

ETH ZURICH

Semi-device-independent self-testing of a single quantum system based on Spekkens contextuality

by

Janek Denzler

Master thesis

Supervisors:

Dr. L. del Rio,
Prof. Dr. R. Renner

Department of Physics

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Abstract

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Acknowledgements

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Chapter 1

Kochen-Specker contextuality

The Kochen-Specker (KS) Theorem CITE is the often overlooked brother of Bell’s monumental no-go theorem CITE. In loose terms, Bell’s Theorem proves that quantum mechanics (QM) cannot be described in terms of a local hidden variable model (HVM). A “Bell inequality” is a condition measurement correlations must obey, for reality to permit a local HVM description. Violations of Bell inequalities have been observed in numerous experiments: The well-known Delft experiment of 2015 was the first loophole-free confirmation of such violations CITE. As will be discussed in Section 1.9, Bell locality and KS non-contextuality are closely related, locality being the weaker assumption and justified by special relativity. Nevertheless, the KS Theorem has several distinct features that deserve to be studied in their own right. Most notably, KS contextuality allows one to examine how even single quantum systems are incompatible with a seemingly orthodox classical description in terms of hidden variables CITE. In contrast, Bell tests require two or more space-like separated quantum systems, in order for the measurement statistics to exhibit the characteristic Bell non-locality. The reason for this is that we observe violations of Bell inequalities for entangled quantum systems. An immediate consequence is that the notion of KS contextuality applies to a much broader class of experiments. [1]

1.1 Introductory example

Specker’s three boxes

Section 1.6 will give a formal definition of KS non-contextuality. Nevertheless, we shall first consider a simple example that illustrates the concept. It is taken from Specker’s 1960 paper “The logic of propositions which are not simultaneously decidable” CITE, which also gives a nice accompanying parable about an overprotective seer that teaches at the school of prophets in the fictitious place Arba’ila. Fairy tales aside, consider three boxes, labelled with 1, 2, 3, that can each be in one of two states: empty or containing a gem. The state of a box

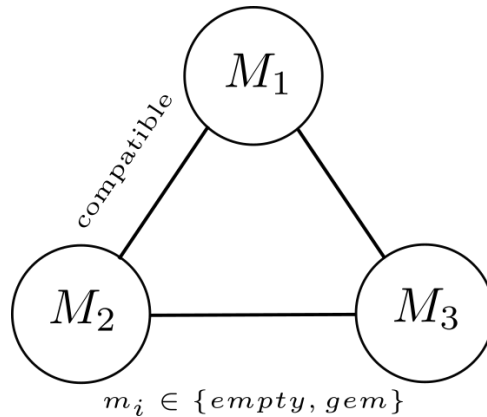


Figure 1.1: Specker’s “three boxes” example. Three boxes 1,2,3 can each either hold a gem or be empty. The measurement M_i corresponds to opening the lid of the i th box and checking its contents. The outcome of measurement M_i is denoted by m_i . A solid line between measurements means that these are compatible and can be measured simultaneously. Despite all measurements being pairwise compatible, only pairs of boxes can be jointly measured. Measuring pairs of boxes always yields anti-correlated outcomes.

can be measured by opening the lid and peeking inside. Let M_1, M_2, M_3 denote the three two-outcome measurements (with outcomes “empty” or “gem”) of the boxes. Further assume, rather unnaturally, that it is impossible to open all three boxes simultaneously. Rather, only pairs of boxes may be jointly measured. Figure 1.1 depicts the setting of Specker’s “three boxes” example. For some repeatable preparation of the system, one curiously observes the following: opening pairs of boxes always yields anti-correlated measurement outcomes (that is, one of the boxes will be empty and the other will contain a gem). How can these statistics be reconciled with a naive model of reality that assumes the three boxes to be in some underlying state that specifies their contents? As we can only ever open two boxes simultaneously, we can never observe this hidden state of reality. Importantly, a valid assignment of gems to the three boxes must be compatible with the observed anti-correlated outcomes. It turns out, as the reader should convince themselves, that any assignment of gems to the boxes is incompatible with the observed statistics. Assuming that the pairs of boxes to be measured are picked at random, in a uniform way, the probability of observing anti-correlated outcomes is bounded by $p(\text{anti-correlated}) \leq \frac{2}{3}$ and we conclude that the naive type of HVM we just considered is unable to account for the experimental data.

The point is that we have in fact considered a deterministic and non-contextual HVM to describe the three boxes. Deterministic just means that the underlying state of the system fixes all measurement outcomes and not just a probability distribution over possible outcomes. More interestingly, we define non-contextuality for the time being as the critical assumption that the outcome of some measurement should be the same, regardless of which other compatible¹ measurement is performed simultaneously. Allowing the outcome of a measurement to depend

¹Two measurements are called *compatible* if they can be performed simultaneously. While this “definition” is sufficient for now, we will refine it in Section 2.1.

on what other compatible measurement is performed together with it, could indeed account for the observed correlations.

1.2 The Kochen-Specker Theorem, informally

Informally, the KS Theorem states the following:

Theorem 1.1 (Kochen-Specker, informal). *A non-contextual deterministic HVM of QM for Hilbert spaces of dimension ≥ 3 is impossible.*

It is tempting to try to recast Specker’s “three boxes” example to the realm of quantum theory. Within the theory of QM, compatible measurements are conventionally represented by commuting Hermitian operators. To implement the example in QM, we would have to find a quantum state $|\Psi\rangle$ and pairwise commuting two-outcome measurements M_1, M_2, M_3 not all simultaneously measurable, such that QM predicts $p(\text{anti-correlated}) > \frac{2}{3}$. For the same reasons, this would imply that non-contextual deterministic HVM are not compatible with the predictions of QM and would prove the KS Theorem, albeit initially only for a fixed dimension that may be greater than 3. However, if all three measurements are pairwise commuting, they necessarily have a joint eigenbasis. Thus, all three measurements could be simultaneously performed and correlations with $p(\text{success}) > \frac{2}{3}$ could not arise. Despite this, Sections 1.7, 1.8 feature examples that are similar in spirit, if a little more involved, and which can indeed be implemented by QM.

Specker hypothesized pairwise compatibility implying global compatibility to be the “fundamental theorem of QM” that shapes the set of quantum correlations CITE. This is supported by the fact that for many correlation experiments the “theorem” singles out correlations that are compatible with QM (and projective measurements) and offers an explanation for why quantum correlations cannot exceed certain mystifying bounds we will encounter in Section 2.3 CITE.

1.3 Outline

Before we can discuss the KS Theorem and its proof in formal terms, as is done in Section 1.7, we will begin by establishing a general mathematical framework of HVM in Section 1.4. Section 1.5 introduces two ways of representing measurements in QM, leading to two equivalent notions of KS non-contextuality, as defined in Section 1.6. Section 1.8 presents a considerably simpler proof of the KS Theorem that holds for four-dimensional systems. The connection between KS non-contextuality and local causality will be the subject of Section 1.9, highlighting the similarities and differences between the KS Theorem and Bell’s Theorem. This discussion will

point out several flaws of the traditional notion of KS contextuality and addresses the need for an operational notion. A proposal of that sort by Spekkens will be introduced in Section 5. In Section 1.10 we will discuss an important class of KS contextuality experiments involving a qutrit, namely the KCBS scenario and generalizations of it. For these experiments one can derive KS non-contextuality inequalities on the convex set of possible correlations, analogously to Bell inequalities, that must hold for all non-contextual behaviours. Interestingly, these inequalities turn out to be incompatible with the predictions of QM for some quantum states. In Section 4, equipped with the graph-theoretic tools presented in Section 2, we will demonstrate that KCBS-type KS contextuality experiments and their corresponding KS non-contextuality inequalities facilitate robust self-testing, after introducing the notion of self-testing in Section 3. These KS non-contextuality inequalities can be lifted to Spekkens non-contextuality inequalities, see Section 5.3, which will serve as more operational tests of non-classicality. We submit two self-testing protocols based on Spekkens contextuality in Section TODO, and discuss their merits compared to previous proposals in Section TODO.

1.4 General framework of hidden variable models

An ontological model of QM², more commonly referred to as a hidden variable model (HVM), presupposes some ontic state space Λ . An ontic state $\lambda \in \Lambda$ is a state of reality or “matter of fact” description of the system. It holds all attributes of the system, regardless of whether they are known or even knowable (hence hidden variables). Importantly, a HVM posits that our system is at each time point in some ontic state with corresponding definite attributes, even when these are not being subjected to measurements. In the case of KS contextuality we will only consider **deterministic** HVM, meaning that an ontic state fixes the outcomes of all possible measurements that may be performed on the system. Uncertainty in the ontic state that was prepared results in a probabilistic distribution of measurement outcomes. It is helpful to formalize this and boil the assumptions down to a simple mathematical model. We follow the notation and terminology of CITE.

With any preparations P one can associate a probability density function $\mu_P(\lambda)$ on the ontic state space Λ . One can think of a preparation as an ensemble of systems, much like in the spirit of statistical mechanics, whereby each individual member of the ensemble is in some well-defined ontic state $\lambda \in \Lambda$. The probability of picking a system from the ensemble at random with a hidden parameter contained in some region $\mathcal{A} \subset \Lambda$, is then given by $\int_{\mathcal{A}} \mu_P(\lambda) d\lambda$. By normalization, we require that $\int_{\Lambda} \mu_P(\lambda) d\lambda = 1$.

²The following applies more generally for ontological models of arbitrary operational theories. An operational theory is one that makes testable predictions.

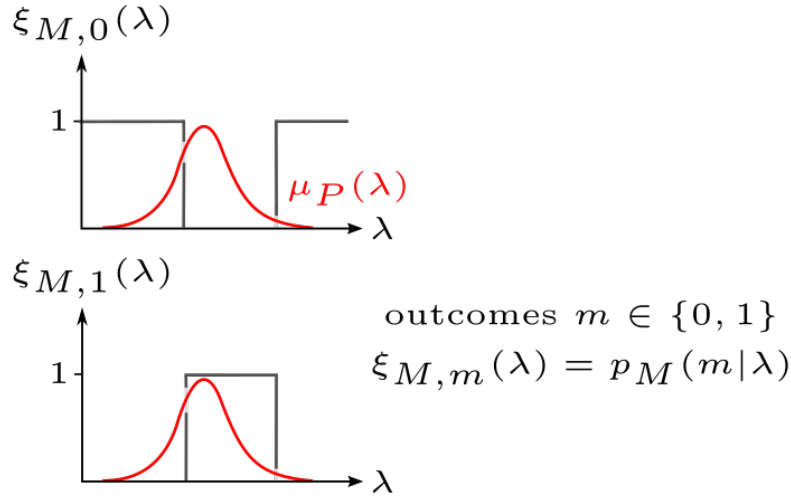


Figure 1.2: Summary of the constituents characterizing a deterministic HVM for the case of a two-outcome measurement.

A system's ontic state space Λ , here depicted as a one-dimensional axis, comprises all possible states of reality. A state preparation P can be associated with a probability density function on Λ , modelling the uncertainty in the prepared ontic state. A deterministic HVM associates with any measurement M , here a two-outcome measurement with outcomes $m \in \{0, 1\}$, a set of indicator functions $\{\xi_{M,m}\}_m$. The indicator function to measurement outcome m , $\xi_{M,m} : \Lambda \rightarrow \{0, 1\}$, maps each ontic state to the probability of observing outcome m , conditioned on the system being in that ontic state.

Measurements M can be associated with sets of indicator functions $\{\xi_{M,k}\}_k$ on Λ , one for each measurement outcome k . The value $\xi_{M,k}(\lambda) \in \{0, 1\}$ is the conditional probability of the measurement M yielding the outcome k , given that the system is in the ontic state λ . Since we are considering only deterministic HVM, meaning that the ontic state of the system fixes all measurement outcomes with certainty, this conditional probability must be either 0 or 1. Naturally, the indicator functions for the different measurement outcomes must obey $\sum_k \xi_{M,k}(\lambda) = 1 \forall \lambda \in \Lambda$. Section 1.5 will introduce two ways of representing measurements in QM. We will apply the framework introduced in this section to these, in order to treat quantum measurements within HVM. Lastly, for the HVM to be consistent with the testable predictions of the operational theory it should model, it must hold that: $p(k|P, M) = \int_{\Lambda} d\lambda \xi_{M,k}(\lambda) \mu_P(\lambda)$, where $p(k|P, M)$ describes the probability of observing the measurement outcome k , given measurement M and preparation P , as proposed by the operational theory. The above quantities are pictorially summarized in Figure 1.2, for the case of a two-outcome measurement M . In the following, when making reference to an arbitrary deterministic HVM of QM, we will implicitly assume an underlying ontic state space Λ and indicator functions $\{\xi_{M,k}\}_k$ for all measurements M with possible outcomes k .

1.5 Measurements in quantum mechanics

Section 1.6 will define the notion of KS non-contextuality that is at the heart of the KS Theorem. It is an assumption about the type of HVM the theorem rules out and, in loose terms, pertains to the measurements within such a HVM. To set the stage for discussing KS non-contextuality, this section introduces two ways of representing (projective) measurements in QM. As will be proved in Section 1.6, the two notions of measurements will give us two equivalent notions of KS non-contextuality, each of which will be particularly useful for certain purposes. Note that within the framework of KS contextuality we assume all quantum measurements to be projective. On the other hand, in quantum information theory a general quantum measurement is given by a set of positive semi-definite operators whose sum is the identity, a so-called positive operator valued measure (POVM) CITE. A projective measurement, or projection-valued measure (PVM), is also a POVM, however the reverse generally does not hold. The two are related in the sense that every POVM can be seen as a PVM acting on an extended Hilbert space, followed by us discarding some degrees of freedom, a consequence of Naimark's dilation theorem CITE. Thus, assuming all quantum measurements to be projective is in principle consistent with the existence of POVMs. In Section 5, we present a revision of the traditional notion of KS contextuality due to Spekkens. Spekkens contextuality is defined largely in terms of operational primitives and is able to accommodate more general POVM quantum measurements. As physical measurements are non-projective CITE, we must account for these in view of our goal to certify quantum properties of a single quantum device under operational assumptions.

The conventional way of representing projective measurements in QM is in terms of Hermitian operators, with every measurement being relatable to a set of pairwise commuting and thus simultaneously measurable operators that share a common eigenbasis. Measurement outcomes are tuples of simultaneous eigenvalues. The spectrum of a Hermitian operator A is denoted by $\sigma(A)$ and the set of all Hermitian operators on a Hilbert space \mathcal{H} is denoted by $\text{Herm}(\mathcal{H})$.

Definition 1.2. Let \mathcal{H} be the system Hilbert space.

A *projective quantum measurement* $\mathcal{M} = (X_1, X_2, \dots) \subset \text{Herm}(\mathcal{H})$ is an ordered set of pairwise commuting Hermitian operators.

An *outcome of a projective quantum measurement* \mathcal{M} is a tuple

$(x_1, x_2, \dots) \in \times_{X_i \in \mathcal{M}} \sigma(X_i)$ of simultaneous eigenvalues to operators in \mathcal{M} , meaning that there exists a common eigenvector $|\Psi\rangle \in \mathcal{H} \setminus \{0\}$ with $X_k |\Psi\rangle = x_k |\Psi\rangle \forall X_k \in \mathcal{M}$.

An ordered set of commuting Hermitian operators like in 1.2 is also referred to as a “measurement context”. We say that X_1 is measured in the context \mathcal{M} . An example of a quantum measurement in this sense is given by “Pauli strings” on the composite Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$, for example

$$\mathcal{M} = (\sigma_x^1 := \sigma_x \otimes \mathbb{1}, \sigma_x^2 := \mathbb{1} \otimes \sigma_x, \sigma_x^1 \sigma_x^2 := \sigma_x \otimes \sigma_x)$$

with outcomes

$$\{(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\}.$$

Each of the four outcome tuples corresponds to one of the four joint eigenstates that together span the four-dimensional Hilbert space. Additionally, each outcome tuple contains three values, corresponding to the three pairwise commuting Hermitian operators that constitute the measurement \mathcal{M} . We will revisit operators of this form in Section 1.8, when discussing a simple, algebraic proof of the KS Theorem for four-dimensional Hilbert spaces.

Let us now apply the framework proposed in Section 1.4 to projective quantum measurements like in Definition 1.2. Let A, B be two commuting Hermitian operators. If we assume the ontic state $\lambda \in \Lambda$ to be fixed, $\xi_{(A,B),(a,b)}(\lambda)$ describes a joint probability distribution $P_{AB}^\lambda(a, b) = \xi_{(A,B),(a,b)}(\lambda)$ for random variables \mathcal{A}, \mathcal{B} , that hold eigenvalues of A, B , respectively. $P_{AB}^\lambda(a, b)$ gives the probability of obtaining outcome (a, b) when performing the joint measurement (A, B) , conditioned on the system being in the ontic state λ . We can go from a joint probability distribution P_{AB}^λ to the “marginal” distribution P_A^λ for random variable \mathcal{A} , by summing over all $b \in \sigma(B)$: $\sum_{b \in \sigma(B)} P_{AB}^\lambda(a, b) = P_A^\lambda(a)$. We will revisit marginals of joint outcome probability distributions when defining the notion of KS non-contextuality in Section REF.

An alternative way of representing measurements in QM is motivated by the spectral decomposition of Hermitian operators. It defines projective quantum measurements in terms of rays in a projective Hilbert space $\mathcal{P}(\mathcal{H})$. The projective Hilbert space $\mathcal{P}(\mathcal{H})$ of a complex Hilbert space \mathcal{H} is defined as the set of equivalence classes $\{[|\Psi\rangle]\}_{|\Psi\rangle \in \mathcal{H} \setminus \{0\}}$ with respect to the equivalence relation $|\Psi\rangle \sim |\Phi\rangle : \Leftrightarrow \exists \alpha \in \mathbb{C} \setminus \{0\} : |\Psi\rangle = \alpha |\Phi\rangle$. The equivalence classes of $\mathcal{P}(\mathcal{H})$ are called “rays”. Intuitively, one can think of $\mathcal{P}(\mathcal{H})$ as the “unit sphere” of normalized vectors in \mathcal{H} , with every $\mathcal{H} \ni |\Psi\rangle \neq 0$ being identified with its normalized counterpart. In this picture, a measurement can be specified by an orthonormal basis (ONB) $\{|\Psi_i\rangle\}_i$ of the Hilbert space, with measurement outcomes being represented by rays.

Definition 1.3. Let \mathcal{H} be the system Hilbert space.

A *projective quantum measurement with respect to a basis* $\mathcal{M} = \{|\Psi_i\rangle\}_i \subset \mathcal{P}(\mathcal{H})$ is an ONB of \mathcal{H} . An *outcome* of such a measurement \mathcal{M} is a ray $|\Psi_k\rangle \in \mathcal{M}$.

As will be discussed in the next section, an ONB is also referred to as a measurement context. A given ray can belong to different measurement contexts.

1.6 Defining KS non-contextuality

We now turn to defining what it means for a HVM of QM to be KS non-contextual. There are two equivalent definitions of KS non-contextuality for deterministic HVM of QM, both

commonly found in literature on the topic. As advertised in 1.5, these stem from the two notions of measurements in QM. Key to both definitions of KS non-contextuality will be interpreting a measurement as “revealing” a true property of the system. “True” means that the system has this property, regardless of whether and how the property is measured. The definitions presented here are adapted from the lecture series CITE.

Let us first consider measurements \mathcal{M} being represented by Hermitian operators, as in Definition 1.2. Given the ontic state of the system, a general deterministic HVM simultaneously fixes all possible measurement outcomes via the indicator functions $\xi_{\mathcal{M},k}(\lambda)$. A priori, for a given ontic state $\lambda \in \Lambda$, the value revealed upon measuring some $A \in \text{Herm}(\mathcal{H})$ may depend on what other compatible operators are measured together with A . These constitute the “measurement context”. For instance, assume A, B and A, C to be pairs of commuting operators in $\text{Herm}(\mathcal{H})$. Further assume B and C to be non-commuting and thus not simultaneously measurable. We can measure A in two different contexts, together with B or together with C . For a given ontic state λ , these measurements may yield a different outcome for A . However, for a HVM to be consistent with the interpretation of measurements revealing “true” properties of the system, the outcome of measuring an operator may not depend on the measurement context. The assumption of KS non-contextuality demands a functional relationship between the Hermitian operators and their “true value”:

Definition 1.4. Let \mathcal{H} be the system Hilbert space and represent measurements like in Definition 1.2. A deterministic HVM is *KS non-contextual* if it satisfies Conditions 1 and 2 for all ontic states $\lambda \in \Lambda$:

1. (context independence)

$$\forall A, B \in \text{Herm}(\mathcal{H}), [A, B] = 0 : \xi_{A,a}(\lambda) = \sum_{b \in \sigma(B)} \xi_{(A,B),(a,b)}(\lambda)$$

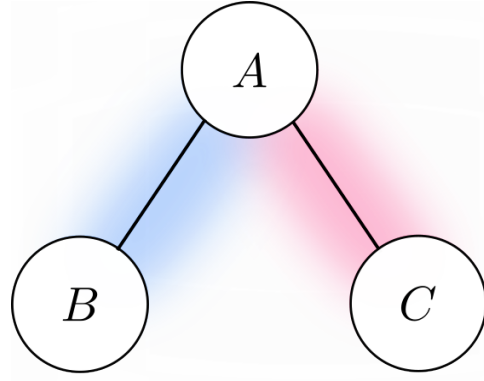
Thus the outcome assignment $\nu_\lambda : \text{Herm}(\mathcal{H}) \rightarrow \mathbb{R}$, defined via

$$\nu_\lambda(A) = a \in \sigma(A) \iff \xi_{A,x}(\lambda) = \delta_{ax}$$

is consistent with all measurement contexts.

2. (commuting observables are assigned simultaneous eigenvalues)

$$\begin{aligned} \forall A, B \in \text{Herm}(\mathcal{H}), [A, B] = 0 : \\ \nu_\lambda(A) = a, \nu_\lambda(B) = b \implies \exists |\Psi\rangle \in \mathcal{H} \setminus \{0\} \text{ with} \\ A|\Psi\rangle = a|\Psi\rangle, \\ B|\Psi\rangle = b|\Psi\rangle \end{aligned}$$



$$\forall \lambda \in \Lambda \forall a \in \sigma(A) : \\ \xi_{A,a}(\lambda) = \sum_b \xi_{(A,B),(a,b)}(\lambda)$$

Figure 1.3: Summary of the notion of KS non-contextuality in terms of Hermitian operators. A, B and B, C are assumed to be commuting operators, whereas B and C do not commute. For every ontic state $\lambda \in \Lambda$, a deterministic non-contextual HVM assigns simultaneous eigenvalues (a, b) to the commuting operators A, B . Context independence requires that taking the marginal of the joint statistics $\xi_{(A,B),(a,b)}$ must reduce to the single operator assignment $\xi_{A,a}$.

Condition 1 in Definition 1.4 requires that, for compatible operators, taking the marginal of the joint probability distribution yields the single operator probability distribution. This just means that $\nu_\lambda(A)$ assigns the correct outcome to measurement A for the ontic state λ , as given by the deterministic HVM, no matter what other compatible measurements are performed jointly. It is crucial that this is required to hold at the ontic level: if we take the ensemble average, as outlined in Section 1.5, this property would always hold in QM. The notion of KS non-contextuality in terms of Hermitian operators is summarized in Figure 1.3.

An important consequence of Condition 2 in Definition 1.4 is often coined “functional consistency” CITE. We will only need and prove the property of “functional consistency” for functional relationships which are polynomial, however it could be extended to smooth functions with a converging (multivariate) Taylor expansion. Note that, strictly speaking, context independence implies Condition 2, as permissible outcomes of joint measurements are by definition simultaneous eigenvalues.

Lemma 1.5 (functional consistency).

Let $\nu_\lambda : \text{Herm}(\mathcal{H}) \rightarrow \mathbb{R}$ be a KS non-contextual value assignment, according to Definition 1.4, and let $f : \mathbb{R}^n \rightarrow \mathbb{R}, (x_1, x_2, \dots, x_n) \mapsto f(x_1, x_2, \dots, x_n)$ be a polynomial function in n real variables. Further assume that $f(X_1, X_2, \dots, X_n) = 0$ holds as an operator identity among the observables of a pairwise commuting set $\{X_1, X_2, \dots, X_n\} \subset \text{Herm}(\mathcal{H})$. Here $f(X_1, X_2, \dots, X_n)$ is the operator expression obtained by formally inserting operators X_i into the real variables x_i . Then it must hold that $f(\nu_\lambda(X_1), \nu_\lambda(X_2), \dots, \nu_\lambda(X_n)) = 0$.

Proof. By definition, the value assignment ν_λ assigns simultaneous eigenvalues to the set of pairwise commuting operators. This means that there exists a joint eigenvector $\mathcal{H} \ni |\Psi\rangle \neq 0$ that

corresponds to the eigenvalues assigned by ν_λ . The operator expression $f(X_1, X_2, \dots, X_n)$ is a sum containing terms of the form $c \prod_{i=1}^n X_i^{\alpha_i}$, with $c \in \mathbb{R}$, $\alpha_i \in \mathbb{N}_0$. Acting with $f(X_1, X_2, \dots, X_n)$ on the joint eigenvector $|\Psi\rangle$ thus “evaluates” f at $(\nu_\lambda(X_1), \nu_\lambda(X_2), \dots, \nu_\lambda(X_n))$:

$$f(X_1, X_2, \dots, X_n) |\Psi\rangle = f(\nu_\lambda(X_1), \nu_\lambda(X_2), \dots, \nu_\lambda(X_n)) |\Psi\rangle = 0.$$

This concludes the proof, as $|\Psi\rangle$ is non-zero. \square

Corollary 1.6. *Let X_1, X_2 be arbitrary commuting observables and $\nu_\lambda : \text{Herm}(\mathcal{H}) \rightarrow \mathbb{R}$ a KS non-contextual value assignment. It holds that:*

1. $\nu_\lambda(X_1 + X_2) = \nu_\lambda(X_1) + \nu_\lambda(X_2)$ ³
2. $\nu_\lambda(X_1 X_2) = \nu_\lambda(X_1) \nu_\lambda(X_2)$

Proof.

1. Define $X = X_1 + X_2$, which commutes with both X_1 and X_2 , and apply Lemma 1.5 to the operator identity $X - X_1 - X_2 = 0$.
2. $X_1 X_2$ is a Hermitian operator that commutes with both X_1 and X_2 . Apply Lemma 1.5 to the operator identity $(X_1 X_2) - (X_1)(X_2) = 0$.

\square

Let us now turn to the definition of KS non-contextuality in terms of rays in a projective Hilbert space.

Definition 1.7. Let \mathcal{H} be the system Hilbert space and represent measurements as in Definition 1.3. A deterministic HVM is *KS non-contextual* if for all ontic states $\lambda \in \Lambda$ there is a value assignment $\omega_\lambda : \mathcal{P}(\mathcal{H}) \rightarrow \{0, 1\}$ satisfying:

1. (context independence)
 $\forall |\Psi\rangle \in \mathcal{P}(\mathcal{H}), \forall \text{ measurements } \mathcal{M} \text{ with } |\Psi\rangle \in \mathcal{M} : \omega_\lambda(|\Psi\rangle) = \xi_{\mathcal{M}, |\Psi\rangle}(\lambda)$
2. (exactly one outcome)
 $\forall \text{ measurements } \{|\Psi_i\rangle\}_i \subset \mathcal{P}(\mathcal{H}) \text{ it holds that } \sum_i \omega_\lambda(|\Psi_i\rangle) = 1$

³Compare this to the much stronger assumption made by Neumann in his faulty no-go theorem, namely that this relation must hold for arbitrary observables. While this produces a contradiction already in two dimensional Hilbert spaces, the unfounded assumption has been mocked as silly by Mermin CITE.

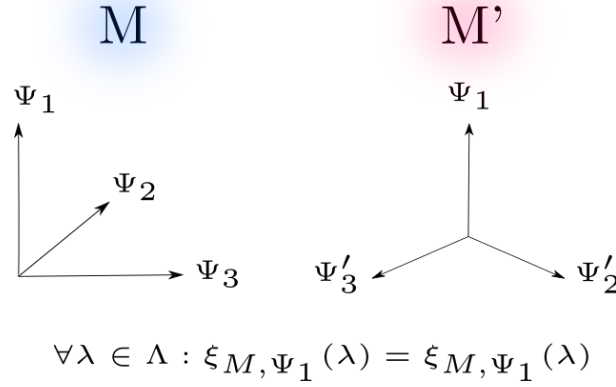


Figure 1.4: Summary of the key points of KS non-contextuality, in terms of vectors in a projective Hilbert space, depicted here to be three-dimensional. The two ONB, $\{|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle\}$ and $\{|\Psi_1\rangle, |\Psi'_2\rangle, |\Psi'_3\rangle\}$ are by a rotation in the plane. As $|\Psi_1\rangle$ appears in both measurements, a non-contextual HVM must assign the same conditional probability of observing the corresponding outcome for every ontic state $\lambda \in \Lambda$.

The interpretation is as follows: think of a measurement $\{|\Psi_i\rangle\}_i \subset \mathcal{P}(\mathcal{H})$ as a set of yes/no questions posed to the system. For each outcome $|\Psi_k\rangle$, a measurement “asks” if this is a true property of the system, fixed by its ontic state, and subsequently “reveals” the answer:

Given the ontic state of the system, a general deterministic HVM simultaneously fixes all possible measurement outcomes via the indicator functions $\xi_{M,k}(\lambda)$. This gives us an assignment similar to that above, with the crucial difference that the value assigned to an outcome $|\Psi_k\rangle$ may a priori depend on the measurement (ONB) it is regarded to be part of. This ONB is referred to as the “measurement context”. For our HVM to be consistent with the interpretation of measurements revealing “true” properties of the system, an outcome being true or false should not depend on what measurement it is a part of. The assumption of KS non-contextuality imposes that the value assigned to any ray be independent of the measurement context, meaning that a ray appearing in multiple measurements must receive the same valuation by the indicator functions, allowing for a functional dependence on only the ray itself. Lastly, Condition 2 ensures that exactly one measurement outcome corresponds to a true property of the system, which is revealed upon performing the measurement. The key points of Definition 1.7 are summarized in Figure 1.4, which depicts two measurement bases, assuming a three dimensional Hilbert space.

As previously stated, both definitions of KS non-contextuality, Definitions 1.4 and 1.7, are equivalent in the following sense:

Lemma 1.8. *Let \mathcal{H} be a Hilbert space. A non-contextual deterministic value assignment $\omega : \mathcal{P}(\mathcal{H}) \rightarrow \{0,1\}$ according to Definition 1.7 induces a non-contextual deterministic value assignment $\nu : \text{Herm}(\mathcal{H}) \rightarrow \mathbb{R}$ according to Definition 1.4, and vice versa.*

Lemma 1.8 implies that proving the impossibility of a non-contextual deterministic value assignment $\omega : \mathcal{P}(\mathcal{H}) \rightarrow \{0, 1\}$ for Hilbert spaces of dimension ≥ 3 , which we will demonstrate in the next section, entails the impossibility of a non-contextual deterministic HVM according to either definition, as such a HVM would come with a mapping ν and ω for every ontic state λ .

Proof. (of Lemma 1.8) We prove both implications separately.

- $\nu \Rightarrow \omega$: Define $\omega(|\Psi\rangle) := \nu(|\Psi\rangle\langle\Psi|)$. Projectors $|\Psi\rangle\langle\Psi|$ have eigenvalues 0,1, therefore ω maps to $\{0, 1\}$. Since orthogonal projectors commute, by the property of “functional consistency”, in particular Corollary 1.6, we get:

$$\{|\Psi_i\rangle\}_i \subset \mathcal{H} \text{ an ONB} \Rightarrow \sum_i \omega(|\Psi_i\rangle) = \sum_i \nu(|\Psi_i\rangle\langle\Psi_i|) = \nu\left(\sum_i |\Psi_i\rangle\langle\Psi_i|\right) = \nu(\mathbb{1}_{\mathcal{H}}) = 1.$$

- $\omega \Rightarrow \nu$: (Adapted from CITE) Define $\nu(A = \sum_{i,j} a_i |\Psi_j^{(i)}\rangle\langle\Psi_j^{(i)}|) := \sum_{i,j} a_i \omega(|\Psi_j^{(i)}\rangle)$, where we have used that every Hermitian operator can be spectrally decomposed. We take the eigenvalues a_i to be distinct, $\{|\Psi_j^{(i)}\rangle\}_j$ being an ONB of the eigenspace corresponding to the eigenvalue a_i . The spectral decomposition of a Hermitian operator is not unique, as we may choose different ONB of the eigenspaces, nonetheless the above assignment is well-defined: Take

$$A = \sum_{i,j} a_i |\Psi_j^{(i)}\rangle\langle\Psi_j^{(i)}| = \sum_{i,j} a_i |\Phi_j^{(i)}\rangle\langle\Phi_j^{(i)}|$$

to be two spectral decompositions of A . Both $\{|\Psi_j^{(i)}\rangle\}_j$ and $\{|\Phi_j^{(i)}\rangle\}_j$ span the eigenspace corresponding to the eigenvalue a_i . We can extend both subsets to ONB of \mathcal{H} by adding an orthonormal set $\{|\Theta_k\rangle\}_k$ to each. By the defining property of ω we get

$$\sum_j \omega(|\Psi_j^{(i)}\rangle) = \sum_j \omega(|\Phi_j^{(i)}\rangle)$$

for all i , which means that ν is well-defined. Suppose $\{A, B, \dots\} \subset \text{Herm}(\mathcal{H})$ is a set of mutually commuting Hermitian operators. There exists a shared ONB of eigenvectors $\{|\Psi_i\rangle\}_i$. Let

$$A = \sum_i a_i |\Psi_i\rangle\langle\Psi_i| \text{ and } B = \sum_i b_i |\Psi_i\rangle\langle\Psi_i|$$

be spectral decompositions of A and B , where the eigenvalues a_i, b_i can now be repeating. The assignment ν maps A and B to

$$\nu(A) = \sum_i a_i \omega(|\Psi_i\rangle) \text{ and } \nu(B) = \sum_i b_i \omega(|\Psi_i\rangle).$$

Thus, commuting operators are mapped to simultaneous eigenvalues, and ν is a KS non-contextual value assignment like in Definition 1.4.

□

1.7 Formal statement and proof of the KS Theorem

The KS Theorem was first proved by Kochen and Specker in 1968 CITE. Previously, Bell independently proved that a deterministic HVM of QM that induces a non-contextual value assignment like in Definition 1.7, must be contextual CITE. However his proof was deemed less convincing, as it used a continuum of vector directions, as opposed to an explicit finite set CITE.

We are now ready to reformulate the KS Theorem in precise mathematical terms (adapted from CITE):

Theorem 1.9 (Kochen-Specker CITE).

For all projective Hilbert spaces $\mathcal{P}(\mathcal{H})$ of dimension ≥ 3 , there exists a finite set $S \subset \mathcal{P}(\mathcal{H})$ of rays for which no valid value assignment $\omega : S \rightarrow \{0, 1\}$ exists, satisfying the conditions for a KS non-contextual model, according to Definition 1.7.

The existence of such a set \mathcal{S} is in contradiction with any attempt of a KS non-contextual deterministic HVM of QM, as KS non-contextuality implies a valid value assignment for every possible ontic state of the system. Thus the KS Theorem effectively states the impossibility of a non-contextual deterministic HVM.

As Pusey notes CITE, all assumptions (outcome determinism, context independence and Hilbert spaces of dimension ≥ 3) of the KS Theorem are necessary for the proof of impossibility. If we allow for context dependence, such a value assignment can be given for all sets \mathcal{S} , as we can assign appropriate values for each measurement contained in \mathcal{S} . Furthermore, relaxing the assumption of outcome determinism to an assignment of probabilities $\tilde{\omega} : \mathcal{S} \rightarrow [0, 1]$ spoils the proof of impossibility, as a valid value assignment $\tilde{\omega}$ is given for instance by the Born rule: $\tilde{\omega}(|\Psi_i\rangle) := \text{tr}(\rho |\Psi_i\rangle \langle \Psi_i|)$ for some non-negative (Hermitian) operator ρ with $\text{tr}(\rho) = 1$. Lastly, for Hilbert spaces of dimension 2, it is possible to give an explicit form of a non-contextual deterministic HVM CITE. The deeper reason for why our proof does not extend to the two-dimensional case is that it makes heavy use of overlapping ONB (measurement contexts) in the construction of \mathcal{S} , meaning different measurements that share at least one ray. These will be key in establishing a contradiction for dimensions ≥ 3 . However, for two-dimensional Hilbert spaces, it is impossible to construct overlapping and distinct ONB, as a single ray fixes the ONB.

The proof of the KS Theorem is by induction with respect to the dimension. The following Lemma (induction step), once proven, reduces the proof of the KS Theorem to the three-dimensional base case. We will prove the three-dimensional base case by giving an explicit set of directions belonging to overlapping orthogonal triads, for which a KS non-contextual value assignment can be shown to be impossible. This will be demonstrated by a chain of geometric arguments.

Lemma 1.10 (CITE).

Assume that all projective Hilbert spaces of dimension d have a finite subset like in the KS Theorem. Let $\mathcal{P}(\mathcal{H})$ be a projective Hilbert space of dimension $d+1$. Then $\exists \mathcal{S} \in \mathcal{P}(\mathcal{H})$ with the same properties.

Proof. We follow CITE. Let $\{|0\rangle, \dots, |d\rangle\}$ be an ONB of \mathcal{H} and let $\mathcal{H}_0, \mathcal{H}_1 \subset \mathcal{H}$ be the d -dimensional subspaces orthogonal to $|0\rangle, |1\rangle$, respectively. There exist subsets $\mathcal{S}_0, \mathcal{S}_1$ of $\mathcal{P}(\mathcal{H}_0), \mathcal{P}(\mathcal{H}_1)$, for which there are no valid value assignments within the respective projective Hilbert space. Let $\mathcal{S}'_0, \mathcal{S}'_1$ denote the embeddings of $\mathcal{S}_0, \mathcal{S}_1$ into \mathcal{H} . We will show that $\mathcal{S}' := \mathcal{S}'_0 \cup \mathcal{S}'_1 \cup \{|0\rangle, \dots, |d\rangle\}$ then defines a subset of $\mathcal{P}(\mathcal{H})$ for which there is no valid value assignment. As $\{|0\rangle, \dots, |d\rangle\}$ constitutes an ONB, a valid value assignment ω' has to map $|0\rangle$ or $|1\rangle$ (or both) to 0. Both $\omega'(|0\rangle) = 0$ and $\omega'(|1\rangle) = 0$ lead to a contradiction: For instance, say $\omega'(|0\rangle) = 0$. For any ONB $M \subset \mathcal{H}_0$ and its embedding $M' \subset \mathcal{H}$ we have

$$\sum_{|\Psi_i\rangle \in M'} \omega'(|\Psi_i\rangle) + \omega'(|0\rangle) = \sum_{|\Psi_i\rangle \in M'} \omega'(|\Psi_i\rangle) = 1.$$

Hence, $\omega : \mathcal{S}_0 \mapsto \{0, 1\}$, $\omega(|\Psi\rangle) := \omega'(|\Psi'\rangle)$ defines a valid value assignment, where $|\Psi'\rangle \in \mathcal{S}'_0$ is the embedding of $|\Psi\rangle \in \mathcal{S}_0$ into \mathcal{H} . This is a contradiction, as such an assignment does not exist for \mathcal{S}_0 . \square

What remains is a proof for the existence of a finite dimensional subset \mathcal{S} that does not permit a valid KS non-contextual value assignment for the case of three dimensions. The following lemma allows us to consider only the three-dimensional complex projective space $\mathcal{P}(\mathbb{C}^3)$ w.l.o.g.:

Lemma 1.11. *Let \mathcal{H}_n be a n -dimensional complex Hilbert space. A valid KS non-contextual value assignment $\omega : \mathcal{P}(\mathcal{H}_n) \rightarrow \{0, 1\}$ according to Definition REF induces a valid value assignment $\tilde{\omega} : \mathcal{P}(\mathbb{C}^n) \rightarrow \{0, 1\}$.*

Proof. Let $\{|\Psi_i\rangle\}_{i \in \{1, \dots, n\}}$ be an ONB of \mathcal{H}_n . We can construct the isomorphism $\mathcal{J} : \mathcal{H}_n \rightarrow \mathbb{C}^n$, $|\Psi_k\rangle \mapsto e_k$, where $\{e_i\}_{i \in \{1, \dots, n\}}$ is the standard basis of \mathbb{C}^n . This isomorphism preserves the inner product structure, as it maps an ONB of \mathcal{H}_n to an ONB of \mathbb{C}^n . Thus, \mathcal{J} preserves all orthogonality relations. A valid non-contextual value assignment according to Definition 1.7 then induces a valid non-contextual value assignment $\tilde{\omega} : \mathcal{P}(\mathbb{C}^n) \rightarrow \{0, 1\}$ via $\tilde{\omega} := \omega \circ \mathcal{J}^{-1}$. \square

Having reduced the proof of the three-dimensional base case to proving the impossibility of a valid non-contextual value assignment $\tilde{\omega} : \mathcal{P}(\mathbb{C}^3) \rightarrow \{0, 1\}$, we will now turn our attention to this task. Lemma sticks to Peres' notation CITE for saving space:

Lemma 1.12. *1.7 Let $\bar{1}$ denote -1 and $\bar{2}$ denote $\sqrt{2}$. For the following set of 33 vectors $\mathcal{S} \subset \mathbb{R}^3 \subset \mathbb{C}^3$ (“Peres configuration” ⁴ CITE) it is impossible to assign values $\{0, 1\}$ to each vector such that for all orthogonal triads contained in \mathcal{S} exactly one member receives the valuation 1:*

$$\begin{aligned} \mathcal{S} = \{ & 100, 010, 001, 110, 101, 011, \bar{1}\bar{1}0, \bar{1}0\bar{1}, 0\bar{1}\bar{1}, \bar{1}\bar{1}\bar{2}, \bar{1}\bar{1}2, \\ & \bar{2}01, 021, 102, \bar{2}11, 211, \bar{1}02, 201, \bar{1}\bar{1}2, 112, 0\bar{2}1, 012, \\ & \bar{1}\bar{2}1, 121, 0\bar{1}2, 12\bar{1}, \bar{1}21, 21\bar{1}, \bar{2}11, \bar{2}10, \bar{1}20, 210\} \end{aligned}$$

Here, the tuple $\bar{2}01$ for instance corresponds to the vector $\begin{pmatrix} -\sqrt{2} \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3$.

We consider unnormalized vectors only for convenience as this does not affect orthogonality relations.

The cubic symmetry of the set \mathcal{S} greatly simplifies the proof. More precisely, the set \mathcal{S} is invariant under interchange of the x, y, z axes and under a reversal of the direction of each axis $x \mapsto \bar{x}$. It is the case that

$$\omega : \mathcal{S} \rightarrow \{0, 1\} \text{ a valid value assignment} \iff \omega \circ T \text{ a valid value assignment,}$$

T being one of the aforementioned symmetry transformations. This is straightforward to see, as the symmetry transformations of \mathcal{S} are orthogonal transformations that leave orthogonality relations unchanged. The following concrete example shows how we can exploit the symmetry of \mathcal{S} to make arbitrary assignments w.l.o.g.: Consider the orthogonal triad $\{001, 100, 010\}$. Assuming there exists a valid value assignment $\omega : \mathcal{S} \rightarrow \{0, 1\}$, then exactly one of the three vectors in the triad is assigned a 1. Due to the symmetry of the set \mathcal{S} , there must be a valid value assignment ω' that assigns 001 the value 1. Analogous considerations apply to symmetry transformations $x \mapsto \bar{x}$.

On a historical note, the original KS paper CITE gives an explicit set of 117 directions in \mathbb{R}^3 that lead to a contradiction. The current record holders are Kochen and Conway, with a set of 31 directions, however their set is not as highly symmetric as the Peres configuration, which complicates the proof CITE.

⁴For a nice visualization of the “Peres configuration”, see Conway and Kochen’s paper “The Strong Free Will Theorem”, which also makes use of it CITE.

Proof. (of Lemma 1.7)

The proof of Lemma REF is given by the table in Figure 1.5 and shows that any attempt to construct a valid KS non-contextual value assignment $\omega : \mathcal{S} \rightarrow \{0, 1\}$ inevitably leads to a contradiction. Finding such an assignment is equivalent to a colouring problem. Identifying $0 \equiv$ “green”, $1 \equiv$ “red”, a valid value assignment corresponds to a colouring of the rays in \mathcal{S} , in accordance with the rule that each orthogonal triad in \mathcal{S} must have exactly one element coloured red and two elements coloured green. Such a colouring of rays is shown to be impossible, making use of the fact that \mathcal{S} contains rays that are part of multiple distinct orthogonal triads. Each line of the table introduces a new orthogonal triad that is coloured, either w.l.o.g. , making use of the symmetry of \mathcal{S} , or according to the values of previous triads. In each line, only the first ray of the triad is coloured green. The “Other rays” in each line are orthogonal to the single green ray in that line and are consequently coloured red. The stepwise colouring of rays leaves the set $\{100, 021, 0\bar{1}2\}$ coloured red. As these rays form an orthogonal triad, this proves that a valid colouring and thus a valid value assignment are impossible. \square

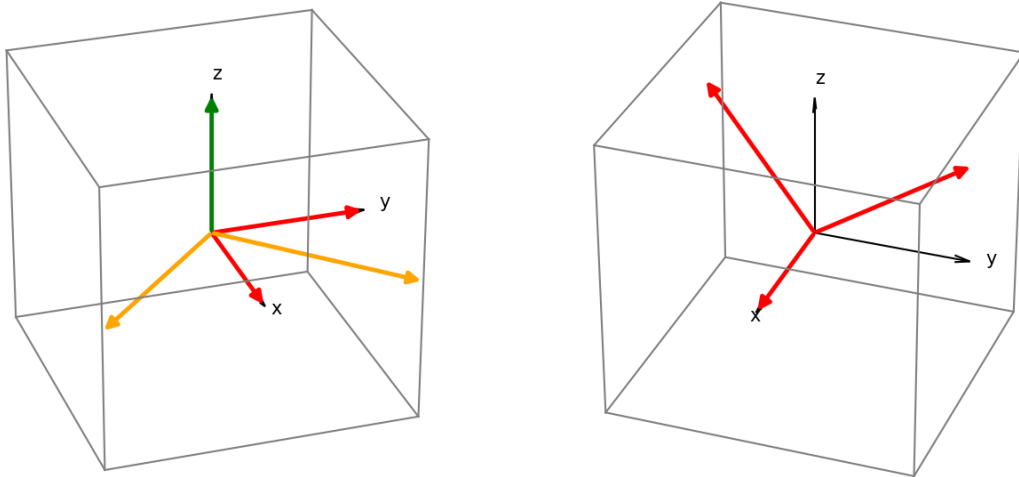
1.8 Algebraic proof in four dimensions

We now reap the fruits of our previous efforts that proved the equivalence of KS non-contextual value assignments to rays and such assignments to Hermitian operators. This allows us to give a far simpler and purely algebraic proof of the KS Theorem in four dimensions, which uses KS non-contextuality in terms of Hermitian operators. For the same reasons as in three dimensions, we can w.l.o.g. assume $\mathcal{H} = \mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ ⁵. Consider the set of operators depicted in Figure 1.6. They form the so-called “Mermin Square” CITE and can be written in terms of Pauli operators. The operators are arranged such that the operators in each of the three rows, columns are mutually commuting. Additionally, the product of the three observables in the rightmost column is $-\mathbb{1}_{\mathbb{C}^2 \otimes \mathbb{C}^2}$. The product of the three observables in all other columns and all rows is $+\mathbb{1}_{\mathbb{C}^2 \otimes \mathbb{C}^2}$. By Corollary 1.6, a KS non-contextual value assignment assigns values to the operators such that the product of values within one row, column is ± 1 , in accordance with the corresponding operator identities. It follows that, by the row identities, the product of all nine values is $+1$, whereas the column identities result in a value of -1 , a contradiction.

⁵This is not to say that we are now considering two remote systems, but only provides a convenient way to define the set of operators used in the proof.

Orthogonal triad			Other rays	The first ray is green because of
001	100	010	110 1 $\bar{1}$ 0	choice of z axis
101	$\bar{1}$ 01	010		choice of x vs -x
011	0 $\bar{1}$ 1	100		choice of y vs -y
1$\bar{1}$2	$\bar{1}$ 12	110	$\bar{2}$ 01 021	choice of x vs y
102	$\bar{2}$ 01	010	$\bar{2}$ 11	orthogonality to second and third rays
211	0 $\bar{1}$ 1	$\bar{2}$ 11	$\bar{1}$ 02	orthogonality to second and third rays
201	010	$\bar{1}$ 02	$\bar{1}$ $\bar{1}$ 2	orthogonality to second and third rays
112	1 $\bar{1}$ 0	$\bar{1}$ $\bar{1}$ 2	0 $\bar{2}$ 1	orthogonality to second and third rays
012	100	0 $\bar{2}$ 1	1 $\bar{2}$ 1	orthogonality to second and third rays
121	$\bar{1}$ 01	1 $\bar{2}$ 1	0 $\bar{1}$ 2	orthogonality to second and third rays

(a) Table containing the proof of the three-dimensional base case of the KS Theorem, in terms of a colouring problem where each orthogonal triad must contain exactly one green and two red rays. Each line of the table introduces a new orthogonal triad that is coloured, either w.l.o.g. or according to the values of previous triads. In each line, only the first ray of the triad is coloured green. The “Other rays” in each line are orthogonal to the single green ray in that line and are consequently coloured red. The stepwise colouring of rays leaves the orthogonal triad {100,021,0 $\bar{1}$ 2} coloured red, proving the impossibility of a valid colouring. Table copied from CITE.



(b) The left plot visualizes the first line of the proof-containing table, “Other rays” being coloured orange. The right plot visualizes the resulting contradiction, refer to the text.

Figure 1.5: Proof of the KS Theorem

σ_x^1	σ_x^2	$\sigma_x^1 \sigma_x^2$	\Rightarrow	$\mathbb{1}_{\mathcal{H}}$
σ_y^2	σ_y^1	$\sigma_y^1 \sigma_y^2$	\Rightarrow	$\mathbb{1}_{\mathcal{H}}$
$\sigma_x^1 \sigma_y^2$	$\sigma_y^1 \sigma_x^2$	$\sigma_z^1 \sigma_z^2$	\Rightarrow	$\mathbb{1}_{\mathcal{H}}$

\Downarrow	\Downarrow	\Downarrow
$\mathbb{1}_{\mathcal{H}}$	$\mathbb{1}_{\mathcal{H}}$	$-\mathbb{1}_{\mathcal{H}}$

Figure 1.6: “Mermin Square”: Operators on $\mathbb{C}^2 \otimes \mathbb{C}^2$ that allow for a proof of the KS Theorem in four dimensions. Tensor products between Pauli operators are omitted for readability. Within each row and each column, the operators commute. A non-contextual value assignment to these operators is contradictory, in the sense that the product of all nine values assigned to the operators would not be well-defined. For each row, the product of the three operators in that row is $\mathbb{1}_{\mathcal{H}}$. The same holds for the columns, with the notable exception that the product of the three operators in the rightmost column is $-\mathbb{1}_{\mathcal{H}}$. These identities on an operator level are key in establishing a contradiction. Figure adapted from CITE.

1.9 Local causality and Bell’s Theorem

Locality as an instance of contextuality

The following is meant to discuss the relationship between the KS Theorem and Bell’s Theorem, in particular the connection between KS non-contextuality and Bell’s notion of local causality.

Consider the well-known CHSH (thought-)experiment CITE that proves QM to be incompatible with a local HVM description. In particular, quantum theory predicts violations of the CHSH Bell inequality, which must be satisfied for all local HVM.

A source S emits pairs of entangled particles in the state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$ (EPR pairs) that split up and travel to two remote labs. In each of the labs, an experimenter can choose between two incompatible two-outcome measurements A_1, A_2 and B_1, B_2 , respectively. One can for instance think of these as measuring different spin components of the particle. We will label the outcomes with ± 1 . The setting of the CHSH experiment is shown in Figure 1.7a. Let X, Y be random variables that describe the measurement outcomes of A_i, B_i , respectively. For

uniformly chosen detector settings, we wish to measure the probability

$$p(\text{success}) = p(X = Y, A_1, B_1) + p(X = Y, A_2, B_1) \\ p(X = Y, A_2, B_2) + p(X \neq Y, A_1, B_2),$$

as is illustrated in Figure 1.7b. All correlations compatible with a general HVM description must be of the form

$$p(X = x, Y = y, A_i, B_j) = \int_{\Lambda} d\lambda \mu(\lambda) p(X = x, Y = y | A_i, B_j, \lambda) p(A_i, B_j | \lambda),$$

where μ is some probability density function on the ontic state space λ . Driven by special relativity, we further assume local causality to be a property of all valid classical descriptions: Rewriting $p(X = x, Y = y | A_i, B_j, \lambda)$ as $p(X = x | A_i, B_j, Y = y, \lambda) p(Y = y | A_i, B_j, \lambda)$, local causality is the assumption that, given a complete specification of the ontic state λ , the outcome of a measurement is independent of what other measurements are performed in a space-like separated region of space-time, as well as their outcomes CITE:

$$p(X = x | A_i, B_j, Y, \lambda) = p(X = x | A_i, \lambda) \\ p(Y = y | A_i, B_j, \lambda) = p(Y = y | B_j, \lambda)$$

For any ontological model of reality, local causality provides the best and most natural explanation for the fact that such superluminal influences have never been observed in experiments. Analogously, KS non-contextuality provides the most natural explanation for why, within the framework of QM, compatible observables do not disturb each other's outcome statistics. We will revisit this point in Section 5, where we introduce a revised notion of contextuality due to Spekkens. All correlations compatible with a local HVM description, i.e. an ontological model obeying local causality, must be of the form

$$p(X = x, Y = y | A_i, B_j) = \int_{\Lambda} d\lambda \mu(\lambda) p_A(X = x | A_i, \lambda) p_B(Y = y | B_j, \lambda).$$

The assumption of local causality for the space-like separated regions A and B manifests itself in the local response functions p_A and p_B . W.l.o.g. , we can assume p_A and p_B to be deterministic response functions, as we may shift all classical randomness into the ontic states and the distribution μ over these. This is sometimes referred to as Fine's Theorem CITE. Importantly, Fine's Theorem does not hold for general contextuality scenarios without space-like separated subsystems, as for two compatible local operators A, A' , the set of simultaneous eigenvalues is in general not given by the Cartesian product $\sigma(A) \times \sigma(A')$. We will discuss the complications that arise without space-like separated subsystems in Section 2.5.

Assuming that the experimenters can choose the detector settings freely (no super-determinism), the choice of measurement should not be correlated with the ontic state λ emitted by the source.

Therefore, $p(A_i, B_j|\lambda) = p(A_i, B_j)$. All in all, for the CHSH experiment⁶ to have a valid classical **HVM description** obeying **local causality** and the assumption of **freely random detector settings**, chosen according to a **uniform distribution**, all correlations must be of the form:

$$p(X = x, Y = y, A_i, B_j) = \frac{1}{4} \int_{\Lambda} d\lambda \mu(\lambda) p_A(X = x|A_i, \lambda) p_B(Y = y|B_j, \lambda), \quad (1.1)$$

where p_A and p_B are deterministic probability assignments.

Given the general expression for correlations, as prescribed by a local HVM description 1.1, it immediately follows that $p(\text{success})$ is upper-bounded by $\frac{3}{4}$ for any such HVM description. This can be verified in the same manner as in Section 1.1, where we showed deterministic and non-contextual HVM descriptions to be incompatible with Specker's three boxes: Examining Figure 1.7b, we realize that there is no deterministic assignment of measurement outcomes ± 1 to the four measurement operators, such that all "winning" correlations are satisfied. In fact, for any such assignment, at most three of the four "winning" correlations are satisfied. The local response functions in Equation 1.1 induce such a deterministic assignment. Therefore, for every ontic state λ , $p(\text{success})$ is upper-bounded by $\frac{3}{4}$. The ensemble average over all ontic states $\lambda \in \Lambda$ is thus upper-bounded by the same quantity. Consequently, $p(\text{success}) \leq \frac{3}{4}$ is a Bell inequality: the so-called **CHSH Bell-inequality** CITE:

$$p(\text{success}) = p(X = Y, A_1, B_1) + p(X = Y, A_2, B_1) + p(X = Y, A_2, B_2) + p(X \neq Y, A_1, B_2) \leq \frac{3}{4} \quad (1.2)$$

In QM, outcome probabilities are computed according to the Born rule. Assuming projective measurements on a pure quantum state, permissible quantum correlations for Bell-type experiments are of the form

$$p(X = x, Y = y|A_i, B_j) = \langle \Psi | P_{A_i}^x \otimes P_{B_j}^y | \Psi \rangle, \quad (1.3)$$

where $|\Psi\rangle \in \mathcal{H}$ is a normalized, bipartite Hilbert state vector and $P_{A_i}^x$ is the projection operator corresponding to the outcome x of the measurement A_i on the subsystem A.

What is remarkable about the CHSH scenario is the fact that one can give explicit projective measurements A_1, A_2, B_1, B_2 , as well as the maximally entangled pure state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$, for which quantum theory predicts $p(\text{success}) = \cos^2(\frac{\pi}{8}) > \frac{3}{4}$: a violation of the CHSH

⁶This can of course be extended to more general correlation experiments in a Bell-like setting.

Bell inequality. A set of Hermitian operators with this property is

$$\begin{aligned}
A_1 &= |0\rangle\langle 0| - |\pi\rangle\langle \pi|, \\
A_2 &= \left|\frac{\pi}{2}\right\rangle\left\langle\frac{\pi}{2}\right| - \left|\frac{3\pi}{2}\right\rangle\left\langle\frac{3\pi}{2}\right|, \\
B_1 &= \left|\frac{\pi}{4}\right\rangle\left\langle\frac{\pi}{4}\right| - \left|\frac{5\pi}{4}\right\rangle\left\langle\frac{5\pi}{4}\right|, \\
B_2 &= \left|\frac{3\pi}{4}\right\rangle\left\langle\frac{3\pi}{4}\right| - \left|\frac{7\pi}{4}\right\rangle\left\langle\frac{7\pi}{4}\right|,
\end{aligned} \tag{1.4}$$

with $|\theta\rangle := \cos(\frac{\theta}{2})|\uparrow\rangle + \sin(\frac{\theta}{2})|\downarrow\rangle$ CITE. Recall that we label the two distinct measurement outcomes with the eigenvalues ± 1 . Each local measurement projects the state of the particle on which it acts onto one of two antipodal surface points in the xz -plane of the single qubit Bloch sphere. As the two particles are entangled, in particular maximally correlated, measuring one particle will “steer” the state of the other particle⁷. Thus, the probability of the measurements (A_1, B_2) yielding anti-correlated outcomes is given by $\frac{1}{2}|\langle 0|\frac{7\pi}{4}\rangle|^2 + \frac{1}{2}|\langle \pi|\frac{3\pi}{4}\rangle|^2 = \cos^2(\frac{\pi}{8})$. In fact, for each of the four possible measurement settings, the probability of the measurement outcomes satisfying the correlations in Figure 1.7b is $\cos^2(\frac{\pi}{8})$. To see this, recall that we may associate with each measurement outcome a point on the Bloch sphere, and by extension with every measurement operator an axis connecting two antipodal points on the Bloch sphere, as shown in Figure 1.8. These axes are rotated by an angle of $\frac{\pi}{4}$ (in real space) relative to one another and lie within the xz -plane. For arbitrary A_i and B_j , the measurement operators are chosen in such a way that the overlap of the Bloch states corresponding to measurement outcomes that obey the correlations in Figure 1.7b is $\cos(\frac{\pi}{8})$.

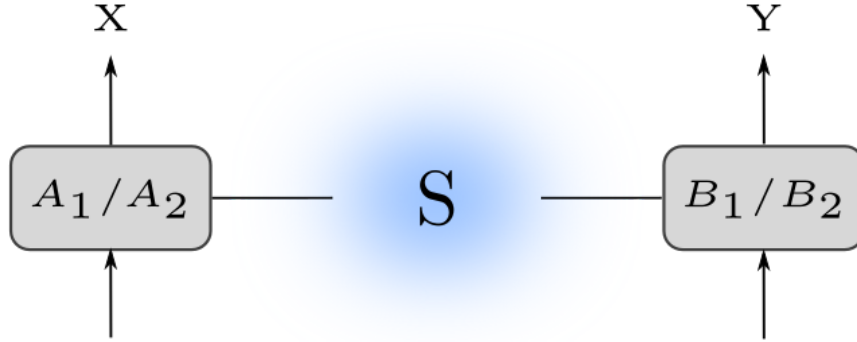
In view of Section 3, important properties of the CHSH Bell test are:

- The choice of quantum state and measurements above leads to a maximal violation of the CHSH inequality, as allowed by QM. We will show this in Section 2. The quantum supremum B_Q of a Bell inequality’s left hand side is called its Tsirelson bound. The Tsirelson bound of the CHSH Bell inequality is $\frac{1}{2}(1 + \frac{1}{\sqrt{2}})$.
- Assuming projective measurements, all quantum experiments $\langle \Psi | \Pi_{A_i}^x \otimes \Pi_{B_j}^y | \Psi \rangle$ producing a maximal violation of the CHSH Bell inequality are essentially equivalent and related by some trivial operations. This property gives rise to the notion of self-testing and will be discussed in more detail in Section 3.

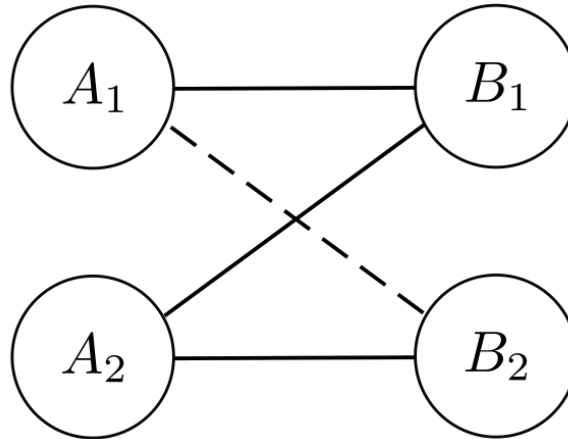
A generic Bell inequality in the bipartite scenario is of the form

$$\sum_i w_i p(\epsilon_i) \leq B_{\mathcal{L}}, \tag{1.5}$$

⁷This does not violate the no-signalling principle, as can be seen by computing the reduced density operator of system B: it describes a maximally mixed state.



(a) CHSH setup leading to violations of the CHSH inequality. A source S emits entangled EPR pairs. Constituent particles are measured in space-like separated regions, with experimenters being able to choose between two incompatible measurements A_1, A_2 or B_1, B_2 . The random variables X, Y hold the measurement outcome for operators A_1, A_2 or B_1, B_2 , respectively.



(b) CHSH correlations for space-like separated operators A_1, A_2 and B_1, B_2 . Lines indicate joint measurability. A solid line further indicates correlated outcomes, whereas a dashed line indicates anti-correlated outcomes. Let X, Y be random variables that describe the measurement outcomes of A_i, B_i , respectively. Thus, we are interested in the probability that $X = Y$ for all choices of measurements but (A_1, B_2) and $X \neq Y$ for the measurements (A_1, B_1) . Adapted from CITE

Figure 1.7: CHSH Bell test

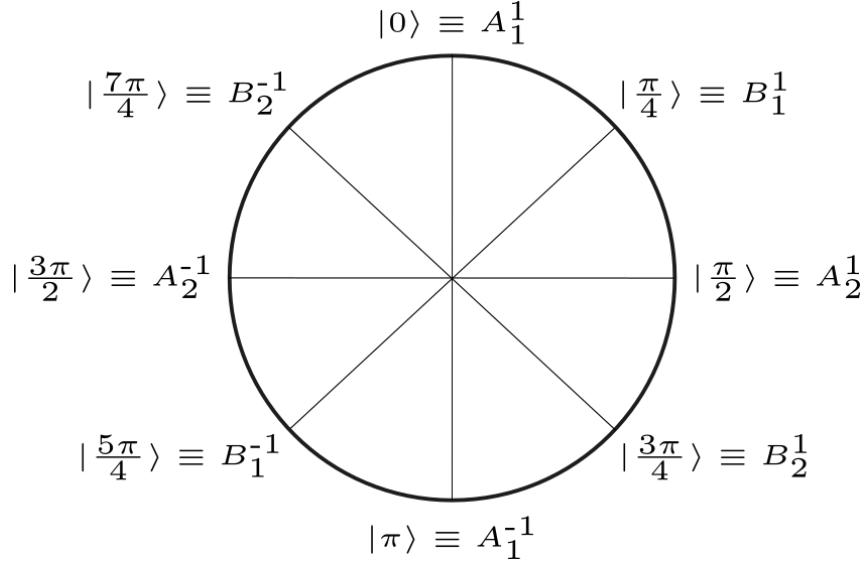


Figure 1.8: The two outcomes of each of the four projective CHSH measurements A_1 , A_2 , B_1 , and B_2 correspond to two antipodal surface points in the xz -plane of the Bloch sphere. For example, the projector associated with the outcome 1 of the measurement A_1 projects onto the state $|0\rangle$. The overlap between two states that are related via a rotation of $\frac{\pi}{4}$ in real space have an overlap of $\cos(\frac{\pi}{8})$.

where $w_i > 0$ and ϵ_i denotes the i -th measurement event. For now, assuming bipartite Bell tests, a measurement event can be thought of as just an initial preparation P of the system (for instance in an EPR state), followed by both parties performing some local measurement a, b , and obtaining outcomes x, y : $a, b|x, y$. We have used the same notation as in CITE and will revisit their proposed graph-theoretic framework in more detail in Section 2, in order to identify the maximal quantum violation of a general KS non-contextuality inequality. Note that all linear combinations of probabilities pertaining to our correlation experiment can be expressed as a positive linear combination with $w_i > 0$, like in 1.5. For any negative w_i we may rewrite

$$w_i p(\epsilon_i) = w_i p(x_i, y_i | a_i, b_i) = w_i - w_i \sum_{(x'_i, y'_i) \neq (x_i, y_i)} p(x'_i, y'_i | a, b),$$

obtaining a positive linear combination of probabilities.

The statistics of a Bell-type correlation experiment are fully characterized by the set $\{p(x, y | a, b)\}_{x, y, a, b}$. For the CHSH experiment, this set contains $2^4 = 16$ values between 0 and 1, which obey some normalization conditions and can be represented by a vector in \mathbb{R}^{16} . Identifying permissible correlations as a subset of $[0, 1]^{16}$, in the case of CHSH, one can characterize the geometry of correlations that are compatible with say local causality, quantum theory, or the no-signalling principle. Each set of permissible correlations is clearly convex: While one can check that convex combination preserves the relevant constraints, there is an intuitive argument as to why this must be the case: Assume that we have two experimental procedure, each producing “allowed” statistics. As we only impose constraints on what correlations we consider to be “allowed”, the

actual existence of such procedures cannot be discarded. We now consider the following new experimental procedure: we generate a random bit and then perform the “old” experimental procedure corresponding to the value of the generated bit. By changing the bias of our “coin”, we can produce all statistics that are convex combinations of the initial, permissible statistics.

As discussed, all correlations $p(x, y|a, b)$ compatible with an ontological model obeying local causality are convex combinations of local, deterministic probability assignments $p_A(x|a, \lambda)p_B(y|b, \lambda)$. There are $2^4 = 16$ distinct such assignments. Thus, the set of local correlations

$$\mathcal{L} = \{\{p(x, y|a, b)\}_{x, y, a, b} \equiv \vec{v} \in \mathbb{R}^{16} : \vec{v} \text{ compatible with local causality}\} \quad (1.6)$$

is a polytope in \mathbb{R}^{16} with 16 extremal vertices. A Bell inequality 1.5 defines a separative hyperplane that splits $[0, 1]^{16}$ in two, such that the closed half-space contains \mathcal{L} . The Bell hyperplane could for instance be a facet of the local polytope, as shown in 1.9. A facet of \mathcal{L} is one of its delimiting hyperplanes. Correlations that maximally violate the CHSH Bell inequality have a maximal distance to the Bell hyperplane. Bell’s Theorem tells us that the set of quantum correlations \mathcal{Q} is strictly larger than \mathcal{L} . The set \mathcal{Q} cannot be characterized in terms of a finite number of delimiting hyperplanes (linear inequalities) CITE.

The no-signalling principle is built into the tensor product representation of traditional QM. As such, the set \mathcal{Q} is contained in the set \mathcal{NS} , which contains all correlations that obey no-signalling. Like \mathcal{L} , \mathcal{NS} is a polytope: \mathcal{NS} is the intersection of $[0, 1]^{16}$, the hyperplanes corresponding to normalization constraints $\sum_{x, y} p(x, y|a, b) = 1$, and those defined by the no-signalling conditions

$$\sum_y p(x, y|a, b) = \sum_y p(x, y|a, b') \text{ and} \quad (1.7)$$

$$\sum_x p(x, y|a, b) = \sum_x p(x, y|a', b) . \quad (1.8)$$

The set \mathcal{NS} is strictly bigger than \mathcal{Q} , as there exist correlations, such as the “PR-box” CITE, that satisfy the no-signalling principle and win the “CHSH game” with certainty.

Analogous to Bell inequalities, KS non-contextuality inequalities define hyperplanes that splits the convex set of all permissible correlations in two, such that the closed half-space contains all correlations compatible with a KS non-contextual HVM description.

We have captured the essence of Bell’s Theorem, cast in a quantum framework, proving that the predictions of QM are incompatible with a local HVM description. Let us now study the relationship between KS non-contextuality and Bell’s notion of local causality, or equivalently local determinism (recall Fine’s Theorem CITE).

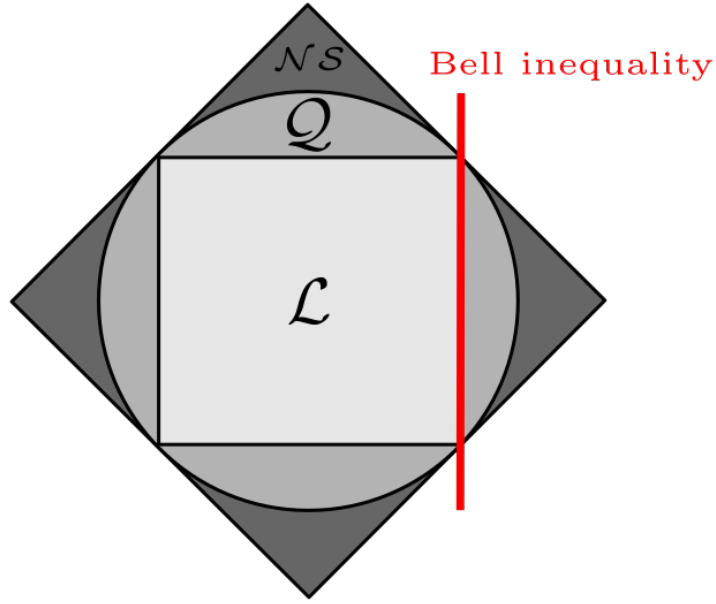


Figure 1.9: Schematic drawing depicting the geometry of the convex sets \mathcal{L} , \mathcal{Q} , and \mathcal{NS} , containing all local, quantum, and no-signalling correlations, respectively. The sets \mathcal{L} , \mathcal{NS} are polytopes and can be characterized by a finite number of facet hyperplanes, whereas the set \mathcal{Q} is more complicated. A Bell inequality, like the CHSH inequality, defines a separative hyperplane such that all local correlations \mathcal{L} lie to one “side” of this hyperplane.

The assumption of local determinism is an assumption of context independence for remote measurement contexts, like for Bell-type setting CITE. Both boil down to measurement outcomes being independent of which remote measurement is simultaneously performed (and its outcome). It is important to keep in mind however that KS non-contextuality is a more general assumption, as compatible measurements are not required to be remote. As Bell’s Theorem implies the impossibility of a local HVM of QM, it in particular also proves the impossibility of a KS non-contextual deterministic HVM of QM. This, together with the fact that locality is accepted as a fundamental pillar of modern physics, is why Bell’s Theorem is considered to be stronger than the KS Theorem.

Informally speaking, Bell’s Theorem, as presented above, is proof of a KS Theorem in four dimensions, with the nice property that KS non-contextuality is only assumed when it can be justified by locality. A caveat that stems from only assuming KS non-contextuality for remote measurement contexts is that the CHSH proof assumes an entangled state preparation $|\Psi\rangle$. In contrast, the KS Theorem, as formally stated in Theorem 1.9, is entirely measurement-dependent. Theorem 1.9 states that any attempt to assign outcomes to measurements in a KS non-contextual manner is inherently contradictory, no matter the state preparation. In Section 1.8 we examined a KS proof in four dimensions that made use of the full extent of KS non-contextuality. Unsurprisingly, “watering down” the assumption of KS non-contextuality,

like in the CHSH proof, requires us to make an additional assumption in order to arrive at a contradiction. For the CHSH proof this assumption is that of a fixed, entangled quantum state.

Mermin gives a GHZ-like proof of an eight-dimensional KS Theorem CITE, which can be recast into a proof of Bell's Theorem, further highlighting the deep connection between both theorems. Again, the information "lost" in assuming KS non-contextuality only when justified by locality is compensated for by a fixed quantum state. Thus, the fact that the KS Theorem is only measurement-dependent and in particular makes no reference to the system preparation is a consequence of the stronger assumption of KS non-contextuality imposed by the KS Theorem.

On a final note, it is worth pointing out that the statement of the KS Theorem is of a fundamentally different nature than that of Bell's Theorem. In particular, it does not propose inequality constraints on correlations that, when found to be violated, would imply the impossibility of a particular classical description. The KS Theorem goes further than that and deduces the impossibility of such an attempt altogether, only by geometric considerations that lead to a contradiction. Nevertheless, this strength is also a key weakness of the theorem. A Bell inequality is operationally testable and does not make reference to the formalism of QM. The general form of correlations compatible with a local HVM description 1.1 makes sense, independent of QM. Therefore, the CHSH Bell inequality also holds for theories beyond QM. An experimental observation of a Bell inequality violation implies that there can be no operational theory of reality (compatible with experimental observations) that admits of a locally causal hidden variable theory, quantum theory only being one possible candidate. Any such theory would be in conflict with the experimental data which violates Bell inequalities CITE. By comparison, the KS Theorem is formulated within the framework of QM, for instance making explicit reference to the system's Hilbert space. Therefore, the scope of the KS Theorem is limited to quantum theory.

In Section 5 we revise the traditional notion of KS non-contextuality. Spekkens proposes a largely operational notion of non-contextuality that addresses these shortcomings CITE. However, for the purposes of self-testing, which assumes the validity of QM, the main advantage of Spekkens contextuality will be that it allows for a consistent treatment of unsharp POVM quantum measurements.

1.10 State-dependent KS contextuality

The KCBS inequality and odd n -cycle scenarios

In Section 1.7 we found that for every projective Hilbert space of dimension ≥ 3 there exists a finite set of rays for which there exists no KS non-contextual value assignment according to Definition 1.7. Identifying these rays with projective measurements of rank one gives us a set of projective measurements for which there exists no KS non-contextual value assignment

according to Definition 1.4. We call such sets of projective measurements (or rays in a projective Hilbert space) KS uncolourable, owing to the fact that one can reduce the problem of finding a valid KS non-contextual value assignment to a colouring problem, as was done in Section 1.7.

Definition 1.13. A set of projective measurements (rays in a projective Hilbert space) is called *KS colourable* if there exists a KS non-contextual assignment of measurement outcomes (values 0,1) to the measurements (rays) in the set. If there exists no such value assignment, the set is called *KS uncolourable*.

KS uncolourable sets embody the “strongest” proofs that QM is incompatible with a KS non-contextual HVM description, in the sense that the structure in the set of measurements alone is enough to arrive at a logical contradiction. Nevertheless, as alluded to in Section 1.9, also KS colourable sets of projective measurements may exhibit KS contextuality. This manifests itself in QM predicting violations of KS non-contextuality inequalities for some KS colourable sets. A KS non-contextuality inequality is, analogously to a Bell inequality, a linear inequality constraint on the convex set of correlations that is satisfied for all correlations compatible with a KS non-contextual HVM description. We will discuss the simplest KS non-contextuality inequality, the KCBS inequality, as well as an infinite class of generalizations thereof. These KS non-contextuality inequalities allow for self-testing, as will be the subject of Section 4. Before that, note that we have already come across one example of a KS non-contextuality inequality, the CHSH inequality, albeit in the context of Bell nonlocality. The set of CHSH measurements given in Section 1.9 that produce a maximal violation of the CHSH inequality when acting on an EPR pair is a KS colourable set of projective measurements. QM predicts state-dependent Bell non-locality, as witnessed by violations of the CHSH inequality. For reasons highlighted in Section 1.9, the CHSH inequality is also a KS non-contextuality inequality and the non-local correlations of the CHSH experiment exhibit state-dependent KS contextuality. Programming the source to emit separable bipartite states, as opposed to EPR pairs, will yield KS non-contextual correlations.

Note that there even exist KS colourable sets of measurements that produce state-independent KS contextuality, in the sense that QM predicts the violation of a KS non-contextuality inequality for all possible quantum state preparations. An example of such a set is one proposed by Yu and Oh CITE, simply called the Yu-Oh set. It is a set of 13 rank one projectors that act on a three-dimensional Hilbert space. In fact, if one identifies the rank one projectors with rays in a projective Hilbert space, the 13 Yu-Oh rays are a subset of the 33 rays of the Peres configuration that we used in proving the KS Theorem in Section 1.7. The Yu-Oh set is the minimal set of rays revealing state-independent KS contextuality, as proved in CITE.

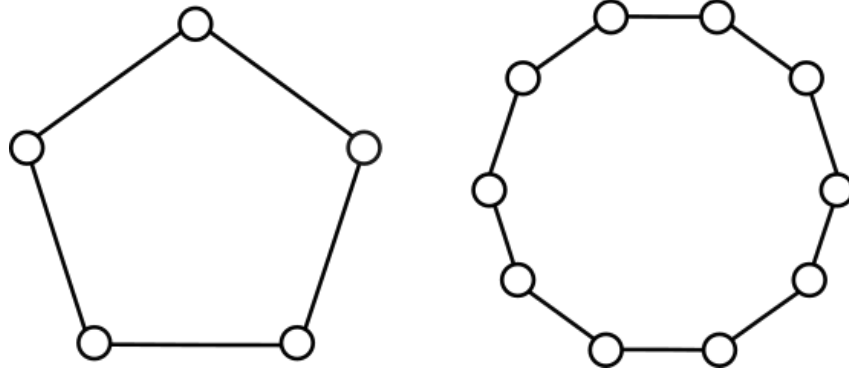


Figure 1.10: Graph showing the compatibility relations for the KCBS (left) and a higher order odd n -cycle scenario (right). Each vertex corresponds to one projective measurement, adjacent measurements are compatible.

The perhaps simplest example of a quantum experiment leading to (state-dependent) KS contextuality for a single quantum system is the KCBS scenario CITE. We will present the corresponding KCBS KS non-contextuality inequality, which will become relevant when we later discuss self-testing of quantum systems in Sections 3 and 4.

For the KCBS KS contextuality scenario we assume five dichotomic projective measurements with outcomes 0 and 1. Additionally, the five measurements should obey cyclic compatibility, as shown in Figure 1.10. We freely choose a pair of compatible measurements (measurement context) at random, each with equal probability, and determine the value of the following linear sum of probabilities:

$$\sum_{i=1}^5 p(0, 1|i, i+1) \leq 2, \quad (1.9)$$

where compatibility with a KS non-contextual HVM imposes an upper bound of 2 for this expression: For any HVM description of the KCBS experiment, $\sum_{i=1}^5 p(0, 1|i, i+1)$ can be rewritten as $\int_{\Lambda} d\lambda \mu(\lambda|i, i+1) \sum_{i=1}^5 p(0, 1|i, i+1, \lambda)$, where μ is some probability density function over the ontic state space Λ . Furthermore $\mu(\lambda|i, i+1) = \mu(\lambda)$ as we assume the choice of measurement context to be uncorrelated with the system. Analogously to Specker's three boxes or the CHSH experiment, one can easily verify that for a KS non-contextual HVM description this quantity has a non-trivial upper bound.

We now present a quantum experiment involving a KS colourable set of five projective measurements on a qutrit for which QM predicts a (maximal) violation of the KCBS KS non-contextuality inequality. The five projective measurements of this reference experiment correspond to five rank one projectors that project onto five pure qutrit states. The components of these five qutrit states with respect to the standard qutrit basis $\{|0\rangle, |1\rangle, |2\rangle\}$ are real, allowing us to represent them as vectors in \mathbb{R}^3 CITE, see Figure 1.11. The angle between each of the five vectors and the $|0\rangle$ qutrit axis is $\cos^2(\theta) = \frac{1}{\sqrt{5}}$. The following is an explicit expression for five

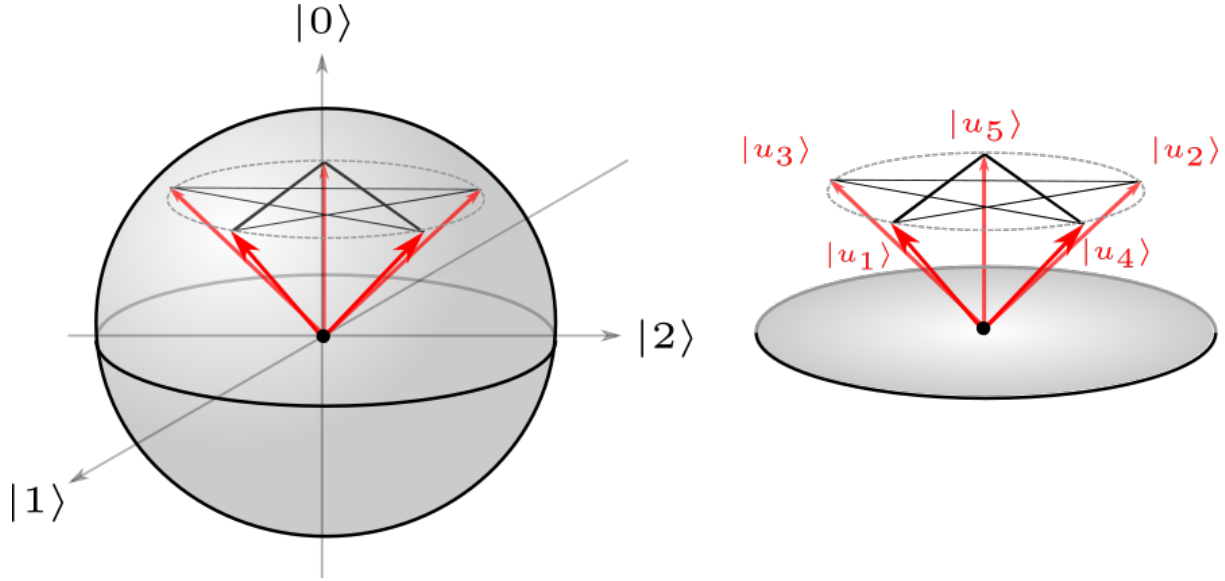


Figure 1.11: Reference KCBS experiment on a qutrit leading to a maximal violation of the KCBS KS non-contextuality inequality. An experimenter can freely choose between five rank one projective measurements $|u_i\rangle$ that satisfy cyclic compatibility: $\langle u_i | u_{i \oplus 1} \rangle = 0$. The qutrit is initially prepared in the state $|0\rangle$ and then measured in one of five measurement contexts $(i, i \oplus 1)$. After the measurement the qutrit state is “reset”.

suitable qutrit states that also satisfy cyclic compatibility:

$$|u_j\rangle = (\cos(\theta), \sin(\theta) \sin(j\phi), \sin(\theta) \cos(j\phi))^T, \quad (1.10)$$

for $j = 1 \dots 5$ with $\phi = \frac{4\pi}{5}$.

For these five projective measurements acting on the $|0\rangle$ qutrit state, QM predicts

$$\sum_{i=1}^5 p(0, 1 | i, i+1) = \frac{5}{\sqrt{5}} = \sqrt{5} > 2.$$

Not only does the reference quantum experiment $|0\rangle, \{|u_i\rangle\}_{i=1}^5$ produce correlations that violate the KCBS non-contextuality inequality, but this violation is also maximal, as we will see in Section 2.

One can generalize the KCBS KS non-contextuality inequality to greater cycle lengths n . For the odd integers $n \geq 5$, where n parametrizes the number of dichotomic projective measurements obeying cyclic compatibility, one finds a class of KS non-contextuality inequalities CITE

$$\sum_{i=1}^n p(0, 1 | i, i+1) \leq \frac{n-1}{2}. \quad (1.11)$$

The quantum supremum for a given cycle length n is given by

$$B_q^{(n)} = \frac{n \cos\left(\frac{\pi}{n}\right)}{1 + \cos\left(\frac{\pi}{n}\right)} \quad (1.12)$$

and can be achieved by measuring the $|0\rangle$ qutrit state with the rank one projective measurements

$$|u_j^{(n)}\rangle = (\cos(\theta_n), \sin(\theta_n) \sin(j\phi_n), \sin(\theta_n) \cos(j\phi_n))^T, \quad (1.13)$$

for $j = 1 \dots n$, $\cos^2(\theta_n) = \frac{\cos\left(\frac{\pi}{n}\right)}{1 + \cos\left(\frac{\pi}{n}\right)}$, and $\phi = \frac{(n-1)}{n}\pi$.

In Section 2, we adopt the graph-theoretic framework proposed in CITE to classify correlations of an arbitrary experiment. It turns out that for a general linear combination of probabilities one can relate the KS non-contextual and quantum suprema to certain graph invariants. Computing these graph invariants corresponds to solving a mathematical optimization problem. In view of self-testing, this will enable us to identify extremal correlations and study whether these can only be realized in an essentially unique manner.

Chapter 2

CSW graph-theoretic approach to quantum correlations

In CITE, the authors propose a graph-theoretic approach to classifying the statistics of an arbitrary correlation experiment that aims to determine the value of some linear combination of probabilities. In particular, CITE identifies a hierarchical structure that classifies correlations as classical (KS non-contextual), quantum, or more generally obeying the exclusivity principle¹. This hierarchical structure is established by associating with a correlation experiment its corresponding “exclusivity graph”. The exclusivity structure obeyed by the experiment imposes non-trivial constraints on the set of permissible correlations compatible with KS non-contextuality, QM, or the exclusivity principle. In particular, upper bounds of the relevant linear combination of probabilities, within these three classes of theories, can be related to graph invariants of the experiment’s exclusivity graph.

2.1 Generic correlation experiments in the CSW framework

The CSW framework takes an operationalist approach to correlation experiments, in the sense that any such experiment is described in terms of a sequence of (reproducible) experimental procedures. For example, one such experimental procedure may be to prepare the system in a certain way. If they provide the experimenter with an output (apart from the transformed system), such experimental procedures can be considered *tests*. For a preparation P , and test M with outcomes $\{k_i\}_i$, an operational theory specifies the probabilities $p(k_i|M, P)$.

There may be multiple equivalent ways of preparing a system, in the sense that we can never tell the preparation procedures apart by experiment, as they yield identical probability distributions

¹The exclusivity applies to pairwise exclusive events and bars their added probabilities from exceeding 1. We will come back to it shortly.

for all tests. The equivalence class of operationally indistinguishable preparation procedures P defines a state. In the same manner, the equivalence class of operationally indistinguishable tests M defines an observable².

We now define what it means for two observables to be compatible, or jointly measurable. While in QM two projective measurements are compatible if and only if the corresponding Hermitian operators commute, we want to define compatibility in operational terms, in order for the definition to be applicable to arbitrary operational theories.

Definition 2.1 (CITE). Two observables M_1, M_2 with outcome sets $\{k_i\}_i$ and $\{k'_i\}_i$, respectively, are *jointly measurable* if there exists an observable M_{12} with outcome set $\{k_i, k'_j\}_{i,j}$ such that:

1. \forall preparations P, \forall outcomes $k \in \{k_i\}_i$:

$$p(k|M_1, P) = \sum_{k'} p(k, k'|M_{12}, P)$$
2. \forall preparations P, \forall outcomes $k' \in \{k'_j\}_j$:

$$p(k'|M_2, P) = \sum_k p(k, k'|M_{12}, P)$$

One may think of M_{12} as measuring both M_1 and M_2 simultaneously, placing the outcome of M_1 into its first and the outcome of M_2 into its second output register. By discarding one of the outcomes, we reduce M_{12} to the corresponding single-outcome measurement. At the level of probability distributions, discarding one of the outcomes corresponds to a summation over all possible outcomes held by the discarded register. For QM as operational theory and projective measurements, this definition is equivalent to operator commutativity. For general quantum measurements, Definition 2.1 implies that the POVM M_1 and M_2 are obtained by coarse graining the POVM M_{12} that jointly realizes both.

An *event* is characterized by a list of compatible tests (M_1, M_2, \dots, M_n) that yield some outcomes (X_1, X_2, \dots, X_n) . Let $X_1, X_2, \dots, X_n | M_1, M_2, \dots, M_n$ denote such event. Some events may be operationally equivalent, in the sense that they have the identical probability of occurring, for all initial preparations of the system. Analogous to our treatment of preparations and tests, operationally equivalent test-outcome tuples define the same event. Within QM, a state would be defined by a density operator, whereas equivalent events correspond to the same positive semi-definite operator or projector.

A correlation experiment, like the KCBS or CHSH experiment, consists of one or multiple experimenters performing subsets of compatible tests on an initial preparation P of the system. For instance, in the KCBS experiment, an experimenter chooses between five measurement contexts $(i, i \oplus 1)$. We assume that an experimenter can freely choose between measurement

²This identification of operationally equivalent tests and preparations parallels the operational equivalence classes we will introduce in Section 5, when introducing the notion of Spekkens contextuality.

contexts, i.e. subsets of compatible tests, meaning that this choice is not correlated with say the system being measured. By recording which of all could-be events occurs for many repetitions of this procedure, the experimenter(s) can determine the value of some relevant linear combination $\sum_i w_i p(\epsilon_i|P)$, where $w_i > 0$, and ϵ_i denotes a possible event i.e. a set of compatible tests that yield some outcomes. Sometimes we will omit the initial preparation P in our formulas. Let us now discuss what conditions and assumptions allow us to impose non-trivial constraints on correlations.

We define two events to be exclusive if they specify different outcomes for identical tests.

Definition 2.2 (CITE). For outcome-repeatable tests, two events

$$X_1, X_2, \dots, X_n \mid M_1, M_2, \dots, M_n \quad \text{and} \quad X'_1, X'_2, \dots, X'_m \mid M'_1, M'_2, \dots, M'_m$$

are *exclusive* if for some i and j

$$M_i = M'_j \quad \text{and} \quad X_i \neq X'_j.$$

An important assumption that underlies the CSW framework is that all measurements are assumed to be outcome-repeatable, meaning that performing two identical tests in a consecutive manner always yields identical outcomes. As such, we can, by performing a subsequent measurement M_i , perfectly distinguish between two exclusive events. The assumption of outcome repeatability compels us to restrict quantum measurements to PVM, as this is not a general feature of unsharp quantum measurements. Dropping the assumption of outcome repeatability and thereby accounting for general POVM quantum measurements will not provide us with any non-trivial constraints distinguishing the set of quantum correlations from the the set of general probabilistic assignments, at least not within the CSW framework. The reason for this that the exclusivity principle, which we invoke in Section 2.3 to obtain constraints on correlations, is in general not true for theories within which pairwise compatibility does not imply joint compatibility (Specker's "fundamental theorem", see Section 1.1). There exist POVM that are pairwise compatible, but fail to satisfy joint compatibility CITE. Section 2.4 will discuss in more detail how the CSW framework falls apart when allowing for general unsharp measurements.

For a correlation experiment with two exclusive could-be measurements events ϵ_1 and ϵ_2 , the sum of their probabilities cannot exceed 1, for all initial preparations P : Take M to be the test that perfectly distinguishes between ϵ_1 and ϵ_2 , like in Definition 2.2, with ϵ_1 specifying the outcome X_1 and ϵ_2 the outcome X_2 . Imagine performing the test M after one input-output

round of the correlation experiment:

$$\begin{aligned} 1 &\geq p(\{X_1, X_2\}|M) \\ &\geq p(\{X_1, X_2\}|M, \epsilon_1) p(\epsilon_1) + p(\{X_1, X_2\}|M, \epsilon_2) p(\epsilon_2) \\ &= p(\epsilon_1) + p(\epsilon_2). \end{aligned}$$

The exclusivity principle extends this property to sets of pairwise exclusive events:

Principle 2.3 (Exclusivity principle).

For a set of pairwise exclusive measurement events of a correlation experiment $\{\epsilon_i\}_i$

$$\sum_i p(\epsilon_i) \leq 1,$$

for all initial preparations.

We denote the set of correlations that are compatible with the exclusivity principle by E_1 .

Not all valid assignments of probabilities to events of a correlation experiment, i.e. probabilistic models of a correlation experiment, are compatible with the exclusivity principle. For example, consider Specker's three boxes, as introduced in Section 1.1. The relevant linear combination of probabilities that this correlation experiment tests is

$$S = \frac{1}{3} \sum_{i=1}^3 p(0, 1 | i, i \oplus 1) + \frac{1}{3} \sum_{i=1}^3 p(1, 0 | i \oplus 1, i).$$

For both sums, the events that appear are pairwise exclusive. Therefore, within theories obeying the exclusivity principle, S is upper-bounded by $\frac{2}{3}$. However, by assigning the probability $\frac{1}{2}$ to each event appearing in S we obtain a probability assignment that is compatible with pairwise exclusivity, but incompatible with the exclusivity principle. Furthermore, this assignment of probabilities is consistent with perfectly anti-correlated outcomes for all measurement contexts, as was considered in Section 1.1. While the exclusivity principle must not be true for arbitrary theories, it does apply to QM, if all measurements are projective, and by extension to KS non-contextual correlations. This can be seen by noting that exclusive events must be assigned orthogonal projectors and noting that for a set of orthogonal projectors $\{\Pi_i\}_i$, $(\mathbb{1} - \sum_i \Pi_i)$ is a positive semi-definite operator. Therefore, in Section 2.3, we may use the exclusivity principle to impose constraints on the set of quantum (and KS non-contextual) correlations.

If we extend quantum measurements to general POVM, the exclusivity principle must no longer hold, as we just demonstrated for correlation experiments like Specker's three boxes (assign the positive operator $\frac{\mathbb{1}}{2}$ to all measurement events).

2.2 The exclusivity graph of a correlation experiment

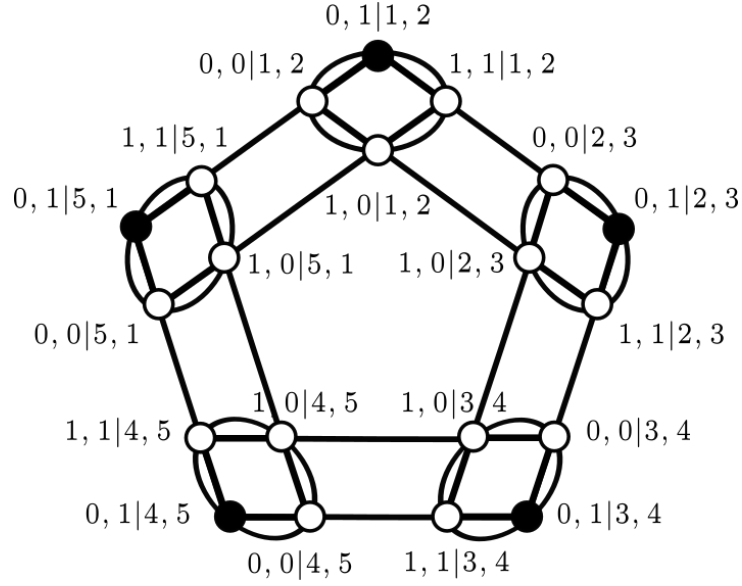
The exclusivity graph of a correlation experiment is a convenient representation of the exclusivity structure obeyed by events that imposes constraints on what correlations are permissible within different classes of theories. The exclusivity graph of a correlation experiment $S = \sum_i w_i p(\epsilon_i)$ has a vertex for each measurement event ϵ_i appearing in S . Furthermore, exclusive measurement events are represented as adjacent vertices, as shown in Figure 2.1 for the KCBS correlation experiment, or in Figure 2.2 for Specker’s three boxes.

2.3 Hierarchical structure of correlations

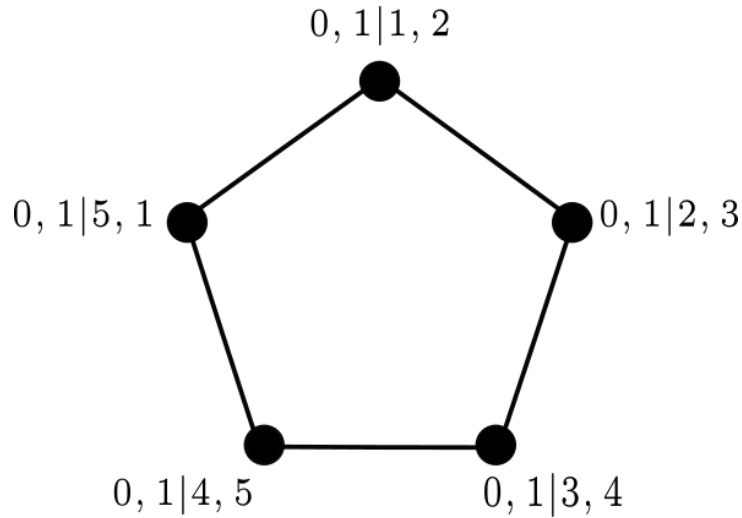
We will now relate upper bounds for the quantity S , for correlations that are compatible with KS non-contextuality, QM, or the exclusivity principle, to graph invariants of the experiment’s exclusivity graph. All three graph invariants can be computed by solving a corresponding mathematical optimization problems. The proofs we present are adapted from CITE:

- **KS non-contextual correlations:** The CSW approach decouples KS non-contextuality from the operational theory that is QM and extends it to arbitrary operational theories. For a given exclusivity graph, classical i.e. KS non-contextual correlations are those that can be written as a convex combination of deterministic assignments $\nu : V \mapsto \{0, 1\}$ that obey the exclusivity principle, where V is the vertex set of the correlation experiment. Let us verify that this extension to arbitrary operational theories is compatible with KS non-contextuality as defined in Definition 1.4: Assuming QM and projective measurements, each event defines a (distinct) projector, as we identify operationally equivalent events. Furthermore, pairwise exclusive events correspond to pairwise orthogonal projectors and can be regarded as part of the same resolution of the identity. The correlation experiment is consistent with a KS non-contextual description according to Definition 1.4 if and only if all correlations are compatible with an assignment ν like above, up to convex combination. Therefore, for the case of QM as operational theory and perfectly sharp measurements, the generalized notion of classicality in CITE reduces to KS non-contextuality like in Definition 1.4.

As will be discussed in Section 5.1, lifting the notion of KS non-contextuality as relevant criterion for non-classicality to arbitrary operational theories, and assigning outcomes in a deterministic fashion to potentially unsharp measurements, gives rise to a number of inconsistencies. Therefore, this “incremental” CITE approach towards an operational notion of non-contextuality seems to be flawed. An alternative will be presented in Section 5.2.



(a) Graph depicting all events of the KCBS correlation experiment and the underlying exclusivity structure. An event is defined in terms of a measurement context and outcome tuple, and is represented by a vertex of the graph. Adjacent vertices correspond to “mutually exclusive” events. Figure copied from CITE.



(b) Simplified exclusivity graph for the KCBS correlation experiment. Contains only those vertices in (a) whose probabilities appear in the KCBS KS non-contextuality inequality. Figure copied from CITE.

Figure 2.1: KCBS exclusivity graph

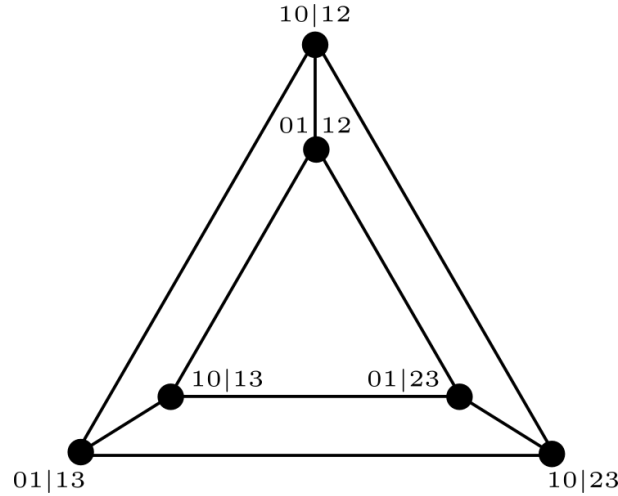


Figure 2.2: Exclusivity graph for Specker's three boxes correlations experiment, see Section 1.1. The outcome 0 corresponds to the opened box being empty, whereas the outcome 1 corresponds to the opened box containing a gem.

We now wish to determine the maximal value of S that is compatible with a classical description like above. For a set of pairwise adjacent vertices, the sum of the probabilities assigned to these cannot exceed 1. Only two independent vertices, meaning two vertices that are non-adjacent can both receive the valuation 1. It follows that the maximal value of S is

$$S_{nc} = \max_U \sum_{i \in U} w_i, \quad (2.1)$$

where the expression is maximized over all independent sets of the exclusivity graph. This is the independence number of the (weighted) exclusivity graph. Section 1.10 presents concrete values for the odd n -cycle scenarios, see Equation 1.11.

- **Quantum correlations:** The set of quantum correlations, restricting to projective measurements, is of the form³

$$p(\epsilon_i) = \langle \Psi | \Pi_i | \Psi \rangle,$$

for some quantum state $|\Psi\rangle$ and Π_i the projector corresponding to the event ϵ_i . Adjacent events must be represented by orthogonal projectors:

$$\Pi_i \Pi_j = 0 \text{ for } \epsilon_i, \epsilon_j \text{ adjacent.}$$

For an arbitrary quantum realization $|\Psi\rangle$, $\{\Pi_i\}_i$, define the vectors

$$|u_0\rangle := |\Psi\rangle \text{ and } |u_i\rangle := \frac{\Pi_i |u_0\rangle}{\sqrt{\langle u_0 | \Pi_i | u_0 \rangle}}$$

³We can w.l.o.g. assume a pure quantum state. For mixed states $|\rho_A\rangle$, consider the purification Ψ_{AE} and projectors $(\Pi_i \otimes \mathbb{1}_E)$.

The $(n+1) \times (n+1)$ Gram matrix⁴ X of the vectors $\{\langle u_i | u_0 \rangle | u_i \rangle\}_{i=0}^n$ obeys the following constraints:

$$\begin{aligned} X_{00} &= 1, \\ X_{0i} &= X_{ii}, \\ \text{and } X_{ij} &= 0 \text{ for } \epsilon_i, \epsilon_j \text{ adjacent} \end{aligned}$$

Additionally, $\sum_i w_i X_{ii} = \sum_i w_i |\langle \Psi | \Pi_i | \Psi \rangle|^2$, the value of the linear combination S for the quantum model $|\Psi\rangle$, $\{\Pi_i\}_i$. As we can construct a Gram matrix that satisfies these constraints and which is related to S like above for all quantum experiments, the maximum value of S , for a given exclusivity graph, that is compatible with QM is upper-bounded by the so-called Lovász semi-definite program (SDP) CITE:

$$\begin{aligned} &\max \sum_{i=1}^n w_i X_{ii} \\ &\text{subject to } X_{ii} = X_{0i}, \quad 1 \leq i \leq n \\ &\quad X_{ij} = 0, \quad \text{for } \epsilon_i, \epsilon_j \text{ adjacent} \\ &\quad X_{00} = 1, \quad X \in \mathcal{S}_+^{n+1}, \end{aligned} \tag{2.2}$$

where \mathcal{S}_+^{1+n} is the set of positive semi-definite $(n+1) \times (n+1)$ matrices.

One might naively think that the Lovász bound is tight: For an optimal solution of the Lovász SDP 2.2, X^* , with Gram decomposition $\{|u_i\rangle\}_{i=0}^n$, $S = \sum_{i=1}^n w_i X_{ii}^*$ can be realized by the quantum experiment which consists of an experimenter performing the rank-one projective measurements $|\tilde{u}_i\rangle \langle \tilde{u}_i|$ on the state $|\tilde{u}_0\rangle$, where the vectors $\{|\tilde{u}_i\rangle\}_{i=0}^n$ are obtained by normalization:

$$p(\epsilon_i) = \text{tr}(|\tilde{u}_i\rangle \langle \tilde{u}_i| \tilde{u}_0 \rangle \langle \tilde{u}_0|) = |\langle \tilde{u}_0 | \tilde{u}_i \rangle|^2 = \frac{|\langle u_0 | u_i \rangle|^2}{\langle u_0 | u_0 \rangle \langle u_i | u_i \rangle} = |\langle u_i | u_i \rangle| = X_{ii},$$

where $\epsilon_i \equiv 1 \mid |\tilde{u}_i\rangle \langle \tilde{u}_i|$ and the last equality follows from the constraints a feasible solution of the Lovász SDP 2.2 must satisfy. However, the Lovász bound on quantum correlations arising from projective measurements is in general not tight. For Bell scenarios with multiple space-like separated subsystems, the concrete physical setting of the correlation experiment imposes additional constraints on what projectors $\{\Pi_i\}_i$ are physical.

⁴The Gram matrix X of a set of vectors v_1, \dots, v_n in an inner product space $(V, \langle \cdot, \cdot \rangle)$ is the $n \times n$ matrix with entries $X_{ij} = \langle v_i | v_j \rangle$. The Gram matrix of a set of vectors is positive semi-definite and all positive semi-definite matrices are the Gram matrix for some non-unique list of vectors. For a $n \times n$ positive semi-definite matrix X , we can always find a suitable list of vectors in $(\mathbb{C}^n, \langle \cdot, \cdot \rangle_{\text{std}})$, where $\langle \cdot, \cdot \rangle_{\text{std}}$ is the standard inner product, by computing the Cholesky decomposition of X . Acting with an arbitrary isometry $V : \mathbb{C}^n \rightarrow \mathbb{C}^d$, $d \geq n$, yields a Gram decomposition in $(\mathbb{C}^d, \langle \cdot, \cdot \rangle_{\text{std}})$.

In particular these have to respect locality $\Pi_i = \Pi_A^{(i)} \otimes \Pi_B^{(i)}$. For contextuality scenarios the Lovász bound is generally tight, as is the case for the class of odd n -cycle scenarios presented in Section 1.10.

- **Correlations in E_1 :** The fractional packing number of the weighted exclusivity graph is defined as

$$\max_{\{p_i\}_i} \sum_{i \in V} w_i p_i,$$

where the expression is maximized over all $\{p_i\}_i$ with $p_i \geq 0$ and $\sum_{i \in C} p_i \leq 1$, for all subsets C of pairwise exclusive events (cliques) of the exclusivity graph. This is just the the maximum value of S for correlations in E_1 . For the KCBS inequality E_1 correlations can achieve $\sum_{i=1}^5 p(0, 1|i, i+1) = \frac{5}{2}$.

2.4 What about unsharp measurements?

The CSW framework falls apart when we allow for outcome unsharp measurements. In particular, as we will see in the next section, the Lovász number 2.2 no longer bounds general quantum behaviours, and even trivial POVM $\{a\mathbb{1}\}_a$ can realize the most general probabilistic models CITE. In particular, they can produce violations of KS non-contextuality inequalities that exceed those achieved by just projective measurements. One arrives at the pathological conclusion that, within the CSW framework, if we do not restrict the set of measurements, all correlations are quantum (compatible with QM), yet all correlations can be produced by trivial POVM, which are intuitively classical⁵. Thus, the hierarchical structure of correlations established in Section 2.3 breaks down for this case. Furthermore, extending our classicality criterion, namely KS non-contextuality, to unsharp measurements is conceptually problematic and raises numerous inconsistencies, as will be discussed in Section 5.1.

2.5 Complications that arise without space-like separation

Let us now comprehend in what sense trivial POVM are able to realize arbitrary probabilistic models, in particular violate KS non-contextuality inequalities, and how come one doesn't encounter the same complications for Bell-type scenarios.

To work with a concrete example, consider the KCBS correlation experiment with exclusivity graph Figure 2.1. We present an argument that parallels the one presented in CITE. Let $(i, i \oplus 1)$ denote two compatible dichotomic measurements an experimenter can perform, which

⁵A trivial POVM generates a random output, totally independent of the measured system's state. One can therefore implement an operationally equivalent measurement by discarding the system and flipping some potentially biased coins.

we will assume to be compatible POVM. Importantly, there generally exist many joint POVM $\{M_{00}^{i,i\oplus 1}, M_{01}^{i,i\oplus 1}, M_{10}^{i,i\oplus 1}, M_{11}^{i,i\oplus 1}\}$, satisfying Definition 2.1, and we cannot rule out any of these without making additional assumptions. For Bell-type correlation experiments, we can impose additional constraints due to locality, namely that compatible measurements act on local Hilbert spaces, and in particular that compatible POVM are commuting. In contrast to projective measurements, commutativity is sufficient but not necessary for two POVM to be compatible: If and only if all elements from both POVM commute, their joint POVM is uniquely determined as

$$M_{ab} = M_a^i M_b^{i\oplus 1}$$

CITE, where $i \equiv \{M_a^i\}_a$ and $i \oplus 1 \equiv \{M_b^{i\oplus 1}\}_b$.

Let us compare the possible correlations that may arise from commuting POVM to those that can arise from non-commuting, compatible POVM:

- **commuting POVM (Bell-type scenarios):** Due to commutativity, the probability of an event $x, y \mid i, i \oplus 1$ can be calculated like

$$p(x, y \mid i, i \oplus 1) = \text{tr}(\rho M_x^i M_y^{i\oplus 1}).$$

For Bell tests with remote measurement contexts, the joint measurement is given by a tensor product.

If we consider all POVM i to be trivial, $i \equiv \{a_0^i \mathbb{1}, a_1^i \mathbb{1}\}$, then we find a joint probability distribution for all measurements i that is compatible with all measurable marginal correlations for the five measurement contexts,

$$p(x, y, z, v, w) = a_x^1 a_y^2 a_z^3 a_v^4 a_w^5 ,$$

where x, \dots, w denote the outcomes of the five measurements i . Finding such a global probability distribution implies that all correlations produced by the experiment are KS non-contextual, and in particular do not violate a KS non-contextuality inequality. To see this, assume an ontic state space that contains one ontic state for each outcome tuple (x, y, z, v, w) and fixes the outcomes of the measurements i accordingly. The preparation that assigns the probability $p(x, y, z, v, w)$ to the ontic state corresponding to that outcome tuple reproduces all correlations of the experiment. The ontological model is KS non-contextual, as it is deterministic and assigns outcomes to the individual operators, independent of the context they are measured in. For Bell scenarios with space-like separated parties, all compatible POVM are commuting, and trivial POVM cannot violate Bell inequalities. As alluded to in Section 1.9, we cannot carry out the same procedure for general contextuality scenarios, as the set of simultaneous eigenvalues for two compatible

operators A, B is in general not given by the Cartesian product of individual eigenvalues $\sigma(A) \times \sigma(B)$.

- **non-commuting POVM (KS-type scenarios):** General compatible POVM, these can be realized jointly in many different ways. A joint realization satisfying Definition 2.1 is of the form

$$\{M_{00}^{i,i\oplus 1}, M_{01}^{i,i\oplus 1}, M_{10}^{i,i\oplus 1}, M_{11}^{i,i\oplus 1}\}.$$

For a single system, $\{M_{xy}^{i,i\oplus 1}\}_{xy}$ must no longer be the product of the individual POVM. As Kunjwal points out CITE, there is one undetermined degree of freedom, i.e. one positive semi-definite operator, say $M_{01}^{i,i\oplus 1}$, of the joint POVM not fixed by the marginal POVM $\{M_a^i\}_a$ and $\{M_b^{i\oplus 1}\}_b$:

$$\begin{aligned} M_{00}^{i,i\oplus 1} &= M_0^i - M_{01}^{i,i\oplus 1} \\ M_{11}^{i,i\oplus 1} &= M_1^{i\oplus 1} - M_{01}^{i,i\oplus 1} \\ M_{10}^{i,i\oplus 1} &= \mathbb{1} - M_{01}^{i,i\oplus 1} - M_{00}^{i,i\oplus 1} - M_{11}^{i,i\oplus 1} \\ &= \mathbb{1} - M_0^i - M_1^{i\oplus 1} + M_{01}^{i,i\oplus 1} \end{aligned}$$

Let us assume $\{M_a^i\}_a = \{\frac{\mathbb{1}}{2}, \frac{\mathbb{1}}{2}\}$ and $\{M_b^{i\oplus 1}\}_b = \{\frac{\mathbb{1}}{2}, \frac{\mathbb{1}}{2}\}$ to be trivial and acting on a qubit system. A possible joint POVM is

$$M_{00}^{i,i\oplus 1} = 0, \quad M_{01}^{i,i\oplus 1} = \frac{\mathbb{1}}{2}, \quad M_{10}^{i,i\oplus 1} = \frac{\mathbb{1}}{2}, \quad M_{11}^{i,i\oplus 1} = 0,$$

which is itself trivial. Furthermore, it produces correlations that exceed the Lovász bound of the KCBS correlations experiment. This shows that, taking KS non-contextuality as the relevant criterion of non-classicality and extending it to unsharp measurements, an intuitively classical procedure can produce correlations that are highly non-classical, even post-quantum, according to the hierarchical structure established in CITE. This example can be generalized, by making use of the one undetermined positive semi-definite operator, to show that trivial POVM can actually realize arbitrary probabilistic assignments.

The above discussion is consistent with the observation that the notion of local causality is formulated entirely theory independent and makes no mention of the sharpness of measurements. Indeed, within QM, Bell inequalities apply to general quantum measurements just as they apply to projective quantum measurements, for reasons just highlighted. The aim of this Section was to motivate why, for general contextuality scenarios without additional locality constraints, extending Bell's notion of local causality, or rather local determinism, to apply to non-remote measurements contexts runs into problems.

Instead of this naive approach, we will follow Spekkens and adopt his proposal of a new notion of contextuality CITE that overcomes these issues. Section 5 will introduce his

proposal. In Section 5.3, we follow CITE and lift KS non-contextuality inequalities to Spekkens non-contextuality inequalities. These revised tests of classicality factor in source-measurement correlations, which one can think of as quantifying the sharpness of the measurements involved in the correlation experiment. In the case of perfectly sharp measurements, the revised inequalities reduce to their KS counterparts. Within QM, the source-measurement correlations capture the “classicality” of trivial POVM measurements, and these in fact do not lead to violations of the new inequalities.

Chapter 3

Self-testing

Self-testing aims to certify properties of unknown quantum devices, only by interacting classically with these, i.e. by passing classical input strings to devices and receiving classical outputs, for instance a binary string. Ideally, we want to make a minimal number of assumptions about the devices themselves, largely treating them as black boxes¹. This is useful for certifying the behaviour of untrusted or noisy quantum devices, even in an adversarial scenario, and for the most part does not require the experimenter to have knowledge about the exact inner workings of the devices. More concretely, many quantum key distribution (QKD) protocols for quantum cryptography, such as CITE, require the generation of a large number of maximally entangled states. To ensure information-theoretic security, one has to confirm the quality of the entanglement source used. Self-testing can provide of measure of this.

A self-testing protocol is essentially a correlation experiment, as introduced in Section 2.1, that allows us to make powerful inferences about the internal workings of the devices. The following example will clarify the concept. We will still assume measurements to be sharp for now:

Consider the CHSH correlation experiment, as introduced in Section 1.9, with two space-like separated systems, \mathcal{A} and \mathcal{B} , as well as two experimenters that can perform local measurements. Assuming no-signalling, i.e. that local measurements on \mathcal{A} do not disturb the outcome statistics of local measurements on \mathcal{B} , remote measurements are compatible according to Definition 2.1. Recall that the CHSH correlation experiment is a test for violations of the inequality

$$\begin{aligned} & \frac{1}{4} (p(x = y|A_1, B_1) + p(x = y|A_2, B_1) \\ & + p(x = y|A_2, B_2) + p(x \neq y|A_1, B_2)) \leq \frac{3}{4} \end{aligned}$$

¹Self-testing schemes that do not require assumptions about the underlying mechanics of the devices, and consider only the input-correlations they generate, are called “device-independent”.

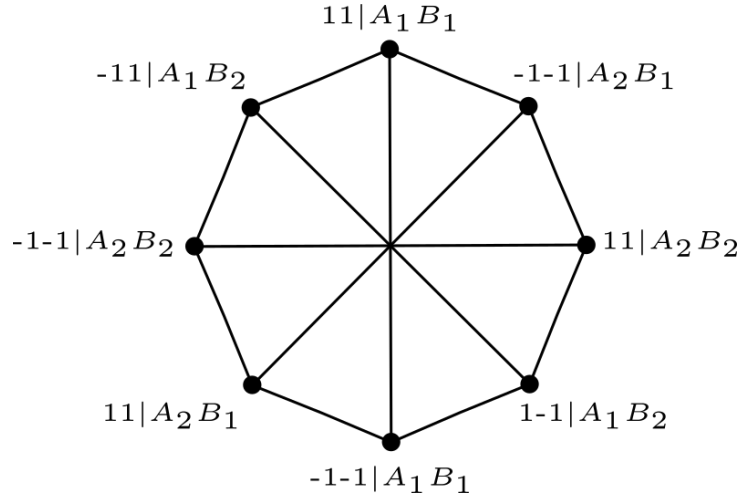


Figure 3.1: Exclusivity graph for the CHSH correlations experiment, see Section 1.9. Like in Section 1.9, the outcomes of local measurements are labelled ± 1 . Vertices correspond to measurement events, exclusive measurement events are represented as adjacent.

The exclusivity graph of the CHSH correlation experiment can be seen in Figure 3.1; the equal weighting $\frac{1}{4}$ each vertex receives is omitted for readability.

The Lovász number of the CHSH exclusivity graph corresponds to the Tsirelson bound $B_q = \frac{1}{2}(1 + \frac{1}{\sqrt{2}})$. Thus, the Lovász bound is tight and the measurements 1.4, together with the maximally entangled EPR state, produce a maximal violation of the CHSH inequality 1.2.

In fact, due to space-like separation, the Lovász bound is a tight upper bound that all correlations compatible with QM have to obey, even if we allow for general POVM measurements. This follows from the Naimark dilation theorem CITE. Informally, we can extend every non-projective quantum measurement $\{M_i\}_i$ acting on the system \mathcal{A} to a projective measurement $\{\Pi_i^{\mathcal{A}}\}_i$ that acts on a composite system $(\mathcal{A} + \text{ancilla})$, where the ancillary system is prepared in some well-defined state $|0\rangle\langle 0|$. The extended projectors acting on the composite system $(\mathcal{A} + \text{ancilla})$ produce the same output correlations as the unsharp measurements acting on the single system \mathcal{A} . The same can be said about local measurements on subsystem \mathcal{B} . Now, due to space-like separation, the projectors $\Pi_i^{\mathcal{A}} \otimes \Pi_j^{\mathcal{B}}$ corresponding to exclusive measurement events commute. As such, we can follow the same line of reasoning as in Section 2.5 to argue that the maximal violation of the CHSH inequality, even with access to unsharp measurements is given by the Tsirelson bound. As we saw in Section 2.5, the same considerations do not apply to contextuality scenarios involving compatible measurements on a single system. While we can extend the individual measurements to projective measurements by adding an ancillary system, thereby associating with each measurement event a projector, we have no grounds to assume that the projectors associated with exclusive measurement events are orthogonal. Take for instance the KCBS correlation experiment with exclusivity graph Figure 2.1. Measurements 1,2 and 2,3 are compatible, however not all three are jointly measurable. We can assume the Naimark projectors associated with the measurement events $0,1 | 1,2$ and $0,0 | 1,2$ to be orthogonal, because

they correspond to distinct outcomes of a single joint measurements. This does not hold for the exclusive measurement events of the KCBS exclusivity graph: $0, 1 \mid 1, 2$ and $0, 1 \mid 2, 3$ can not be distinct outcomes of a single joint measurement, as the measurements $1, 2, 3$ are not even pairwise compatible.

Back to the CHSH correlation experiment: A violation of the CHSH inequality implies that the observed correlations are not compatible with a classical description of the form 1.1, but require quantum resources, in particular entanglement, to generate. What is remarkable about, but not unique to the CHSH Bell inequality², is that the quantum model presented in 1.4, leading to a maximal violation of the inequality, i.e. corresponding to the extremal point $\vec{v}_{max} \in \mathcal{Q}$ of the set of quantum correlations, see Figure 3.2, is “essentially unique”. By this we mean that the only quantum model $|\Psi\rangle, \{\Pi_i\}_i$ predicting a maximal violation of the CHSH inequality is 1.4, modulo trivial changes to the reference experiment which we will specify shortly. A comprehensive proof is given in CITE. Thus, by interacting only classically with our preparation and measurement devices we can characterize the quantum state and measurements our devices implement. We can only hope to self-test extremal correlations, as other correlations $\vec{v} \in \mathcal{Q}$ correspond to a non-unique convex combination of distinct extremal correlations.

It is apparent that we can never certify exact states and measurements, as we will always be blind to a number of trivial degrees of freedom. For instance, we cannot tell if there are any additional ancillary systems, $\mathcal{A}', \mathcal{B}'$, prepared in some joint state $\rho_{\mathcal{A}'\mathcal{B}'}$, in the remote labs that we are not probing:

$$\begin{aligned} & |\Psi\rangle_{\mathcal{AB}}, & A_1 &= \sigma_{z,\mathcal{A}} \otimes \mathbb{1}_{\mathcal{B}}, \dots \\ \equiv & |\Psi\rangle\langle\Psi|_{\mathcal{AB}} \otimes \rho_{\mathcal{A}'\mathcal{B}'}, & A_1 &= \sigma_{z,\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}'\mathcal{B}'}, \dots \end{aligned}$$

Furthermore, we can never detect local unitaries $U = U_{\mathcal{AA}'} \otimes U_{\mathcal{BB}'}$, i.e. local basis changes:

$$\begin{aligned} & |\Psi\rangle\langle\Psi|_{\mathcal{AB}} \otimes \rho_{\mathcal{A}'\mathcal{B}'}, & A_1 &= \sigma_{z,\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}'\mathcal{B}'}, \dots \\ \equiv & U(|\Psi\rangle\langle\Psi|_{\mathcal{A},\mathcal{B}} \otimes \rho_{\mathcal{A}'\mathcal{B}'})U^\dagger, & A_1 &= U_{\mathcal{AA}'}(\sigma_{z,\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}'\mathcal{B}'})U_{\mathcal{AA}'}^\dagger, \dots \end{aligned}$$

According to Stinespring’s Dilation Theorem CITE, these trivial degrees of freedom can account for arbitrary local isometries $\mathcal{A} \rightarrow \mathcal{AA}'$, $\mathcal{B} \rightarrow \mathcal{BB}'$. Furthermore, the previous two examples suggest that in order to define a sensible notion of self-testing, we must first define some notion of equivalence for quantum models. We then aim to construct self-testing protocols that single out a unique quantum model, modulo this notion of equivalence. This notion of equivalence should account for some trivial degrees of freedom, like the ones above, but should also not be too broad, in order for self-testing to be a powerful property of select correlation experiments.

²Other Bell inequalities may self-test different reference experiments.

Ideally, the equivalence of quantum realizations should have an operational meaning. The following definition applies to Bell-type correlation experiments and does justice to these three criteria.

Definition 3.1 (CITE). A quantum experiment involving $|\Psi'\rangle_{\mathcal{A}'\mathcal{B}'\mathcal{P}}$ and local measurements $\{P'_x{}^a\}_x, \{Q'_y{}^b\}_y$, where $|\Psi'\rangle_{\mathcal{A}'\mathcal{B}'\mathcal{P}}$ is an arbitrary purification and $P'_x{}^a$ is the positive semi-definite operator corresponding to the outcome x of the local measurement a on system \mathcal{A} , is *equivalent* to a reference experiment $|\Psi\rangle_{\mathcal{AB}}, \{P_x^a\}_x, \{Q_y^b\}_y$, if there exists a local isometry $\Phi = \Phi_{\mathcal{A}} \otimes \Phi_{\mathcal{B}}$, $\Phi_{\mathcal{A}/\mathcal{B}}: \mathcal{H}_{\mathcal{A}'/\mathcal{B}'} \rightarrow \mathcal{H}_{\mathcal{A}/\mathcal{B}} \otimes \mathcal{H}_{\mathcal{A}/\mathcal{B}}$, such that

$$\begin{aligned} (\Phi \otimes \mathbb{1}_{\mathcal{P}}) |\Psi'\rangle_{\mathcal{A}'\mathcal{B}'\mathcal{P}} &= |\text{junk}\rangle_{\mathcal{A}'\mathcal{B}'\mathcal{P}} \otimes |\Psi\rangle_{\mathcal{AB}} \\ (\Phi \otimes \mathbb{1}_{\mathcal{P}})(P'_x{}^a \otimes Q'_y{}^b) |\Psi'\rangle_{\mathcal{A}'\mathcal{B}'\mathcal{P}} &= |\text{junk}\rangle_{\mathcal{A}'\mathcal{B}'\mathcal{P}} \otimes (P_x^a \otimes Q_y^b) |\Psi\rangle_{\mathcal{AB}}. \end{aligned}$$

Operationally, this means that a physical experiment is equivalent to some reference experiment if by means of local operations alone³, one can recover the reference experiment, up to some “junk”. In particular, the experimenters can for instance extract the reference state, i.e. the maximally entangled EPR state for the CHSH correlation experiment, to their ancillary systems, by means of local operations alone. As local operations cannot generate entanglement, their initial state must have been maximally entangled. Note that the notion of equivalence, as defined in Definition 3.1, is not an equivalence relation, as it fails to be symmetric CITE. While the experimenters can extract the reference state to ancillary systems using only local operations, if they have access to an equivalent experiment, the converse direction must not hold: Consider the reference CHSH experiment 1.4 producing a maximal violation of the CHSH inequality 1.2. Further, consider the equivalent quantum experiment where both experimenters control an additional ancillary system and both ancillae are prepared in an entangled state. While this extended experiment is equivalent to the reference experiment, the converse does not hold, as we cannot create entanglement by local operations alone.

Definition 3.1 accounts for all of the following statistics-preserving trivial degrees of freedom, as presented in CITE:

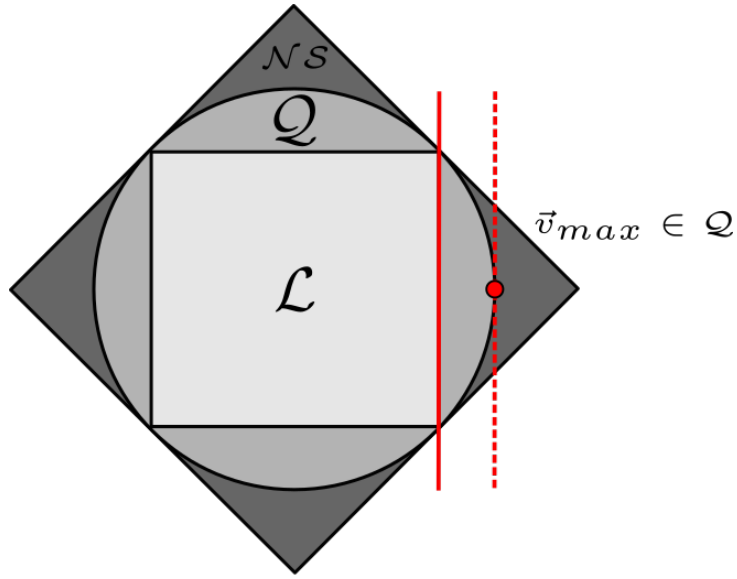
1. Local changes of basis
2. adding ancillae to the physical systems, prepared in any joint state (measurements do not act on these)
3. changing the action of an observable outside the support of the state
4. locally embedding the state and operators into a larger (or smaller) Hilbert space

³Note that the two experimenters can realize an arbitrary local isometry $\Phi = \Phi_{\mathcal{A}} \otimes \Phi_{\mathcal{B}}$ by preparing ancillary systems in some well-defined state $|00\rangle$ and subsequently applying local unitary operations.

Claim 3.2 (CITE). *Any finite number of changes 1-4 transforms a quantum experiment into an equivalent quantum experiment like in Definition 3.1. Furthermore, any quantum experiment equivalent to some reference experiment can be constructed from that reference experiment by applying a finite number of changes 1-4.*

Let us contrast self-testing via Bell inequalities with self-testing via KS non-contextuality inequalities. Section 4 will examine in detail a self-testing protocol based on the class of odd n -cycle KS non-contextuality inequalities CITE. Importantly, without space-like separated subsystems, we have to give up device-independence: While relaxing the assumption of space-like separation means that we no longer require the device to be split up among remote subsystems, or that it can generate a large number of entangled states, we can no longer make use of quantum non-locality to certify that the correlations were not simulated by some pre-programmed classical computer CITE. In principle, we can never discard this possibility without imposing additional assumptions about the devices. In Section 4.3, we examine how bounding the information carrying capacity or memory of the device can be helpful in this regard. Additionally, as KS non-contextuality inequalities are derived from given compatibility relations, using them to certify quantumness requires assumptions about operator compatibility. This is illustrated in Figure 3.3. Since contextuality scenarios no longer require remote subsystems, we need to amend our notion of “equivalence” for quantum experiments, see Definition 3.1, accordingly. We will come back to this in Section 4, but already note that contextuality-based self-testing protocols aim to single out a quantum realization from correlations alone, up to a **global** isometry.

Finally, self-testing results should be robust to noise. Informally, this means that a near-optimal violation of the Bell or KS non-contextuality inequality should imply that all compatible quantum models are, up to equivalence, “close” to a reference quantum experiment. The self-testing protocol we will examine Section 4, and which is based on KS contextuality, is robust, see Theorem 4.4.



maximal violation $\vec{v}_{max} \in \mathcal{Q}$
 $\implies |\Psi\rangle, \{A_i\}_i, \{B_j\}_j$

Figure 3.2: Schematic drawing depicting the geometry of the convex sets \mathcal{L} , \mathcal{Q} , and \mathcal{NS} , containing all local, quantum, and no-signalling correlations, respectively. A Bell inequality, like the CHSH inequality, defines a separative hyperplane such that all local correlations \mathcal{L} lie to one “side” of this hyperplane, here represented by the unbroken red line. The extremal point $\vec{v}_{max} \in \mathcal{Q}$ of the set of quantum correlations \mathcal{Q} corresponds to a maximal violation of the CHSH inequality. The CHSH inequality can be used for self-testing, as the quantum model $|\Psi\rangle, \{\Pi_i\}_i$ compatible with a maximal violation of the inequality is unique, up to trivial degrees of freedom.

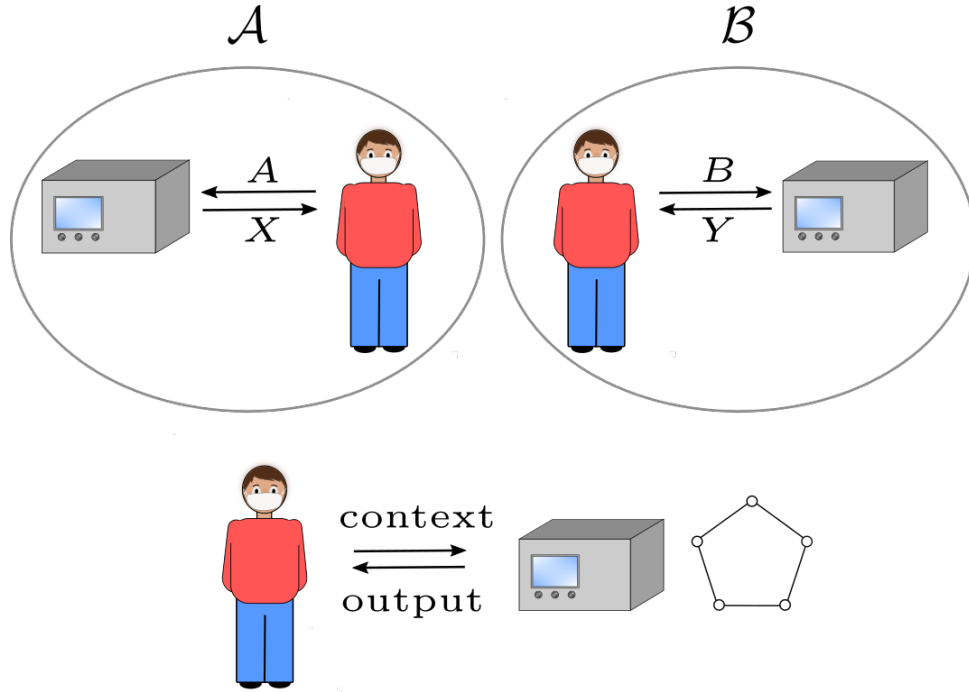


Figure 3.3: Top: Typical device-independent Bell-type self-testing scenario. The two subsystems \mathcal{A} and \mathcal{B} are space-like separated. Accordingly, the black box device, which can be interpreted as a correlation experiment testing a Bell inequality, is split into two non-communicating parts. An experimenter can interact classically with a local sub-device, by passing an input value and recording the outcome returned by the device. A violation of the tested Bell inequality implies that the corresponding correlations were obtained by quantum means, i.e. joint measurements on an entangled state shared between \mathcal{A} , \mathcal{B} . For some Bell inequalities, a maximal violation is only compatible with an essentially unique quantum realization, i.e. quantum state and set of measurements. **Bottom:** In contrast, self-testing protocols based on KS non-contextuality inequalities as tests of classicality do not assume the device to be split into multiple non-communicating parts. Instead, a single experimenter interacts classically with the device by freely choosing between measurement contexts and determining the corresponding outcome probabilities. Dropping the unnatural assumption of space-like separation complicates the problem of self-testing considerably and we are required to make additional assumptions about the device, for example regarding compatibility relations. Thus, contextuality-based self-testing protocols are only semi-device-independent.

Chapter 4

Self-testing via KS non-contextuality inequalities

4.1 Preceding results

In Section 2 we considered general correlation experiments testing some linear combination of probabilities S and found that any quantum realization $|\Psi\rangle, \Pi_i$ can be translated into a feasible solution X of the Lovász SDP 2.2, such that the value of S , as predicted by QM, corresponds to the objective function the SDP aims to maximize, evaluated at the feasible solution X . Conversely, if there are no additional physical constraints, for example due to space-like separated subsystems, one can construct a quantum realization for every feasible solution X that achieves $S = \sum_{i=1}^n X_{ii}$. Such quantum realization was constructed from an arbitrary Gram decomposition of X . This connection between feasible solutions to the Lovász SDP and quantum models of the correlation experiment will be central to proving robust self-testing based on KS non-contextuality inequalities.

As outlined in Section 3, the essence of self-testing is that a maximal violation of suitable Bell or KS non-contextuality inequalities is only compatible with an essentially unique quantum realization $|\Psi\rangle, \{\Pi_i\}_i$, up to statistics-preserving trivial degrees of freedom. Furthermore, the inequalities are tests of non-classicality. If our experiment produces correlations that violate these, this certifies that the correlations were not simulated by some classical mechanism, which would render all proofs of information-theoretic security superfluous. Extra care has to be taken in the case of contextuality scenarios, as will be the topic of Section 4.3. Ideally, such self-tests should be robust to noise, i.e. a ϵ -suboptimal violation should imply that the only compatible quantum realizations are “close” to the ideal reference implementation, up to trivial degrees of freedom. For sufficiently small ϵ , we would still observe a Bell or KS non-contextuality inequality violation.

We will now sketch the proof in CITE, which demonstrates that the class of odd n -cycle contextuality scenarios allows for robust self-testing. Recall that an odd n -cycle correlation experiment assumes cyclic compatibility relations, like in Figure 1.10. The self-testing protocol consists of an experimenter choosing freely between the n measurement contexts $(i, i \oplus 1)$ and determining the outcome correlations $p(0, 1 \mid i, i \oplus 1)$ that constitute the test for non-classicality S . The main “ingredients” of the proof are the following three lemmas:

Lemma 4.1 (CITE). *For the class of odd n -cycle exclusivity graphs, $n \geq 5$, the Lovász SDP 2.2 has a unique optimal solution X_n^* .*

From Section 2.3 we know that the reference quantum experiment $\{|u_j^{(n)}\rangle\}_{j=0}^n$ presented in Section 1.10 produces a maximal violation of the odd n -cycle KS non-contextuality inequality, because it attains the Lovász bound. The unique optimal solution X_n^* is obtained from this reference quantum realization by the usual procedure:

$$X_n^* = \text{Gram}(|u_0^{(n)}\rangle, \langle u_1^{(n)}|u_0^{(n)}\rangle|u_1^{(n)}\rangle, \dots, \langle u_n^{(n)}|u_0^{(n)}\rangle|u_n^{(n)}\rangle).$$

The following two lemmas will be important to prove robustness:

Lemma 4.2 (CITE). *Let (G_{ex}, w) be a weighted exclusivity graph with n vertices and assume that 2.2 has a unique optimal solution $X^* \in S_+^{n+1}$. Further, let \tilde{X} be a feasible solution of 2.2 that is ϵ -suboptimal, i.e.*

$$\sum_{i=1}^n w_i \tilde{X}_{ii} \geq B_q(G_{ex}, w) - \epsilon,$$

where $B_q(G_{ex}, w) = \sum_{i=1}^n w_i X_{ii}^*$ is the optimal solution of the Lovász SDP for the weighted exclusivity graph (G_{ex}, w) . Then

$$\|\tilde{X} - X^*\|_F \leq \mathcal{O}(\epsilon),$$

where $\|A\|_F = \text{tr}(A^\dagger A)$ is the Frobenius norm.

Lemma 4.3. *Let \tilde{X} and $X^* \in S_+^n$ be two positive semi-definite matrices, ϵ -close in Frobenius distance:*

$$\|\tilde{X} - X^*\|_F \leq \epsilon$$

Let $\{|\tilde{u}_i\rangle\}_{i=1}^n \subset \mathbb{C}^d$ and $\{|u_i\rangle\}_{i=1}^n \subset \mathbb{C}^d$ be Gram decompositions of \tilde{X} and X^* , respectively. Then there exists a $d \times d$ unitary matrix U such that

$$\| |\tilde{u}_i\rangle - U|u_i\rangle \|_2 \leq \mathcal{O}(\epsilon),$$

for all $i \in \{1, \dots, n\}$

We will omit the proofs of Lemmas 4.1 and 4.2, as they do not provide much conceptual insight, and offer only a few comments. We include the proof of Lemma 4.3, as CITE proves only a

weaker version for $d = n$, but implicitly uses the stronger statement of Lemma 4.3. Furthermore, the proof we present establishes a tighter error bound of the order $\mathcal{O}(\epsilon)$, compared to $\mathcal{O}(\sqrt{\epsilon})$ in CITE.

The proof of Lemma 4.1 utilizes SDP duality theory and shows a sufficient condition for the unicity of X_n^* to be satisfied by explicit construction. It applies only to the class of odd n -cycle scenarios, as it makes use of the cyclic exclusivity constraints that enter the Lovász SDP 2.2. Lemma 4.2 is a general result for the optimization problem 2.2, and as such applies to arbitrary exclusivity graphs. Like Lemma 4.1, the proof of Lemma 4.2 makes heavy use of SDP duality.

Proof. (of Lemma 4.3)

The proof is a generalization of the proof of Theorem 7.3.11 in CITE, which only considers the noise-less case $\epsilon = 0$, and we for the most part will adopt their notation.

Define $A, B \in \mathbb{C}^{n,d}$ as the matrices with rows $\{|u_i\rangle\}_{i=1}^n \subset \mathbb{C}^d$ and $\{|\tilde{u}_i\rangle\}_{i=1}^n \subset \mathbb{C}^d$, respectively. The Gram matrices \tilde{X} and X^* are given by $X^* = AA^\dagger$ and $\tilde{X} = BB^\dagger$.

Let $A^\dagger = V\Sigma W^\dagger$ be a singular value decomposition (SVD) of A^\dagger , $V \in \mathbb{C}^{d,d}$ and $W \in \mathbb{C}^{n,n}$ being unitary matrices, and Σ a rectangular diagonal matrix containing the singular values of A^\dagger . By definition,

$$X^* = AA^\dagger = (A^\dagger)^\dagger A^\dagger = W\Sigma^\dagger \Sigma W^\dagger = W\Lambda W^\dagger,$$

where we have defined $\mathbb{C}^{n,n} \ni \Lambda := \Sigma^\dagger \Sigma$. Denote the non-zero singular values of A^\dagger , ordered according to magnitude, by $\sigma_1 \geq \dots \geq \sigma_r > 0$, where r is the rank of A^\dagger . Note that A and A^\dagger have identical singular values. As acting with permutation matrices on V , W^\dagger preserves unitarity, we can w.l.o.g assume

$$\Lambda = \begin{pmatrix} \sigma_1^2 & & & \\ & \ddots & & \\ & & \sigma_r^2 & \\ & & & 0_{n-r} \end{pmatrix},$$

where empty spaces indicate zero entries. Further, let

$$\Sigma_r = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix} \in \mathbb{C}^{r,r}$$

and partition $W \in \mathbb{C}^{n,n}$ like $\begin{pmatrix} W_1 & W_2 \end{pmatrix}$, with $W_1 \in \mathbb{C}^{n,r}$ and $W_2 \in \mathbb{C}^{n,n-r}$. Theorem 7.3.2 in CITE implies that there exists an isometry $V_1 \in \mathbb{C}^{d,r}$ such that

$$A^\dagger = V_1 \Sigma_r W_1^\dagger.$$

The first step towards proving Lemma 4.3 is to generalize the result of Theorem 7.3.2 in CITE. Concretely, we want to show that B^\dagger , obeying $(B^\dagger)^\dagger B^\dagger \approx X^* = W\Lambda W^\dagger$, can be approximately written like $B^\dagger \approx V_2 \Sigma_r W_1^\dagger$, where $V_2 \in \mathbb{C}^{d,r}$ is an isometry:

By definition,

$$(B^\dagger)^\dagger B^\dagger = \tilde{X} \stackrel{\epsilon}{\approx} X^* = W\Lambda W^\dagger.$$

Let

$$D := \Sigma_r \oplus \mathbb{1}_{n-r} \in \mathbb{C}^{n,n}$$

be the $n \times n$ diagonal matrix with diagonal entries corresponding to the non-zero singular values of A^\dagger . The remaining $n - r$ diagonal entries are set to 1, such that D is invertible. Further, define

$$X := B^\dagger W D^{-1} \in \mathbb{C}^{d,n}.$$

We partition X like $X = \begin{pmatrix} \tilde{V}_2 & Z \end{pmatrix}$, where $\tilde{V}_2 \in \mathbb{C}^{d,r}$. It follows that

$$\begin{aligned} X^\dagger X &= D^{-1} W^\dagger B B^\dagger W D^{-1} \\ &= D^{-1} W^\dagger A A^\dagger W D^{-1} + D^{-1} W^\dagger (B B^\dagger - A A^\dagger) W D^{-1} \\ &= D^{-1} \Lambda D^{-1} + D^{-1} W^\dagger (\tilde{X} - X^*) W D^{-1} \\ &= \begin{pmatrix} \mathbb{1}_r & \\ & 0_{n-r} \end{pmatrix} + D^{-1} W^\dagger (\tilde{X} - X^*) W D^{-1}. \end{aligned}$$

Therefore,

$$X^\dagger X = \begin{pmatrix} \tilde{V}_2^\dagger \\ Z^\dagger \end{pmatrix} \begin{pmatrix} \tilde{V}_2 & Z \end{pmatrix} = \begin{pmatrix} \tilde{V}_2^\dagger \tilde{V}_2 & * \\ * & Z^\dagger Z \end{pmatrix} = \begin{pmatrix} \mathbb{1}_r & \\ & 0_{n-r} \end{pmatrix} + D^{-1} W^\dagger (\tilde{X} - X^*) W D^{-1}.$$

We now aim to bound the Frobenius distance

$$\left\| \begin{pmatrix} \tilde{V}_2^\dagger \tilde{V}_2 & * \\ * & Z^\dagger Z \end{pmatrix} - \begin{pmatrix} \mathbb{1}_r & \\ & 0_{n-r} \end{pmatrix} \right\|_F.$$

As the Frobenius norm is submultiplicative, i.e. $\|AB\|_F \leq \|A\|_F \|B\|_F$, we find

$$\left\| \begin{pmatrix} \tilde{V}_2^\dagger \tilde{V}_2 & * \\ * & Z^\dagger Z \end{pmatrix} - \begin{pmatrix} \mathbb{1}_r & \\ & 0_{n-r} \end{pmatrix} \right\|_F \leq \|D^{-1}\|_F^2 \|\tilde{X} - X^*\|_F \leq \epsilon \|D^{-1}\|_F^2 \stackrel{(\triangle)}{\leq} \epsilon \frac{n}{\sigma_{\min}^2},$$

provided that the smallest singular value $\sigma_{\min} \leq 1$. For the KCBS scenario, $n = 5 + 1$, we find $\sigma_{\min}^2 = \frac{1}{2}(\sqrt{5} - 1) \approx 0.618$. The inequality (Δ) holds because

$$\|D^{-1}\|_F^2 = \left(\sqrt{\sum_{i=1}^r \frac{1}{\sigma_i^2} + (n-r)} \right)^2 = \sum_{i=1}^r \frac{1}{\sigma_i^2} + (n-r) \leq \frac{n}{\sigma_{\min}^2}.$$

Therefore,

$$\begin{aligned} \|\tilde{V}_2^\dagger \tilde{V}_2 - \mathbb{1}_r\|_F &\leq \epsilon \frac{n}{\sigma_{\min}^2} \quad \text{and} \\ \|Z\|_F &\stackrel{(\square)}{\leq} \|Z^\dagger Z\|_F \leq \epsilon \frac{n}{\sigma_{\min}^2}, \end{aligned}$$

where the inequality (\square) holds because $\|A\|_F \geq \text{tr}(A)$ in general.

By definition, $X = B^\dagger W D^{-1}$, and hence

$$B^\dagger = X D W^\dagger = \begin{pmatrix} \tilde{V}_2 & Z \end{pmatrix} \begin{pmatrix} \Sigma_r & \\ & \mathbb{1}_{n-r} \end{pmatrix} \begin{pmatrix} W_1^\dagger \\ W_2^\dagger \end{pmatrix} = \begin{pmatrix} \tilde{V}_2 \Sigma_r & Z \end{pmatrix} \begin{pmatrix} W_1^\dagger \\ W_2^\dagger \end{pmatrix} = \tilde{V}_2 \Sigma_r W_1^\dagger + Z W_2^\dagger$$

Unitaries leave the Frobenius norm invariant, as $\|A\|_F^2 = \sum_k \|A e_k\|_2^2 = \sum_k \|U A e_k\|_2^2 = \|U A\|_F^2$, where U is a unitary. Therefore,

$$\|B^\dagger - \tilde{V}_2 \Sigma_r W_1^\dagger\|_F = \|Z W_2^\dagger\|_F = \|Z\|_F \leq \epsilon \frac{n}{\sigma_{\min}^2}.$$

It remains to show that \tilde{V}_2 is approximately isometric, which follows from the previously derived bound $\|\tilde{V}_2^\dagger \tilde{V}_2 - \mathbb{1}_r\|_F \leq \epsilon \frac{n}{\sigma_{\min}^2}$, as we will now prove:

Consider a SVD of \tilde{V}_2 ,

$$\tilde{V}_2 = S \tilde{\Sigma} T^\dagger,$$

and define V_2 as the isometry

$$V_2 := S \begin{pmatrix} \mathbb{1}_r \\ 0_{d-r,r} \end{pmatrix} T^\dagger.$$

V_2 is isometric because

$$V_2^\dagger V_2 = T \begin{pmatrix} \mathbb{1}_r & 0_{r,d-r} \end{pmatrix} \begin{pmatrix} \mathbb{1}_r \\ 0_{d-r,r} \end{pmatrix} T^\dagger = T \mathbb{1}_r T^\dagger = \mathbb{1}_r,$$

and its Frobenius distance to \tilde{V}_2 is

$$\|\tilde{V}_2 - V_2\|_F = \left\| S \left[\tilde{\Sigma} - \begin{pmatrix} \mathbb{1}_r \\ 0_{d-r,r} \end{pmatrix} \right] T^\dagger \right\|_F = \text{tr} \left(\left[\tilde{\Sigma}^\dagger - \begin{pmatrix} \mathbb{1}_r & 0_{r,d-r} \end{pmatrix} \right] \left[\tilde{\Sigma} - \begin{pmatrix} \mathbb{1}_r \\ 0_{d-r,r} \end{pmatrix} \right] \right)$$

Now, let r' be the rank of \tilde{V}_2 , and $\tilde{\sigma}_1 \geq \dots \tilde{\sigma}_{r'} > 0$ the non-zero singular values of \tilde{V}_2 . Again, w.l.o.g.

$$\tilde{\Sigma}^\dagger = \begin{pmatrix} \tilde{\sigma}_1 & & & \\ & \ddots & & \\ & & \tilde{\sigma}_{r'} & \\ & & & 0_{r-r'} \\ & 0_{d-r,r} & & \end{pmatrix}.$$

Hence, $\|\tilde{V}_2 - V_2\|_F$ can be re-written like

$$\begin{aligned} \|\tilde{V}_2 - V_2\|_F &= \text{tr} \left(\begin{pmatrix} \tilde{\sigma}_1^2 & & \\ & \ddots & \\ & & \tilde{\sigma}_{r'}^2 \\ & & & 0_{r-r'} \end{pmatrix} - 2 \begin{pmatrix} \tilde{\sigma}_1 & & \\ & \ddots & \\ & & \tilde{\sigma}_{r'} \\ & & & 0_{r-r'} \end{pmatrix} + \mathbb{1}_r \right) \\ &= \sum_{i=1}^{r'} (\tilde{\sigma}_i^2 - 2\tilde{\sigma}_i + 1) + (r - r') = \sum_{i=1}^{r'} (\tilde{\sigma}_i - 1)^2 + (r - r'). \end{aligned}$$

The condition $\|\tilde{V}_2^\dagger \tilde{V}_2 - \mathbb{1}_r\|_F \leq \epsilon \frac{n}{\sigma_{\min}^2}$ implies

$$\begin{aligned} \left\| T \begin{pmatrix} \tilde{\sigma}_1^2 & & \\ & \ddots & \\ & & \tilde{\sigma}_{r'}^2 \\ & & & 0_{r-r'} \end{pmatrix} T^\dagger - \mathbb{1}_r \right\|_F &= \sum_{i=1}^{r'} (\tilde{\sigma}_i^4 - 2\tilde{\sigma}_i^2 + 1) + (r - r') \\ &= \sum_{i=1}^{r'} (\tilde{\sigma}_i - 1)^2 (\tilde{\sigma}_i + 1)^2 + (r - r') \leq \epsilon \frac{n}{\sigma_{\min}^2}, \end{aligned}$$

which in particular implies that, for $\epsilon < \frac{\sigma_{\min}^2}{n}$, the rank of \tilde{V}_2 must match that of A^\dagger , i.e. $r = r'$. Additionally, \tilde{V}_2 is approximately isometric:

$$\|\tilde{V}_2 - V_2\|_F = \sum_{i=1}^r (\tilde{\sigma}_i - 1)^2 \leq \epsilon \frac{n}{\sigma_{\min}^2}.$$

Together, this gives us the desired intermediate result:

$$\begin{aligned} \|B^\dagger - V_2 \Sigma_r W_1^\dagger\|_F &\leq \|B^\dagger - \tilde{V}_2 \Sigma_r W_1^\dagger\|_F + \|(\tilde{V}_2 - V_2) \Sigma_r W_1^\dagger\|_F \leq \epsilon \frac{n}{\sigma_{\min}^2} + \epsilon \frac{n}{\sigma_{\min}^2} \|\Sigma_r\|_F \\ &\leq \epsilon \frac{n}{\sigma_{\min}^2} (1 + \sqrt{n} \sigma_{\max}), \end{aligned}$$

where σ_{\max} is the maximum singular value of A^\dagger . For the KCBS scenario, $n = 5 + 1$, $\sigma_{\max} = \sqrt{2}$, and consequently $\|B^\dagger - V_2 \Sigma_r W_1^\dagger\|_F \leq \kappa \epsilon$, where $\kappa \approx 43.34$.

To recap, we have just shown that

$$\begin{aligned} A^\dagger &= V_1 \Sigma_r W_1^\dagger \\ B^\dagger &\stackrel{\mathcal{O}(\epsilon)}{\approx} V_2 \Sigma_r W_1^\dagger, \end{aligned}$$

where V_2 is an isometry. The next proof step is to show that there exists a unitary $U \in \mathbb{C}^{d,d}$ such that $B^\dagger \approx UA^\dagger$.

According to Theorem 2.1.18 in CITE, there exists a unitary $U \in \mathbb{C}^{d,d}$ such that $V_2 = UV_1$. This unitary U also relates the matrices B^\dagger and A^\dagger , as

$$B^\dagger \stackrel{\mathcal{O}(\epsilon)}{\approx} V_2 \Sigma_r W_1^\dagger = UV_1 \Sigma_r W_1^\dagger = UA^\dagger.$$

Finally, we can use this result to bound the distance between the two Gram decompositions $\{u_i\}_{i=1}^n$ and $\{\tilde{u}_i\}_{i=1}^n$, and thereby conclude the proof:

$$\begin{aligned} \|B^\dagger - UA^\dagger\|_F &= \left\| \begin{pmatrix} \tilde{u}_1 & \dots & \tilde{u}_n \end{pmatrix} - U \begin{pmatrix} u_1 & \dots & u_n \end{pmatrix} \right\|_F \\ &= \left\| \begin{pmatrix} \tilde{u}_1 - Uu_1 & \dots & \tilde{u}_n - Uu_n \end{pmatrix} \right\|_F \leq \mathcal{O}(\epsilon), \end{aligned}$$

where vertical lines delimit columns. Since the Frobenius norm of an $n \times m$ matrix is just the two norm of the $(n \cdot m)$ -long vector containing all matrix entries, we have

$$\forall i \in \{1, \dots, n\} : \|\tilde{u}_i - Uu_i\|_2 \leq \mathcal{O}(\epsilon).$$

□

Having distilled the main proof components into three auxiliary theorems, Lemmas 4.1-4.3, we will now demonstrate that these imply the desired self-testing result. CITE for the most part omits these final proof steps. Section 4.2 will provide an in-depth discussion about all of the assumptions that go into the self-testing protocol.

Theorem 4.4 (Odd n -cycle KS non-contextuality inequalities facilitate self-testing CITE).

Let $n \geq 5$ be an odd integer. The only quantum model $|\Psi\rangle$, $\{\Pi_i\}_{i=1}^n$ compatible with a maximal violation of the odd n -cycle KS non-contextuality inequality 1.11 with exclusivity graph $\mathcal{G}_{ex}^{(n)}$ is 1.13, up to a global isometry:

$$\begin{aligned} \exists \text{ isometry } V : |\Psi\rangle &= V|u_0^{(n)}\rangle \\ \forall i \in \{1, \dots, n\} : \Pi_i |\Psi\rangle &= V|u_i^{(n)}\rangle \langle u_i^{(n)}| |u_0^{(n)}\rangle \end{aligned}$$

Furthermore, all quantum models $|\Psi\rangle, \{\Pi_i\}_{i=1}^n$ compatible with an ϵ -suboptimal violation of 1.11, i.e.

$$\sum_{i=0}^n |\langle \Psi | \Pi_i | \Psi \rangle|^2 > B_q(\mathcal{G}_{ex}^{(n)}) - \epsilon,$$

where $B_q(\mathcal{G}_{ex}^{(n)})$ is the quantum supremum 1.12, are $\mathcal{O}(\epsilon)$ close in 2-norm to 1.13, up to a global isometry:

$$\exists \text{ isometry } V : \| |\Psi\rangle - V|u_0^{(n)}\rangle \|_2 \leq \mathcal{O}(\epsilon)$$

$$\forall i \in \{1, \dots, n\} : \|\Pi_i |\Psi\rangle - V|u_i^{(n)}\rangle \langle u_i^{(n)}| u_0^{(n)}\rangle\| \leq \mathcal{O}(\epsilon)$$

Here, the global isometry V expresses the fact that the statistics of a correlation experiment are always blind to embeddings into a higher-dimensional space and a global basis change. It is too optimistic to hope to determine these trivial degrees of freedom.

Proof. We denote the optimal solution of the Lovász SDP by X_n^* . Due to the unicity of X_n^* , see Lemma 4.1, one possible Gram decomposition of X_n^* is

$$X_n^* = \text{Gram}(|u_0^{(n)}\rangle, \langle u_1^{(n)} | u_0^{(n)} \rangle |u_1^{(n)}\rangle, \dots, \langle u_n^{(n)} | u_0^{(n)} \rangle |u_n^{(n)}\rangle),$$

with $\{|u_i^{(n)}\rangle\}_{i=0}^n$ the set of vectors defined in 1.13.

Say the odd n -cycle correlation experiment produces an ϵ -suboptimal violation of the KS non-contextuality inequality 1.11. Any quantum model $|\Psi\rangle, \{\Pi_i\}_{i=1}^n$ consistent with these statistics can be translated into a Gram matrix \tilde{X} that satisfies the assumptions of Lemma 4.2:

$$\tilde{X}_n = \text{Gram}(|\tilde{u}_0\rangle, \langle \tilde{u}_1 | \tilde{u}_0 \rangle |\tilde{u}_1\rangle, \dots, \langle \tilde{u}_n | \tilde{u}_0 \rangle |\tilde{u}_n\rangle),$$

where

$$\begin{aligned} |\tilde{u}_0\rangle &:= |\Psi\rangle \\ |\tilde{u}_i\rangle &:= \frac{\Pi_i |\Psi\rangle}{\sqrt{\langle \Psi | \Pi_i | \Psi \rangle}}. \end{aligned}$$

If ϵ is sufficiently small, and the quantum model violates the KS non-contextuality inequality 1.11, then the underlying Hilbert space \mathcal{H} must be at least three-dimensional, since there exists an explicit KS non-contextual hidden variable model for a qubit CITE. Let $d \geq 3$ denote the dimension of \mathcal{H} . If \mathcal{J} is an inner product isomorphism $\mathcal{H} \mapsto \mathbb{C}^d$, define $|\tilde{\mathbf{u}}_i^{(n)}\rangle := \mathcal{J}|\tilde{u}_i^{(n)}\rangle$. For the vectors $\{|u_i^{(n)}\rangle\}_{i=0}^n \subset \mathbb{C}^3$, define $\{|\mathbf{u}_i^{(n)}\rangle\}_{i=0}^n$ to be their embedding into the larger space \mathbb{C}^d .

The vectors $|u_i^{(n)}\rangle$ and $|\mathbf{u}_i^{(n)}\rangle$ are related by the isometric embedding

$$\begin{aligned} V' : \mathbb{C}^3 &\rightarrow \mathbb{C}^d \\ e_i^{\mathbb{C}^3} &\mapsto e_i^{\mathbb{C}^d}, \end{aligned}$$

where $\{e_i^{\mathbb{C}^d}\}$ is the standard basis of \mathbb{C}^d .

Lemma 4.3 implies that

$$\begin{aligned} \forall i \in \{0, \dots, n\}, \exists \text{ unitary } U \in \mathbb{C}^{d,d} : & \| \langle \tilde{\mathbf{u}}_i^{(n)} | \tilde{\mathbf{u}}_0^{(n)} \rangle | \tilde{\mathbf{u}}_i^{(n)} \rangle - \langle \mathbf{u}_i^{(n)} | \mathbf{u}_0^{(n)} \rangle U | \mathbf{u}_i^{(n)} \rangle \|_2 \\ &= \| \langle \tilde{u}_i^{(n)} | \tilde{u}_0^{(n)} \rangle | \tilde{u}_i^{(n)} \rangle - \langle u_i^{(n)} | u_0^{(n)} \rangle \mathcal{J}^{-1} U V' | u_i^{(n)} \rangle \|_2 \leq \mathcal{O}(\epsilon) \end{aligned} \quad (4.1)$$

For convenience, we define $V := \mathcal{J}^{-1} U V'$. V is an isometry because the inner product isomorphism \mathcal{J} is unitary.

We want 4.1 to hold true for the normalized vectors $|\tilde{u}_i^{(n)}\rangle, |u_i^{(n)}\rangle$, without pre-factors, as this would imply Theorem 4.4. This is a direct consequence of the following, final Lemma 4.5. \square

Lemma 4.5 (CITE). *Let $\| |a\rangle - |b\rangle \|_2 \leq \delta$ for vectors $|a\rangle$ and $|b\rangle$, such that $\| |a\rangle \|_2 \geq 2\delta$. Let $|\hat{a}\rangle$ and $|\hat{b}\rangle$ be $|a\rangle$ and $|b\rangle$, normalized to have unit norm. Then, we have that $\| |\hat{a}\rangle - |\hat{b}\rangle \|_2 \leq \frac{2\delta}{\| |a\rangle \|_2}$.*

The proof is found in CITE (Lemma 9). As $|\langle u_i^{(n)} | u_0^{(n)} \rangle| = 5^{\frac{1}{4}}$, the assumption in Lemma 4.5 is true for sufficiently small ϵ .

Having proved that the only quantum model $|\Psi\rangle, \Pi_i$ compatible with a maximal violation of 1.11 is 1.13, and that this self-testing result is noise robust, we will now analyze what assumptions the protocol in CITE requires for practical purposes.

4.2 Assumptions

The following is an as comprehensive as possible list of explicit or implicit assumptions that are required to self-test the reference quantum experiment 1.13, on the basis of Theorem 4.4:

1. i.i.d. rounds $x, y | i, i \oplus 1$
2. freely random choice of measurement context
3. sharp (outcome-repeatable and minimally disturbing) measurements
4. cyclic compatibility

5. sequential measurements

- system not disturbed between measurements
- each measurement device is used only once
- information carrying capacity of the unknown system does not exceed “simulable” bound
- alternatively: Landauer’s principle, ability to detect Landauer heat of the order $k_B T$ per measurement

Let us go through Assumptions 1-5 in order:

Assumptions 1, namely that the devices behave in the same way for every input-output round of the correlation experiment, is made in a large majority of current self-testing works CITE. It allows us to lift relative frequencies observed in a correlation experiment to well-defined probability distributions $p(x, y | i, i \oplus 1)$ that is valid for all rounds of the protocol, a round being characterized by one input-output cycle for a given measurement context. When checking for compatible quantum models of the observed correlations, this assumption lets us relate the observed input-output sequence to quantum expectation values.

Assumption 2 is required to ensure the integrity of whatever Bell or KS inequality is used in the self-testing protocol. For a second, assume that Assumption 2 is not true. It follows that violations of say the KCBS KS non-contextuality inequality are indeed compatible with a KS non-contextual HVM description. Thus, violations of the KCBS inequality could be simulated by a classical mechanism, voiding the self-testing certificate. This can be seen as follows: The KCBS correlations experiment is a test for the expression $\sum_{i=1}^5 p(0, 1 | i, i \oplus 1)$. Assuming compatibility with an ontological model and a finite number of ontic states $\lambda \in \Lambda$, we can rewrite this as

$$\sum_{\lambda \in \Lambda} \sum_{i=1}^5 p(0, 1 | i, i \oplus 1, \lambda) p(\lambda | i, i \oplus 1) = \sum_{\lambda \in \Lambda} \sum_{i=1}^5 p(0, 1 | i, i \oplus 1, \lambda) \mu_i(\lambda),$$

where μ_i corresponds to the state preparation, conditioned on the measurement context $(i, i \oplus 1)$: $\mu_i(\lambda) = p(\lambda | i, i \oplus 1)$. Let Λ consist of five ontic states λ_i , one for every measurement context $(i, i \oplus 1)$. Furthermore, let these ontic states λ_i fix measurement outcomes in a way such that $p(0, 1 | i, i \oplus 1, \lambda_i) = 1$. If we allow for the choice of measurement context to be correlated with the ontic state of the system, such that an experimenter can no longer make a free choice of measurement context, and that the ontic state of the system in fact fixes the measurement context like $\mu_i(\lambda) = \delta_{\lambda, \lambda_i}$, then

$$\sum_{\lambda \in \Lambda} \sum_{i=1}^5 p(0, 1 | i, i \oplus 1, \lambda) \mu_i(\lambda) = 5 > 2.$$

Thus, even the algebraic maximum of the KCBS inequality is compatible with a KS non-contextual HVM description, if we do not make Assumption 2.

The necessity of Assumption 3 for the soundness of the self-testing protocol was already highlighted in Sections 2.4 and 2.5. For unsharp quantum measurements, the class of odd n -cycle inequalities no longer comprises conceptually well-grounded tests of classicality. Relatedly, the CSW hierarchy falls apart and the Lovász SDP no longer characterizes optimal quantum violations of KS non-contextuality inequalities. Assumption 3 was critical in the above approach. Nevertheless, we would like to relax it - in a realistic setting, measurements will never be truly projective.

Assumption 4 is required as cyclic compatibility went into deriving the odd n -cycle inequalities 1.11 (recall that KS non-contextuality restricts the ontological representation of compatible measurements). Also, the CSW framework assumes compatible measurements. If the measurements $i, i \oplus 1$ were not compatible, the measurement events in Figure 2.1 could not be related to well-defined projectors Π_i , even for sequential measurements, as the product of two non-commuting projectors is not necessarily a projector.

As Assumption 5 is extensive, we devote an entire section to it.

4.3 Sequential measurements and the memory assumption

As described in Section 3, self-testing allows us to make powerful inferences about the quantum properties of a system. For applications of self-testing, the prime example being quantum key distribution (QKD), it is of great importance that the certificate not only encompasses the quantum state and measurements consistent with the maximal violation, but that the optimal input-output correlations also bear witness to the quantumness of the system. This means that we must exclude the possibility of the correlations being simulated by some classical mechanism, as this would for instance allow the copying of information and render the protocol information-theoretically insecure. To do so, we wish to study the resources a classical device requires, in order to simulate the ideal reference quantum realization. The odd n -cycle contextuality scenarios will be of particular interest to us. In the case of Bell inequalities, the relevant resources have been identified and studied extensively CITE. As highlighted in Section 1.9, assuming local causality and freely chosen detector settings, correlations that violate a Bell inequality are incompatible with a local hidden variable model description. Thus, classically simulating correlations that exhibit Bell non-locality requires superluminal communication (“communication cost”) CITE and would violate special relativity. For self-testing via Bell inequalities this means that, assuming any adversary is bounded by the no-signalling principle, the violation of a

Bell inequality certifies quantumness, provided that both subsystems are indeed space-like separated¹. Analogously to Bell non-locality and the no-signalling principle, we wish to identify a physical principle that enables us to differentiate between a quantum system exhibiting KS non-contextuality and a classical mechanism simulating KS contextual correlations. One strength of contextuality as a competing notion of nonclassicality is that contextuality experiments comprise a much broader class of experiments. In particular, generic contextuality experiments don't presuppose multiple space-like separated systems, but can consist of sequences of state preparations and measurements of a single system. However, this generality also complicates the analysis of the relevant resources required for simulating KS contextuality. Moreover, we can in principle never distinguish between a single-system quantum experiment exhibiting some input-output correlations and a classical pre-programmed computer simulating the statistics CITE. For Bell-type scenarios this possibility is circumvented by extending the experiment to multiple space-like separated subsystems. In Section 4.3 we examine the “memory cost” of KS contextuality CITE: As we will see, the minimal internal entropy of a classical system emulating KS contextuality in many cases exceeds the information-carrying capacity of the quantum system it aims to emulate. To weed out the possibility of a classical mechanism producing the correlations, in CITE it is assumed that “[...] the measured system has no more memory than its information carrying capacity”. However, for self-testing scenarios the information carrying capacity of the physical device is unknown. One therefore has to justify imposing an upper bound on the unknown device's information carrying capacity to conclude quantumness. As we will see, these upper bounds have to be of the order of a few bits, which means that the required memory assumption is a very strong one indeed.

In Section REF, where we propose two novel self-testing protocols based on Spekkens contextuality, the assumption of cyclic compatibility will be replaced by assumptions about (approximate) operational equivalences of certain experimental procedures. Both protocols will also assume an upper bound on the information carrying capacity of the unknown system. We will discuss their features in Section REF.

Alternatively, Cabello proposes using Landauer's principle CITE to translate the memory cost of KS contextuality into a “thermodynamical cost” CITE, in terms of heat the system dissipates when measured. His proposal will be subject of Section 4.3.4. As the minimal unit of heat a classical simulation must dissipate per measurement will be of the order $\sim k_B T$, this proposal is hardly viable in a realistic experimental setting.

¹Note that it is highly non-trivial to verify that two space-time points are space-like separated in a device independent manner.

4.3.1 Setting and preliminary considerations

As covered in Section 1.7, the KS Theorem states that a deterministic assignment of outcomes to measurements in a way that is compatible with quantum theory must be context-dependent. It is not at all apparent why context-independence is a reasonable demand that captures the essence of what it means for a system to behave classically. After all, Bell points out that “[t]he result of an observation may reasonably depend not only on the state of the system (including hidden variables), but also on the complete disposition of the apparatus” CITE. Perhaps correlations exhibiting KS contextuality should not come as a surprise and may indeed be compatible with a classical interpretation. Addressing this issue, Peres suggests KS contextuality experiments should consist of sequential measurements, allowing for measurements to be performed in an identical manner, regardless of context. Assuming sequential measurements and compatibility relations like in Figure 1.3, Peres argues that KS contextuality would imply backwards-in-time signalling: “Suppose that we measure A first and only at a later time decide whether to measure B or C or none of them. How can the outcome of the measurement A depend on this future decision?” CITE. Thus, the assumption of KS non-contextuality becomes compelling in the case of sequential measurements, and we will assume KS contextuality experiments to consist of a series of sequential measurements for different contexts. Note that while sequential measurements are crucial for the interpretation of the experiment, within quantum theory it does not matter whether we consider joint or sequential measurements, as both produce identical correlations. To that effect we will assume that our system is not perturbed between successive measurements. If one measurement is performed at time t and another at $t' > t$, then we want the system’s state to remain unchanged in the time interval (t, t') .

Let us now examine how, in an adversarial setting where correlations may not describe the outcomes of compatible and ideal quantum measurements, a classical device with an internal memory can simulate the statistics of sequential measurements.

4.3.2 Input-output processes as information transducers

In most general terms, a KS contextuality experiment is an input-output process. To verify that an experiment producing a maximal violation of say the KCBS KS non-contextuality inequality is compatible with the ideal reference experiment, as mandated by self-testing, we may perform an infinite sequence of measurements $\overleftrightarrow{X} = \{\dots, X_{-1}, X_0, X_1, \dots\}$, yielding an infinite sequence of outputs $\overleftrightarrow{Y} = \{\dots, Y_{-1}, Y_0, Y_1, \dots\}$ CITE. Here, \overleftrightarrow{X} and \overleftrightarrow{Y} are bi-infinite sequences of random variables and define stochastic processes. The random variable X_i represents the freely chosen random measurement at time $t = i \in \mathbb{Z}$, whereas the random variable Y_i represents the output obtained when performing the measurement X_i at time i . We assume that all X_i and Y_i take values from finite alphabets \mathcal{X} and \mathcal{Y} ; most of the measurements we will consider are

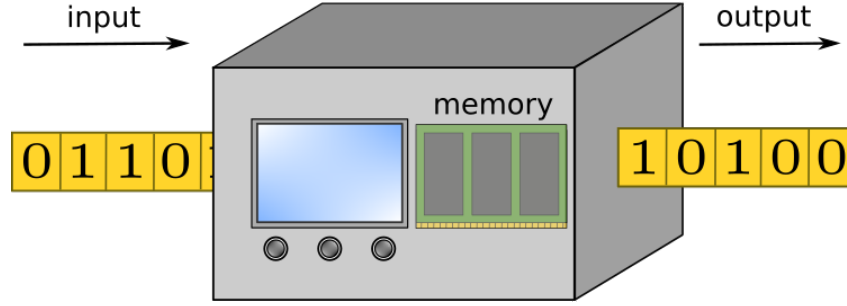


Figure 4.1: A classical device simulating the ideal reference statistics of some KS-type experiment, consisting of sequential measurements. The device is fed a sequence of inputs (measurement choices) and produces outputs (measurement outcomes), based on its internal state prior to an input. The internal state of the device, which one can equate with a memory, stores relevant historical information about the past input-output process, on the basis of which the device generates the correct input-output statistics. After one input-output cycle, the device can update its memory state.

dichotomic, i.e. $|\mathcal{Y}| = 2$, as is the case for the KCBS KS contextuality scenario. Furthermore, we can distinguish the present $t = 0$ and partition the processes \vec{X}, \vec{Y} into past and future: $\overleftarrow{X} := \dots, X_{t-2}, X_{t-1}$, $\overrightarrow{X} := X_t, X_{t+1}, \dots$. The reference time is in fact arbitrary, as the stochastic processes we will consider turn out to be stationary. Any classical input-output mechanism producing a maximal violation that we cannot “unmask” on the basis of correlations alone, must be consistent with the ideal reference input-output process.

Any classical apparatus simulating the reference correlations has the form of a general information transducer, transforming an input stochastic process into an output stochastic process CITE. As such, a classical simulation is at all times i in some well-defined internal state S_i that holds information about the past input-output sequence $\overleftarrow{Z} := (\overleftarrow{X}, \overleftarrow{Y})$, stored inside a memory. As the future process may be correlated with the past, the internal memory of the device allows it to produce statistically correct predictions. If we pass a valid input symbol from \mathcal{X} to the transducer it will generate an output symbol in \mathcal{Y} , based on the information about the past encoded in the internal state prior to the input. The output may be accompanied by an update of the transducer’s internal memory. One can think of this input-output process as feeding the transducer an infinite tape, whereby each tape cell contains a valid input symbol (choice of measurement). The transducer then generates an output, overwrites the corresponding tape cell with said output, and updates its internal state, see Figure 4.1. Note that we will only consider the RAM memory cost of such a simulation, which relates to the number of memory bits the device needs to store the relevant historical information about the past process², and in particular not to the memory required to store say output and transition probabilities.

Definition 4.6 (CITE). An *information transducer* is a tuple $(\mathcal{X}, \mathcal{Y}, \mathcal{S}, \mathcal{T})$ where

- \mathcal{X}, \mathcal{Y} are the transducer’s (finite) input and output alphabets,

²alternatively, the RAM memory cost can be regarded as the uncertainty (in bits) in the internal state of the transducer.

- \mathcal{S} is a (finite or countably infinite) set of internal memory states, and
- $\mathcal{T} \equiv \{p(S_{i+1} = s', Y_i = y | S_i = s, X_i = x)\}_{x,y,s,s'}$ is the set of conditional output-transition probabilities.

More concretely, $p(S_{i+1} = s', Y_i = y | S_i = s, X_i = x)$ denotes the probability of the transducer outputting y and transitioning to the internal state s' after the i th input-output cycle, conditioned on the input x and internal state s .

We require a valid classical model to produce statistics that are consistent with the infinite family of stochastic processes describing the quantum probabilities of the ideal reference experiment: $\{P(\vec{Y} | \vec{x})\}_{\vec{x}}$, \vec{x} being an instantiation of the input sequence \vec{X} .

Input-output transducers have been extensively studied within the field of computational mechanics. Of interest to us are memory-optimal classical models that produce the correct statistics $\{P(\vec{Y} | \vec{x})\}_{\vec{x}}$. A transducer is memory-optimal if it stores the minimal amount of information about the past (in terms of number of bits) to make statistically correct future predictions. It has been shown that for stationary input-output processes there exists a unique memory-optimal transducer producing the correct statistics CITE: the process' ϵ -transducer. The memory-optimal transducer does not store all information about the past \vec{Z} , as this may be highly redundant. In particular, the memory-optimal transducer does not differentiate between pasts that predict statistically identical futures. As such, the internal states of an ϵ -transducer are given by the causal states of the process.

Definition 4.7. Let \vec{X}, \vec{X} denote the past and future stochastic input processes, respectively. The *causal states* $[\vec{z}]$ of an input-output process $\vec{Y} | \vec{X}$ are the equivalence classes on the set of pasts \vec{z} with respect to the equivalence relation

$$\vec{z} \sim_{\epsilon} \vec{z}' :\Leftrightarrow P(\vec{Y} | \vec{X}, \vec{Z}=\vec{z}) = P(\vec{Y} | \vec{X}, \vec{Z}=\vec{z}').$$

4.3.3 Memory-optimal classical simulation of quantum correlations

In CITE it is shown that, for a “causally complete” KS contextuality reference experiment, the causal states of the process' ϵ -transducer correspond one-to-one to the possible quantum states $\{|\Phi_{\vec{z}}\rangle\}_{\vec{z}}$ the system can occupy after all past measurements \vec{x} . The assumption of causal completeness holds for many KS contextuality scenarios and in the following we will examine the odd n -cycle KS contextuality scenarios, in particular the KCBS scenario for $n = 5$. In CITE the same analysis is performed for the Peres-Mermin (see Section 1.8) and Yu-Oh (see Section 1.10) sets of measurements.

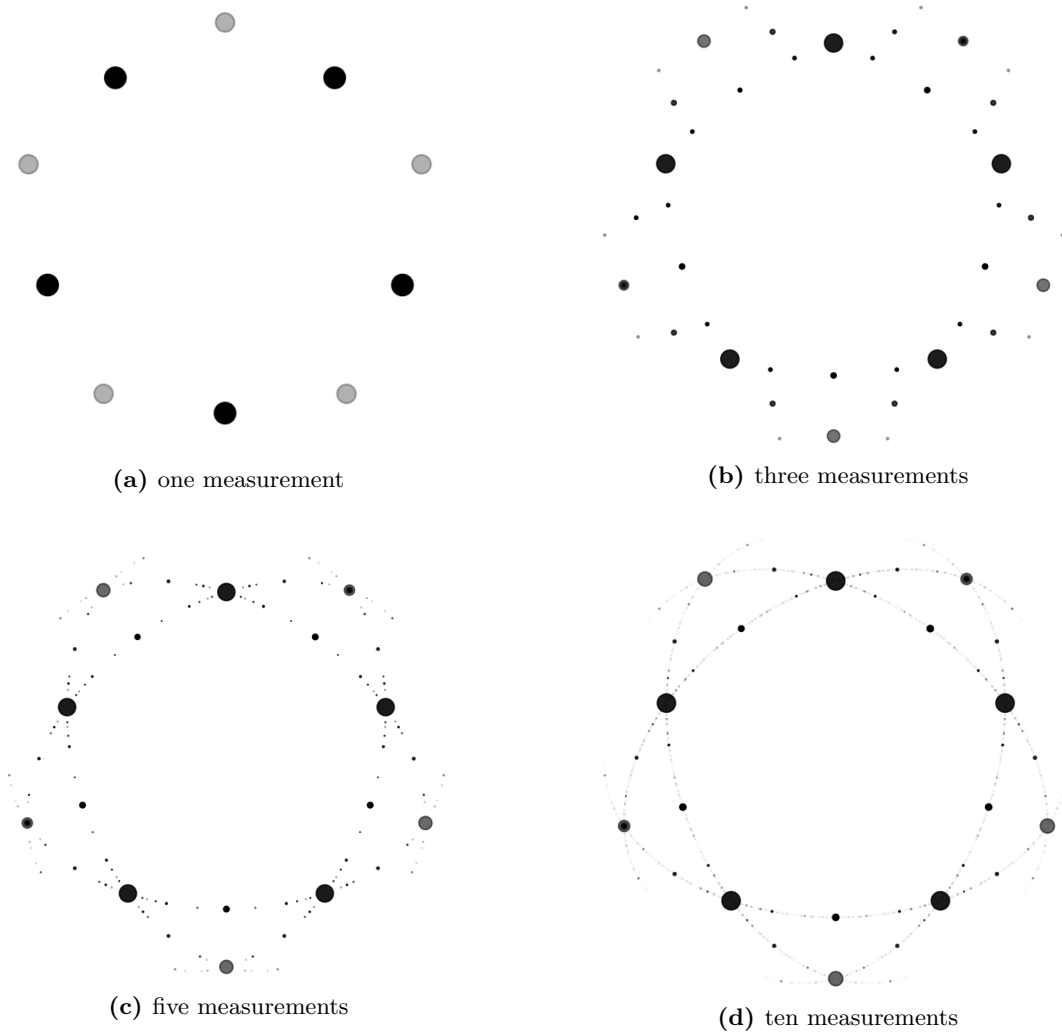


Figure 4.2: Possible post-measurement quantum states after one, three, five, and ten measurements for the ideal KCBS experiment. The components of the post-measurement quantum states w.r.t. the standard qutrit basis are all real. The phases (signs) of the real qutrit states are chosen such that the corresponding vectors lie on a joint semisphere of \mathbb{R}^3 . What is plotted are centered views of the three-dimensional plots along the negative x direction. As can be seen, the number of possible post-measurement states increases with the number of sequential measurements. The volume of a scattered point is proportional to the probability of the system being in that state after the measurements. The probability distribution over possible post-measurement quantum states becomes approximately stationary for large measurement sequences. The fact that all states lie on one of five semi-circles is a consequence of the compatibility structure of the five KCBS measurements. Analogous plots are obtained in CITE for the Peres Mermin and Yu-Oh KS contextuality scenarios.

For all ideal odd n -cycle reference experiments, the number of possible post-measurement states tends to infinity when we increase the number of measurements in our sequence. However, the post-measurement states are not all equally probable. Figure 4.2 illustrates this behaviour for the KCBS scenario $n = 5$.

Problematically, the process describing the ideal quantum experiment is not stationary, i.e. not invariant under translations in time, as can be seen from Figure 4.2. However, if we “truncate”

the process and consider only input-output cycles after sufficiently many initial measurements, the probability distribution over quantum states and by extension the stochastic process describing the experiment becomes stationary for all practical purposes. We only have a notion of the memory-optimal statistics-emulating classical model for the case of stationary processes. For this reason we will always consider the truncated input-output process, noting that the ϵ -transducer's memory cost refers to the minimal number of bits needed to generate correct statistics, ignoring some of the initial behaviour. The true minimal number of memory bits a classical simulation requires to produce the correct statistics for all input-output cycles may therefore be greater.

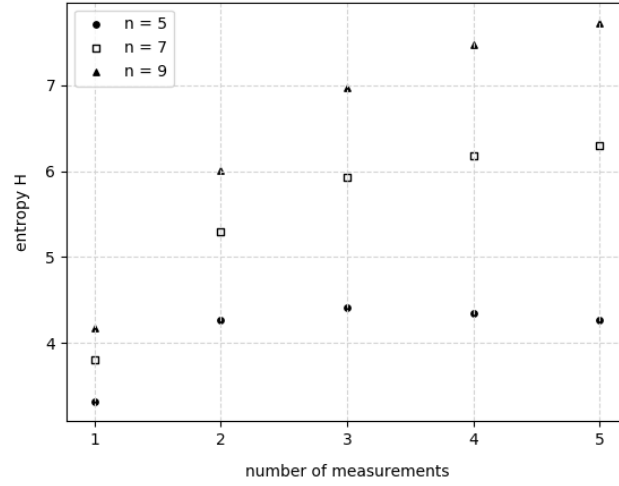
To numerically compute the memory cost, we consider all possible measurement sequences of a given length k . The statistical complexity of the process (RAM memory cost) is then the Shannon entropy $H = -\sum_i p_i \log_2 p_i$ of the probability distribution over all achievable post-measurement quantum states after k measurements, in the limit of $k \rightarrow \infty$ CITE. Recall that this probability distribution becomes approximately stationary, reflected by the above expression converging. We find the number of bits (RAM) needed by a memory-optimal simulation of the ideal reference KCBS experiment to be just over 4. The statistical complexity thus exceeds the information carrying capacity $\mathcal{C} = \log_2(3)$ of the reference qutrit. In CITE the same qualitative behaviour is observed for the Peres-Mermin and Yu-Oh sets of measurements, both of which exhibit state-independent KS contextuality.

An interesting feature we observe is that the memory cost of simulating the ideal reference experiment for odd n -cycle contextuality scenarios with $n \geq 5$ odd increases with the number of exclusivity graph vertices n , as is shown in Figure 4.3. However, this increase is only logarithmic. Furthermore, what does not change is the fact that, no matter the cycle length, we need to assume an upper bound on the unknown device's information carrying capacity in order to conclude quantumness. Additionally, we need to assume that each measurement device is used only once, requiring a potentially infinite supply, as otherwise the measurement device, and not the system whose information carrying capacity we bound, could retain memory about the past stochastic process.

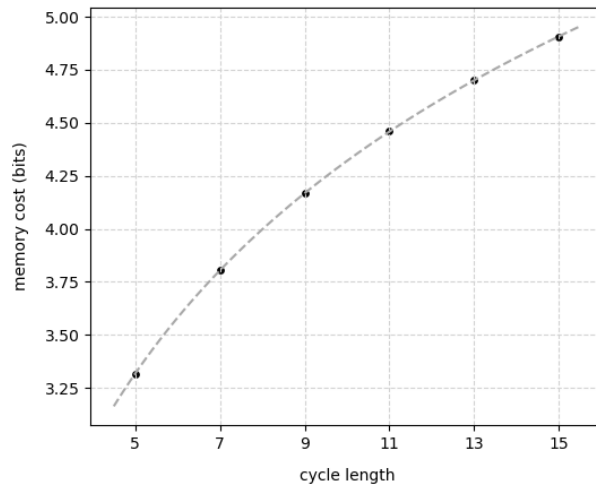
4.3.4 Classical simulations dissipate heat

As advertised, let us briefly discuss the alternative approach to assuming an upper bound on the unknown device's information carrying capacity. It is based on Landauer's principle and we will make the following assumption CITE:

1. sequential measurements can be chosen freely at random (as before),



(a) Plotted is the Shannon entropy of the probability distribution over all achievable post-measurement quantum states after a given number of measurements for the 5, 7, and 9-cycle KS contextuality scenarios. The probability distribution over possible post-measurement quantum states becomes approximately stationary for a sufficiently large number of measurements and the entropies thus saturate. This stationary limit of H is the process' RAM memory cost, and it increases with an increasing cycle length



(b) The RAM memory cost of classically simulating the ideal reference experiment for n -cycle KS contextuality scenarios increases logarithmically with the number of ideal measurements obeying cyclic compatibility that are performed.

Figure 4.3: The RAM memory cost of classically simulating the ideal reference experiment for odd n -cycle KS contextuality scenarios in terms of the cycle length n .

2. any classical simulation of the input-output process has limited memory³
3. Landauer's principle holds

First an intuitive description of how Landauer's principle aids us in certifying quantumness. Assumptions 1 and 2 imply that any classical simulation must erase information during its operation: The simulation cannot know in advance the measurement sequence the experimenter chooses freely at random. Additionally, due to its finite capacity, the simulation's memory cannot contain instructions about what to do for all possible measurement sequences, i.e. the device cannot store in memory the internal states the system will be in for all possible measurement sequences. Therefore, the memory-bounded simulation has to erase memory bits, so it can store the updating historical information about the process, which enables it to emulate the ideal reference realization. Landauer's principle relates this erasure to a thermodynamical cost, meaning that the classical simulation will have to dissipate a minimal amount of heat to the environment for every measurement. If we do not detect this extra Landauer heat dissipated to the environment during measurement sequences, then we can be sure that our system is quantum, assuming that any adversary is bounded by finite memory ... in principle. The unit of extra Landauer heat emitted per measurement by a classical device is of the order $\sim k_B T$, rendering this method of certifying quantumness highly unrealistic. The following is a quantitative treatment of the intuition just presented.

It can be proven that the classical simulation of an input-output process that on average dissipates the minimal unit of heat per measurement is the process' memory-optimal ϵ -transducer CITE. Consider the ϵ -transducer of a stationary input-output process. The ϵ -transducer is unifilar CITE, due to the causal structure of its internal states. Unifilar describes the property that the present internal state, input x' , and subsequent output y' uniquely determine the next internal state of the machine. The subsequent internal state is the causal state $[\overleftarrow{z}']$ corresponding to the updated past input-output process \overleftarrow{z}' , where \overleftarrow{z}' is the concatenation of \overleftarrow{z} and (x', y') . Importantly, the reverse is generally not true CITE. The present internal state, previous input, and previous output in general do not determine the previous internal state uniquely. This means that the memory-optimal simulation "forgets" information about its previous internal state and is as such a logically irreversible computation CITE.

Landauer's principle states that whenever classical information is processed in a logically irreversible manner, for example when erasing a classical bit of information, the entropy of the environment increases by a corresponding amount. For the erasure (setting to zero) of a classical bit this entropy increase is at least $\Delta S_{env} \geq -\Delta S_{sys} = -(0 - k_B \log(2)) = k_B \log(2)$. This entropy increase is related to the heat dissipated to the environment reservoir via $\Delta Q = (\Delta S) T$,

³Note that the Bekenstein bound gives a fundamental upper bound for the internal entropy contained in a system, in terms of its size and energy, albeit impractically huge for our purposes CITE

meaning that the minimal amount of dissipated heat for the erasure of a single classical bit is given by $\Delta Q = k_B T \log(2)$.

For the ϵ -transducer that simulates the ideal reference statistics, the Shannon entropy

$$I_{erased} = H(S_{t-1}|X_t, Y_t, S_t) \quad (4.2)$$

characterizes the amount of erased information for one step of the computation CITE. The erased entropy can be re-written as $I_{erased} = H(S_t|X_t, S_{t-1}) + I(S_t : X_t)$ CITE, which simplifies to $I_{erased} = H(S_t|S_{t-1})$. Identifying the causal states of the optimal transducer with the post-measurement quantum states gives us an explicit expression for the erased information per measurement for odd n -cycle contextuality scenarios:

$$I_{erased} = -\frac{1}{n} \sum_{s, (x,y)} \pi(s) \langle s|P_x^y|s \rangle \log_2\left(\frac{\langle s|P_x^y|s \rangle}{n}\right), \quad (4.3)$$

where we sum over all causal states s and $(x, y) \in \{1, \dots, n\} \times \{0, 1\}$. π is the stationary probability distribution over the causal states and P_x^y is the projector corresponding to the measurement-outcome pair (x, y) .

We find that any classical simulation that emulates the ideal KCBS experiment must erase at least ~ 2.75 bits for each measurement (in the stationary regime). Therefore, the heat per measurement any classical simulation dissipates is at least $\sim 2.75 k_B T \log(2)$. The heat dissipated by classical devices emulating higher order odd n -cycle scenarios increases with the cycle length n : By paying a price in terms of the number of ideal measurements satisfying cyclic compatibility we must implement, we can increase the minimal unit of heat that a classical simulation must dissipate per measurement cycle. Once again, this dependence is only sub-linear, as can be seen in Figure 4.4.

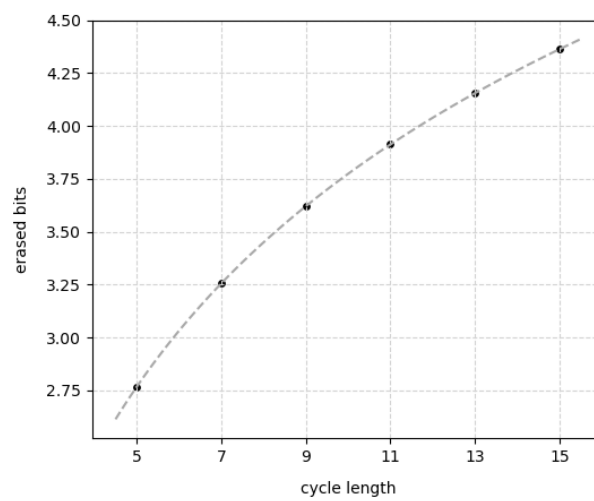


Figure 4.4: The minimal number of bits a classical simulation of the ideal n -cycle reference experiment must erase per measurement, in terms of the cycle length n . The number of erased bits increases logarithmically with the cycle length.

Chapter 5

A revised notion of classicality

Spekkens non-contextuality

5.1 Outcome determinism for unsharp measurements (ODUM)

We want a notion of classicality that accomodates unsharp measurements and is robust to noise. A first approach, assuming the validity of quantum theory, might entail simply extending KS non-contextuality to general POVM measurements, in the sense that a classical HVM assigns a conditional probability in $\{0, 1\}$ to all positive semi-definite operators, such that for every resolution of the identity only one operator receives the valuation 1. This incremental approach has been given the name “ODUM”, which stands for “outcome determinism for unsharp measurements”. The problem with ODUM is that it runs into numerous inconsistencies, as convincingly argued in CITE. We confine ourselves to the most immediate inconsistency: Consider the quantum experiment consisting of an experimenter repeatedly performing the trivial POVM $\{\frac{1}{2}, \frac{1}{2}\}$ on some arbitrary preparation ρ . Any classical value assignment to positive semi-definite operators would be in conflict with the normalization conditions for general ontological models in Section 1.4, i.e. either the added probabilities of the two outcomes would be zero or exceed one. Thus, either this trivial POVM is non-classical according to the new notion, or the new notion is not meaningful. The trivial POVM corresponds to a fair coin flip - it can be implemented by simply throwing away the system and generating a random bit. Such a “measurement” should by all means be considered classical. For this implementation, the only sensible choice is to assign every ontic state λ the conditional probability $\frac{1}{2}$, as measurement and system are completely decoupled. The same line of reasoning leads to the conclusion that the approach in Section 2.3 cannot be extended to general unsharp measurements, as this produces inconsistencies when treating QM.

5.2 Operational approach due to Spekkens

The previous examples highlight the need for a new notion of contextuality that captures the spirit of KS non-contextuality, but is applicable to realistic experimental scenarios. Let us take a step back and identify common features of the contextuality scenarios discussed so far, which led to behaviour we deemed non-classical. The setting was in all cases for the most part identical: we considered sets of projective measurements, with the key property that a given projector was shared amongst multiple measurements. Identical projectors appearing in multiple measurement contexts was crucial for proving a contradiction. Spekkens' approach captures this key feature and operationalizes the assumption. In QM, a positive semi-definite operator defines an operational equivalence class, in the sense that all measurement events which are operationally indistinguishable, for all states, must be represented by the identical operator. An analogous observation applies to density operators describing the preparation of a system. On that account, instead of assuming a projector to be part of several measurements, Spekkens assumes certain measurement events to be operationally equivalent; we will formalize this shortly. Assuming KS non-contextuality, the implication of a projector appearing as part of multiple measurements is that it must be assigned the same conditional probability $\{0, 1\}$ by the ontological model, for all ontic states λ . In other words, operational equivalence implies an identical ontological representation. We will find that Spekkens' revised notion of non-contextuality follows exactly this prescription.

In CITE, Spekkens introduces a largely operational notion of contextuality for preparations, measurements, and transformations. It is a revision of KS contextuality that addresses several shortcomings of the traditional definition. Contextuality is generalized to apply to arbitrary operational theories. In comparison, KS contextuality is limited to the scope of QM, as it is defined within the framework of quantum theory. Furthermore, the revised definition will not assume outcome determinism at an ontic level, accommodating HVM whose indicator functions are more general mappings $\Lambda \mapsto [0, 1]$. Taking QM as operational theory, Spekkens contextuality extends to arbitrary measurements, including physical, non-projective ones. In fact, within the Spekkens framework, measurements, preparations, and transformations can be considered black box devices, i.e. primitives of which one can compose an experiment. An operational theory makes predictions about these operational primitives, i.e. assigns probabilities to measurement outcomes that are consistent with experimental observations. In QM, for instance, outcome probabilities are given by the Born rule, where we posit that every preparation is operationally fully specified by a density operator and every measurement by a POVM. A Spekkens non-contextual operational theory is one that is compatible with a Spekkens non-contextual ontological model, which we will define shortly. Spekkens' revised concept of non-contextuality will serve as a new, more suitable notion of classicality that is in particular compatible with unsharp measurements.

This section only discusses Spekkens non-contextuality for preparations and measurements, as transformations will not be relevant to our discussion¹. Preparations and measurements will be represented as outlined in Section 1.4. In the following, if not states otherwise, the term “contextuality” and all variant forms will refer to Spekkens contextuality.

The essence of what it means for an ontological model to be non-contextual can be summarized as follows: An ontological model of an operational theory is *non-contextual* if: The operational equivalence of two experimental procedures, i.e. two preparations or two measurements, implies that they have an equivalent representation in the ontological model. Two experimental procedures are operationally equivalent if they cannot be distinguished by any measurement statistics.

Let us formalize the above, starting with preparation non-contextuality. We can define an equivalence relation on the set of all preparation procedures P that partitions the set into operational equivalence classes $[P]$. By the guiding principle above, we require equivalent preparations to have the same ontological representation in a non-contextual HVM.

Definition 5.1 (CITE).

Two preparation procedures P, P' are *equivalent*, denoted $P \sim P'$, if

$$\forall \text{ measurements } M, \forall \text{ measurement outcomes } k : p(k|P, M) = p(k|P', M).$$

An ontological model is *preparation non-contextual* if equivalent preparations have identical associated probability density functions on the ontic state space:

$$\forall \text{ preparations } P : \mu_P = \mu_{[P]}.$$

Measurement non-contextuality is defined in an analogous manner:

Definition 5.2 (CITE).

Two measurement procedures M, M' are *equivalent*, denoted $M \sim M'$, if their outcomes can be associated one-to-one and

$$\forall \text{ outcomes } k, \forall \text{ preparations } P : p(k|P, M) = p(k|P, M').$$

¹Consider a prepare-and-measure type experiment that consists of an initial state preparation, followed by some transformation of the system, and finally a measurement. We may consider the transformation as part of the initial preparation procedure, or as part of the final measurement, and describe the experiment solely in terms of preparation procedures and measurements.

An ontological model is *measurement non-contextual* if equivalent measurements have identical associated indicator functions:

$$\forall \text{ measurements } M, \forall \text{ outcomes } k : \xi_{M,k} = \xi_{[M],k}.$$

Spekkens proves three no-go theorems for ontological models of QM CITE. These rule out preparation non-contextual, measurement non-contextual, and transformation non-contextual models, respectively. All three proofs apply to two-dimensional Hilbert spaces, making them stronger than traditional proofs of contextuality.

Whenever referring to a non-contextual ontological model, without specifying whether we are referring to preparations or measurements, we will assume the ontological model to be both preparation and measurement non-contextual. As the reasons for assuming classical correlations to be compatible with a measurement non-contextual HVM description are the same as those for preparation non-contextuality, the combination of both is the only natural assumption of non-contextuality.

We now discuss to what extent non-contextuality is a sensible notion of classicality. A contextual ontological model implies a difference in reality that cannot be observed. Such a model is in conflict with Leibniz' "Identity of Indiscernibles" CITE. Leibniz's principle states that two empirically indistinguishable scenarios are to be ontologically equivalent. As such, it rejects ontological models for which there exist ontologically distinct, but empirically indistinguishable scenarios. Recall that, within QM, KS non-contextuality can be seen as a generalization of Bell's notion of local determinism to non-remote measurement contexts. One can also link Leibniz' principle, which is at the heart of Spekkens' revised notion of contextuality to Bell's notion of local causality. Like before, local causality is the weaker assumption, which is owed to the fact that Bell-type experiments impose additional constraints on the experiment's causal structure:

In principle, it might well be the case that an agent's free choice of measurement can have a causal influence on a remote system and alter the ontic or "matter of fact" state of that system. However, physicists have yet to devise an experiment producing correlations that are in conflict with the no-signalling principle, which is reinforced by special relativity. Therefore, to the best of our knowledge, such causal influence, if it exists, is not detectable. This in turn means that an agent in the remote frame of reference can detect no difference in the physical properties of his system for different free choices of measurement. Local causality is the assumption that there is in fact no difference at the ontic level, and thus provides the most natural explanation for our inability to detect a causal influence. Another, more hair-raising explanation could be that all possible ensemble preparations in principle accessible to an experimenter are too spread out to resolve minor differences in the conditional probabilities. The assumption of local causality just corresponds to Leibniz' principle applied to Bell-type experiments with space-like separated

measurements. The assumption of Spekkens non-contextuality simply extends the applicability of Leibniz’s principle to arbitrary prepare-and-measure type experiments, in particular those without remote subsystems.

Just as local causality is motivated by the no-signalling principle, local causality providing the most natural explanation for the impossibility of superluminal signalling, a point can be made that non-contextuality is perhaps equally well motivated by Leibniz’ Identity of indiscernibles. Analogously to local causality, non-contextuality is the most natural explanation for operational equivalences. Spekkens points out that the credentials of Leibniz’ principle parallel those of no-signalling and that it was critical in Einstein’s conception of relativity CITE. Take for instance the equivalence principle CITE which states that one cannot distinguish between being at rest in a uniform gravitational field and accelerating uniformly through free space. Within Newtonian mechanics the two scenarios are ontologically distinct, as one’s absolute acceleration differs in both cases. On his way to general relativity, Einstein reasoned that the empirical indistinguishability of both scenarios implies that they should be treated as ontologically equivalent by a physical theory, a powerful invocation of Leibniz’s principle that reshaped our perception of reality. An almost identical line of reasoning is found in Einstein’s 1905 paper “On the electrodynamics of moving bodies” CITE. He notes that the predictions of Maxwell’s equations depend only on relative motion. For example, the current induced in a coil will have the same magnitude and direction, no matter if we consider a magnet to be moving through the coil or the coil moving through the field of the magnet with equal but opposite velocity. To Einstein, this empirical indistinguishability was in conflict with the prevailing aether theories of the time. An aether would define a distinguished frame of reference, rendering the two empirically equivalent induction experiments ontologically distinct. Once again, Einstein’s invocation of Leibniz’ principle lead him to abandon aether theory and conceive relativity theory, in which both scenarios are equivalent.

In Section 5.3 we will see how to lift KS non-contextuality inequalities to Spekkens non-contextuality inequalities by substituting assumptions about operator compatibility for assumptions about operational equivalence. Like before, these lifted non-contextuality inequalities, when violated, bear witness to the quantumness of the system, in the Spekkens sense. Violations of non-contextuality inequalities that prove nature’s incompatibility with a non-contextual HVM description have been observed in experiments involving photonic qubit systems CITE. While the lifted non-contextuality inequalities no longer make unphysical assumptions about the measurements themselves, they introduce two new practical complications: the issue of tomographic completeness, and that of exact operational equivalence. Experimental tests of non-contextuality inequalities assume that we can implement a tomographically complete (TC) set of measurements and state preparations. The reason for this is that, according to Definitions 5.1, 5.2, two experimental procedures can only be asserted operationally equivalent if the two

procedures are operationally indistinguishable, for all possible prepare-and-measure-type experiment that utilize them. For instance, two preparations are only operationally equivalent if they produce the same statistics for all possible measurements, requiring in principle an infinite number of tests. In practice, we aim to identify a TC set of measurements, such that for an arbitrary measurement and preparation there is a functional relationship between the statistics of that measurement and those of the TC set. As such, if two preparations are operationally equivalent with respect to a TC set of measurements, they are operationally equivalent with respect to all measurements. Within QM, a TC set of measurements for a qubit system is given by the three Pauli measurements σ_x , σ_y , and σ_z , that together determine the three Bloch vector components of an arbitrary state. Problematically, without additional assumptions, we can in practice never declare a set of measurements to be TC. What we can do is try our best to refute such a claim. Additionally, there are tests of contextuality that account for some number of unknown measurements in a TC complete set CITE. By implementing more preparations and measurements one can in principle account for an arbitrary number of unknown measurements in a TC set. We will make use of the results in CITE in Section REF, and they will allow us to relax the assumption of TC. Just like in the case of KS contextuality, where had to assume operator compatibility, we cannot expect to devise a fully device-independent self-testing protocol based on non-contextuality.

Another issue with experimental tests of contextuality is that Definitions 5.1, 5.2 assume exact operational equivalences, requiring infinite precision. This problem has been tackled in CITE. Let us for the moment consider operational equivalences amongst preparations. The key is noticing that performing measurements on some set of preparations defines the statistics for all preparations in the convex hull. As part of post-processing, we can define new preparations in the convex hull, whose statistics we know, and that are exactly operationally equivalent. The price to pay is that these new preparations may be less optimal than the original ones, in the sense that they may only produce a lesser violation of the non-contextuality inequality. This “fitting” process that establishes exact operational equivalences is done within the framework of generalized probabilistic theories (GPT). Operational equivalences amongst measurements are treated in an analogous manner.

Finally, we wish to understand how KS contextuality and Spekkens contextuality are related. It turns out that a Spekkens non-contextual ontological model of QM assign deterministic outcomes to projective quantum measurements and induces a KS non-contextual outcome assignment, like in Definition 1.4, if we restrict measurements to projective ones. KS non-contextuality is an assumption that restricts the ontological representation of compatible (commuting) projective measurements. Let us therefore examine what conditions Spekkens non-contextuality imposes on the ontological representation of compatible measurements. Spekkens

non-contextuality covers arbitrary, not necessarily sharp, measurements and applies to arbitrary operational theories. Therefore, recall Definition 2.1, which defines what it means for two measurements to be compatible, purely in operational terms.

Let M_1 and M_2 be compatible measurements according to Definition 2.1, and M_{12} a dual outcome measurement that jointly realizes M_1 and M_2 . By definition, the measurement procedure M_1 is operationally equivalent to measuring M_{12} and discarding register two. Analogously, the measurement procedure M_2 is operationally equivalent to measuring M_{12} and discarding register one. What can be said about the ontological representation of compatible measurements within a Spekkens non-contextual model? In a Spekkens non-contextual ontological model, operationally equivalent measurement procedures have identical associated indicator functions. For the measurements M_1 , M_2 , and M_{12} like above, this implies:

$$\begin{aligned}\forall \lambda \in \Lambda, \forall P, \forall a_k : p(a_k | M_1, P, \lambda) &= \sum_{b_j} p(a_k, b_j | M_{12}, P, \lambda) \\ \forall \lambda \in \Lambda, \forall P, \forall b_k : p(b_k | M_2, P, \lambda) &= \sum_{a_i} p(a_i, b_k | M_{12}, P, \lambda)\end{aligned}$$

Consequently, a Spekkens measurement non-contextual ontological model is one for which the probability of measuring say M_1 and obtaining the outcome a_k , conditioned on the system being in any ontic state, is independent of what other compatible measurements are simultaneously performed. If we consider QM as operational theory, this is the essence of KS non-contextuality, as discussed in Section 1.6, with the notable difference that we have decoupled outcome determinism from context independence.

Interestingly, it can be proven that a Spekkens non-contextual ontological model must fix the outcomes of sharp measurements deterministically CITE. This result is robust, in the sense that “almost sharp” measurements are assigned outcomes “almost deterministically”. Thus, for projective measurements, the notion of Spekkens non-contextuality reduces to KS non-contextuality. For Spekkens non-contextual ontological models, outcome determinism for sharp measurements can thus be derived from within the framework itself, and is not an assumption introduced ad-hoc, as was the case for KS non-contextuality.

5.3 Spekkens non-contextuality inequalities

KCBS revisited

Chapter 6

Two protocols based on Spekkens contextuality

We now generalize the self-testing result in CITE, which was discussed in Section REF, to the framework of Spekkens contextuality. In particular, this will lift the usual restriction to projective measurements. Additional features will be discussed in Section REF. The first protocol we will consider assumes approximately operationally equivalent preparations, as well as a bounded information carrying capacity, much like the protocol in Section REF. In Section REF, we will study how imposing additional assumptions on the measurement devices allows us to relax this memory assumption.

6.1 Self-testing from approximately operationally equivalent preparations and bounded memory

Let $n \geq 5$ be an odd integer. The protocol we consider is reminiscent of the ideal n -cycle reference experiment 1.13. As such, we assume an experimenter to be able to freely choose between n three-outcome measurements $\{M_i\}_{i=1}^n$ with outcomes m_1, m_2, m_3 . We assume measurements to be chosen according to a uniform distribution and denote measurement events by $m_k|M_i, P_0 \equiv m_k|M_i$, or $m_k, m_l|M_i, M_j, P_0 \equiv m_k, m_l|M_i, M_j$ for two sequential measurements. As shown in Figure REF, for ideally operating devices, the measurement events $m_1|M_i$ correspond to the rank-one projectors $|u_i^{(n)}\rangle\langle u_i^{(n)}|$. Further, in the ideal case, the measurement events $m_3|M_i \equiv m_1|M_{i\oplus 1} \equiv |u_{i\oplus 1}^{(n)}\rangle\langle u_{i\oplus 1}^{(n)}|$ and the events $m_2|M_i$ correspond to the projectors $\mathbb{1} - |u_i^{(n)}\rangle\langle u_i^{(n)}| - |u_{i\oplus 1}^{(n)}\rangle\langle u_{i\oplus 1}^{(n)}|$, projecting onto the pure qutrit state completing the orthogonal triad $\subset \mathbb{R}^3$. In Figure REF, the events $m_3|M_i$ and $m_1|M_{i\oplus 1}$ are non-overlapping, despite corresponding to the same ideal projectors, to highlight we do not need to assume the operational equivalence of these.

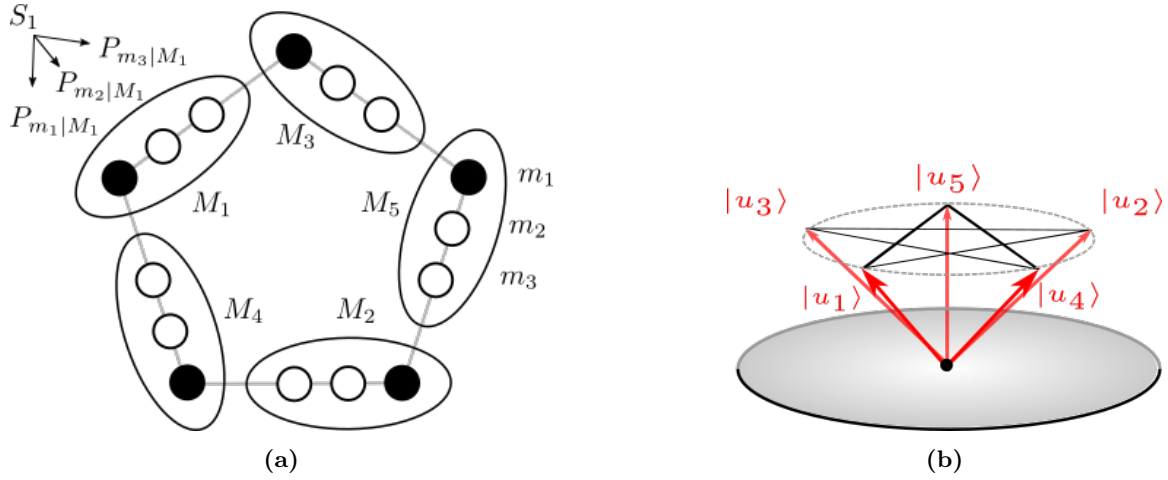


Figure 6.1

Like in Section REF, we assume i.i.d. rounds, i.e. the devices to operate in an identical manner for all rounds of the protocol, in particular independently of previous input-output cycles. This allows us to lift relative frequencies to probabilities. We will discuss all other assumptions in due time. The protocol consists of the following steps:

1. The experimenter prepares the system in a distinguished preparation P_0 by selecting the appropriate setting on the corresponding device.
2. The experimenter samples from a uniform distribution and selects two integers $(i, j) \in \{1, \dots, n\}^2$ at random.
3. The experimenter performs the measurements M_i, M_j in sequence, first M_i , then M_j . He does so by selecting the correct settings on the corresponding devices and records the outcomes.
4. Steps 1-3 are repeated to obtain estimates for the probability distributions $p((m_k, m_l)|(M_i, M_j), P_0)$.
5. Additionally, the experimenter performs single measurements on two auxilliary preparations P_1 and P_2 he can choose from and obtains estimates for the probability distributions $p(m_k|M, i, P_j)$, for $j \in 1, 2$.
6. Post-processing
 - (a) Certify quantumness
 - (b) Bound compatible quantum models

We write the free choice of measurements (M_i, M_j) and subsequent outcomes (m_k, m_l) as ordered lists to indicate that the distributions $p((m_k, m_l)|(M_i, M_j), P_0)$ are in general not independent of the order, since we do not assume any underlying compatibility relations, like we did in Section

REF. In the ideal case, the distinguished preparation P_0 corresponds to the $|0\rangle$ qutrit state, which produces a maximal violation of the KCBS inequality. Additionally, we assume access to two additional auxilliary preparations, P_1 and P_2 , which we require to establish approximate operational equivalences during post-processing. In the ideal case the preparations P_1 and P_2 correspond to the $|1\rangle$ and $|2\rangle$ qutrit states, forming an orthogonal triad with $P_0 \equiv |0\rangle\langle 0|$. The convex combination $\frac{1}{3} \sum_{i=0}^2 P_i \equiv \frac{1}{3} \sum_{i=0}^2 |i\rangle\langle i| = \frac{\mathbb{1}}{3}$ corresponds to the fully mixed state $\frac{\mathbb{1}}{3}$, as does the convex combination with equal weights of the ideal projectors corresponding to the measurement events $\{m_k|M_i\}_{k=1}^3$ for all $i \in \{1, \dots, n\}$. Here, we implicitly defined the convex combination of two preparations to be the preparation that is implemented by generating a random number and performing the preparation procedure associated with that outcome. We note that Steps 1-4 of the protocol correspond almost one-to-one to the steps of the protocol in CITE, which was discussed in Section REF, the only difference being that the experimenter now freely chooses between n^2 , as opposed to n measurement settings.

We now elaborate on Step 6 of the protocol, detailing how an agent may post-process the acquired data, with the goal of self-testing the apparatus. The experimenter retrospectively defines $3n$ preparations, which we denote $\{P_{m_k|M_i}\}_{i \in \{1, \dots, n\}, k \in \{1, 2, 3\}}$. As suggested by the notation, $P_{m_k|M_i}$ is prepared by performing the measurement M_i on a system initially prepared like P_0 , and conditioning on the outcome m_k . We refer to these preparations as retrospective, since they are not directly implemented by the experimenter. However, one can infer the relevant statistics $p(m_k|M_i, P_{m_l|M_j})$ for these preparations from the distributions $p((m_k, m_l)|(M_i, M_j))$:

$$p(m_k|M_i, P_{m_l|M_j}) = \frac{p((m_k, m_l)|(M_i, M_j))}{\sum_{k' \in \{1, 2, 3\}} p((m_{k'}, m_l)|(M_i, M_j))}. \quad (6.1)$$

These retrospective preparations will be used in Section REF, where we propose a criterion to certify the quantumness of the apparatus.

Additionally, the experimenter can infer the statistics $p(m_k|M_i, P_0)$ from the correlations $p((m_k, m_l)|(M_i, M_j), P_0)$,

$$\sum_{l,j} \frac{1}{n} p((m_k, m_l)|(M_i, M_j), P_0). \quad (6.2)$$

These will become relevant in Section REF, when bounding the compatible quantum models.

6.1.1 Step 6a: Certifying quantumness

By coarse-graining, we obtain $3n$ binary measurements from the n three-outcome measurements M_i . In total, we define $3n$ binary measurements and $3n$ preparations during post-processing. We denote the outcomes of the $3n$ binary measurements as 0,1, where the outcome 0 corresponds to the two coarse-grained measurement events. If we consider the ideal reference experiment

REF and take the outcomes 0, 1 to be the eigenvalues of the binary projective measurements, one can think of these as rank-one projectors. There are $2n$ ideal binary measurements that probe the probability of finding the system in one of the n one-dimensional subspaces spanned by the n cycle states. The other n ideal binary measurements correspond to rank-one projectors onto states in an orthogonal triad containing two cycle states. For the extent of this subsection we will for simplicity index the $3n$ binary measurements and preparations like $\{M_i\}_{i \in \{1, \dots, 3n\}}$ and $\{P_i\}_{i \in \{1, \dots, 3n\}}$. For ideal measurements and preparations, we can choose the order such that the ideal measurement M_i is the rank-one projector onto the ideal pure state preparation P_i . Therefore, for ideal devices, the retrospective preparations and binary measurements, if ordered in this manner, satisfy

$$\epsilon := \max_i p(0|M_i, P_i) \stackrel{\text{ideal}}{=} 0. \quad (6.3)$$

For noisy devices, we expect to find an ordering for which REF is approximately satisfied, with the parameter ϵ characterizing the amount of noise. The parameter ϵ can be determined from the observed statistics during post-processing.

Apart from the noise-characterizing parameter ϵ , the second relevant parameter for certifying quantumness is

$$\eta := \min_{\substack{i,j \\ i \neq j}} p(0|M_i, P_j). \quad (6.4)$$

For quantum devices, the parameter η can be thought of as a measuring the maximum overlap or closeness of distinct states. As such, η decreases as the cycle length n increases. Let $m = 2^k$. In Theorem REF of CITE it is proven that for a set of preparations $\{P_i\}_{i=1}^m$ and measurements $\{M_i\}_{i=1}^m$, producing statistics obeying $\epsilon < \frac{1}{4}\eta^2$, to be compatible with a (preparation) non-contextual ontological description, requires at least k binary measurements in any TC set. The intuition behind this is the following: From the statistics of the m preparations $\{P_i\}_{i=1}^m$ one can infer the statistics of any preparation in the convex hull of the $\{P_i\}_{i=1}^m$. If the m preparations are not very “close”, as measured by η , relative to their sharpness, as measured by ϵ , (this is enforced by the condition $\epsilon < \frac{1}{4}\eta^2$) then the associated probability density functions $\{\mu_i\}_{i=1}^m$ will have non-significant overlap. In particular, if we regard the m probability density functions on the ontic state space Λ as vectors $\{v_i\}_{i=1}^m$ in $\mathbb{R}^{|\Lambda|}$, then the proof of Theorem REF in CITE establishes that the $m = 2^k$ vectors vary in k linearly independent directions. Demanding preparation non-contextuality, every distinct preparation in the convex hull of the $\{P_i\}_{i=1}^m$ must produce different predictions for at least one of the measurements in a TC set. Finally, CITE notes that the function mapping a probability density function μ_j , or rather the corresponding vector in $\mathbb{R}^{|\Lambda|}$ to the probability $p(m_k|M_i, \mu_j)$ of obtaining the outcome m_k when performing the measurement M_i for an initial preparation μ_j is linear. Hence, such mapping $\mathbb{R}^{|\Lambda|} \rightarrow \mathbb{R}$ is of the form $(x_1, \dots, x_{|\Lambda|}) \mapsto a_1x_1 + \dots + a_nx_n = \vec{a} \cdot \vec{x}$ with $0 \leq a_i \leq 1$, and x_i the components w.r.t the basis $\{v_i\}_{i=1}^m$. Binary measurements can therefore only distinguish between preparations along a single direction, \vec{a} . Preparations whose ontological representations differ

w.r.t. directions orthogonal to \vec{a} , are assigned the same probabilities by the binary measurement characterized by \vec{a} . A TC set of measurement must contain at least k binary measurements for k linearly independent preparations, if we demand compatibility with a preparation non-contextual ontological model.

We now turn to the bounded memory assumption required by the protocol, which relies on the results in CITE, the relevant ones of which were sketched above. Assume that the experimenter has access to $n \geq 5$ three-outcome measurement settings, like we considered at the beginning of Section REF, where n is odd (think of n as the cycle-length). Further, let k be the largest integer satisfying $3n \geq 2^{k+1}$. To certify quantumness, we assume that there exists a TC set of measurements with k binary measurements, i.e. that k binary measurements are sufficient to fully characterize the statistics of the system at hand. If we find the condition $\epsilon < \frac{1}{4}\eta^2$ to be satisfied, we consider the system to be quantum. How does this assumption relate to the assumption of bounded memory in Section REF? If we define the information carrying capacity as

$$\mathcal{I} := \log_2(\text{number of states that can be perfectly distinguished}) \quad (6.5)$$

, then assuming a TC set with k measurements implies bounding the information carrying capacity $\mathcal{I} \leq k\text{bits}$. In this sense, assumptions regarding the minimal size of TC sets of measurements directly relate to bounds on the information carrying capacity. In particular, bounding the information carrying capacity $\mathcal{I} \leq k\text{bits}$ implies a finite-dimensional Hilbert space \mathcal{H} , $\dim(\mathcal{H}) \leq k$.

If we wish to allow for an information carrying capacity of k bits, we require an apparatus consistent with a cycle length n such that

$$3n \geq 2^{k+1} \quad (6.6)$$

$$\iff \log_2(3n) - 1 \geq k. \quad (6.7)$$

The relationship between cycle length and simulable bound on the information carrying capacity is logarithmic, as was the case for the class of protocols in Section REF, which assumed sharp measurements and cyclic compatibility relations. The price one has to pay for accounting for a greater information carrying capacity is two-fold:

- the apparatus must include additional measurement settings to be consistent with a greater cycle length n
- for a greater cycle-length n , a greater fidelity to the reference experiment is required, since for the ideal odd n -cycle scenario the relevant parameter $\frac{1}{4}\eta^2$, which bounds how sharp the measurement implementations must be for self-testing, decreases with an increasing cycle length n :

$$\frac{1}{4}\eta^2 = \frac{1}{2}\left(\frac{\pi}{n}\right)^4. \quad (6.8)$$

6.1.2 Step 6b: Bounding compatible quantum models

As discussed in Section REF, we assume a Hilbert space of bounded dimension, say dimension $\dim(\mathcal{H}) \leq d$. We can w.l.o.g. take \mathcal{H} to be \mathbb{C}^k , $k \leq d$, where k is the true dimension of \mathcal{H} .

We will now introduce the second key assumption of the protocol, which allows us to bound the quantum models compatible with the observed correlations: We assume that for all $i \in \{1, \dots, n\}$ there exists a convex combination $\{p_j^{(i)}\}_j$ of the preparations $\{P_{m_j|M_i}\}_j$ and a convex combination $\{q_j^{(*)}\}_j$ of the preparations $\{P_0, P_1, P_2\}$ such that these are (approximately) operationally equivalent:

$$\forall i \in \{1, \dots, n\} S_i := \sum_{j=1}^3 p_j^{(i)} P_{m_j|M_i} \sim S_* := \sum_{j=0}^2 q_j^{(*)} P_j. \quad (6.9)$$

Although assuming approximate operational equivalences suffices, we will for now take the equivalences in REF to be exact. Once we establish bounds on the compatible quantum models for exact equivalences, this restriction may be relaxed. We consider two quantum preparations to be approximately operationally equivalent if the corresponding density operators are “close” w.r.t. the Frobenius norm. Recall that for ideal devices and equal weights $p_j^{(i)} = q_j^{(*)} = \frac{1}{3}$, both S_i and S_* correspond to the fully mixed state $\frac{1}{3}$. Note that the assumption of (approximate) operational equivalence cannot be verified on the basis of statistics alone, rendering the protocol semi-device-independent. What one can do is check whether the claim holds w.r.t. the available measurement settings. The reason for assuming the convex combinations S_i and S_* to be (approximately) operationally equivalent is to relate the action of the measurements M_i on the preparations $P_{m_j|M_i}$ to how the measurements act on the initial preparation P_0 . In particular, we want that if the measurements M_i are almost projective when acting on the preparations $P_{m_j|M_i}$, then this also applies to the preparation P_0 .

Denote the density operator corresponding to the preparation $P_{m_k|M_i}$ by

$$\rho_{m_k|M_i} = \sum_m p_m^{m_k|M_i} |\Psi_m^{m_k|M_i}\rangle \langle \Psi_m^{m_k|M_i}|, \quad (6.10)$$

$|\Psi_m^{m_k|M_i}\rangle \in \mathbb{C}^k$, and the positive semi-definite operator corresponding to the measurement event $m_k|M_i$ by

$$F_{m_k|M_i} = \sum_l \lambda_l^{m_k|M_i} |\alpha_l^{m_k|M_i}\rangle \langle \alpha_l^{m_k|M_i}|, \quad (6.11)$$

$|\alpha_l^{m_k|M_i}\rangle \in \mathbb{C}^k$. By REF,

$$p(m_k|M_i, P_{m_k|M_i}) \geq 1 - 2\epsilon \quad (6.12)$$

$$p(m_k|M_i, P_{m_{k'}|M_i}) \leq \epsilon, \quad (6.13)$$

for $k' \neq k$. Therefore,

$$\sum_{l,m} \lambda_l^{m_k|M_i} p_m^{m_k|M_i} |\langle \alpha_l^{m_k|M_i} | \Psi_m^{m_k|M_i} \rangle|^2 \geq 1 - 2\epsilon \quad (6.14)$$

and analogously, for $k' \neq k$,

$$\sum_{l,m} \lambda_l^{m_k|M_i} p_m^{m_{k'}|M_i} |\langle \alpha_l^{m_k|M_i} | \Psi_m^{m_{k'}|M_i} \rangle|^2 \leq \epsilon. \quad (6.15)$$

Later, we will consider the behaviour for $\epsilon \rightarrow 0$

We want to show that the measurement M_i acts “almost projectively” on the preparations $\{P_{m_j|M_i}\}_{j=1}^3$, and by extension on P_0 . Consider the density operator $\rho_{m_k|M_i}$, like in REF. For eigenvalues $p_m^{m_k|M_i}$ of the noisy density operator that are very small, say of the order ϵ , we cannot infer from the statistics $\{p(m_j|M_i, P_{m_k|M_i})\}_{j,k}$ alone whether the measurement M_i acts almost projectively on the corresponding subspace $\mathbb{C}|\Psi_m^{m_k|M_i}\rangle$, since this subspace is insignificant in terms of statistics. In order to sidestep this issue, we consider only a “statistically relevant” subspace of \mathbb{C}^k : Define $\Pi_{\text{relev}}^{(i)}$ as the projector onto the subspace

$$V_{\text{relev}}^{(i)} = \text{span}\left(\bigcup_{k=1}^3 \{|\Psi_m^{m_k|M_i}\rangle\}_{m: p_m^{m_k|M_i} \geq \mathcal{X}}\right). \quad (6.16)$$

For now, \mathcal{X} is an arbitrary cutoff, characterizing the minimum magnitude of eigenvalues for which the corresponding eigenvector in REF spans a statistically relevant one-dimensional subspace. We will later set \mathcal{X} to an appropriate value in terms of ϵ . Implementing this technical workaround, the condition REF and REF become

$$\sum_{m: p_m^{m_k|M_i} \geq \mathcal{X}} \left(1 - \sum_l \lambda_l^{m_k|M_i} |\langle \alpha_l^{m_k|M_i} | \Psi_m^{m_k|M_i} \rangle|^2\right) \leq \frac{2\epsilon}{\mathcal{X}} \quad (6.17)$$

and

$$\sum_{l, m: p_m^{m_{k'}|M_i} \geq \mathcal{X}} \lambda_l^{m_k|M_i} |\langle \alpha_l^{m_k|M_i} | \Psi_m^{m_{k'}|M_i} \rangle|^2 \leq \frac{\epsilon}{\mathcal{X}}, \quad (6.18)$$

for $k' \neq k$.

6.1.2.1 Overlap between statistically relevant states

We begin by bounding the overlap $|\langle \Psi_k^{m_j|M_i} | \Psi_l^{m_{j'}|M_i} \rangle|$, for $j' \neq j$ and $|\Psi_k^{m_j|M_i}\rangle, |\Psi_l^{m_{j'}|M_i}\rangle \in V_{\text{relev}}^{(i)}$. In the ideal case these vectors are perfectly orthogonal. For notational simplicity, we will write $\sum_{k: p_k^{m_j|M_i} \geq \mathcal{X}}$ as \sum'_k , and analogously for two summation indices. Using the fact that

$\sum_{\tilde{j}} F_{m_{\tilde{j}}|M_i} = \mathbb{1}$, we can re-write

$$\sum_{k,l}' |\langle \Psi_k^{m_j|M_i} | \Psi_l^{m_{j'}|M_i} \rangle| = \sum_{k,l}' \left| \sum_{\tilde{j}} \text{tr}(F_{m_{\tilde{j}}|M_i} | \Psi_k^{m_j|M_i} \rangle \langle \Psi_l^{m_{j'}|M_i} |) \right|. \quad (6.19)$$

This is upper-bounded by

$$\sum_{k,l}' |\langle \Psi_k^{m_j|M_i} | \Psi_l^{m_{j'}|M_i} \rangle| \leq \sum_{\tilde{j},m} \sum_{k,l}' \lambda_m^{m_{\tilde{j}}|M_i} |\langle \alpha_m^{m_{\tilde{j}}|M_i} | \Psi_k^{m_j|M_i} \rangle| |\langle \Psi_l^{m_{j'}|M_i} | \alpha_m^{m_{\tilde{j}}|M_i} \rangle|. \quad (6.20)$$

Note that since $j \neq j'$, $\tilde{j} \neq j$ or $\tilde{j} \neq j'$. We now use condition REF. REF bounds the square of the 2-norm of the vector with entries

$$\left((\lambda_m^{m_{\tilde{j}}|M_i})^{\frac{1}{2}} |\langle \alpha_m^{m_{\tilde{j}}|M_i} | \Psi_k^{m_j|M_i} \rangle| \right)_{m,l: p_l^{m_j|M_i} \geq \mathcal{X}},$$

for $\tilde{j} \neq j$. Such vector has at most d^2 components, where d is the upper bound on the Hilbert space dimension we imposed in Section REF. Since $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, with $\|\vec{x}\|_1 \leq \sqrt{k}\|\vec{x}\|_2$ for $\vec{x} \in \mathbb{C}^k$, we find

$$\sum_{k,m}' \lambda_m^{m_{\tilde{j}}|M_i} |\langle \alpha_m^{m_{\tilde{j}}|M_i} | \Psi_k^{m_j|M_i} \rangle| \leq d \sqrt{\frac{\epsilon}{\mathcal{X}}}, \quad (6.21)$$

for $\tilde{j} \neq j$, and

$$\sum_{k,l}' |\langle \Psi_k^{m_j|M_i} | \Psi_l^{m_{j'}|M_i} \rangle| \leq 3d^2 \sqrt{\frac{2\epsilon}{\mathcal{X}}}. \quad (6.22)$$

6.1.2.2 Orthonormal set $\subset \mathbb{C}^k$ close to the set of relevant eigenvectors

In the following subsection, we will bound the Frobenius distance

$$\|F_{m_j|M_i} \Pi_{\text{relev}}^{(i)} - \sum_k' |\Psi_k^{m_j|M_i} \rangle \langle \Psi_k^{m_j|M_i}|\|_F \quad (6.23)$$

Recall that the Frobenius norm of an operator $A \in \mathbb{C}^{k,k}$ is defined in terms of the trace of $A^\dagger A$. The trace of $A^\dagger A$ is conveniently expressed in terms of an orthonormal basis $\{|\phi_j\rangle\}_j \subset \mathbb{C}^k$: $\sum_j \langle \phi_j | A^\dagger A | \phi_j \rangle$. However, the conditions REF are all expressed in terms of the states $\bigcup_{j=1}^3 \{|\Psi_k^{m_j|M_i}\rangle\}_k$, which are in general not orthogonal. Therefore, we want to find an orthonormal set $\bigcup_{j=1}^3 \{|\Phi_k^{m_j|M_i}\rangle\}_{k:p_k^{m_j|M_i} \geq \mathcal{X}} \subset \mathbb{C}^k$ closest to $\mathcal{E} := \bigcup_{j=1}^3 \{|\Psi_k^{m_j|M_i}\rangle\}_{k:p_k^{m_j|M_i} \geq \mathcal{X}} \subset \mathbb{C}^k$, in the sense that the quantity

$$\sum_j \sum_k' \| |\Psi_k^{m_j|M_i}\rangle - |\Phi_k^{m_j|M_i}\rangle \|_2. \quad (6.24)$$

is small for all j, k .

Let k' be the cardinality of the set \mathcal{E} . Define $A \in \mathbb{C}^{k, k'}$ as the matrix with columns $\mathcal{E} \subset \mathbb{C}^k$. Note that $k \geq k'$ for sufficiently small ϵ , since all columns are linearly independent. The entries of the matrix $A^\dagger A$ are essentially just the scalar products whose magnitude we bounded in Section REF. Some off-diagonal entries, namely those corresponding to the scalar product of two vectors that are part of the same spectral decomposition, are 0. Finally, the diagonal entries of $A^\dagger A$ are all 1, due to normalization. Like in the proof of Lemma REF, we consider a spectral decomposition of A , $A = V\Sigma W^\dagger$, and denote the rank of A by r . Analogously to the proof of Lemma REF, we consider the matrix \tilde{A} , where we have set all diagonal entries of Σ to 1. The columns of \tilde{A} are orthonormal. It remains to show that $A \approx_F \tilde{A}$:

$$\|A - \tilde{A}\|_F = \left\| \begin{pmatrix} \mathbb{1}_{k'} \\ 0_{k-k', k'} \end{pmatrix} - \Sigma \right\|_F = \sqrt{(k' - r) + \sum_{i=1}^r (\sigma_i - 1)^2}. \quad (6.25)$$

We now use REF to bound the singular values and rank of A :

$$\|A^\dagger A - \mathbb{1}\|_F = \sqrt{(k' - r) + \sum_{i=1}^r (\sigma_i^2 - 1)^2} \leq \mathcal{O}(d^2 \sqrt{\frac{\epsilon}{\mathcal{X}}}), \quad (6.26)$$

where the inequality follows from REF. Thus, for sufficiently small ϵ the rank of A is $r = k'$, and

$$\|A - \tilde{A}\|_F = \sqrt{\sum_{i=1}^{k'} (\sigma_i - 1)^2} \leq \sqrt{\sum_{i=1}^{k'} (\sigma_i - 1)^2 (\sigma_i + 1)^2} \leq \mathcal{O}(d^2 \sqrt{\frac{\epsilon}{\mathcal{X}}}). \quad (6.27)$$

As such, the orthonormal set $\bigcup_{j=1}^3 \{|\Phi_k^{m_j|M_i}\rangle\}_{k: p_k^{m_j|M_i} \geq \mathcal{X}} \subset \mathbb{C}^k$, given by the columns of \tilde{A} , satisfies

$$\sum_{j=1}^3 \sum_k' \|\Psi_k^{m_j|M_i}\rangle - |\Phi_k^{m_j|M_i}\rangle\|_2^2 = \|A - \tilde{A}\|_F^2 \leq \mathcal{O}(d^4 \frac{\epsilon}{\mathcal{X}}) \quad (6.28)$$

Such a set can be found for all $i \in \{1, \dots, n\}$.

6.1.2.3 The measurement M_i acts almost projectively on the subspace $V_{\text{relev}}^{(i)}$

Appendix A

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Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

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