UNIVERSITÄT
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Asymmetric Numeral System

December 24, 2024

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Outline

1. Introduction
2. Entropy Coding
3. Asymmetric Numeral System
4. rANS
5. Streaming-rANS
6. Summary
7. Reference

Introduction

Introduction

- Hi there, this is a self inspired latex beamer template for Leipzig university.

Entropy Coding

Entropy Coding

- The theoretical limit of compression is given by Shannon entropy, which measures the minimum average number of bits needed to encode a symbol from a data source.
- Lossless compression algorithms aim to approach this limit but cannot go below it.
- The entropy depends on the probability distribution of the symbols, with more skewed distributions being more compressible.
- Shannon Entropy [1]

$$H(X) = -\sum_{i=1}^{n} p(x_i) \log_2 p(x_i)$$

- Shannon's Source Coding Theorem [1]

$$N \cdot H(X)$$

Asymmetric Numeral System

Table of Contents for Asymmetric Numeral System

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Motivation

ANS combines the compression ratio of arithmetic coding (which uses a nearly accurate probability distribution), with a processing cost similar to that of Huffman coding [4].

Basic Idea

- Symbol spread function

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- Representation of information
  (represents an entire message as a single number, similar to Arithmetic Coding) –> example: representation of {2,0,2,5,1,8}

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Basic Idea

- Symbol spread function
- Representation of information
  (represents an entire message as a single number, similar to Arithmetic Coding) –> example: representation of {2,0,2,5,1,8}
- Symmetric or Asymmetric
- rANS and tANS

Symbol spread function

- the function that determines the split of the $\mathbb{N}$ subsets corresponding to different symbols [1]

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- $\bar{s} : \mathbb{N} \rightarrow \mathcal{A}, \bar{s}(x) = \text{mod}(x, b) = s$

Symbol spread function

- the function that determines the split of the $\mathbb{N}$ subsets corresponding to different symbols [1]
- $\bar{s} : \mathbb{N} \to \mathcal{A}, \bar{s}(x) = \text{mod}(x, b) = s$
- In the standard binary system example: $\bar{s}(x) = \text{mod}(x, 2)$
  e.g. splits natural numbers into subsets of even/odd numbers [2]

Symbol spread function (Exa. 1)

$S = \{0, 1\}$

Exa. 1: Standard Binary System

N	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
s=0																			
s=1																			

Fig.1

Symbol spread function (Exa. 2)

$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$x' = 10x + s$

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		202		2025
$s = 0$																									
$s = 1$																									
$s = 2$																									
$s = 3$																									
$s = 4$																						...		...	
$s = 5$																									
$s = 6$																									
$s = 7$																									
$s = 8$																									
$s = 9$																									

Fig.2

Representation of information (Exa. 2)

- a finite sequence of symbols/digits:

$$\mathcal{A} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

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- the sequence of symbols/digits to be encoded:

$$S_{in} = (2, 0, 2, 5)$$

- How can we represent this sequence as a single number?
- The easiest way is to concatenate the symbols/digits:

$$x_{out} = 2025$$

Representation of Information (Exa. 2)

- encoding the sequence of digits (2,0,2,5,1,8)

$$
\begin{aligned}
x_{\text{init}} &= 0 \\
x_1 &= x_{\text{init}} \times 10 + 2 = 2 \\
x_2 &= x_1 \times 10 + 0 = 20 \\
x_3 &= x_2 \times 10 + 2 = 202 \\
x_{\text{out}} &= x_3 \times 10 + 5 = 2025
\end{aligned}
$$

Representation of Information (Exa. 2)

– What is the encoding function?

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$$x_{i+1} = x_i \times 10 + s_i$$

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$$x_{i+1} = x_i \times 10 + s_i$$

- x' is the x-th appearance of symbol s

$$x' = C(s, x) = x \cdot 10 + s$$

Representation of Information (Exa. 2)

$x' = 10x + s$

$s = 0$ $s = 2$ $s = 2$ $s = 5$

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		202		2025
$s = 0$	0										1										2				
$s = 1$		0										1													
$s = 2$			0										1												
$s = 3$				0										1											
$s = 4$					0										1							...		...	
$s = 5$						0										1									
$s = 6$							0										1								
$s = 7$								0										1							
$s = 8$									0										1						
$s = 9$										0										1					

Fig.3

Representation of Information (Exa. 1)

Exa. 1: Standard Binary System

X' = 2X + s

s=1 s=0 s=0 s=1

X'	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
X s=0	0		1		2		3		4		5		6		7		8		9
X s=1		0		1		2		3		4		5		6		7		8	

Fig.4

Asymmetric Numeral System | Representation of information |

Exa. 1: Standard Binary System

X' = 2X + s

s=1 s=0 s=0 s=1

X'	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
x s=0	0		1		2		3		4		5		6		7		8		9
x s=1		0		1		2		3		4		5		6		7		8	

Exa. 2: Standard Decimal System

X' = 10X + s

s=2 s=0 s=2 s=5

N	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		202		2025
s=0	0										1										2				
s=1		0										1													
s=2			0										1												
s=3				0										1											
s=4					0										1							...		...	
s=5						0										1									
s=6							0										1								
s=7								0										1							
s=8									0										1						
s=9										0										1					

Fig.5

Asymmetric Numeral System | Symmetric or Asymmetric |

Exa. 1: Standard Binary System

X' = 2X + s

s=1 s=0 s=0 s=1

X'	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
X s=0	0		1		2		3		4		5		6		7		8		9
X s=1		0		1		2		3		4		5		6		7		8	

Exa. 2: Standard Decimal System

X' = 10X + s

s=2 s=0 s=2 s=5

N	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		202		2025
s=0	0										1										2				
s=1		0										1													
s=2			0										1												
s=3				0										1											
s=4					0										1							...		...	
s=5						0										1									
s=6							0										1								
s=7								0										1							
s=8									0										1						
s=9										0										1					

Fig.6

Exa. 3: Asymmetric Binary System

- Let $\mathcal{A} = \{0, 1\}$
- $Pr(0) = 1/4, Pr(1) = 3/4$

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- $Pr(0) = 1/4, Pr(1) = 3/4$
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- $x' = C(s, x) \approx x/p_s$

Exa. 3: Asymmetric Binary System

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$x' \approx x/Pr(s)$ | $Pr(0) = 1/4, Pr(1) = 3/4$

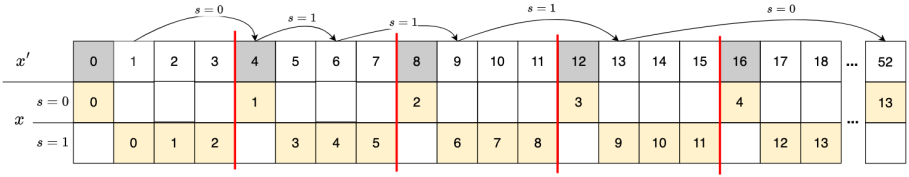


Fig.7

We refine even/odd number by changing their densities

Symmetric or Asymmetric

- Symmetric: uniform distribution
- Asymmetric: non-uniform distribution

tANS

- d

Performance

- f

rANS

rANS

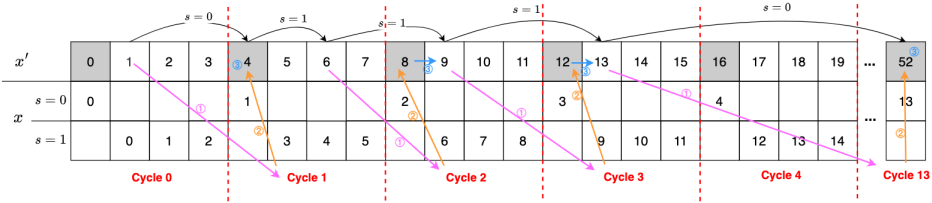
- ”r” = ”range”: similar to range coding (RC)
- symbol spread function
- representation of information
- asymmetric

Exa.3

How to encode 101110? What is the encoding function $C(s, x)$?

$x' = C(s, x) \approx x/p_s$

$x' \approx x/Pr(s)$ $\boxed{Pr(0) = 1/4, Pr(1) = 3/4}$



Given current state x and next symbol s. How to find next state x', which is the x-th appearance of symbol s ?

	$s = 0$	$s = 1$	0 and 1
Step ①: Find cycle number k	$k = \lfloor x/1 \rfloor = x$	$k = \lfloor x/3 \rfloor$	$k = \lfloor x/F_s \rfloor$ F_s:
Step ②: Navigate to the beginning (first number) of that cycle	$4 \cdot k$	$4 \cdot k$	$m \cdot k$ m: size of the cycle
Step ③: Find the position of the symbol s in that cycle - part 1: position of the symbol s in the sequence of the same symbol - part 2: how many other symbols before the symbol s	part 1: first of one + 0 part 2: no other symbols + 0	part 1: + mod$(x, 3)$ part 2: only one other symbol (0) and with one of that symbol in one cycle + 1	part 1: + mod(x, F_s) part 2: only one other symbol (0) and with one of that symbol in one cycle $+c_s$
	$x' = 4 \cdot x$	$x' = 4 \cdot \lfloor x/3 \rfloor + \mathrm{mod}(x, 3) + 1$	$x' = m \cdot \lfloor x/F_s \rfloor + \mathrm{mod}(x, F_s) + c_s$

$$x' = C(s, x) = m \cdot \lfloor x/F_s \rfloor + \mathrm{mod}(x, F_s) + c_s$$

Fig.8

Encoding Function (general):

$$x' = m \cdot \left\lfloor \frac{x}{F_s} \right\rfloor + \mathrm{mod}(x, F_s) + c_s$$ [5]

- **Alphabet set:** $\mathcal{A} = \{a_1, a_2, \ldots, a_k\}$
- **Absolute frequency of symbol** s**:** F_s
- **Total frequency:** $m = \sum_{i=1}^{k} F_i$
- **Probability of symbol** s**:** $p_s = \frac{F_s}{m}$
- **Cumulative frequency before** s**:** $c_s := F_0 + F_1 + \cdots + F_{s-1}$

Encoding Function (binary):

$$x' = C(s, x) = \left\lfloor \frac{x}{F_s} \right\rfloor << n + \mathrm{mod}(x, F_s) + c_s$$ [2]

- **Alphabet set:** $\mathcal{A} = \{a_1, a_2, \ldots, a_k\}$
- **Absolute frequency of symbol** s**:** F_s
- **Total frequency:** $m = \sum_{i=1}^{k} F_i$
- **Probability of symbol** s**:** $p_s = \frac{F_s}{m}$
- **Cumulative frequency before** s**:** $c_s := F_0 + F_1 + \cdots + F_{s-1}$
- $<<$ n is a left shift by n bits

Decoding Function (binary):

$$D(x) = (s, F_s \cdot (x >> n) + (x \& mask) - c_s)^{[2]}$$

Implementation

- **Python Implementation:**

Streaming-rANS

Streaming-rANS

To avoid using large number arithmetic, in practice stream variants are used: which enforce by renormalization: sending the least significant bits of x to or from the bitstream (usually L and b are powers of 2). [4]

Summary

Summary

- To summarize, this may help you to make a Leipzig university beamer presentation.

Reference

Reference

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2. J. Duda, K. Tahboub, N. J. Gadgil and E. J. Delp, ”The use of asymmetric numeral systems as an accurate replacement for Huffman coding,” 2015 Picture Coding Symposium (PCS), Cairns, QLD, Australia, 2015, pp. 65-69, doi: 10.1109/PCS.2015.7170048.
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