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Linear Time Variance Inference for PEP 695

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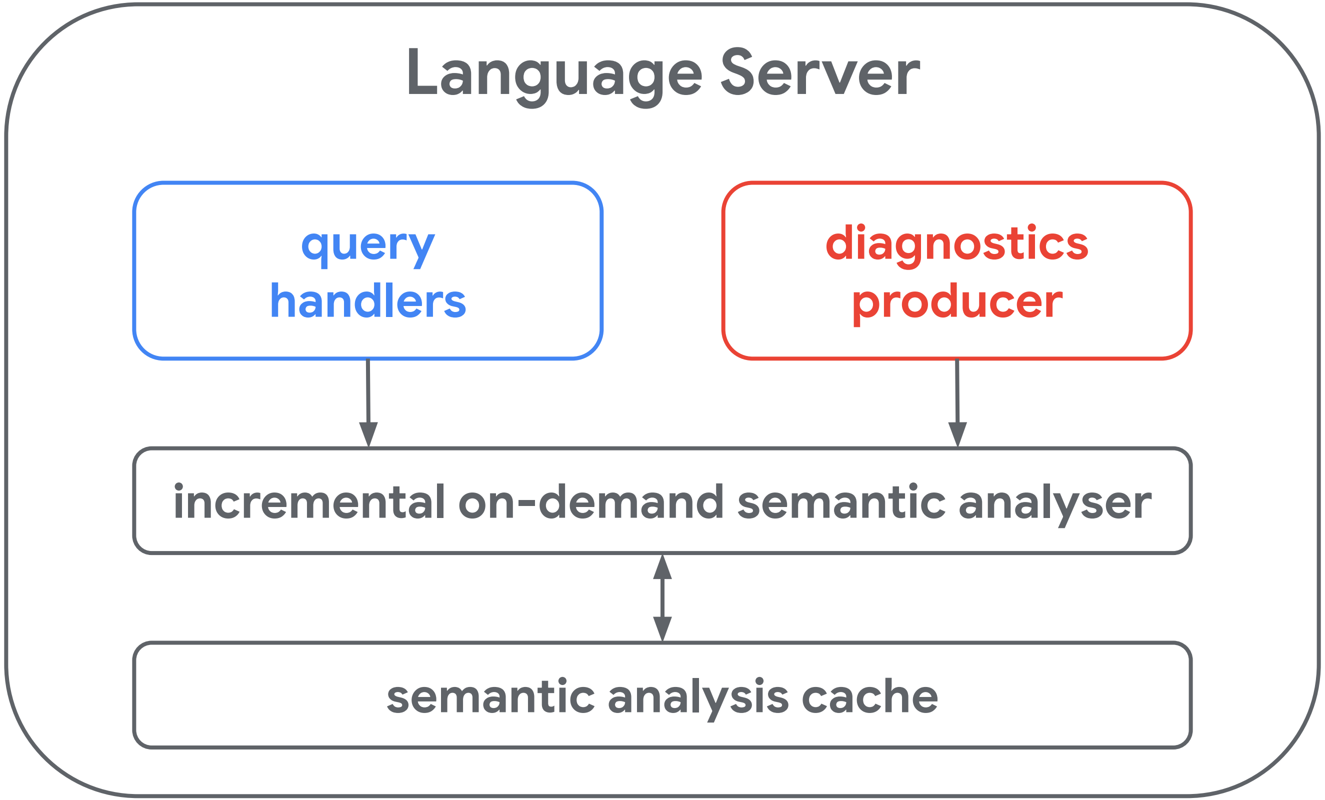
Context



Language Server
query handlers
diagnostics producer
incremental on-demand semantic analyser
semantic analysis cache

Warm-up

Question: What was variance again?

Assumption:

`Snake <: Animal`

Invariance:

`list[Snake] ⊥ list[Animal]`

Covariance:

`Sequence[Snake] <: Sequence[Animal]`

Contravariance:

`Callable[[Animal], bool] <: Callable[[Snake], bool]`

PEP 695 and generic classes

```python
# Python 3.11 and before

X = TypeVar("X", bound=Animal,
        covariant=True)

class Cage(Generic[X]):
    animal: Final[X]

    def __init__(self, animal: X):
        self.animal = animal

    def open(self) -> X:
        return self.animal
```

```python
# Python 3.12 and after




# How to make Cage covariant?
class Cage[X: Animal]:
    animal: Final[X]

    def __init__(self, animal: X):
        self.animal = animal

    def open(self) -> X:
        return self.animal
```

PEP 695 and generic classes

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# Python 3.11 and before

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```python
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# How to make Cage covariant?
class Cage[X: Animal]:
    animal: Final[X]

    def __init__(self, animal: X):
        self.animal = animal

    def open(self) -> X:
        return self.animal
```

VARIANCE INFERENCE!

Variance inference à la PEP 695

Definition: Let C be a generic class.

1. C is <u>covariant</u> if `S <: T` implies `C[S] <: C[T]` for all types `S` and `T`.
2. C is <u>contravariant</u> if `S <: T` implies `C[T] <: C[S]` for all types `S` and `T`.

Algorithm: Reduce to subtyping using parametricity and `X <: object`.

1. Return that C is <u>covariant</u> if *structurally* `C[X] <: C[object]`.
2. Return that C is <u>contravariant</u> if *structurally* `C[object] <: C[X]`.

```
class Cage[X](Protocol):
    animal: Final[X]
    def __init__(self, animal: X): ...
    def open(self) -> X: ...
```

WHAT ABOUT (MUTUALLY) RECURSIVE CLASSES?

Mutually recursive classes

```
class C[X]:
    def f(self) -> D[X]:
        ...
    def g(self, x: X) -> None:
        ...


class D[Y]:
    def h(self) -> C[Y]:
        ...


def cast(c: C[bool]) -> C[int]:
    return c  # Check this!
```

```
IsSubtype(C[bool], C[int])
├─IsSubtype(bool, int)──return True
├─IsCovariant(C)
│ ├─IsStructuralSubtype(C[X], C[object])
│ │ ├─IsSubtype(D[X], D[object])
│ │ │ ├─IsSubtype(X, object)──return True
│ │ │ ├─IsCovariant(D)
│ │ │ │ ├─IsStructuralSubtype(D[Y], D[object])
│ │ │ │ │ ├─IsSubtype(C[Y], C[object])
│ │ │ │ │ │ ├─IsSubtype(Y, object)──return True
│ │ │ │ │ │ ├─IsCovariant(C)
│ │ │ │ │ │ │ ├─LOOP DETECTED: USE COINDUCTION!
│ │ │ │ │ │ │ └─return True
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│ │ │ │ └─return True
│ │ │ └─return True
│ │ ├─IsSubtype(object, X)──return False
│ │ └─return False
│ └─return False
├─IsSubtype(int, bool)──return False
└─return False
```

Mutually recursive classes

```
class C[X]:
    def f(self) -> D[X]:
        ...
    def g(self, x: X) -> None:
        ...


class D[Y]:
    def h(self) -> C[Y]:
        ...


def cast(c: C[bool]) -> C[int]:
    return c  # Check this!
```

```
IsSubtype(C[bool], C[int])
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│ ├─IsStructuralSubtype(C[X], C[object])
│ │ ├─IsSubtype(D[X], D[object])
│ │ │ ├─IsSubtype(X, object)──return True
│ │ │ ├─IsCovariant(D)
│ │ │ │ ├─IsStructuralSubtype(D[Y], D[object])
│ │ │ │ │ ├─IsSubtype(C[Y], C[object])
│ │ │ │ │ │ ├─IsSubtype(Y, object)──return True
│ │ │ │ │ │ ├─IsCovariant(C)
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│ └─return False
├─IsSubtype(int, bool)──return False
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```

Mutually recursive classes

```
class C[X]:
    def f(self) -> D[X]:
        ...
    def g(self, x: X) -> None:
        ...


class D[Y]:
    def h(self) -> C[Y]:
        ...


def cast(c: C[bool]) -> C[int]:
    return c  # Check this!
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│ │ │ ├─IsCovariant(D)
│ │ │ │ ├─IsStructuralSubtype(D[Y], D[object])
│ │ │ │ │ ├─IsSubtype(C[Y], C[object])
│ │ │ │ │ │ ├─IsSubtype(Y, object)──return True
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Mutually recursive classes

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    def f(self) -> D[X]:
        ...
    def g(self, x: X) -> None:
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    def h(self) -> C[Y]:
        ...


def cast(c: C[bool]) -> C[int]:
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│ │ │ ├─IsCovariant(D)
│ │ │ │ ├─IsStructuralSubtype(D[Y], D[object])
│ │ │ │ │ ├─IsSubtype(C[Y], C[object])
│ │ │ │ │ │ ├─IsSubtype(Y, object)──return True
│ │ │ │ │ │ ├─IsCovariant(C)
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```

Mutually recursive classes

```python
class C[X]:
    def f(self) -> D[X]:
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    def h(self) -> C[Y]:
        ...


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│ │ ├─IsSubtype(D[X], D[object])
│ │ │ ├─IsSubtype(X, object)──return True
│ │ │ ├─IsCovariant(D)
│ │ │ │ ├─IsStructuralSubtype(D[Y], D[object])
│ │ │ │ │ ├─IsSubtype(C[Y], C[object])
│ │ │ │ │ │ ├─IsSubtype(Y, object)──return True
│ │ │ │ │ │ ├─IsCovariant(C)
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Mutually recursive classes

```
class C[X]:
    def f(self) -> D[X]:
        ...
    def g(self, x: X) -> None:
        ...


class D[Y]:
    def h(self) -> C[Y]:
        ...


def cast(c: C[bool]) -> C[int]:
    return c  # Check this!
    ~~~~~~~~~
      Not C[bool] <: C[int].
```

```
IsSubtype(C[bool], C[int])
├─IsSubtype(bool, int)──return True
├─IsCovariant(C)
│ ├─IsStructuralSubtype(C[X], C[object])
│ │ ├─IsSubtype(D[X], D[object])
│ │ │ ├─IsSubtype(X, object)──return True
│ │ │ ├─IsCovariant(D)
│ │ │ │ ├─IsStructuralSubtype(D[Y], D[object])
│ │ │ │ │ ├─IsSubtype(C[Y], C[object])
│ │ │ │ │ │ ├─IsSubtype(Y, object)──return True
│ │ │ │ │ │ ├─IsCovariant(C)
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│ │ ├─IsSubtype(object, X)──return False
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│ └─return False
├─IsSubtype(int, bool)──return False
└─return False
```

Mutually recursive classes

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class C[X]:
    def f(self) -> D[X]:
        ...
    def g(self, x: X) -> None:
        ...


class D[Y]:
    def h(self) -> C[Y]:
        ...


def cast(c: C[bool]) -> C[int]:
    return c  # Check this!
    ~~~~~~~~
      Not C[bool] <: C[int].
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IsSubtype(C[bool], C[int])
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```

QUADRATIC RUNTIME!

Linear time algorithm

Input: Signatures of classes `C1, ..., Cn` that are closed under dependencies.

1. Traverse the signatures of `C1, ..., Cn` and emit boolean equations.
2. Solve the system of boolean equations.
3. Read variances off assignments to certain boolean variables.

Output: *Optimal* variances for classes `C1, ..., Cn`.

Complexity:

1. Time for generating all boolean equations = O(input size)
2. Time for solving equations = O(size of equation system) = **O(input size)**

Generalised variance

Definition: Let T be a type and X a type variable.

1. T is <u>covariant in X</u> if $U <: V$ implies $T[X \mapsto U] <: T[X \mapsto V]$ for all U and V.
2. T is <u>contravariant in X</u> if $U <: V$ implies $T[X \mapsto V] <: T[X \mapsto U]$ for all U and V.

Observations:

1. If X does not appear in T, T is covariant and contravariant in X.
2. Let C be a generic class. If C is covariant/contravariant, C[X] is covariant/contravariant in X.
3. X is covariant in X but not contravariant in X.

Mutually recursive classes revisited

```
class C[X]:
    def f(self) -> D[X]: ...
    def g(self, x: X) -> None: ...

class D[Y]:
    def h(self) -> C[Y]: ...
```

Equations:

$C^{+} = F^{+X} \wedge G^{+X}$

$C^{-} = F^{-X} \wedge G^{-X}$

$D^{+} = H^{+Y}$

$D^{-} = H^{-Y}$

$F^{+X} = D^{+}$

$F^{-X} = D^{-}$

$G^{+X} = X^{-X} \wedge N^{+X}$

$G^{-X} = X^{+X} \wedge N^{-X}$

$H^{+Y} = C^{+}$

$H^{-Y} = C^{-}$

$X^{+X} = \mathtt{true}$

$X^{-X} = \mathtt{false}$

$N^{+X} = \mathtt{true}$

$N^{-X} = \mathtt{true}$

Boolean variables:

C^{+}/C^{-} ~ C is covariant/contravariant

D^{+}/D^{-} ~ D is covariant/contravariant

F^{+X}/F^{-X} ~ f's signature is covariant/contravariant in X

G^{+X}/G^{-X} ~ g's signature is covariant/contravariant in X

H^{+Y}/H^{-Y} ~ h's signature is covariant/contravariant in Y

X^{+X}/X^{-X} ~ X is covariant/contravariant in X

N^{+X}/N^{-X} ~ None is covariant/contravariant in X

Mutually recursive classes revisited

```
class C[X]:
    def f(self) -> D[X]: ...
    def g(self, x: X) -> None: ...

class D[Y]:
    def h(self) -> C[Y]: ...
```

Boolean variables:

C^+ / C^- ~ C is covariant/contravariant

D^+ / D^- ~ D is covariant/contravariant

F^{+X} / F^{-X} ~ f's signature is covariant/contravariant in X

G^{+X} / G^{-X} ~ g's signature is covariant/contravariant in X

H^{+Y} / H^{-Y} ~ h's signature is covariant/contravariant in Y

X^{+X} / X^{-X} ~ X is covariant/contravariant in X

N^{+X} / N^{-X} ~ None is covariant/contravariant in X

Equations:

$C^+ = F^{+X} \wedge G^{+X}$

$C^- = F^{-X} \wedge G^{-X}$

$D^+ = H^{+Y}$

$D^- = H^{-Y}$

$F^{+X} = D^+$

$F^{-X} = D^-$

$G^{+X} = X^{-X} \wedge N^{+X}$

$G^{-X} = X^{+X} \wedge N^{-X}$

$H^{+Y} = C^+$

$H^{-Y} = C^-$

$X^{+X} = \mathtt{true}$

$X^{-X} = \mathtt{false}$

$N^{+X} = \mathtt{true}$

$N^{-X} = \mathtt{true}$

Mutually recursive classes revisited

```
class C[X]:
    def f(self) -> D[X]: ...
    def g(self, x: X) -> None: ...

class D[Y]:
    def h(self) -> C[Y]: ...
```

Boolean variables:

C^+/C^- ~ C is covariant/contravariant

D^+/D^- ~ D is covariant/contravariant

F^{+X}/F^{-X} ~ f's signature is covariant/contravariant in X

G^{+X}/G^{-X} ~ g's signature is covariant/contravariant in X

H^{+Y}/H^{-Y} ~ h's signature is covariant/contravariant in Y

X^{+X}/X^{-X} ~ X is covariant/contravariant in X

N^{+X}/N^{-X} ~ None is covariant/contravariant in X

Equations:

$C^+ = F^{+X} \wedge G^{+X}$

$C^- = F^{-X} \wedge G^{-X}$

$D^+ = H^{+Y}$

$D^- = H^{-Y}$

$F^{+X} = D^+$

$F^{-X} = D^-$

$G^{+X} = X^{-X} \wedge N^{+X}$

$G^{-X} = X^{+X} \wedge N^{-X}$

$H^{+Y} = C^+$

$H^{-Y} = C^-$

$X^{+X} = \mathtt{true}$

$X^{-X} = \mathtt{false}$

$N^{+X} = \mathtt{true}$

$N^{-X} = \mathtt{true}$

Mutually recursive classes revisited

```
class C[X]:
    def f(self) -> D[X]: ...
    def g(self, x: X) -> None: ...

class D[Y]:
    def h(self) -> C[Y]: ...
```

Boolean variables:

C^{+}/C^{-} ~ C is covariant/contravariant

D^{+}/D^{-} ~ D is covariant/contravariant

F^{+X}/F^{-X} ~ f's signature is covariant/contravariant in X

G^{+X}/G^{-X} ~ g's signature is covariant/contravariant in X

H^{+Y}/H^{-Y} ~ h's signature is covariant/contravariant in Y

X^{+X}/X^{-X} ~ X is covariant/contravariant in X

N^{+X}/N^{-X} ~ None is covariant/contravariant in X

Equations:

$C^{+} = F^{+X} \wedge G^{+X}$

$C^{-} = F^{-X} \wedge G^{-X}$

$D^{+} = H^{+Y}$

$D^{-} = H^{-Y}$

$F^{+X} = D^{+}$

$F^{-X} = D^{-}$

$G^{+X} = X^{-X} \wedge N^{+X} = \text{false}$

$G^{-X} = X^{+X} \wedge N^{-X} = \text{true}$

$H^{+Y} = C^{+}$

$H^{-Y} = C^{-}$

$X^{+X} = \text{true}$

$X^{-X} = \text{false}$

$N^{+X} = \text{true}$

$N^{-X} = \text{true}$

Mutually recursive classes revisited

```
class C[X]:
    def f(self) -> D[X]: ...
    def g(self, x: X) -> None: ...

class D[Y]:
    def h(self) -> C[Y]: ...
```

Boolean variables:

C^{+}/C^{-} ~ C is covariant/contravariant

D^{+}/D^{-} ~ D is covariant/contravariant

F^{+X}/F^{-X} ~ f's signature is covariant/contravariant in X

G^{+X}/G^{-X} ~ g's signature is covariant/contravariant in X

H^{+Y}/H^{-Y} ~ h's signature is covariant/contravariant in Y

X^{+X}/X^{-X} ~ X is covariant/contravariant in X

N^{+X}/N^{-X} ~ None is covariant/contravariant in X

Equations:

$C^{+} = F^{+X} \wedge G^{+X} = \texttt{false}$

$C^{-} = F^{-X} \wedge G^{-X} = F^{-X}$

$D^{+} = H^{+Y}$

$D^{-} = H^{-Y}$

$F^{+X} = D^{+}$

$F^{-X} = D^{-}$

$G^{+X} = X^{-X} \wedge N^{+X} = \texttt{false}$

$G^{-X} = X^{+X} \wedge N^{-X} = \texttt{true}$

$H^{+Y} = C^{+}$

$H^{-Y} = C^{-}$

$X^{+X} = \texttt{true}$

$X^{-X} = \texttt{false}$

$N^{+X} = \texttt{true}$

$N^{-X} = \texttt{true}$

Mutually recursive classes revisited

```
class C[X]:
    def f(self) -> D[X]: ...
    def g(self, x: X) -> None: ...

class D[Y]:
    def h(self) -> C[Y]: ...
```

Boolean variables:

C^+ / C^- ~ C is covariant/contravariant

D^+ / D^- ~ D is covariant/contravariant

F^{+X} / F^{-X} ~ f's signature is covariant/contravariant in X

G^{+X} / G^{-X} ~ g's signature is covariant/contravariant in X

H^{+Y} / H^{-Y} ~ h's signature is covariant/contravariant in Y

X^{+X} / X^{-X} ~ X is covariant/contravariant in X

N^{+X} / N^{-X} ~ None is covariant/contravariant in X

Equations:

$C^+ = F^{+X} \wedge G^{+X} = \mathtt{false}$

$C^- = F^{-X} \wedge G^{-X} = F^{-X}$

$D^+ = H^{+Y} = \mathtt{false}$

$D^- = H^{-Y}$

$F^{+X} = D^+ = \mathtt{false}$

$F^{-X} = D^-$

$G^{+X} = X^{-X} \wedge N^{+X} = \mathtt{false}$

$G^{-X} = X^{+X} \wedge N^{-X} = \mathtt{true}$

$H^{+Y} = C^+ = \mathtt{false}$

$H^{-Y} = C^-$

$X^{+X} = \mathtt{true}$

$X^{-X} = \mathtt{false}$

$N^{+X} = \mathtt{true}$

$N^{-X} = \mathtt{true}$

Mutually recursive classes revisited

```
class C[X]:
    def f(self) -> D[X]: ...
    def g(self, x: X) -> None: ...

class D[Y]:
    def h(self) -> C[Y]: ...
```

Boolean variables:

C^{+}/C^{-} ~ C is covariant/contravariant

D^{+}/D^{-} ~ D is covariant/contravariant

F^{+X}/F^{-X} ~ f's signature is covariant/contravariant in X

G^{+X}/G^{-X} ~ g's signature is covariant/contravariant in X

H^{+Y}/H^{-Y} ~ h's signature is covariant/contravariant in Y

X^{+X}/X^{-X} ~ X is covariant/contravariant in X

N^{+X}/N^{-X} ~ None is covariant/contravariant in X

Equations:

$C^{+} = F^{+X} \wedge G^{+X} = \text{false}$

$C^{-} = F^{-X} \wedge G^{-X} = \text{true}$

$D^{+} = H^{+Y} = \text{false}$

$D^{-} = H^{-Y} = \text{true}$

$F^{+X} = D^{+} = \text{false}$

$F^{-X} = D^{-} = \text{true}$

$G^{+X} = X^{-X} \wedge N^{+X} = \text{false}$

$G^{-X} = X^{+X} \wedge N^{-X} = \text{true}$

$H^{+Y} = C^{+} = \text{false}$

$H^{-Y} = C^{-} = \text{true}$

$X^{+X} = \text{true}$

$X^{-X} = \text{false}$

$N^{+X} = \text{true}$

$N^{-X} = \text{true}$

Mutually recursive classes revisited

```
class C[X]:
    def f(self) -> D[X]: ...
    def g(self, x: X) -> None: ...
```

CONTRAVARIANT!

```
class D[Y]:
    def h(self) -> C[Y]: ...
```

CONTRAVARIANT!

Boolean variables:

C^{+}/C^{-} ~ C is covariant/contravariant

D^{+}/D^{-} ~ D is covariant/contravariant

F^{+X}/F^{-X} ~ f's signature is covariant/contravariant in X

G^{+X}/G^{-X} ~ g's signature is covariant/contravariant in X

H^{+Y}/H^{-Y} ~ h's signature is covariant/contravariant in Y

X^{+X}/X^{-X} ~ X is covariant/contravariant in X

N^{+X}/N^{-X} ~ None is covariant/contravariant in X

Equations:

$C^{+} = F^{+X} \wedge G^{+X} = \text{false}$

$C^{-} = F^{-X} \wedge G^{-X} = \text{true}$

$D^{+} = H^{+Y} = \text{false}$

$D^{-} = H^{-Y} = \text{true}$

$F^{+X} = D^{+} = \text{false}$

$F^{-X} = D^{-} = \text{true}$

$G^{+X} = X^{-X} \wedge N^{+X} = \text{false}$

$G^{-X} = X^{+X} \wedge N^{-X} = \text{true}$

$H^{+Y} = C^{+} = \text{false}$

$H^{-Y} = C^{-} = \text{true}$

$X^{+X} = \text{true}$

$X^{-X} = \text{false}$

$N^{+X} = \text{true}$

$N^{-X} = \text{true}$

Equations for classes

```
class C[X](B):
    a: Final[S]
    b: T
    def f(self, u: U, v: V) -> W: ...
```

Equations:

$$C^{+} = B^{+X} \wedge S^{+X} \wedge T^{+X} \wedge T^{-X} \wedge F^{+X}$$

$$C^{-} = B^{-X} \wedge S^{-X} \wedge T^{+X} \wedge T^{-X} \wedge F^{-X}$$

$$F^{+X} = U^{-X} \wedge V^{-X} \wedge W^{+X}$$

$$F^{-X} = U^{+X} \wedge V^{+X} \wedge W^{-X}$$

Intuition for fields:

- `a: Final[S]` ≈ `def get_a(self) -> S`
- `b: T` ≈ `def get_b(self) -> T` + `def set_b(self, b: T) -> None`

Equations for builtins

$$(S \mid T)^{+X} = S^{+X} \wedge T^{+X}$$

$$(S \mid T)^{-X} = S^{-X} \wedge T^{-X}$$

$$\texttt{tuple}[S,\ T]^{+X} = S^{+X} \wedge T^{+X}$$

$$\texttt{tuple}[S,\ T]^{-X} = S^{-X} \wedge T^{-X}$$

$$\texttt{tuple}[T,\ \ldots]^{+X} = T^{+X}$$

$$\texttt{tuple}[T,\ \ldots]^{-X} = T^{-X}$$

$$\vdots$$

Equations for specialisations

Theorem: Let C be a generic class, T a type, and X a type variable. The type `C[T]` is covariant/contravariant in X if one of the following conditions is met:

1. C is covariant and T is covariant/contravariant in X,
2. C is contravariant and T is contravariant/covariant in X,
3. C is covariant and contravariant,
4. T is covariant and contravariant in X.

Equations:

$$C[T]^{+X} = (C^{+} \wedge T^{+X}) \vee (C^{-} \wedge T^{-X}) \vee (C^{+} \wedge C^{-}) \vee (T^{+X} \wedge T^{-X})$$

$$C[T]^{-X} = (C^{+} \wedge T^{-X}) \vee (C^{-} \wedge T^{+X}) \vee (C^{+} \wedge C^{-}) \vee (T^{+X} \wedge T^{-X})$$

Linear time algorithm

Input: Signatures of classes `C1, ..., Cn` that are closed under dependencies.

1. Traverse the signatures of `C1, ..., Cn` and emit boolean equations:
   a. **Incremental:** If `Ci`'s variances already known, emit $\mathtt{Ci}^{\pm}$ `= true/false`.
2. Solve the system of boolean equations:
   a. Propagate constants and simplify as long as possible.
   b. Set all unconstrained boolean variables to `true`: sound and optimal.
   c. Yields the *greatest fixed point* of the equation system (Knaster–Tarski).
3. Read variances off assignments to boolean variables $\mathtt{C1}^{\pm}$`, ..., `$\mathtt{Cn}^{\pm}$.

Output: *Optimal* variances for classes `C1, ..., Cn`.

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Open questions

1. How do we best integrate short circuiting techniques into this approach?
2. Can we use this approach to improve incrementality/caching for subtyping?
3. Can we support immutable data structures better?

```
class Pair[X]:
    fst: Final[X]
    snd: Final[X]
    def __init__(self, fst: X, snd: X):
        self.fst = fst; self.snd = snd
    def replace_fst(self, fst: X) -> Pair[X]:
        return Pair(fst, self.snd)
```

`Pair` is inferred as invariant but covariant would be safe too.

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THANK YOU!

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Appendix

Interesting read

[Taming the Wildcards: Combining Definition- and Use-Site Variance]

Example for quadratic runtime

```
class A₀[X]:
    def g(self, x: X) -> None: ...

# for i = 1, ..., n:
class Aᵢ[X]:
    def f(self) -> Aᵢ₊₁[X]: ...
    def g(self) -> Aᵢ₋₁[X]: ...

class Aₙ₊₁[X]:
    def f(self) -> X: ...
```

Queries:

$\mathrm{IsCovariant}(A_1), \ldots, \mathrm{IsCovariant}(A_n)$

Equations:

$$
\begin{aligned}
A_0 &= \text{false} \\
A_1 &= A_2 \land A_0 \\
A_2 &= A_3 \land A_1 \\
A_3 &= A_4 \land A_2 \\
&\vdots \\
A_i &= A_{i+1} \land A_{i-1} \\
&\vdots \\
A_n &= A_{n+1} \land A_{n-1} \\
A_{n+1} &= \text{true}
\end{aligned}
$$