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# OPTIMAL DATA SPLITTING IN DISTRIBUTED OPTIMIZATION FOR MACHINE LEARNING

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TECHNICAL REPORT

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## ABSTRACT

The distributed optimization problem has become increasingly relevant recently. It has a lot of advantages such as processing a large amount of data in less time compared to undistributed methods. However, most distributed approaches suffer from a significant bottleneck - the cost of communications. Therefore, a large amount of research has recently been directed at reducing these costs. One such approach uses function similarity. A method has now been developed that produces lower estimates. But this method does not take into account the various power of network devices and the different relationship between communication time and server capacity. We consider this problem and the objective of this study is to achieve an optimal ratio of distributed data between the server and local machines for any communication costs. The running times of the network are compared between uniform and optimal distributions. The superior theoretical performance of our solutions is experimentally validated.

## 1 Introduction

### 1.1 Distributed optimization

We consider optimization problems of the following form:

$$\min_{x \in \mathbb{R}^d} f(x) = \frac{1}{n} \sum_{i=1}^n f_i(x), \quad (1)$$

$x \in \mathbb{R}^d$  collects the parameters of a statistical model to be trained,  $n$  is the number of devices/nodes, and  $f_i$  represents the convex loss-function of agent  $i$ , which is unknown to the other agents. This is a direct formulation of a distributed optimization problem. Nowadays, there are some reasons to consider this.

To achieve the best results in modern machine learning and minimization tasks, researchers and practitioners face various challenges. Training modern machine learning models remains an extremely challenging task, also because models are trained on increasingly large datasets. Having more data in

the dataset increases the robustness and generalizability of the trained model. In this case, the data is typically processed using a network of devices, i.e., collected in a distributed manner and stored in edge nodes of the network, such as in federated learning. This has been discussed in [17].

Several solution methods have been proposed to solve (1). The prototype approach involves interleaving edge devices calculations (nodes  $i = 2, \dots, n$ ) with communications to and from the server ( $i = 1$ ), which maintains and updates the authoritative copy of the optimization variables, eventually producing the final solution estimate. In distributed learning of complex models, the communication overhead between devices in the network often becomes a bottleneck. Such a problem makes it necessary to develop more efficient distributed learning methods, some of which have been described in [7, 3, 14, 4].

## 1.2 Distributed optimization under similarity

It's trendy today in machine learning to use moment-based methods. One of the methods for solving a distributed optimisation problem is the application of Nesterov acceleration [10], which is an optimal method for non-distributed optimization problems. This method can be transferred to distributed networks as follows. Consider  $f = f_1 + \frac{\delta}{2}\|x\|_2^2$ , where  $f_1$  is the server loss function. Then the number of communications will be equal to the number of iterations. Local computations are performed using Nesterov acceleration. In this case, we obtain optimal estimates for local computations -  $\sqrt{\kappa}$ ,  $\kappa = L/\mu$ . In case  $\kappa$  is small, this approach is acceptable. However, for ill-conditioned functions with a large  $\kappa$ , the polynomial dependence on  $\kappa$  may be unsatisfactory, due to the high cost of communications. This is often the case for many empirical risk minimization (ERM) problems where the optimal regularization parameter for test predictive performance is very small.

To further improve communication complexity, we can exploit the additional structure typically found in ERM problems, known as function similarity [1, 13, 9]. One can define it as the difference of function gradients, i.e.,  $\|\nabla f_i(x) - \nabla f_j(x)\| < \delta \ \forall x$ . But this approach is not "natural", since if the problem is not bounded, such a  $\delta$  cannot exist. Consider for example a quadratic problem:  $\nexists \delta : \|(A_i - A_j)x\| < \delta \text{ if } x \rightarrow \infty$ . Therefore, we will consider a different approach: hessian similarity. Specifically, for all  $x$  in a suitable domain of interest and all  $i \neq j$ ;  $i, j \in \{1, \dots, n\}$ , the difference between the Hessian matrices of local losses, denoted by  $\|\nabla^2 f_i(x) - \nabla^2 f_j(x)\|$ , is bounded by  $\delta$ , where  $\delta > 0$  measures the degree of similarity. Under this assumption, we can estimate  $\delta \sim \mathcal{O}(1/\sqrt{N})$ , where  $N$  is the sample size per device. This approach was used for the first time in [13]. After that, lower estimates were proved for this problem in [1], where communication costs are proportional to  $\sqrt{\delta/\mu}$ . Then for a long time researchers tried to find methods that would reach these estimates. Algorithms such as [16, 15, 12, 5, 2] have been obtained. In 2022, it was possible to obtain the optimal method which is described in [8]. But all of them do not take into account in any way the fact that devices in the network may have different capacities, i.e., process a unit of information in different times, and relationship between communication time and server capacity also can be different. In this paper, we reply the question:

*Can we find such a distribution of data among the devices in the network to reduce the actual running time of the optimal algorithm [8] for any communication costs ?*

## 1.3 Contributions

In general, our contribution is as follows:

- **Generalization of the computation model.** We build a general model for computing time in networks under distributed optimization. The model is based on the optimal algorithm [8] and takes into account the different capacity of edge devices, various communication costs.
- **Comprehensive analysis.** We pay special attention to the limiting cases and obtain results in them. The case where communications are too expensive is not of practical interest as the whole idea of distributed learning is lost, but the case of inexpensive communications

(not so expensive that the communication takes longer than processing all data by just one device) is of great interest.

- **Different techniques for obtaining a solution.** We obtain results in different cases, including for different estimates of  $\delta$ . We use different techniques: Cardano's formula, upper estimates in limiting cases, finding the zero of the function using the simplest numerical methods.

We also conducted experiments confirming that with the obtained distribution it takes less time to solve the selected problem.

## 2 Problem Statement

To achieve lower communication and local gradient complexity, we can refer to Algorithm 1 from [8]. For this purpose, the function  $f$  needs to be transformed into the following form:

$$f(x) = \underbrace{f_1(x)}_{q(x)} + \underbrace{\frac{1}{n} \sum_{i=1}^n [f_i(x) - f_1(x)]}_{p(x)}. \quad (2)$$

Here,  $r$  is assumed to be convex and decomposed as the sum of a smooth, potentially non-convex function  $p$  and a smooth convex function  $q$ . For our purposes, let us write down a more general statement, taking  $p = f - f_1$ . Then the algorithm will be rewritten in the following more general form:

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### Algorithm 1 Accelerated Extragradient

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- 1: **Input:**  $x^0 = x_r^0 \in \mathbb{R}^d$
  - 2: **Parameters:**  $\tau \in (0, 1), \eta, \theta, \alpha > 0, K \in \{1, 2, \dots\}$
  - 3: **for**  $k = 0, 1, 2, \dots, K - 1$  **do**
  - 4:      $x_g^k = \tau x^k + (1 - \tau)x_r^k$
  - 5:      $x_r^{k+1} \approx \arg \min_{x \in \mathbb{R}^d} [A_\theta^k(x) := \langle \nabla(f - f_1)(x_g^k), x - x_g^k \rangle + \frac{1}{2\theta} \|x - x_g^k\|^2 + f_1(x)]$
  - 6:      $x^{k+1} = x^k + \eta\alpha(x_r^{k+1} - x^k) - \eta\nabla f(x_r^{k+1})$
  - 7: **end for**
  - 8: **Output:**  $x^K$
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We need to analyse the work of this algorithm, namely, find out how many operations this algorithm performs per iteration. In line 5, when searching for  $\arg \min$ , one local computation is performed, followed by one communication to transmit these results, a computation on the server, and additional computations that also take place there. Then in line 6 there is one local computation, one communication, and one computation on the server. Let us introduce the following notations:  $\tau_i$  - time of one local computation on the  $i$ -th device,  $K$  - number of iterations,  $\tau_{comm}$  - time of one communication,  $k_{some}$  - additional computation of the central node,  $n$  - number of nodes in the network. Taking this into account, the total running time of the algorithm can be written as:

$$T_{sum} = 2 \cdot \max(\tau_1, \tau_2, \dots, \tau_n) \cdot K + 2 \cdot K \cdot \tau_{comm} + \tau_1 \cdot k_{some}. \quad (3)$$

Our task is to minimize the time  $T_{sum}$ . Let us represent the time  $\tau_i$  as  $\tau_i = \tau_i^{loc} \cdot b_i$ , where  $\tau_i^{loc}$  is capacity, i.e., the time spent by the  $i$ -th device to process a unit of information submitted to its input, and  $b_i$  is the size of dataset submitted to the  $i$ -th device.  $b_i$  must satisfy the following constraints:  $\sum_{i=1}^n b_i = N$ , where  $N$  is the size of the whole dataset,  $\delta = \frac{L}{\sqrt{b_i}}$  or  $\delta = \frac{L}{b_i}$  (this estimate is given in

[8]). We obtained the following optimization problem:

$$\min_{\sum_{i=1}^n b_i = N; \delta = \frac{L}{b_1 \gamma}} [2 \cdot \max(\tau_1^{loc} \cdot b_1, \tau_2^{loc} \cdot b_2, \dots, \tau_n^{loc} \cdot b_n) \cdot K + 2 \cdot K \cdot \tau_{comm} + \tau_1 \cdot k_{some}], \gamma \in \{\frac{1}{2}, 1\}. \quad (4)$$

### 3 How to solve (4)

#### 3.1 The primary problem of minimization

In [8] the estimates of  $K$  and  $k_{some}$  are found, namely:

$$2 \cdot K = \mathcal{O}(\max\{1, \sqrt{\frac{L_p}{\mu}}\} \log(\frac{1}{\varepsilon})), \tau_1 \cdot k_{some} = \mathcal{O}(\max\{1, \sqrt{\frac{L_q}{L_p}}, \sqrt{\frac{L_p}{\mu}}, \sqrt{\frac{L_q}{\mu}}\} \log(\frac{1}{\varepsilon})).$$

The value of  $\tau_{comm}$  will be determined later.

Thus, (4) is reduced to:

$$\begin{aligned} \min_{\sum_{i=1}^n b_i = N; \delta = \frac{L}{b_1 \gamma}} & [(\max(\tau_1^{loc} \cdot b_1, \tau_2^{loc} \cdot b_2, \dots, \tau_n^{loc} \cdot b_n) + \tau_{comm}) \cdot \mathcal{O}(\max\{1, \sqrt{\frac{L_p}{\mu}} \log(\frac{1}{\varepsilon})\}) \\ & + \mathcal{O}(\max\{1, \sqrt{\frac{L_q}{L_p}}, \sqrt{\frac{L_p}{\mu}}, \sqrt{\frac{L_q}{\mu}}\} \log(\frac{1}{\varepsilon}))], \gamma \in \{\frac{1}{2}, 1\}. \end{aligned} \quad (5)$$

#### 3.2 Auxiliary problem

Consider an auxiliary problem:

$$\min_{\sum_{i=2}^n b_i = N} [\max(\tau_2^{loc} \cdot b_2, \tau_3^{loc} \cdot b_3, \dots, \tau_n^{loc} \cdot b_n)]. \quad (6)$$

**Lemma 1.** *The solution of problem (6) is  $\vec{b}$  satisfying  $\tau_2^{loc} \cdot b_2 = \tau_3^{loc} \cdot b_3 = \dots = \tau_n^{loc} \cdot b_n$ .*

*Proof.* Without loss of generality, let us assume fixed values for  $\tau_2^{loc} \leq \tau_3^{loc} \leq \dots \leq \tau_n^{loc}$ .

Then let us arbitrarily choose  $b_2 \geq b_3 \geq \dots \geq b_n$ .

This is indeed the case, otherwise we would have a situation where  $\exists i \neq j : i, j \in \{2, \dots, n\} : \max(\tau_i^{loc} \cdot b_i, \tau_j^{loc} \cdot b_j) > \max(\tau_i^{loc} \cdot b_j, \tau_j^{loc} \cdot b_i)$ , and therefore the distribution would be suboptimal.

Our goal is to minimize the function  $g(b) = \max(\tau_2^{loc} \cdot b_2, \tau_3^{loc} \cdot b_3, \dots, \tau_n^{loc} \cdot b_n)$ .

Suppose that there exists a distribution such that  $\exists i \in \{2, \dots, n\} : g(\vec{b}^0) = \tau_i^{loc} \cdot b_i^0$  is the minimum,

and  $\forall j : j \geq 2, j \neq i \hookrightarrow \tau_i^{loc} \cdot b_i^0 > \tau_j^{loc} \cdot b_j^0$ .

It follows that  $b_i^0 > \frac{\tau_j^{loc}}{\tau_i^{loc}} b_j^0 > \frac{\tau_{j_1}^{loc}}{\tau_i^{loc}} b_{j_1}^0 > \dots > \frac{\tau_{j_k}^{loc}}{\tau_i^{loc}} b_{j_k}^0$ .

Then, considering  $\sum_{i=2}^n b_i = N \hookrightarrow b_i^0 + \frac{\tau_j^{loc}}{\tau_i^{loc}} b_j^0 + \frac{\tau_{j_1}^{loc}}{\tau_i^{loc}} b_{j_1}^0 + \dots + \frac{\tau_{j_k}^{loc}}{\tau_i^{loc}} b_{j_k}^0 > N$ , we obtain

$$b_i^0 > N(1 + \tau_i^{loc} \sum_{\substack{j=2 \\ j \neq i}}^n \frac{1}{\tau_j^{loc}})^{-1}.$$

Next, let us consider  $b_i = N(1 + \tau_i^{loc} \sum_{\substack{j=2 \\ j \neq i}}^n \frac{1}{\tau_j^{loc}})^{-1}$ ,  $b_j = \frac{\tau_i^{loc}}{\tau_j^{loc}} \cdot b_i \forall j \in \{2, \dots, n\}$ . This distribution

yields a minimum of  $g(\vec{b}) = \tau_i^{loc} \cdot b_i = \tau_j^{loc} \cdot b_j \forall j \in \{2, \dots, n\}$ , and  $g(\vec{b}) < g(\vec{b}^0)$ . This contradicts the assumption of minimality.

Thus, for the distribution that minimizes the function  $g(b) = \max(\tau_2^{loc} \cdot b_2, \tau_3^{loc} \cdot b_3, \dots, \tau_n^{loc} \cdot b_n)$ , it holds that  $\tau_2^{loc} \cdot b_2 = \tau_3^{loc} \cdot b_3 = \dots = \tau_n^{loc} \cdot b_n$ .

□

Let us return to the problem (5). In addition to the minimum expression already studied in the problem (6), there are additional terms in the problem (5). Note that  $L_q = L, L_p = \delta = \frac{L}{\sqrt{b_1}}$  or  $L_p = \delta = \frac{L}{b_1}$  (this estimate is given in [8]). They depend on the value of  $b_1$ , but do not depend on  $b_i, i \in \overline{2, n}$ . From this and Lemma 1, it follows that in the original problem (5), the data sharing between the 2nd, 3rd, and subsequent devices should be proportional. Thus, the problem (5) is reduced to a new problem with additional constraints:

$$\begin{aligned} \min_{\substack{\sum_{i=1}^n b_i = N; \delta = \frac{L}{b_1} \tau; \\ \tau_2^{loc} \cdot b_2 = \dots = \tau_n^{loc} \cdot b_n}} & [(\max(\tau_1^{loc} \cdot b_1, \tau_2^{loc} \cdot b_2, \dots, \tau_n^{loc} \cdot b_n) + \tau_{comm}) \cdot \mathcal{O}(\max\{1, \sqrt{\frac{L_p}{\mu}} \log(\frac{1}{\varepsilon})\}) \quad (7) \\ & + \mathcal{O}(\max\{1, \sqrt{\frac{L_q}{L_p}}, \sqrt{\frac{L_p}{\mu}}, \sqrt{\frac{L_q}{\mu}} \log(\frac{1}{\varepsilon})\}), \gamma \in \{\frac{1}{2}, 1\}. \end{aligned}$$

### 3.3 Define the final minimization problem

It follows from Lemma 1 that  $b_i \tau_i^{loc} = \text{const} \forall i \in \overline{2, n}$ . Therefore,

$$N - b_1 = \sum_{i=2}^n b_i = \sum_{i=2}^n \frac{\tau_2^{loc} \cdot b_2}{\tau_i^{loc}} = \tau_2^{loc} \cdot b_2 \cdot \sum_{i=2}^n \frac{1}{\tau_i^{loc}} \Rightarrow b_2 = \frac{N - b_1}{\tau_2^{loc}} \left( \sum_{i=2}^n \frac{1}{\tau_i^{loc}} \right)^{-1}.$$

As mentioned above, we will consider the case of  $\delta = \frac{L}{b_1}$  and case of  $\delta = \frac{L}{\sqrt{b_1}}$ .

#### 3.3.1 Case of $\delta = \frac{L}{b_1}$

There the following relations are fulfilled:

$$L_p = \delta, L_q = L, \mu \leq \delta \leq L \Rightarrow \begin{cases} 2 \cdot K = \mathcal{O}(\sqrt{\frac{L_p}{\mu}} \log(\frac{1}{\varepsilon})) = \mathcal{O}(\sqrt{\frac{L}{\mu b_1}} \log(\frac{1}{\varepsilon})) \\ \tau_1 \cdot k_{some} = \mathcal{O}(\sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon})) \end{cases}.$$

Substituting the estimates into (7) the problem will take the following form:

$$\min_{\sum_{i=1}^n b_i = N} [(\max\{\tau_1^{loc} \cdot b_1, \tau_2^{loc} \cdot b_2\} + \tau_{comm}) \cdot \mathcal{O}(\sqrt{\frac{L}{\mu b_1}} \log(\frac{1}{\varepsilon})) + \tau_1^{loc} \cdot b_1 \cdot \mathcal{O}(\sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}))].$$

As a result, leaving the only variable  $b_1$  in the function we pass to the final form of minimization problem:

$$\begin{aligned} \min_{0 < b_1 \leq N} & [(\max\{\tau_1^{loc} \cdot b_1; (N - b_1) \cdot (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1}\} + \tau_{comm}) \cdot \mathcal{O}(\sqrt{\frac{L}{\mu b_1}} \log(\frac{1}{\varepsilon})) \quad (8) \\ & + \tau_1^{loc} \cdot b_1 \cdot \mathcal{O}(\sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}))]. \end{aligned}$$

Let us investigate the problem further. To do this, find the point at which the expressions under the maximum coincide.

$$b_1^0 \cdot (\tau_1^{loc} + (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1}) = N(\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1} \Rightarrow b_1^0 = \frac{N(\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1}}{\tau_1^{loc} + (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1}}.$$

Thus, we obtained two half-intervals, on each of which we can formulate a different minimization problem:

$$\begin{cases} (a) & 0 < b_1 \leq b_1^0 \Rightarrow \max\{\tau_1^{loc} \cdot b_1; (N - b_1) \cdot (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1}\} = (N - b_1) \cdot (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1} \\ (b) & b_1^0 < b_1 \leq N \Rightarrow \max\{\tau_1^{loc} \cdot b_1; (N - b_1) \cdot (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1}\} = \tau_1^{loc} \cdot b_1 \end{cases} \quad (9)$$

We construct functions of one variable  $\mathcal{F}_1(b_1), \mathcal{F}_2(b_1)$  on the corresponding half-intervals that need to be minimized according to problem (8). Besides immediately find their derivatives for further analysis.

$$(a) : \mathcal{F}_1(b_1) = [N(\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1} + \tau_{comm}] \cdot c_1 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) b_1^{-\frac{1}{2}} - c_1 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1} b_1^{\frac{1}{2}} + \tau_1^{loc} \cdot c_2 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) b_1$$

$$(b) : \mathcal{F}_2(b_1) = \tau_{comm} \cdot c_1 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) b_1^{-\frac{1}{2}} + c_1 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) \tau_1^{loc} b_1^{\frac{1}{2}} + \tau_1^{loc} \cdot c_2 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) b_1$$

$$(a) : \mathcal{F}'_1(b_1) = -\frac{1}{2} c_1 b_1^{-\frac{3}{2}} [N(\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1} + \tau_{comm}] \cdot \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) - \frac{1}{2} c_1 b_1^{-\frac{1}{2}} \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1} + \tau_1^{loc} \cdot c_2 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon})$$

$$(b) : \mathcal{F}'_2(b_1) = -\frac{1}{2} c_1 b_1^{-\frac{3}{2}} \tau_{comm} \cdot \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) + \frac{1}{2} c_1 b_1^{-\frac{1}{2}} \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) \tau_1^{loc} + \tau_1^{loc} \cdot c_2 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon})$$

### 3.3.2 Case of $\delta = \frac{L}{\sqrt{b_1}}$

We will proceed similarly to the previous point. First, let us present the necessary relations in this case.

$$L_p = \delta, L_q = L, \mu \leq \delta \leq L \Rightarrow \begin{cases} 2 \cdot K = \mathcal{O}(\sqrt{\frac{L_p}{\mu}} \log(\frac{1}{\varepsilon})) = \mathcal{O}(\sqrt{\frac{L}{\mu \sqrt{b_1}}} \log(\frac{1}{\varepsilon})) \\ \tau_1 \cdot k_{some} = \mathcal{O}(\sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon})) \end{cases}.$$

Substituting these relations into (7) we obtain

$$\min_{\sum_{i=1}^n b_i = N} [(\max\{\tau_1^{loc} \cdot b_1, \tau_2^{loc} \cdot b_2\} + \tau_{comm}) \cdot \mathcal{O}(\sqrt{\frac{L}{\mu \sqrt{b_1}}} \log(\frac{1}{\varepsilon})) + \tau_1^{loc} \cdot b_1 \cdot \mathcal{O}(\sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}))].$$

Again, getting rid of all variables except  $b_1$  we write the final minimization problem in this case

$$\begin{aligned} \min_{0 < b_1 \leq N} [ & (\max\{\tau_1^{loc} \cdot b_1; (N - b_1) \cdot (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1}\} + \tau_{comm}) \cdot \mathcal{O}(\sqrt{\frac{L}{\mu \sqrt{b_1}}} \log(\frac{1}{\varepsilon})) \\ & + \tau_1^{loc} \cdot b_1 \cdot \mathcal{O}(\sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}))]. \end{aligned} \quad (10)$$

Similarly, we select the point  $b_1^0$ , it turns out to be the same as in the previous paragraph. After we can obtained two half-intervals, on each of which we can formulate a different minimization problem:

$$\begin{cases} (a) \ 0 < b_1 \leq b_1^0 \Rightarrow \max\{\tau_1^{loc} \cdot b_1; (N - b_1) \cdot (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1}\} = (N - b_1) \cdot (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1} \\ (b) \ b_1^0 < b_1 \leq N \Rightarrow \max\{\tau_1^{loc} \cdot b_1; (N - b_1) \cdot (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1}\} = \tau_1^{loc} \cdot b_1 \end{cases} \quad (11)$$

We construct functions of one variable  $\mathcal{F}_1(b_1), \mathcal{F}_2(b_1)$  on the corresponding half-intervals that need to be minimized according to problem (10). Besides immediately find their derivatives for further analysis.

$$\begin{aligned} (a) : \mathcal{F}_1(b_1) &= [N(\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1} + \tau_{comm}] \cdot c_1 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) b_1^{-\frac{1}{4}} - c_1 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1} b_1^{\frac{3}{4}} + \\ &\tau_1^{loc} \cdot c_2 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) b_1 \\ (b) : \mathcal{F}_2(b_1) &= \tau_{comm} \cdot c_1 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) b_1^{-\frac{1}{4}} + c_1 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) \tau_1^{loc} b_1^{\frac{3}{4}} + \tau_1^{loc} \cdot c_2 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) b_1 \\ (a) : \mathcal{F}'_1(b_1) &= -\frac{1}{4} c_1 b_1^{-\frac{5}{4}} [N(\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1} + \tau_{comm}] \cdot \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) - \frac{3}{4} c_1 b_1^{-\frac{1}{4}} \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1} + \\ &\tau_1^{loc} \cdot c_2 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) \\ (b) : \mathcal{F}'_2(b_1) &= -\frac{1}{4} c_1 b_1^{-\frac{5}{4}} \tau_{comm} \cdot \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) + \frac{3}{4} c_1 b_1^{-\frac{1}{4}} \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) \tau_1^{loc} + \tau_1^{loc} \cdot c_2 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) \end{aligned}$$

### 3.4 Final solution in limiting cases

#### 3.4.1 Case of $\delta = \frac{L}{b_1}$

Our goal is to find the minimum of the already obtained functions  $\mathcal{F}_1(b_1), \mathcal{F}_2(b_1)$ . To do this, we will look for the zeros of  $\mathcal{F}'_1(b_1), \mathcal{F}'_2(b_1)$ . Here we obtain the cubic equation. To solve it, we can use the Cardano formula.

Consider the equation  $ax^{-\frac{1}{2}} + bx^{-\frac{3}{2}} + c = 0$ ,

where in cases (a) :  $0 < b_1 \leq b_1^0$  and (b) :  $b_1^0 < b_1 \leq N$  we assume:

$$\begin{cases} (a) : a = \frac{1}{2} c_1 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1}; \ b = -\frac{1}{2} c_1 [N(\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1} + \tau_{comm}] \cdot \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}); \\ \quad c = \tau_1^{loc} \cdot c_2 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) \\ (b) : a = \frac{1}{2} c_1 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) \tau_1^{loc}; \ b = -\frac{1}{2} c_1 \tau_{comm} \cdot \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}); \quad c = \tau_1^{loc} \cdot c_2 \sqrt{\frac{L}{\mu}} \log(\frac{1}{\varepsilon}) \end{cases}$$

Then on the condition that

$$N \geq \frac{a^2}{3c^2} + \frac{\sqrt[3]{2a^6 + 3\sqrt{3}\sqrt{4a^3b^3c^6 + 27b^4c^8 + 18a^3bc^2 + 27b^2c^4}}}{3\sqrt[3]{2c^2}} - \frac{\sqrt[3]{2}(-a^4 - 6abc^2)}{3c^2\sqrt[3]{2a^6 + 3\sqrt{3}\sqrt{4a^3b^3c^6 + 27b^4c^8 + 18a^3bc^2 + 27b^2c^4}}},$$

we get a solution:

$$x = \frac{a^2}{3c^2} + \frac{\sqrt[3]{2a^6 + 3\sqrt{3}\sqrt{4a^3b^3c^6 + 27b^4c^8 + 18a^3bc^2 + 27b^2c^4}}}{3\sqrt[3]{2}c^2} - \frac{\sqrt[3]{2}(-a^4 - 6abc^2)}{3c^2\sqrt[3]{2a^6 + 3\sqrt{3}\sqrt{4a^3b^3c^6 + 27b^4c^8 + 18a^3bc^2 + 27b^2c^4}}}.$$

Hence the desired solution is trivially obtained. Since we have obtained one value of  $b_1$  on each of the half-intervals, which is the minimum of the function on its, so by choosing the one on which the function is smaller, we obtain the desired  $b_1$ .

### 3.4.2 Case of $\delta = \frac{L}{\sqrt{b_1}}$

Proceed similarly to 3.4.1 does not work, since we cannot write out the solution of these equations in analytic form due to their powers. Therefore, let us consider the following limiting cases:

1.  $\forall i \hookrightarrow \tau_{comm} \ll \tau_i^{loc}$ ;
2.  $\forall i \hookrightarrow \tau_{comm} \gg \tau_i^{loc}, \forall i \neq j \hookrightarrow \tau_i^{loc} = \tau_j^{loc}$ .

Establish  $\alpha = c_1 \cdot \sqrt{\frac{L}{\mu}} \cdot \log(\frac{1}{\varepsilon}), \beta = c_2 \cdot \sqrt{\frac{L}{\mu}} \cdot \log(\frac{1}{\varepsilon})$ .

**Consider case 1:**

a)  $0 < b_1 \leq b_1^0$

$$\mathcal{F}_1(b_1) = [N(\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1} + \tau_{comm}] \cdot \alpha b_1^{-\frac{1}{4}} - \alpha(\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1} b_1^{\frac{3}{4}} + \tau_1^{loc} \cdot \beta b_1$$

Consider

$$\tau_1^{loc} \leq \tau_2^{loc} \leq \dots \leq \tau_n^{loc}. \quad (12)$$

Make the following estimate:

$$\begin{aligned} (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1} &= \frac{1}{\frac{1}{\tau_1^{loc}} + \dots + \frac{1}{\tau_n^{loc}}} = \\ &= \frac{\tau_2^{loc} \cdot \dots \cdot \tau_n^{loc}}{\tau_3^{loc} \cdot \dots \cdot \tau_n^{loc} + \tau_2^{loc} \cdot \tau_4^{loc} \cdot \dots \cdot \tau_n^{loc} + \dots + \tau_2^{loc} \cdot \dots \cdot \tau_{n-1}^{loc}} \stackrel{(12)}{\geq} \frac{\tau_2^{loc}}{n-1} \gg \tau_{comm}. \end{aligned} \quad (13)$$

Given the estimate (13), the functions  $\mathcal{F}_1(b_1), \mathcal{F}'_1(b_1)'$  are transformed as follows:

$$\begin{aligned} \mathcal{F}_1(b_1) &= \alpha(\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1} \cdot b_1^{-\frac{1}{4}} (N - b_1) + \tau_1^{loc} \beta \cdot b_1 \\ \mathcal{F}'_1(b_1) &= \alpha(\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1} \cdot (-\frac{1}{4} b_1^{-\frac{5}{4}} N - \frac{3}{4} b_1^{-\frac{1}{4}}) + \tau_1^{loc} \beta. \end{aligned}$$

We get the equation in the same powers, and so again we cannot write out an analytic solution.

b)  $b_1^0 \leq b_1 \leq N$

$$\mathcal{F}_2(b_1) = \tau_{comm} \cdot \alpha b_1^{-\frac{1}{4}} + \alpha \tau_1^{loc} b_1^{\frac{3}{4}} + \tau_1^{loc} \cdot \beta b_1 = \alpha \cdot b_1^{-\frac{1}{4}} (\tau_{comm} + \tau_1^{loc} b_1) + \beta \cdot \tau_1^{loc} \cdot b_1$$

Make the following estimate

$$\tau_1^{loc} b_1 \stackrel{b_1 \geq b_1^0, (12)}{\geq} \frac{\tau_1^{loc} N \tau_2^{loc}}{\tau_1^{loc} + \frac{\tau_2^{loc}}{n-1}} \geq \frac{\tau_1^{loc} \tau_2^{loc} N}{(n-1)(\tau_1^{loc} + \tau_n^{loc})} \geq \frac{\tau_1^{loc} \tau_2^{loc} N}{2(n-1)\tau_n^{loc}} \gg \quad (14)$$



$$\gg \tau_{comm} \frac{N}{2(n-1)} \gg \tau_{comm}.$$

Then taking into account (14):

$$\begin{aligned} \mathcal{F}_2(b_1) &= \alpha \cdot \tau_1^{loc} \cdot b_1^{\frac{3}{4}} + \beta \tau_1^{loc} \cdot b_1 \\ \mathcal{F}'_2(b_1) &= \frac{3}{4} \alpha \cdot \tau_1^{loc} \cdot b_1^{-\frac{1}{4}} + \beta \cdot \tau_1^{loc} > 0. \end{aligned}$$

Since the derivative of the function is positive, the function is increasing, and therefore the

minimum will be taken at  $b_1 = b_1^0 = \frac{N(\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1}}{\tau_1^{loc} + (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1}}$ . Thus, in the case of small  $\tau_{comm}$

we obtained the following result:

$$b_{1\min} \leq b_1^0 = \frac{N(\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1}}{\tau_1^{loc} + (\sum_{i=2}^n \frac{1}{\tau_i^{loc}})^{-1}}.$$

### Consider case 2:

Establish:  $\tau := t_i^{loc} \forall i \in 1, \dots, n$ .

$$\text{Then } \mathcal{F} = (\max\{\tau b_1; (N - b_1) \frac{\tau}{n-1}\} + \tau_{comm}) \cdot \frac{\alpha}{\sqrt[4]{b_1}} + \tau \beta b_1. \quad (15)$$

Consider the case  $\tau_{comm} = N^2 \tau$ .  $N$  can be considered large, so  $\tau_{comm} \gg N\tau$ . Then:

$$\max\{\tau b_1; (N - b_1) \frac{\tau}{n-1}\} < \tau N \ll \tau_{comm} \Rightarrow \mathcal{F} \approx \frac{\alpha \tau_{comm}}{\sqrt[4]{b_1}} + \beta \tau b_1$$

$$\mathcal{F}'(b_1) = -\frac{\alpha \tau_{comm}}{4b_1 \sqrt[4]{b_1}} + \beta \tau = 0 \Rightarrow b_{1\min}^{\frac{5}{4}} = \frac{\tau_{comm} \alpha}{4\beta \tau} \Rightarrow b_{1\min} = \left(\frac{\tau_{comm} \alpha}{4\beta \tau}\right)^{\frac{4}{5}}.$$

Assuming that the found value  $b_1$  lies on the interval  $(0, N)$ , that is, at  $0 < \left(\frac{\tau_{comm} \alpha}{4\beta \tau}\right)^{\frac{4}{5}} < N$ , it will be the point of minimum function  $\mathcal{F}$ . Then:

$$\mathcal{F}(b_{1\min}) = (\alpha \tau_{comm})^{\frac{4}{5}} \cdot (4\beta \tau)^{\frac{1}{5}} + (\beta \tau)^{\frac{1}{5}} \cdot \left(\frac{\alpha \tau_{comm}}{4}\right)^{\frac{4}{5}} = (\alpha \tau_{comm})^{\frac{4}{5}} \cdot (4\beta \tau)^{\frac{1}{5}} (4^{\frac{1}{5}} + 4^{-\frac{4}{5}}).$$

Otherwise, the minimum will be reached at the right boundary, since at zero we can say that the function is increasing.

Summarizing all of the above in this case, it is worth noting that for very large values of  $N$  the second special case generalizes to the following condition:

$$\forall i \hookrightarrow \tau_{comm} = \mathcal{O}(N^k \tau_i^{loc}), k > 1, \forall i \neq j \hookrightarrow \tau_i^{loc} = \tau_j^{loc} \quad (16)$$

$$\min \mathcal{F}(b_1) = \begin{cases} (\alpha \tau_{comm})^{\frac{4}{5}} \cdot (4\beta \tau)^{\frac{1}{5}} (4^{\frac{1}{5}} + 4^{-\frac{4}{5}}), & 0 < \left(\frac{\tau_{comm} \alpha}{4\beta \tau}\right)^{\frac{4}{5}} < N \\ \frac{\alpha \tau_{comm}}{N} + \beta \tau N, & \left(\frac{\tau_{comm} \alpha}{4\beta \tau}\right)^{\frac{4}{5}} \geq N \end{cases}. \quad (17)$$

### 3.5 Practical solution

Since an analytical solution is not found for all cases, we give a general numerical solution to our problem. In order to determine the minimum of these functions on the respective half-intervals, we will examine points where the derivatives of  $\mathcal{F}'_1(b_1)$  and  $\mathcal{F}'_2(b_1)$  approach zero. It should be noted that, given the nature of these functions, their derivatives can only be zero once on the desired half-interval. Hence, by employing Newton's method [11] for  $\mathcal{F}'_1(b_1)$  and  $\mathcal{F}'_2(b_1)$ , we can locate its zeros. Subsequently, we need to compare the values of the corresponding function at these points with the value at the extreme point of the interval. One of these points will provide the minimum solution, thereby serving as the ultimate solution to the problem (8) and (10).

## 4 Experiments

### 4.1 Description of experiments

For experimental verification of the theoretical results we consider the ridge regression problem:

$$\min_{\omega} \left[ \frac{1}{2N} \|X\omega - y\|_2^2 + \frac{\lambda}{2} \|\omega\|_2^2 \right], \quad (18)$$

where  $\omega$  is the vector of weights of the model,  $\{x_i, y_i\}_{i=1}^N$  is the training dataset, and  $\lambda > 0$  is the regularization parameter. We consider a network with 21 workers simulated on a single-CPU machine. We use dataset from LIBSVM. Value  $\tau_1^{loc} = 1$ , values for other  $i$   $\tau_i^{loc}, i \neq 1$  were taken conditionally and generated uniformly from 3 to 7.  $\tau_{comm}$  were chosen so that  $\frac{\tau_{comm}}{\tau_1^{loc}} = 10^l, l = -6, 12$

We implemented Algorithm 1 in Python 3.9.6 using the iterative OGM-G method from [6] to find the  $\arg \min$  in 1. That's what the original article [8] recommends. After calculating the required number of iterations to achieve a certain accuracy we find the values of constants  $c_1, c_2$ , and, respectively,  $\alpha, \beta$ . With their help, we were able to distribute the data from the dataset to the devices according to the above formulas.

Next, we ran the algorithm and measured the running time on the resulting distribution of data across devices and uniform distribution. The cases of large and small communications were considered.

In the end, two cases were considered:

1.  $\delta = \frac{L}{\sqrt{b_1}}$ ;
2.  $\delta = \frac{L}{b_1}$ .

For case 1, following setups were considered:

1. small communications ( $b_1 = b_1^0$ ) (3.4.2);
2. large communications (3.4.2);
3. search for optimal allocation using Newton's method (3.5).

For case 2, following setups were considered:

1. search for the optimal solution using the Cardano formula (3.4.1);
2. search for optimal allocation using Newton's method (3.5).

For all cases, acceleration was found and plots were shown at 1

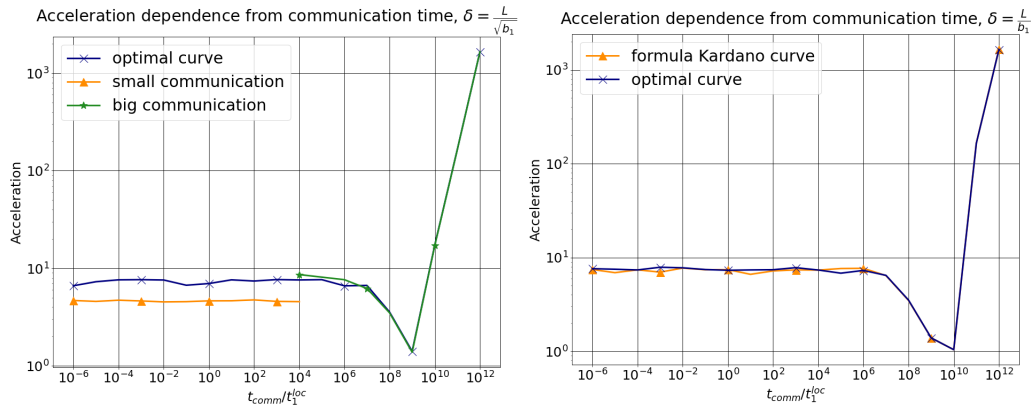


Figure 1: Final results

## 4.2 Analyze

Let us analyze the obtained plots. The formula for the case of large communications and the Cardano formula practically coincided with the optimal solution search. The case of small communications showed worse results. This is explained by the fact that the formula was obtained in rough approximation. But if we take into account the constants  $\alpha, \beta$ , we can get a better result, which is shown below.

$$F = \left( \max \left\{ \tau_1^{loc} \cdot b_1; (N - b_1) \cdot \left( \sum_{i=2}^n \frac{1}{\tau_i^{loc}} \right)^{-1} \right\} + \tau_{comm} \right) \cdot \frac{\alpha}{\sqrt[4]{b_1}} + \tau_1^{loc} b_1 \cdot \beta$$

It has already been evaluated that  $b_1 \leq b_1^0 \Rightarrow F = (N - b_1) \cdot \left( \sum_{i=2}^n \frac{1}{\tau_i^{loc}} \right)^{-1} \cdot \frac{\alpha}{\sqrt[4]{b_1}} + \tau_1^{loc} b_1 \cdot \beta$ .

$$F = N \left( \sum_{i=2}^n \frac{1}{\tau_i^{loc}} \right)^{-1} \alpha b^{-\frac{1}{4}} - \left( \sum_{i=2}^n \frac{1}{\tau_i^{loc}} \right)^{-1} \alpha b_1^{\frac{3}{4}} + \tau_1^{loc} \beta b_1.$$

Consider that  $\alpha \sim 10^6, \beta \sim 10^9 \Rightarrow$

$$\begin{aligned} F &\cong 10^6 N \left( \sum_{i=2}^n \frac{1}{\tau_i^{loc}} \right)^{-1} \cdot b^{-\frac{1}{4}} - 10^6 \left( \sum_{i=2}^n \frac{1}{\tau_i^{loc}} \right)^{-1} \cdot b_1^{\frac{3}{4}} + 10^9 \tau_1^{loc} b_1 \\ \frac{1}{4} \cdot 10^6 N \left( \sum_{i=2}^n \frac{1}{\tau_i^{loc}} \right)^{-1} \cdot b_1^{-\frac{5}{4}} &\leq 10^5 \tau_1^{loc} \Rightarrow b_1 \leq \frac{4 \cdot 10^3 \tau_1^{loc}}{N \left( \sum_{i=2}^n (\tau_i^{loc})^{-1} \right)^{-1}}. \end{aligned}$$

## 5 Conclusion

In this paper we presented a new way to partition the data for the distributed optimization problem. Our solution is based on separating convex and non-convex functions and applying Algorithm 1 as well as the OGM-G algorithm from [6]. Our method works well on networks with various communication costs between the server and the local devices. The theoretical results have been confirmed experimentally. This indicates that our method gives acceleration on tasks of this type.

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