Non-equilibrium steady state and induced currents of a mesoscopically-glassy system: interplay of resistor-network theory and Sinai physics

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We introduce an explicit solution for the non-equilibrium steady state (NESS) of a ring that is coupled to a thermal bath, and is driven by an external hot source with log-wide distribution of couplings. Having time scales that stretch over several decades is similar to glassy systems. Consequently there is a wide range of driving intensities where the NESS is like that of a random walker in a biased Brownian landscape. We investigate the resulting statistics of the induced current I. For a single ring we discuss how sign(I) fluctuates as the intensity of the driving is increased, while for an ensemble of rings we highlight the fingerprints of Sinai physics on the abs(I) distribution.

The transport in a chain due to random non-symmetric transition probabilities is a fundamental problem in statistical mechanics [1–7]. This type of dynamics is of great relevance for surface diffusion [8], thermal ratchets [9–12] and was used to model diverse biological systems, such as molecular motors, enzymes, and unidirectional motion of proteins along filaments [13–16]. Of particular interest are applications that concern the conduction of DNA segments [17, 18], and thin glassy electrolytes under high voltages [19–23].

Mathematically one can visualize the dynamics as a *a random-walk in a random environment*: a particle that makes incoherent jumps between "sites" of a network. In an unbounded quasi-one-dimensional network we might have either diffusion or sub-diffusive Sinai spreading [6], depending on whether the transitions rates form a symmetric matrix or not. In contrast, when the system is bounded (and without disjoint components) it eventually reaches a well-defined steady state. This would be an equilibrium *canonical* (Boltzmann) state if the transition rates were detailed-balanced, else it is termed non-equilibrium steady state (NESS).

We consider the NESS of a mesoscopically glassy system. *Our working hypothesis is that glassiness might lead to a novel NESS with fingerprints of Sinai physics.* By "glassiness" we mean that the rates that are induced by a bath, or by an external source, have a log-wide distribution, hence many time scales are involved [24] as in spin-glass models [25]. Having a log-wide distribution of time scales is typical for hopping in a random energy landscape, where the rates depend exponentially on the barrier heights. It also arises in driven quasi-integrable systems, where due to approximate selection-rules there is a "sparse" fraction of large coupling-elements, while the majority become very small [26].

We consider a geometrically closed mesoscopic system that has a non-trivial topology. The system is immersed in a finite temperature "cold" bath, and additionally it is coupled to a driving source. The latter can be regarded as a "hot bath" of infinite temperature. Consequently detailed-balance is spoiled, and after a transient a NESS is reached. Specifically we consider the simplest possible model: a mesoscopic ring that is made up of N sites. See [b] for a graphical illustration. Due to the lack of detailed-balance a circulating current is induced. We shall see that the value of the current depends in a non-linear way on the intensity of the driving source. Our interest is in the statistical aspects of this dependence.

The emergence of Sinai physics in a system that is described by a rate equation with asymmetric transition probabilities is naturally expected, but not self-evident [27]. An experimental observation of Sinai diffusion regarding the unzipping transition of DNA molecules has been reported [28], and other applications have been considered [29, 30]. The non-linear current depedence of a mesoscopic rings has been theoretically studied in the past [19, 23], with references to experiments [20–22], but the statistical aspects, and the possible relevance of Sinai physics, have not been considered. In previous publications, we have pointed out that due to "glassiness" Sinai physics becomes a relevant ingredient in the analysis of energy absorption [31] and transport [32] in such a ring system.

Scope.– Below we introduce an explicit NESS solution for a minimal model that has all the essential ingredients of the problem, involving transitions between sites on a ring and a log-wide distribution of couplings to an external driving source. The induced steady state current I is the central quantity used to characterize the NESS in actual experiments. The purpose of the present study is to investigate its statistics. Specifically, for a single ring we discuss how sign(I) fluctuates as the intensity of the driving is increased, while for an ensemble of rings we highlight the fingerprints of Sinai physics on the abs(I) distribution. Our model construction is physically motivated and significantly differs from the standard setup of e.g.[7], see [a].

The model.– Consider a ring that consists of sites labeled by n with positions $x = n$ that are defined modulo N. The bonds are labeled as $\overrightarrow{n} \equiv (n{-}1 \rightsquigarrow n)$. The inverse bond is $\overleftarrow{n}$, and if direction does not matter we label both by $\bar{n}$. The position of the nth bond is defined as $x_n \equiv n-(1/2)$. The on-site energies E_n are normally distributed over a range Δ, and the transitions rates are between nearest-neighboring sites:

$$w_{\overrightarrow{n}} \ = \ w^{\beta}_{\overrightarrow{n}} + \nu g_{\bar{n}} \qquad (1)$$

Here w^{β} are the rates that are induced by a bath that has a finite temperature T_B. The $g_{\bar{n}}$ are couplings to a driving source that has an intensity ν. These couplings are log-box distributed within $[g_{\min}, g_{\max}]$. This means that $\ln(g_{\bar{n}})$ are distributed uniformly over a range $\sigma = \ln(g_{\max}/g_{\min})$. The bath transition rates satisfy detailed-balance, namely $w^{\beta}_{\overrightarrow{n}}/w^{\beta}_{\overleftarrow{n}} = \exp[-(E_n - E_{n-1})/T_B]$. The driving spoils the detailed-balance. We define the resulted stochastic field as follows:

$$\mathcal{E}(x_n) \ \equiv \ \ln\left[\frac{w_{\overrightarrow{n}}}{w_{\overleftarrow{n}}}\right] \ \approx \ -\left[\frac{1}{1+g_{\bar{n}}\nu}\right]\frac{E_n - E_{n-1}}{T_B} \qquad (2)$$

where the last equality assumes $\Delta \ll T_B$ (see [b]) and without loss of generality the $g_{\bar{n}}$ have been re-scaled such that all the bath-induced transitions have the same average transition rate $\bar{w}^{\beta} = 1$.

The direction of the current.– $\text{sign}(I)$ is determined by the stochastic motive force (SMF), also known as the affinity, or as the entropy production [33–36]:

$$\mathcal{E}_{\circlearrowleft} \ \equiv \ \ln\left[\frac{\prod_n w_{\overrightarrow{n}}}{\prod_n w_{\overleftarrow{n}}}\right] \ = \ \oint \mathcal{E}(x)\, dx \qquad (3)$$

In the second equality we formally regard x as a continuous variable. This will make the later mathematics more transparent. Using Eq. (2) one observes that for $\nu \ll g_{\max}^{-1}$ the SMF is linear $\mathcal{E}_{\circlearrowleft} \propto \nu$, while for $\nu \gg g_{\min}^{-1}$ it vanishes $\mathcal{E}_{\circlearrowleft} \propto 1/\nu$. In the intermediate regime, which we call below *the Sinai regime*, the SMF changes sign several times, see Fig.1. Using the notations

$$\tau \ \equiv \ \frac{1}{\sigma}\ln(g_{\max}\nu) \qquad (4)$$

and $\tau_n = (1/\sigma)\ln(g_{\max}/g_{\bar{n}})$, the expression for the SMF takes the following form:

$$\mathcal{E}_{\circlearrowleft}(\tau) = -\sum_{n=1}^{N} f_{\sigma}(\tau - \tau_n)\frac{E_n - E_{n-1}}{T_B} \qquad (5)$$

where $f_{\sigma}(t) \equiv [1 + \mathrm{e}^{\sigma t}]^{-1}$ is like a step function. If $f(t)$ were a sharp step function it would follow that in the Sinai regime $\mathcal{E}_{\circlearrowleft}(\tau)$ is formally like a random walk [37–39]. The number of sign reversals equals the number of times the random walker crosses the origin. We have here a coarse-grained random walk: the τ_n are distributed uniformly over a range $[0, 1]$, and each step is smoothed by $f_{\sigma}(t)$ such that the effective number of coarse-grained steps is σ. Hence we expect the number of sign changes to be not $\sim \sqrt{\pi N}$ but $\sim \sqrt{\pi\sigma}$, reflecting the log-width of the distribution.

Adding bonds in series.– The NESS equations are quite simple and can be solved using elementary algebra as in [19, 20, 23, 32], or optionally using the network formalism for stochastic systems [40–42]. Below we propose a generalized resistor-network approach that allows to obtain a more illuminating version for the NESS, that will provide better insight for the statistical analysis. Let us assume that we have a NESS with a current I. The steady state equations for two adjacent bonds are

$$I \ = \ w_{\overrightarrow{1}} p_0 - w_{\overleftarrow{1}} p_1 \qquad (6)$$
$$I \ = \ w_{\overrightarrow{2}} p_1 - w_{\overleftarrow{2}} p_2 \qquad (7)$$

We can combine them into one equation:

$$I = \overrightarrow{G} p_0 - \overleftarrow{G} p_2, \qquad \overrightarrow{G} \equiv \left[\frac{1}{w_{\overrightarrow{1}}} + \frac{1}{w_{\overrightarrow{2}}}\left(\frac{w_{\overleftarrow{1}}}{w_{\overrightarrow{1}}}\right)\right]^{-1} \qquad (8)$$

and similarly for $\overleftarrow{G}$, see [b]. We can repeat this procedure iteratively. If we have N bonds in series we get

$$\overrightarrow{G} \ = \ \left[\sum_{m=1}^{N}\frac{1}{w_{\overrightarrow{m}}}\exp\left(-\int_0^{m-1}\mathcal{E}(x)dx\right)\right]^{-1} \qquad (9)$$

Coming back to the ring, we can cut it at an arbitrary site n, and calculate the associated Gs. It follows that $I = (\overrightarrow{G}_n - \overleftarrow{G}_n)\, p_n$. Consequently the NESS is

$$p_n \ = \ \frac{I}{\overrightarrow{G}_n - \overleftarrow{G}_n} \qquad (10)$$

and I can be regarded as the normalization factor:

$$I \ = \ \left[\sum_{n=1}^{N}\frac{1}{\overrightarrow{G}_n - \overleftarrow{G}_n}\right]^{-1} \qquad (11)$$

In the next paragraph we show how to write these results in an explicit way that illuminates the relevant physics.

The NESS formula.– We define the conductance of a bond as the geometric mean of the clockwise and anticlockwise transmission rates:

$$w(x_n) \ = \ \sqrt{w_{\overrightarrow{n}} w_{\overleftarrow{n}}} \qquad (12)$$

Hence $w_{\overrightarrow{n}} = w(x_n)\exp[(1/2)\mathcal{E}(x_n)]$. Accordingly

$$\overrightarrow{G}_n = \left[\sum_{m=n+1}^{N+n}\frac{1}{w(x_m)}\exp\left(-\int_n^{x_m}\mathcal{E}(x)dx\right)\right]^{-1} \qquad (13)$$

With the implicit understanding that the summation and the integration are anticlockwise modulo N. With the new notations it is easy to see that $\overleftarrow{G}_n = \exp(-\mathcal{E}_{\circlearrowleft})\,\overrightarrow{G}_n$. We use the notation G_n for the geometric mean. Consequently the formula for the current takes the form

$$I \ = \ \left[\sum_{n=1}^{N}\frac{1}{G_n}\right]^{-1} 2\sinh\left(\frac{\mathcal{E}_{\circlearrowleft}}{2}\right) \qquad (14)$$

while $p_n \propto 1/G_n$. Our next task is to find a tractable expression for the latter. Regarding x as an extended coordinate, the potential $V(x)$ that is associated with the field $\mathcal{E}(x)$ is a tilted periodic potential. Adding $[\mathcal{E}_\circlearrowleft/N]x$ we get a periodic potential $U(x)$, see Fig.2. Accordingly

$$\int_{x'}^{x''} \mathcal{E}(x)dx \ = \ U(x') - U(x'') + \frac{\mathcal{E}_\circlearrowleft}{N}(x''-x') \qquad (15)$$

With any function $A(x)$ we can associate a smoothed version using the following definition

$$\sum_{r=1}^{N} A(x+r)\, \mathrm{e}^{U(x+r)-(1/N)\mathcal{E}_\circlearrowleft r} \ \equiv \ A_\varepsilon(x)\, \mathrm{e}^{U_\varepsilon(x)} \qquad (16)$$

In particular the smoothed potential $U_\varepsilon(x)$ is defined by this expression with $A=1$. Note that without loss of generality it is convenient to have in mind $\mathcal{E}_\circlearrowleft > 0$. (One can always flip the x direction). Note also that the smoothing scale $N/\mathcal{E}_\circlearrowleft$ becomes larger for smaller SMF. With the above definitions we can write the NESS expression as follows:

$$p_n \ \propto \ \left(\frac{1}{w(x_n)}\right)_\varepsilon \mathrm{e}^{-(U(n)-U_\varepsilon(n))} \qquad (17)$$

This expression is physically illuminating, see Fig.2. In the limit of zero SMF it coincides, as expected, with the canonical (Boltzmann) result. For finite SMF the smoothed pre-factor and the smoothed potential are not merely constants. Accordingly the pre-exponential factor becomes important and the "slow" modulation by the Boltzmann factor is flattened. If we take the formal limit of infinite SMF the Boltzmann factor disappears and we are left with $p_n \propto 1/w_n$ as expected from the continuity equation for a resistor-network.

Statistics of the current.– From the preceding analysis it should become clear that the formula for the current can be written schematically as

$$I(\nu) \ \sim \ \frac{1}{N}\, w_\varepsilon\, \mathrm{e}^{-B}\, 2\sinh\left(\frac{\mathcal{E}_\circlearrowleft}{2}\right) \qquad (18)$$

In the absence of a potential landscape ($U(x)=0$) the formula becomes equivalent to Ohm law: it is a trivial exercise to derive it if all anticlockwise and clockwise rates are equal to the same values $\overrightarrow{w}$ and $\overleftarrow{w}$ respectively, hence $w_\varepsilon = (\overrightarrow{w}\overleftarrow{w})^{1/2}$, and $\mathcal{E}_\circlearrowleft = N\ln(\overrightarrow{w}/\overleftarrow{w})$. In the presence of a potential landscape we have an activation barrier. Assuming that the current is dominated by the highest peak a reasonable estimate would be

$$B = \max\{U(x) - U_\varepsilon(x)\} \approx \ \tfrac{1}{2}\Big[\max\{U\} - \min\{U\}\Big] \qquad (19)$$

The implication of Eq.(18) with Eq.(19) for the *statistics* of the current is as follows: in the Sinai regime we expect that it will reflect the *log-wide* distribution of the activation factor, as discussed below, while outside of the Sinai regime we expect it to reflect the *normal* distributions of the total resistance w_ε^{-1}, and of the SMF.

Statistics in the Sinai regime.– We now focus on the statistics in the Sinai regime. In order to unfold the log-wide statistics it is not a correct procedure to plot blindly the distribution of $\ln(|I|)$. Rather one should look on the joint distribution $(\mathcal{E}_\circlearrowleft, I)$. See Fig.3a. The non-trivial statistics is clearly apparent. In order to describe it analytically we use the single-barrier estimate of Eq.(19), which is tested in Fig.3b. We see that it overestimates the current for small B values (flat landscape) as expected, but it can be trusted for large B where the Sinai physics becomes relevant.

In an actual experiment it would be desired to extract the statistics from the $I(\nu)$ measurements without referring to the SMF. See Fig.4. Either way this figure confirms that the I statistics is the same as the barrier $\exp(-B)$ statistics. We therefore turn to find an explicit expression for the latter. The probability to have a random walk trajectory $X_n = U(x_n)$ within $[X_a, X_b]$ equals the survival probability in a diffusion process that starts as a delta function at $X=0$ with absorbing boundary conditions at X_a and X_b. Integrating over all possible positions of the walls such that $X_b - X_a = 2B$ is like starting with a uniform distribution between the walls. From here it is straightforward to deduce [b]

$$\mathrm{Prob}\{\mathrm{barrier} < B\} \ \sim \ \exp\left[-\frac{1}{2}\left(\frac{\pi\sigma_U}{2B}\right)^2\right] \qquad (20)$$

where $\sigma_U^2 = 2DN$ is the variance of the diffusing 'points', which is determined ny the diffusion coefficient $D \propto \Delta^2$. Taking into account that for a given ν a fraction of the elements in Eq.(5) are effectively zero we get

$$\sigma_U^2 \ \ = \ \ 2\Delta^2 N\, \frac{\ln(g_{\max}\nu)}{\sigma} \qquad (21)$$

The validity of the exact version of Eq.(20), see [b], has been verified in Fig.3. No fitting parameters are required.

Summary.– We have introduced a generalized "random-resistor-network" approach for the purpose of obtaining the NESS current due to nonsymmetric transition rates. Specifically our interest was focused on the NESS of a "glassy" mesoscopic system. The NESS expression clearly interpolates the canonical (Boltzmann) result that applies in equilibrium, with the resistor-network result, that applies at infinite temperature. Due to the "glassiness" the current has novel dependence on the driving intensity, and it posseses unique statistical properties that reflect the Brownian landscape of the stochastic potential. This statistics is related to Sinai's random walk problem, and would not arise if the couplings to the driving source were merely disordered.

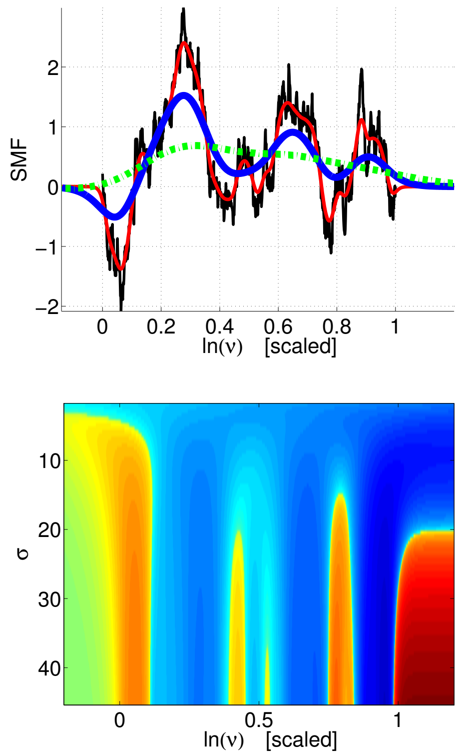

FIG. 1: We consider a ring with $N = 1000$ sites whose energies are normally distributed with dispersion $\Delta = 1$. The bath temperature is $T_B = 10$. In the upper panel the SMF of Eq. (5) is plotted for $\sigma = \infty$, and for $\sigma = 50, 10, 4$. The smaller σ, the smoother ν dependence. This is reflected in the current $I(\nu)$, which is colored imaged in the lower panel: each row is for a different σ, blue and red are for positive and negative (clockwise) circulating current respectively. In both panels the horizontal axis is the scaled driving intensity as defined in Eq. (4).

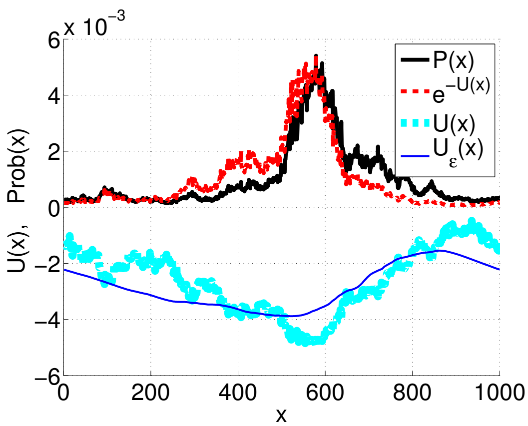

FIG. 2: The NESS profile of Eq. (17) (solid black) is similar but not identical to the quasi-equilibrium distribution (dashed red line). Also shown (lower curves) is the potential landscape $U(x)$ and its smoothed version $U_\varepsilon(x)$. The parameters are the same as in Fig. 1, with $\sigma = 10$, and driving intensity that corresponds to $\tau = 0.3$. The bonds were re-arranged to have a larger SMF, namely $\mathcal{E}_\circlearrowleft = 7.4$.

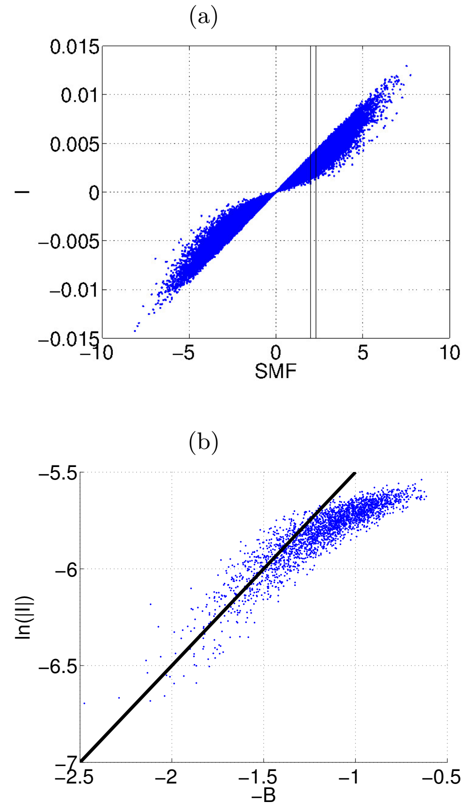

FIG. 3: (a) Scatter diagram of the current versus the SMF in the Sinai regime. Note that in the linear regime, see [b], it looks like a perfect linear correlation with *negligible* transverse dispersion. (b) The correlation between the current I and the barrier B, within the slice $\mathcal{E}_\circlearrowleft \in [2.0, 2.1]$. One deduces that the single-barrier approximation is valid for small currents.

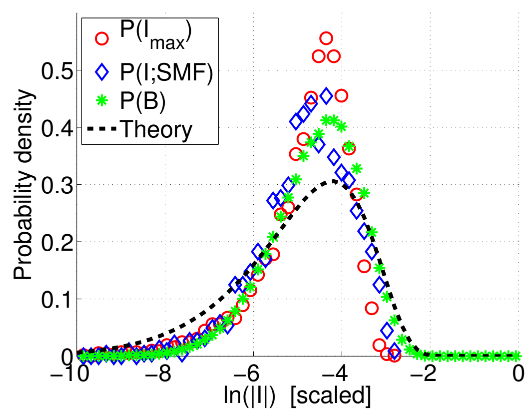

FIG. 4: The log-wide distribution $P(I)$ of the current in the Sinai regime is revealed provided a proper procedure is adopted. For theoretical analysis it is convenient to plot an histogram of the I values for a given SMF: the blue diamonds refer to the data of Fig. 3b. In an actual experiment it is desired to extract statistics from $I(\nu)$ measurements without referring to the SMF: the red empty circles show the statistics of the first maximum of $I(\nu)$. Both distributions look the same, and reflect the barrier statistics (full green circles). The line is the exact version [b] of Eq. (21).

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[a] Previous study of Sinai-type disordered systems [7], has considered an open geometry with uncorrelated transition rates that have the same coupling everywhere. Consequentially the random-resistor-network aspect (which is related to local variation of the couplings) has not emerged. Furthermore, in the physically motivated setup that we have defined above (ring+bath+driving) Sinai physics would not arise if the couplings to the driving source were merely disorderly random. The log-wide distribution is a crucial ingredient. Finally, in a closed (ring) geometry, unlike an open (two terminal) geometry, the statistics of I is not only affected by the distribution of transition rates, but also by the spatial profile of the NESS. This is like "canonical" as opposed to "grand canonical" setting, leading to remarkably different results.

[b] See supplementary material at URL for a graphical illustration of the model, and some extra technical details regarding: the stochastic field expression Eq. (2); the normal statistics of the current outside of the Sinai regime; the random-walk occupation-range statistics; the random-walk maximal-distance statistics; and the serial addition of bonds.

Supplementary Material

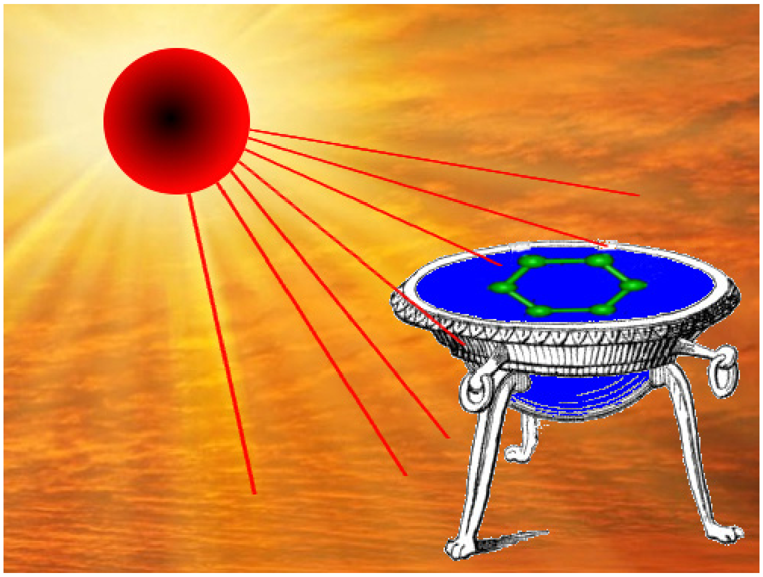

FIG. 5: A ring made up of N sites is immersed in a "cold" bath and subjected to a "hot" driving source. As a result a current is induced. In the numerics the the driving source induces rates that are log-box distributed over 6 decades. This illustration combines pieces of images that were taken from `http://en.wikipedia.org/wiki/Aeolipile` and `http://thehealthyhavenblog.com/2012/06/18/sun-safety`.

THE EXPRESSION FOR THE STOCHASTIC FIELD

Form the detailed balance condition it follows that to leading order

$$w^{\beta}_{\overrightarrow{n}} \approx \left[1 - \frac{1}{2}\left(\frac{E_n - E_{n-1}}{T_B}\right)\right] \bar{w}^{\beta}_{\bar{n}} \tag{22}$$

$$w^{\beta}_{\overleftarrow{n}} \approx \left[1 + \frac{1}{2}\left(\frac{E_n - E_{n-1}}{T_B}\right)\right] \bar{w}^{\beta}_{\bar{n}} \tag{23}$$

Hence

$$\frac{w_{\overrightarrow{n}}}{w_{\overleftarrow{n}}} = \frac{w^{\beta}_{\overrightarrow{n}} + \nu g_{\bar{n}}}{w^{\beta}_{\overleftarrow{n}} + \nu g_{\bar{n}}} \approx 1 + \frac{(E_n - E_{n-1})/T_B}{1 + (g_{\bar{n}}/\bar{w}^{\beta}_{\bar{n}})\nu} \tag{24}$$

Absorbing the bath couplings into the definition of the $g_{\bar{n}}$ we get

$$\mathcal{E}(x_n) \equiv \ln\left[\frac{w_{\overrightarrow{n}}}{w_{\overleftarrow{n}}}\right] \approx -\left[\frac{1}{1 + g_{\bar{n}}\nu}\right] \frac{E_n - E_{n-1}}{T_B} \tag{25}$$

The SMF is obtained by integrating the stochastic field along the entire ring

$$\mathcal{E}_{\circlearrowleft} \approx -\sum_{n=1}^{N} \left[\frac{1}{1 + g_{\bar{n}}\nu}\right] \frac{\Delta_n}{T_B} \tag{26}$$

STATISTICS OF CURRENT OUTSIDE OF THE SINAI REGIME

As the driving intensity is increased one observes a crossover from a linear regime, to a Sinai regime, and finally a saturation regime:

$$\text{Linear regime:} \qquad \nu < g_{max}^{-1} \tag{27}$$
$$\text{Sinai regime:} \qquad g_{max}^{-1} < \nu < g_{min}^{-1} \tag{28}$$
$$\text{Saturation regime:} \qquad \nu > g_{min}^{-1} \tag{29}$$

Consequently we get for the SMF the following approximations:

$$\mathcal{E}_{\circlearrowleft} \quad \approx \quad \frac{1}{T_B}\begin{cases} \Delta^{(0)}\nu, & \text{Linear regime} \\ -\Delta^{(\infty)}/\nu, & \text{Saturation regime} \end{cases} \tag{30}$$

where

$$\Delta^{(0)} \;\equiv\; \sum_n g_{\bar{n}}\Delta_n \quad\sim\quad \pm\Big[2N\,\mathrm{Var}(g)\Big]^{1/2}\Delta \tag{31}$$
$$\Delta^{(\infty)} \;\equiv\; \sum_n \frac{1}{g_{\bar{n}}}\Delta_n \quad\sim\quad \pm\Big[2N\,\mathrm{Var}(g^{-1})\Big]^{1/2}\Delta \tag{32}$$

The estimates for $\Delta^{(0)}$ and for $\Delta^{(\infty)}$ follow from the observation that we have sums of independent random variables. For example $\Delta^{(0)}$ can be re-arranged as $\sum_{n=1}^{N}(g_{\bar{n}+1}-g_{\bar{n}})E_n$. Furthermore, we conclude that both $\Delta^{(0)}$ and $\Delta^{(\infty)}$ have *normal* statistics as implied by the central limit theorem. Consequently we expect *normal* statistics for the SMF, and hence for the current, as verified in Fig.6.

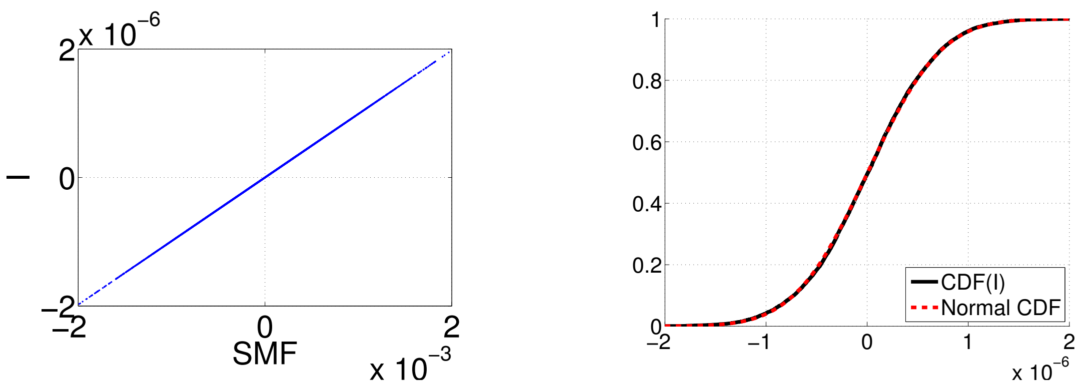

FIG. 6: In the linear regime, the current is strongly correlated with the SMF (left panel), and consequently it has *normal* statistics (right panel). For the statistical analysis we have generated 10^5 realizations of the ring with $\sigma = 6$.

RANDOM-WALK OCCUPATION-RANGE STATISTICS

In this section we derived the probability density function $f(R)$ to have a random walk process $x(\cdot)$ of t steps that occupies a range R. This is determined by the probability

$$P_t(x_a, x_b) \quad \equiv \quad \text{Prob}\Big(x_a < x(t') < x_b \ \ \text{for any } t' \in [0,t]\Big) \tag{33}$$

Accordingly the joint probability density that a random walker would occupy an interval $[x_a, x_b]$ is

$$f(x_a, x_b) \quad = \quad -\frac{d}{dx_a}\frac{d}{dx_b}P_t(x_a, x_b) \tag{34}$$

It is convenient to use the coordinates

$$X \ = \ \frac{x_a + x_b}{2} \tag{35}$$

$$R \ = \ x_b - x_a \tag{36}$$

Consequently the expression for $f(R)$ is

$$f(R) \quad = \quad \int_{-\infty}^{0}\int_{0}^{\infty} dx_a dx_b \ f(x_a, x_b)\ \delta\left(R - (x_b - x_a)\right) \tag{37}$$

$$f(R) \quad = \quad -\int_{-R/2}^{R/2}\left(\frac{1}{4}\partial_X^2 - \partial_R^2\right)\ P_t(R, X)\ dX \tag{38}$$

Taking into account that $P_t(R, X)$ and its derivatives vanish at the endpoints $X = \pm(R/2)$ we get

$$f(R) \ = \ \int_{-R/2}^{R/2} \partial_R^2 \ P_t(R, X)\ dX \quad = \quad \partial_R^2\Big[R\ P_t(R)\Big] \tag{39}$$

where $P_t(R)$ is the survival probability of a diffusion process that starts with an initial *uniform* distribution, instead of a random walk that starts as a delta distribution. Optionally we can write

$$\text{Prob}(\text{range} < R) \quad = \quad \partial_R\Big[R\ P_t(R)\Big] \tag{40}$$

We now turn to find an explicit expression for $P_t(R)$. This is done by solving the diffusion equation. Using Fourier expansion the solution is

$$\rho_t(x) \ = \ \sum_{n=1,3,5,...}^{\infty} \exp\left[-D\left(\frac{\pi n}{R}\right)^2 t\right]\frac{4}{\pi n R}\sin\left(\frac{\pi n}{R}x\right) \tag{41}$$

For simplicity we have shifted above the domain to $x \in [0, R]$. For the survival probability we get

$$P_t(R) \quad = \quad \int_0^R \rho_t(x)\ dx \quad = \quad \sum_{n=1,3,5,...}^{\infty}\frac{8}{\pi^2 n^2}\exp\left[-D\left(\frac{\pi n}{R}\right)^2 t\right] \tag{42}$$

Using Eq.(42) in Eq.(39) we get

$$f(R) \quad = \quad \frac{8\sigma^2}{R^3}\sum_{n=1,3,5,...}^{\infty}\left[\left(\frac{\pi\sigma n}{R}\right)^2 - 1\right]\exp\left[-\frac{1}{2}\left(\frac{\pi\sigma n}{R}\right)^2\right] \tag{43}$$

This result is in perfect agreement with the numerical simulation of Fig.7. Still we would like to have a more compact expression. One possibility is to keep only the first term. The other possibility is to approximate the summation by an integral:

$$\text{Prob}(\text{range} < R) \quad \approx \quad \frac{2}{\pi^2}\frac{\partial}{\partial R}\left[R\int_1^\infty \frac{dx}{x^2}\exp\left(-\frac{\pi^2 Dt}{R^2}x^2\right)\right] \quad = \quad \exp\left(-\frac{\pi^2 Dt}{R^2}\right) \tag{44}$$

Either way we get

$$\text{Prob}(\text{range} < R) \quad \sim \quad \exp\left(-\frac{1}{2}\left(\frac{\pi\sigma}{R}\right)^2\right) \tag{45}$$

where $\sigma^2 = 2Dt$. This asymptotic expression is illustrated in Fig.7. Though it does not work very well, it has the obvious advantage of simplicity.

RANDOM-WALK MAXIMAL-DISTANCE STATISTICS

The occupation-range statistics of the previous section should not be confused with the maximal-distance statistics. The maximal distance from the initial point is defined as follows:

$$K = \max[x(t)], \qquad \text{where } 0 < t < N \tag{46}$$

Naively, one might think that the probability distribution of K is similar to the probability distribution of R that has been discussed in the previous section. But this is not true. Furthermore, it is also very sensitive to whether the random walk is constrained to end up at the origin, $x(N) = x(0) = 0$. Without the latter constraint $f(K)$ is finite for small K, but if the constraint is taken into account, it vanishes linearly in this limit.

It is the constrained random walk process that describes the potential $U(x)$. The exact result for the the K statistics in this case is known [39]:

$$\text{Prob}(K \geq k; N) = \frac{\begin{pmatrix} 2N \\ N-k \end{pmatrix}}{\begin{pmatrix} 2N \\ N \end{pmatrix}}, \qquad k = 0, 1, 2 \cdots N \tag{47}$$

Switching variables to $\kappa = k/N$ and taking the large N limit, one obtains the probability density function

$$f(\kappa) = N \left[\frac{(1-\kappa)^{\kappa-1}}{(1+\kappa)^{\kappa+1}}\right]^N \ln\left[\frac{1+\kappa}{1-\kappa}\right] \tag{48}$$

which has a peak at $\kappa \sim 1/\sqrt{2N}$. For $\kappa \ll 1$ this expression can be approximated by the simple function. Switching back to K it takes the form

$$f(K) \approx \frac{2K}{N} \exp\left[-\frac{K^2}{N}\right] \tag{49}$$

In Fig.8a we illustrate this distribution and demonstrate its applicability to the $U(x)$ of the ring model. In Fig.8b we illustrate the joint distribution of the extreme values $x_{\min} = \min[x(\cdot)]$ and $x_{\max} = \max[x(\cdot)]$. The $f(R)$ distribution of the previous section corresponds to its projection along the diagonal direction, while the $f(K)$ distribution of the present section is its projection along the horizontal or vertical directions.

MORE DETAILS ON SERIAL ADDITION

Adding two bonds in series we have obtained Eq.(8) for $\overrightarrow{G}$. The formula for $\overleftarrow{G}$ is similarly obtained:

$$\overleftarrow{G} \equiv \left[\frac{1}{w_{\overleftarrow{2}}} + \frac{1}{w_{\overleftarrow{1}}}\left(\frac{w_{\overrightarrow{2}}}{w_{\overleftarrow{2}}}\right)\right]^{-1} \tag{50}$$

We can repeat this procedure iteratively. If we have N bonds in series we get

$$\overrightarrow{G} = \left[\sum_{m=1}^{N} \frac{1}{w_{\overrightarrow{m}}} \exp\left(-\int_0^{m-1} \mathcal{E}(x)dx\right)\right]^{-1} \tag{51}$$

$$\overleftarrow{G} = \left[\sum_{m=1}^{N} \frac{1}{w_{\overleftarrow{m}}} \exp\left(\int_m^{N} \mathcal{E}(x)dx\right)\right]^{-1} \tag{52}$$

One should notice the miss-match between m and $m-1$, that prevents of us treating the two formulas on equal footing. For this reason we have introduced an improved convention for the description of the bonds in terms of $w(x)$ leading to Eq.(13).

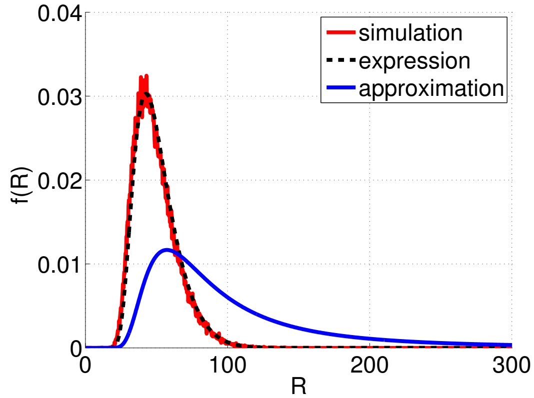

FIG. 7: Plot of $f(R)$. Red line is the outcome of a random walk simulation with $t = 1000$ steps that are Gaussian distributed with unit dispersion. The black dashed line is the exact result Eq.(43), while the blue solid line is from the simple asymptotic approximation Eq.(45).

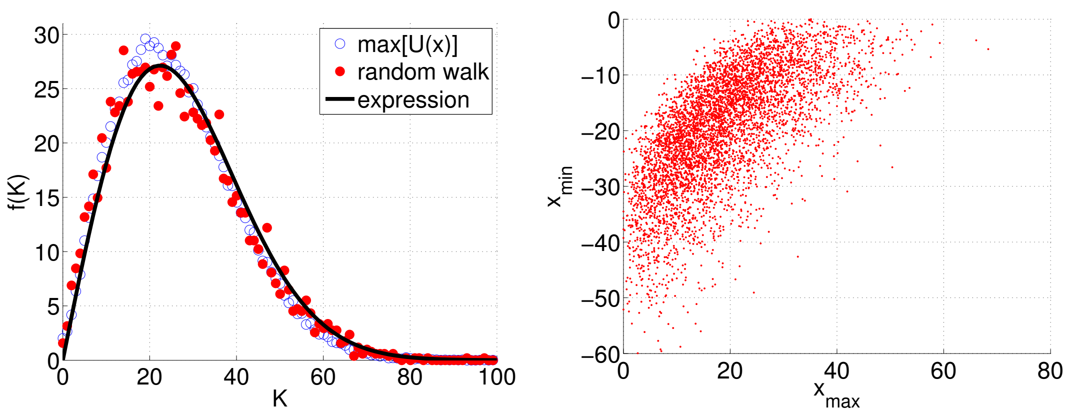

FIG. 8: [Left panel] Plot of $f(K)$. The histogram of $\max[U(x)]$ values over many ring realizations (blue circles) is compared with the K statistics in a constrained random walk process (red points). The analytical result Eq.(49) is represented by a black line. [Right panel] Scatter plot of $(x_{\min}, x_{\max})$ for the same random walk simulation illustrating the strong correlation.

Appeal

The referees have based their recommendations on the "general interest" of the paper and on a superficial similarity of the Ms to a previous submission. There was no scientific criticism concerning the content of the Ms. Comments with regard to some minor presentation issues were handled (see list of changes below).

We believe that an additional independent reviewer would agree that the arguments that have been used by the referees to determine PRL suitability are not valid. Accordingly we treat this re-submission as an appeal. We hope to convince the reviewer that the Ms is in fact of *general interest* and possesses the critical mass for a PRL publication.

Relation to a previous publication.– RefereeA identifies this Ms with a past submission that he/she had refereed two years ago. The present submission focuses on a subject that was not covered in the past submission, namely the statistics of the current in the Sinai regime. Apparently he/she did not notice that this past submission had been published in Europhysics Letters [32], and had been cited explicitly in the original version of the Ms as Ref[26]. Consequently the referee has concluded that "the authors could have informed their scientific community two years ago already about their findings [forming] a very nice contribution to Physical Review."

This failure to make a distinction between the two manuscripts is apparently the main reason for the recommendations of refereeA. We believe that the recommendation whether to accept or reject our Ms should be based on the new results presented in it, and not on under-estimating the novelty of a particular equation that is required for the subsequent analysis.

Novelty of the present submission.– The novelty of the present work is not related to the derivation of Eq(11). This had been stated explicitly. Citing the original Ms: "The NESS equations are quite simple and can be solved using elementary algebra as in [19,20,24,26]". The novelty of the present work is related to the observation that Sinai physics is reflected in the statistics of the current, as illustrated in the major figure that is now labeled as Fig.4. *This statement is highly non-trivial as explained in the text: with superficial analysis the statistics looks normal.*

To say that Physical Reviews Letters should not publish several Letters that concern the same model is manifestly not an acceptable standpoint. To make a relevant analogy: if one Letter explains how to calculate the "conductance" of a ring, does it mean that one should not allow the publication of a subsequent Letter that reports "universal conductance fluctuations"? Indeed our previous Letter [32] and this Ms concern the same model. But clearly the subject of the present Ms, namely, reflection of glassiness in the *statistics* of the current, is novel, and has never been addressed to the best of our knowledge.

Response to a secondary statement of referee A.– The referee writes: "The same formula for the current is derived by using, as far as I can see, the same method, just rewritten in a different way." As an answer we simply cite what we wrote in the Ms with regard to this formula: "Adding bonds in series: The NESS equations are quite simple and can be solved using elementary algebra as in [19,20,24,26], [...] Below we propose a generalized resistor-network approach that allows to obtain a more illuminating version for the NESS, that will provide better insight for the statistical analysis."

We emphasize again that the novelty of our Ms is not based on having introduced a resistor-network version of the formula for the current. Rather, the novelty is primarily related to the analysis of its statistics. In the text we were very careful to make it clear what is *old* and what is *new*. This distinction should not be blurred.

General interest.– When we wrote our previous Letter [32] we were not aware of theoretical and experimental work that concerns essentially the same model. We first found out about the paper by Einax, Korner, Maass, and Nitzan [23] called Nonlinear hopping transport in ring systems [...], and from it learned about the earlier works on this model [19–22] and about further motivation that comes from more loosely related works that we cite as well. We emphasize - *by now we found out that this type of model is of "general interest", including experimental work.* RefereeA spreads over many items his objection to the general interest in the physics of our model. We find his/her criticism very strange.

Our *rate equation* model is evidently very common as detailed in the introduction. To say that the references that we cite do not establish this point does not make sense. In particular we explicitly specify references to both theoretical and experimental work that contain the same(!) solution for the current. see [19, 20, 23, 32]. Still we note that our new version illuminates some resistor-network aspects that were not explicit in past versions.

We also note that *log-wide* distributions are common in hopping processes because $g \sim \exp(-B)$ and also since matrix elements of quasi-integrable systems are subject to approximate selection rules that cause most matrix elements to be small, resulting in a very wide distribution, see [26] for an overview.

RefereeA has a strange attitude towards Sinai spreading: “In my view, Sinai diffusion is a very special case due to asymmetric transition rates.” We have no idea even how to address such criticism: Do we have to establish the interest of the Chemistry / Statistical / Physics community in asymmetric transition rates? Or do we have to establish the novelty or the general interest in Sinai diffusion? We are convinced that our introduction provides the proper “zoom in” that motivates our analysis. Clearly previous experiments did not address the statistics, and emphasized only the non-linearity of current vs bias. Evidently our Ms might provide a new perspective for experimental study: not only ”signal” but also the ”statistics” are informative.

We point out that in the electrical context theoretical work had indeed motivated experiments that were aimed at exploring statistics of currents in mesoscopic rings [Reulet, Ramin, Bouchiat, Mailly, PRL 75, 124 (1995)]. We see no reasons to insist that a similar perspective should not be encouraged with regard to non-coherent transport.

Glassiness and Sinai physics.– Per the requirement of refereeA we further relate to the notion of glassiness and try to establish the related interest in Sinai physics.

Glassiness means “log wide distribution of time scales”. It typically arises in studies of models with a random energy landscape (the list of papers is endless). We now cite the reference that has been mentioned by refereeA [24]. We also looked at Ritort's website where he writes that "Scientists start to agree that the physics of glasses... characterized by a wide distribution of relaxational timescales (heterogeneous kinetics)", the standard example being spin-glass models [25].

Sales, Bouchaud and Ritort [27] discuss Sinai physics in the context of glassy systems. An experimental observation of Sinai diffusion regarding the unzipping transition of DNA has been reported by Lubensky and Nelson [28] pointing out that ”polynucleotide unzipping provides an experimental realization of the famous Sinai problem of thermally activated diffusion in a quenched random force field”. Sinai physics is relevant for the analysis of the motion of a domain wall in the random field Ising model, see Corberi, Candia, Lippiello, and Zannetti [29]. and also to the description of some tapping experiments in sand piles, see Luding, Nicolas and Pouliquen [30].

Response to the major argument of referee B.– Essentially the argument is that some mathematical approximations would be understood only by experts. “In the end, the math tricks get in the way [of the general reader]". We would like to claim that the existence of a stressing stage in the algebra should not be regarded as a valid argument against PRL publication.

The main message of this paper, as demonstrated in Fig.4, is that the statistics of the current contains fingerprints of ”glassiness” related to Sinai physics. This message should be understood and appreciated by any ”general reader”, irrespective of technicalities.

Response to a secondary item of referee B.– The referee complains that he/she cannot make sense of our mathematics, namely, why to write sums as integrals. But then he/she gives the answer: "Apparently because this makes possible the trick that appears in Eq.(13)...". So there is nothing for us to answer here, just to point out that the previous versions of the same formula, [19, 20, 23, 32], are not tractable for the purpose of our statistical analysis. Furthermore, we believe that a simple comparison with the cited references will convince the reader that our notations help to gain intuition into the mathematics, and better connect it to the physical picture. In any case we followed the referee suggestion to better clarify the model, and added remark regarding notation issue.

Presentation issues / List of changes.– The introduction has been improved in order to reflect better some “motivation” issues that have been debated by refereeA. Additional relevant references have been added. Otherwise there were only 4 action items with regard to the presentation: **[A5,B1]:** Both referees wrote that former Fig1, the ”artistic” illustration of the model, is not helpful. We therefore moved it to the supplementary and used the space for a more detailed explanation of the model as suggested by referee B. **[A7]:** Color code in Fig2b (now Fig1b) is now explained in the caption. **[B2]:** The word ”tangible” has been replaced by ”tractable”.