UNIVERSITÄT
LEIPZIG

# Asymmetric Numeral System

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# Outline

1. Introduction

2. Asymmetric Numeral System

3. rANS

4. Streaming-rANS

5. Summary

6. Reference

# Introduction

## Outline

1. Introduction
   1.1 Background
   1.2 Information Theory
   1.3 Source Coding Theorem
   1.4 Entropy Coding

- 1948: Claude Shannon introduced the concept of information theory in his paper ”A Mathematical Theory of Communication” [1].
- 1950: David Huffman introduced Huffman coding [3].
- 1977: Arithmetic coding was introduced by Robert M. Fano [4].
- 2014: Asymmetric Numeral System (ANS) was introduced by Jarosław Duda [2].
- 2016: Zstandard (Zstd) was introduced by Yann Collet [5].
- 2018: AV1 Image File Format (AVIF) was introduced by Alliance for Open Media [6].
- 2020: JPEG XL was introduced by JPEG Committee [7].

## Information Theory[1]

- Claude Shannon: ”A Mathematical Theory of Communication”
- Self-information
  a symbol of probability $p$ contains $log_2 \frac{1}{p}$ bits of information.
- Shannon Entropy
  Let $X$ be a random variable, which takes values in the set $\mathcal{X}$ and is distributed according to $p$

$$H(X) = \sum_{x \in \mathcal{X}} p(x) log_2 \frac{1}{p(x)}$$

## Source Coding Theorem (Shannon) [1]

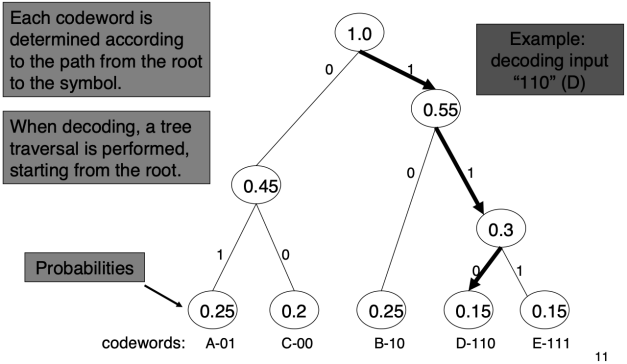
- prerequisite: lossless data compression
- $H(X)$ is compression limit, i.e. lower bound $L_{\text{avg}} \geq H(X)$
  which measures the minimum average number of bits needed to encode a symbol from a data source
- $H(X)$ is achievable, $L_{\text{avg}} \rightarrow H(X)$
  for a sufficiently long sequence of symbols it is possible to achieve an average code length close to $H(X)$
- Shannon's Source Coding Theorem [1]

$$N \cdot H(X)$$

## Entropy Coding

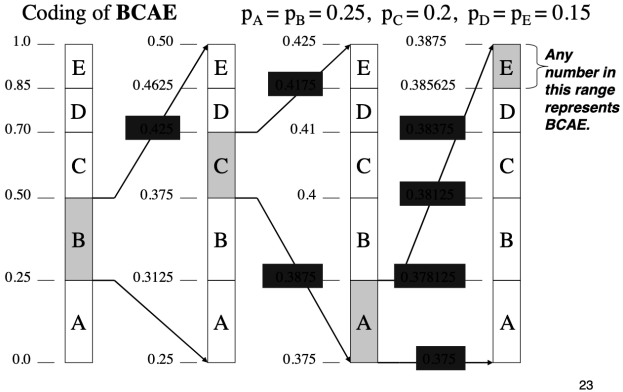
- Implementation of Shannon's Source Coding Theorem
- High-probability symbols get shorter codes
- Low-probability symbols get longer codes.
- A method that attempts to approach the lower bound $H(X)$
- Huffman Coding: ZIP, PNG
- Arithmetic Coding: JPEG, H.264, HEVC.
- Asymmetric Numeral System (ANS): Zstandard, AVIF, JPEG XL

## Huffman Coding




source: http://www.math.tau.ac.il/ dcor/Graphics/adv-slides/entropy.pdf

## Arithmetic Coding

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# Asymmetric Numeral System

# Outline

2. Asymmetric Numeral System
- 2.1 Motivation
- 2.2 Symbol spread function
- 2.3 Representation of information
- 2.4 Symmetric or Asymmetric
- 2.5 Performance

## Motivation

ANS combines the compression ratio of arithmetic coding (which uses a nearly accurate probability distribution), with a processing cost similar to that of Huffman coding [4].

## Basic Idea

- Symbol spread function

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- Representation of information
  (represents an entire message as a single number, similar to Arithmetic Coding) –> example: representation of {2,0,2,5,1,8}

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## Basic Idea

- Symbol spread function
- Representation of information
  (represents an entire message as a single number, similar to Arithmetic Coding) –> example: representation of {2,0,2,5,1,8}
- Symmetric or Asymmetric
- rANS and tANS

## Symbol spread function

- the function that determines the split of the $\mathbb{N}$ subsets corresponding to different symbols [1]

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- $\bar{s} : \mathbb{N} \to \mathcal{A}, \bar{s}(x) = \text{mod}(x, b) = s$

## Symbol spread function

- the function that determines the split of the $\mathbb{N}$ subsets corresponding to different symbols [1]
- $\bar{s} : \mathbb{N} \to \mathcal{A}, \bar{s}(x) = \text{mod}(x, b) = s$
- In the standard binary system example: $\bar{s}(x) = \text{mod}(x, 2)$
  e.g. splits natural numbers into subsets of even/odd numbers [2]

## Symbol spread function (Exa. 1)

$S = \{0, 1\}$

### Exa. 1: Standard Binary System

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| s=0 | | | | | | | | | | | | | | | | | | | |
| s=1 | | | | | | | | | | | | | | | | | | | |

**Fig.1**

## Symbol spread function (Exa. 2)

$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$x' = 10x + s$

| $\mathbb{N}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | | 202 | | 2025 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $s = 0$ | | | | | | | | | | | | | | | | | | | | | | | | | |
| $s = 1$ | | | | | | | | | | | | | | | | | | | | | | | | | |
| $s = 2$ | | | | | | | | | | | | | | | | | | | | | | | | | |
| $s = 3$ | | | | | | | | | | | | | | | | | | | | | | | | | |
| $s = 4$ | | | | | | | | | | | | | | | | | | | | | | ... | | ... | |
| $s = 5$ | | | | | | | | | | | | | | | | | | | | | | | | | |
| $s = 6$ | | | | | | | | | | | | | | | | | | | | | | | | | |
| $s = 7$ | | | | | | | | | | | | | | | | | | | | | | | | | |
| $s = 8$ | | | | | | | | | | | | | | | | | | | | | | | | | |
| $s = 9$ | | | | | | | | | | | | | | | | | | | | | | | | | |

**Fig.2**

## Representation of information (Exa. 2)

- a finite sequence of symbols/digits:

$$\mathcal{A} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

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$$S_{in} = (2, 0, 2, 5)$$

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$$\mathcal{A} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

- the sequence of symbols/digits to be encoded:

$$S_{in} = (2, 0, 2, 5)$$

- How can we represent this sequence as a single number?
- The easiest way is to concatenate the symbols/digits:

$$x_{out} = 2025$$

## Representation of Information (Exa. 2)

- encoding the sequence of digits (2,0,2,5,1,8)

$$
\begin{aligned}
x_{\text{init}} &= 0 \\
x_1 &= x_{\text{init}} \times 10 + 2 = 2 \\
x_2 &= x_1 \times 10 + 0 = 20 \\
x_3 &= x_2 \times 10 + 2 = 202 \\
x_{\text{out}} &= x_3 \times 10 + 5 = 2025
\end{aligned}
$$

## Representation of Information (Exa. 2)

– What is the encoding function?

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- Encoding function

$$x_{i+1} = x_i \times 10 + s_i$$

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- What is the encoding function?
- Encoding function

$$x_{i+1} = x_i \times 10 + s_i$$

- $x'$ is the $x$-th appearance of symbol $s$

$$x' = C(s, x) = x \cdot 10 + s$$

# Representation of Information (Exa. 2)

$x' = 10x + s$

$s = 0$ $s = 2$ $s = 2$ $s = 5$

| $\mathbb{N}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | | 202 | | 2025 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $s = 0$ | 0 | | | | | | | | | | 1 | | | | | | | | | | 2 | | | | |
| $s = 1$ | | 0 | | | | | | | | | | 1 | | | | | | | | | | | | | |
| $s = 2$ | | | 0 | | | | | | | | | | 1 | | | | | | | | | | | | |
| $s = 3$ | | | | 0 | | | | | | | | | | 1 | | | | | | | | | | | |
| $s = 4$ | | | | | 0 | | | | | | | | | | 1 | | | | | | | ... | | ... | |
| $s = 5$ | | | | | | 0 | | | | | | | | | | 1 | | | | | | | | | |
| $s = 6$ | | | | | | | 0 | | | | | | | | | | 1 | | | | | | | | |
| $s = 7$ | | | | | | | | 0 | | | | | | | | | | 1 | | | | | | | |
| $s = 8$ | | | | | | | | | 0 | | | | | | | | | | 1 | | | | | | |
| $s = 9$ | | | | | | | | | | 0 | | | | | | | | | | 1 | | | | | |

**Fig.3**

# Representation of Information (Exa. 1)

**Exa. 1: Standard Binary System**

**X' = 2X + s**

s=1 s=0 s=0 s=1

| X' | | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| X | s=0 | 0 | | 1 | | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | | 9 |
| | s=1 | | 0 | | 1 | | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | |

**Fig.4**

# **Asymmetric Numeral System** | Representation of information |

**Exa. 1: Standard Binary System**

**X' = 2X + s**

s=1 s=0 s=0 s=1

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| X' | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| x s=0 | 0 | | 1 | | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | | 9 |
| x s=1 | | 0 | | 1 | | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | |

**Exa. 2: Standard Decimal System**

**X' = 10X + s**

s=2 s=0 s=2 s=5

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | | 202 | | 2025 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| s=0 | 0 | | | | | | | | | | 1 | | | | | | | | | | 2 | | | | |
| s=1 | | 0 | | | | | | | | | | 1 | | | | | | | | | | | | | |
| s=2 | | | 0 | | | | | | | | | | 1 | | | | | | | | | | | | |
| s=3 | | | | 0 | | | | | | | | | | 1 | | | | | | | | | | | |
| s=4 | | | | | 0 | | | | | | | | | | 1 | | | | | | | ... | | ... | |
| s=5 | | | | | | 0 | | | | | | | | | | 1 | | | | | | | | | |
| s=6 | | | | | | | 0 | | | | | | | | | | 1 | | | | | | | | |
| s=7 | | | | | | | | 0 | | | | | | | | | | 1 | | | | | | | |
| s=8 | | | | | | | | | 0 | | | | | | | | | | 1 | | | | | | |
| s=9 | | | | | | | | | | 0 | | | | | | | | | | 1 | | | | | |

**Fig.5**

# **Asymmetric Numeral System** | Symmetric or Asymmetric |

**Exa. 1: Standard Binary System**

**X' = 2X + s**

s=1 s=0 s=0 s=1

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| X' | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| X s=0 | 0 | | 1 | | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | | 9 |
| X s=1 | | 0 | | 1 | | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | |

**Exa. 2: Standard Decimal System**

**X' = 10X + s**

s=2 s=0 s=2 s=5

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | | 202 | | 2025 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| s=0 | 0 | | | | | | | | | | 1 | | | | | | | | | | 2 | | | | |
| s=1 | | 0 | | | | | | | | | | 1 | | | | | | | | | | | | | |
| s=2 | | | 0 | | | | | | | | | | 1 | | | | | | | | | | | | |
| s=3 | | | | 0 | | | | | | | | | | 1 | | | | | | | | | | | |
| s=4 | | | | | 0 | | | | | | | | | | 1 | | | | | | | ... | | ... | |
| s=5 | | | | | | 0 | | | | | | | | | | 1 | | | | | | | | | |
| s=6 | | | | | | | 0 | | | | | | | | | | 1 | | | | | | | | |
| s=7 | | | | | | | | 0 | | | | | | | | | | 1 | | | | | | | |
| s=8 | | | | | | | | | 0 | | | | | | | | | | 1 | | | | | | |
| s=9 | | | | | | | | | | 0 | | | | | | | | | | 1 | | | | | |

**Fig.6**

## Exa. 3: Asymmetric Binary System

- Let $\mathcal{A} = \{0, 1\}$
- $Pr(0) = 1/4, Pr(1) = 3/4$

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- Let $\mathcal{A} = \{0, 1\}$
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- a cycle of length 4, which contains 1 zero and 3 ones

## Exa. 3: Asymmetric Binary System

- Let $\mathcal{A} = \{0, 1\}$
- $Pr(0) = 1/4, Pr(1) = 3/4$
- a cycle of length 4, which contains 1 zero and 3 ones
- $x' = C(s, x) \approx x/p_s$

# Exa. 3: Asymmetric Binary System

**Exa. 3: Asymmetric Binary System**

$x' \approx x/Pr(s)$ | $Pr(0) = 1/4, Pr(1) = 3/4$

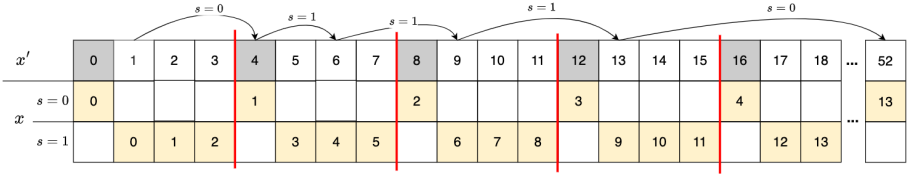



**Fig.7**

We refine even/odd number by changing their densities

## Symmetric or Asymmetric

- Symmetric: uniform distribution
- Asymmetric: non-uniform distribution
  - To change the symmetric behavior so that the information added to $x$ by coding a new symbol depends on the probability distribution of the symbols
  - $log_2(x)$ bits
  - A symbol of probability pcarries $log_2(1/p)$ bits of information
  - How many bits of new information? $log_2 x + log_2(1/p) \rightarrow log_2 x'$
  - $log_2 x' = log_2 x + log_2(1/p) = log_2(x/p) \Rightarrow x' \rightarrow x/p$

## Performance

- f

# rANS

# Outline

3. rANS
- 3.1 Key Concepts
- 3.2 Exa. 3
- 3.3 Notations
- 3.4 Encoding Function
- 3.5 Implementation of Encoder
- 3.6 Decoding Function
- 3.7 Implemenation of Decoder

## rANS

- ”r” = ”range”: similar to range coding (RC)
- symbol spread function
- representation of information
- asymmetric

## Exa.3

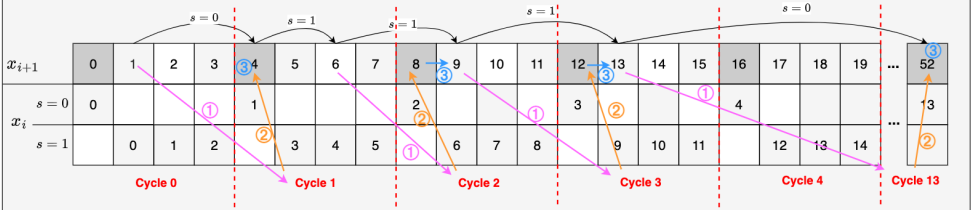
How to encode 101110? What is the encoding function $C(s, x)$?

$x' = C(s, x) \approx x/p_s$

## Some notations

- $\mathcal{A} = \{s_1, s_2, \ldots, s_k\}$ is the alphabet set
- $s$ is the symbol, if it is used in subscript: $s$ is the current symbol, $s-1$ is the previous symbol, $s+1$ is the next symbol
- $F_s$ is the absolute frequency of symbol $s$
- $m = \sum_{s \in \mathcal{A}} F_s$ is the total frequency
- $c_s = \sum_{s \in \mathcal{A}} F_{s-1}$ is the cumulative frequency ($c_{s+1}$ is the cumulative frequency of the next symbol)
- $p_s = F_s / m$ is the probability of symbol $s$
- $x_i$ is the current state (next state: $x_{i+1}$, previous state: $x_{i-1}$)

$x_{i+1} \approx x_i / Pr(s)$ | $Pr(0) = 1/4, Pr(1) = 3/4$




Given current state $x_i$ and next symbol $s$. How to find next state $x_{i+1}$, which is the $x_i$-th appearance of symbol $s$ ?

| STEPS / EXAMPLE | $Pr(0) = 1/4, Pr(1) = 3/4$ | | General |
|---|---|---|---|
| | $s = 0$ | $s = 1$ | |
| Step ①: Find cycle number $k$ | $k = \lfloor x_i/1 \rfloor$ | $k = \lfloor x_i/3 \rfloor$ | $k = \lfloor x_i/F_s \rfloor$ |
| Step ②: Navigate to the beginning (first number) of that cycle | $4 \cdot k$ | $4 \cdot k$ | $m \cdot k$ |
| Step ③: Find the position of the symbol $s$ in that cycle | part 1: first of one<br>+ 0 | part 1:<br>+ mod$(x_i, 3)$ | part 1:<br>+ mod$(x_i, F_s)$ |
| - part 1: position of the symbol $s$ in the sequence of the same symbol<br>- part 2: how many other symbols before the symbol $s$ | part 2: no other symbols<br>+ 0 | part 2: only one other symbol (0) and with one of that symbol in one cycle<br>+ 1 | part 2: only one other symbol (0) and with one of that symbol in one cycle<br>$+c_s$ |
| | $x_{i+1} = m \cdot k$ | $x_{i+1} = 4 \cdot \lfloor x_i/3 \rfloor + \text{mod}(x_i, 3) + 1$ | $x_{i+1} = m \cdot \lfloor x/F_s \rfloor + \text{mod}(x, F_s) + c_s$ |

$$x_{i+1} = C(s, x_i) = m \cdot \lfloor x_i/F_s \rfloor + \text{mod}(x_i, F_s) + c_s$$

**Fig.8**

## Encoding Function [2] [5]:

$$x_{i+1} = m \cdot \lfloor x_i / F_s \rfloor + \mathrm{mod}(x_i, F_s) + c_s$$

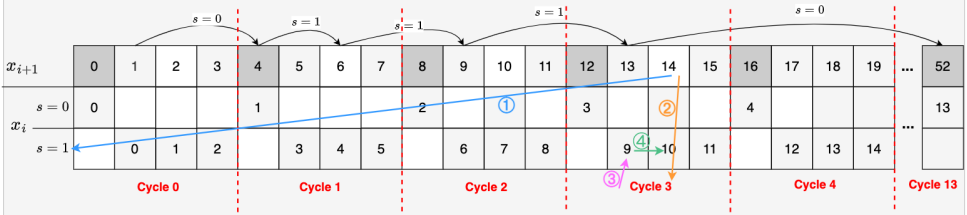
## implementation with python (dash app)

```python
def rANS_encoder(symbol_counts, s_input):
    cumul_freq = [] # c_s
    sum_freq = 0 # m
    x = 0 # initial state
    output = [] # list to store encoded (symbols, state)

    for count in symbol_counts:
        cumul_freq.append(sum_freq)
        sum_freq += count
    for s in s_input:
        F_s = symbol_counts[s]
        c_s = cumul_freq[s]
        x = sum_freq * (x // F_s) + (x % F_s) + c_s
        output.append((s, x))

    L_avg = math.ceil(math.log(x) / math.log(2.0)) / len(s_input)
    return output, x, L_avg, entropy(symbol_counts)
```

$x_{i+1} \approx x_i / Pr(s)$ $\boxed{Pr(0) = 1/4, Pr(1) = 3/4}$




Given current state $x_i$. How to find previous state $x_{i-1}$, where $x_i$ is the $x_{i-1}$-th appearance of symbol $s$ ?

| STEPS / EXAMPLE | $Pr(0) = 1/4, Pr(1) = 3/4$ | | General |
|---|---|---|---|
| | $s = 0$ | $s = 1$ | |
| Step ① : Find the symbol $s$ for the state $x_i$ | if $x_i \bmod 4 = 0$ | otherwise or if $x_i \bmod 4 = 1, 2, 3$ | $s$ such that $c_s \leq \mathrm{mod}(x_i, m) < c_{s+1}$ |
| Step ②: Find the cycle $k$ of the current state $x_i$ | $k = x_i/4$ | $k = \lfloor x_I/4 \rfloor$ | $k = \lfloor x_i/m \rfloor$ |
| Step ③: Find the first number of that symbol $s$ in that cycle $k$ | $x_i/4 \cdot 1$ | $\lfloor x_i/4 \rfloor \cdot 3$ | $\lfloor x_i/m \rfloor \cdot F_s$ |
| Step ④: Allocate to the exact position | $x_{i-1} = x_i/4 \cdot 1 + 0 - 0$ | $x_{i-1} = \lfloor x_i/4 \rfloor \cdot 3 + 2 - 1$ | $x_{i+1} = \lfloor x_i/m \rfloor \cdot F_s + \mathrm{mod}(x_i, F_s) - c_s$ |

1. $s$ such that $c_s \leq \mathrm{mod}(x_1, m) < c_{s+1}$
2. $x_{i-1} = D(x_i) = \lfloor x_i/m \rfloor \cdot F_s + \mathrm{mod}(x_i, F_s) - c_s$

**Fig.9**

## Decoding Function [2] [5]:

- $s$

$$s \text{ such that } c_s \leq \text{mod}(x_i, m) < c_{s+1}$$

- $x_{i-1}$

$$x_{i-1} = D(x_i) = \lfloor x_i / m \rfloor \cdot F_s + \text{mod}(x_i, F_s) - c_s$$

## implementation with python (dash app)

```python
def rANS_decoder(symbol_counts, x):
    cumul_freq = []  # c_s
    sum_freq = 0 # m
    output = []  # List to store decoded symbols and states
    # compute m and c_s
    for count in symbol_counts:
        cumul_freq.append(sum_freq)
        sum_freq += count
    while x > 0:
        slot = x % sum_freq  # Find the slot in the cumulative frequency range
        s = bisect_right(cumul_freq, slot) - 1 # Binary search to find the symbol s (slot).
        F_s = symbol_counts[s]          # Frequency of the symbol
        c_s = cumul_freq[s]             # Cumulative count of the symbol
        output.append((s,x))            # Append decoded symbol to the output
        x = (x // sum_freq) * F_s + x % sum_freq - c_s  # Update the state
    return output, state
```

# Streaming-rANS

# Outline

4. Streaming-rANS
- 4.1 Problem
- 4.2 Idea
- 4.3 Solution
- 4.4 Implementation of Streaming-rANS

## Streaming-rANS

To avoid using large number arithmetic, in practice stream variants are used: which enforce by renormalization: sending the least significant bits of x to or from the bitstream (usually L and b are powers of 2). [4]

Example of the problem //TODO: add example

- ANS (Asymmetric Numeral Systems) uses a single integer x, known as the "state," to represent the encoded data.
- Over time, as more symbols are encoded, the state x can grow arbitrarily large. This can lead to:
  - Overflow: If the value exceeds the precision limits of your system (e.g., a 32-bit or 64-bit register).
  - Inefficient Encoding: A larger x means you need more memory or bits to represent it.
- Using the C function from the previous section to encode succeeding symbols, we can encode a sequence into a very large natural number $x$, which can be stored as a length $\lfloor lg(x) \rfloor + 1$ bit sequence. [2]

## Renormalization in Range Coding

- Range Coding
  - 1
  - 2
- we use the similar idea for rANS
  - range $[L, H]$
  - Encoding: shrink the current state $x$ until it is in the range $[L, H]$
  - Decoding: expand the current state $x$ until it is in the range $[L, H]$
- In AC, renormalization is used to allow finite precision, an analogous approach should be used for ANS. Specifically, we will enforce xto remain in some fixed range I by transferring the least significant bits to the stream. For generality, let us transfer a base $b$ digit, for example $b = 2$ means transferring single bits, while $b = 2^8$ means transferring complete bytes.

## Methods from Jarek Duda: Method 1 - Decoding [2]

**ANS decoding step from state *x***

$(s, x) = D(x)$ *# decode the current state x*

useSymbol($s$)

**while** $x < L$ **do** *# if x is too small that not in the range [L,H]*

*# expand the x by reading the outstreamed digits back from the stream (an array)*

expand$(x, b) = b \cdot x +$ readDigit$() \rightarrow x$

**end while**

*# return the new state*

## What is $L, H$ ?

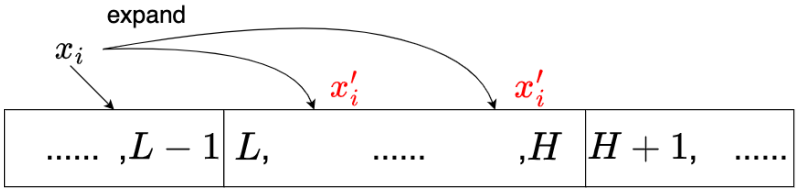



**Fig 11**

## What is $L, H$ ?

- $x_i = b \cdot x_{i-1} +$ bit
- Either $x_i$ in $[L, H]$ or $x_{i-1}$ in $[L, H]$. Assume $x_{i-1}$ in $[L, H]$.
  - $x_{i-1} >= L$
  - $x_i >= b \cdot L + bit >= b \cdot L$
  - $x_i$ can't be in $[L, H]$
  - derive: $H <= b \cdot L - 1$
- $[L, H]$, where $H <= b \cdot L - 1$ (Constraint 1)
- $I = \{L, ..., b \cdot L - 1\}$, which is defined by Jarek Duda $b$-**unique interval** and $L \in \mathbb{N}$
- This will ensure that the loop in Method 1 will always return to the original value

## Methods from Jarek Duda: Method 2 - Encoding [2]

shrink the current state $x$ by stream out the least significant bit ($x$ will be stored in binary and we take out the bit from right side)

**ANS encoding step of symbol sfrom state $x$**

*# once the current state x is greater than max value that we define later*

**while** $x > maxX[s]$ **do**

*# write the least significant digit to the stream (stream could be an array in implementaion )*

writeDigit(mod($x$, $b$))

shrink$(x, b) = \lfloor x/b \rfloor \rightarrow x$ *# shrink the x*

**end while**

$x = C(s, x)$ *# encode with the shrunk x*

## What is $maxX[s]$?

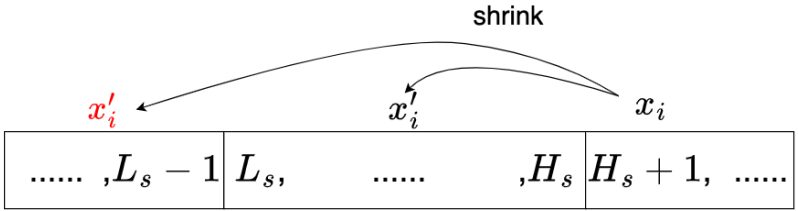



**Fig 10**

## What is $maxX[s]$?

- to make sure not to jump over the range
- Assume that $x_i = H_s + 1$ and it will be shrinked to $x'_i = \lfloor x_i/b \rfloor$
- $x'_i = \lfloor x_i/b \rfloor = \lfloor (H_s + 1)/b \rfloor >= L_s$
- $L_s <= (H_s + 1)/b <= \lfloor (H_s + 1)/b \rfloor < (H_s + 1)/b + 1$
- $H_s >= b \cdot L_s - 1, \forall s \in \mathcal{A}$ (Constraint 2)

## Final Form of $[L, H]$ in practice

- Contraints

$$[L, H]: \text{ where } H <= b \cdot L - 1$$
$$[L_s, H_s] : H_s >= b \cdot L_s - 1, \forall s \in \mathcal{A}$$

- In practice ($b = 2^1$): we choose $L = k \cdot m$, $L_s = k \cdot F_s$ ($k \in \mathbb{N}$)

$$\mathcal{I} = [km, 2 \cdot km - 1]$$
$$\mathcal{I}_s = [kF_s,\ 2 \cdot kF_s - 1] \text{ for } \forall s \in \mathcal{A}$$

## implementation with python (dash app)

```python
def rANS_stream_encoder(symbol_counts, s_input):
    # Init: c_s, m, x, output, bit_stream (omit the code here)
    for s in s_input:
        F_s = symbol_counts[s]
        c_s = cumul_freq[s]
        L_s = F_s # k = 1
        maxX_s = 2 * L_s - 1
        out_bits = ""

        while x > maxX_s:
            out_bits += str(x % 2)
            x = x // 2
        x = (x // F_s) * sum_freq + (x % F_s) + c_s
        bit_stream += out_bits
        output.append((s, x, out_bits, bit_stream))

    return output, x, bit_stream, L_avg, entropy(symbol_counts)
```

## implementation with python (dash app)

```python
def rANS_stream_decoder(symbol_counts, x, bit_stream, num_symbols):
    # Init: c_s, m, x, output, bit_stream (omit the code here)
    bit_stream = list(map(int, bit_stream.split(',')))  # Convert bit_stream to a list of integers
    for _ in range(num_symbols):
        slot = x % sum_freq
        s = bisect_right(cumul_freq, slot) - 1
        F_s = symbol_counts[s]
        c_s = cumul_freq[s]
        output.append((s, x))
        x = (x // sum_freq) * F_s + slot - c_s

        while x < sum_freq and len(bit_stream) > 0:
            x = x * 2 + bit_stream.pop()

    return output, x
```

# Summary

## Summary

- To summarize, this may help you to make a Leipzig university beamer presentation.

# Reference

# Reference


1. C. Shannon, “A mathematical theory of communication,” The Bell Sys- tem Technical Journal, vol. 27, pp. 379–423, 623–656, July, October
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3. J. Duda, “Asymmetric numeral systems: entropy coding combining speed of huffman coding with compression rate of arithmetic coding,” arXiv:1311.2540.
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