Non-equilibrium steady state and induced currents of a mesoscopically-glassy system: interplay of resistor-network theory and Sinai physics

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We introduce an explicit solution for the non-equilibrium steady state (NESS) of a ring that is coupled to a thermal bath, and is driven by an external hot source with log-wide distribution of couplings. Having time scales that stretch over several decades is similar to glassy systems. Consequently there is a wide range of driving intensities where the NESS is like that of a random walker in a biased Brownian landscape. We investigate the resulting statistics of the induced current I. For a single ring we discuss how $\text{sign}(I)$ fluctuates as the intensity of the driving is increased, while for an ensemble of rings we highlight the fingerprints of Sinai physics on the $\text{abs}(I)$ distribution.

The transport in a chain due to random non-symmetric transition probabilities is a fundamental problem in statistical mechanics [1–5]. It can be regarded as *a random walk in a random environment* [6, 7]. This type of dynamics is of great relevance for surface diffusion [8], thermal ratchets [9–12] and was used to model diverse biological systems, such as molecular motors, enzymes, and unidirectional motion of proteins along filaments [13–16]. Of particular interest are applications that concern the conduction of DNA segments [17, 18], and thin glassy electrolytes under high voltages [19–24].

Mathematically one can visualize the dynamics as a random-walk of a particle that makes incoherent jumps between "sites" in a network. In an unbounded quasi-one-dimensional network we might have either diffusion or sub-diffusive Sinai spreading [6], depending on whether the transitions rates form a symmetric matrix or not. In contrast, when the system is bounded it eventually reaches a well-defined steady state. This would be an equilibrium *canonical* state if the transition rates were detailed-balanced, else it is termed non-equilibrium steady state (NESS).

We would like to consider the NESS of a mesoscopically glassy system. By "glassiness" we mean that the rates that are induced by a bath, or by an external source, have a log-wide distribution, hence many time scales are involved. Specifically we consider a common way of maintaining a NESS (Fig. 1): a system that is coupled to a driving source ("hot bath") that spoils the detailed-balance of the environment ("cold bath"). The log wide distribution of the transition rates leads to a novel NESS. In previous publications we have pointed out that due to "glassiness" (also termed "sparsity") the physics of Sinai-type disorder is a relevant ingredient in the analysis of energy absorption [25] and transport [26].

FIG. 1: A ring made up of N sites is immersed in a "cold" bath and subjected to a "hot" driving source [a]. As a result a current is induced. In the numerics the the driving source induces rates that are log-box distributed over 6 decades.

Scope.– Below we introduce an explicit NESS solution for a minimal model that has all the essential ingredients of the problem, involving transitions between sites on a ring and a log-wide distribution of couplings to an external driving source. The induced steady state current I is the central quantity used to characterize the NESS in actual experiments. The purpose of the present study is to investigate its statistics. Specifically, for a single ring we discuss how $\text{sign}(I)$ fluctuates as the intensity of the driving is increased, while for an ensemble of rings we highlight the fingerprints of Sinai physics on the $\text{abs}(I)$ distribution.

Remarks.– Previous study of Sinai-type disordered systems [7], has considered an open geometry with uncorrelated transition rates that have the same coupling everywhere. Consequentially the random-resistor-network aspect (which is related to local variation of the couplings) has not emerged. Furthermore, in the physically motivated setup that we have defined above (ring+bath+driving) Sinai physics would not arise if the couplings to the driving source were merely disorderly random. The log-wide distribution is a crucial ingredient. Finally, in a closed (ring) geometry, unlike an open (two terminal) geometry, the statistics of I is not only affected by the distribution of transition rates, but also by the spatial profile of the NESS. This is like "canonical" as opposed to "grand canonical" setting, leading to remarkably different results.

The model.– Consider a ring that consists of sites labeled by n with positions $x = n$ that are defined modulo N. The bonds are labeled as $\vec{n} \equiv (n-1 \rightsquigarrow n)$. The inverse bond is $\overleftarrow{n}$, and if direction does not matter we label it $\bar{n}$. The position of the nth bond is defined as $x_n \equiv n-(1/2)$. The on-site energies E_n are normally distributed over a range Δ, and the transitions rates are between near-neighbor sites:

$$w_{\vec{n}} \;=\; w^{\beta}_{\vec{n}} + \nu g_{\bar{n}} \tag{1}$$

Here w^{β} are the rates that are induced by a bath that has a finite temperature T_B. The $g_{\bar{n}}$ are couplings to a driving source that has an intensity ν. These coupling are log-box distributed within $[g_{\min}, g_{\max}]$. This means that $\ln(g_{\bar{n}})$ are distributed uniformly over a range $\sigma = \ln(g_{\max}/g_{\min})$. The bath transition rates satisfy detailed-balance, namely $w^{\beta}_{\vec{n}}/w^{\beta}_{\overleftarrow{n}} = \exp[-(E_n-E_{n-1})/T_B]$. The driving spoils the detailed-balance. We define the resulted stochastic field as follows:

$$\mathcal{E}(x_n) \;\equiv\; \ln\left[\frac{w_{\vec{n}}}{w_{\overleftarrow{n}}}\right] \;\approx\; -\left[\frac{1}{1+g_{\bar{n}}\nu}\right]\frac{E_n-E_{n-1}}{T_B} \tag{2}$$

where the last equality assumes $\Delta \ll T_B$, and without loss of generality the $g_{\bar{n}}$ have been re-scaled such that all the bath-induced transitions have the same average transition rate $\bar{w}^{\beta} = 1$.

The direction of the current.– The stochastic motive force (SMF), also known as the affinity, or as the entropy production [27–30] determines $\text{sign}(I)$. It is defined as follows:

$$\mathcal{E}_{\circlearrowleft} \;\equiv\; \ln\left[\frac{\prod_n w_{\vec{n}}}{\prod_n w_{\overleftarrow{n}}}\right] \;=\; \oint \mathcal{E}(x)\, dx \tag{3}$$

Using Eq.(2) one observes that for $\nu \ll g_{\max}^{-1}$ the SMF is linear $\mathcal{E}_{\circlearrowleft} \propto \nu$, while for $\nu \gg g_{\min}^{-1}$ it vanishes $\mathcal{E}_{\circlearrowleft} \propto 1/\nu$. In the intermediate regime, which we call below *the Sinai regime*, the SMF changes sign several times, see Fig. 2. Using the notations

$$\tau \;\equiv\; \frac{1}{\sigma}\ln(g_{\max}\nu) \tag{4}$$

and $\tau_n = (1/\sigma)\ln(g_{\max}/g_{\bar{n}})$, the expression for the SMF takes the following form:

$$\mathcal{E}_{\circlearrowleft}(\tau) = -\sum_{n=1}^{N} f_{\sigma}(\tau-\tau_n)\frac{E_n-E_{n-1}}{T_B} \tag{5}$$

where $f_{\sigma}(t) \equiv [1+\mathrm{e}^{\sigma t}]^{-1}$ is like a step function. If $f(t)$ were a sharp step function it would follows that in the Sinai regime $\mathcal{E}_{\circlearrowleft}(\tau)$ is formally like a random walk. The number of sign reversals equals the number of times the random walker cross the origin. We have here a coarse-grained random walk: the τ_n are distributed uniformly over a range $[0, 1]$, and each step is smoothed by $f_{\sigma}(t)$ such that the effective number of coarse-grained steps is σ. Hence we expect the number of sign change to be not $\sim \sqrt{\pi N}$ [31, 32] but $\sim \sqrt{\pi\sigma}$, reflecting the log-width of the distribution.

Adding bonds in series.– The NESS equations are quite simple and can be solved using elementary algebra as in [19, 20, 24, 26], or optionally using the network formalism for stochastic systems [34–36]. Below we propose a generalized resistor-network approach that allows to obtain a more illuminating version for the NESS, that will provide a better insight for the statistical analysis. Let us assume that we have a NESS with a current I. The steady state equations for two adjacent bonds are

$$I \;=\; w_{\vec{1}}p_0 - w_{\overleftarrow{1}}p_1 \tag{6}$$

$$I \;=\; w_{\vec{2}}p_1 - w_{\overleftarrow{2}}p_2 \tag{7}$$

We can combine them into one equation:

$$I = \overrightarrow{G}p_0 - \overleftarrow{G}p_2, \qquad \overrightarrow{G} \equiv \left[\frac{1}{w_{\vec{1}}} + \frac{1}{w_{\vec{2}}}\left(\frac{w_{\overleftarrow{1}}}{w_{\vec{1}}}\right)\right]^{-1} \tag{8}$$

and similar expression for $\overleftarrow{G}$. We can repeat this procedure iteratively. If we have N bonds in series we get

$$\overrightarrow{G} \;=\; \left[\sum_{m=1}^{N}\frac{1}{w_{\vec{m}}}\exp\left(-\int_0^{m-1}\mathcal{E}(x)dx\right)\right]^{-1} \tag{9}$$

Coming back to the ring, we can cut it at an arbitrary site n, and calculate the associated Gs. It follows that $I = (\overrightarrow{G}_n - \overleftarrow{G}_n)\, p_n$. Consequently the NESS is

$$p_n \;=\; \frac{I}{\overrightarrow{G}_n - \overleftarrow{G}_n} \tag{10}$$

and I can be regarded as the normalization factor:

$$I \;=\; \left[\sum_{n=1}^{N}\frac{1}{\overrightarrow{G}_n - \overleftarrow{G}_n}\right]^{-1} \tag{11}$$

In the next paragraph we show how to write these results in an explicit way that illuminates the relevant physics.

The NESS formula.– We define the conductance of a bond as the geometric mean of the clockwise and anticlockwise transmission rates:

$$w(x_n) \;=\; \sqrt{w_{\vec{n}}w_{\overleftarrow{n}}} \tag{12}$$

Hence $w_{\vec{n}} = w(x_n)\exp[(1/2)\mathcal{E}(x_n)]$. Accordingly

$$\overrightarrow{G}_n = \left[\sum_{m=n+1}^{N+n}\frac{1}{w(x_m)}\exp\left(-\int_n^{x_m}\mathcal{E}(x)dx\right)\right]^{-1} \tag{13}$$

With the implicit understanding that the summation and the integration are anticlockwise modulo N. With the new notations it is easy to see that $\overleftarrow{G}_n = \exp(-\mathcal{E}_{\circlearrowleft})\,\overrightarrow{G}_n$.

We use the notation G_n for the geometric mean. Consequently the formula for the current takes the form

$$I \;=\; \left[\sum_{n=1}^{N}\frac{1}{G_n}\right]^{-1} 2\sinh\left(\frac{\mathcal{E}_{\circlearrowleft}}{2}\right) \qquad (14)$$

while $p_n \propto 1/G_n$. Our next task is to work out a tangible expression for the latter. Regarding x as an extended coordinate the potential $V(x)$ that is associated with the field $\mathcal{E}(x)$ is a tilted periodic potential. Adding $[\mathcal{E}_{\circlearrowleft}/N]x$ we get a periodic potential $U(x)$, see Fig.3. Accordingly

$$\int_{x'}^{x''}\mathcal{E}(x)dx \;=\; U(x')-U(x'') + \frac{\mathcal{E}_{\circlearrowleft}}{N}(x''-x') \qquad (15)$$

With any function $A(x)$ we can associate a smoothed version using the following definition

$$\sum_{r=1}^{N} A(x+r)\, \mathrm{e}^{U(x+r)-(1/N)\mathcal{E}_{\circlearrowleft} r} \;\equiv\; A_{\varepsilon}(x)\, \mathrm{e}^{U_{\varepsilon}(x)} \qquad (16)$$

In particular the smoothed potential $U_{\varepsilon}(x)$ is defined by this expression with $A=1$. Note that without loss of generality it is convenient to have in mind $\mathcal{E}_{\circlearrowleft} > 0$. (One can always flip the x direction). Note also that the smoothing scale $N/\mathcal{E}_{\circlearrowleft}$ becomes larger for smaller SMF. With the above definitions we can write the NESS expression as follows:

$$p_n \;\propto\; \left(\frac{1}{w(x_n)}\right)_{\varepsilon} \mathrm{e}^{-(U(n)-U_{\varepsilon}(n))} \qquad (17)$$

This expression is physically illuminating, see Fig.3. In the limit of zero SMF it coincides, as expected, with the canonical result. For finite SMF the smoothed pre-factor and the smoothed potential are not merely constants. Accordingly the pre-exponential factor becomes important and the "slow" modulation by the Boltzmann factor is flattened. If we take the formal limit of infinite SMF the Boltzmann factor disappears and we are left with $p_n \propto 1/w_n$ as expected from the continuity equation for a resistor-network.

Statistics of the current.– Form the preceding analysis it should become clear that the formula for the current can be written schematically as

$$I \;\sim\; \frac{1}{N}\, w_{\varepsilon}\, \mathrm{e}^{-B}\, 2\sinh\left(\frac{\mathcal{E}_{\circlearrowleft}}{2}\right) \qquad (18)$$

In the absence of a potential landscape ($U=0$) the formula becomes equivalent to Ohm law: it is a trivial exercise to derive it if all anticlockwise and clockwise rates are equal to the same values $\overrightarrow{w}$ and $\overleftarrow{w}$ respectively, hence $w_{\epsilon} = (\overrightarrow{w}\overleftarrow{w})^{1/2}$, and $\mathcal{E}_{\circlearrowleft} = N\ln(\overrightarrow{w}/\overleftarrow{w})$. In the presence of a potential landscape we have an activation barrier. Assuming that the current is dominated by the highest peak a reasonable estimate would be

$$B = \max\left\{U(x)-U_{\varepsilon}(x)\right\} \approx\; \tfrac{1}{2}\Big[\max\{U\} - \min\{U\}\Big] \qquad (19)$$

The implication of Eq.(18) with Eq.(19) for the *statistics* of the current is as follows: in the Sinai regime we expect that it will reflect the log-wide distribution of the activation factor, as discussed below, while outside of the Sinai regime we expect it to reflect the *normal* distributions of the total resistance w_{ε}^{-1}, and of the SMF.

Statistics in the Sinai regime.– We now focus on the statistics in the Sinai regime. In order to unfold the log-wide statistics it is not a correct procedure to plot blindly the distribution of $\ln(I)$. Rather one should look on the joint distribution $(\mathcal{E}_{\circlearrowleft}, I)$. See Fig.4. The non-trivial statistics is clearly apparent. In order to describe it analytically we use the single-barrier estimate of Eq.(19), which is tested in Fig.4b. We see that it overestimates the current for small B values (flat landscape) as expected, but it can be trusted for large B where the Sinai physics becomes relevant.

We turn to find an expression for the probability distribution of B, which is displayed in Fig.5. This is done as follows: the probability to have a random walk trajectory $X_n = U(x_n)$ within $[X_a, X_b]$ equals the survival probability of a diffusing process that starts as a delta function at $X=0$ with absorbing boundary conditions at X_a and X_b. Integrating over all possible positions of the walls such that $X_b - X_a = 2B$ is like starting with a uniform distribution between the walls. From here it is straightforward to deduce that in the lowest-mode approximation

$$\mathrm{Prob}\left\{\mathrm{barrier} < B\right\} \;\approx\; \exp\left[-\frac{1}{2}\left(\frac{\pi\sigma_U}{2B}\right)^2\right] \qquad (20)$$

where $\sigma_U^2 = 2DN$ is the variance of the diffusing 'points', which is determined ny the diffusion coefficient $D \propto \Delta^2$. Taking into account that for a given ν a fraction of the elements in Eq.(5) are effectively zero we get

$$\sigma_U^2 \;=\; 2\Delta^2 N\,\frac{\ln(g_{\max}\nu)}{\sigma} \qquad (21)$$

In Fig.5 the implied prediction is verified.

Summary.– We have introduced a generalized "random-resistor-network" approach for the purpose of obtaining the NESS current due to nonsymmetric transition rates. Specifically our interest was focused on the NESS of a "glassy" mesoscopic system. The NESS expression clearly interpolates the canonical result (that applies in equilibrium) with the resistor-network result (that applies at infinite temperature). Due to the "glassiness" the current has novel dependence on the driving intensity, and it posses unique statistical properties that reflect the Brownian landscape of the stochastic potential. This statistics is related to Sinai's random walk problem, and would not arise if the couplings to the driving source were merely disordered.

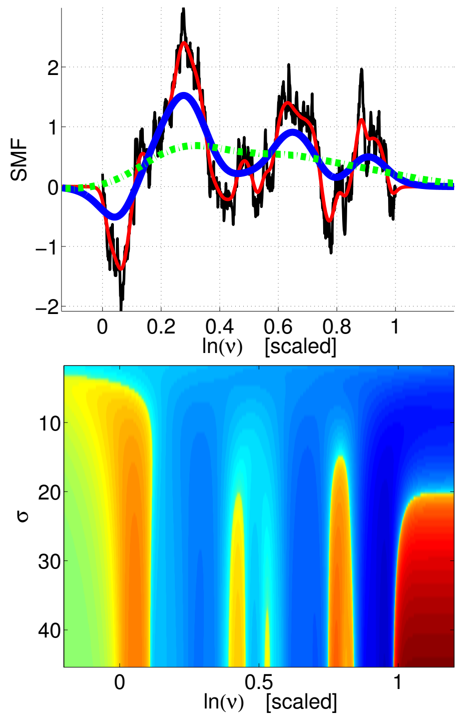

FIG. 2: We consider a ring with $N = 1000$ sites whose energies are normally distributed with dispersion $\Delta = 1$. The bath temperature is $T_B = 10$. In the upper panel the SMF of Eq. (5) is plotted for $\sigma = \infty$, and for $\sigma = 50, 10, 4$. The smaller σ, the smoother ν dependence. This is reflected in the current (lower panel). The horizontal axis is the scale $\ln(\nu)$ as defined in Eq.(4).

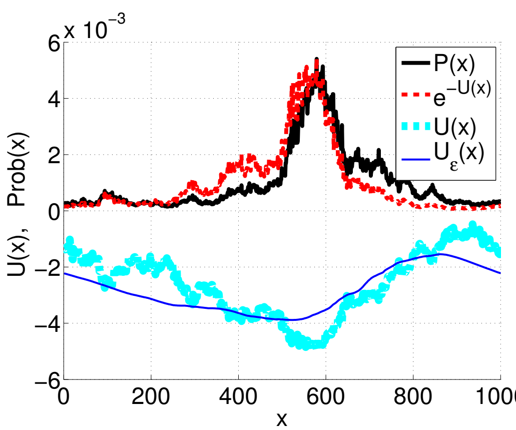

FIG. 3: The NESS distribution of Eq. (17) (solid black) is similar but not identical to the quasi-equilibrium distribution (dashed red line). Also shown, in the background is the potential landscape $U(x)$ and its smoothed version $U_\varepsilon(x)$. The parameters as the same as in Fig.2, however the bonds were rearranged to have a larger SMF, $\mathcal{E}_\circlearrowleft = 7.4$. The driving intensity corresponds to $\tau = 0.3$.

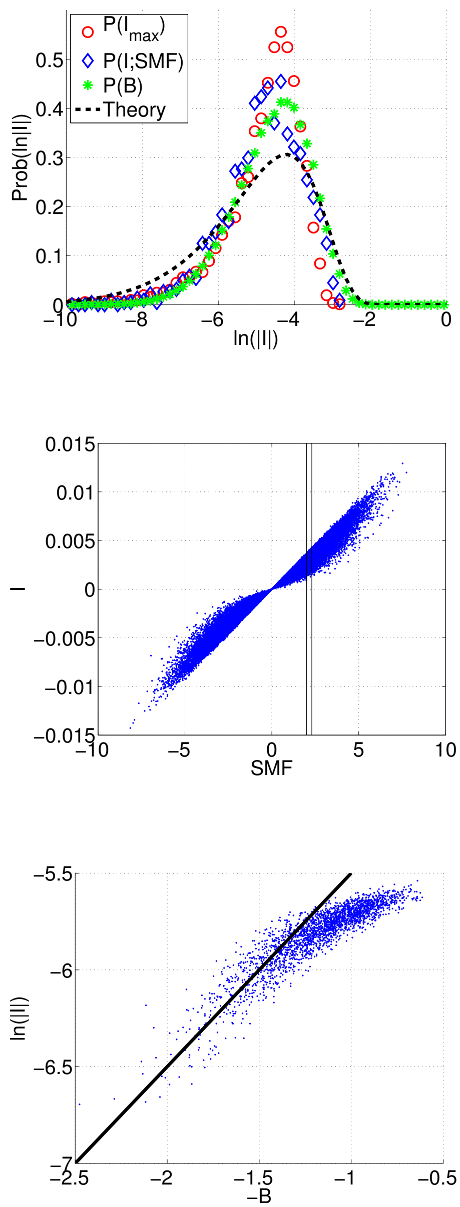

FIG. 4: Inset: scatter diagram of the current versus the SMF in the Sinai regime. Note that in the linear regime, see [b], it looks like a perfect linear correlation with negligible transverse dispersion. The distribution in the main upper panel corresponds to the slice $\mathcal{E}_\circlearrowleft \in [2.0, 2.1]$. The long tails of small I values reflects the likelihood of having a large activation barrier B. The lower panel displays the correlation between I and B, hence the single-barrier approximation is established for small currents.

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FIG. 5: The cumulative distribution of B compared with the analytical result of Eq.(20). There are no fitting parameters. The inset shows how the distribution density looks in normal scale. this is the most urgent action item.

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Supplementary Material

DETAILED DEFINITION OF THE MODEL

Including detailed explanation of Eq(2)

Form the detailed balance condition it follows that to leading order

$$w^{\beta}_{\overrightarrow{n}} \approx \left[1 - \frac{1}{2}\left(\frac{E_n - E_{n-1}}{T_B}\right)\right]\bar{w}^{\beta}_{\bar{n}} \tag{22}$$

$$w^{\beta}_{\overleftarrow{n}} \approx \left[1 + \frac{1}{2}\left(\frac{E_n - E_{n-1}}{T_B}\right)\right]\bar{w}^{\beta}_{\bar{n}} \tag{23}$$

It follows that

$$\frac{w_{\overrightarrow{n}}}{w_{\overleftarrow{n}}} = \frac{w^{\beta}_{\overrightarrow{n}} + \nu g_{\bar{n}}}{w^{\beta}_{\overleftarrow{n}} + \nu g_{\bar{n}}} \approx 1 + \frac{(E_n - E_{n-1})/T_B}{1 + (g_{\bar{n}}/\bar{w}^{\beta}_{\bar{n}})\nu} \tag{24}$$

Absorbing the bath couplings into the definition of the $g_{\bar{n}}$ we get

$$\mathcal{E}(x_n) \equiv \ln\left[\frac{w_{\overrightarrow{n}}}{w_{\overleftarrow{n}}}\right] \approx -\left[\frac{1}{1 + g_{\bar{n}}\nu}\right]\frac{E_n - E_{n-1}}{T_B} \tag{25}$$

DEFINITION OF REGIMES

There are 3 regimes, depending on the driving intensity and sparsity

$$\text{Linear} : \nu < g^{-1}_{max} \tag{26}$$

$$\text{Sinai} : g^{-1}_{max} < \nu < g^{-1}_{min} \tag{27}$$

$$\text{Saturation} : \nu > g^{-1}_{min} \tag{28}$$

The main body of the paper dealt with the Sinai regime. In the linear and saturation regimes, it is possible to obtain expressions for the current and SMF. To calculate the SMF, we integrate Eq.(25) along the entire ring

$$\mathcal{E}_{\circlearrowleft} \approx -\sum_{n=1}^{N}\left[\frac{1}{1 + g_{\bar{n}}\nu}\right]\frac{\Delta_n}{T_B}$$

Recall that we have $\sum_n \Delta_n = 0$. Additionally we define

$$\Delta^{(0)} \equiv \sum_n g_{\bar{n}}\Delta_n \sim \pm\Big[2N\,\mathrm{Var}(g)\Big]^{1/2}\Delta \tag{29}$$

$$\Delta^{(\infty)} \equiv \sum_n \frac{1}{g_{\bar{n}}}\Delta_n \sim \pm\Big[2N\,\mathrm{Var}(g^{-1})\Big]^{1/2}\Delta \tag{30}$$

The RMS-based estimate of the sums follows from the observation that, say, $\Delta^{(0)}$ can be rearranged as $\sum_n (g_{\bar{n}+1} - g_{\bar{n}})E_n$, which is a sum of N independent random variables. Consequently we get for the SMF the following approximation

$$\mathcal{E}_{\circlearrowleft} \approx \frac{1}{T_B}\begin{cases} \Delta^{(0)}\nu, & \text{Linear} \\ -\Delta^{(\infty)}/\nu, & \text{Saturation} \end{cases} \tag{31}$$

since $\Delta^{(0)}$ and $\Delta^{(\infty)}$ are both sums of independent random variables, they may be expected to behave according to the central limit theorem. This is verified in Fig.9, where it is shown that Δ_0 follows a normal distribution.

FIG. 6: Histogram of maxima of $u = \mathsf{max}\{U(x)\}$ (red points) and distribution of maxima of random walk (blue line) of equivalent length, Eq.(34). Also shown is the approximation, Eq.(35) which holds for the region of interest.

MORE DETAILS ON SERIAL ADDITION

e.g. expression for $\overleftarrow{G}$ analogous to Eq(8).

STATISTICS RELATED TO RANDOM WALK

Naively, one might think that the distribution of I is determined by the maxima of the energy landscape $\mathsf{max}\{U(x)\}$. To test this hypothesis, we notice first that $\mathsf{max}\{U(x)\}$ is the same as the point furthest from the origin that a random walker reaches. The distribution of this extreme point K in random walk of N steps is given by (see for example [33])

$$\begin{aligned} P(K \geq k; N) &= \binom{2N}{N-k} \Big/ \binom{2N}{N}, & (32) \\ k &= 0, 1, 2 \cdots N & (33) \end{aligned}$$

Switching variables to $u = k/N$ and taking the large N limit, one obtains the probability distribution function $P(\mathsf{max}\{U\} = u) = f(u)$

$$f(u) = \ln\left[\frac{1+u}{1-u}\right]\left[\frac{(1-u)^{u-1}}{(1+u)^{u+1}}\right]^N \frac{N}{1-4^{-N}} \tag{34}$$

Which has a peak at $u = 1/\sqrt{2N}$. For values $u \ll 1$, this expression can be approximated by the simple function

$$f(u) \approx 2ue^{-Nu^2}\frac{N}{1-e^{-N}}, \; u \geq 0 \tag{35}$$

In Fig.6 we plot Eq.(34), Eq.(35) and the maxima of $U(x)$.

DERIVATION OF THE B STATISTICS

The derivation of the distribution of barrier heights is as follows. We consider a random walk of N steps, $Y(N) = X_1 + X_2 + \cdots X_N$ and calculate the survival probability within the boundaries x_a, x_b:

$$\mathrm{Prob}\left(x_a < Y(N) < x_b\right) \tag{36}$$

The joint probability distribution of extreme points is

$$f(x_a, x_b) = -\frac{d}{dx_a}\frac{d}{dx_b}P \tag{37}$$

FIG. 7: Histogram of barriers (red points) and naive distribution of barriers in a random walk (blue line) of equivalent length, Eq.(??).

Define the barrier R and center of mass X via the coordinate transformation

$$R = \frac{x_a + x_b}{2} \tag{38}$$

$$X = x_b - x_a \tag{39}$$

So $dXdR = dx_a dx_b$. The distribution of barriers is obtained by calculating the marginal distribution

$$\begin{aligned}
f(R) &= -\int_{-\infty}^{\infty} dx_a \int_{x_a}^{\infty} dx_b f(x_a, x_b)\delta\left(R - (x_b - x_a)\right) && (40)\\
&= -\int_0^{\infty} dR' \int_0^{R'} dX f(X, R')\delta(R - R') = \int_0^R f(R, X) dX = && (41)\\
&= -\int_0^R (\frac{1}{4}\partial_X^2 - \partial_R^2) P_t(R, X) dX = && (42)\\
&= \int_0^R \partial_R^2 P_t(R, X) dX = && (43)\\
&= \partial_R^2 R P_t(R|\text{uniform}) && (44)
\end{aligned}$$

In the last step we use the fact that (a) $P_t(R, X = 0) = P_t(R, X = R) = 0$ and (b) the linearity of the diffusion equation: instead of calculating $P_t(R, x)$ for various initial points x and summing over x, we solve for an initially uniform distribution

$$P_t(R|\text{uniform}) = \rho_t(x), \quad \rho_{t=0}(x) = \frac{1}{R} \tag{45}$$

Where $\rho_t(x)$ is the solution to the 1D diffusion equation on the interval $[0, R]$:

$$\rho_t(x) = \sum_{n=1,3,5,\dots}^{\infty} \exp\left[-D\left(\frac{\pi n}{R}\right)^2 t\right] A_n \sin(\frac{\pi n}{R} x) \tag{46}$$

Starting with a uniform distribution $\rho_0(x) = 1/R$, we obtain the coefficients A_n

$$\rho_t(x) = \sum_{n=1,3,5,\dots}^{\infty} \exp\left[-D\left(\frac{\pi n}{R}\right)^2 t\right] \frac{4}{\pi n R} \sin(\frac{\pi n}{R} x) \tag{47}$$

So the survival probability within a region R, given an initially uniform distribution, is

$$P_t(R|\text{uniform}) = \int_0^R \rho_t(x) dx = \sum_{n=1,3,5,\dots}^{\infty} \exp\left[-D\left(\frac{\pi n}{R}\right)^2 t\right] \frac{8}{\pi^2 n^2} \tag{48}$$

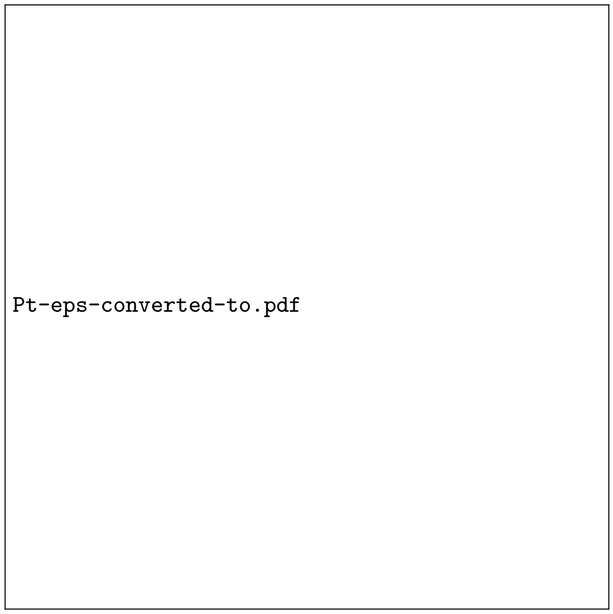

FIG. 8: The function increases from zero to 1, however for small R, the function is convex while for large R it is concave, so $f(R)$ changes sign...

And we get for $f(R)$

$$f(R) \;=\; \frac{\partial^2}{\partial R^2} R \sum_{n=1,3,5,\ldots}^{\infty} \exp\left[-D\left(\frac{\pi n}{R}\right)^2 t\right] \frac{8}{\pi^2 n^2} \tag{49}$$

The sum may be approximated by an integral (Fig.8):

$$P_t(R|\text{uniform}) \;=\; \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \exp\left[-\frac{\alpha}{R^2} n^2\right] \approx \frac{1}{2}\int_1^{\infty} \frac{1}{x^2} \exp\left[-\frac{\alpha}{R^2} x^2\right] dx = \tag{50}$$

$$= \exp\left(-\frac{\alpha}{R^2}\right) - \sqrt{\pi\frac{\alpha}{R^2}}\,\mathsf{erfc}\left(\sqrt{\frac{\alpha}{R^2}}\right) \tag{51}$$

Where $\alpha = \pi^2 D t$. Finally,

$$f(R) \;=\; \frac{2}{\pi^2}\frac{\partial^2}{\partial R^2}\left[R\int_1^{\infty} \frac{1}{x^2} \exp\left(-\frac{\alpha}{R^2} x^2\right)\right] dx \tag{52}$$

$$= \frac{2}{\pi^2}\frac{2\alpha}{R^3} \exp\left(-\frac{\alpha}{R^2}\right) \tag{53}$$

And the CDF is

$$P(\text{barrier} < R) = \exp\left(-\frac{\pi^2 D N}{R^2}\right) \tag{54}$$

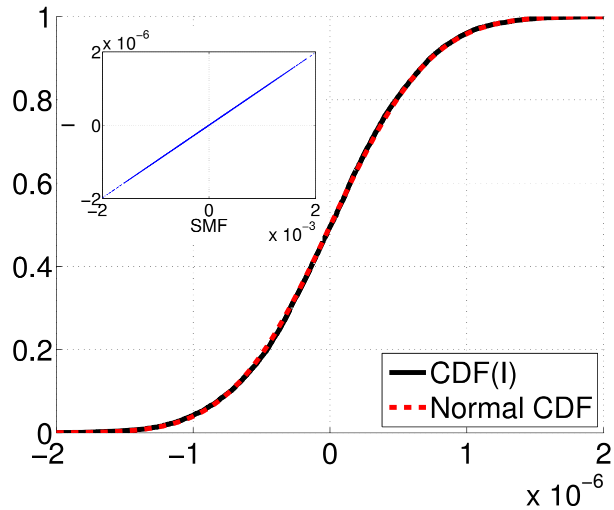

FIG. 9: In the linear regime, the current has a normal distribution. Since, $I \propto w_\epsilon \mathcal{E}_\circlearrowleft$, for a given value of $\mathcal{E}_\circlearrowleft$ the dispersion is very small (scatter plot of current vs. SMF in the inset) due to the central limit theorem. For the statistical analysis we generated 10^5 realisations with $\sigma = 6$.