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Asymmetric Numeral System

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Outline

1. Introduction
2. Entropy Coding
3. Asymmetric Numeral System
4. rANS
5. Streaming-rANS
6. Summary
7. Reference

Introduction

Introduction

- Hi there, this is a self inspired latex beamer template for Leipzig university.

Entropy Coding

Entropy Coding

- The theoretical limit of compression is given by Shannon entropy, which measures the minimum average number of bits needed to encode a symbol from a data source.
- Lossless compression algorithms aim to approach this limit but cannot go below it.
- The entropy depends on the probability distribution of the symbols, with more skewed distributions being more compressible.
- Shannon Entropy [1]

$$H(X) = -\sum_{i=1}^{n} p(x_i) \log_2 p(x_i)$$

- Shannon's Source Coding Theorem [1]

$$N \cdot H(X)$$

Asymmetric Numeral System

Motivation

ANS combines the compression ratio of arithmetic coding (which uses a nearly accurate probability distribution), with a processing cost similar to that of Huffman coding [4].

Basic Idea

- Symbol spread function

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- Representation of information
  (represents an entire message as a single number, similar to Arithmetic Coding) –> example: representation of {2,0,2,5,1,8}

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- Symbol spread function
- Representation of information
  (represents an entire message as a single number, similar to Arithmetic Coding) –> example: representation of {2,0,2,5,1,8}
- Symmetric or Asymmetric
- rANS and tANS

Symbol spread function

- the function that determines the split of the $\mathbb{N}$ subsets corresponding to different symbols [1]

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- $\bar{s} : \mathbb{N} \rightarrow \mathcal{A}, \bar{s}(x) = \text{mod}(x, b) = s$

Symbol spread function

- the function that determines the split of the $\mathbb{N}$ subsets corresponding to different symbols [1]
- $\bar{s} : \mathbb{N} \to \mathcal{A}, \bar{s}(x) = \text{mod}(x, b) = s$
- In the standard binary system example: $\bar{s}(x) = \text{mod}(x, 2)$
  e.g. splits natural numbers into subsets of even/odd numbers [2]

Symbol spread function (Exa. 1)

$S = \{0, 1\}$

Exa. 1: Standard Binary System

N	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
s=0																			
s=1																			

Fig.1

Symbol spread function (Exa. 2)

$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$x' = 10x + s$

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		202		2025
$s = 0$																									
$s = 1$																									
$s = 2$																									
$s = 3$																									
$s = 4$																						...		...	
$s = 5$																									
$s = 6$																									
$s = 7$																									
$s = 8$																									
$s = 9$																									

Fig.2

Representation of information (Exa. 2)

- a finite sequence of symbols/digits:

$$\mathcal{A} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

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$$S_{in} = (2, 0, 2, 5)$$

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- How can we represent this sequence as a single number?

Representation of information (Exa. 2)

- a finite sequence of symbols/digits:

$$\mathcal{A} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

- the sequence of symbols/digits to be encoded:

$$S_{in} = (2, 0, 2, 5)$$

- How can we represent this sequence as a single number?
- The easiest way is to concatenate the symbols/digits:

$$x_{out} = 2025$$

Representation of Information (Exa. 2)

- encoding the sequence of digits (2,0,2,5,1,8)

$$
\begin{aligned}
x_{\text{init}} &= 0 \\
x_1 &= x_{\text{init}} \times 10 + 2 = 2 \\
x_2 &= x_1 \times 10 + 0 = 20 \\
x_3 &= x_2 \times 10 + 2 = 202 \\
x_{\text{out}} &= x_3 \times 10 + 5 = 2025
\end{aligned}
$$

Representation of Information (Exa. 2)

– What is the encoding function?

Representation of Information (Exa. 2)

- What is the encoding function?
- Encoding function

$$x_{i+1} = x_i \times 10 + s_i$$

Representation of Information (Exa. 2)

- What is the encoding function?
- Encoding function

$$x_{i+1} = x_i \times 10 + s_i$$

- x' is the x-th appearance of symbol s

$$x' = C(s, x) = x \cdot 10 + s$$

Representation of Information (Exa. 2)

$x' = 10x + s$

$s = 0$ $s = 2$ $s = 2$ $s = 5$

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		202		2025
$s = 0$	0										1										2				
$s = 1$		0										1													
$s = 2$			0										1												
$s = 3$				0										1											
$s = 4$					0										1							...		...	
$s = 5$						0										1									
$s = 6$							0										1								
$s = 7$								0										1							
$s = 8$									0										1						
$s = 9$										0										1					

Fig.3

Representation of Information (Exa. 1)

Exa. 1: Standard Binary System

X' = 2X + s

s=1 s=0 s=0 s=1

X'	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
X s=0	0		1		2		3		4		5		6		7		8		9
X s=1		0		1		2		3		4		5		6		7		8	

Fig.4

Asymmetric Numeral System | Representation of information |

Exa. 1: Standard Binary System

X' = 2X + s

s=1 s=0 s=0 s=1

X'	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
x s=0	0		1		2		3		4		5		6		7		8		9
x s=1		0		1		2		3		4		5		6		7		8	

Exa. 2: Standard Decimal System

X' = 10X + s

s=2 s=0 s=2 s=5

N	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		202		2025
s=0	0										1										2				
s=1		0										1													
s=2			0										1												
s=3				0										1											
s=4					0										1							...		...	
s=5						0										1									
s=6							0										1								
s=7								0										1							
s=8									0										1						
s=9										0										1					

Fig.5

Asymmetric Numeral System | Symmetric or Asymmetric |

Exa. 1: Standard Binary System

X' = 2X + s

s=1 s=0 s=0 s=1

X'	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
X s=0	0		1		2		3		4		5		6		7		8		9
X s=1		0		1		2		3		4		5		6		7		8	

Exa. 2: Standard Decimal System

X' = 10X + s

s=2 s=0 s=2 s=5

N	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		202		2025
s=0	0										1										2				
s=1		0										1													
s=2			0										1												
s=3				0										1											
s=4					0										1							...		...	
s=5						0										1									
s=6							0										1								
s=7								0										1							
s=8									0										1						
s=9										0										1					

Fig.6

Exa. 3: Asymmetric Binary System

- Let $\mathcal{A} = \{0, 1\}$
- $Pr(0) = 1/4, Pr(1) = 3/4$

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- $x' = C(s, x) \approx x/p_s$

Exa. 3: Asymmetric Binary System

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$x' \approx x/Pr(s)$ | $Pr(0) = 1/4, Pr(1) = 3/4$

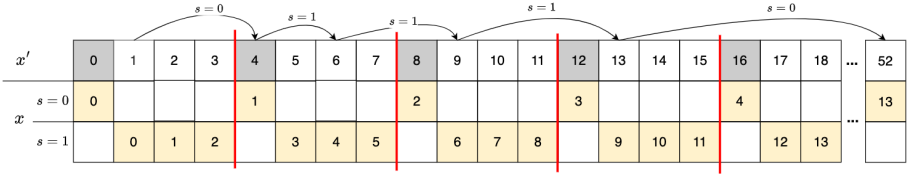


Fig.7

We refine even/odd number by changing their densities

Symmetric or Asymmetric

- Symmetric: uniform distribution
- Asymmetric: non-uniform distribution

tANS

- d

Performance

- f

rANS

rANS

- ”r” = ”range”: similar to range coding (RC)
- symbol spread function
- representation of information
- asymmetric

Exa.3

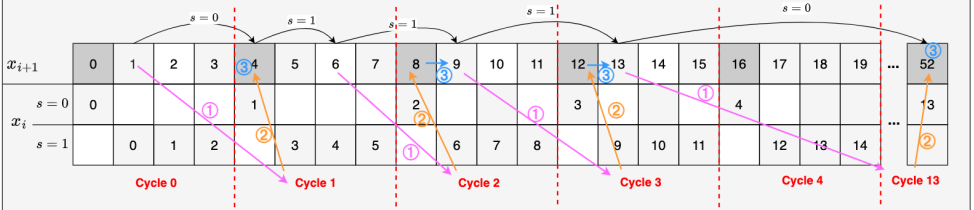
How to encode 101110? What is the encoding function $C(s, x)$?

$x' = C(s, x) \approx x/p_s$

Some notations

- $\mathcal{A} = \{s_1, s_2, \ldots, s_k\}$ is the alphabet set
- s is the symbol, if it is used in subscript: s is the current symbol, $s-1$ is the previous symbol, $s+1$ is the next symbol
- F_s is the absolute frequency of symbol s
- $m = \sum_{s \in \mathcal{A}} F_s$ is the total frequency
- $c_s = \sum_{s \in \mathcal{A}} F_{s-1}$ is the cumulative frequency (c_{s+1} is the cumulative frequency of the next symbol)
- $p_s = F_s / m$ is the probability of symbol s
- x_i is the current state (next state: x_{i+1}, previous state: x_{i-1})

$x_{i+1} \approx x_i / Pr(s)$ | $Pr(0) = 1/4, Pr(1) = 3/4$



Given current state x_i and next symbol s. How to find next state x_{i+1}, which is the x_i-th appearance of symbol s ?

STEPS / EXAMPLE	$Pr(0) = 1/4, Pr(1) = 3/4$		General
	$s = 0$	$s = 1$	
Step ①: Find cycle number k	$k = \lfloor x_i/1 \rfloor$	$k = \lfloor x_i/3 \rfloor$	$k = \lfloor x_i/F_s \rfloor$
Step ②: Navigate to the beginning (first number) of that cycle	$4 \cdot k$	$4 \cdot k$	$m \cdot k$
Step ③: Find the position of the symbol s in that cycle	part 1: first of one + 0	part 1: + mod$(x_i, 3)$	part 1: + mod(x_i, F_s)
- part 1: position of the symbol s in the sequence of the same symbol - part 2: how many other symbols before the symbol s	part 2: no other symbols + 0	part 2: only one other symbol (0) and with one of that symbol in one cycle + 1	part 2: only one other symbol (0) and with one of that symbol in one cycle $+c_s$
	$x_{i+1} = m \cdot k$	$x_{i+1} = 4 \cdot \lfloor x_i/3 \rfloor + \text{mod}(x_i, 3) + 1$	$x_{i+1} = m \cdot \lfloor x/F_s \rfloor + \text{mod}(x, F_s) + c_s$

$$x_{i+1} = C(s, x_i) = m \cdot \lfloor x_i/F_s \rfloor + \text{mod}(x_i, F_s) + c_s$$

Fig.8

Encoding Function [2] [5]:

$$x_{i+1} = m \cdot \lfloor x_i / F_s \rfloor + \mathrm{mod}(x_i, F_s) + c_s$$

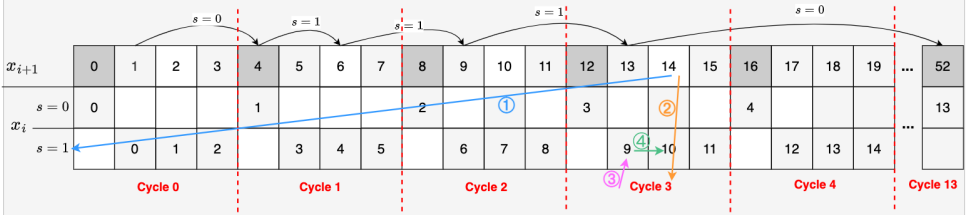
implementation with python (dash app)

```python
def rANS_encoder(symbol_counts, s_input):
    cumul_freq = [] # c_s
    sum_freq = 0 # m
    x = 0 # initial state
    output = [] # list to store encoded (symbols, state)

    for count in symbol_counts:
        cumul_freq.append(sum_freq)
        sum_freq += count
    for s in s_input:
        F_s = symbol_counts[s]
        c_s = cumul_freq[s]
        x = sum_freq * (x // F_s) + (x % F_s) + c_s
        output.append((s, x))

    L_avg = math.ceil(math.log(x) / math.log(2.0)) / len(s_input)
    return output, x, L_avg, entropy(symbol_counts)
```

$x_{i+1} \approx x_i / Pr(s)$ $\boxed{Pr(0) = 1/4, Pr(1) = 3/4}$



Given current state x_i. How to find previous state x_{i-1}, where x_i is the x_{i-1}-th appearance of symbol s ?

STEPS / EXAMPLE	$Pr(0) = 1/4, Pr(1) = 3/4$		General
	$s = 0$	$s = 1$	
Step ① : Find the symbol s for the state x_i	if $x_i \bmod 4 = 0$	otherwise or if $x_i \bmod 4 = 1, 2, 3$	s such that $c_s \leq \mathrm{mod}(x_i, m) < c_{s+1}$
Step ②: Find the cycle k of the current state x_i	$k = x_i/4$	$k = \lfloor x_I/4 \rfloor$	$k = \lfloor x_i/m \rfloor$
Step ③: Find the first number of that symbol s in that cycle k	$x_i/4 \cdot 1$	$\lfloor x_i/4 \rfloor \cdot 3$	$\lfloor x_i/m \rfloor \cdot F_s$
Step ④: Allocate to the exact position	$x_{i-1} = x_i/4 \cdot 1 + 0 - 0$	$x_{i-1} = \lfloor x_i/4 \rfloor \cdot 3 + 2 - 1$	$x_{i+1} = \lfloor x_i/m \rfloor \cdot F_s + \mathrm{mod}(x_i, F_s) - c_s$

1. s such that $c_s \leq \mathrm{mod}(x_1, m) < c_{s+1}$
2. $x_{i-1} = D(x_i) = \lfloor x_i/m \rfloor \cdot F_s + \mathrm{mod}(x_i, F_s) - c_s$

Fig.9

Decoding Function [2] [5]:

- s

$$s \text{ such that } c_s \leq \text{mod}(x_i, m) < c_{s+1}$$

- x_{i-1}

$$x_{i-1} = D(x_i) = \lfloor x_i / m \rfloor \cdot F_s + \text{mod}(x_i, F_s) - c_s$$

implementation with python (dash app)

```python
def rANS_decoder(symbol_counts, x):
    cumul_freq = []  # c_s
    sum_freq = 0 # m
    output = []  # List to store decoded symbols and states
    # compute m and c_s
    for count in symbol_counts:
        cumul_freq.append(sum_freq)
        sum_freq += count
    while x > 0:
        slot = x % sum_freq  # Find the slot in the cumulative frequency range
        s = bisect_right(cumul_freq, slot) - 1 # Binary search to find the symbol s (slot).
        F_s = symbol_counts[s]          # Frequency of the symbol
        c_s = cumul_freq[s]             # Cumulative count of the symbol
        output.append((s,x))            # Append decoded symbol to the output
        x = (x // sum_freq) * F_s + x % sum_freq - c_s  # Update the state
    return output, state
```

Streaming-rANS

Streaming-rANS

To avoid using large number arithmetic, in practice stream variants are used: which enforce by renormalization: sending the least significant bits of x to or from the bitstream (usually L and b are powers of 2). [4]

Example of the problem //TODO: add example

- ANS (Asymmetric Numeral Systems) uses a single integer x, known as the "state," to represent the encoded data.
- Over time, as more symbols are encoded, the state x can grow arbitrarily large. This can lead to:
  - Overflow: If the value exceeds the precision limits of your system (e.g., a 32-bit or 64-bit register).
  - Inefficient Encoding: A larger x means you need more memory or bits to represent it.
- Using the C function from the previous section to encode succeeding symbols, we can encode a sequence into a very large natural number x, which can be stored as a length $\lfloor lg(x) \rfloor + 1$ bit sequence. [2]

Renormalization in Range Coding

- Range Coding
  - 1
  - 2
- we use the similar idea for rANS
  - range $[L, H]$
  - Encoding: shrink the current state x until it is in the range $[L, H]$
  - Decoding: expand the current state x until it is in the range $[L, H]$
- In AC, renormalization is used to allow finite precision, an analogous approach should be used for ANS. Specifically, we will enforce xto remain in some fixed range I by transferring the least significant bits to the stream. For generality, let us transfer a base b digit, for example $b = 2$ means transferring single bits, while $b = 2^8$ means transferring complete bytes.

Methods from Jarek Duda: Method 1 - Decoding [2]

ANS decoding step from state *x*

$(s, x) = D(x)$ *# decode the current state x*

useSymbol(s)

while $x < L$ **do** *# if x is too small that not in the range [L,H]*

 # expand the x by reading the outstreamed digits back from the stream (an array)

expand$(x, b) = b \cdot x +$ readDigit$() \rightarrow x$

end while

 # return the new state

What is L, H ?

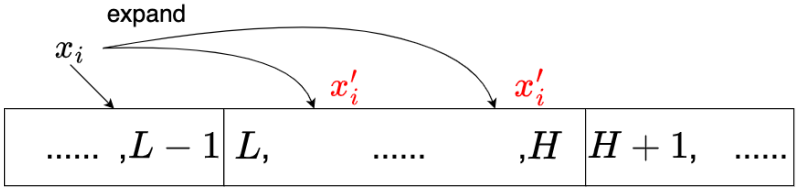


Fig 11

What is L, H ?

- $x_i = b \cdot x_{i-1} +$ bit
- Either x_i in $[L, H]$ or x_{i-1} in $[L, H]$. Assume x_{i-1} in $[L, H]$.
  - $x_{i-1} >= L$
  - $x_i >= b \cdot L + bit >= b \cdot L$
  - x_i can't be in $[L, H]$
  - derive: $H <= b \cdot L - 1$
- $[L, H]$, where $H <= b \cdot L - 1$ (Constraint 1)
- $I = \{L, ..., b \cdot L - 1\}$, which is defined by Jarek Duda b-**unique interval** and $L \in \mathbb{N}$
- This will ensure that the loop in Method 1 will always return to the original value

Methods from Jarek Duda: Method 2 - Encoding [2]

shrink the current state x by stream out the least significant bit (x will be stored in binary and we take out the bit from right side)

ANS encoding step of symbol sfrom state x

 # once the current state x is greater than max value that we define later

while $x > maxX[s]$ **do**

 # write the least significant digit to the stream (stream could be an array in implementaion)

writeDigit(mod(x, b))

shrink$(x, b) = \lfloor x/b \rfloor \rightarrow x$ *# shrink the x*

end while

$x = C(s, x)$ *# encode with the shrunk x*

What is $maxX[s]$?

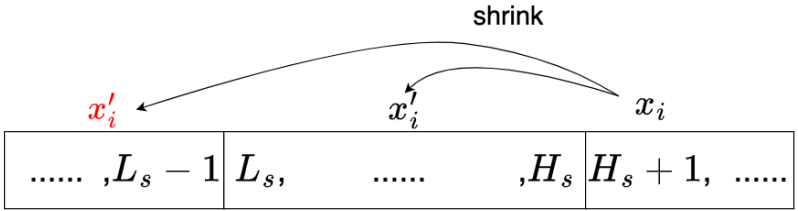


Fig 10

What is $maxX[s]$?

- to make sure not to jump over the range
- Assume that $x_i = H_s + 1$ and it will be shrinked to $x'_i = \lfloor x_i/b \rfloor$
- $x'_i = \lfloor x_i/b \rfloor = \lfloor (H_s + 1)/b \rfloor >= L_s$
- $L_s <= (H_s + 1)/b <= \lfloor (H_s + 1)/b \rfloor < (H_s + 1)/b + 1$
- $H_s >= b \cdot L_s - 1, \forall s \in \mathcal{A}$ (Constraint 2)

Final Form of $[L, H]$ in practice

- Contraints

$$[L, H]: \text{ where } H <= b \cdot L - 1$$

$$[L_s, H_s] : H_s >= b \cdot L_s - 1, \forall s \in \mathcal{A}$$

- In practice ($b = 2^1$): we choose $L = k \cdot m$, $L_s = k \cdot F_s$ ($k \in \mathbb{N}$)

$$\mathcal{I} = [km, 2 \cdot km - 1]$$

$$\mathcal{I}_s = [kF_s,\ 2 \cdot kF_s - 1] \text{ for } \forall s \in \mathcal{A}$$

implementation with python (dash app)

```python
def rANS_stream_encoder(symbol_counts, s_input):
    # Init: c_s, m, x, output, bit_stream (omit the code here)
    for s in s_input:
        F_s = symbol_counts[s]
        c_s = cumul_freq[s]
        L_s = F_s # k = 1
        maxX_s = 2 * L_s - 1
        out_bits = ""

        while x > maxX_s:
            out_bits += str(x % 2)
            x = x // 2
        x = (x // F_s) * sum_freq + (x % F_s) + c_s
        bit_stream += out_bits
        output.append((s, x, out_bits, bit_stream))

    return output, x, bit_stream, L_avg, entropy(symbol_counts)
```

implementation with python (dash app)

```python
def rANS_stream_decoder(symbol_counts, x, bit_stream, num_symbols):
    # Init: c_s, m, x, output, bit_stream (omit the code here)
    bit_stream = list(map(int, bit_stream.split(',')))  # Convert bit_stream to a list of integers
    for _ in range(num_symbols):
        slot = x % sum_freq
        s = bisect_right(cumul_freq, slot) - 1
        F_s = symbol_counts[s]
        c_s = cumul_freq[s]
        output.append((s, x))
        x = (x // sum_freq) * F_s + slot - c_s

        while x < sum_freq and len(bit_stream) > 0:
            x = x * 2 + bit_stream.pop()

    return output, x
```

Summary

Summary

- To summarize, this may help you to make a Leipzig university beamer presentation.

Reference

Reference

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