One dimensional hopping model - interplay of resistor-network and Sinai spreading

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The conventional view of variable-range-hopping is based on the assumption that the rates of transitions are *symmetric* but have log-wide distribution as in glassy systems. Namely, the rate at the nth bond is written as

$$g_n \;=\; \text{rate of transition} \;=\; \exp(-\mathcal{B}_n) \tag{1}$$

where the barriers $\mathcal{B}_n$ have (say) a box distribution. Consequently a resistor-network analysis is most appropriate for the analysis,leading to an Ohmic result for the current, namely $I \propto 1/L$, where L is the length of the sample, but the prefactor is exponentially small.

Contrary to that Sinai has considered a model where the rates are *asymmetric*. The asymmetry of transitions in the nth bond is characterized by

$$\mathcal{E}_n \;=\; \ln\left(\frac{\overrightarrow{w}_n}{\overleftarrow{w}_n}\right) \tag{2}$$

which we call "stochastic field". At equilibrium $\mathcal{E}_n = (E_n - E_{n+1})/T$, where T is the temperature. More generally we can define a stochastic potential

$$V_n \;=\; \sum_{n'=0}^{n} \mathcal{E}_{n'} \tag{3}$$

In non-equilibrium circumstances the telescopic correlations are likely to be broken, the sum becomes of order $\sqrt{L}$, and therefore $I \propto \exp(-\sqrt{L})$ due to this buildup of an activation barrier.

I. THE MODEL, AND MAIN OBSERVATIONS

Below we consider a 1D model were we have an interplay of resistor-network physics (due to random couplings) and Sinai physics (due to random stochastic field). Specifically we consider an N site ring with near-neighbor transitions. Each bond is characterized by two numbers:

$$\mathcal{B}_n \;=\; \text{barriers} \;=\; \text{random} \in [-\Delta, \Delta] \tag{4}$$

$$\mathcal{E}_n \;=\; \text{stochastic field} \;=\; \text{random} \in [s-\sigma, s+\sigma] \tag{5}$$

Accordingly the transition rates across the n^{th} bond are

$$\overrightarrow{w}_n \;=\; w_{n+1,n} \;=\; g_n \mathrm{e}^{\mathcal{E}_n/2} \;=\; \mathrm{e}^{-\mathcal{B}_n+\mathcal{E}_n/2} \tag{6}$$

$$\overleftarrow{w}_n \;=\; w_{n,n+1} \;=\; g_n \mathrm{e}^{-\mathcal{E}_n/2} \;=\; \mathrm{e}^{-\mathcal{B}_n-\mathcal{E}_n/2} \tag{7}$$

The average stochastic field s is called in the literature "affinity". It can be written as $s = \text{Force}/T$, where the force is defined as the slope of the potential, and T is the temperature. The term "slope" is appropriate if we view our model as describing a periodic lattice with N-site unit cell. But we prefer to have in mind a ring geometry so the more appropriate notion is "stochastic motive force":

$$S_{\circlearrowleft} \;=\; \sum_{n\in\text{ring}}^{N} \mathcal{E}_n \;=\; Ns \tag{8}$$

We calculate the drift velocity v and the diffusion coefficient D as a function of the affinity s. In the absence of disorder ($\Delta = \sigma = 0$) they are related by a *generalized* Einstein relations:

$$\frac{v}{D} \;=\; \frac{2}{a}\tanh\left(\frac{as}{2}\right) \tag{9}$$

where $a = 1$ is the lattice constant. The traditional Einstein relation $v/D = s$ holds if s is sufficiently small. This is usually rephrased as mobility/diffusion $= 1/T$. The other extreme limit of very large s implies unidirectional transitions with $v/D = 2/a$. The latter ratio reflects Poisson statistics.

Taking the disorder into account the Poisson limit becomes $v/D = 2/a_\infty$, where a_∞ depends on the disorder:

$$a_\infty = \left(\frac{2D}{v}\right)_{s\to\infty} = \left[\frac{\langle(1/\overrightarrow{w})^2\rangle}{\langle(1/\overrightarrow{w})\rangle^2}\right] = \frac{\sigma\Delta}{4}\coth\left(\frac{\sigma}{2}\right)\coth\left(\frac{\Delta}{2}\right) \tag{10}$$

If we increase s from zero to infinity the ratio v/D has a crossover from $v/D = s$ to $v/D = 2/a_\infty$, which is illustrated in Fig. 1. Due to the disorder σ we get what we call a "Sinai step". This means that for small s the transport is suppressed due to the buildup of a potential barrier, while large enough s flattens the barrier leading to "sliding" with large drift velocity.

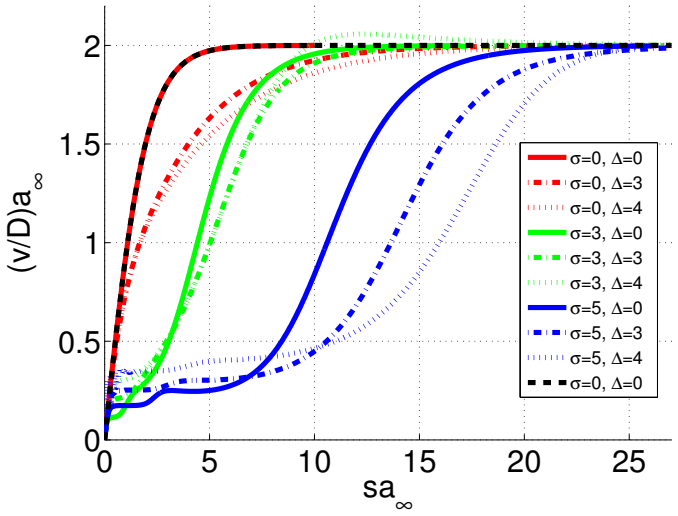

FIG. 1: The scaled v/D ratio versus the scaled s for various values of σ and Δ. Different colors correspond to different σ, while different line styles correspond to different Δ. The dashed black line $2\tanh(x)$ describes the result that would be obtained for a non-disordered model. The number of sites is $N = 30$.

The additional aspect that we want to analyze is the implication of having log-wide distribution of couplings, i.e. having finite Δ. We observe in Fig. 1 that by increasing Δ the Sinai step is somewhat smeared: the crossover from activated transport to sliding becomes wider.

In order to get the "big picture" we calculate the ratio between the scaled v/D and the reference curve $2\tanh(x)$ and displays the results as images in Fig. 2. The small s results imply that Δ has little effect on the beginning of the Sinai step. Namely it hardly affects the crossover from the linear-response (Einstein) regime to the activated transport (Sinai) regime. But the effect of Δ becomes qualitativley similar to the effect of σ as we cross from activated transport to "sliding".

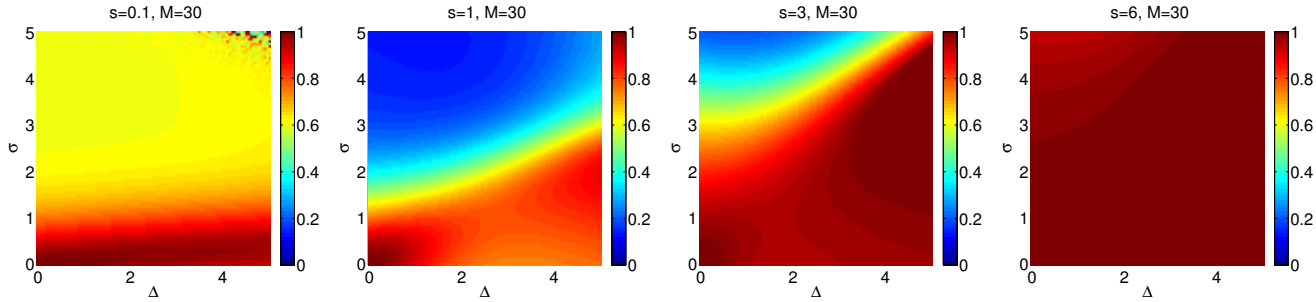

FIG. 2: The ratio between $a_\infty v/D$ and $2\tanh(x)$ is imaged as a function of Δ and σ for representative values of the affinity ($s = 0.1, 1, 3, 6$). The number of sites is $N = 30$.

II. A MORE DETAILED ANALYSIS

To determine the roles played by σ and Δ, we study the probability distribution of the displacement x. In the ring context it is defined as the winding number times N. We look at the cumulant generating function

$$g(\lambda) \;=\; \lim_{t\to\infty}\left[-\frac{1}{t}\ln\langle \mathrm{e}^{-\lambda x}\rangle_t\right] \tag{11}$$

which completely determines the probability distribution in the long time limit. Note that $g(\lambda)$ satisfies the non equilibrium fluctuation theorem (NFT)

$$g(\lambda) \;=\; g(s-\lambda) \tag{12}$$

If the distribution is Gaussian, then $g(\lambda)$ is a parabola

$$g(\lambda) \;=\; v\lambda - D\lambda^2 \tag{13}$$

and the NFT implies $v/D = s$. This parabola has a maximum at $\lambda = (1/2)v/D$ with height

$$h \;=\; \max[g(\lambda)] \;=\; \frac{v^2}{4D} \;=\frac{1}{4}vs \tag{14}$$

Accordingly we use the following dimensionless parameters in order to characterize the deviation from Gaussian statistics:

$$\frac{v/D}{s} \;\text{ and }\; 4\frac{h/v}{s} \qquad \text{[both equal unity for Gaussian statistics]} \tag{15}$$

In Fig. 3 these two parameters are imaged as a function of Δ and σ for representative values of s. For small values of s, the ratio v/Ds is quite independent of Δ, while as s increases, the effect of Δ becomes more significant.

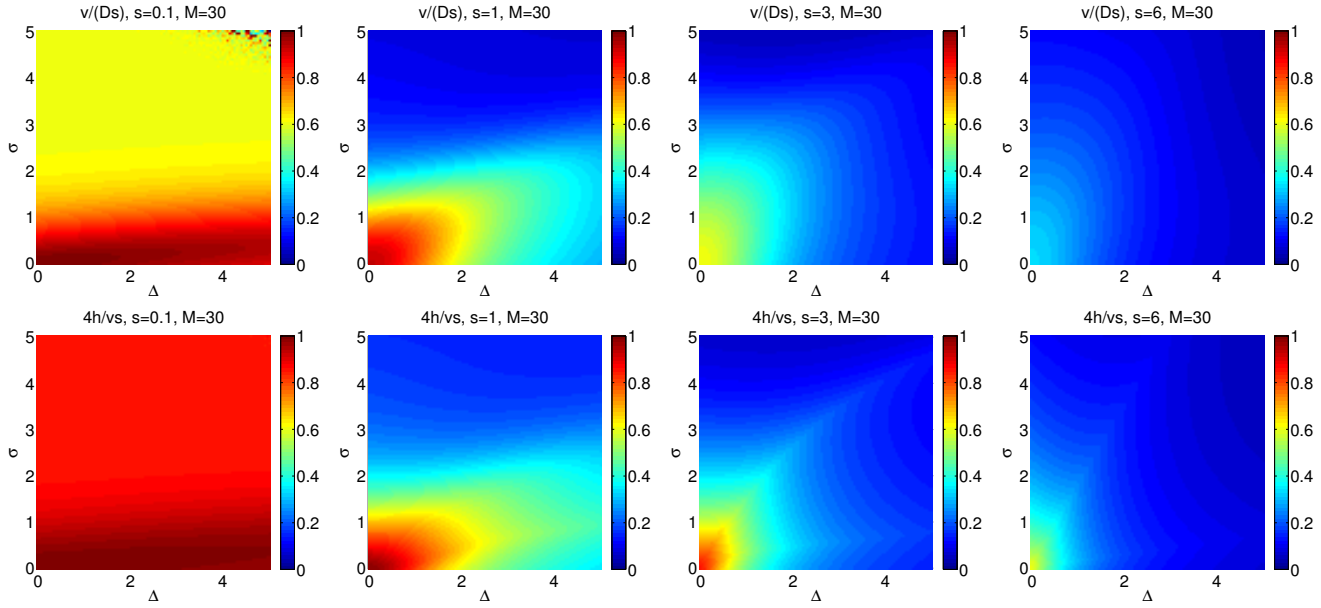

FIG. 3: The measures for Gaussian statistics are imaged as a function of Δ and σ for representative values of s.

III. THE SHAPE OF $g(\lambda)$

Given v and D, the parabola is defined and it has a peak value of

$$h[\text{Gaussian}] = v^2/4D \tag{16}$$

In general, however, the peak value is different In the Poisson limit, $s \to \infty$, it is easy to show that the minimum of the generating function is given by the smallest transition rate

$$h[\text{Poisson}] = \min[\overrightarrow{w}_n] \tag{17}$$

If there is no disorder, the cumulant generating function can easily be shown to be

$$g(\lambda) \;=\; \overrightarrow{w}\mathrm{e}^{\lambda a} + \overleftarrow{w}\mathrm{e}^{-\lambda a} - (\overleftarrow{w} + \overrightarrow{w}) \tag{18}$$

For zero bias ($s = 0$) this reduces to

$$g(\lambda) \;=\; 2w(\cosh(\lambda a) - 1) \tag{19}$$

We emphasize that even in the absence of disorder and zero bias, the distribution is not Gaussian, due to having a discrete lattice (Fig. 4).

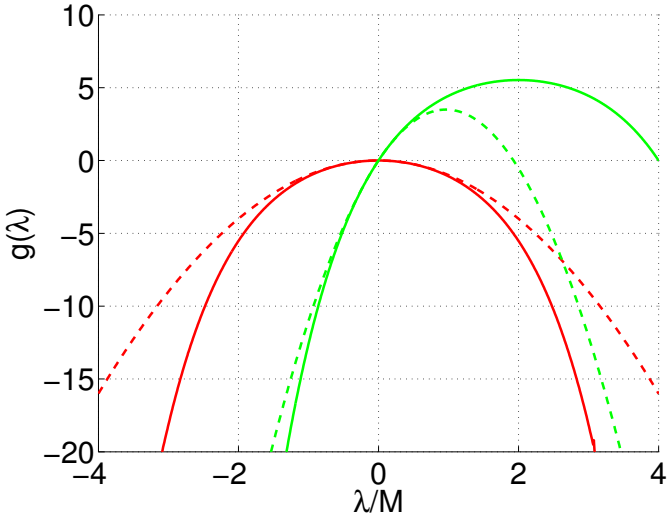

FIG. 4: The effect of discretization on the generating function $g(\lambda)$.The red line is for $s = 0$, the green line is for $s = 4$. In both cases $\sigma = 0$ and $\Delta = 0$. Dashed lines are parabolas $v\lambda - D\lambda^2$.

IV. GAUGING OUT THE FIELD DISORDER

The field disorder can be removed from the off diagonals by a simple gauge transformation. After the gauge transformation we have

$$\tilde{W} = -\text{diag}[\Gamma_{n-1}] + \text{offdiag}[g_n e^{\pm s/2}] \tag{20}$$

where $\Gamma_n = -(\overrightarrow{w}_{n+1} + \overleftarrow{w}_n)$ due to conservatively, depends on σ and Δ, but the off diagonals are now free of σ. So, surprisingly, the field disorder is diagonal disorder. Note that in the gauged basis, $\tilde{W}$ is not conservative.

V. THE SLIDING TRANSITION

It is know from Derrida that there is a sliding transition where the velocity exhibits a phase transition from $v = 0$ to finite v. This transition occurs at the value of s for which

$$\left\langle \left(\frac{\overrightarrow{w}}{\overleftarrow{w}} \right) \right\rangle = 1 \tag{21}$$

This transition has fingerprints on the spectrum. Below we plot the number of complex eigenvalues as a function of s. An interesting observation is that not all of the eigenvalues become complex. This effect is due to the conservativity, as we shall explain. We call this "complexity saturation".

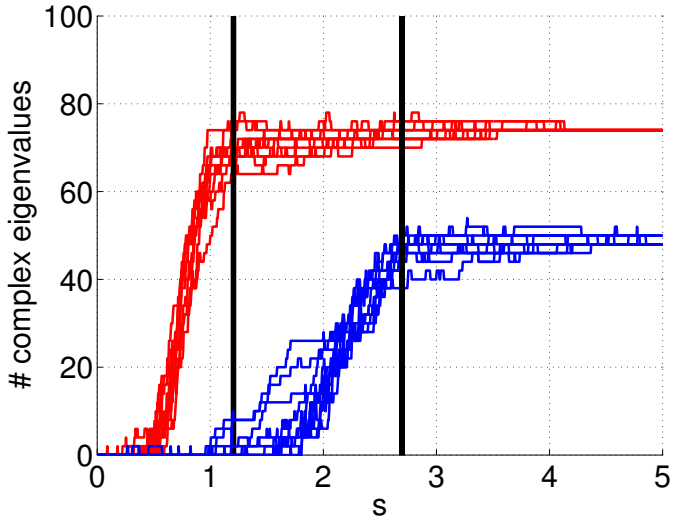

FIG. 5: Number of complex eigenvalues vs. s for $N = 100$ sites. Each red line corresponds to a different realisation of field disorder with $\sigma = 3$(red) and $\sigma = 5$ (blue), but $\Delta = 0$. The black vertical lines are at values of s at which the sliding transition occurs.

VI. THE SPECTRUM

We define $\Gamma^{(s)} \equiv -\tilde{W}$,

$$\Gamma^{(s)} \;=\; \mathrm{diag}[\gamma] + \mathrm{offdiag}[-ge^{\pm s/2}] \tag{22}$$

The eigenvalues of Γ are solutions to the equation

$$\det(\Gamma^{(s)} - \varepsilon\mathcal{I}) = 0 \tag{23}$$

This determinant can be calculated using a formula due to Molinari. We define the matrix [check the indices... go over the convention with Doron]

$$T^{(n)} \;=\; \begin{pmatrix} \gamma_{n-1} - \varepsilon & -g_n^2 \\ 1 & 0 \end{pmatrix} \tag{24}$$

where γ_n is the escape rate of the n^{th} site. And obtain

$$\det(\Gamma^{(s)} - \varepsilon\mathcal{I}) \;=\; \mathrm{tr}\left[\prod_{n=1}^{N} T^{(n)}\right] - \left[\prod_{n=1}^{N} g_n\right] 2\cosh\left(\frac{sN}{2}\right) = \tag{25}$$

$$= \det\Gamma^{(0)} - \left[\prod_{n=1}^{N} g_n\right] 2\left(\cosh\left(\frac{sN}{2}\right) - 1\right) = \tag{26}$$

$$= \prod_{j=1}^{N}(\varepsilon - \varepsilon_j^{(0)}) - \left[\prod_{n=1}^{N} g_n\right] 2\left(\cosh\left(\frac{sN}{2}\right) - 1\right) \tag{27}$$

Where $\Gamma^{(0)}$ means that on the off diagonals we set $s = 0$, but do not touch the diagonal. So, in fact the eigenvalues $\varepsilon_j^{(0)}$ depend on s. According to Thouless, for the corresponding Hermitian problem

$$e^{\kappa N} = \prod_{j=1}^{N} \frac{(\varepsilon - \varepsilon_j^{(0)})}{w} \tag{28}$$

where $\kappa = 1/\xi$ is the inverse localization length. In order for ϵ to be an eigenvalue of $\Gamma^{(s)}$, it must be a solution to the equation

$$\sum_{j=1}^{N} \left[\log \left(\frac{\varepsilon - \varepsilon_j}{w_j} \right) \right] = \log \left[2 \left(\cosh \left(\frac{sN}{2} \right) - 1 \right) \right] \tag{29}$$

The left hand side oscillates and the right hand side is a straight line. The number of intersections is the number of real eigenvalues, see Fig. 5. In the nonconservative case, one can raise the horizontal line (right hand side of equation (29))

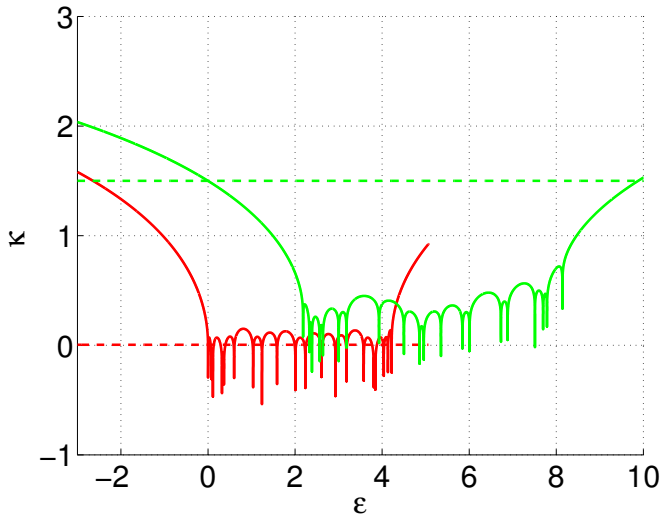

FIG. 6: The inverse localization length κ vs. ε for $N = 20$, $\sigma = 1$, $s = 0.1$ (red) and $s = 3$ (green). The dashed lines are the right hand side of equation (29). For small s all eigenvalues are real, for large s, only two are real. Notice that $\varepsilon = 0$ is always a solution.

by increasing s, thus making the entire spectrum complex. For a conservative matrix, however, κ is also a function of s, so increasing s raises κ more or less at the same rate, leaving the number of intersections unchanged, se Fig. 7.

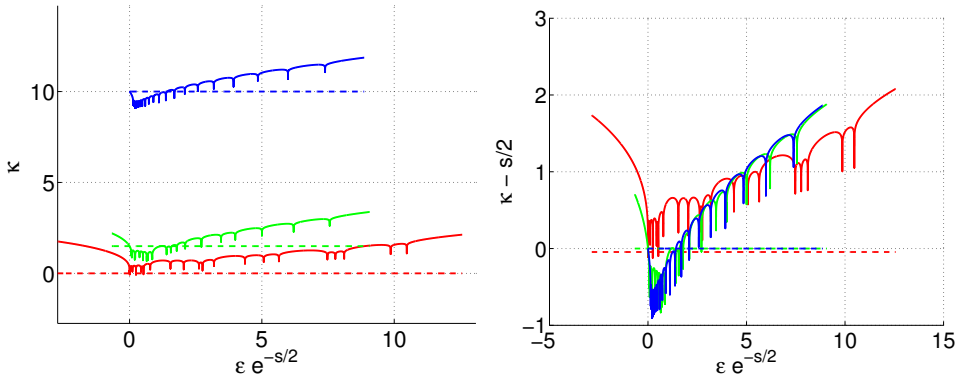

FIG. 7: The inverse localization length κ vs. ε for $N = 20$, $\sigma = 4$, $s = 0.1$ (red), $s = 3$ (green) and $s = 20$ (blue). The dashed lines are the right hand side of equation (29). In the right panel, the horizontal axis is $\varepsilon e^{-s/2}$ and the vertical axis is $\kappa - s/2$.