Percolation, sliding, localization and relaxation in glassy circuits

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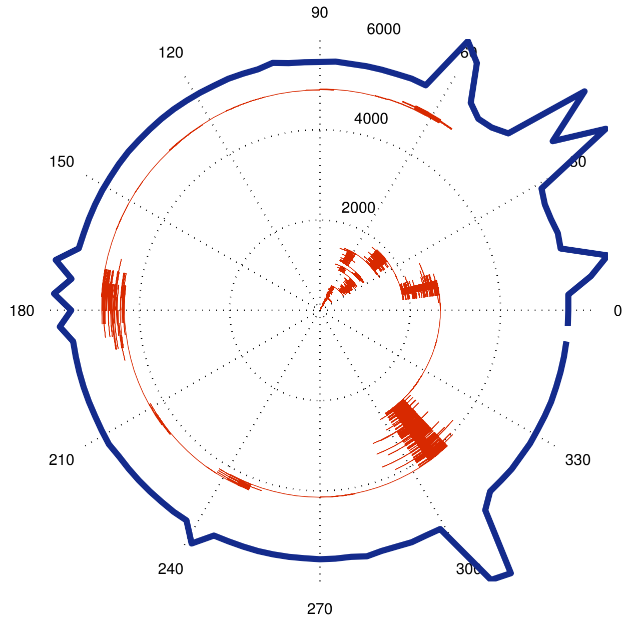
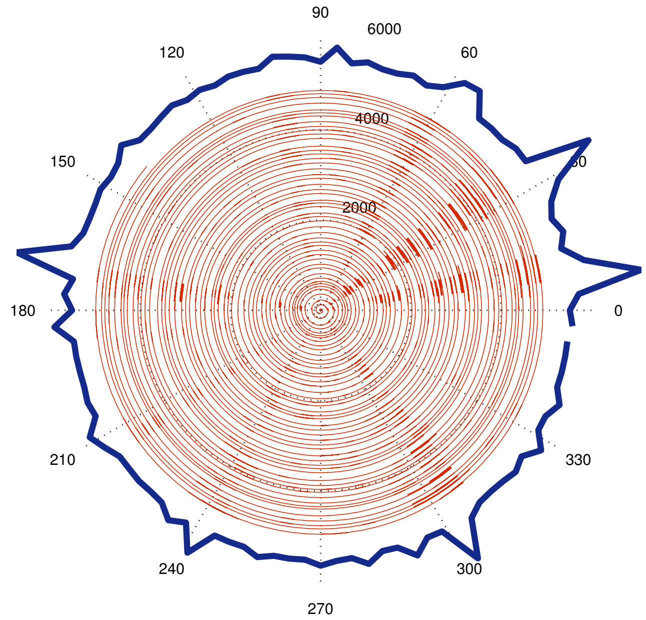


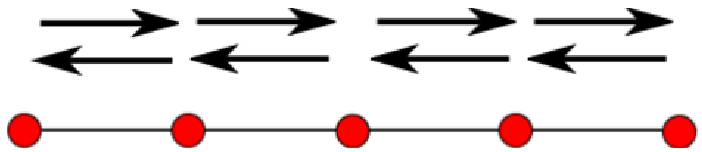
"Localized"



Sliding

Brownian motion

Simple random walk [Einstein]

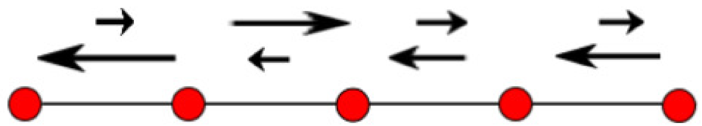


Uniform lattice - all rates are equal w,
$D = a^2 w$

Random walk on disordered lattice [Alexander et al.]

Random lattice - random, symmetric transition rates w_n
$P(w) \propto w^{\alpha-1}$ (for small w)
$D =$ resistor network calculation
Percolation related transition to subdiffusion for
$D = 0$ for $\alpha < 1$

Random walk in random environment [Sinai, Derrida, ...]



Rates allowed to be asymmetric $\overrightarrow{w}_n / \overleftarrow{w}_n = e^{\mathcal{E}_n}$
Stochastic field: $\mathcal{E}_n \sim [s-\sigma,\ s+\sigma]$
Sliding transitions for $\left\langle e^{-\mathcal{E}\mu} \right\rangle = 1$ (defines s_μ)
$D = 0$ for $s < s_{1/2}$
$v = 0$ for $s < s_1$

Implications of the percolation and sliding transitions on relaxation modes of closed ring

Dynamics

Conservative rate equation

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{W}\boldsymbol{p}, \qquad \boldsymbol{W} = \begin{bmatrix} -\gamma_1 & w_{1,2} & 0 & ... \\ w_{2,1} & -\gamma_2 & w_{2,3} & ... \\ 0 & w_{3,2} & -\gamma_3 & ... \\ ... & ... & ... & ... \end{bmatrix},$$

$$\sum_{n=1}^{N} w_{nm} = 0$$

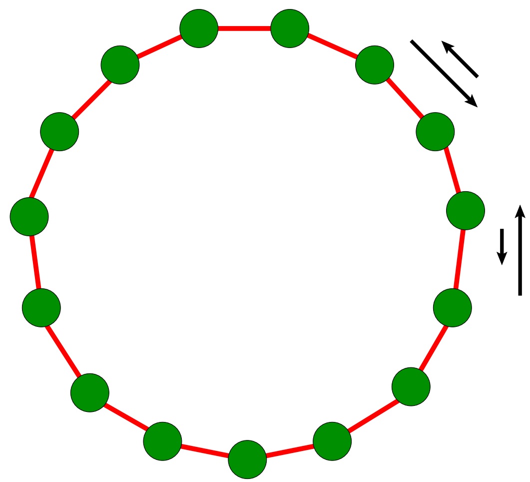
i.e. $\gamma_2 = w_{1,2} + w_{3,2}$

Transition rates across n^{th} bond $w_n \mathrm{e}^{\pm \mathcal{E}_n/2}$

Stochastic field $\mathcal{E}_n \sim [s-\sigma,\ s+\sigma]$

Resistor network $P(w) \propto w^{\alpha-1}$

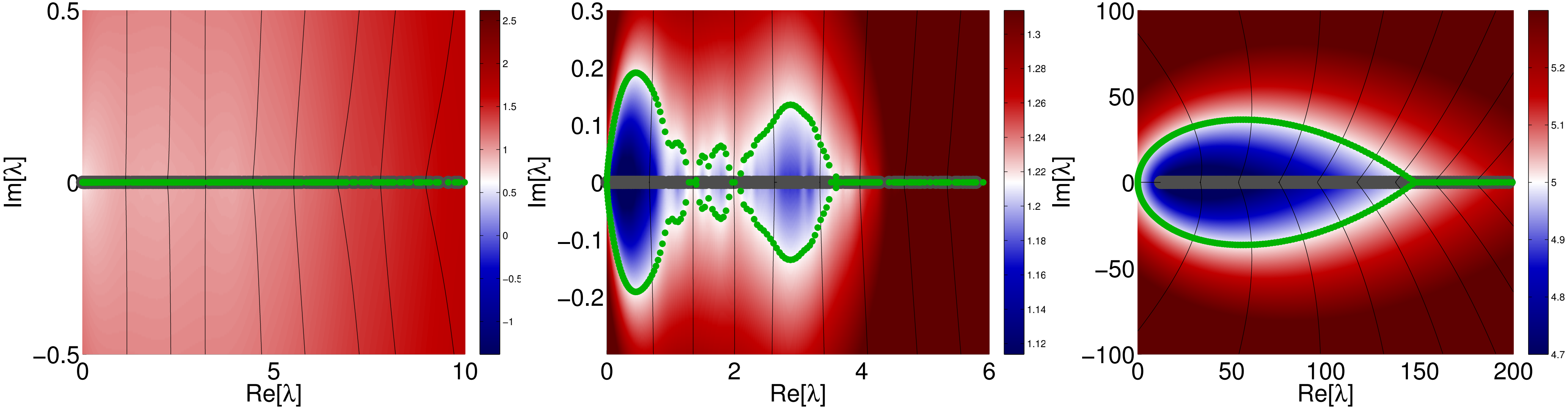
Affinity (closed ring) $S_{\circlearrowleft} = \sum_{n=1}^{N} \mathcal{E}_n = Ns$



How do spectral properties of $\boldsymbol{W}$ depend on (α, σ, s)?

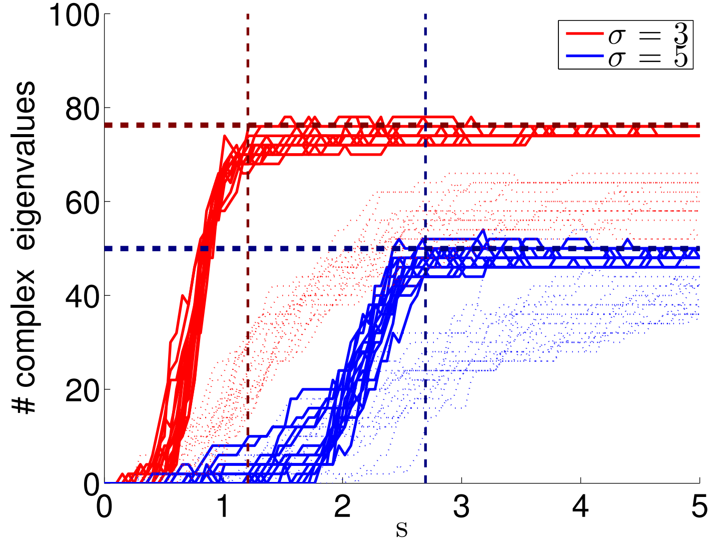
- **What is the threshold bias s_c for complex eigenvalues (delocalization)?**
- **How is s_c related to the percolation transition? to the sliding transition?**
- **Implications of conservativity?**

The spectrum



Observations

- $\lambda_0 = 0$ due to conservativity
- Complex eigenvalues $\rightsquigarrow$ oscillating density
- Complex bubble at bottom of band
- Complexity saturation



The spectral equation

Equation for eigenvalues

$$\det(z - \boldsymbol{W}) = \prod_{k=0}^{N-1} (z - \lambda_k) \quad = \quad 0$$

Gauge away disorder (imaginary AB flux)

$\tilde{\boldsymbol{W}} = \mathrm{e}^{\boldsymbol{U}/2} \boldsymbol{W} \mathrm{e}^{-\boldsymbol{U}/2}$

$\tilde{\boldsymbol{W}} \;=\; \text{diagonal}\Big\{ -\gamma_n \Big\} + \text{offdiagonal}\Big\{ w_n \mathrm{e}^{\pm \frac{S_\circlearrowleft}{2N}} \Big\}$

- **Cannot gauge away asymmetry $\mathcal{S}_\circlearrowleft$** (cannot gauge away flux in closed loop)
- Off diagonal elements depend on $(\alpha, S_\circlearrowleft)$
- Diagonal elements depend on (α, σ, s) (conservativity)

Associated Hermitian matrix $\boldsymbol{H}$

Set $S_\circlearrowleft = 0$ for off diagonal elements

$\boldsymbol{H}$ is symmetric with real eigenvalues $\epsilon_k(s)$

Express spectral equation in terms of ϵ_k [1]

$$\prod_{k=0}^{N-1} \left(\frac{z + \epsilon_k(s)}{\overline{w}} \right) \quad = \quad 2 \left[\cosh\left(\frac{S_\circlearrowleft}{2} \right) - 1 \right]$$

[1] N.M. Shnerb, D.R. Nelson, Phys. Rev. Lett. 80, 5172 (1998)

Spectrum of $\boldsymbol{H}$

$\boldsymbol{H}$ is a symmetric matrix with real eigenvalues $\epsilon_k(s) \in [\epsilon_s,\ \epsilon_\infty]$

Clean ring - gapped spectrum $\epsilon_{s,\infty} = 2\left[\cosh(s/2) \mp 1\right]$

Sparse disorder ($M \ll N$ defects) - Few eigenvalues isolated from the continuum

Fully disordered ring - Gap closes, densifty of states $\rho(\epsilon) \propto \epsilon^{\mu-1}$ (for small ϵ)

Stochastic field disorder

Gaussian disorder [1] $s = \frac{1}{2}\sigma^2\mu$

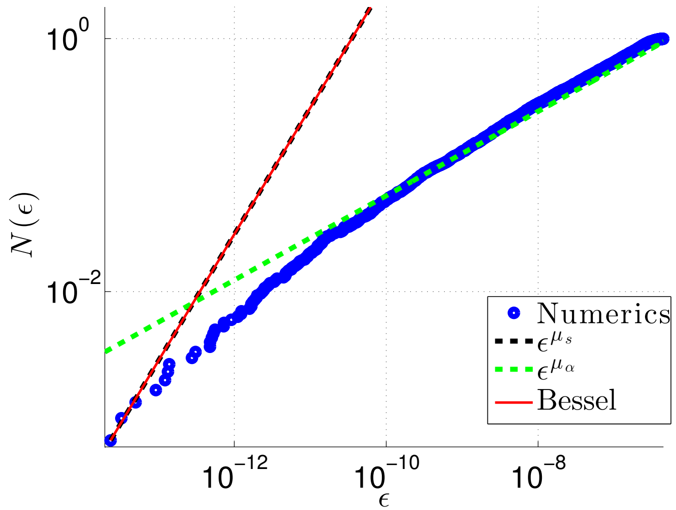
Box disorder $s = s_\mu = \frac{1}{\mu}\ln\left(\frac{\sinh(\sigma\mu)}{\sigma\mu}\right)$

For $s > s_\infty = \sigma$, gap opens, $\epsilon_s = \mathrm{e}^{(s-\sigma)/2}$

Resistor network disorder [2]

$\mu = \min\left\{\frac{\alpha}{1+\alpha},\ \frac{1}{2}\right\}$

For large bias, $\boldsymbol{H}$ is trivially localized, $\mu = \alpha$



Resistor network modifies spectral density at high energies

[1] J. Bouchaud, A. Comtet, A. Georges, and P. L. Doussal, Annals of Physics 201, 285 (1990)

[2] S. Alexander, J. Bernasconi, W. R. Schneider, and R. Orbach, Rev. Mod. Phys. 53, 175 (1981).

Electrostatic picture

2D Electrostatic potential $\Psi(z) \;=\; \sum_k \ln(z-\epsilon_k) \;\equiv\; V(x,y) + iA(x,y)$

The secular equation $\qquad V(x,y) = V(0) \qquad A(x,y) = 2\pi * \text{integer}$

Condition for complexity $V(\epsilon) < V(0) = \ln\left[2(\cosh(S_\circlearrowleft/2) - 1)\right];$

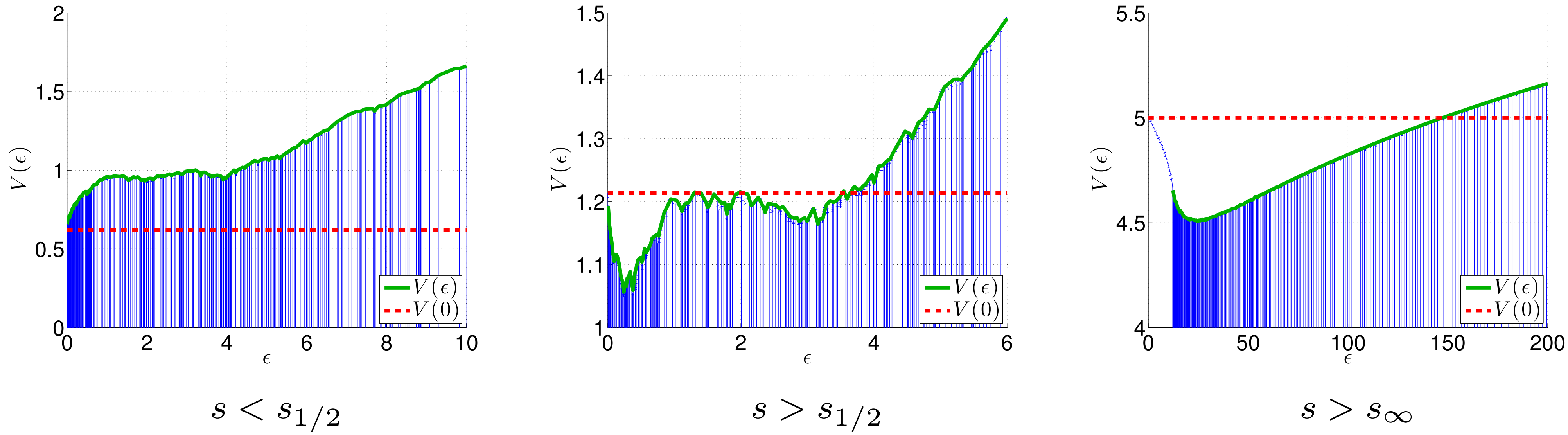
Continuum approximation

Density of states $\iff$ charge density $\rho \propto \epsilon^{\mu-1}$ along real axis

Potential along real axis $V(\epsilon) \;=\; \int \ln(|\epsilon - x'|)\rho(x')dx'$

Derivative at origin $\qquad V'(\epsilon) \;\approx\; \dfrac{\epsilon^{\mu-1}}{\epsilon_c^{\mu}}\pi\mu\cot(\pi\mu)$, **changes sign at** $\mu = 1/2$

Condition for complexity $V'(\epsilon) < 0$



Complexity threshold bias

Stochastic field, $\mu = \mu_s(\sigma)$

$s_c = s_{1/2} < s_1$

Resistor network, $\mu = \mu_\alpha$

$$\begin{cases} s = 0 & \text{resistor network} \quad \Longrightarrow \quad \mu = \frac{\alpha}{1+\alpha} \\ \text{large } s & \text{trivial localization} \quad \Longrightarrow \quad \mu = \alpha \end{cases}$$

$\alpha < 1/2 \Rightarrow s_c = \infty$

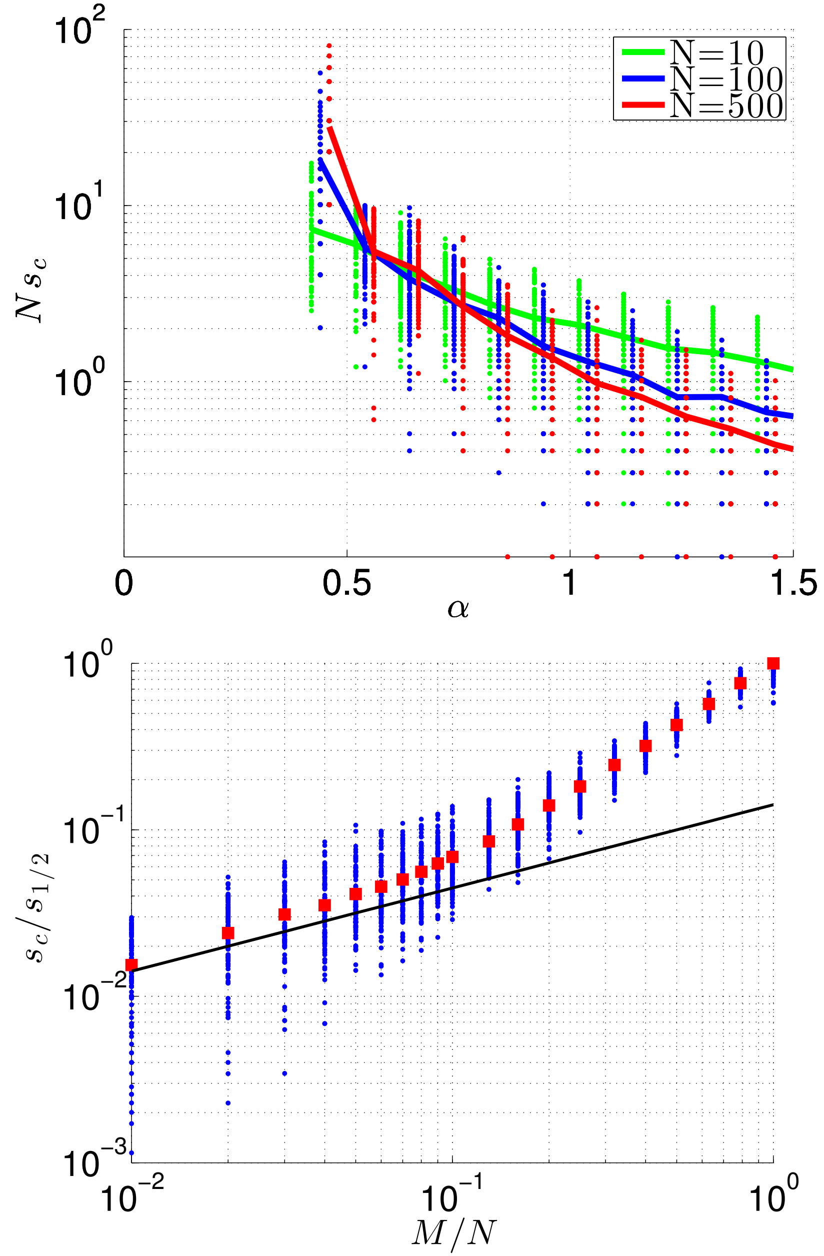
$\alpha > 1/2 \Rightarrow s_c \sim 1/N$ [Numerically verified]

Sparse disorder

Clean ring with $M \ll N$ defects

$s_c \sim 1/N \ll s_1$

For M field defects $s_c = \sigma\sqrt{M}/N$



Complexity saturation

Recall the spectral determinant

$$\prod_{k=0}^{N-1}\left(\frac{z+\epsilon_k(s)}{\overline{w}}\right) \;=\; 2\left[\cosh\left(\frac{S_{\circlearrowleft}}{2}\right)-1\right], \qquad S_{\circlearrowleft}=sN$$

For large s

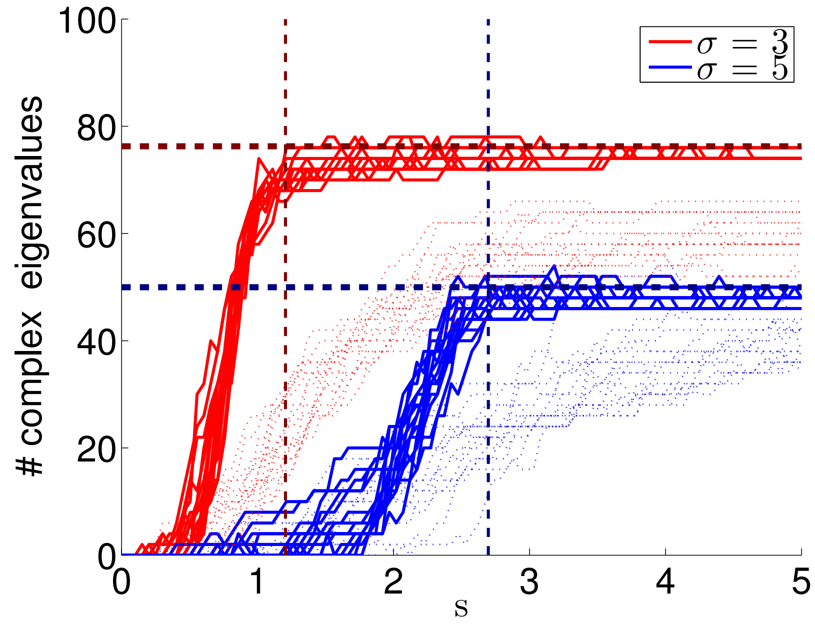
Non conservative

LHS, RHS are independent $\rightsquigarrow$ entirely complex spectrum

Conservative

$\epsilon_n \;=\; \gamma_n \;=\; w_n \mathrm{e}^{\mathcal{E}_n/2} \sim \mathrm{e}^{s}$

LHS, RHS $\sim \mathrm{e}^{sN/2} \rightsquigarrow$ complexity saturation



The saturation fraction

Upper cutoff ϵ_c determined by $\overline{\ln\left[\epsilon - w\mathrm{e}^{\mathcal{E}/2}\right]} \;=\; s/2$

Count number of eigenvalues up to ϵ_c

Gaussian disorder

$V(\epsilon\to\infty)=\mathrm{const}$

Enitrely complex spectrum

Sharp transition at $s=s_{1/2}$

Summary

Type of disorder	Parameters	s_c	Remarks
Resistor-network disorder	$\alpha<\frac{1}{2},\ \sigma=0$	$s_c=\infty$	non-percolating
Resistor-network disorder	$\frac{1}{2}<\alpha<1,\ \sigma=0$	$s_c\sim(1/N)$	residual percolation
Sparse disorder	$(M/N)\ll 1$	$s_c\sim(1/N)$	both disorder types
Stochastic field disorder	$\alpha>1,\ \sigma\neq 0$	$s_c\approx s_{1/2}$	percolating

- Relaxation properties of a closed circuit, whose dynamics is generated by a conservative rate-equation, is dramatically different from that of a biased non-hermitian Hamiltonian.
- Random resistor network - Transition to complexity happens for $\alpha>1/2$ **before** the percolation transition.
- Random walk in random environment - Transition to complexity happens for $\mu>1/2$ **before** the sliding transition.
- Sparse disorder - s_c diminishes as $1/N$ for .
- Increasing the bias does not lead to full delocalization, instead **"complexity saturation"** is observed.

Motivation

- **Glassines - log wide distribution of rates**

1. A. Amir, Y. Oreg, and Y. Imry, Proceedings of the National Academy of Sciences 109, 1850 (2012)
2. A. Vaknin, Z. Ovadyahu, and M. Pollak, Phys. Rev. Lett. 84, 3402 (2000).
3. A. Amir, Y. Oreg, and Y. Imry, Phys. Rev. Lett. 103, 126403 (2009).

- **Non Hermitian QM**

Vortex depinning in type II superconductors (s = applied transverse magnetic field)

1. N. Hatano and D. R. Nelson, Phys. Rev. Lett. 77, 570 (1996), Phys. Rev. B 56, 8651 (1997).

Follow ups

1. P. W. Brouwer, P. G. Silvestrov, and C. W. J. Beenakker, Phys. Rev. B 56, R4333 (1997).
2. I. Y. Goldsheid and B. A. Khoruzhenko, Phys. Rev. Lett. 80, 2897 (1998).
3. J. Feinberg and A. Zee, Phys. Rev. E 59, 6433 (1999).
4. L. G. Molinari, Linear Algebra and its Applications 429, 2221 (2008).

- **Biophysics**

Pulling pinned polymers, DNA denaturation (s = pulling force)

1. D. K. Lubensky and D. R. Nelson, Phys. Rev. Lett. 85, 1572 (2000), Phys. Rev. E 65, 031917 (2002).

Population biology (s = convective flow of bacteria relative to the nutrients)

1. D. R. Nelson and N. M. Shnerb, Phys. Rev. E 58, 1383 (1998).
2. K. A. Dahmen, D. R. Nelson, and N. M. Shnerb, in Statistical mechanics of biocomplexity (Springer, 1999) pp. 124151.

Molecular motors (s = affinity of chemical cycle)

1. M. E. Fisher, A. B. Kolomeisky, The force exerted by a molecular motor. PNAS 96, 65976602 (1999).
2. Rief, M. et al. Myosin-v stepping kinetics: a molecular model for processivity. PNAS 97, 94829486 (2000).
3. Y. Kafri, D. K. Lubensky, and D. R. Nelson, Biophysical Journal 86, 3373 (2004), Phys. Rev. E 71, 041906 (2005).