UNIVERSITÄT
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Asymmetric Numeral System

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Outline

1. Introduction
2. Entropy Coding
3. Asymmetric Numeral System
4. rANS
5. Streaming-rANS
6. Summary
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Introduction

Introduction

- Hi there, this is a self inspired latex beamer template for Leipzig university.

Entropy Coding

Entropy Coding

- The theoretical limit of compression is given by Shannon entropy, which measures the minimum average number of bits needed to encode a symbol from a data source.
- Lossless compression algorithms aim to approach this limit but cannot go below it.
- The entropy depends on the probability distribution of the symbols, with more skewed distributions being more compressible.
- Shannon Entropy [1]

$$H(X) = -\sum_{i=1}^{n} p(x_i) \log_2 p(x_i)$$

- Shannon's Source Coding Theorem [1]

$$N \cdot H(X)$$

Asymmetric Numeral System

Motivation

ANS combines the compression ratio of arithmetic coding (which uses a nearly accurate probability distribution), with a processing cost similar to that of Huffman coding [4].

Basic Idea

- Symbol spread function

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- Representation of information
  (represents an entire message as a single number, similar to Arithmetic Coding) –> example: representation of {2,0,2,5,1,8}

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- Symbol spread function
- Representation of information
  (represents an entire message as a single number, similar to Arithmetic Coding) –> example: representation of {2,0,2,5,1,8}
- Symmetric or Asymmetric
- rANS and tANS

Symbol spread function

- the function that determines the split of the $\mathbb{N}$ subsets corresponding to different symbols [1]

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Symbol spread function

- the function that determines the split of the $\mathbb{N}$ subsets corresponding to different symbols [1]
- $\bar{s} : \mathbb{N} \rightarrow \mathcal{A}, \bar{s}(x) = \text{mod}(x, b) = s$
- In the standard binary system example: $\bar{s}(x) = \text{mod}(x, 2)$
  e.g. splits natural numbers into subsets of even/odd numbers [2]

Symbol spread function (Exa. 1)

$S = \{0, 1\}$

Exa. 1: Standard Binary System

N	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
s=0																			
s=1																			

Fig.1

Symbol spread function (Exa. 2)

$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$x' = 10x + s$

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		202		2025
$s = 0$																									
$s = 1$																									
$s = 2$																									
$s = 3$																									
$s = 4$																						...		...	
$s = 5$																									
$s = 6$																									
$s = 7$																									
$s = 8$																									
$s = 9$																									

Fig.2

Representation of information (Exa. 2)

- a finite sequence of symbols/digits:

$$\mathcal{A} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

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$$S_{in} = (2, 0, 2, 5)$$

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- the sequence of symbols/digits to be encoded:

$$S_{in} = (2, 0, 2, 5)$$

- How can we represent this sequence as a single number?
- The easiest way is to concatenate the symbols/digits:

$$x_{out} = 2025$$

Representation of Information (Exa. 2)

- encoding the sequence of digits (2,0,2,5,1,8)

$$
\begin{aligned}
x_{\text{init}} &= 0 \\
x_1 &= x_{\text{init}} \times 10 + 2 = 2 \\
x_2 &= x_1 \times 10 + 0 = 20 \\
x_3 &= x_2 \times 10 + 2 = 202 \\
x_{\text{out}} &= x_3 \times 10 + 5 = 2025
\end{aligned}
$$

Representation of Information (Exa. 2)

– What is the encoding function?

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- Encoding function

$$x_{i+1} = x_i \times 10 + s_i$$

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$$x_{i+1} = x_i \times 10 + s_i$$

- x' is the x-th appearance of symbol s

$$x' = C(s, x) = x \cdot 10 + s$$

Representation of Information (Exa. 2)

$x' = 10x + s$

$s = 0$, $s = 2$, $s = 2$, $s = 5$

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		202		2025
$s = 0$	0										1										2				
$s = 1$		0										1													
$s = 2$			0										1												
$s = 3$				0										1											
$s = 4$					0										1							...		...	
$s = 5$						0										1									
$s = 6$							0										1								
$s = 7$								0										1							
$s = 8$									0										1						
$s = 9$										0										1					

Fig.3

Representation of Information (Exa. 1)

Exa. 1: Standard Binary System

X' = 2X + s

s=1 s=0 s=0 s=1

X'	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
X s=0	0		1		2		3		4		5		6		7		8		9
X s=1		0		1		2		3		4		5		6		7		8	

Fig.4

Asymmetric Numeral System | Representation of information |

Exa. 1: Standard Binary System

X' = 2X + s

s=1 s=0 s=0 s=1

X'	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
x s=0	0		1		2		3		4		5		6		7		8		9
x s=1		0		1		2		3		4		5		6		7		8	

Exa. 2: Standard Decimal System

X' = 10X + s

s=2 s=0 s=2 s=5

N	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		202		2025
s=0	0										1										2				
s=1		0										1													
s=2			0										1												
s=3				0										1											
s=4					0										1							...		...	
s=5						0										1									
s=6							0										1								
s=7								0										1							
s=8									0										1						
s=9										0										1					

Fig.5

Asymmetric Numeral System | Symmetric or Asymmetric |

Exa. 1: Standard Binary System

X' = 2X + s

s=1 s=0 s=0 s=1

X'	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
X s=0	0		1		2		3		4		5		6		7		8		9
X s=1		0		1		2		3		4		5		6		7		8	

Exa. 2: Standard Decimal System

X' = 10X + s

s=2 s=0 s=2 s=5

N	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		202		2025
s=0	0										1										2				
s=1		0										1													
s=2			0										1												
s=3				0										1											
s=4					0										1							...		...	
s=5						0										1									
s=6							0										1								
s=7								0										1							
s=8									0										1						
s=9										0										1					

Fig.6

Exa. 3: Asymmetric Binary System

- Let $\mathcal{A} = \{0, 1\}$
- $Pr(0) = 1/4, Pr(1) = 3/4$

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- $x' = C(s, x) \approx x/p_s$

Exa. 3: Asymmetric Binary System

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$x' \approx x/Pr(s)$ | $Pr(0) = 1/4, Pr(1) = 3/4$

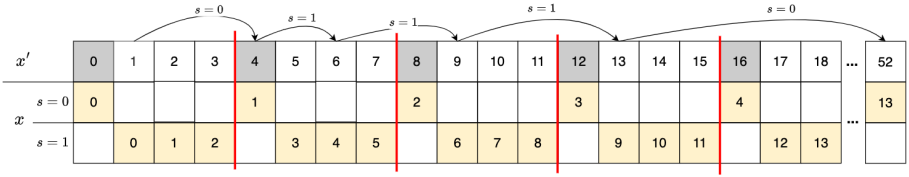


Fig.7

We refine even/odd number by changing their densities

Symmetric or Asymmetric

- Symmetric: uniform distribution
- Asymmetric: non-uniform distribution

tANS

- d

Performance

- f

rANS

rANS

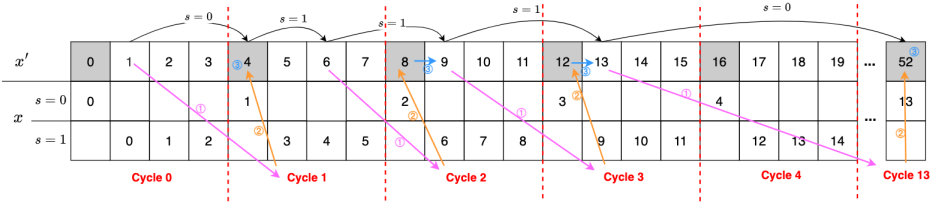
- ”r” = ”range”: similar to range coding (RC)
- symbol spread function
- representation of information
- asymmetric

Exa.3

How to encode 101110? What is the encoding function $C(s, x)$?

$x' = C(s, x) \approx x/p_s$

$x' \approx x/Pr(s)$ $\boxed{Pr(0) = 1/4, Pr(1) = 3/4}$



Given current state x and next symbol s. How to find next state x', which is the x-th appearance of symbol s ?

	$s = 0$	$s = 1$	0 and 1
Step ①: Find cycle number k	$k = \lfloor x/1 \rfloor = x$	$k = \lfloor x/3 \rfloor$	$k = \lfloor x/F_s \rfloor$ F_s:
Step ②: Navigate to the beginning (first number) of that cycle	$4 \cdot k$	$4 \cdot k$	$m \cdot k$ m: size of the cycle
Step ③: Find the position of the symbol s in that cycle - part 1: position of the symbol s in the sequence of the same symbol - part 2: how many other symbols before the symbol s	part 1: first of one + 0 part 2: no other symbols + 0	part 1: + mod$(x, 3)$ part 2: only one other symbol (0) and with one of that symbol in one cycle + 1	part 1: + mod(x, F_s) part 2: only one other symbol (0) and with one of that symbol in one cycle $+c_s$
	$x' = 4 \cdot x$	$x' = 4 \cdot \lfloor x/3 \rfloor + \mathrm{mod}(x, 3) + 1$	$x' = m \cdot \lfloor x/F_s \rfloor + \mathrm{mod}(x, F_s) + c_s$

$$x' = C(s, x) = m \cdot \lfloor x/F_s \rfloor + \mathrm{mod}(x, F_s) + c_s$$

Fig.8

Encoding Function (general):

$$x' = m \cdot \left\lfloor \frac{x}{F_s} \right\rfloor + \mathrm{mod}(x, F_s) + c_s$$ [5]

- **Alphabet set:** $\mathcal{A} = \{a_1, a_2, \ldots, a_k\}$
- **Absolute frequency of symbol** s**:** F_s
- **Total frequency:** $m = \sum_{i=1}^{k} F_i$
- **Probability of symbol** s**:** $p_s = \frac{F_s}{m}$
- **Cumulative frequency before** s**:** $c_s := F_0 + F_1 + \cdots + F_{s-1}$

Encoding Function (binary):

$$x' = C(s, x) = \left\lfloor \frac{x}{F_s} \right\rfloor << n + \mathrm{mod}(x, F_s) + c_s$$ [2]

- **Alphabet set:** $\mathcal{A} = \{a_1, a_2, \ldots, a_k\}$
- **Absolute frequency of symbol** s**:** F_s
- **Total frequency:** $m = \sum_{i=1}^{k} F_i$
- **Probability of symbol** s**:** $p_s = \frac{F_s}{m}$
- **Cumulative frequency before** s**:** $c_s := F_0 + F_1 + \cdots + F_{s-1}$
- $<<$ n is a left shift by n bits

Decoding Function (binary):

$$D(x) = (s, F_s \cdot (x >> n) + (x \& mask) - c_s)^{[2]}$$

```python
def rANS_encoder(symbol_counts, s_input):
    # compute cumulative frequencies
    cumul_counts = []
    sum_counts = 0
    for count in symbol_counts:
        cumul_counts.append(sum_counts)
        sum_counts += count
    state = 0
    output = []
    for s in s_input:
        Fs = symbol_counts[s]
        Cs = cumul_counts[s]
        state = (state // Fs) * sum_counts + Cs + (state % Fs)
        output.append((s, state))

    L_avg = math.ceil(math.log(state) / math.log(2.0)) / len(s_input)
    return output, state, L_avg, entropy(symbol_counts)
```

Implemenation in Dash app with interactive visualization

```python
def rANS_decoder(symbol_counts, num_symbols, state):
    cumul_counts = [0]  # Start with 0 as the first cumulative frequency
    total_frequency = sum(symbol_counts)
    for count in symbol_counts:
        cumul_counts.append(cumul_counts[-1] + count)
    def c_inv(y):
        return bisect_right(cumul_counts, y) - 1
    output = []  # List to store decoded symbols
    for _ in range(num_symbols):
        slot = state % total_frequency  # Find the slot in the cumulative frequency range
        s = c_inv(slot)                 # Get the symbol corresponding to the slot
        Fs = symbol_counts[s]           # Frequency of the symbol
        Cs = cumul_counts[s]            # Cumulative count of the symbol
        output.append((s,state))                # Append decoded symbol to the output
        state = (state // total_frequency) * Fs + (slot - Cs)  # Update the state
    return output, state
```

Implementation in Dash app with interactive visualization

Streaming-rANS

Streaming-rANS

To avoid using large number arithmetic, in practice stream variants are used: which enforce by renormalization: sending the least significant bits of x to or from the bitstream (usually L and b are powers of 2). [4]

Example of the problem //TODO: add example

- ANS (Asymmetric Numeral Systems) uses a single integer x, known as the "state," to represent the encoded data.
- Over time, as more symbols are encoded, the state x can grow arbitrarily large. This can lead to:
  - Overflow: If the value exceeds the precision limits of your system (e.g., a 32-bit or 64-bit register).
  - Inefficient Encoding: A larger x means you need more memory or bits to represent it.
- Using the Cfunction from the previous section to encode succeeding symbols, we can encode a sequence into a very large natural number x, which can be stored as a length $\lfloor lg(x) \rfloor + 1$ bit sequence. [2]

$[L, H]$

- How to determine L and H?

$[L, H]$

- How to determine L and H?
-

$[L, H]$

- How to determine L and H?
-
-

Method 1 [2]: ANS encoding step of symbol sfrom state x [2]

while $x > maxX[s]$ **do** {fff}

  writeDigit(mod(x,b)); $x = \lfloor x/b \rfloor$ {fff}

end while

$x = C(s, x)$ {fff}

Method 2: ANS decoding step from state x [2]

$(s, x) = D(x)$ {fff}

useSymbol(s) {fff}

while $x < L$ **do** {fff}

  $x = b \cdot x +$ readDigit() {fff}

end while

```
    for i in range(10):
        print("Hello, World!")
```

Summary

Summary

- To summarize, this may help you to make a Leipzig university beamer presentation.

Reference

Reference

1. C. Shannon, “A mathematical theory of communication,” The Bell Sys- tem Technical Journal, vol. 27, pp. 379–423, 623–656, July, October
2. J. Duda, K. Tahboub, N. J. Gadgil and E. J. Delp, ”The use of asymmetric numeral systems as an accurate replacement for Huffman coding,” 2015 Picture Coding Symposium (PCS), Cairns, QLD, Australia, 2015, pp. 65-69, doi: 10.1109/PCS.2015.7170048.
3. J. Duda, “Asymmetric numeral systems: entropy coding combining speed of huffman coding with compression rate of arithmetic coding,” arXiv:1311.2540.
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