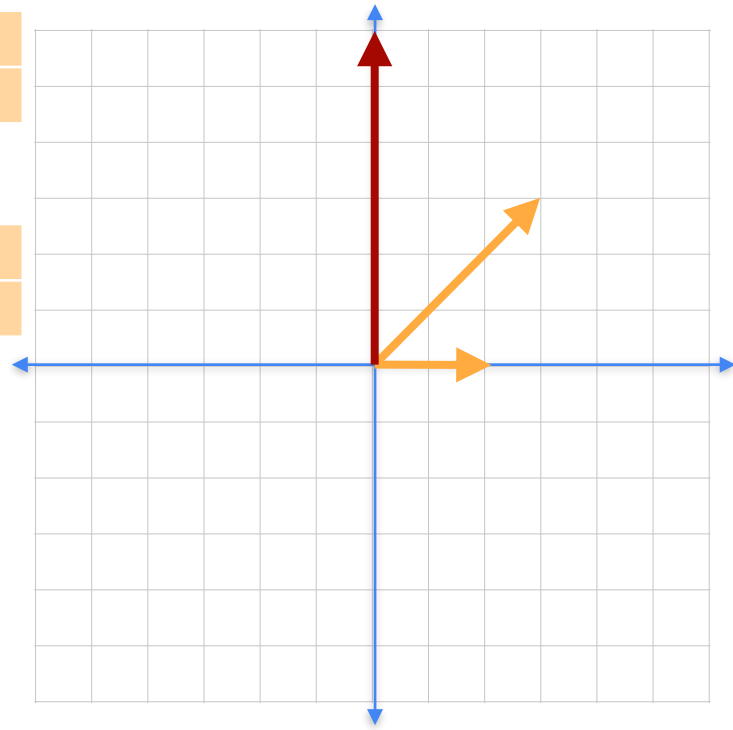


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

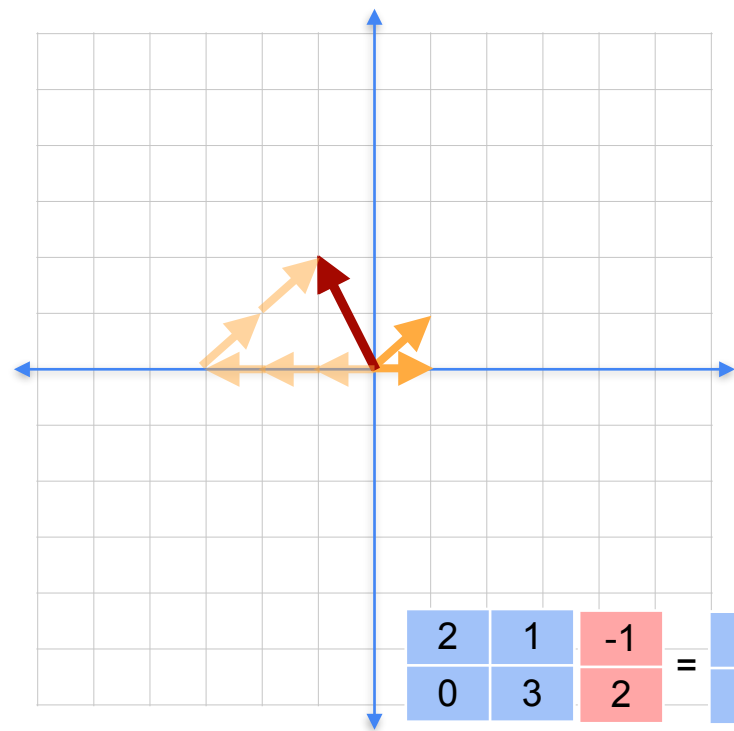
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$   
form a basis

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$



# Eigenvalues and eigenvectors

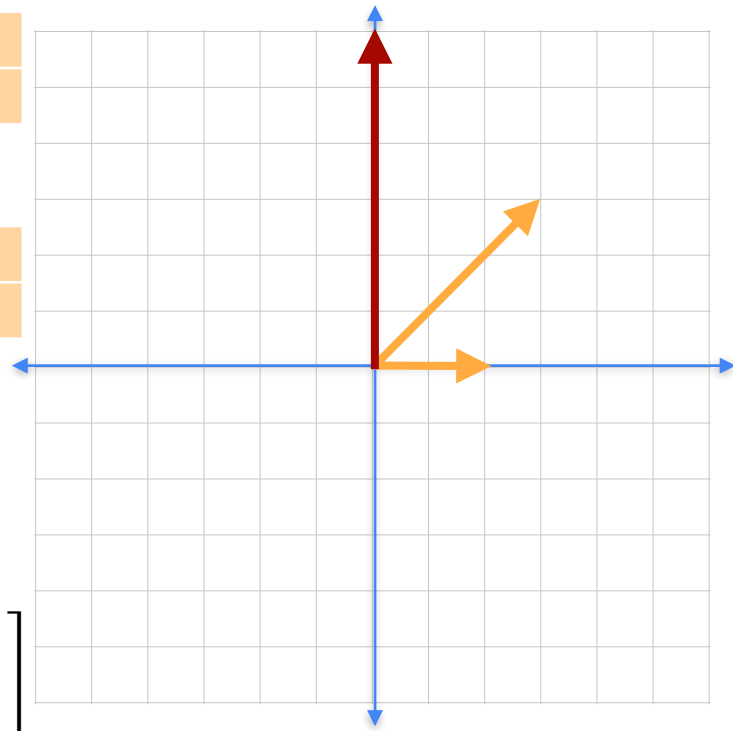


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

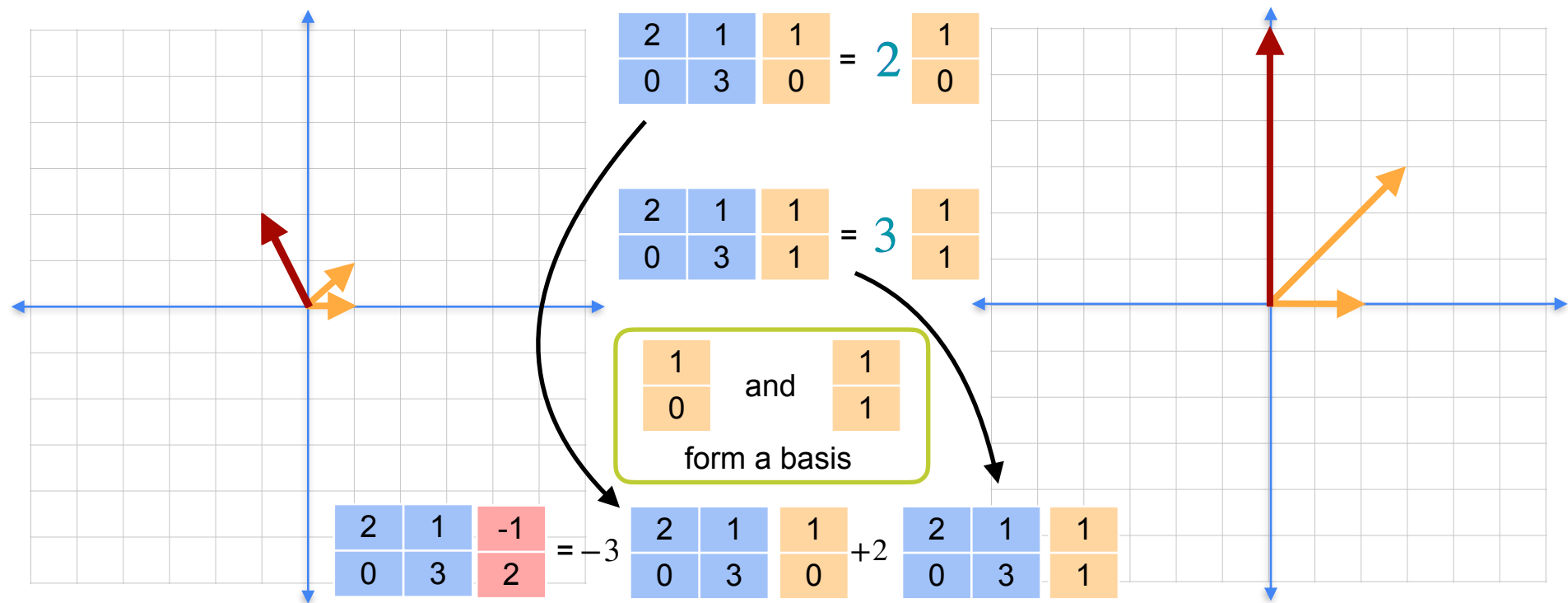
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$   
 form a basis

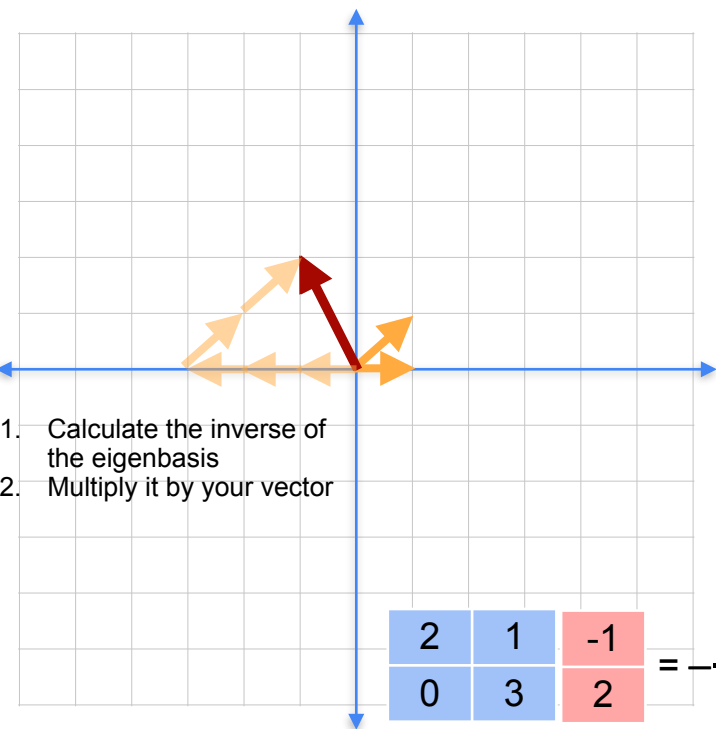
$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -3 & +2 \end{bmatrix}$$



# Eigenvalues and eigenvectors



# Eigenvalues and eigenvectors

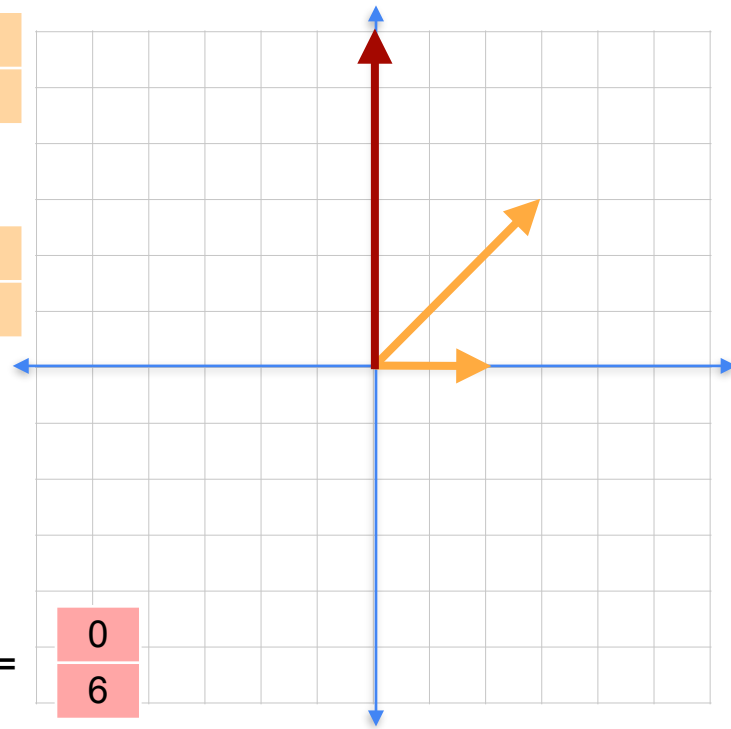


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ form a basis}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$



# Eigenvalues and eigenvectors

- $Av = \lambda v$  for each eigenvector / eigenvalue
- Eigenvectors: direction of stretch
- Eigenvalues: how much stretch
- Eigenbasis: the set of a matrix's eigenvectors, can be arranged as a matrix with one eigenvector in each column
- Save work and characterize a transformation



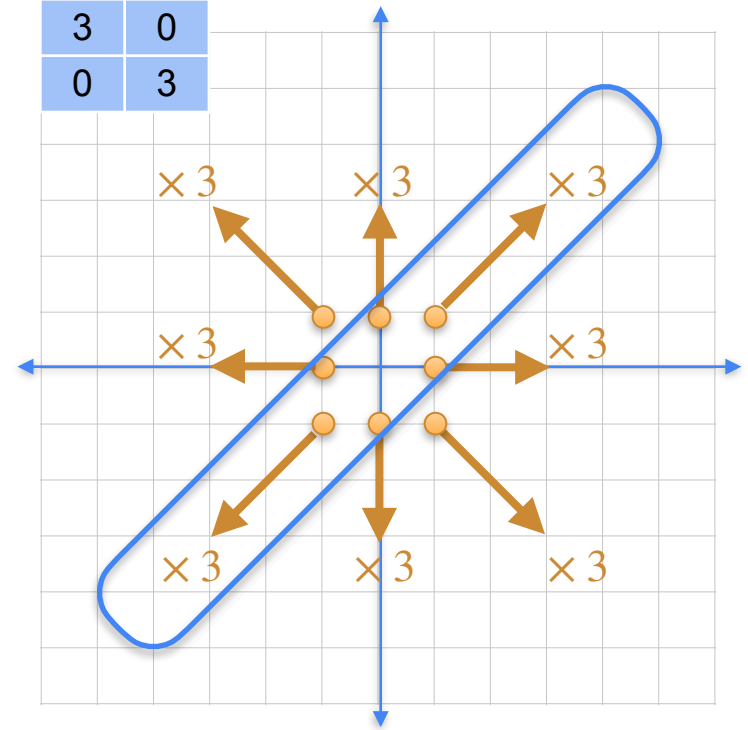
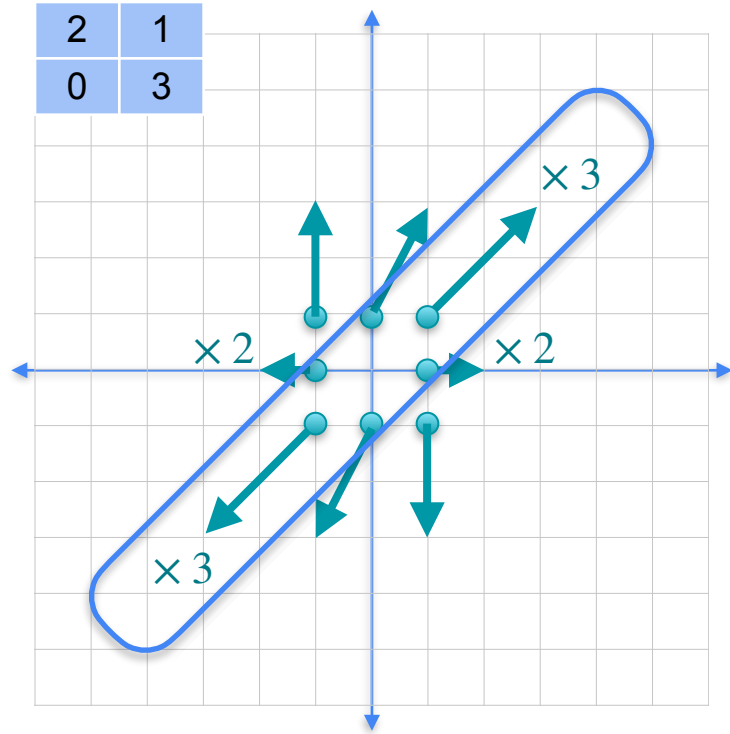
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# Determinants and Eigenvectors

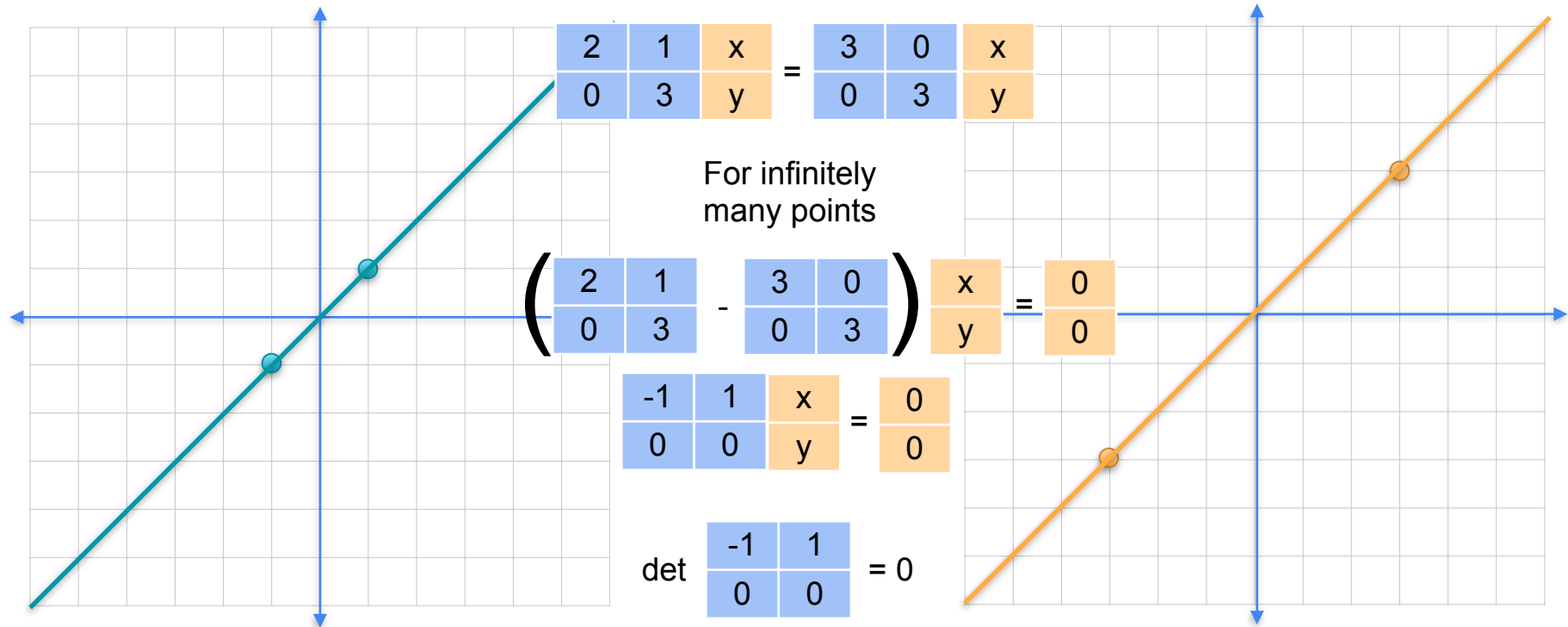
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## **Calculating eigenvalues and eigenvectors**

# Finding eigenvalues

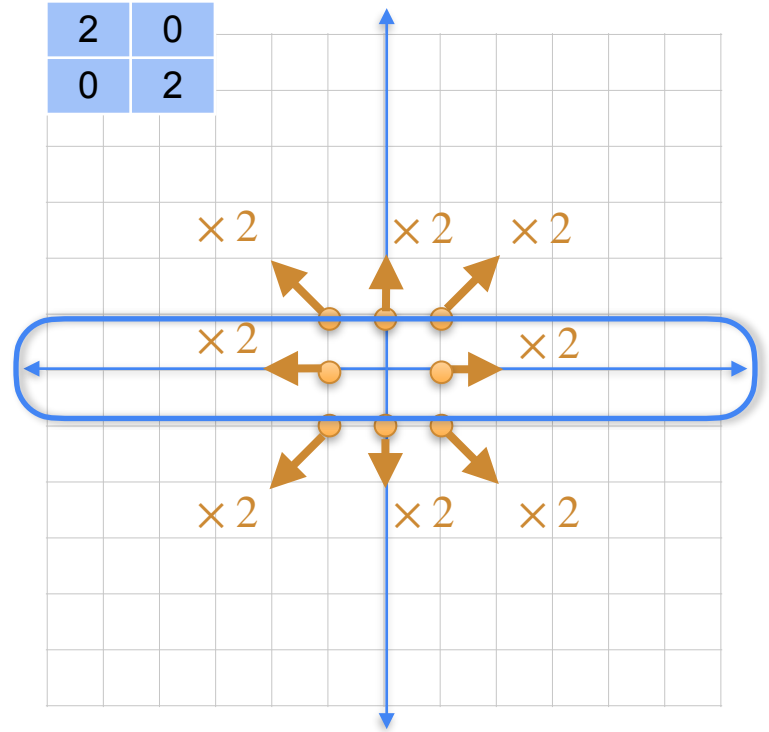
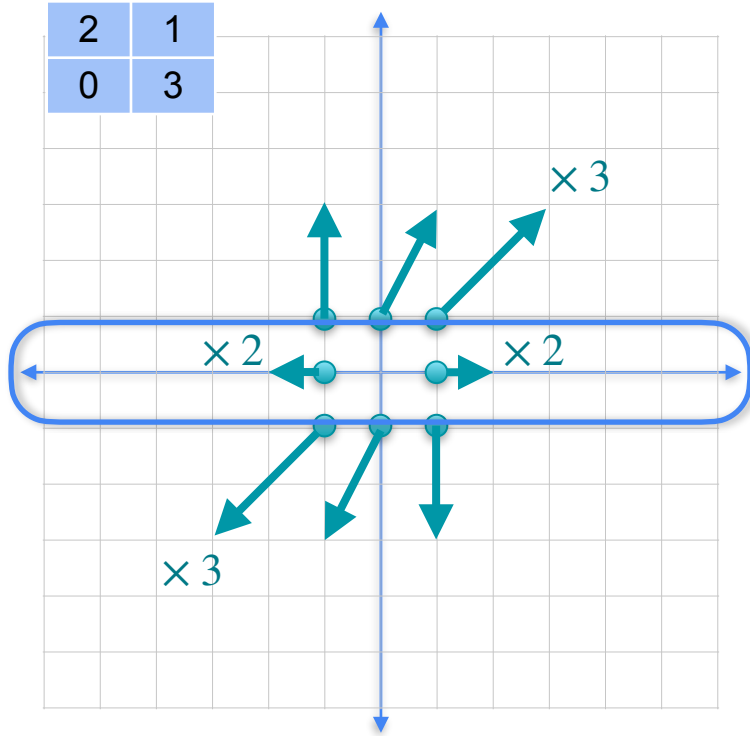


# Finding eigenvalues





# Finding eigenvalues



# Finding eigenvalues

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

For infinitely many points

$$\left( \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = 0$$

# Finding eigenvalues

If  $\lambda$  is an eigenvalue:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For infinitely many (x,y)

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Has infinitely many solutions

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

Characteristic polynomial

$$(2 - \lambda)(3 - \lambda) - 1 \cdot 0 = 0$$

$$\begin{aligned} \lambda &= 2 \\ \lambda &= 3 \end{aligned}$$

# Finding eigenvectors

Eigenvalues:  $\lambda = 2$   
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 2x$$

$$0x + 3y = 2y$$

$$x = 1$$

$$y = 0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 3x$$

$$0x + 3y = 3y$$

$$x = 1$$

$$y = 1$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

# Quiz

- Find the eigenvalues and eigenvectors of this matrix:

9	4
4	3

# Solution

- Eigenvalues: 11, 1
- Eigenvectors: (2,1), (-1,2)

9	4
4	3

- The characteristic polynomial is

$$\det \begin{array}{|c|c|} \hline 9-\lambda & 4 \\ \hline 4 & 3-\lambda \\ \hline \end{array} = (9-\lambda)(3-\lambda) - 4 \cdot 4 = 0$$

- Which factors as  $\lambda^2 - 12\lambda + 11 = (\lambda - 11)(\lambda - 1)$

- The solutions are  $\lambda = 11$   
 $\lambda = 1$

# Finding eigenvalues

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

$$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

**Characteristic polynomial:**  $\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} 2 - \lambda & 1 & -1 \\ 1 & -\lambda & -3 \\ -1 & -3 & -\lambda \end{bmatrix} = 0$$

$$(2 - \lambda)\lambda^2 + 3 - 3 - 9(2 - \lambda) + \lambda + \lambda = -\lambda^3 + 2\lambda^2 + 11\lambda - 12 = 0$$

$$-(\lambda + 3)(\lambda - 1)(\lambda - 4) = 0$$

**Eigenvalues:**  $-3, 1, 4$

# Finding eigenvalues

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix} \quad \text{Eigenvalues: } -3, 1, 4$$

$$Av = \lambda v$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + x_2 - x_3 \\ x_1 - 3x_3 \\ -x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$



# Finding eigenvalues

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

**Eigenvalues:**  $-3, 1, 4$

$$Av = \lambda v$$

$$\begin{bmatrix} 2 & 1 & -1 & x_1 \\ 1 & 0 & -3 & x_2 \\ -1 & -3 & 0 & x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + x_2 - x_3 \\ x_1 - 3x_3 \\ -x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 4x_1 \\ x_1 - 3x_3 &= 4x_2 \\ -x_1 - 3x_2 &= 4x_3 \end{aligned}$$

$$\begin{aligned} R_1 \quad -2x_1 + x_2 - x_3 &= 0 \\ R_2 \quad x_1 - 4x_2 - 3x_3 &= 0 \\ R_3 \quad -x_1 - 3x_2 - 4x_3 &= 0 \end{aligned}$$

$$\begin{aligned} R_2 + R_3 \quad -7x_2 - 7x_3 &= 0 & 3R_1 + R_3 \quad -7x_1 - 7x_3 &= 0 \\ x_2 = -x_3 & & x_1 = -x_3 & \end{aligned}$$

$$\begin{aligned} x_1 &= k \\ x_2 &= k \\ x_3 &= -k \end{aligned}$$

infinite solutions  
of this form

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= -1 \end{aligned}$$

this works!

$$\begin{aligned} x_1 &= 2 \\ x_2 &= 2 \\ x_3 &= -2 \end{aligned}$$

so does this!

**Eigenvector:**

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

# Finding eigenvalues

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

**Eigenvalues**

$\lambda_1 = 4$

$\lambda_2 = 1$

$\lambda_3 = -3$

**Eigenvectors**

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

# Note on dimensions

Eigenvalues  $\longrightarrow$  Determinant  $\longrightarrow$  Square Matrix

9	4
4	3



9	4	5
4	3	-2





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# Determinants and Eigenvectors

---

**On the number of  
eigenvectors**

# Number of eigenvectors

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

3 by 3  
matrix

?

3 distinct  
eigenvalues

?

3 distinct  
eigenvectors

Eigenvalues

$$\lambda_1 = 4 \quad \lambda_2 = 1 \quad \lambda_3 = -3$$

Eigenvectors

1	0	2
1	1	-1
-1	1	1



# Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

Characteristic polynomial =  $\det(A - \lambda I) = \det$

$$\begin{bmatrix} 2 - \lambda & 0 & 0 \\ 1 & 4 - \lambda & 0.5 \\ 0 & 0 & 2 - \lambda \end{bmatrix}$$

$$(2 - \lambda)^2(4 - \lambda) + 0 + 0 - 0 - 0 - 0 = 0$$

Eigenvalues: 4, 2, 2 Repeated eigenvalue

# Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{Eigenvalue: } 4$$

$$Av = 4v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

The diagram illustrates the matrix multiplication  $Av = 4v$ . The matrix  $A$  is multiplied by the vector  $v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$  to produce the vector  $\begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$ . A curved arrow points from the result vector to a detailed breakdown of the first three rows of the matrix multiplication:

$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{bmatrix}$$

# Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{Eigenvalue: } 4$$

$$Av = 4v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{bmatrix}$$

$$\begin{aligned} 2x_1 &= 4x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 4x_2 \\ 2x_3 &= 4x_3 \end{aligned}$$

$$\begin{aligned} -2x_1 &= 0 \\ -x_1 - 0.5x_3 &= 0 \\ -2x_3 &= 0 \end{aligned}$$

$$\begin{aligned} x_1 &= 0 \\ x_2 &= \text{any number} \\ x_3 &= 0 \end{aligned}$$

Eigenvector

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$



# Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{Eigenvalue: } 2$$

$$Av = 2v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$
$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{bmatrix}$$

# Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 2

$$Av = 2v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{bmatrix}$$

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 2x_3 &= 2x_3 \end{aligned}$$

$$\begin{aligned} 0 &= 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

$$x_1 = 2x_2 - 0.5x_3$$

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 2 \\ x_2 &= 1 \\ x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= 2 \end{aligned}$$

Point in different directions

Different eigenvectors

# Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigenvalues  $\lambda_1 = 4$   $\lambda_2 = 2$   $\lambda_3 = 2$

Eigenvectors 

0	2	1
1	1	1
0	0	2

# Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

# Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Characteristic polynomial =  $\det(A - \lambda I) = \det$

$$\begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 4-\lambda & 0.5 \\ -4 & 0 & 2-\lambda \end{bmatrix}$$

$$(2 - \lambda)^2(4 - \lambda) + 0 + 0 - 0 - 0 - 0$$

Eigenvalues: 4, 2, 2 Repeated eigenvalue

# Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \quad \text{Eigenvalue: } 4$$

$$Av = 4v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$
  
$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

# Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 4

$$\begin{aligned} 2x_1 &= 4x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 4x_2 \\ 4x_1 + 2x_3 &= 4x_3 \end{aligned}$$

$$Av = 4v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}$$

$$\begin{aligned} -2x_1 &= 0 \\ -x_1 - 0.5x_3 &= 0 \\ 4x_1 - 2x_3 &= 0 \end{aligned}$$

$$x_1 = 0 \quad x_3 = 0 \quad x_2 = \text{any number}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{aligned} x_1 &= 0 \\ x_2 &= 1 \\ x_3 &= 0 \end{aligned}$$

Same as before!

# Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \quad \text{Eigenvalue: } 2$$

$$Av = 2v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$
$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}$$



# Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 2

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 4x_1 + 2x_3 &= 2x_3 \end{aligned}$$

$$Av = 2v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}$$

$$\begin{aligned} 0 &= 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 \\ 4x_1 &= 0 \end{aligned}$$

$$\begin{bmatrix} 0 \\ k \\ 4k \end{bmatrix}$$

$$x_1 = 0 \quad x_3 = 4x_2$$

$$\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 1 \\ x_3 &= 4 \end{aligned}$$

$$\begin{bmatrix} 0 \\ 0.5 \\ 2 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0.5 \\ x_3 &= 2 \end{aligned}$$

On the same line  
Same eigenvector

# Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 2

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 4x_1 + 2x_3 &= 2x_3 \end{aligned}$$

$$Av = 2v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}$$

$$\begin{aligned} 0 &= 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 \\ 4x_1 &= 0 \end{aligned}$$

$$x_1 = 0 \quad x_3 = 4x_2$$

$$\begin{bmatrix} 0 \\ k \\ 4k \end{bmatrix}$$

# Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalues

$\lambda_1 = 4$

$\lambda_2 = 2$

$\lambda_3 = 2$

Eigenvectors

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$


Can't create an eigenbasis  
from this matrix



# Summary

a	b
c	d

Eigenvalues

$\lambda_1, \lambda_2$



If  $\lambda_1 \neq \lambda_2$   2 eigenvectors  
(2 different directions)




If  $\lambda_1 = \lambda_2$   1 eigenvector  
(1 direction)  
 2 eigenvectors  
(2 different directions)

a	b	c
d	e	f
g	h	i

$\lambda_1, \lambda_2, \lambda_3$

If  $\lambda_1 \neq \lambda_2 \neq \lambda_3$   3 eigenvectors  
(3 different directions)

If  $\lambda_1 = \lambda_2 \neq \lambda_3$   2 eigenvectors  
(2 different directions)  
 3 eigenvectors  
(3 different directions)

If  $\lambda_1 = \lambda_2 = \lambda_3$   1 eigenvector  
(1 direction)  
 2 eigenvectors  
(2 different directions)  
 3 eigenvectors  
(3 different directions)



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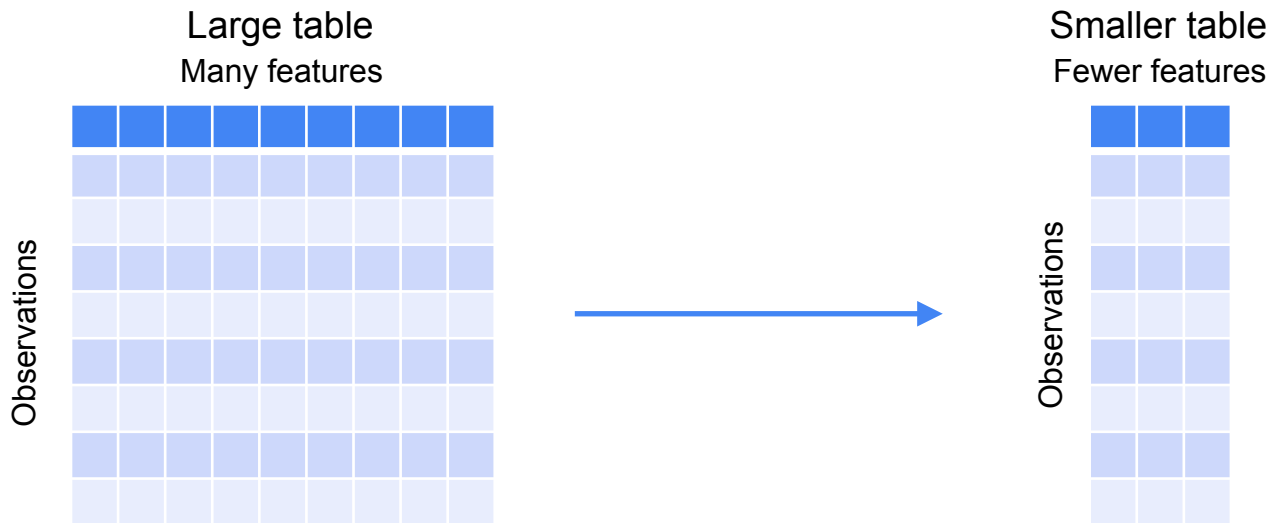
# Determinants and Eigenvectors

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**Dimensionality reduction  
and projection**

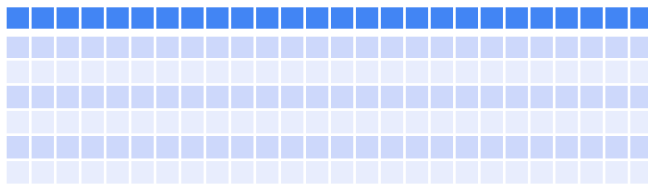
# Dimensionality Reduction

- Reduce dimensions (# of columns) of dataset
- Preserve as much information as possible



# Dimensionality Reduction

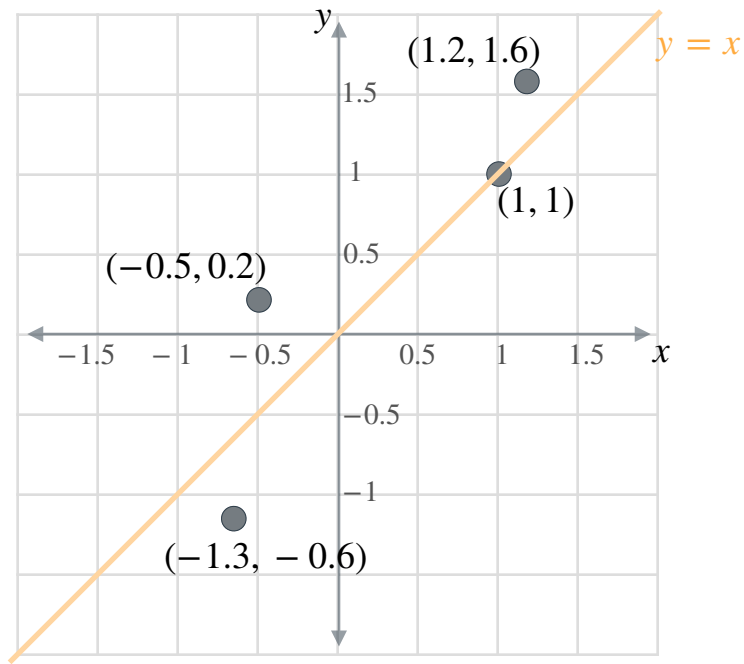
- Leads to smaller datasets
- Easier to visualize



Customer Age	Account Age	Days Since Login	Total Purchases	Total \$ Spent
23	1 month	10 days	1	\$100
71	45 months	2 days	Easy approach - just delete columns Loses valuable information	
54	30 months	15 days	2	\$70
36	22 months	12 days	4	\$210

# Projections

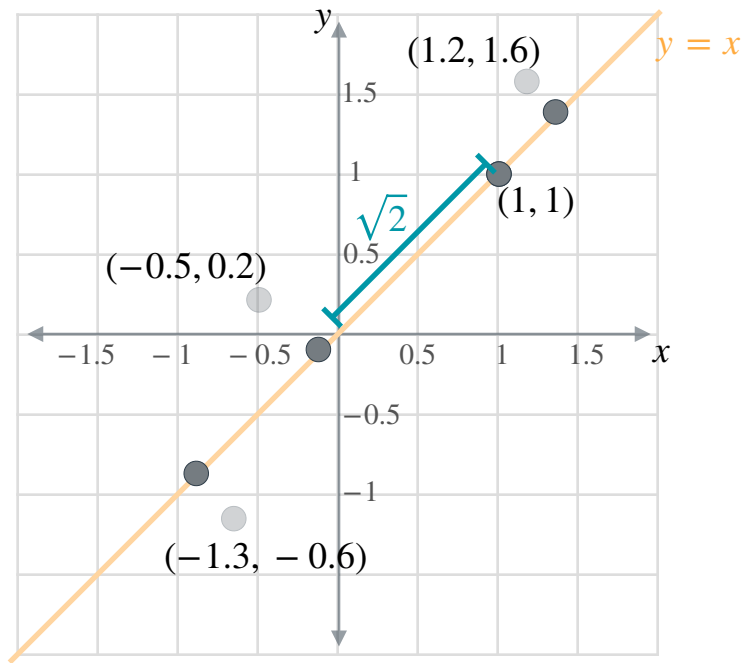
x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6





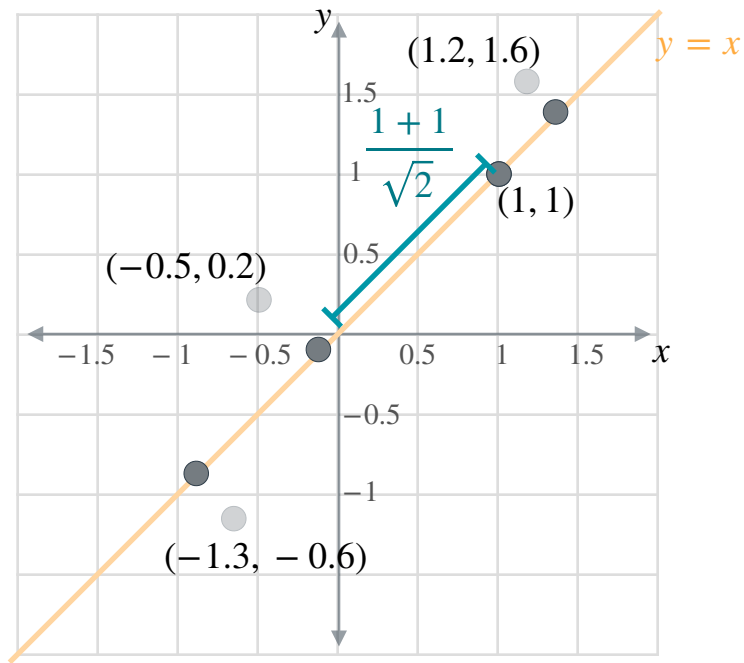
# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6



# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6



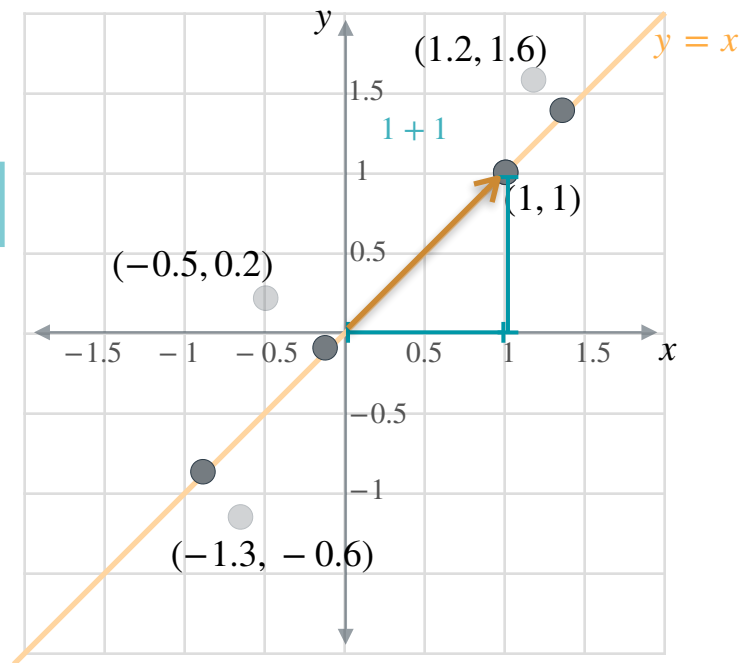
# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

1  
1

=

(1 + 1)



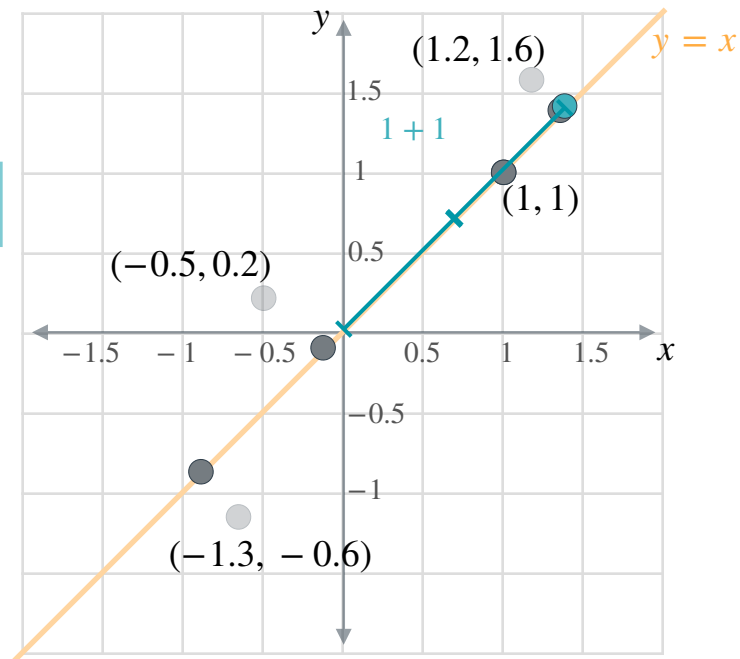
# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

1  
1

=

(1 + 1)

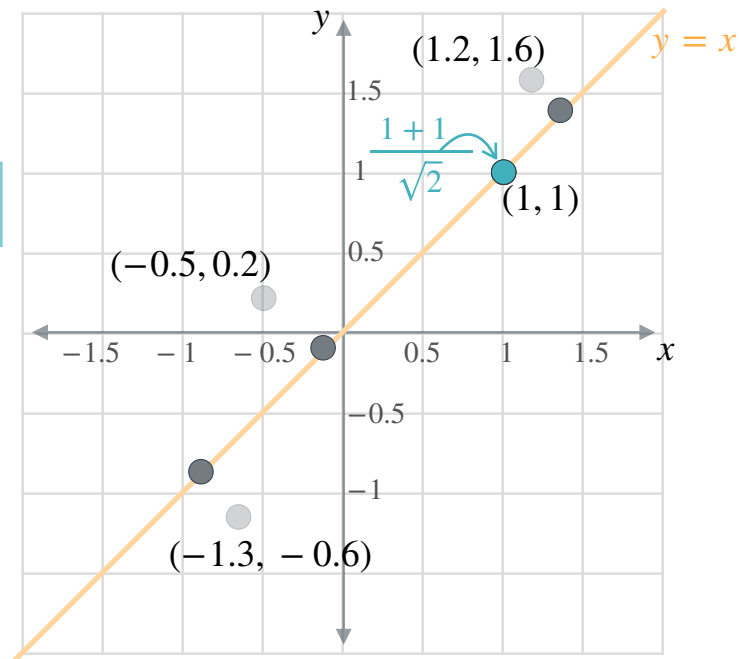


# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

$$(1 + 1) / \sqrt{2}$$



# Projections

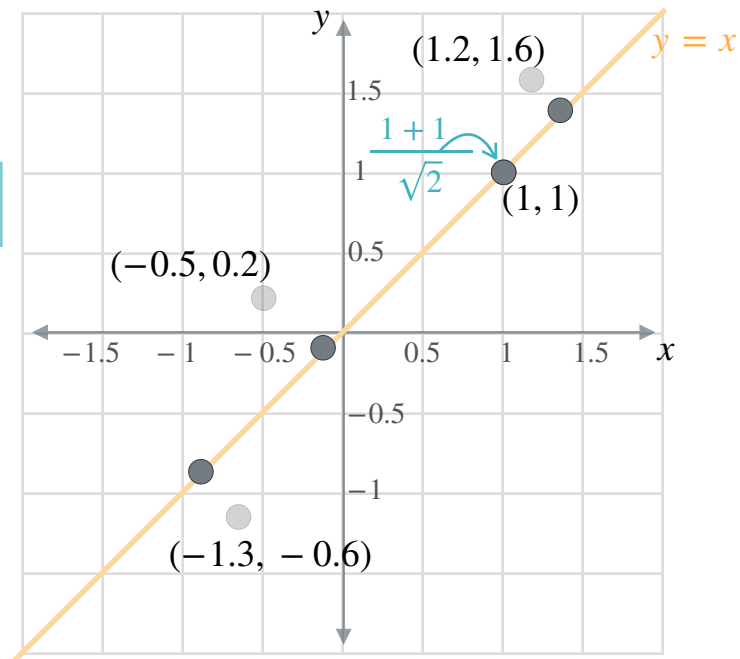
x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

Norm of 1

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

$$(1 + 1) / \sqrt{2}$$

$$\frac{1}{\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_2}$$



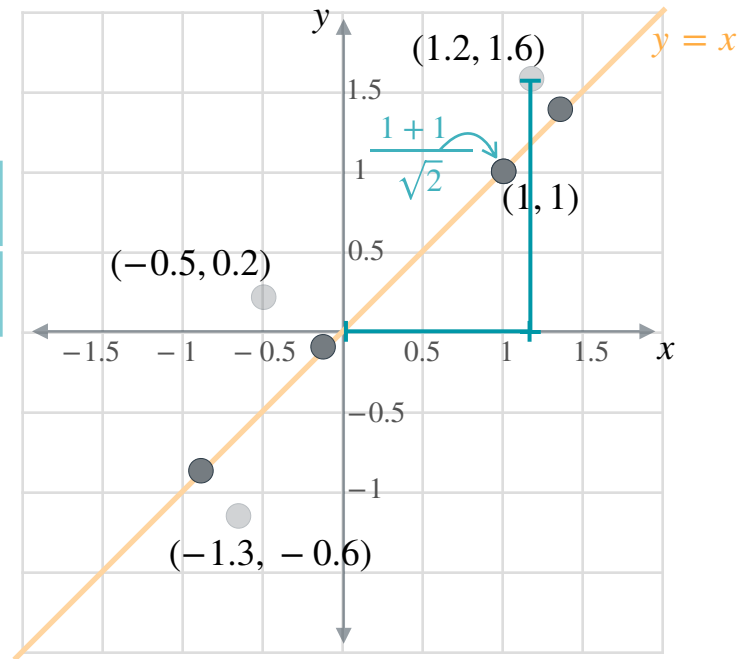
# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

1  
1

$$\frac{1}{\sqrt{2}} =$$

$$(1 + 1) / \sqrt{2}$$



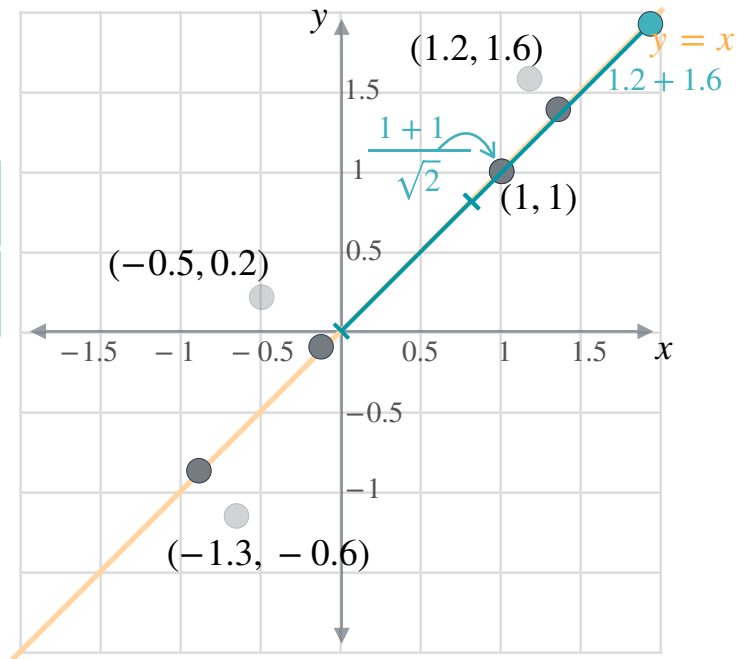
# Projections



$$\frac{1}{1} \cdot \frac{1}{\sqrt{2}} =$$

$$(1 + 1) / \sqrt{2}$$

$$(1.2 + 1.6)$$





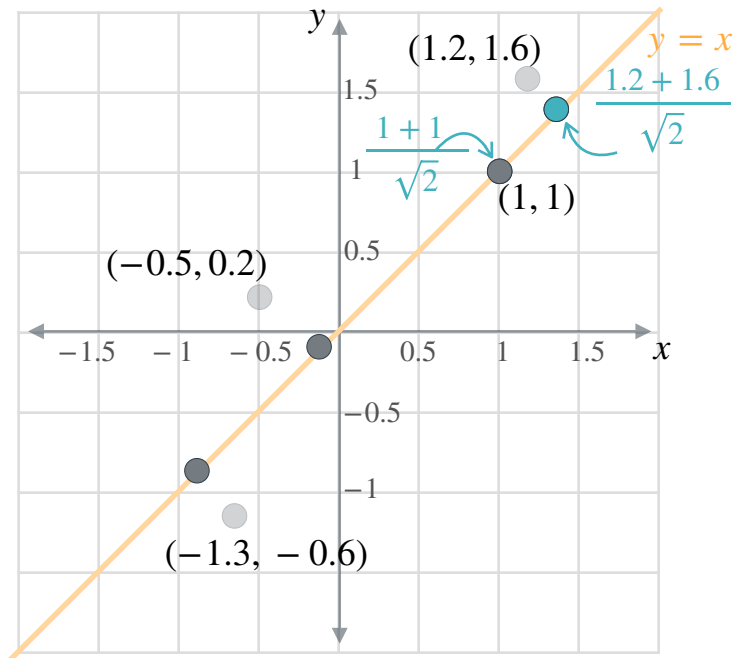
# Projections



$$\frac{1}{1} \cdot \frac{1}{\sqrt{2}} =$$

$$(1 + 1)/\sqrt{2}$$

$$(1.2 + 1.6)/\sqrt{2}$$



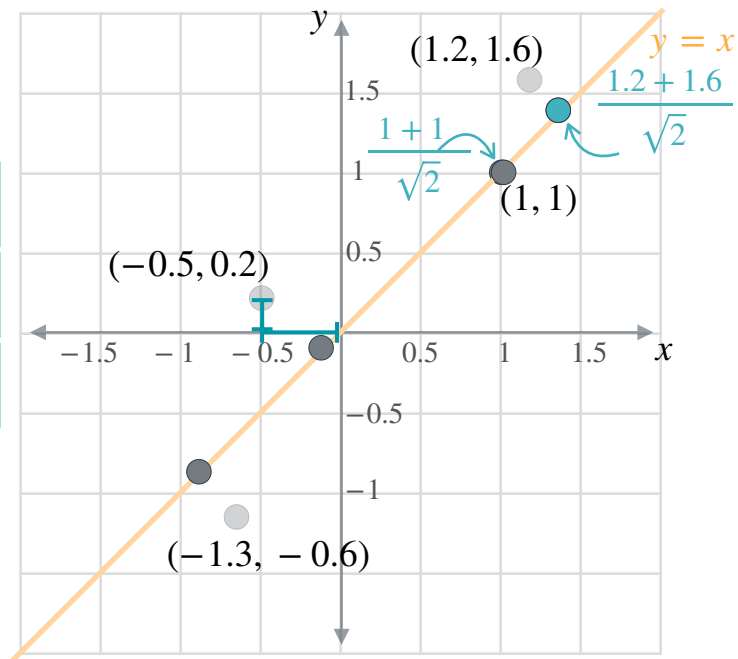
# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}} =$$

$$\begin{bmatrix} (1 + 1) / \sqrt{2} \\ (1.2 + 1.6) / \sqrt{2} \end{bmatrix}$$



# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

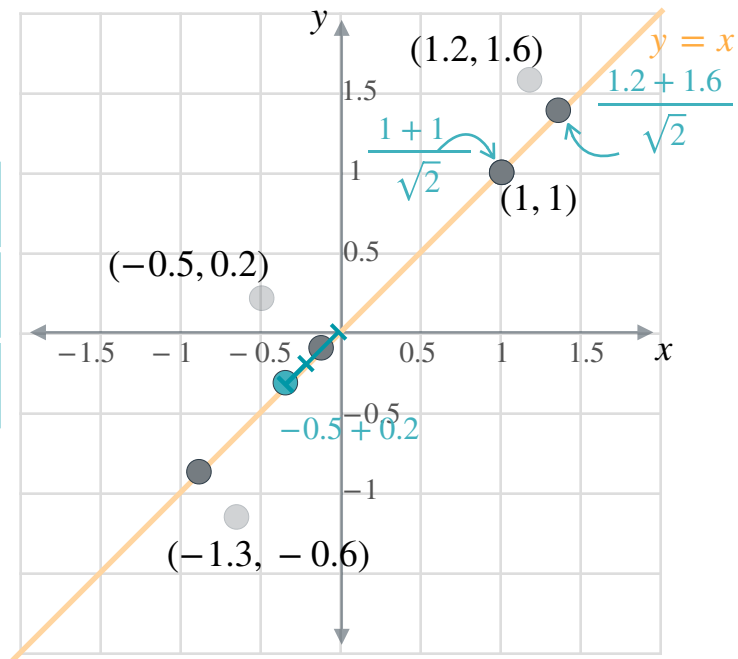
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}} =$$

$$(1 + 1) / \sqrt{2}$$

$$(1.2 + 1.6) / \sqrt{2}$$

$$(-0.5 + 0.2)$$



# Projections

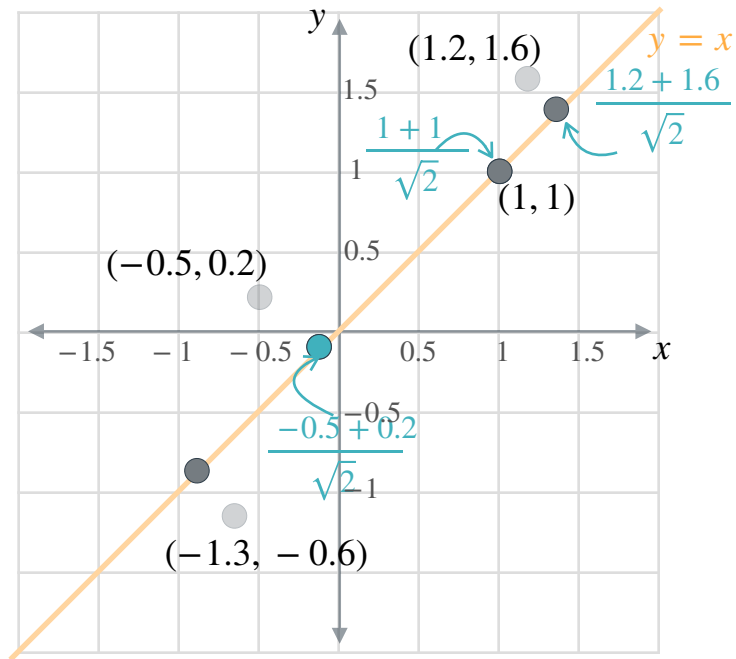


$$\frac{1}{1} \cdot \frac{1}{\sqrt{2}} =$$

$$(1 + 1) / \sqrt{2}$$

$$(1.2 + 1.6)/\sqrt{2}$$

$$(-0.5 + 0.2)/\sqrt{2}$$



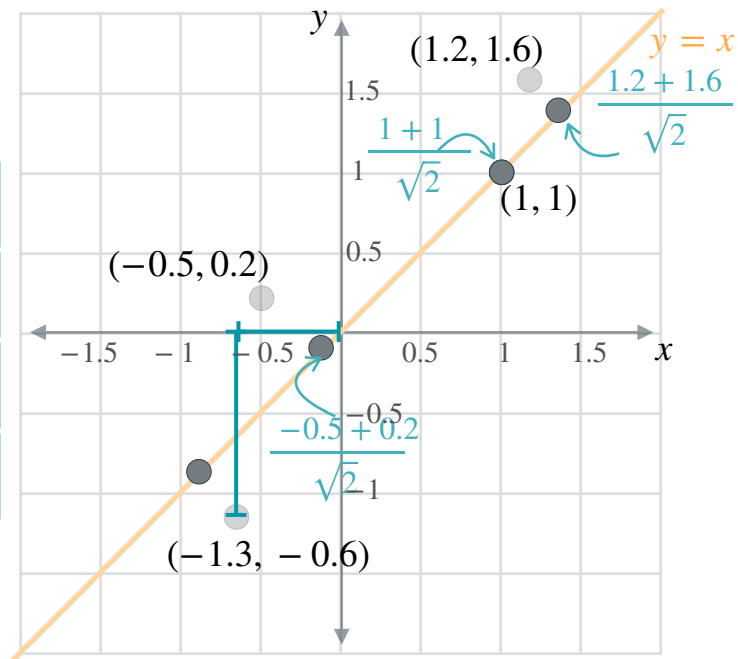
# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}} =$$

$$\begin{bmatrix} (1 + 1) / \sqrt{2} \\ (1.2 + 1.6) / \sqrt{2} \\ (-0.5 + 0.2) / \sqrt{2} \\ \end{bmatrix}$$

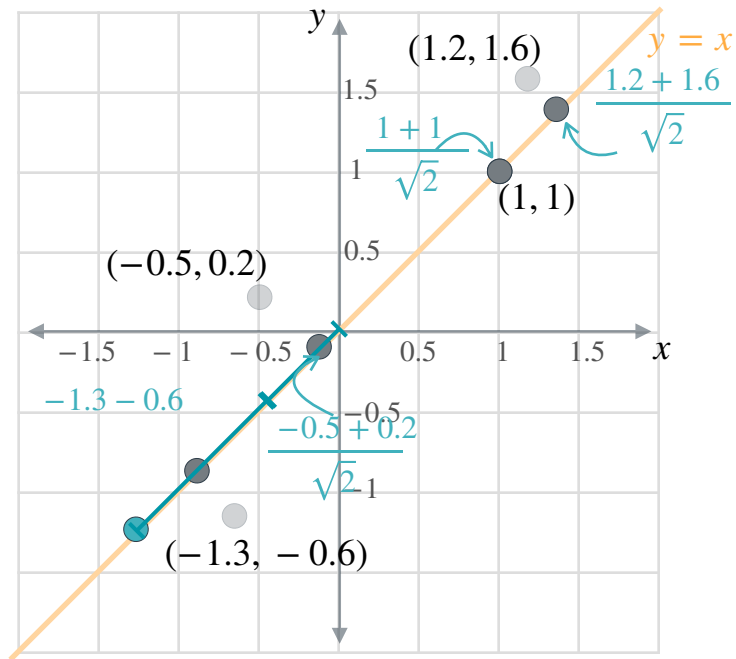


# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

$$\begin{aligned} & (1 + 1) / \sqrt{2} \\ & (1.2 + 1.6) / \sqrt{2} \\ & (-0.5 + 0.2) / \sqrt{2} \\ & (-1.3, -0.6) \end{aligned}$$



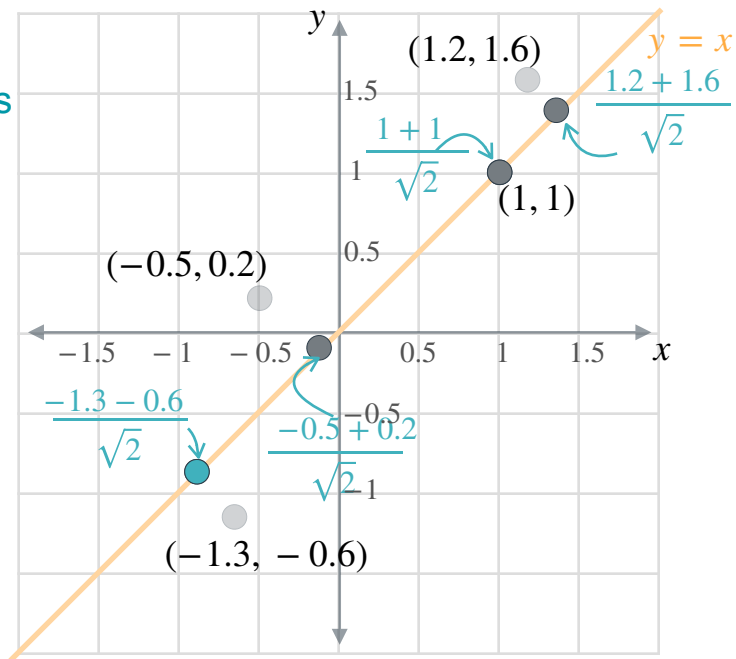
# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

Final coordinates

$$\begin{aligned} & (1 + 1) / \sqrt{2} \\ & (1.2 + 1.6) / \sqrt{2} \\ & (-0.5 + 0.2) / \sqrt{2} \\ & (-1.3 - 0.6) / \sqrt{2} \end{aligned}$$



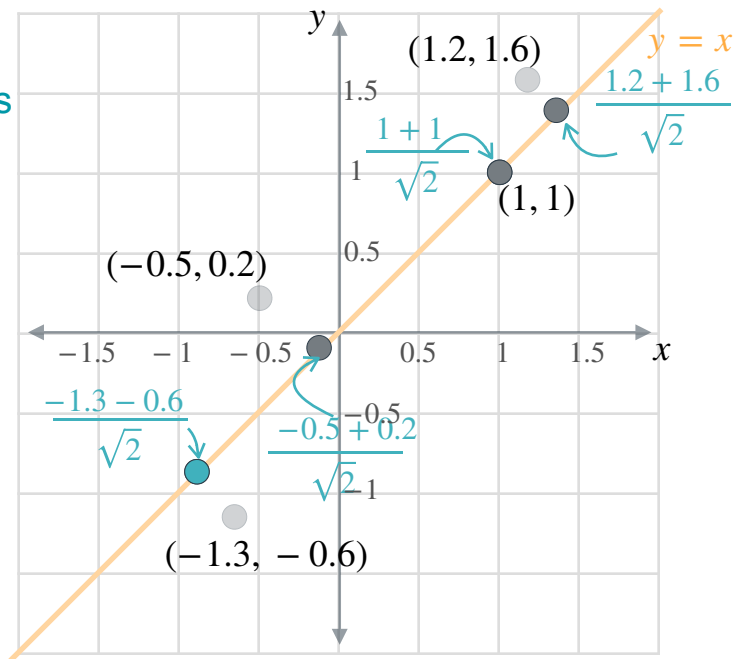
# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

Final coordinates

1.4142
1.9799
-0.2121
-1.344





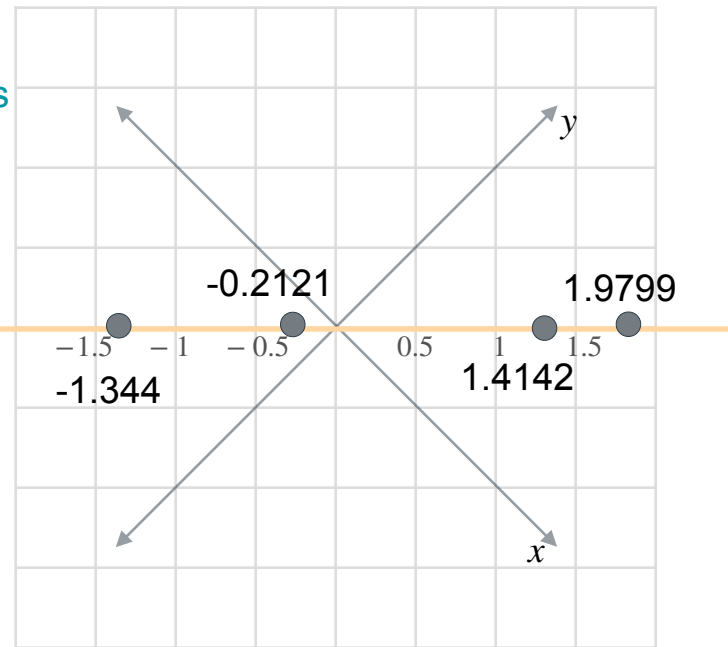
# Projections

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

Final coordinates

1.4142
1.9799
-0.2121
-1.344



# Projections

To project a matrix  $A$  onto a vector  $v$

$$A_P = A \frac{v}{\|v\|_2}$$

$r \times 1 \qquad r \times c \qquad c \times 1$

# Projections

To project a matrix  $A$  onto vectors  $v_1$  and  $v_2$

$$A_P = A \overbrace{\begin{bmatrix} \frac{v_1}{\|v_1\|_2} & \frac{v_2}{\|v_2\|_2} \end{bmatrix}}^V$$

$r \times 2$        $r \times c$        $c \times 2$

# Projections

To project a matrix  $A$  onto vectors  $v_1$  and  $v_2$

$$A_P = A \overbrace{\begin{bmatrix} \frac{v_1}{\|v_1\|_2} & \frac{v_2}{\|v_2\|_2} \end{bmatrix}}^V$$

$r \times 2$        $r \times c$        $c \times 2$

# Projections

To project a matrix  $A$  onto vectors  $v_1$  and  $v_2$

$$A_P = AV$$

$$r \times 2 \quad r \times c \quad c \times 2$$



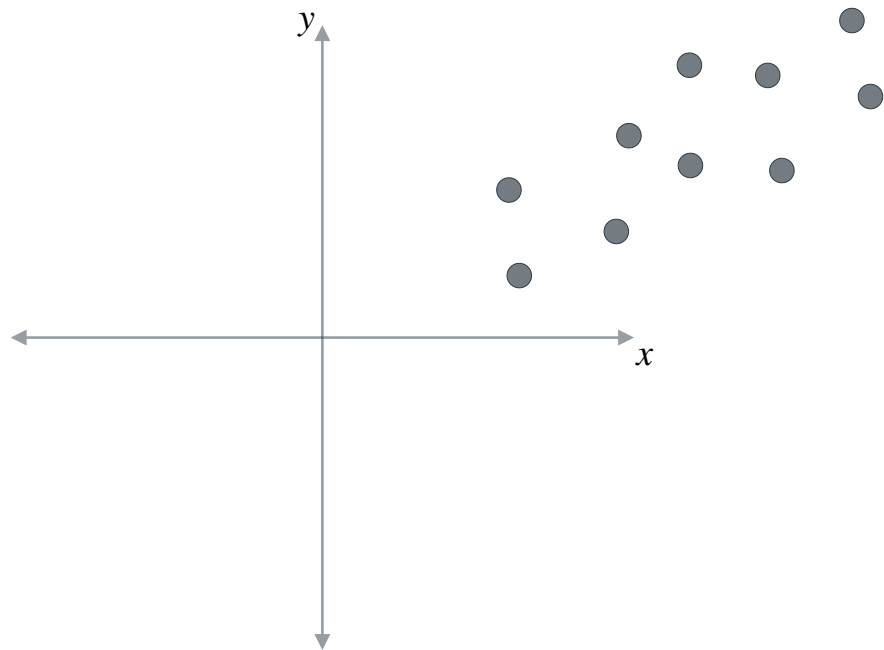
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# Determinants and Eigenvectors

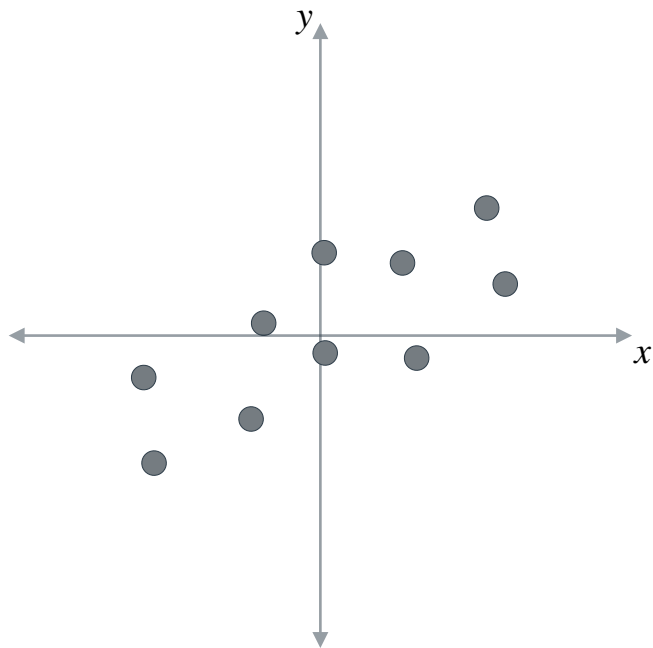
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## Motivating PCA

# Dimensionality Reduction

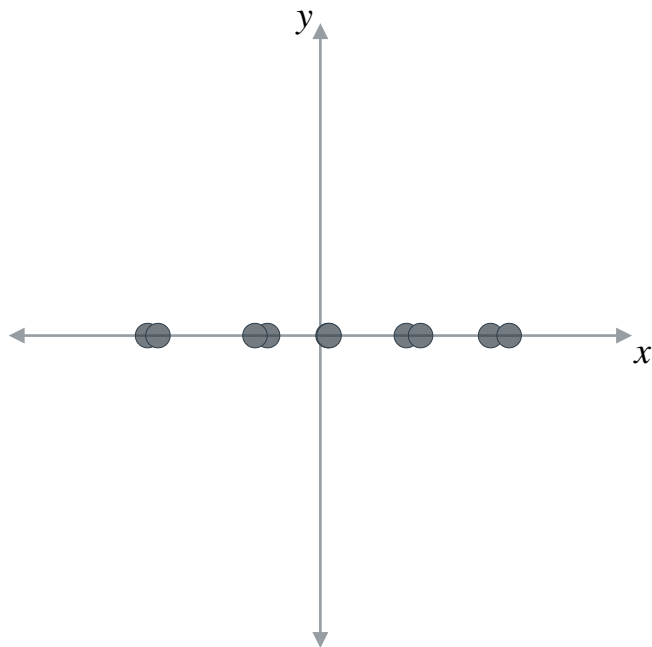


# Principal Component Analysis (PCA)

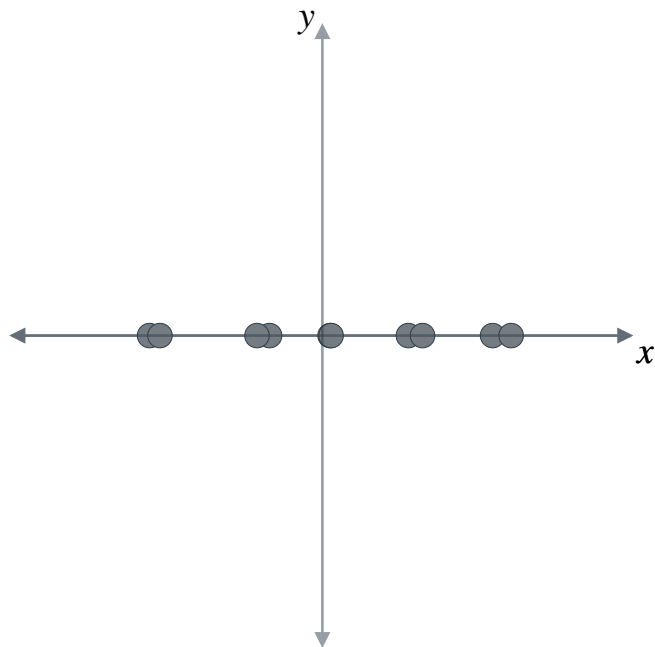




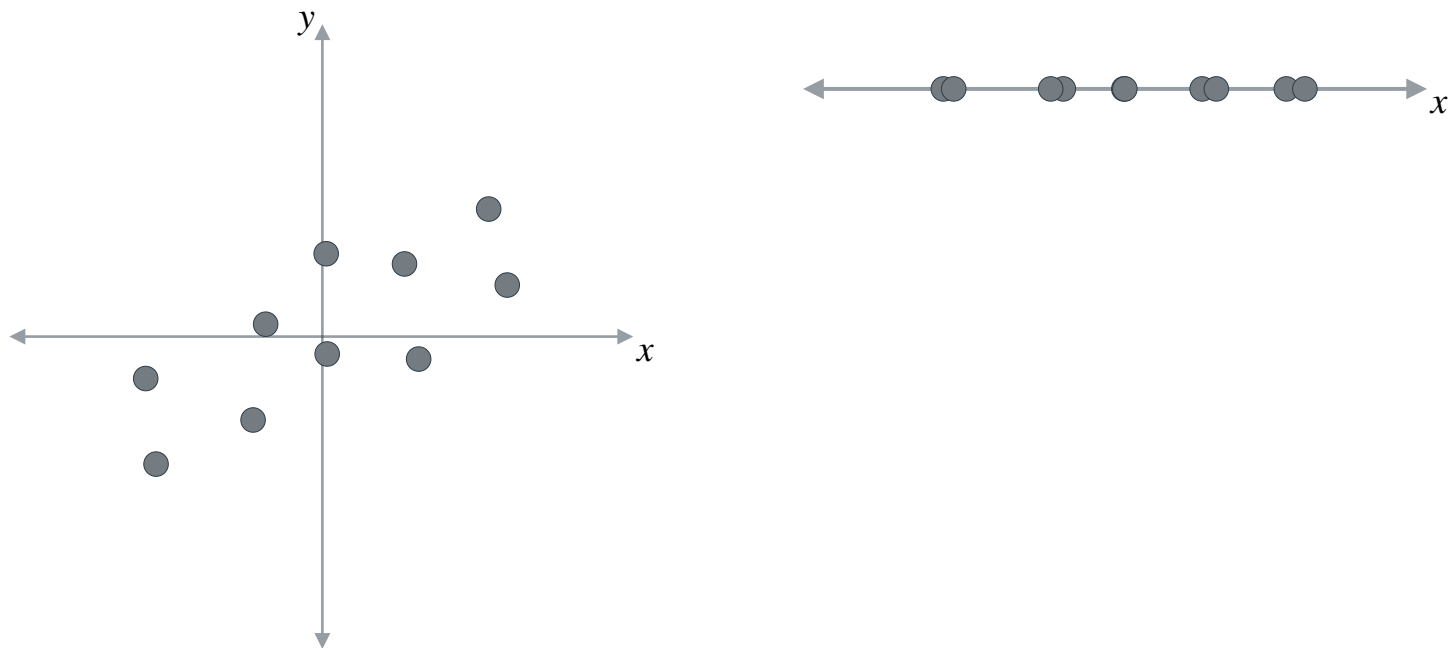
# Principal Component Analysis (PCA)



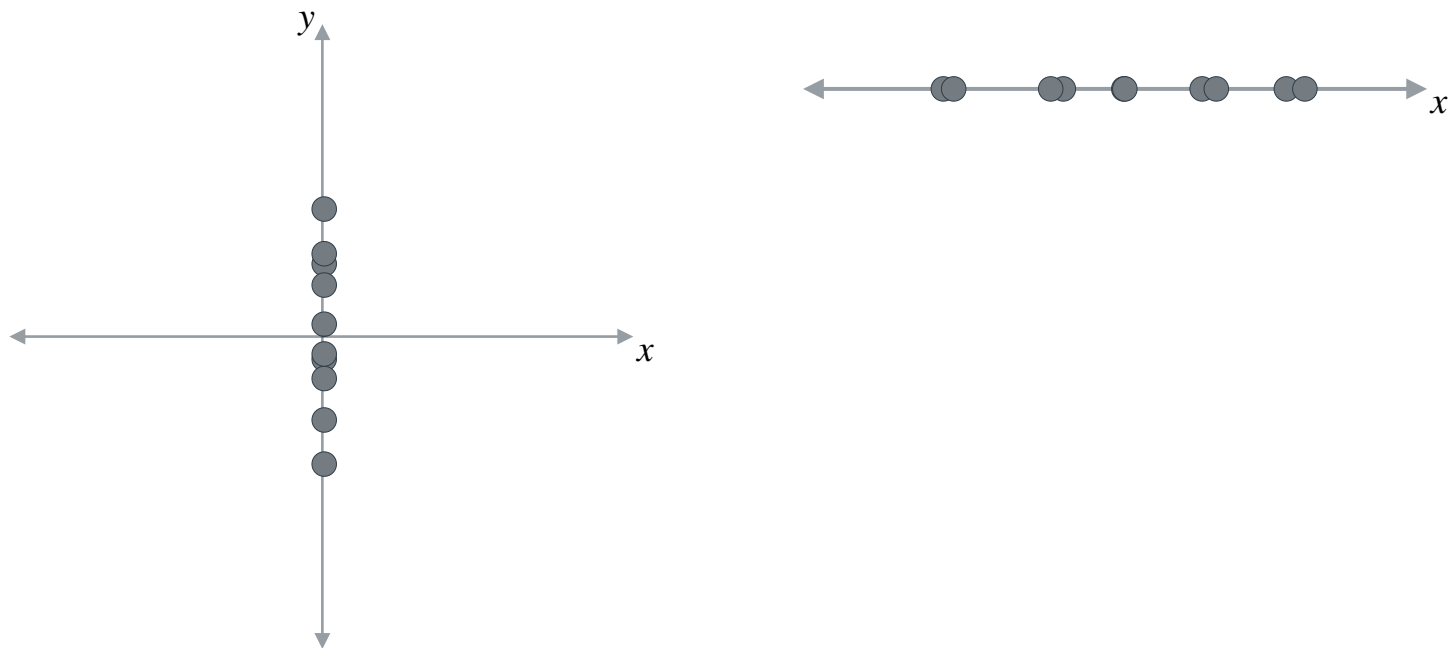
# Principal Component Analysis (PCA)



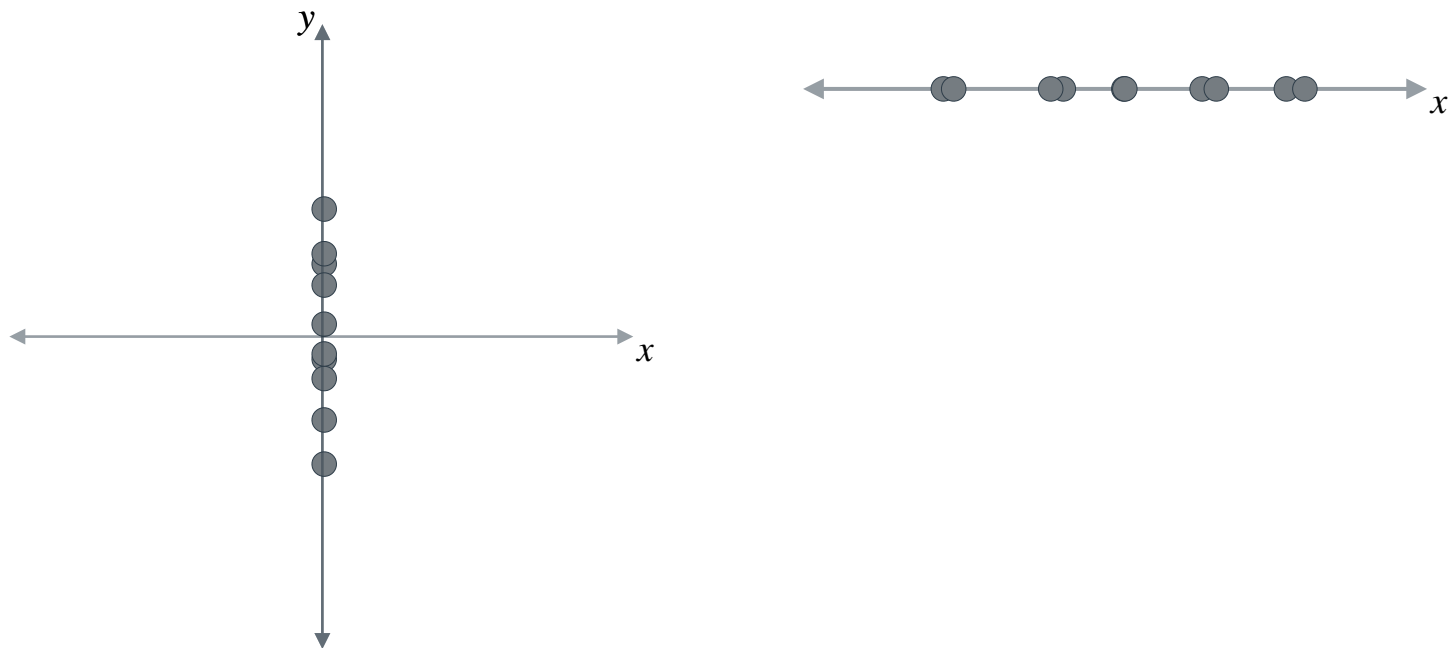
# Principal Component Analysis (PCA)



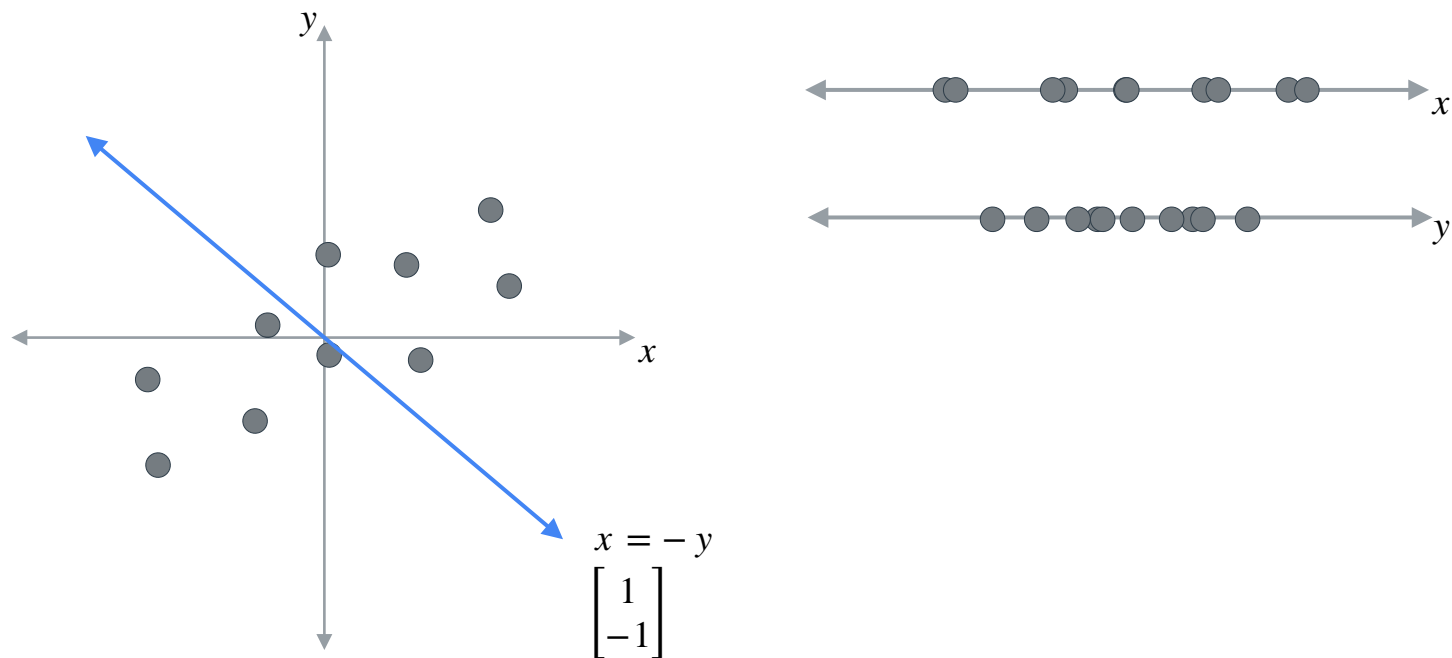
# Principal Component Analysis (PCA)



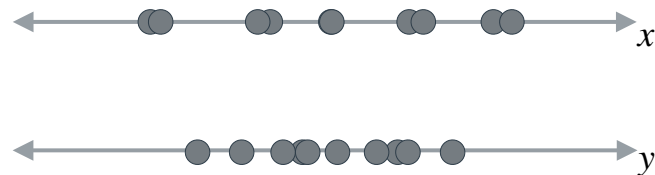
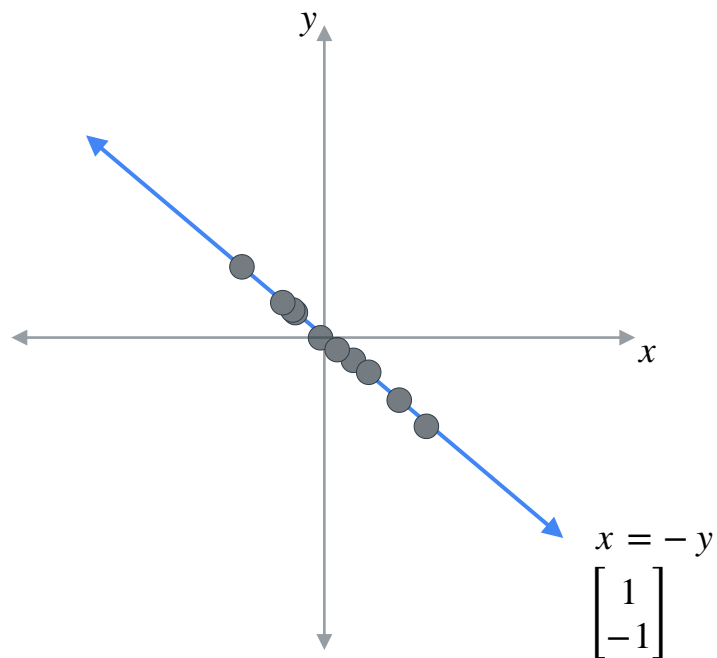
# Principal Component Analysis (PCA)



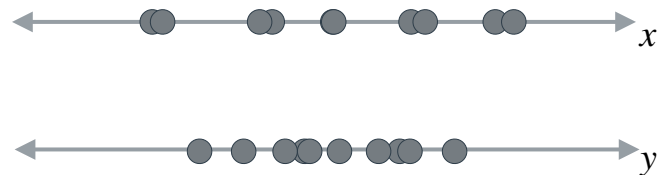
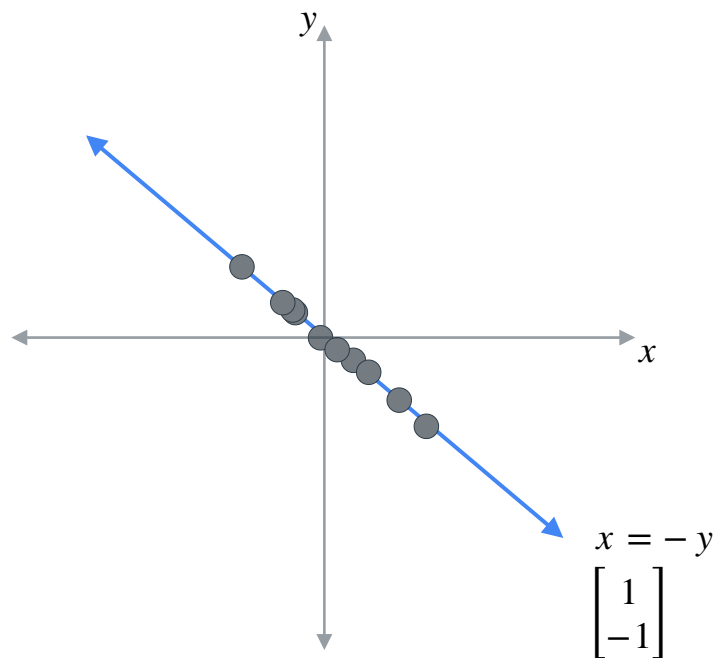
# Principal Component Analysis (PCA)



# Principal Component Analysis (PCA)

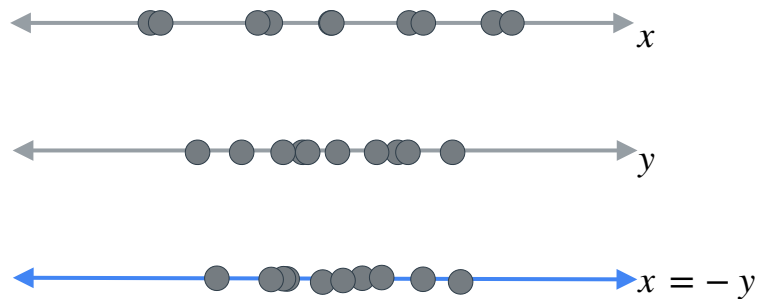
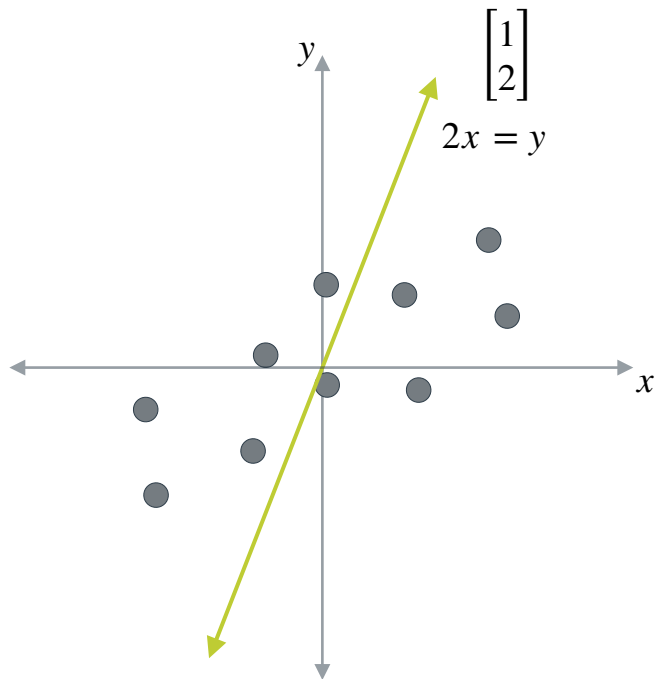


# Principal Component Analysis (PCA)

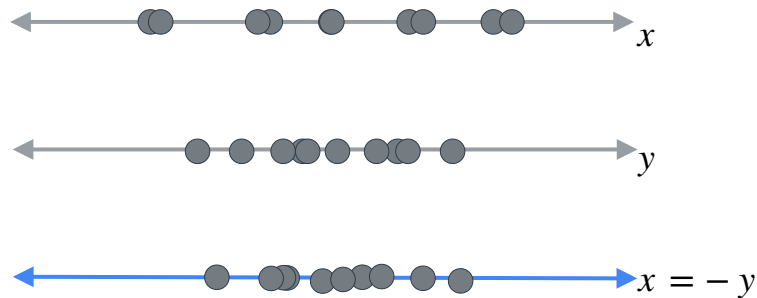
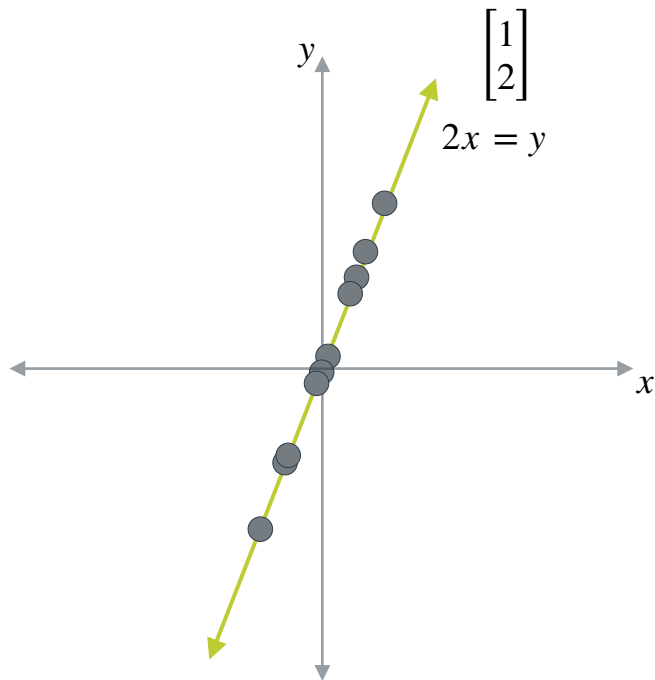




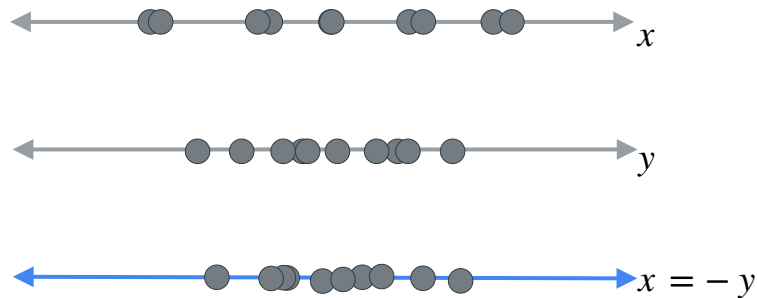
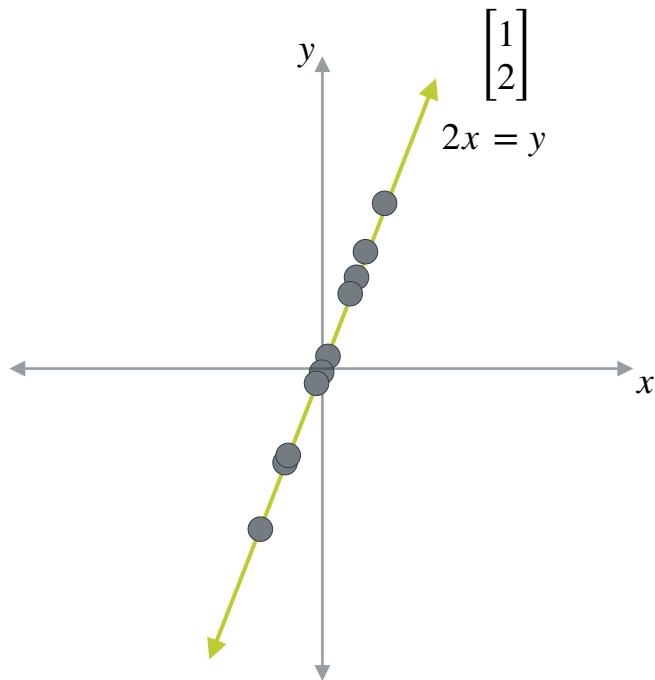
# Principal Component Analysis (PCA)



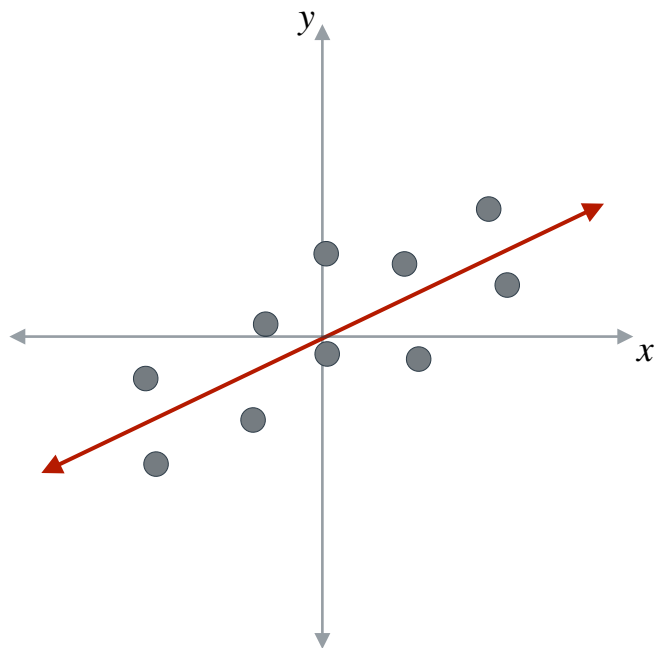
# Principal Component Analysis (PCA)



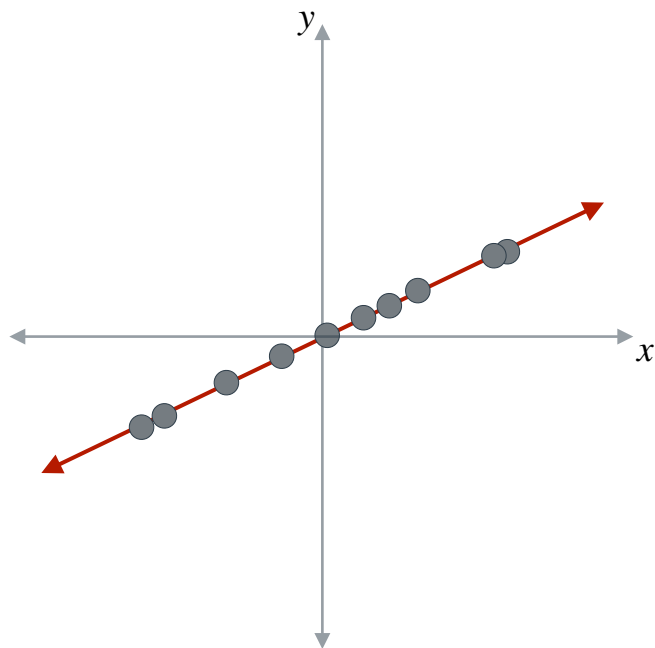
# Principal Component Analysis (PCA)



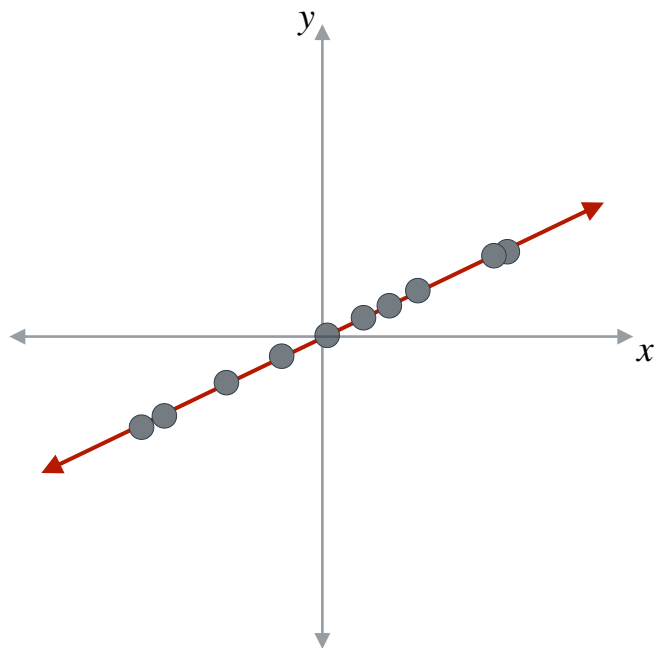
# Principal Component Analysis (PCA)



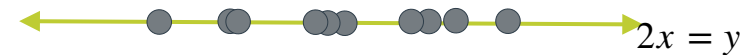
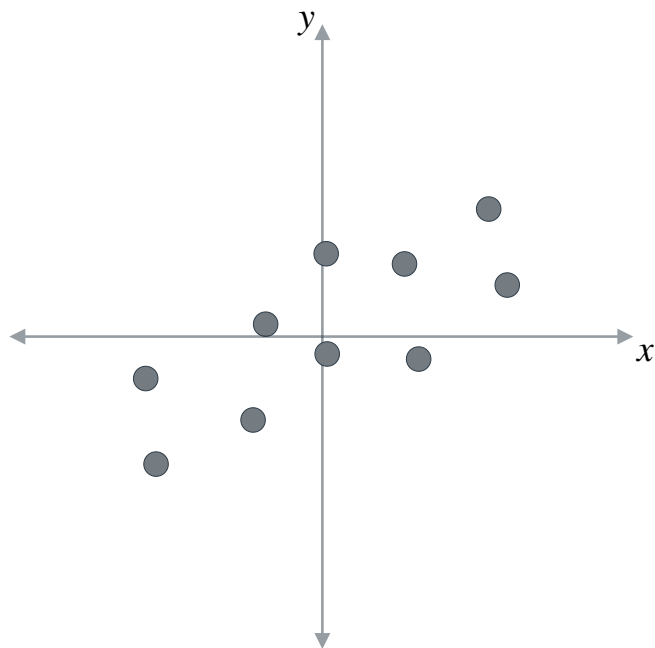
# Principal Component Analysis (PCA)



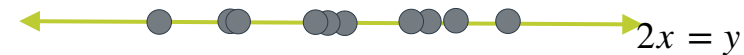
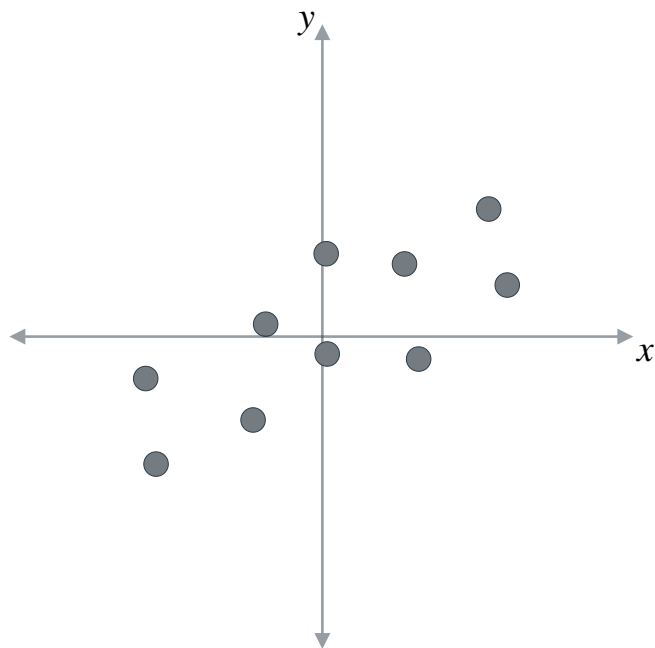
# Principal Component Analysis (PCA)



# Principal Component Analysis (PCA)

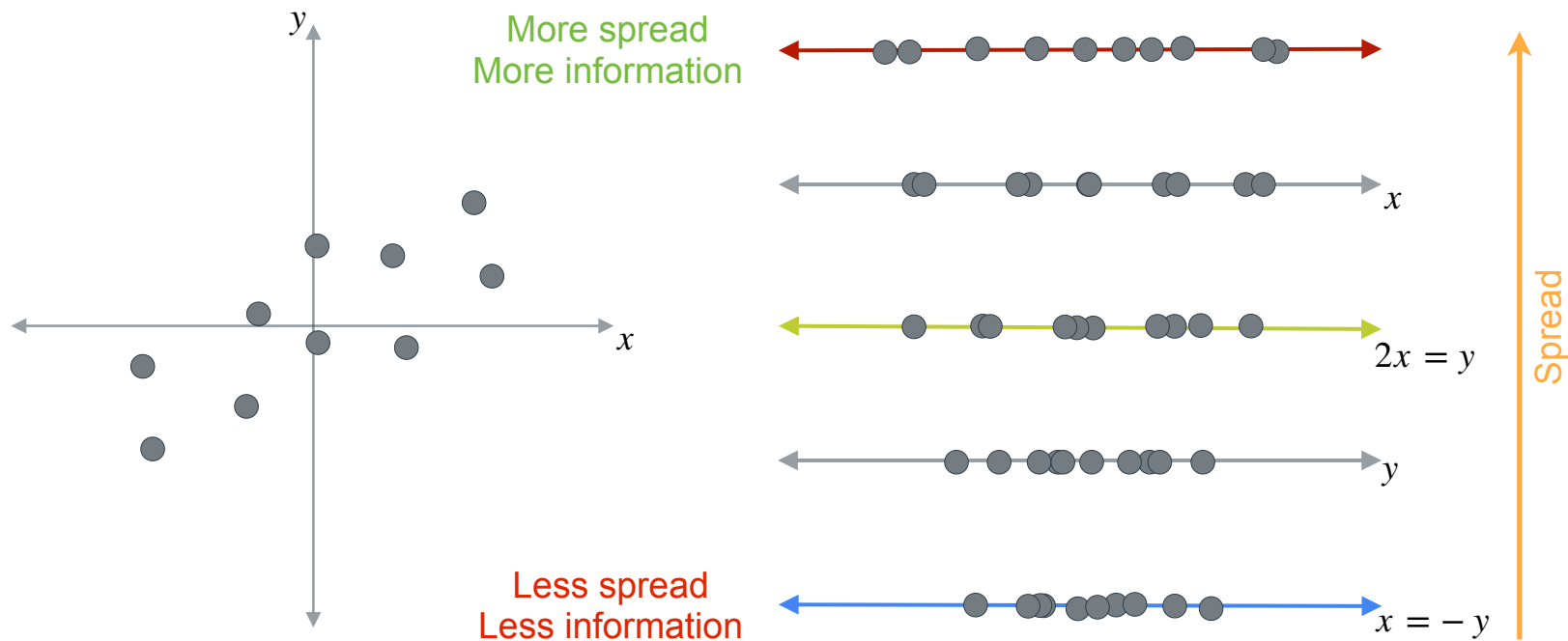


# Principal Component Analysis (PCA)



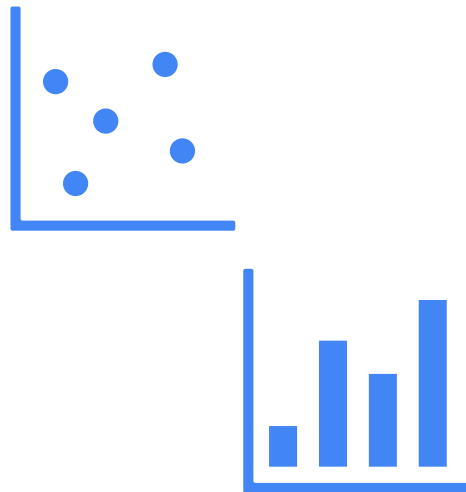
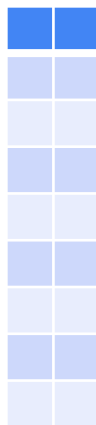
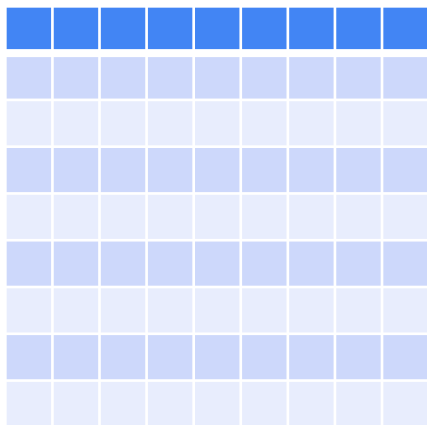


# Principal Component Analysis (PCA)



# Benefits of Dimensionality Reduction

- Easier dataset to manage
- PCA reduces dimensions while minimizing information loss
- Simpler visualization





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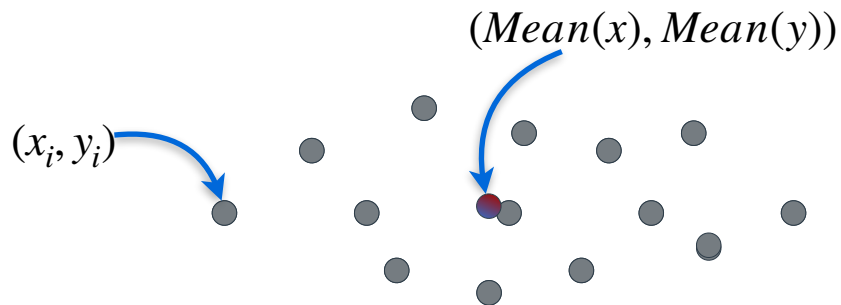
# Determinants and Eigenvectors

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## **Variance and covariance**

# Mean

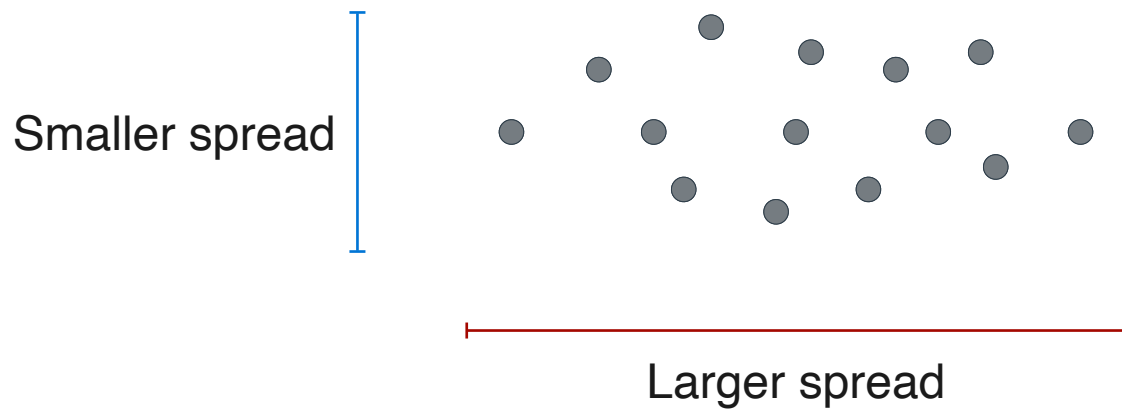
“The average of the data”



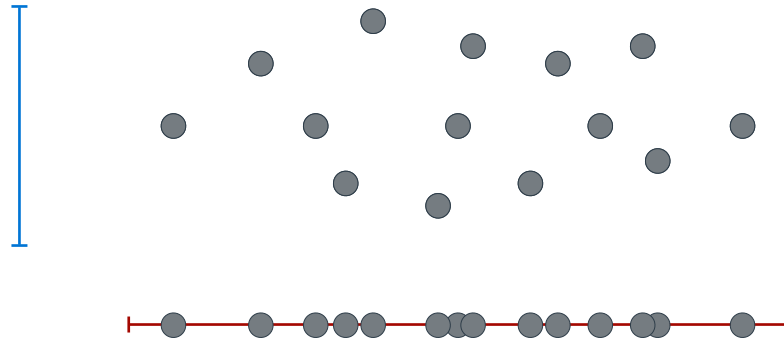
$$Mean(x) = \frac{1}{n} \sum_{i=1}^n x_i$$

$$Mean(y) = \frac{1}{n} \sum_{i=1}^n y_i$$

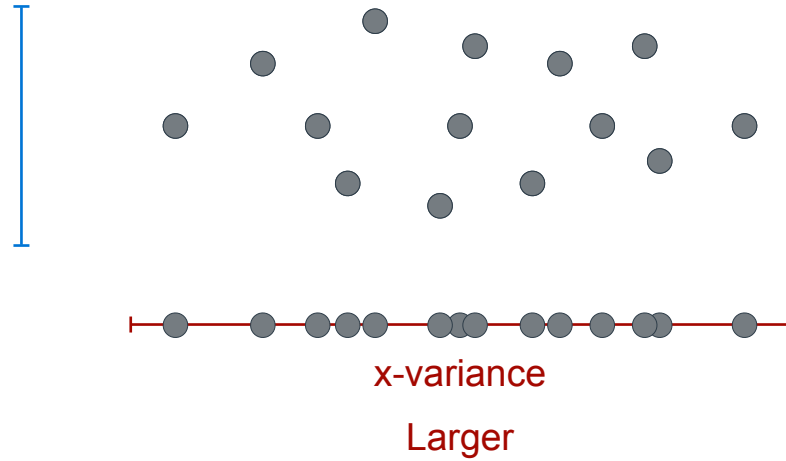
# Variance



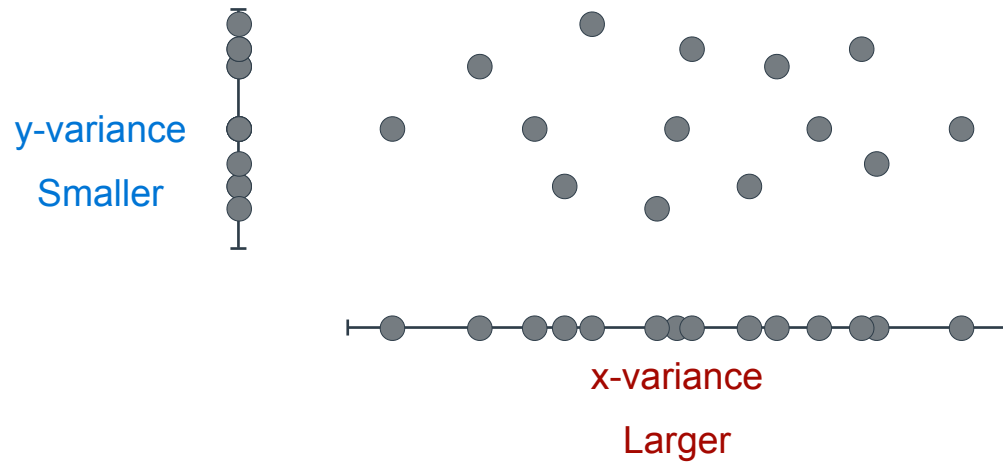
# Variance



# Variance



# Variance





# Variance

$$\text{Variance}(x) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \text{Mean}(x))^2 = 16$$

	$x_i$	$x_i - \text{Mean}(x)$	$(x_i - \text{Mean}(x))^2$
1	10	1	1
2	4	-5	25
3	11	2	4
4	14	5	25
5	6	-3	9

→ 64

$$\text{Mean}(x) = 9$$

# Variance

$$\text{Variance}(x) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \text{Mean}(x))^2$$

$$\text{Var}(x)$$

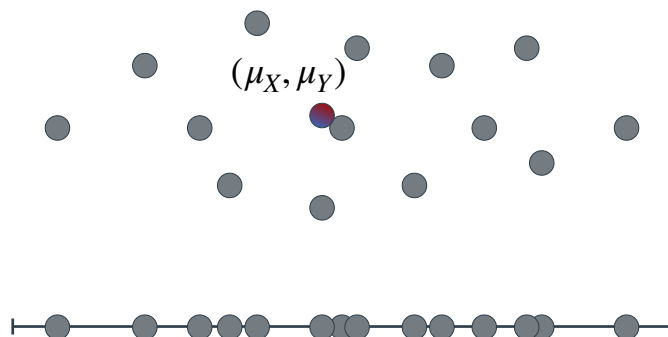
$$\mu_x$$

“The average squared distance from the mean”

# Variance

$$Var(y) = \frac{1}{n-1} \sum_{i=1}^n (y_i - \mu_Y)^2$$

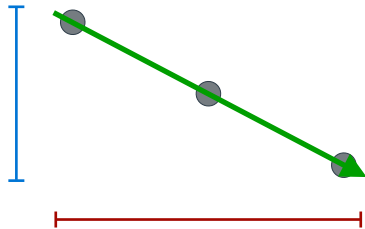
y-variance  
Smaller



x-variance  
Larger

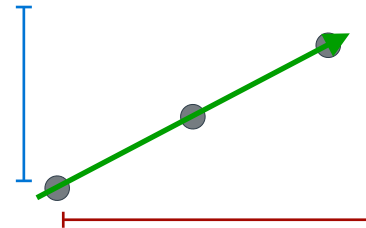
$$Var(x) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)^2$$

# Problem



Negative covariance

Solution: Covariance



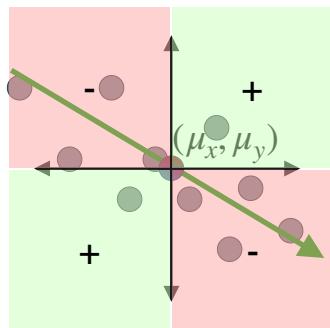
Positive covariance

# Covariance

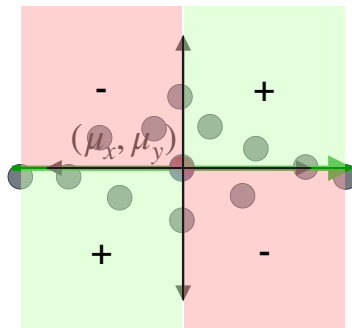
“Take the average”

$$Cov(x, y) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y)$$

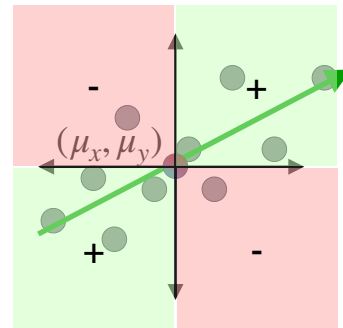
$$Var(x) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)^2$$



negative  
covariance



covariance zero  
(or very small)

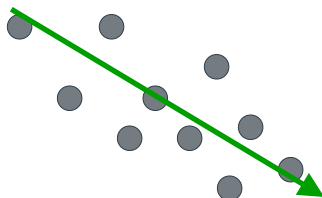


positive  
covariance

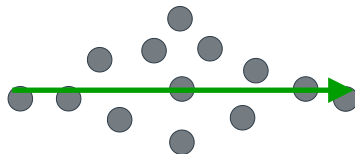
# Covariance

$$\text{Cov}(x, y) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y)$$

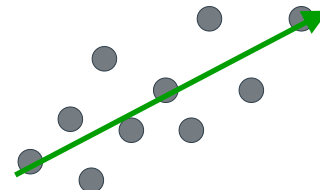
“The direction of the relationship between two variables”



negative  
covariance



covariance zero  
(or very small)



positive  
covariance



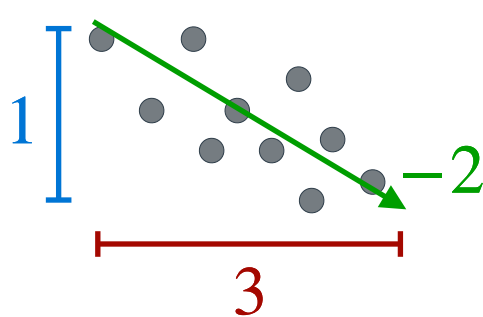
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# Determinants and Eigenvectors

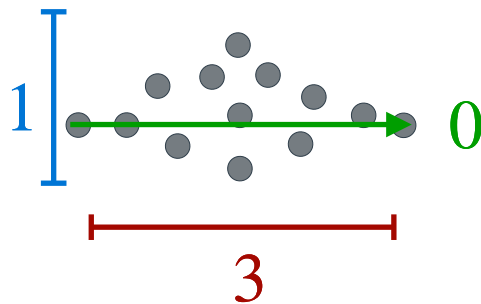
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## **The covariance matrix**

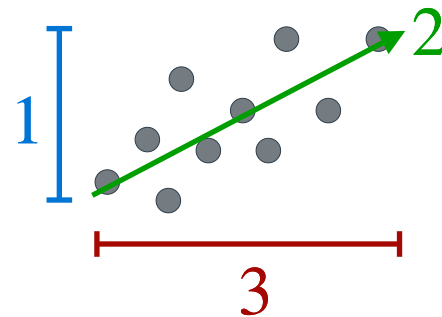
# Covariance matrix



$$\begin{bmatrix} & \\ & \end{bmatrix}$$



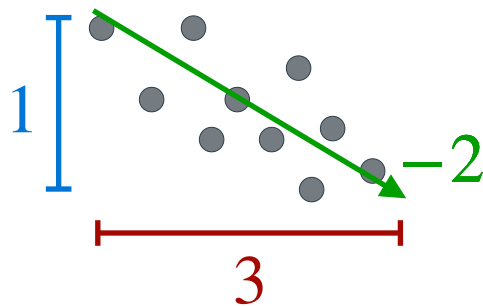
$$\begin{bmatrix} & \\ & \end{bmatrix}$$



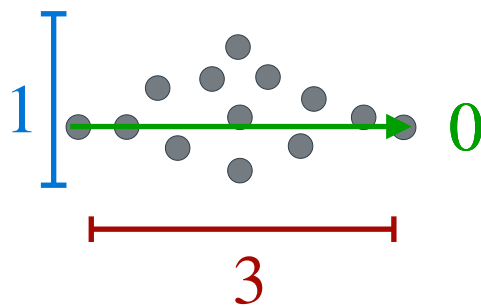
$$\begin{bmatrix} & \\ & \end{bmatrix}$$



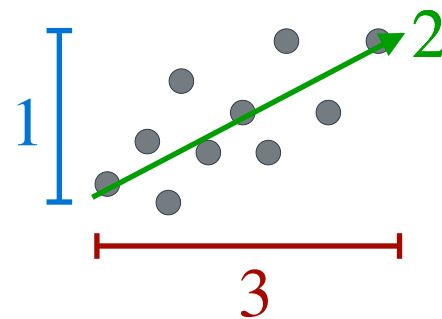
# Covariance matrix



$$\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$$

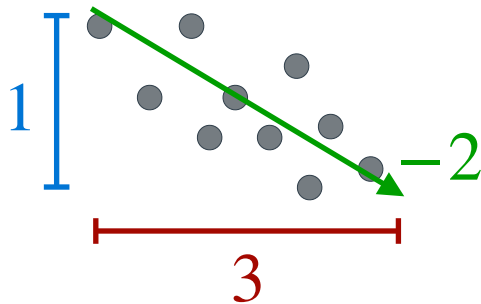


$$\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$$

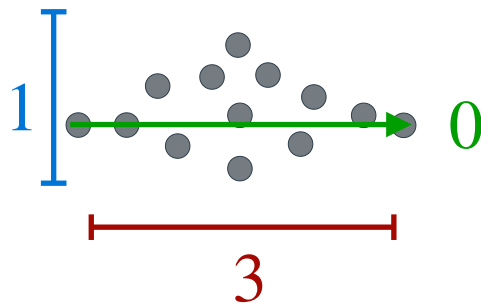


$$\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$$

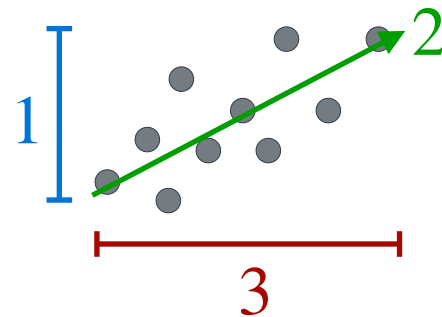
# Covariance matrix



$$\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$$

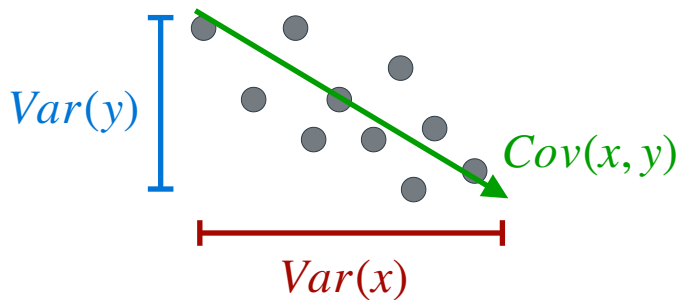


$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

# Covariance matrix



$$C = \begin{matrix} & \begin{matrix} x & y \end{matrix} \\ \begin{matrix} x \\ y \end{matrix} & \begin{bmatrix} \text{Var}(x) & Cov(x, y) \\ Cov(y, x) & \text{Var}(y) \end{bmatrix} \end{matrix}$$

$$Cov(x, x) = Var(x)$$

# Covariance matrix

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

# Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

$$\mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n-1} \left( \begin{array}{c} \phantom{A} \\ \phantom{A} \\ \phantom{A} \\ \phantom{A} \end{array} - \begin{array}{c} \phantom{A} \\ \phantom{A} \\ \phantom{A} \\ \phantom{A} \end{array} \right)^T \left( \begin{array}{c} \phantom{A} \\ \phantom{A} \\ \phantom{A} \\ \phantom{A} \end{array} - \begin{array}{c} \phantom{A} \\ \phantom{A} \\ \phantom{A} \\ \phantom{A} \end{array} \right)$$

# Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

$$\mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n-1} (A - \mu)^T (A - \mu)$$

# Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left( \begin{array}{cc} & \\ & \\ & \\ & \end{array} \right)^T \left( \begin{array}{cc} & \\ & \\ & \\ & \end{array} \right)$$
$$= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}$$

# Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$\begin{aligned} C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \end{aligned}$$



# Covariance matrix

$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &\quad \quad \quad \boxed{2} \times n \quad \quad \quad n \times \boxed{2}
 \end{aligned}$$

# Covariance matrix

$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}
 \end{aligned}$$

$$(x_1 - \mu_x)(x_1 - \mu_x) + (x_2 - \mu_x)(x_2 - \mu_x) + \dots + (x_n - \mu_x)(x_n - \mu_x)$$

# Covariance matrix

$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}
 \end{aligned}$$

$\sum_{i=1}^n (x_i - \mu_x)^2 = \text{Var}(x)$

# Covariance matrix

$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \text{Var}(x) & \text{Cov}(x, y) \\ \text{Cov}(y, x) & \text{Var}(y) \end{bmatrix} \\
 &\quad \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)^2 = \text{Var}(x)
 \end{aligned}$$

# Covariance matrix

$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \text{Var}(x) & \text{Cov}(x, y) \\ \text{Cov}(y, x) & \text{Var}(y) \end{bmatrix} \\
 &\quad (x_1 - \mu_x)(y_1 - \mu_y) + (x_2 - \mu_x)(y_2 - \mu_y) + \dots + (x_n - \mu_x)(y_n - \mu_y)
 \end{aligned}$$

# Covariance matrix

$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \text{Var}(x) & \text{Cov}(x, y) \\ \text{Cov}(y, x) & \text{Var}(y) \end{bmatrix}
 \end{aligned}$$

$\sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y) = \text{Cov}(x, y)$

# Covariance matrix

$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \text{Var}(x) & \text{Cov}(x, y) \end{bmatrix} \\
 &= \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y) = \text{Cov}(x, y)
 \end{aligned}$$

# Covariance matrix

$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \text{Var}(x) & \text{Cov}(x, y) \\ \text{Cov}(y, x) & \text{Var}(y) \end{bmatrix}
 \end{aligned}$$

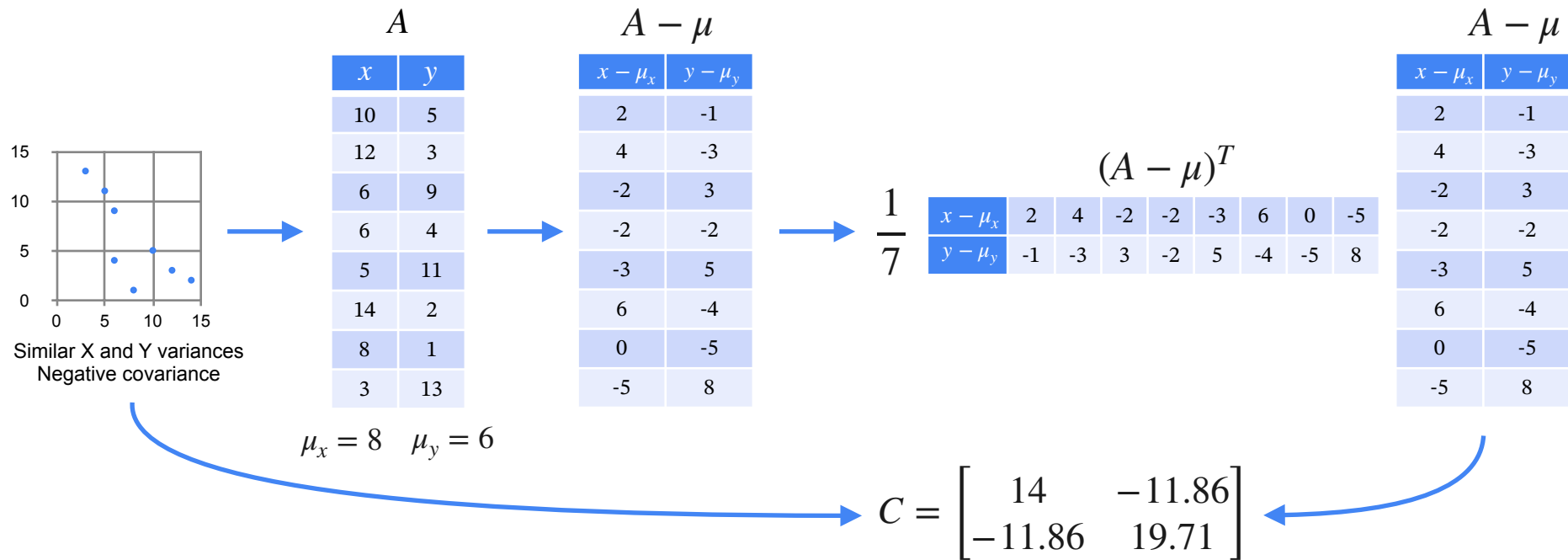


# Covariance matrix

$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left( \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \text{Var}(x) & \text{Cov}(x, y) \\ \text{Cov}(y, x) & \text{Var}(y) \end{bmatrix}
 \end{aligned}$$

# Matrix formula

$$A - \mu = \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \quad C = \frac{1}{n-1} (A - \mu)^T (A - \mu)$$



# Matrix formula

$$A = \begin{bmatrix} x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & z_n \end{bmatrix} \quad C = \frac{1}{n-1} (A - \mu)^T (A - \mu)$$

1. Arrange data with a different feature in each column
2. Calculate column averages
3. Subtract each average from their respective column to generate  $A - \mu$
4.  $\frac{1}{n-1} (A - \mu)^T (A - \mu)$  gives the covariance matrix  $C$



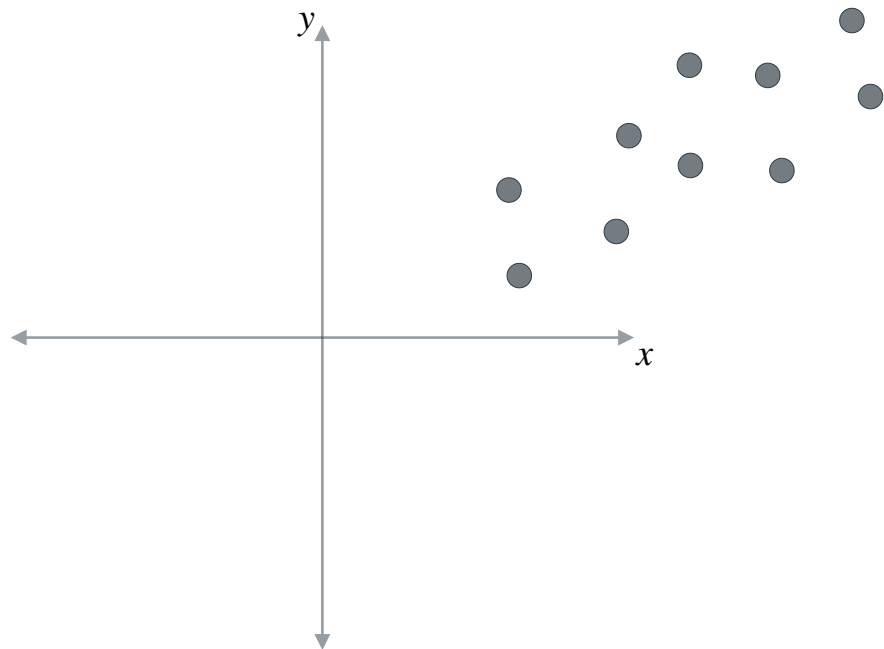
DeepLearning.AI

# Determinants and Eigenvectors

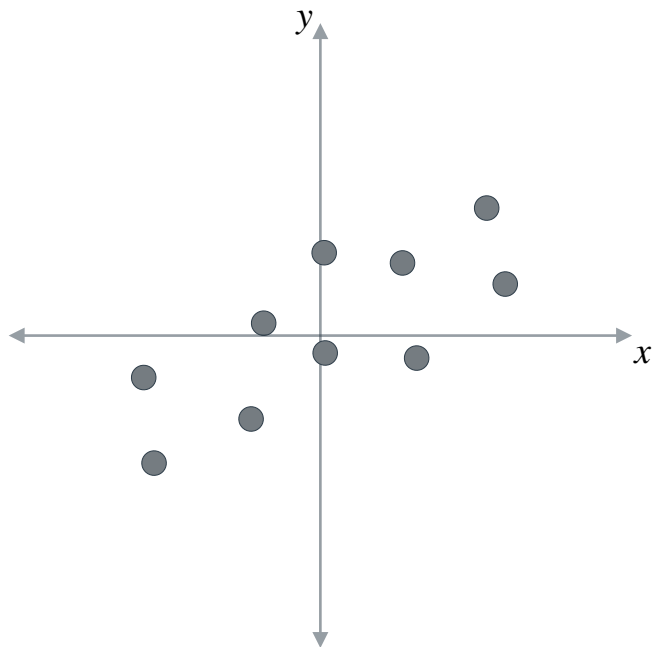
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## PCA - Overview

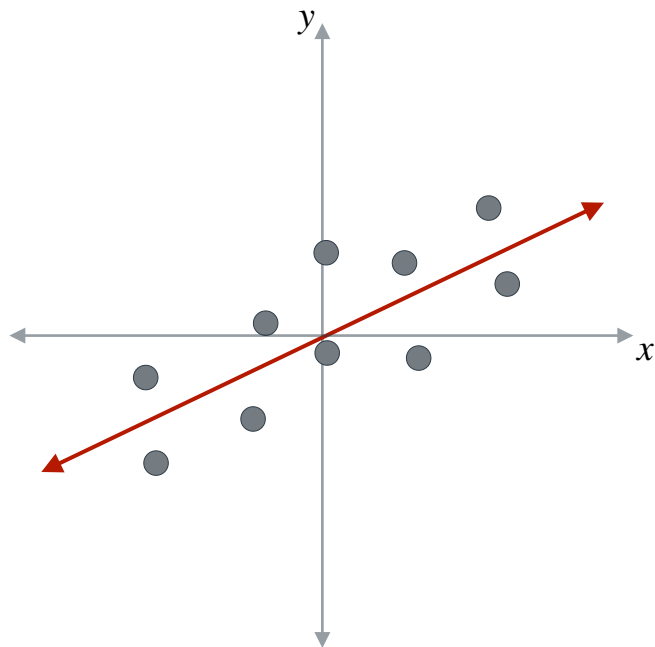
# Principal Component Analysis (PCA)



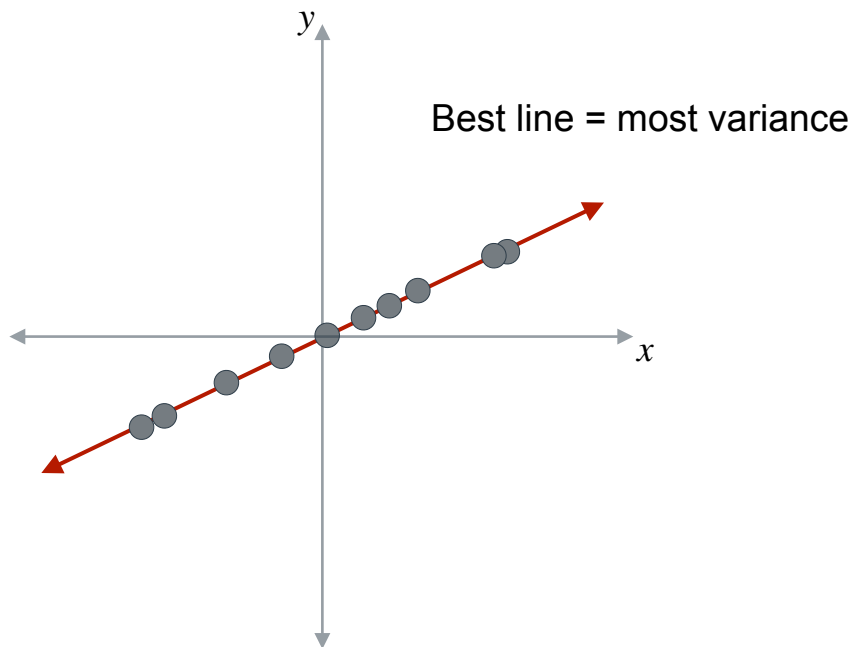
# Principal Component Analysis (PCA)



# Principal Component Analysis (PCA)

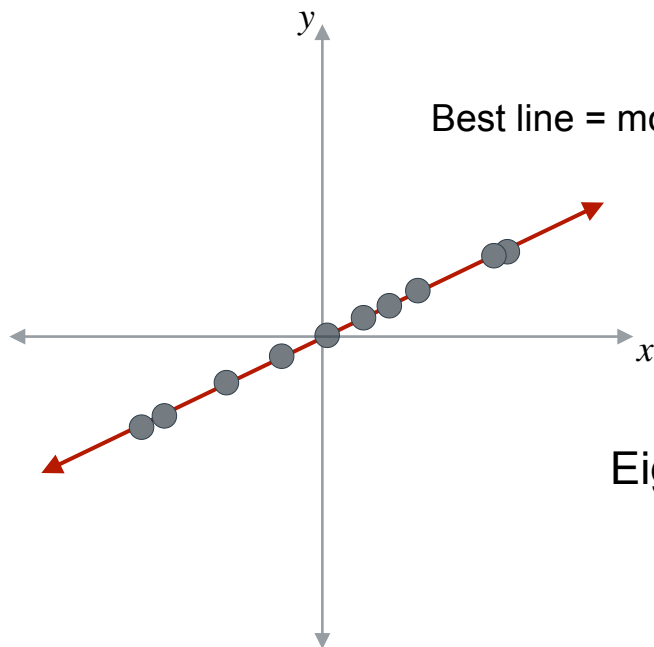


# Principal Component Analysis (PCA)





# Principal Component Analysis (PCA)



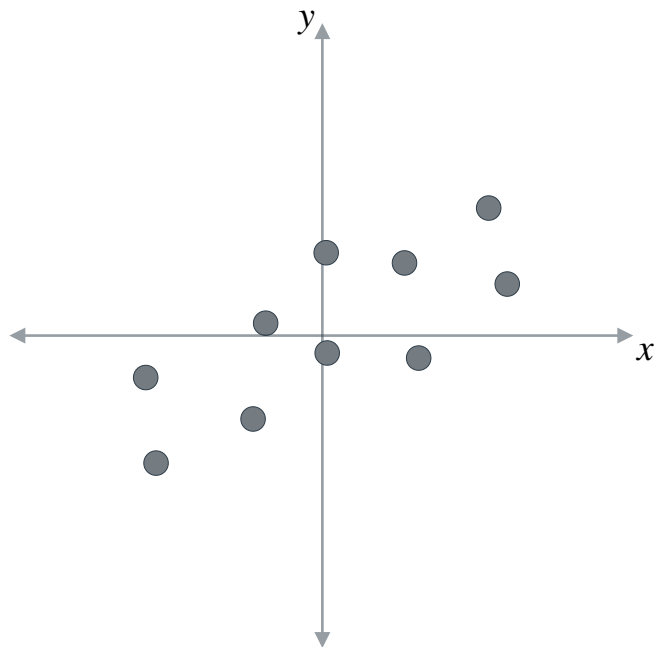
Projections

Eigenvalues / Eigenvectors

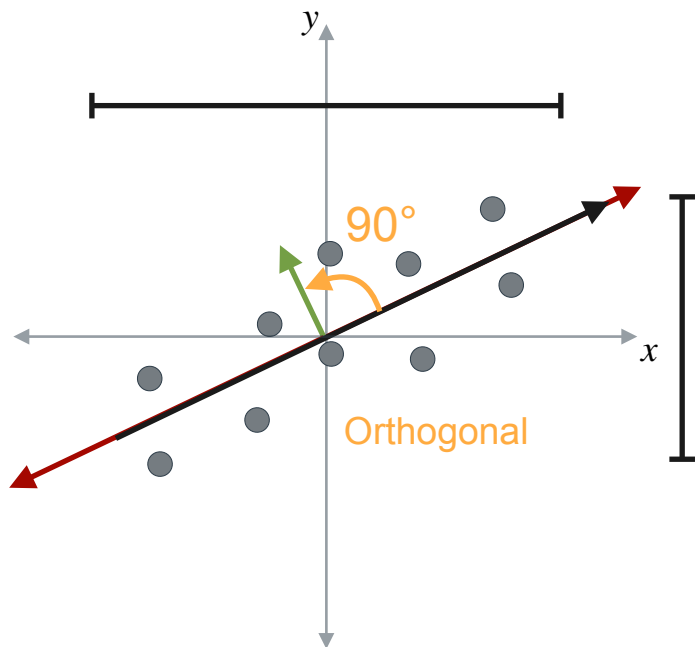
Covariance Matrix

PCA

# Principal Component Analysis (PCA)



# Principal Component Analysis (PCA)



$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

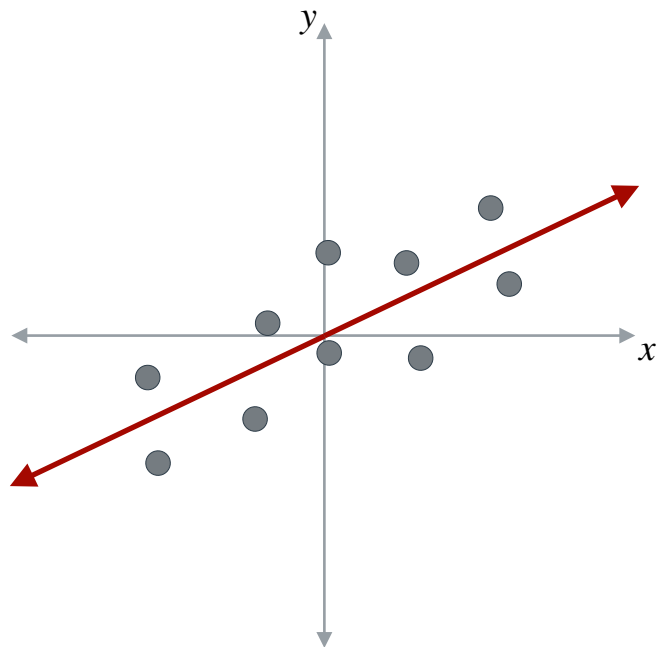
Eigenvectors  
(direction)

$$11$$


$$1$$

Eigenvalues  
(magnitude)

# Principal Component Analysis (PCA)



$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

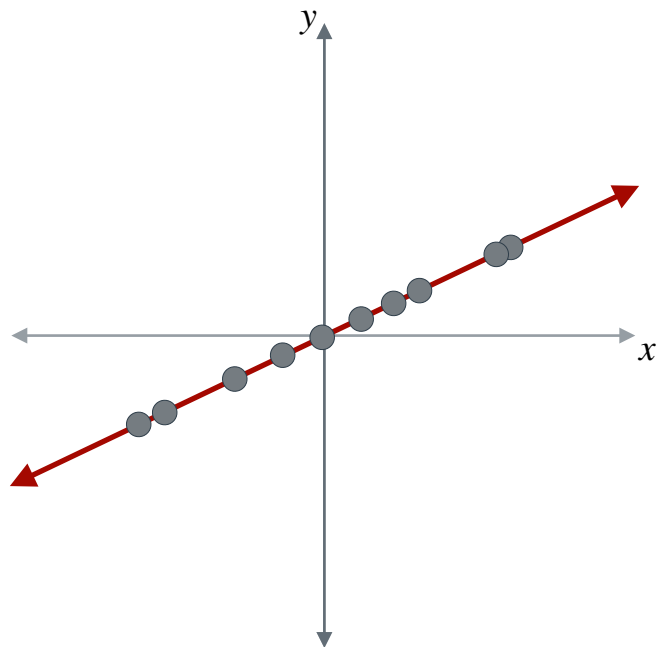
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvectors  
(direction)

$$11$$

Eigenvalues  
(magnitude)

# Principal Component Analysis (PCA)



$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

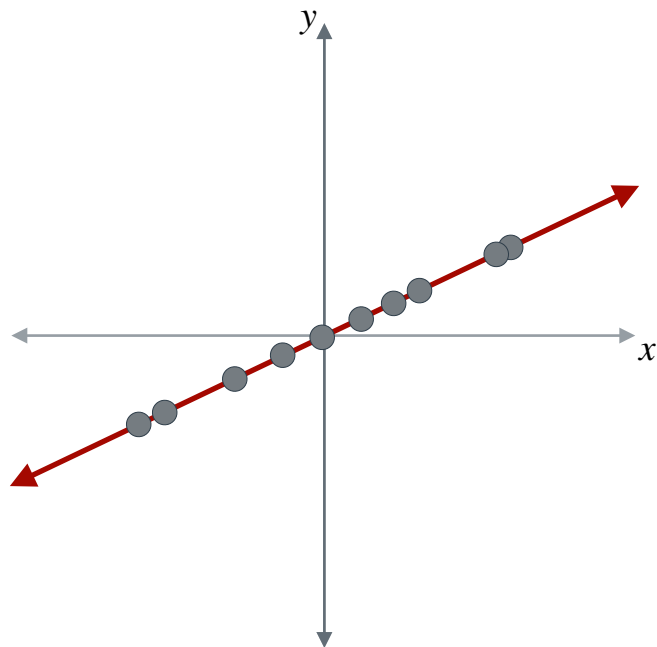
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvectors  
(direction)

$$11$$

Eigenvalues  
(magnitude)

# Principal Component Analysis (PCA)



$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvectors  
(direction)

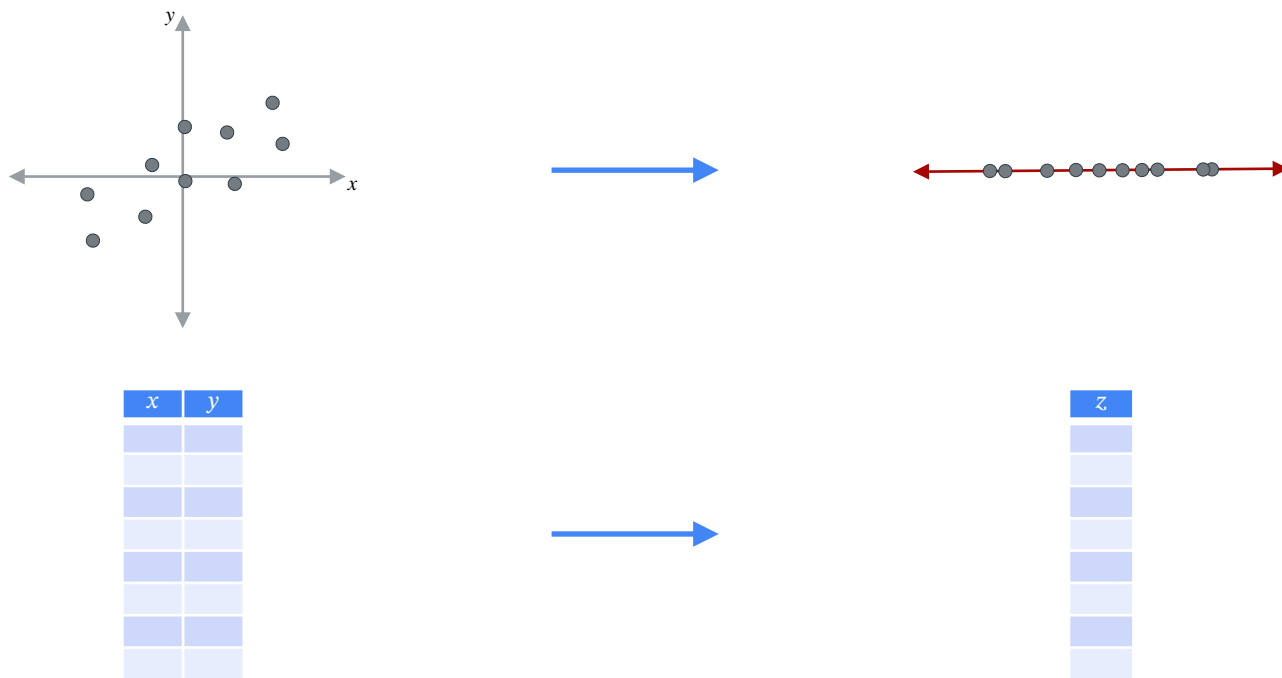
$$11$$

Eigenvalues  
(magnitude)

# Principal Component Analysis (PCA)

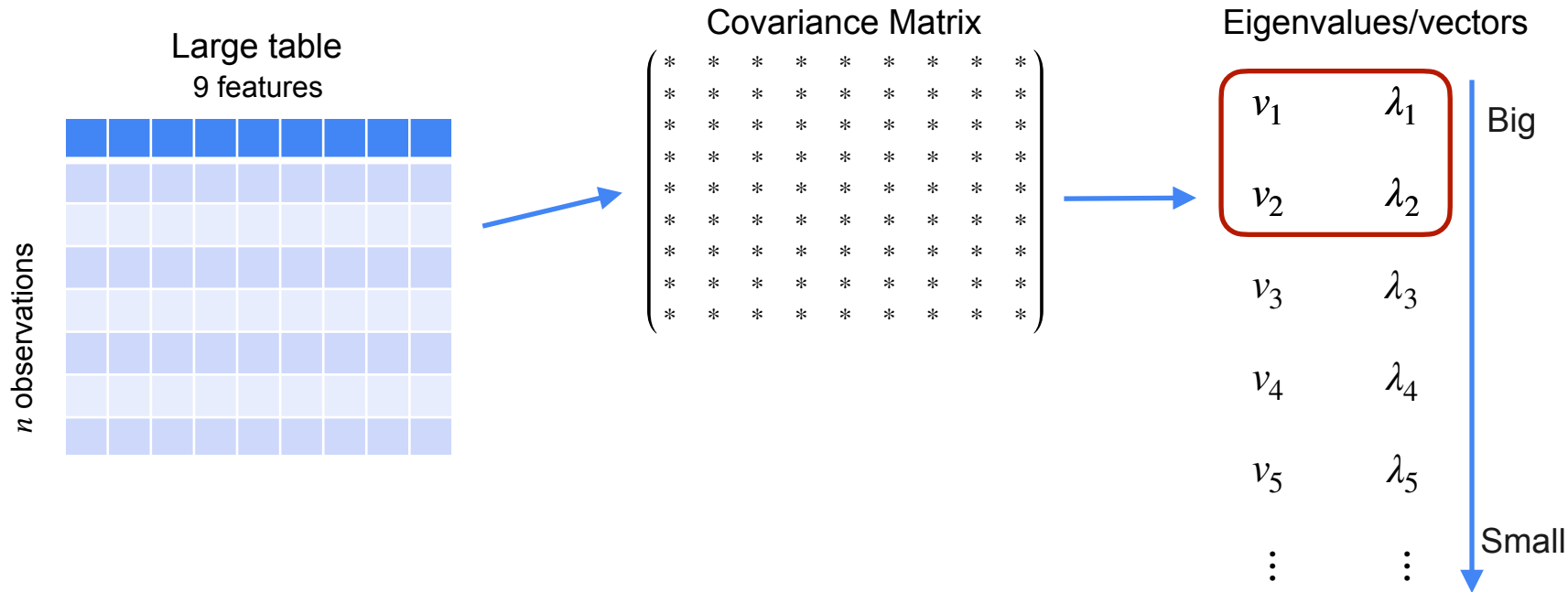


# Principal Component Analysis (PCA)

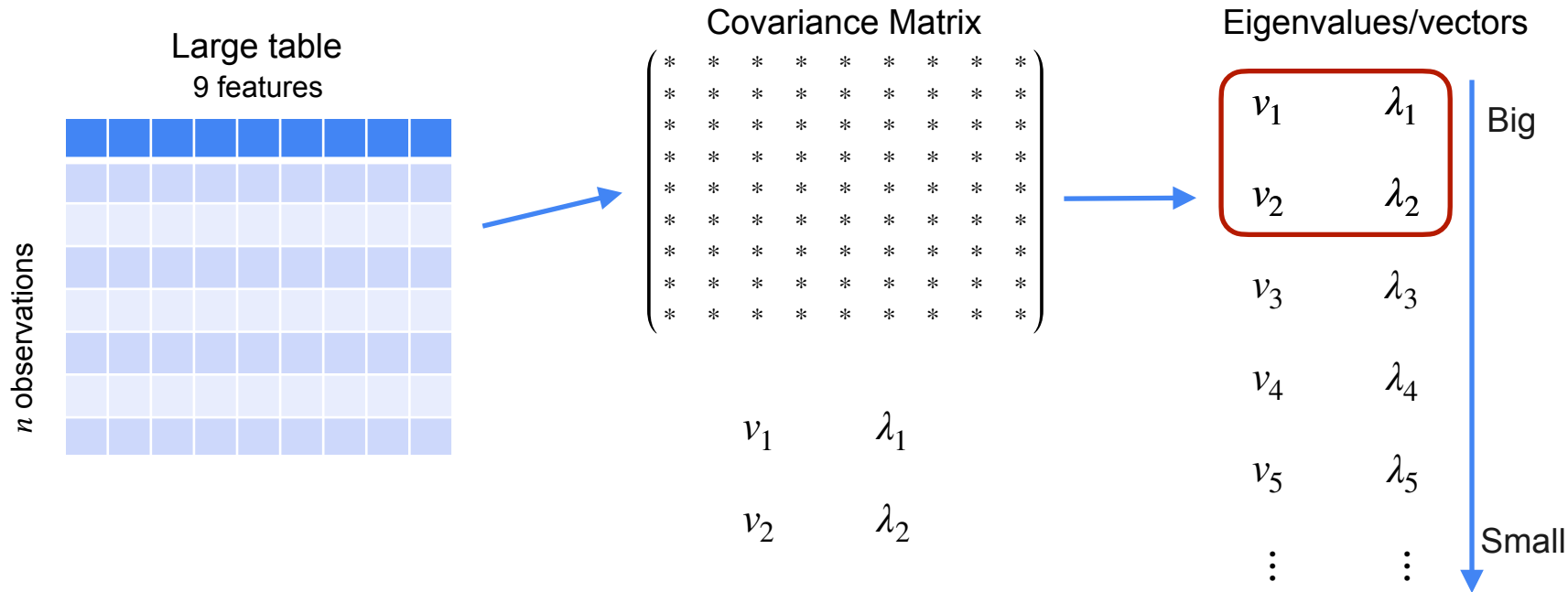




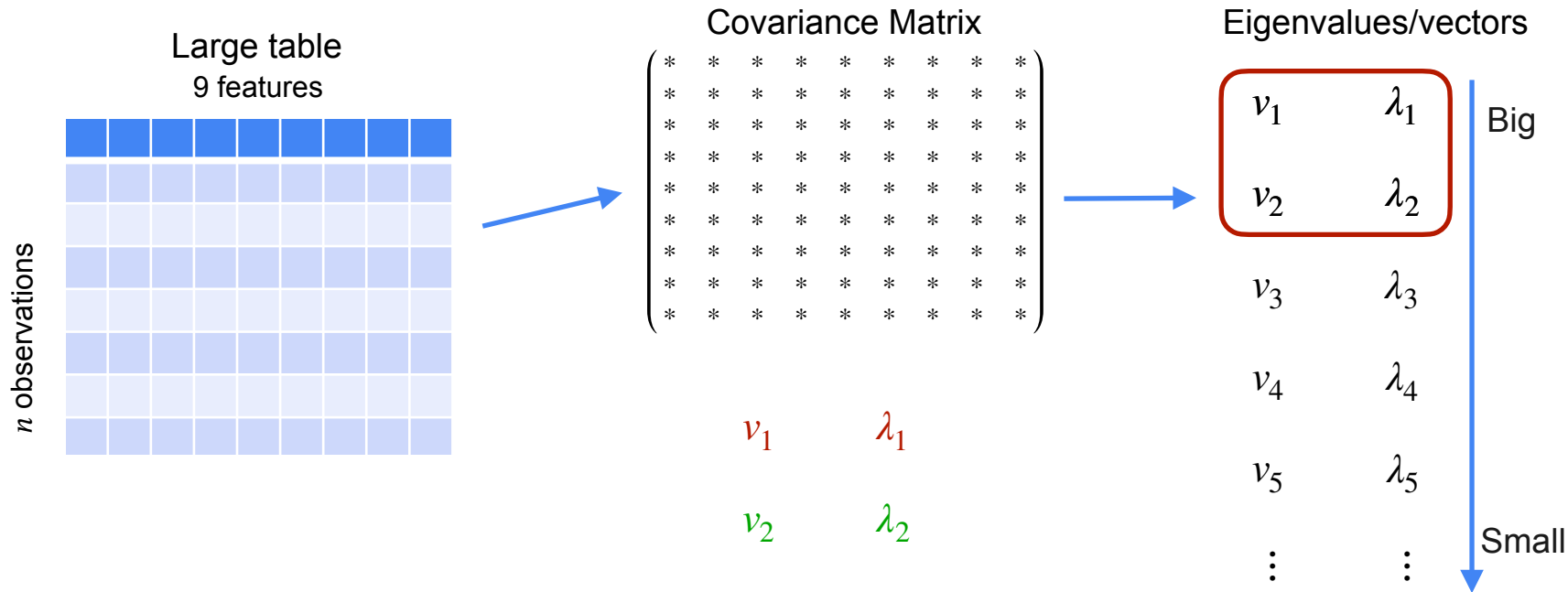
# PCA: Principal Component Analysis



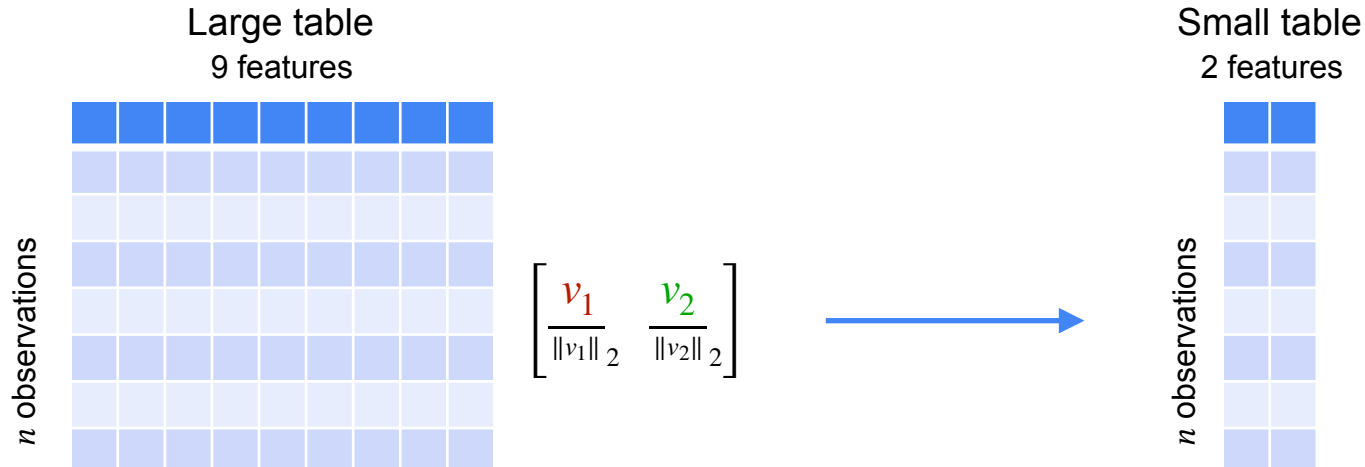
# PCA: Principal Component Analysis



# PCA: Principal Component Analysis



# PCA: Principal Component Analysis





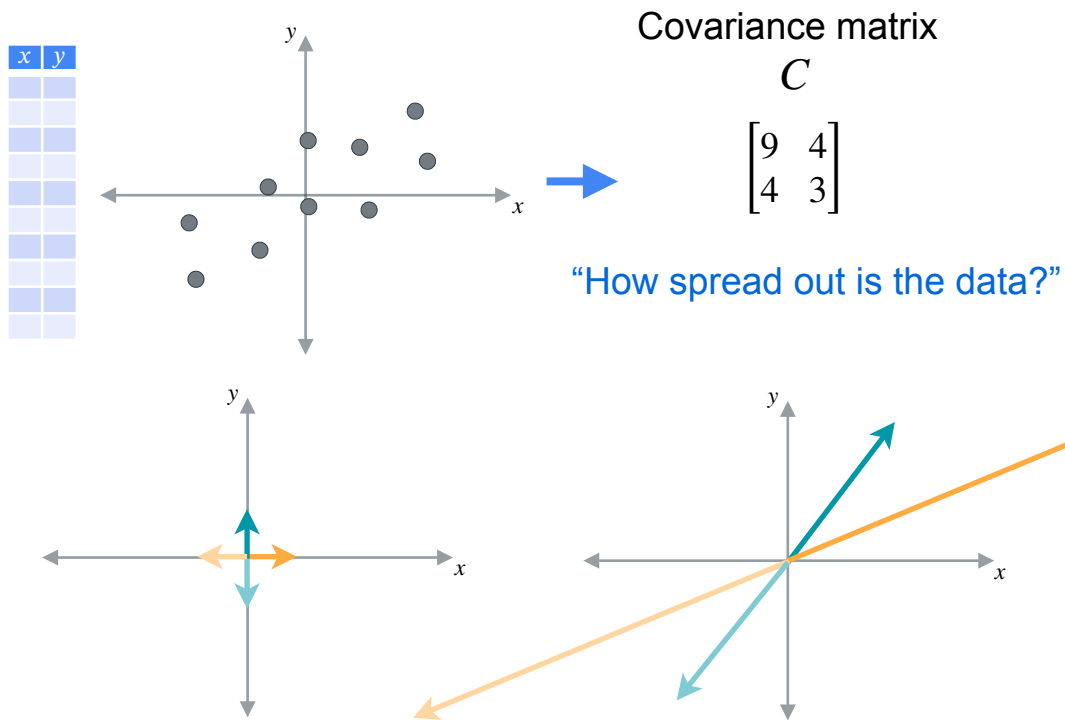
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# Determinants and Eigenvectors

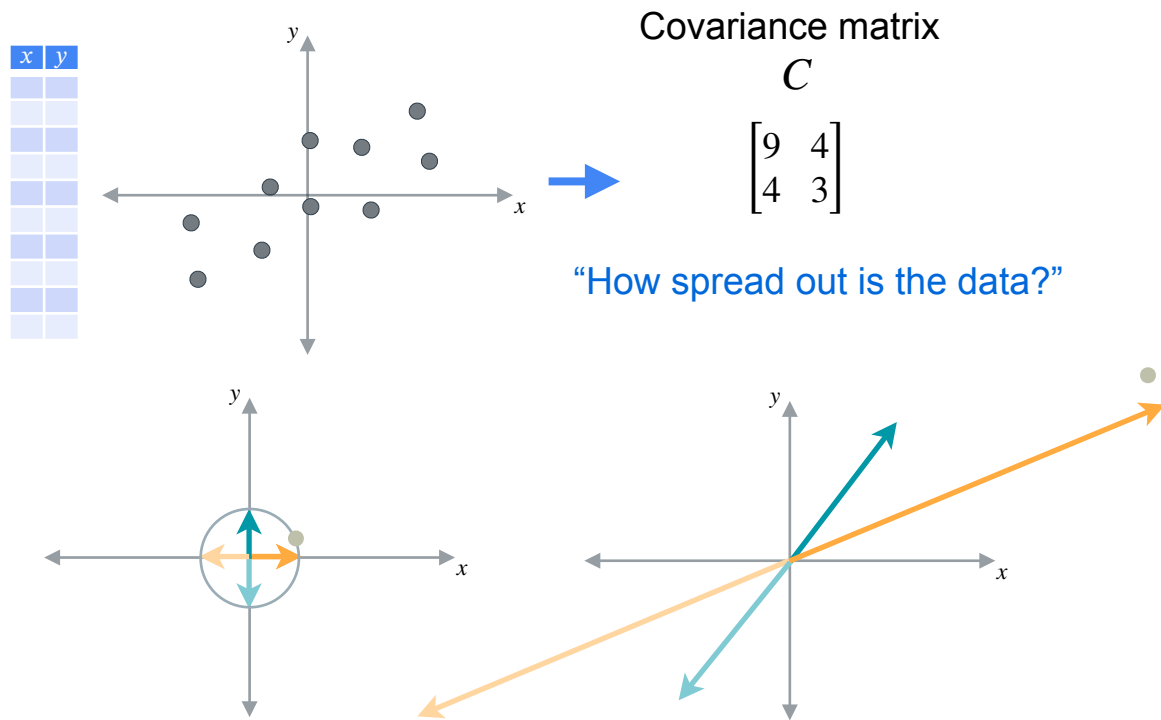
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## **PCA - Why it works**

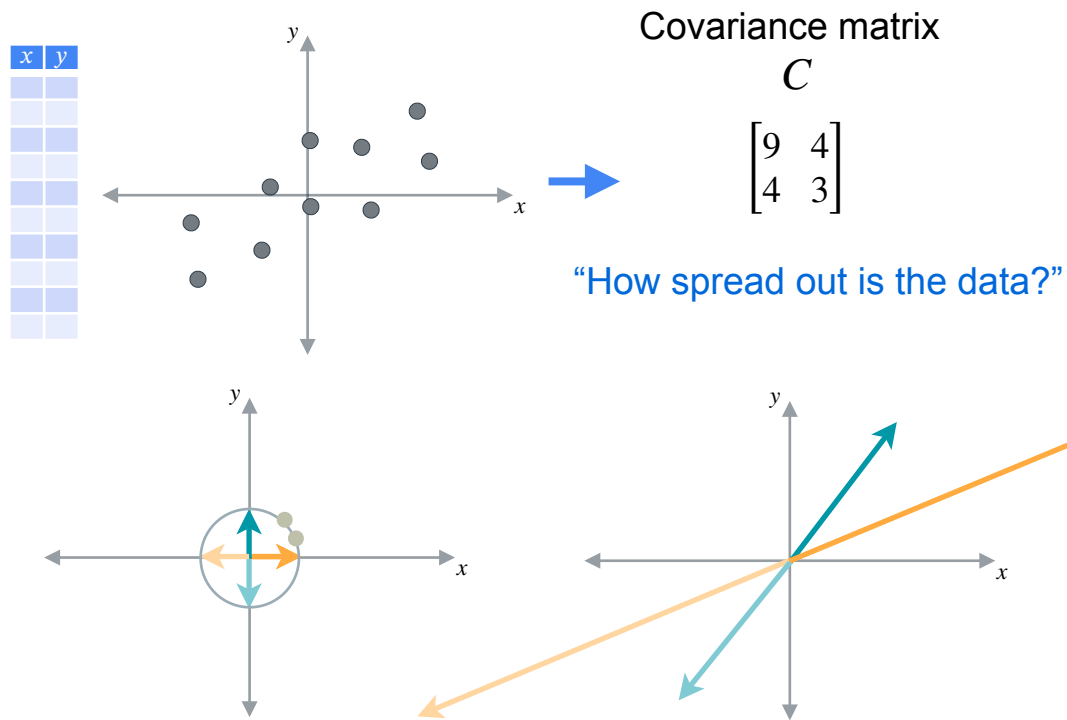
# PCA: Why It Works



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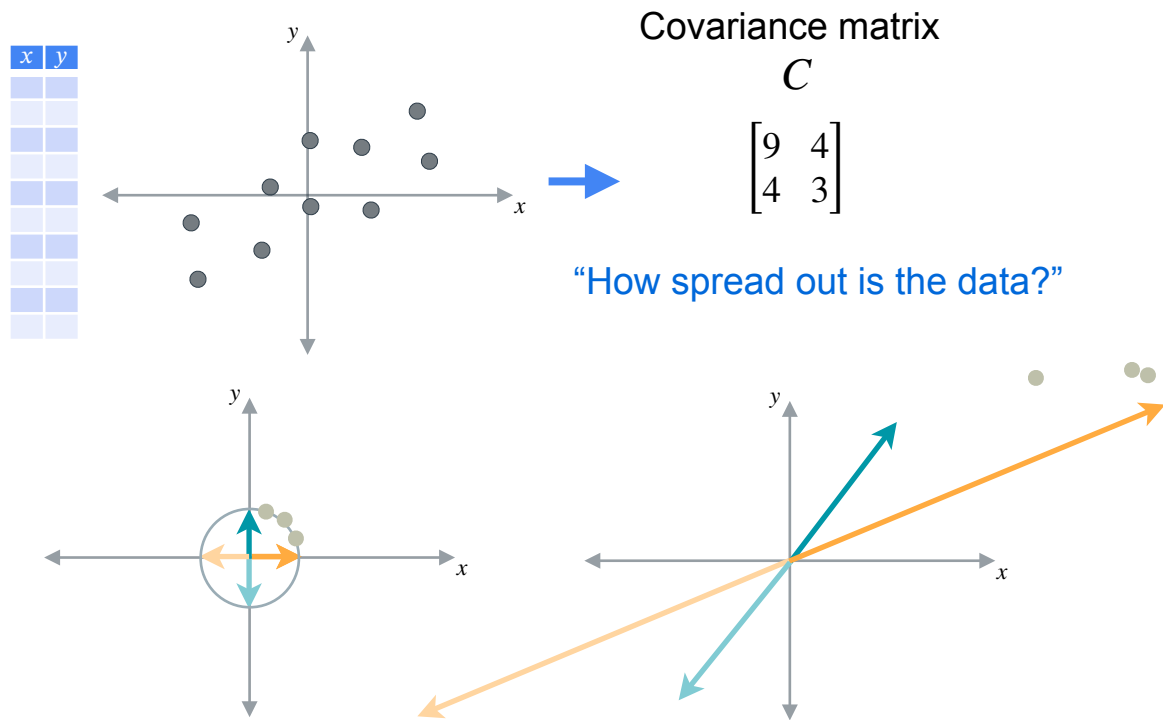


# PCA: Why It Works

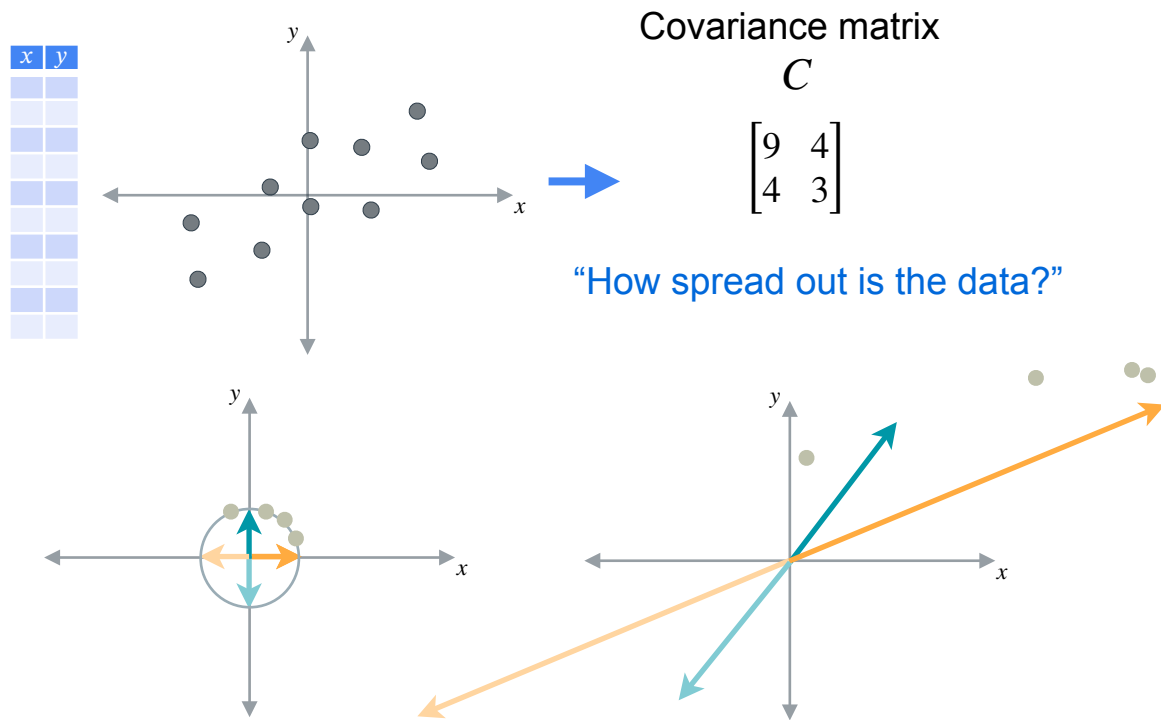




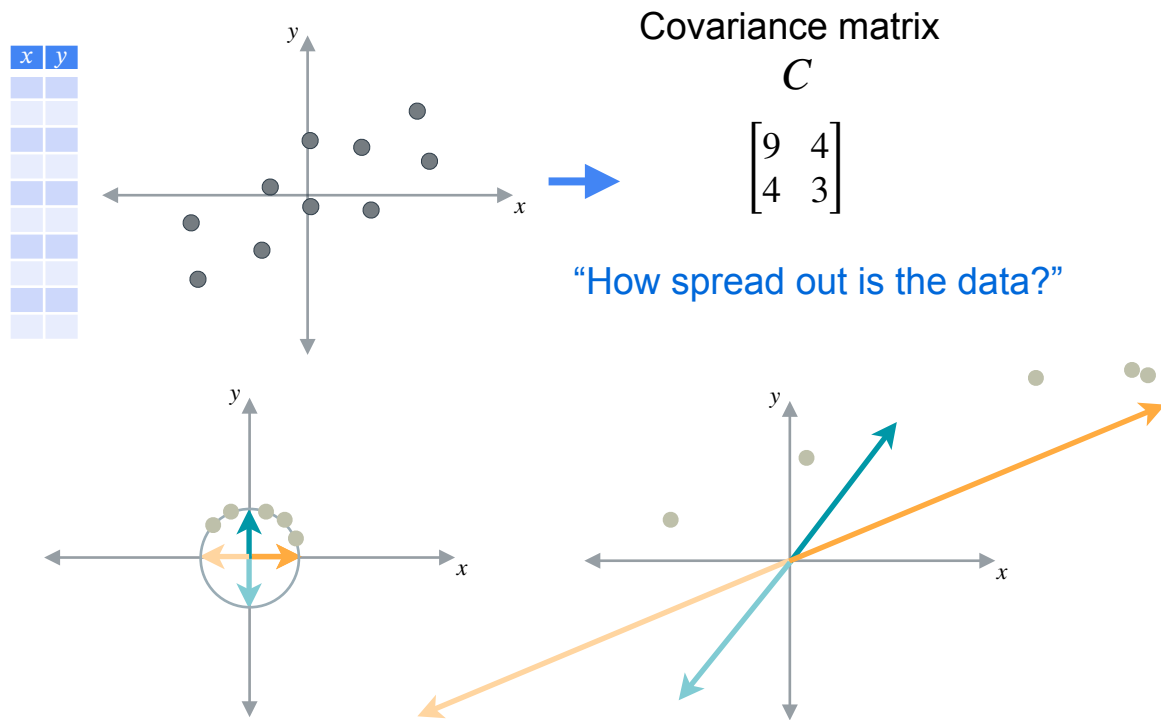
# PCA: Why It Works



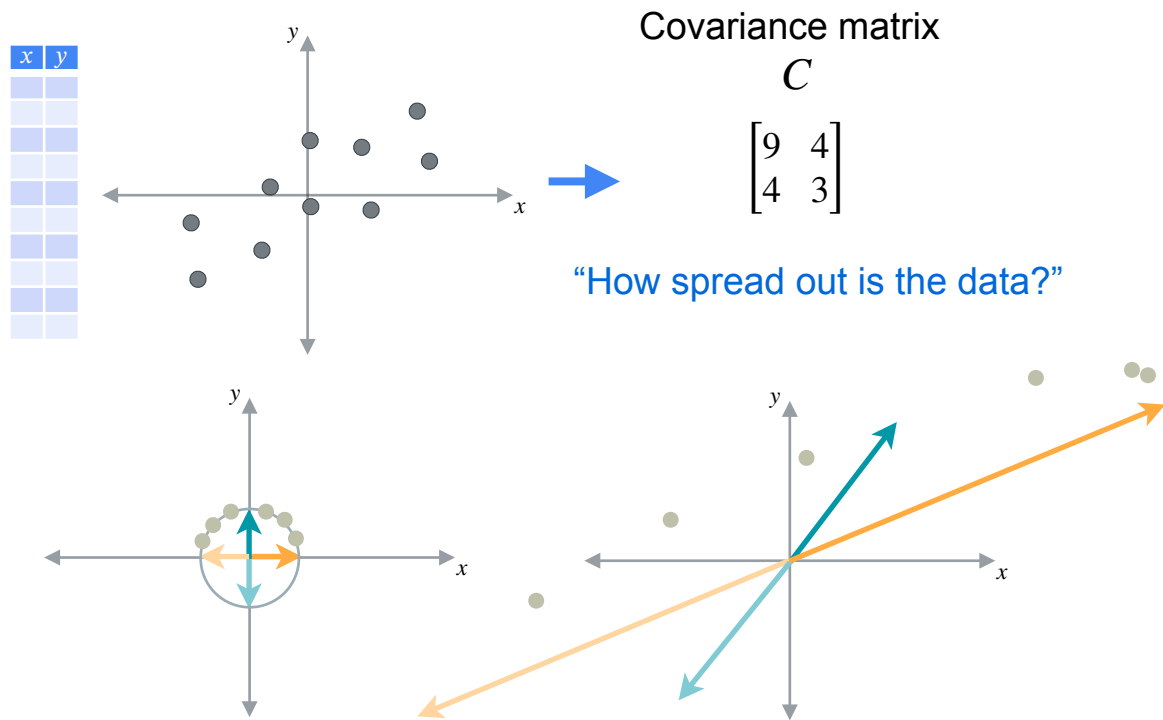
# PCA: Why It Works



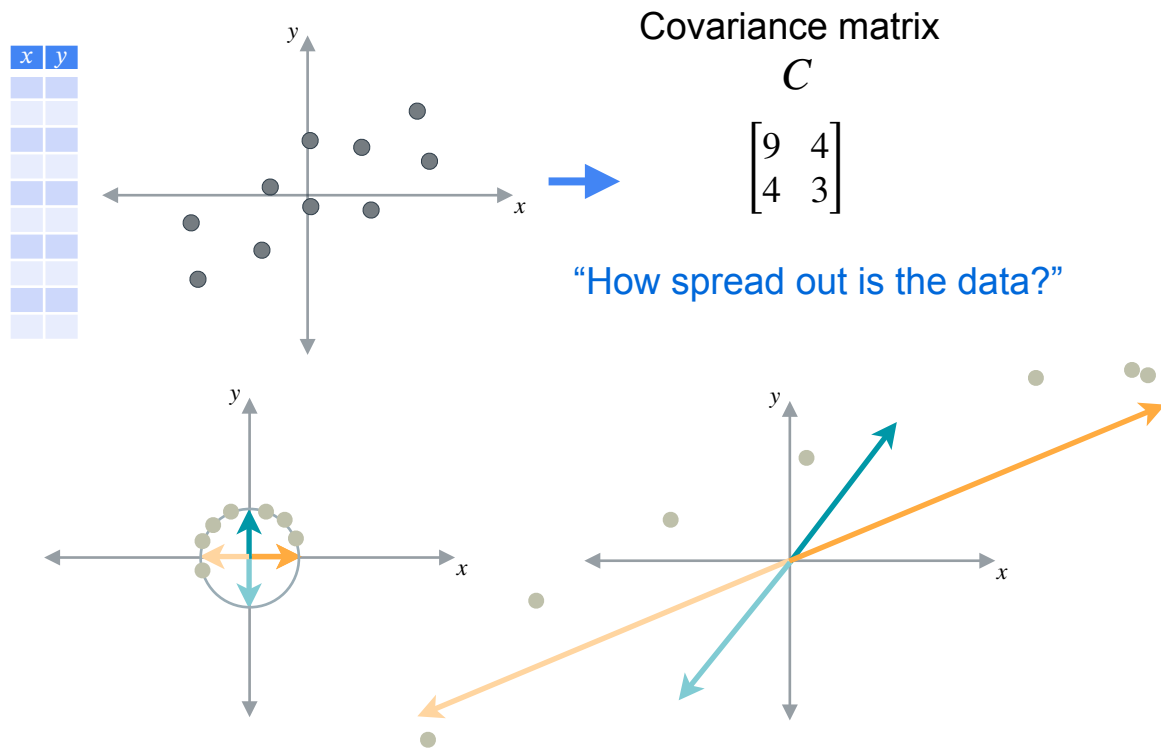
# PCA: Why It Works



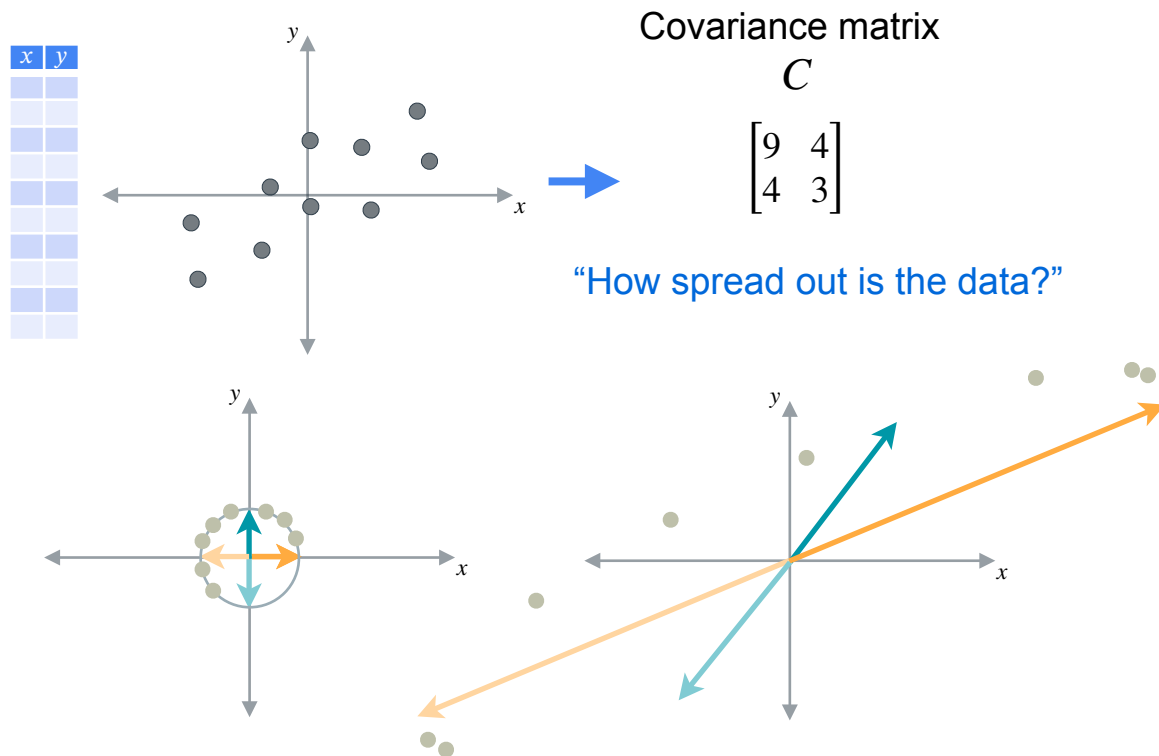
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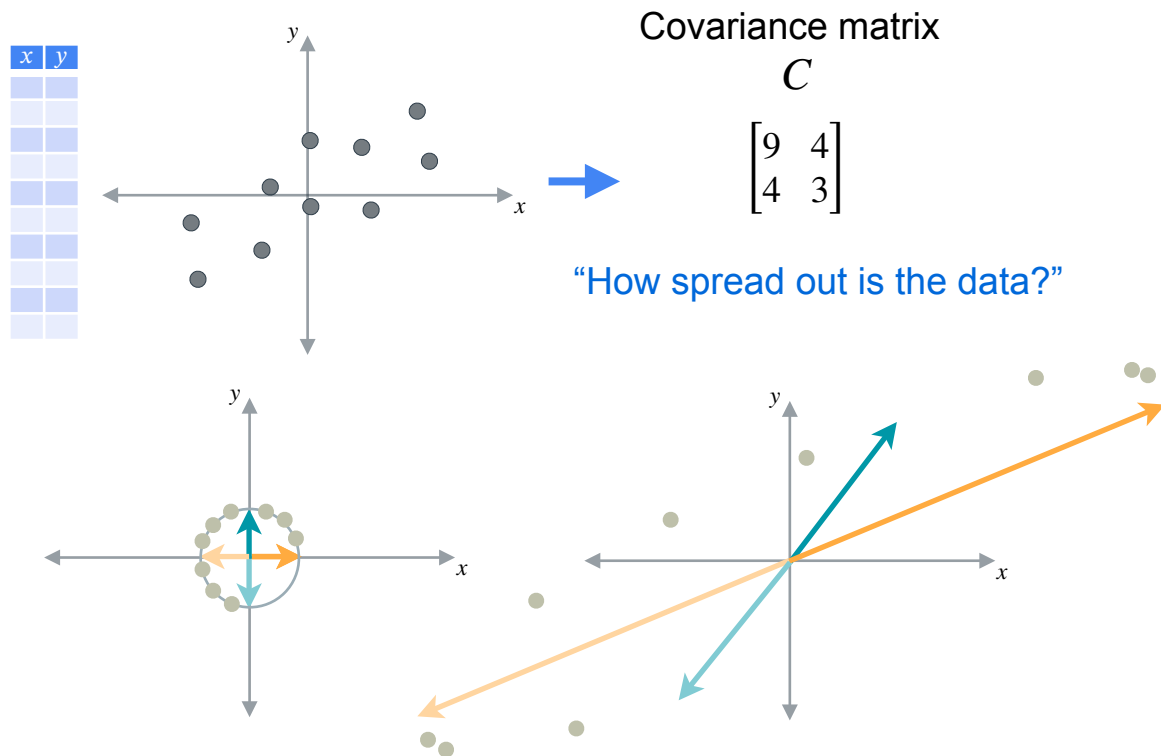
# PCA: Why It Works



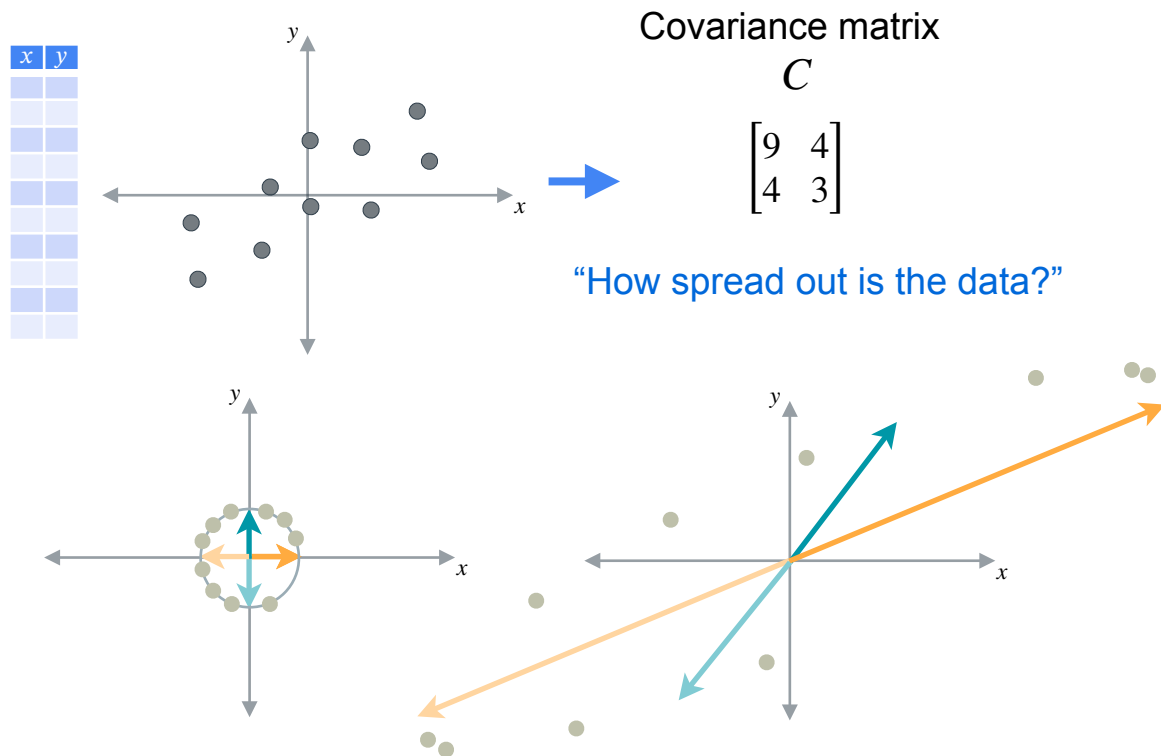
# PCA: Why It Works



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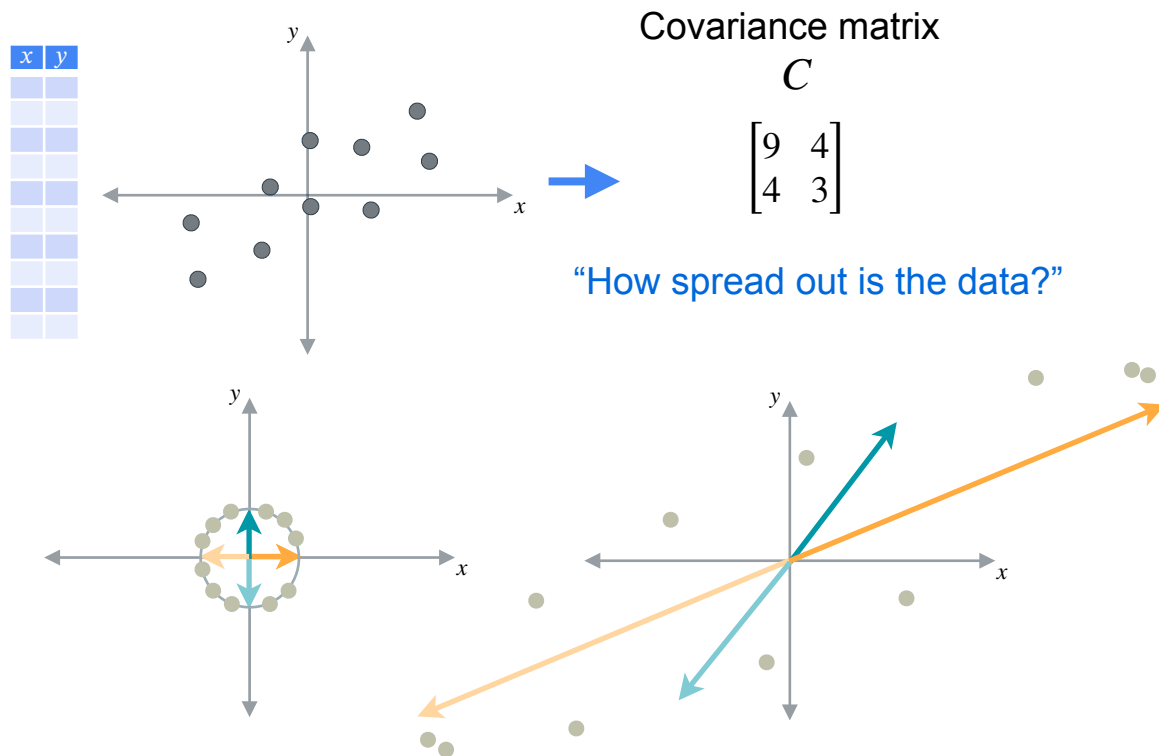


# PCA: Why It Works

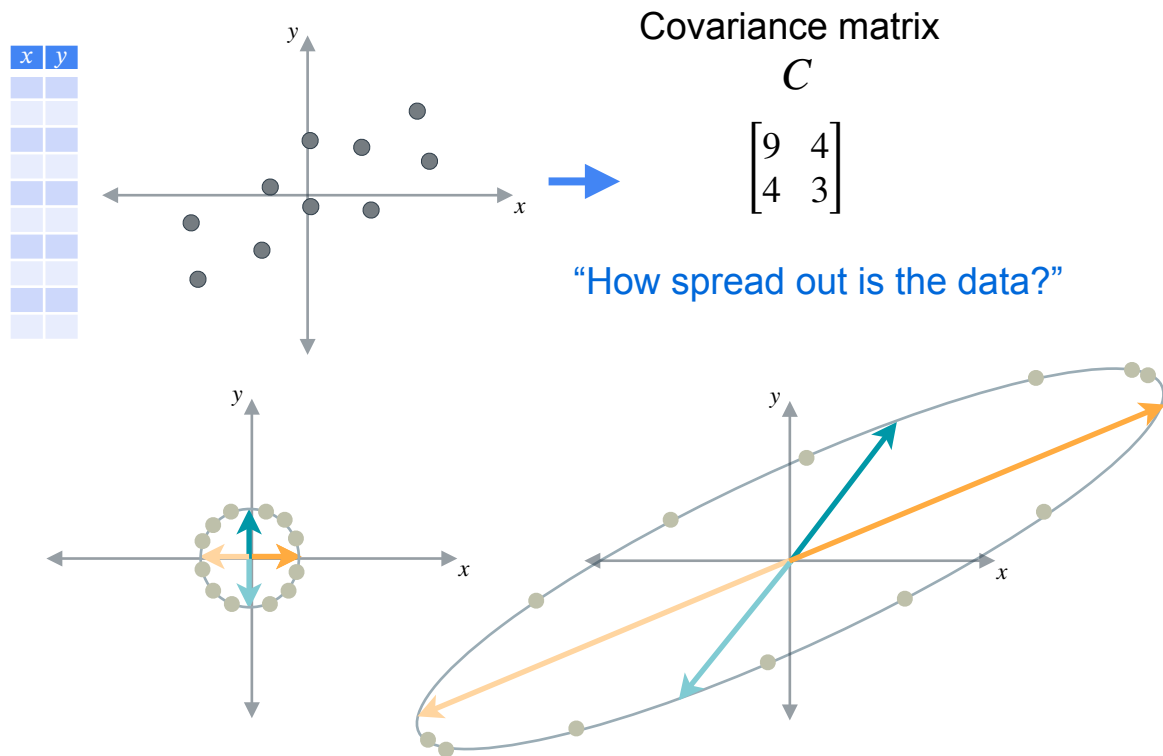




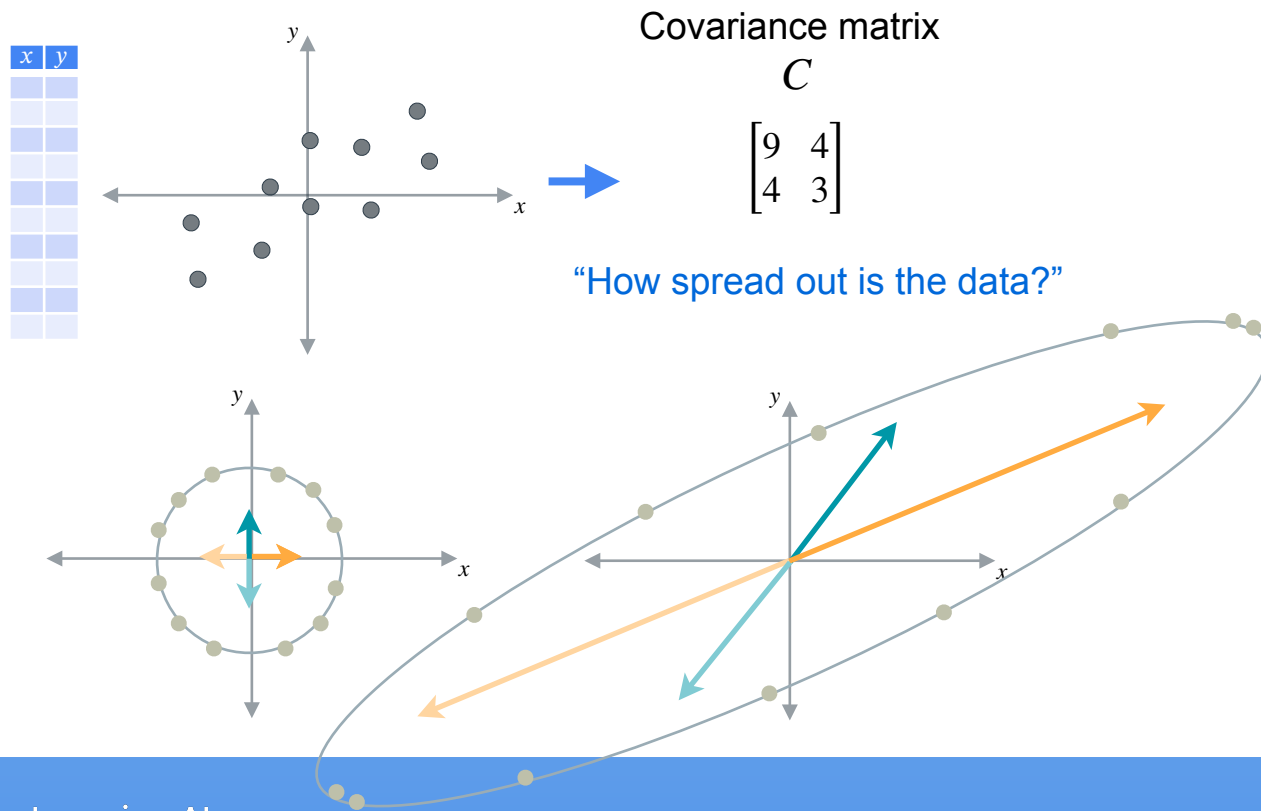
# PCA: Why It Works



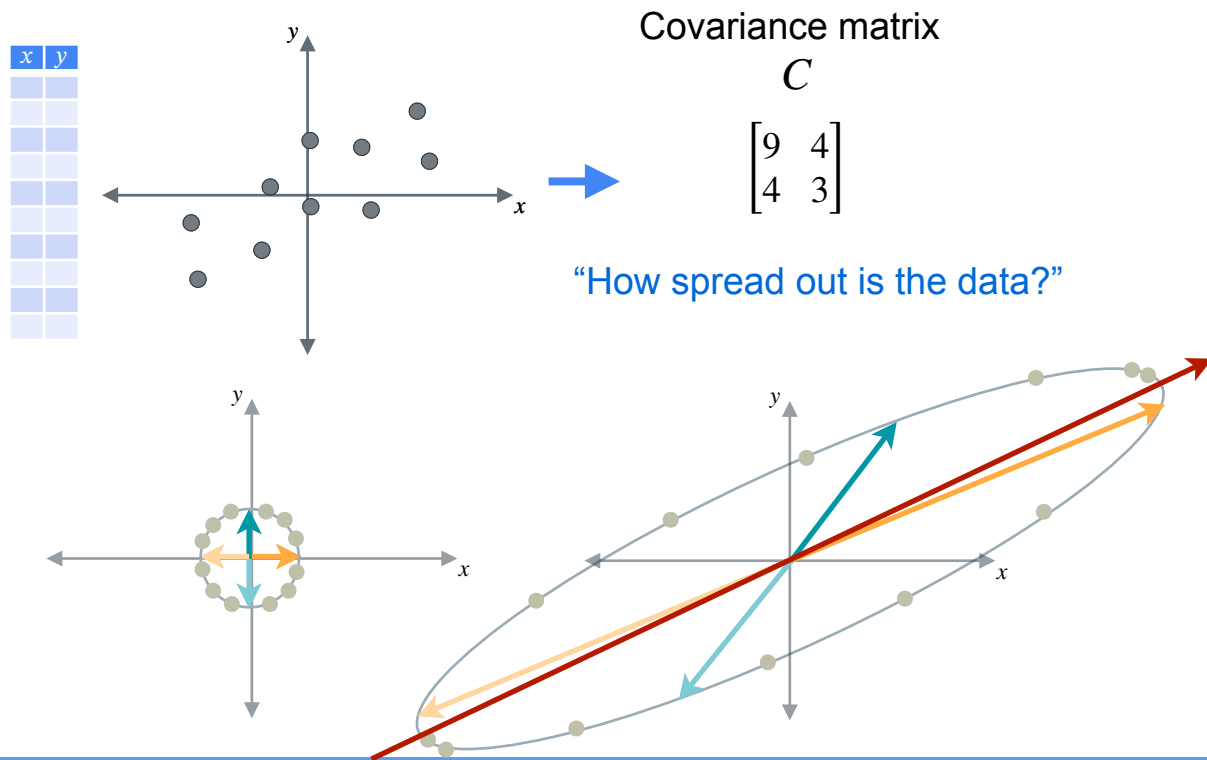
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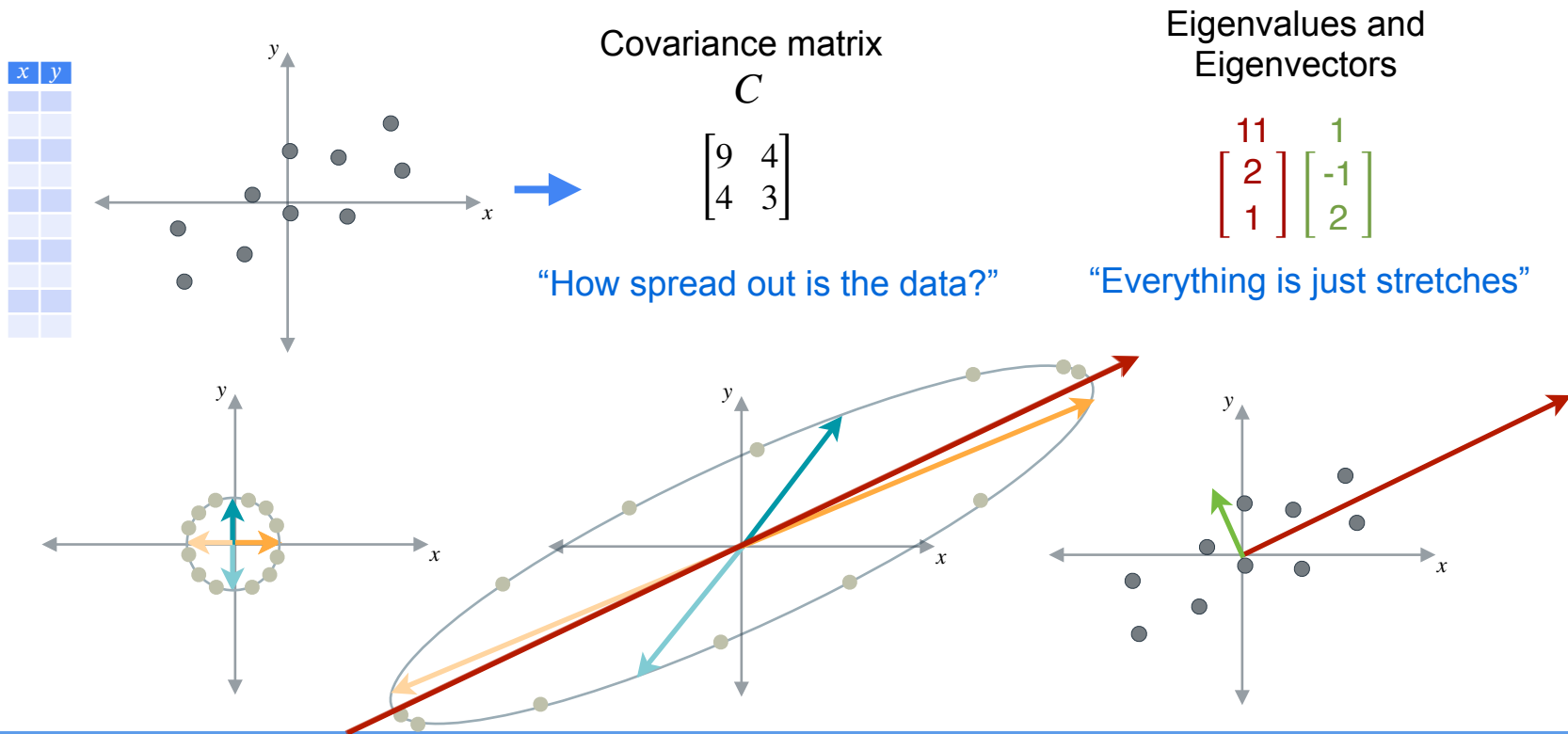
# PCA: Why It Works



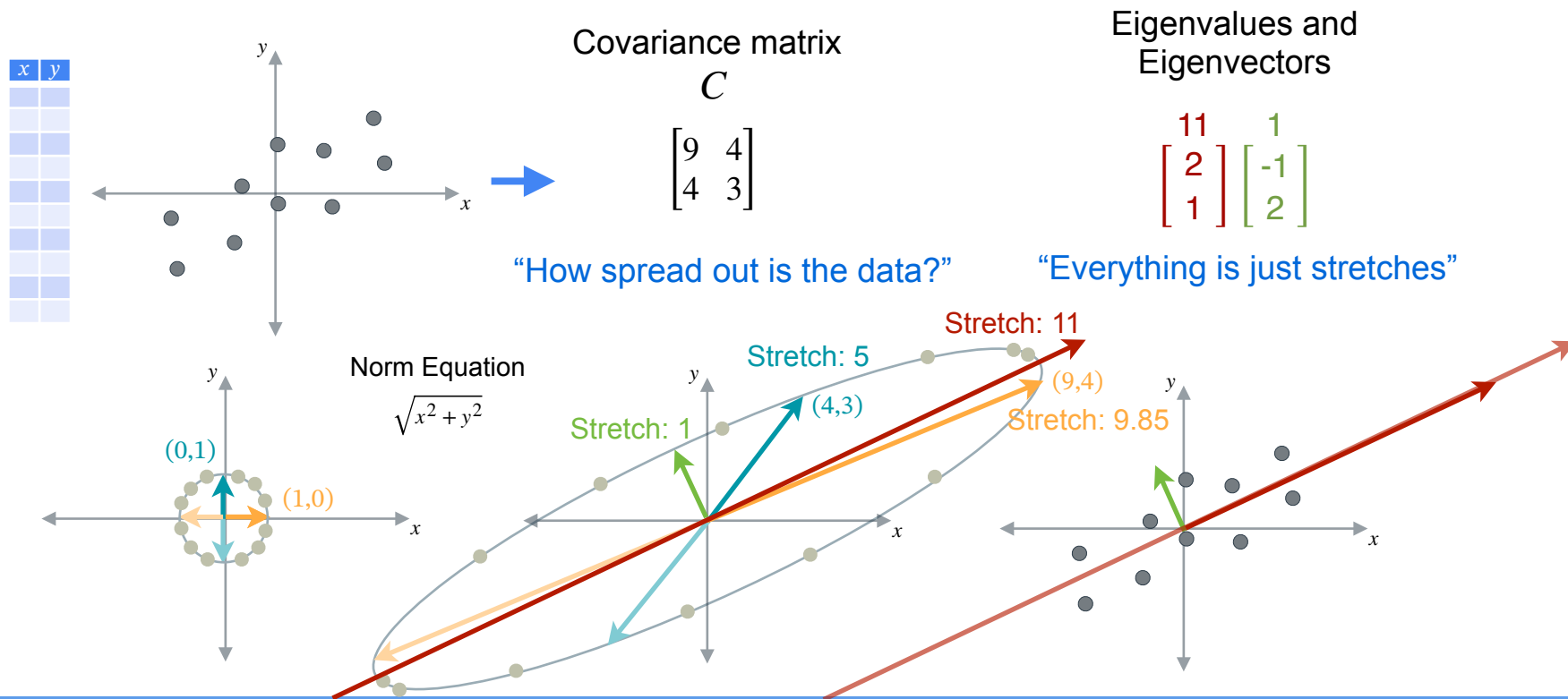
# PCA: Why It Works



# PCA: Why It Works



# PCA: Why It Works





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# Determinants and Eigenvectors

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## **PCA - Mathematical formulation**

# PCA Mathematical formulation

You have  $n$  observations of 5 variables ( $x_1, x_2, x_3, x_4, x_5$ )

Goal: Reduce to 2 variables

1 Create matrix

$$X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{15} \\ x_{21} & x_{22} & \dots & x_{25} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{n5} \end{bmatrix}$$

5 variables

$n$  Observations

2 Center the data

$$X - \mu = \begin{bmatrix} x_{11} - \mu_1 & x_{12} - \mu_2 & \dots & x_{15} - \mu_5 \\ x_{21} - \mu_1 & x_{22} - \mu_2 & \dots & x_{25} - \mu_5 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} - \mu_1 & x_{n2} - \mu_2 & \dots & x_{n5} - \mu_5 \end{bmatrix}$$



# PCA Mathematical formulation

You have  $n$  observations of 5 variables ( $x_1, x_2, x_3, x_4, x_5$ )

Goal: Reduce to 2 variables

3 Calculate Covariance Matrix

$$C = \frac{1}{n-1}(X - \mu)^T(X - \mu) = \begin{bmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_2) & \text{Cov}(X_1, X_3) & \text{Cov}(X_1, X_4) & \text{Cov}(X_1, X_5) \\ \text{Cov}(X_1, X_2) & \text{Var}(X_2) & \text{Cov}(X_2, X_3) & \text{Cov}(X_2, X_4) & \text{Cov}(X_2, X_5) \\ \text{Cov}(X_1, X_3) & \text{Cov}(X_2, X_3) & \text{Var}(X_3) & \text{Cov}(X_3, X_4) & \text{Cov}(X_3, X_5) \\ \text{Cov}(X_1, X_4) & \text{Cov}(X_2, X_4) & \text{Cov}(X_3, X_4) & \text{Var}(X_4) & \text{Cov}(X_4, X_5) \\ \text{Cov}(X_1, X_5) & \text{Cov}(X_2, X_5) & \text{Cov}(X_3, X_5) & \text{Cov}(X_4, X_5) & \text{Var}(X_5) \end{bmatrix}$$

# PCA Mathematical formulation

You have  $n$  observations of 5 variables ( $x_1, x_2, x_3, x_4, x_5$ )

Goal: Reduce to 2 variables

4 Calculate Eigenvectors and Eigenvalues

Big  $\uparrow$

$\lambda_1$	$v_1$
$\lambda_2$	$v_2$
$\lambda_3$	$v_3$
$\lambda_4$	$v_4$
Small $\lambda_5$	$v_5$

5 Create Projection Matrix

$$V = \begin{bmatrix} \overline{\|v_1\|_2} & \overline{\|v_2\|_2} \end{bmatrix}$$

6 Project Centered Data

$$X_{PCA} = (X - \mu)V$$



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# Determinants and Eigenvectors

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## Conclusion