Glassiness vs. disorder in a one dimensional hopping model

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Below we consider the effect of disorder and glassiness on the non equilibrium steady state of a one dimensional hopping model on a ring.

I. THE MODEL

We have N discrete sites on a ring. Transitions are allowed between nearest neighbours, with transition rates that are random and asymmetric. The transition rates across the n^{th} bond are written as

$$\overrightarrow{w}_n = w_{n+1,n} = g_n e^{s_n/2} = e^{-b_n+s_n/2} \tag{1}$$

$$\overleftarrow{w}_n = w_{n,n+1} = g_n e^{-s_n/2} = e^{-b_n-s_n/2} \tag{2}$$

where $g_n = e^{-b_n}$ and s_n are random numbers. The barriers b_n are uniform random variables on the interval $[-\Delta, \Delta]$. If g_n has a log-wide distribution, it is said to be glassy, so we call Δ "glassiness". We define $s_n = \ln \frac{\overrightarrow{w}_n}{\overleftarrow{w}_n}$ as the stochastic field. The stochastic field is a uniform random variable on the interval $[s-\sigma, s+\sigma]$. The field induces asymmetry. In general, detailed balance is violated, such that

$$s = \frac{1}{N} \ln \left[\prod_{n=1}^{N} \frac{\overrightarrow{w}_n}{\overleftarrow{w}_n} \right] = \frac{1}{N} \sum_{n=1}^{N} s_n \neq 0 \tag{3}$$

II. ANALYSIS

To determine the roles played by σ and Δ, we study the probability distribution of the winding number n. In particular, we look at the cumulant generating function

$$g(\lambda) = \lim_{t\to\infty} -\frac{1}{t} \ln\langle e^{-\lambda n} \rangle \tag{4}$$

which completely determines the probability distribution in the long time limit. Note that $g(\lambda)$ satisfies the non equilibrium fluctuation theorem

$$g(\lambda) = g(sN - \lambda) \tag{5}$$

If the distribution is gaussian, then $g(\lambda)$ is a parabola

$$g(\lambda) = v\lambda - D\lambda^2 \tag{6}$$

otherwise there are additional terms of order $\mathcal{O}(\lambda^3)$in $g(\lambda)$. These terms can be "packaged" in the maximum value of the generating function,

$$h = \max[g(\lambda)] \tag{7}$$

We consider two aspects of the distribution: The dependence of v/D on σ and Δ and the shape of $g(\lambda)$ beyond the first and second moments. The problem is defined by 4 parameters v, D, s and h, from which we construct the dimensionless parameters

$$\frac{v}{Ds}, \text{ and } \frac{h}{vs} \tag{8}$$

which of course depend on σ and Δ. Notice that the ratio of these parameters is $v^2/4Dh$, which is the ratio of the peak of the parabola to the peak of $g(\lambda)$. The first quantity is determined by the first and second moments of the distribution, while in some sense the second quantity determines the shape of the distribution, because of the information contained in h.

III. v/D VS. σ AND Δ

We begin with the quantity v/Ds. We take the zero disorder case as a reference (black line of Fig. ??), for which it can be shown that

$$\frac{v}{Ds}[\text{no disorder}] \;=\; \frac{2}{as}\tanh\frac{as}{2} \tag{9}$$

In the general case where σ and Δ are non zero, we define an effective length scale a_s, such that

$$\frac{v}{Ds} \;=\; \frac{2}{a_s s}\tanh\frac{a_s s}{2} \tag{10}$$

We define an effective "disorder limited" length scale, obtained by taking the limit $s \to \infty$,

$$a_\infty \;=\; \left(\frac{2D}{v}\right)_{s\to\infty} \;=\; \left[\frac{\left\langle(1/\overrightarrow{w})^2\right\rangle}{\left\langle(1/\overrightarrow{w})\right\rangle^2}\right] \;=\; \frac{\sigma\Delta}{4}\coth\left(\frac{\sigma}{2}\right)\coth\left(\frac{\Delta}{2}\right) \tag{11}$$

In Fig. ?? We plot the dimensionless quantity $a_\infty v/D$ vs. s for various values of σ and Δ for a given realization of the ring. We observe that as σ is increased the Sinai step becomes higher and wider. On the other hand as Δ increases, the step behaviour is suppressed, or smoothed over. Another interesting feature is seen in the images of v/Ds for a continuous range of σ and Δ at various values of s (Fig. ??, row 1). For small values of s, v/Ds is mostly independent of Δ and as s increases, the weight of Δ increases and for very large s, σ and Δ contribute equally, as expected from equation (??). These are our two main observations as to the dependence and relevance of σ and Δ.

IV. THE SHAPE OF $g(\lambda)$

Given v and D, the parabola is defined and it has a peak value of

$$h[\text{Gaussian}] = v^2/4D \tag{12}$$

In general, however, the peak value is different

$$h = \max[g(\lambda)] \tag{13}$$

In the Poisson limit, $s \to \infty$, it is easy to show that the minimum of the generating function is given by the smallest transition rate

$$h[\text{Poisson}] = \min[\overrightarrow{w}_n] \tag{14}$$

We emphasize that even for no disorder and zero bias, the distribution is non gaussian, due to the discrete lattice (Fig. ??). If there is no disorder, the cumulant generating function can easily be shown to be

$$g(\lambda) \;=\; \overrightarrow{w}\mathrm{e}^{\lambda a} + \overleftarrow{w}\mathrm{e}^{-\lambda a} - (\overleftarrow{w} + \overrightarrow{w}) \tag{15}$$

If the bias $s = 0$, this reduces to

$$g(\lambda) \;=\; 2w(\cosh(\lambda a) - 1) \tag{16}$$

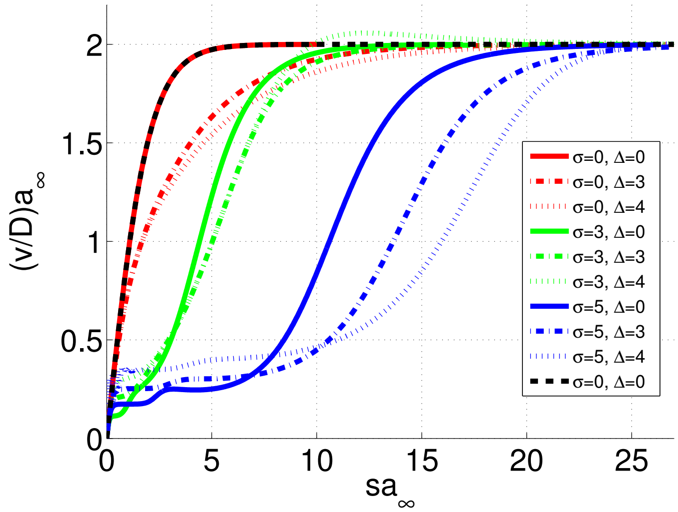

FIG. 1: The Einstein relation scaled by the effective lattice constant $a_\infty v/D$ vs. the scaled affinity $x = a_\infty s$ for various values of σ and Δ. Different colors correspond to different σ, different line styles correspond to different Δ. The dashed black line is $2\tanh(x)$ corresponding to no disorder.

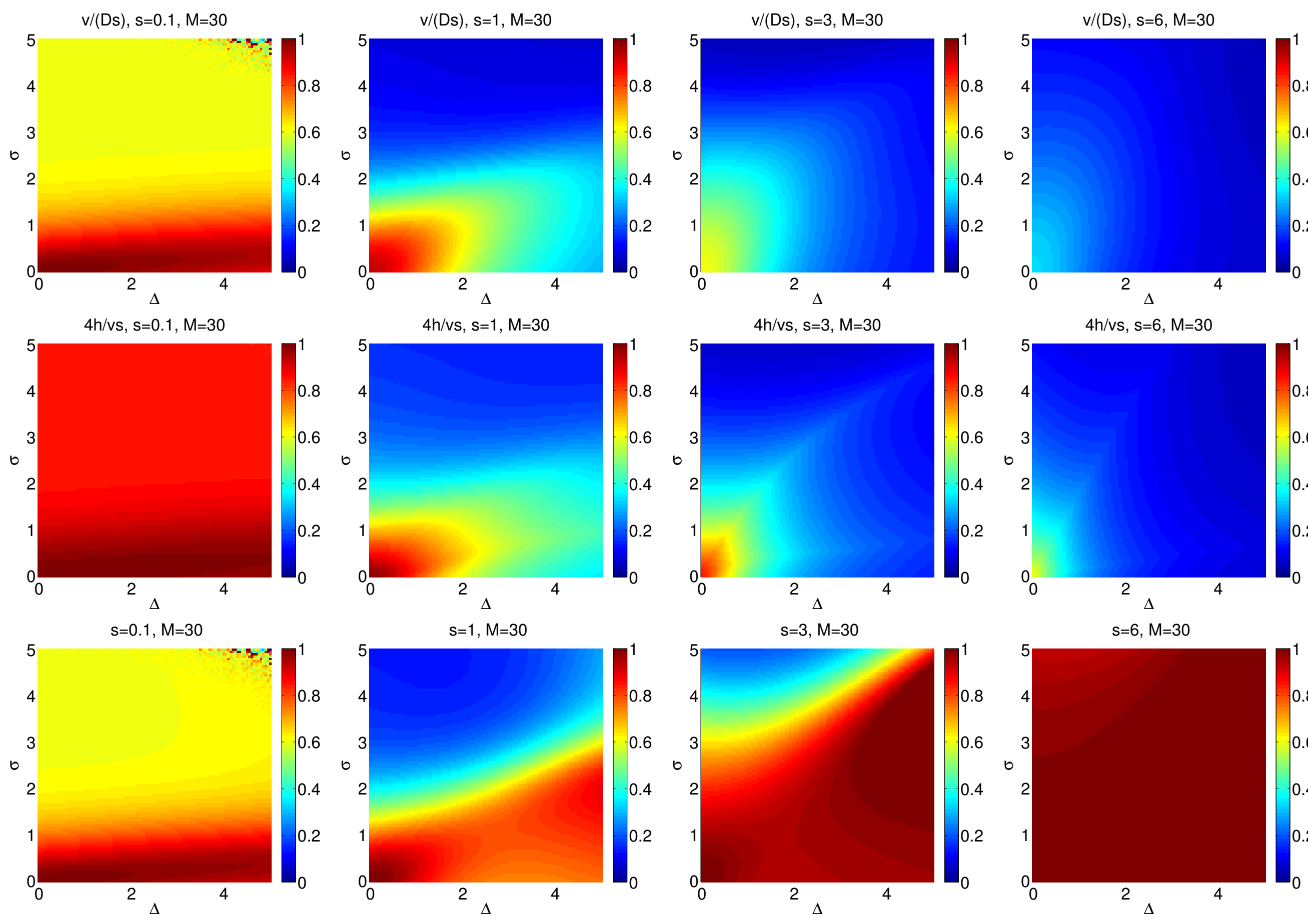

FIG. 2: First row: The ratio of first to second moments, normalised by s. Second row: Ratio of velocity to peak value h, normalised by s. Third row: The ratio v/D divided by $2a_\infty^{-1}\tanh(a_\infty s/2)$.

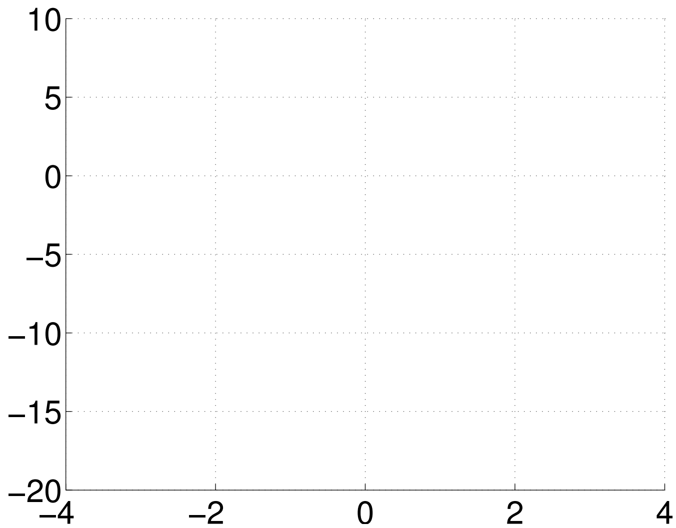

FIG. 3: The effect of discretization on the generating function $g(\lambda)$.The red line is for $s = 0$, the green line is for $s = 4$. In both cases $\sigma = 0$ and $\Delta = 0$. Dashed lines are parabolas $v\lambda - D\lambda^2$.