Percolation, sliding, localization and relaxation in glassy circuits

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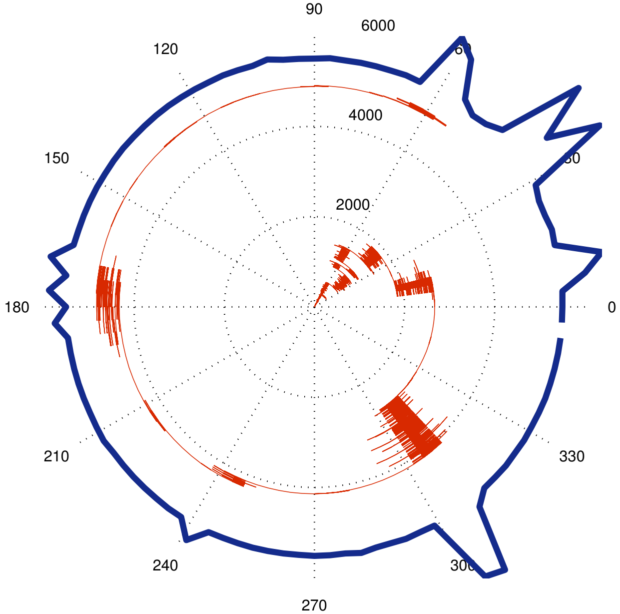
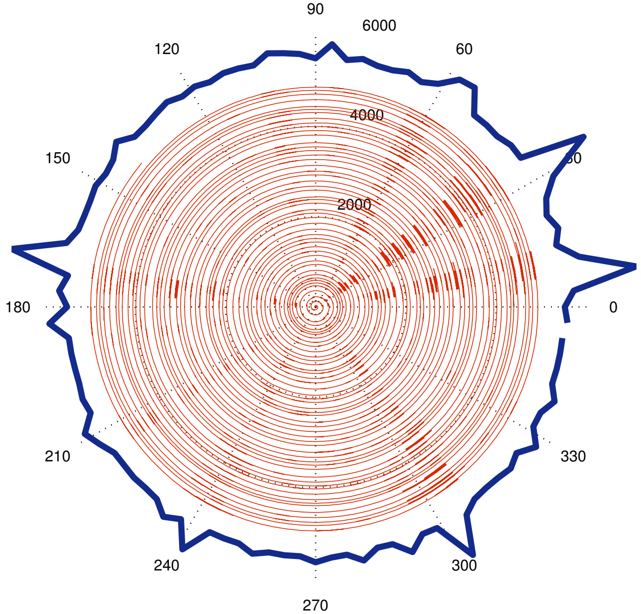
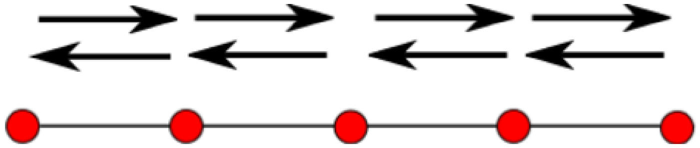




D. Hurowitz and D. Cohen, `arXiv` (2014)

Brownian motion



Simple random walk [Einstein]

uniform lattice - all rates are equal w,

$D = a^2 w$

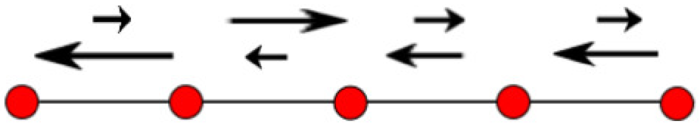
Random walk on disordered lattice [Alexander et. al]

Random lattice - random, symmetric transition rates w_n

$P(w) \propto w^{\alpha-1}$

$D =$ resistor network calculation

Percolation related transition to subdiffusion for $\alpha < 1$



Random walk in random environment [Sinai, Derrida]

Rates allowed to be asymmetric $\overrightarrow{w}_n / \overleftarrow{w}_n = e^{\mathcal{E}_n}$

Stochastic field: $\mathcal{E}_n \sim [s - \sigma,\ s + \sigma]$

Sliding transitions for $\left\langle e^{-\mathcal{E}\mu} \right\rangle = 1$, defines s_μ

$D = 0$ for $s < s_{1/2}$

$v = 0$ for $s < s_1$

Dynamics

Stochastic rate equation

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{W}\boldsymbol{p}, \qquad \boldsymbol{W} = \begin{bmatrix} -\gamma_1 & w_{1,2} & 0 & \dots \\ w_{2,1} & -\gamma_2 & w_{2,3} & \dots \\ 0 & w_{3,2} & -\gamma_3 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

with transition rates across n^{th} bond $w_n \mathrm{e}^{\pm \mathcal{E}_n/2}$

Probability conservation $\sum_n w_{nm} = 0$

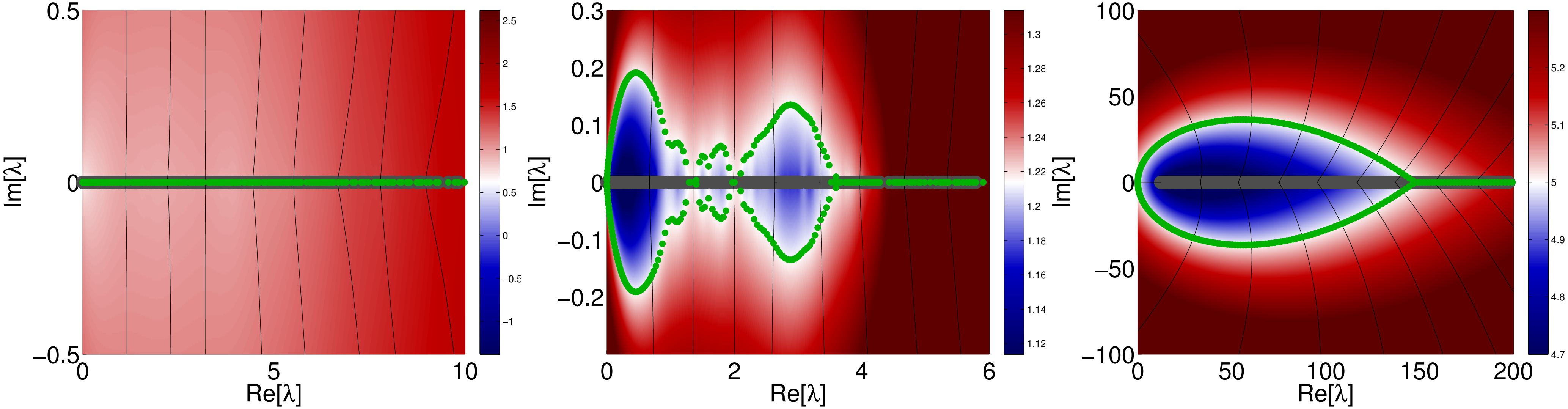
Affinity (closed ring) $S_{\circlearrowleft} = \sum_{n=1}^{N} \log\left(\frac{\overrightarrow{w}_n}{\overleftarrow{w}_n}\right) = Ns$

Relaxation modes of closed ring λ_k

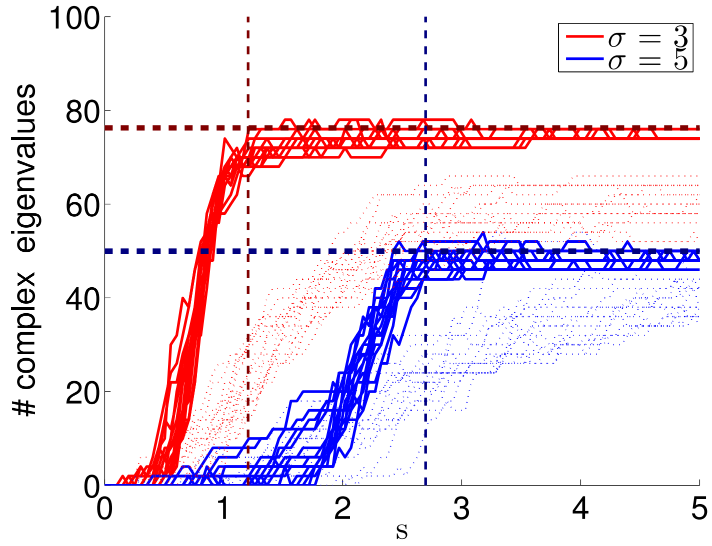
How do spectral properties of $\boldsymbol{W}$ depend on (α, σ, s)?

- **What is the threshold bias s_c for complex eigenvalues (delocalization)?**
- **How is s_c related to the percolation transition? to the sliding transition?**
- **Implications of conservativity?**

The spectrum



- $\lambda_0 = 0$ due to conservativity
- Complex eigenvalues $\rightsquigarrow$ oscillating density
- Complex bubble at bottom of band
- Complexity saturation



The spectral equation

Characteristic polynomial $\prod_{k=0}^{N-1}(z-\lambda_k) \;=\; 0$

Hatano, Nelson form $\prod_{k=0}^{N-1}\left(\frac{z+\epsilon_k(s)}{\overline{w}}\right) \;=\; 2\left[\cosh\left(\frac{S_{\circlearrowleft}}{2}\right)-1\right]$

The associated Hermitian matrix $\boldsymbol{H}$

Open chain

Gauge away bias $\boldsymbol{H} = \mathrm{e}^{\boldsymbol{U}/2}\boldsymbol{W}\mathrm{e}^{-\boldsymbol{U}/2}$

$\boldsymbol{H}$ symmetric matrix with real eigenvalues $\epsilon_k(s)$

Density of states $\rho(\epsilon) \;\propto\; \epsilon^{\mu-1}$ (for small ϵ)

Field disorder

$$s \;=\; s_\mu \;=\; \frac{1}{\mu}\ln\left(\frac{\sinh(\sigma\mu)}{\sigma\mu}\right)$$

For $s > s_\infty = \sigma$, gap opens

Resistor network disorder

$$\mu = \min\left\{\frac{\alpha}{1+\alpha},\; \frac{1}{2}\right\}$$

For large bias, $\boldsymbol{H}$ is trivially localized, $\mu = \alpha$

Closed ring

Gauge away disorder $\tilde{\boldsymbol{W}} = \mathrm{e}^{\boldsymbol{U}/2}\boldsymbol{W}\mathrm{e}^{-\boldsymbol{U}/2}$ (cannot gauge away asymmetry)

Associated hermitian matrix $\boldsymbol{H}$ with real eigenvalues $\epsilon_k(s)$ by setting $\mathcal{S}_{\circlearrowleft} = 0$

Electrostatic picture

2D Electrostatic potential $\Psi(z) \;=\; \sum_k \ln\left(z-\epsilon_k\right) \;\equiv\; V(x,y)+iA(x,y)$

The secular equation $\quad V(x,y) = V(0) = \ln\left[2(\cosh(S_{\circlearrowleft}/2)-1)\right]\,; \qquad A(x,y) = 2\pi * \text{integer}$

Condition for complexity $V(\epsilon) < V(0)$

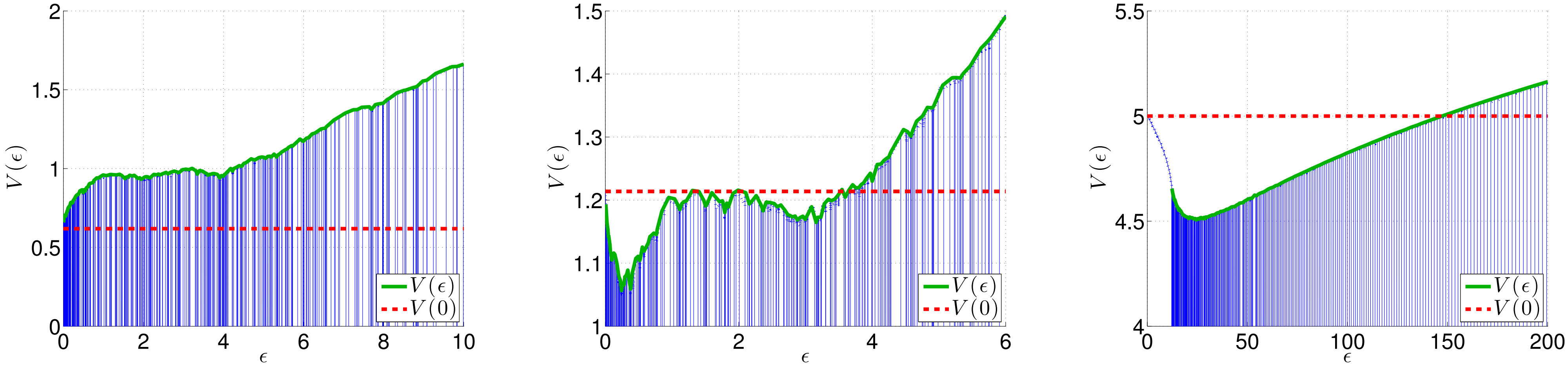
Continuum approximation

Density of states $\Longleftrightarrow$ charge density $\rho \propto \epsilon^{\mu-1}$

Potential along real axis $V(\epsilon) \;=\; \int \ln\left(|\epsilon - x'|\right)\rho(x')dx'$

Derivative at origin $\quad V'(\epsilon) \;\approx\; \dfrac{\epsilon^{\mu-1}}{\epsilon_c^{\mu}}\pi\mu\cot(\pi\mu)$, changes sign at $\mu = 1/2$

Condition for complexity $V'(\epsilon) < 0$



$s < s_{1/2}$ $\qquad$ $s > s_{1/2}$ $\qquad$ $s > s_\infty$

Examples

Stochastic field, $\mu = \mu_s(\sigma)$

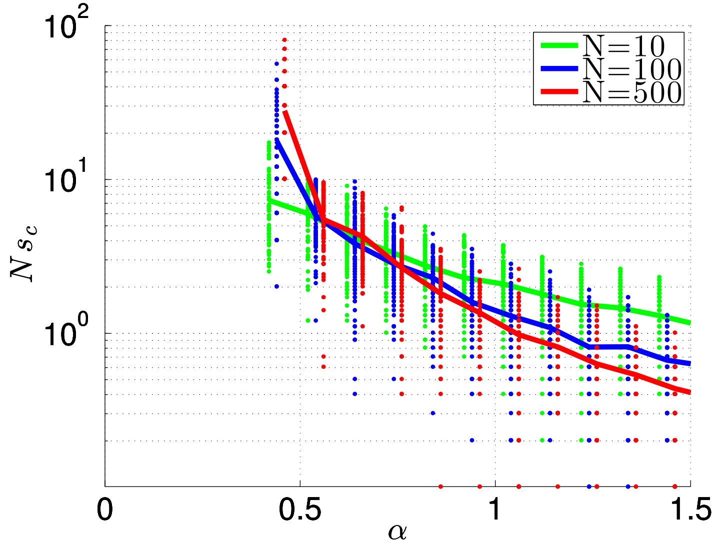
$s_c = s_{1/2} < s_1$

Resistor network, $\mu = \mu_\alpha$

$$\begin{cases} s = 0 & \text{resistor network} & \rightarrow & \mu & = & \frac{\alpha}{1+\alpha} \\ \text{large } s & \text{trivial localization} & \rightarrow & \mu & = & \alpha \end{cases}$$

$\alpha < 1/2 \Rightarrow s_c = \infty$

$\alpha > 1/2 \Rightarrow s_c \sim 1/N$ [Numerically verified]

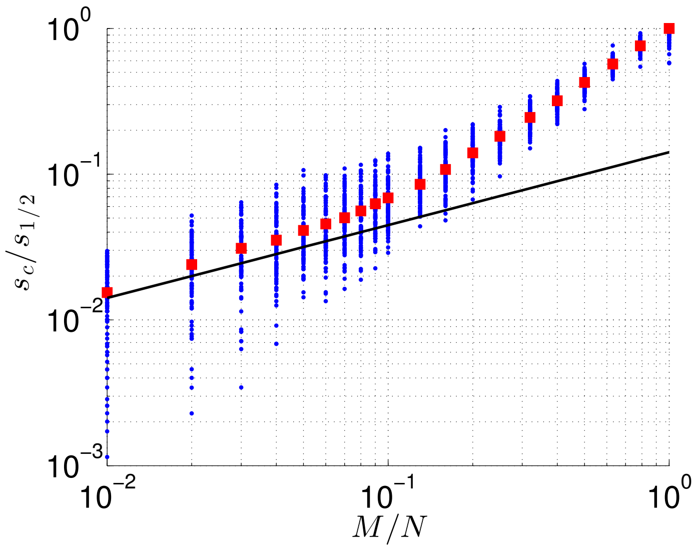


Sparse disorder

Clean ring with $M \ll N$ defects

$s_c \sim 1/N \ll s_1$

For M field defects $s_c = \sigma\sqrt{M}/N$



Complexity saturation

Recall the spectral determinant $$\prod_{k=0}^{N-1}\left(\frac{z+\epsilon_k(s)}{\overline{w}}\right) \;=\; 2\left[\cosh\left(\frac{S_{\circlearrowleft}}{2}\right)-1\right]$$

For large s

Non conservative $\rightsquigarrow$ entire spectrum becomes complex

Conservative $\rightsquigarrow$ complexity saturation

Trivial localization $\rightsquigarrow$ $\epsilon_n = w_n \mathrm{e}^{\mathcal{E}_n/2}$

Upper cutoff ϵ_c determined by $\overline{\ln\left[\epsilon - w\mathrm{e}^{\mathcal{E}/2}\right]} \;=\; s/2$